# Evolution of Quantum Correlations in Quantum Algorithms

**A Thesis**

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BS-MS Dual Degree Programme

by

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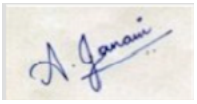
# Certificate

This is to certify that this dissertation entitled Evolution of Quantum Correlations in Quantum Algorithmstowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Janani A at Indian Institute of Science Education and Research under the supervision of Dr. T S Mahesh, Professor, Department of Physics , during the academic year 2019-2020.



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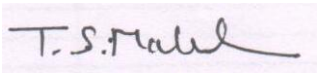

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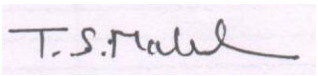

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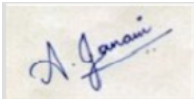
*To my family who always have my back so that I can face forward against every challenge*

# Declaration

I hereby declare that the matter embodied in the report entitled Evolution of Quantum Correlations in Quantum Algorithms are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. T S Mahesh and the same has not been submitted elsewhere for any other degree.



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# Abstract

Quantum systems are often correlated in classically inaccessible ways and hence non-classical correlations in a system are signatures of it being genuinely quantum. Quantum correlations are precious resources for quantum information processing and need to be preserved. Hence it's useful to study their dynamics and evolution, and employ methods to protect them from incoherence and decoherence during computational processes. Some of the commonly studied quantum correlations include entanglement, contextuality, Leggett-Garg inequality, quantum discord, etc. Quantum discord is a quantum correlation that measures the maximum quantum mutual information that can be extracted from a system, by measuring classical mutual information in two different ways, one without measurement and the other via measurement.

Grover's algorithm or the quantum search algorithm introduced by Lou Grover in 1996 [14], executes a search in an unstructured database to find solution elements that satisfy particular conditions. It provides a quadratic speed up over all its classical counterparts and is widely used as a subroutine to speed up many classical algorithms, and in accordance with phase estimation, forms the basis for Quantum Counting.

Dynamical Decoupling (DD) is a technique used to reduce decoherences generated during quantum information processing tasks. It performs a series of rapid flips on the system to decouple it from the environment and refocuses their interactions. DD can be integrated into quantum gates and operations using optimal control techniques to make them more robust. It is advantageous compared to other fault tolerant schemes and decoherence free subspaces in that they don't need separate ancilla qubits or separate manual splicing of gates[25].

In this project, the evolution of quantum discord in Grover's algorithm for a two qubit system has been studied. Then, a dephasing noise is introduced into the system to realistically observe the evolution of discord during various stages of the algorithm. DD is integrated into the gate operations of the Grover's algorithm, and the evolution of discord for different DD schemes based on $\pi$ and non-$\pi$ pulses are simulated. Realistic pulses are seldom perfect, so this is taken into account by simulating the evolution of discord with DD in the presence of radiofequency field inhomogeneties (RFI).

It was experimentally found that some DD sequences based on $\pi/2$ pulses performed

as well as, if not, better than DD based on $\pi$ pulses [1]. DD sequences based on different non-$\pi$ pulse angles were tested for their performance in decoherence suppression in Grover's algorithm through simulations. Unlike the conventional notion that dynamical decoupling is achieved only by $\pi$ pulses, we show by numerical analysis that if properly incorporated into control sequences, even non-$\pi$ pulses can bring about efficient dynamical decoupling, for low noise amplitudes.

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# Chapter 1

# Introduction

## 1.1 Quantum Correlations

### 1.1.1 Relevance

In information processing, certain processors that displayed genuinely quantum behaviour were faster and more efficient than their classical counterparts. The relevance of quantum correlations in current day quantum information processing emerges from the quest to understand this feature. There was no definitive way to identify and categorize a system as "quantum", especially for instances like composite systems, and the correlations in their subsystems.

The concept of superpositions previously existed in major classical theories like waves and electromagnetism, and was insufficient to be considered as an exclusive characteristic of a system exhibiting quantum mechanical behaviour. Consequently, in the early development of quantum information processing, entanglement was viewed as the feature that gave the distinct "quantum" advantage for some processors over classical ones. This had several implications. Separable states, for instance, that have no entanglement, would be of no use in quantum information processing.

However, Knill and Laflamme in 1998 proposed the DQC1 algorithm [8]. It could estimate the trace of a unitary matrix quicker than any classical algorithm known as of then. It

did not invoke the necessity of entanglement, and provided an exponential improvement in the efficiency compared to a classical computer. These results spurred debates about the necessity of entanglement as a critical and indispensable resource for quantum information processing. Furthermore, NMR quantum information processing was found to be an efficient test bed for quantum algorithms [10], despite the purity of typically used spin systems being too low to show sufficient entanglement.

The resolution to the question of entanglement being the source for all quantum enhancement, came from the proposal of another kind of quantum correlation named quantum discord [4] [5] [6]. It showed that there were non-classical correlations not accounted for by entanglement, and that even separable states could contain non-classical correlations. The comprehensive and mathematical explanation was given by Datta [26]. They calculated the entanglement for the DQC1 algorithm and showed it was vanishingly small, and it decayed further as we increase the number of qubits. The discord also consistently scaled with the efficiency displayed by the algorithm. This conclusively proved that it was discord rather than entanglement that attributed to the quantum speed up of the algorithm [3].

This led to the application of quantum discord to various different algorithms and problems in quantum computation, and set about the search for non-classical correlations other than entanglement as a resource for quantum information processing. These non-classical correlations are manifestations of the "quantumness" in systems, and can be measures that attribute to the efficiency of quantum processors over the classical ones. Hence, they are regarded as very important resources for quantum information processing. Today, there are various measures of quantum correlations that are extensively studied, like discord, contextuality, deficit, Leggett-Garg inequality, etc [7][12]. They are of interest in many major directions of research like quantum information, quantum algorithms, quantum thermodynamics, many-body physics and open system dynamics[7]. Their evolution during the flow of quantum algorithms can help us understand how decoherence and inhomogeneities in the environment can affect the functioning of the algorithm. It can also serve as a test bed for investigating the effectiveness of different techniques employed to suppress the decoherences and incoherence that deter the smooth and ideal functioning of quantum algorithms.

### 1.1.2 Quantum correlations in a composite system

Pure states are usually uncorrelated if they are separable and entangled if not separable. However, mixed states have several layers of correlations, both classical and non-classical. They can be broadly classified as shown in Figure 1.1, and enable quantum information processing on various levels. Among the quantum correlations, a fraction of the entangled correlations are steerable - a steerable state can be manipulated indirectly, or "steered" by operating on a state it is entangled with. If "steering" the two entangled states is possible even when the states are far-away, then the correlations are said to be non-local.

Each of these non-classical correlations have a wide range of applications that can enable classically impossible tasks, like quantum cryptography, quantum teleportation, dense coding, etc. [7][18]

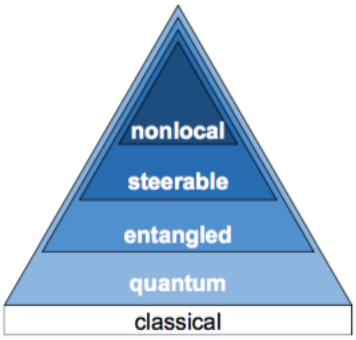


Figure 1.1: The hierarchy of correlations in quantum systems. Source [7]

In a composite system, there exist observables that are not direct combinations of those of their constituent subsystems. A system with $M$ states that is represented by a Hermitian matrix of order $M$, has $M^2$ real parameters, and $M^2 - 1$ linearly independent observables.

Consider two systems with $M$ and $N$ states that are combined to make a composite system with MN states. Then the composite system has $M^2N^2 - 1$ linearly independent observables and $M^2N^2$ parameters. But the total sum of linearly independent observables in the constituent systems $M$ and $N$ are only $(M^2 - 1) + (N^2 - 1) = M^2 + N^2 - 2$. This shows that a composite system has numerous more non-trivial observables than the constituent subsystems combined. This shows how a composite system involves more information than the sum of its constituents [22].

This excess information which is non-classical resides in the *quantum correlations* and involves the phases of the states. In density matrices, they exist in the off-diagonal elements.

## 1.2 Quantum Computation and Quantum Algorithms

The inception of quantum computation was in early 1980s. Paul Bienoff and Richard Feynman proposed that for quantum systems to be simulated accurately and efficiently, it was necessary to have quantum computers that were built with quantum mechanical parts governed by the laws of quantum mechanics [18]. One simple reason of many, could be attributed to the exponential growth of quantum systems. Storing the encoded data and its exponentially growing components would overwhelm classical computers. However, if they were to be encoded in quantum computers that are built to exploit the unique, non-classical features of the quantum mechanical constructs that they are governed by, then they might be capable of processing large amounts of data in polynomial time.

The intentions to efficiently simulate quantum systems might have commenced the need to develop quantum computers, but the computational power they promised progressed their study to greater heights. Quantum algorithms were developed for a multitude of uses in fields like information theory, cryptography, mathematics and language theory.

One of the most fundamental and attractive features of using quantum algorithms is *Quantum Parallelism*. Quantum Parallelism is the happy consequence of the capacity of quantum states to exist in superpositions. This is highly advantangeous because it allows simultaneous but independent processing of multiple states present in a superposition by only acting on the single superposition state. This ability of a quantum register to simultaneously perform a task on multiple states at once, contributes significantly to the efficiency of quantum information processing, and is termed, quantum parallelism. However, the unhappy consequence of this is that the processed states are also superposed, and the efficiently obtained information needs to be carefully extracted using measurements that do not destroy it as the superposition collapses.

Quantum computation needs more than just quantum parallelism to be useful. One of the earliest quantum algorithms to be proposed resolved this dilemma - The Deutsch-Jozsa algorithm (1992) combines quantum parallelism with *interference* [8]. It also proved that

it was possible to successfully use a quantum algorithm to solve a problem, and obtain the result much faster than a classical computer could simulate it. The principles behind the conception and working of this algorithm sowed the seeds for many other impressive quantum algorithms.

There was a turning point to the relevance of quantum algorithms when Peter Shor gave efficient algorithms to solve prime factorization of integers and discrete logarithms [11]. Both these problems were thought to be computationally hard and hence formed the basis of many cryptographic techniques and cybersecurity measures. But Shor's algorithm provided a solution that gave an exponential increase in speed over the fastest known classical algorithms. This raised the stakes of developing a full-scale, reliable quantum computer, both for the efficiency it offered and its implications on the field of cybersecurity [21].

But the study of quantum computation algorithms are motivated by reasons apart from the speedy computation it offers. Miniaturization has made current day processors in classical computers powerful. Their development has made them available in sizes of the order of microns, where quantum properties start taking effect. Chip-makers go to great lengths to prevent and suppress these effects, but it would be more beneficial to take advantage of them instead. Quantum computation has shown potential to raise the power and limits of modern day computation. Probing into its finer workings may also provide insight into the finer workings of quantum mechanics.

Broadly classifying quantum algorithms, some of the significant ones are as listed below[18]:

- Quantum Search Algorithm

  The quantum search algorithm[14] is a search algorithm implemented in an unstructured database. It provides a quadratic speed up over its classically known counterparts. The algorithm implements a search over an unordered set of $N = 2^n$ elements to find solutions that satisfy certain conditions. This algorithm utilizes features of quantum systems that demonstrate how the speed-up over classical computation can be achieved, and hence serves as a good model for introduction to quantum algorithms. The two major principles behind the steps of this algorithm involves the usage of *quantum parallelism* and *amplitude amplification*. This topic is discussed in further detail in Section 2.2.

- Quantum Fourier Transform and related algorithms

  Quantum Fourier Transform is a crucial tool in the formulation of many quantum algorithms like the Deutsch-Jozsa Algorithm and Shor's Algorithm. It enables *phase estimation*, which allows us to calculate the approximate eigenvalue of a given corresponding eigenvector of a unitary operator. Problems with high classical complexity, like the order-finding and factoring problem can be solved by being reduced to phase estimation problems.

- Quantum Simulations

  Quantum simulations try to simulate naturally occurring quantum systems so that we may gain insight into their dynamics. The simulation of quantum systems is an important problem in many fields. In quantum chemistry, computational constraints caused by classical computers make it difficult to accurately simulate the behaviour of even moderately sized molecules, much less the larger ones that occur in biological systems. Quantum simulation can also offer insight into the finer workings of quantum algorithms.

Despite all these advantages, the usage of quantum computers is nowhere nearly as widespread as classical computers are. The progress with physical implementations of quantum computing is steady but slow, and the engineering of quantum computers is still in its infancy. Engineers and architects have to tackle some formidable practical challenges while making quantum computers - and foremost of them is to make the quantum computers resilient to noise and decoherence, that arise from interactions between the system and its environment. Many techniques have been developed and actively employed to protect quantum information from decoherence and incoherence, and they are discussed in further detail in Section 1.3.

## 1.3 Errors and Decoherence

While quantum computation enables many advantageous and classically impossible tasks, this section highlights some of the obstacles and challenges that have held back its implementation on a larger scale. One of the biggest advantages of quantum information is that it can exist in superposition states. But this also lends to one of its biggest disadvantages -

the fragility of quantum information. A qubit is potentially more valuable than a classical bit, but it is also more vulnerable. This is because it can exist in states other than just 0 or 1, and can get affected more non-trivially. Even simple environmental disturbances can cause the superpositions to collapse to a classical state and this risks the loss of all precious information stored. An accidental flip on one qubit may affect all the states contained in the superposition.

However, with the introduction of *quantum error correction principles* [9] and *threshold theorem* [15] it was shown that reliable quantum information processing was indeed possible.

To physically implement an input into an output, an algorithm prescribes appropriate steps to break the computation into suitable components that are carried out by available hardware. The hardware is designed such that it optimizes efficiency and precision in carrying out these tasks. However, any real physical device will show some amount of deviation from ideal behaviour, leading to a mismatch between the theoretically calculated results and experimentally computed results. This is the case for both classical and quantum computers.

In the case of quantum computers, the input is encoded in exponentially many complex amplitudes of the initial state. During a computation, they need to be steered along a specific designated path in the Hilbert space. The dimension of the Hilbert space grows exponentially with the number of qubits. The final state which is the result of the computation is stored in the amplitudes and phases of the constituent states as continuous variables. Hence, it is extremely important to make sure that these constituent components are *phase coherent*, to perform a proper quantum computation.

There are different sources of errors that can affect a quantum computational process, and they need to be given due consideration. These sources dictate the kind of error and how they disturb the states, which in turn helps us identify and develop the appropriate method to correct them:

- The implementation of gate operations may not ideal due to hardware limitations and other imperfections.
- Interactions between the environment and the quantum register may cause various disturbances to the state - like *relaxation* (unwanted transitions) or *decoherence* (decay of phase coherence, i.e., dephasing).

- The quantum system being used while designing the computer, may not be the idealized version in application. This can be in the form of coupling constants that are different from the ideal values, or presence of quantum states that have not been included in the computational Hilbert space.

Understanding and characterizing errors is key to figuring out how to correct them during the course of a computational process. In addition, it is also necessary to measure how the errors cause deviations to the quantum state, and quantitatively measure the difference between the idealized and actual evolution.*Fidelity* and related measures quantify the overlap between the ideal and the actual states for various parameters [12][13][23].

Though it is important to minimize the errors, there is a limit to the precision that can be achieved by error correcting schemes. Hence, it is just as important to prevent and lessen the imperfections in processing, that cause these errors. This can be achieved by:

- Optimizing the classical apparatus controlling the quantum system to minimize errors in the gate operations.
- Making gate operations robust by making it such that the errors in the parameters correct themselves rather than amplify.
- Storing the information in the Hilbert space in areas where the interactions between the system and the environment are affected the least.
- Using error correcting schemes and codes.
- Using schemes to decouple the system and the environment, like dynamical decoupling.

Interference of quantum states is one of the most fundamental causes for the plethora of quantum phenomena seen in quantum systems. For interference to be possible, the states need to have a well defined phase relationship, i.e., they are *coherent*. If the coherence between the states is destroyed by their interactions with the environment, then the states have undergone *decoherence*. In case the decoherence is so rapid that it makes interference impossible, then the behaviour of quantum systems will collapse to their classical limits.

If two states are affected by the environment in different ways, they will decohere. But if they are affected in the same way, then in spite of being coupled to the environment, they

can preserve their coherence. The rate at which decoherence occurs for a given quantum system can determine its viability for being implemented in a quantum computer.

If the environment is also quantum mechanical in nature, the interactions between the system and the environment manifests as correlations between their degrees of freedom. This can lead to the two being highly entangled. If this occurs, then in general, the quantum state is no longer pure but mixed. This mixing of the environmental degrees of freedom and the quantum state hints at loss of information that can be measured. The change in entropy of the density matrix of this product state can be given by *von Nuemann entropy*, which is discussed in further detail in Section 2.1.

In a quantum computational process, the information is distributed over all the qubits of a quantum register unlike in classical computation. Hence, the rate at which information is lost is not determined by the rate of decoherence of a single qubit. The information is affected by decoherence acting on all constituent qubits, and decays equivalently faster.

The rate at which decoherence grows with the number of qubits is dependent on how it is coupled to the environmental system; specifically how the system undergoes decoherence by the environment. *Decoherence free subspaces* in some large quantum systems may contain states that are relatively immune, and undergo decoherence due to environmental noise much slower than the average quantum state.

In practical systems, there is a finite degree of correlations between qubits and the external fields. Based on this, clusters of qubits that are more strongly correlated than the system average can be identified. These clusters and their correlations allow for effective encoding of appropriate schemes to reduce information loss [24].

There are many kinds of errors that can cause decoherence in a quantum system during a computational process. Some of them are stated below:

- Spin-flip errors
- Projection errors
- Continuous errors
- Qubit-related errors

- Phase errors
- Projection errors

Of the above mentioned, phase errors are a topic of relevance for this project. Phase errors are a form of continuous errors, and correspond to a random rotation about the $z$ axis. For a random angle $\theta$ in the range $[0, 2\pi]$, the random rotation is given by

$$P(\eta) = e^{i\eta\theta Z} \tag{1.1}$$

where $\eta$ is the "strength parameter" which controls the mean phase spread by the operator $Z$. Other kinds of errors, error correction mechanisms and error prevention schemes are detailed in [23].

# Chapter 2

# Theory

## 2.1 Quantum Discord

Quantum entanglement was thought to be the key resource that enabled enhanced processing in systems that displayed non-classical behaviour (further detail in Section 1.1.1). However, there were non-classical systems that displayed enhanced processing but had vanishingly small entanglement. This led to the discovery of quantum discord, which was non-zero even for systems with separable states (zero entanglement), and could account for the enhanced processing power in instances when entanglement was absent. Since then, many new quantum correlations with unique features that can be exploited in different ways have been discovered and extensively studied.

Quantum discord is only one of the many quantum correlations that can measure the non-classicality of a system. Two systems that contain more information together than when they are taken separately, are correlated. Quantum discord is the difference between the correlations in a system measured in two classically equivalent ways.

If the system is purely classical, the information shared by the systems would be equal, irrespective of how they are being measured. This is because classical information is locally accessible and information can be obtained without perturbing the system. In other words, it is possible to perform a measurement without altering the density matrix, and the classical correlations can be recovered. However, if a system is not purely classical, there will be a

mismatch in the information, because measurements are basis dependent and can change the state of the system.

Classically, the amount of information contained in a random variable is given by the *Shannon entropy*. It can be interpreted as either the uncertainty before measuring a random variable or the information gained from measuring it. For a random variable $X$, the Shannon entropy is given by:

$$H(X) = -\sum_x p_x log_2 p_x \tag{2.1}$$

where $p_x$ is the probability of the event $X$. $H(X) = 0$ implies that $X$ is determinate, and no new information obtained by measuring it.

Consider a bipartite system containing two subsystems $A$ and $B$. Conditional entropy of $B$ quantifies the uncertainty in the measurement of $B$ when $A$ is known, and is represented by $H(B|A)$. If the information in the entire system is $H(A, B)$ and $H(A)$ is the information contained in the subsystem A, then conditional probability of $B$ can be written as:

$$H(B|A) = H(A, B) - H(A) \tag{2.2}$$

An alternate, equivalent way of defining the conditional entropy would be

$$H(B|A) = \sum_i^a H(B|a = i) \tag{2.3}$$

where

$$H(B|a = i) = -\sum_j p(b_j|a_i) log_2 p(b_j|a_i) \tag{2.4}$$

Given that event $a_i$ has occurred, the probability of the event $b_j$ occurring is the conditional probability $p(b_j|a_i)$. Unlike the previous case, this definition of the conditional entropy requires measurement of one of the subsystems.

We use *von Neumann entropy* as the quantum equivalent of Shannon entropy to define the information contained in a density matrix $\rho$:

$$H(\rho) = -\sum_x \lambda_x log_2 \lambda_x \tag{2.5}$$

where $\lambda_x$ are the eigenvalues of the density matrix $\rho$.

The amount of information shared by the subsystems $A$ and $B$ of the bipartite system is called *mutual information*. It is given by:

$$I(A:B) = H(A) + H(B) - H(A,B) \tag{2.6}$$

This definition of mutual information is both symmetric and measurement independent, i.e, $I(A:B) = I(B:A)$.

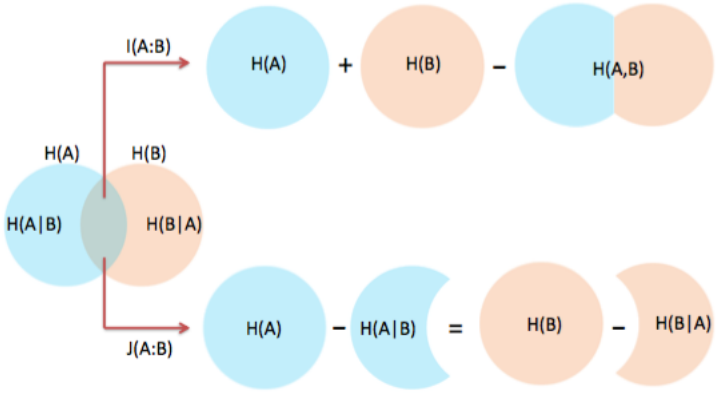


Figure 2.1: Mutual information calculated in two different classically equivalent ways. Source [12]

Another classically equivalent way of defining mutual information as shown in Figure 2.1 would be:

$$J(A:B) = H(B) - H(B|A) = H(B) - \sum_i p^a H(B|a=i) \tag{2.7}$$

but this is basis dependent. To obtain the maximum classical correlation possible from this measure, we maximize it over all possible orthonormal measurement bases $\Pi_i^a$ for $A$, with $\Pi_i^a$ satisfying the orthonormality conditions $\sum_i \Pi_i^a = \mathbb{I}$ and $\Pi_i^a \Pi_j^a = \delta_{ij} \Pi_i^a$

Hence, the non-classical correlation extracted from the total system can be quantified as:

$$D(B|A) = I(A:B) - \max(\prod_i^a J(A:B)). \tag{2.8}$$

This difference was named *discord* by Olliver and Zurek [6]. Even for separable states that have zero entanglement, discord may be non-zero. It can function as the resource that provides quantum enhancement for quantum information processing, like in the case of the DQC1 algorithm. Zero-discord states or "classical" states are hence states in which maximal information about a subsystem can be obtained without any projective measurements that can disturb the correlations in the rest of the system.

However, analytically evaluating the maximized correlation $J(A:B)$ over every possible orthonormalized basis state is non-trivial. The formula for discord has been described for some special classes of states [16], but a closed analytical formula describing quantum discord for a general state does not exist. The brute force method would be to consider as many orthonormal bases as possible, while considering all the symmetries and constraints present in the system.

In general, discord is not a symmetric quantity, i.e. $D(B|A)$ and $D(A|B)$ may differ. Calculation of discord and other related measures have been reviewed in [13][16].

## 2.2 Quantum Search Algorithm

The quantum search algorithm, commonly known as Grover's algorithm, is one of the earliest proposed quantum algorithms. Along with Shor's algorithm, it illustrated how quantum algorithms can significantly speed up the solutions to computationally complex problems for classical computers. It was proposed by Grover in 1996 [14] and conducts a search in an unstructured database to find a set of solutions that may satisfy certain conditions. A classical computer requires minimum $N$ steps to find a particular element in a database

with $N = 2^n$ elements. Compared to that, the quantum search algorithm requires only $\sqrt{N}$ queries to execute the search, and offers a quadratic speed up.

The algorithm accomplishes this in three major steps: preparation of the state, marking out the solutions, and amplitude amplification. All elements in the unstructured database are first put in a uniform superposition. Then the *oracle* operator marks out all the solution states of the search by flipping their phase. Then the *diffusion* operator flips the amplitude of all the elements in the database about the mean, amplifying the amplitude of the solutions. This cycle of the oracle and diffusion operator continues to run the course of the algorithm until it is successful, and repeats until the amplitudes of the solution states are amplified to the extent that it can be measured to a desirably successful probability.

### 2.2.1 The oracle

We start the search with a quantum register of $n$ qubits, where $n$ is the minimum number of qubits necessary to represent the database of elements. Without loss of generality, let $N = 2^n$, and contains $M$ solutions, where $1 \leq M \leq N$. To initialize the algorithm, we use the Hadamard Operator to put all the elements of the search in a uniform superposition.

Let us call this uniform superposition $|\psi\rangle$. $|\psi\rangle$ is a superposition of both the solution and non-solution states. Let the normalized superposition of the solution states be $|\beta\rangle$ and that of the non-solution states $|\alpha\rangle$. Then, $|\psi\rangle$ can be written as:

$$|\psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle \tag{2.9}$$

An oracle is a black box function which modifies the system into a required configuration. The quantum oracle $U_w$ is a quantum black-box operator, which means it can observe and operate on states conditionally without collapsing them to a classical state. This is possible by making use of an *oracle qubit*, which can recognize solutions to the search problem. $U_w$ is a unitary, and it can be represented as

$$|x\rangle \overset{U_w}{\rightarrow} (-1)^{f(x)}|x\rangle \tag{2.10}$$

The oracle operator marks the solution to the search by flipping their phase. It performs a conditional phase shift of $\pi$ on the solutions, which is equivalent to their amplitudes being multiplied by a factor of -1. This changes the amplitude of the states without altering their probability density. The oracle is a unitary operator, and this conditional phase shift can be written as:

$$U_w = \mathbb{I} - 2(\frac{M}{N})|\beta\rangle\langle\beta| \tag{2.11}$$

For an $N$ item search problem with $M$ solutions, the oracle needs to make only $O(\sqrt{N/M})$ queries in order to mark out all the solutions in the database [18].

### 2.2.2 The Diffusion Operator

The purpose of the diffusion operator is *amplitude amplification*. It performs an inversion on all the states about their mean. Following the oracle, the marked states in $|\beta\rangle$ have a negative amplitude compared to all the other states. Hence, when amplified about the mean, the amplitude of the solutions will be higher than those of the non-solution states. This implies that the solutions can be measured with a higher probability density than the non-solution states. The diffusion operator given by $U_d$ can be written as

$$U_d = 2|\psi\rangle\langle\psi| - \mathbb{I} \tag{2.12}$$

### 2.2.3 The Grover Iterate

The oracle and diffusion operator together form the *Grover iterate*, also known as the *Grover operator*, denoted by $G$ . The quantum search algorithm now consists of repeated iterations of this quantum subroutine. It can be written as:

$$G = U_d U_w = [\ 2|\psi\rangle\langle\psi| - \mathbb{I}\ ][\ \mathbb{I} - 2(\frac{M}{N})|\beta\rangle\langle\beta|\ ] \tag{2.13}$$

Let us consider a worked example for the three qubit case, with $|011\rangle$ as the marked state. The superposition contains 8 basis states, and after initialization to a uniform superposition, is as depicted in Figure 2.2:

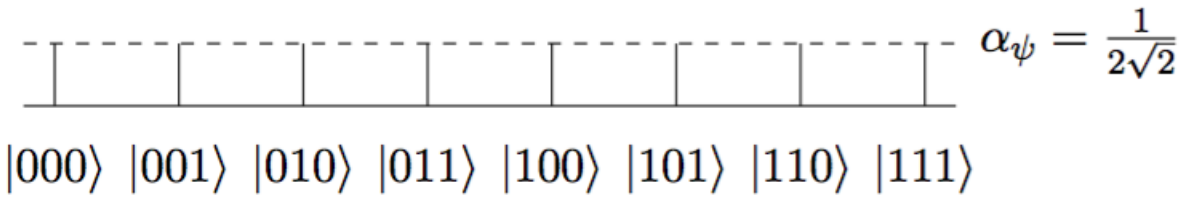


Figure 2.2: Source [21]

Next, the oracle flips the phase of the marked state (Figure 2.3) . Following the oracle,

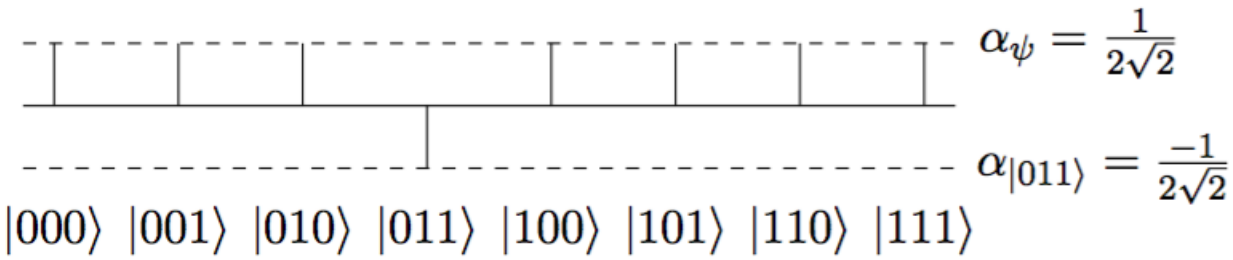


Figure 2.3: Source [21]

the diffusion operator flips all the states about their mean (Figure 2.4) .

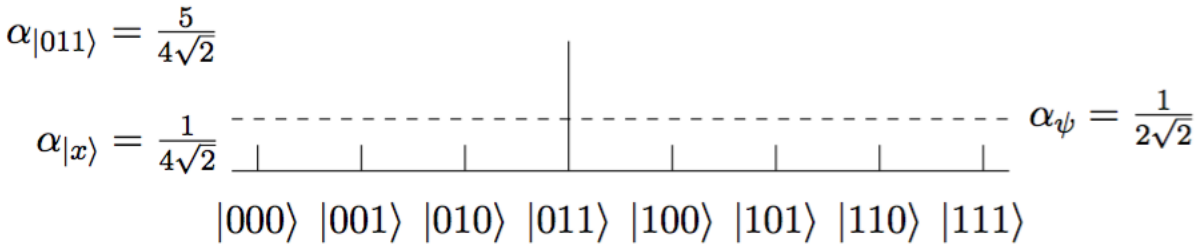


Figure 2.4: Source [21]

The grover iterate is repeated over and over until the marked solution states have amplified upto a desired probability measure (Figure 2.5) .

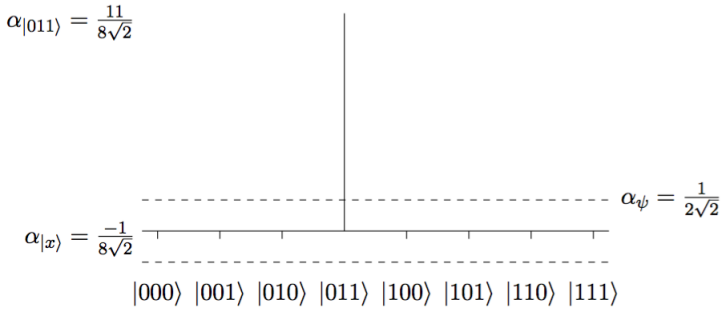


Figure 2.5: Source [21]

### 2.2.4 Geometric Picture

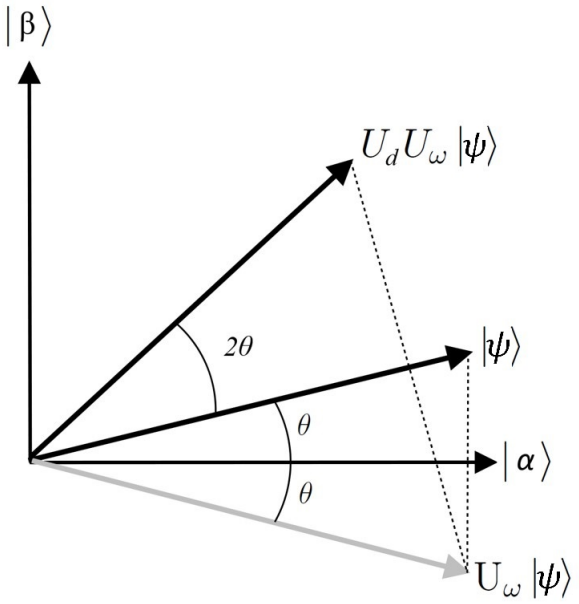


Figure 2.6: Geometric Representation of a Grover Iterate (Source [18])

The dynamics of the Grover Iterate in the basis formed by the normalized superposition states of the solutions and non-solutions $\{|\beta\rangle, |\alpha\rangle\}$ can be visualized to give a geometric interpretation:

$$|\psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle = \cos\theta|\alpha\rangle + \sin\theta|\beta\rangle \tag{2.14}$$

where $\cos\theta = \sqrt{\frac{N-M}{N}} \Rightarrow \theta = \arccos(\sqrt{\frac{N-M}{N}})$. The Grover operator $G = U_d U_w$ rotates the uniform superposition state $|\psi\rangle$ in the $\{|\beta\rangle, |\alpha\rangle\}$ basis, with each iteration rotating $|\psi\rangle$ closer to $|\beta\rangle$. The oracle $U_w$ reflects $|\psi\rangle$ about $|\beta\rangle$. Then $U_d$ rotates $U_w|\psi\rangle$ about $|\psi\rangle$, and rotating closer to $|\beta\rangle$. The search is successful when the final state $U_d U_w|\psi\rangle$ coincides with the $|\beta\rangle$ axis.

Grover's algorithm was one of the earliest developed quantum algorithms, and had a significant role in making quantum algorithms relevant in quantum computation. It has many applications. It is used as a subroutine in many classical problems to speed up the algorithm. It is possible to estimate the number of solutions to any search problem, if they exist, by using phase estimation in accordance with Grover's algorithm. This is called *Quantum Counting*. Quantum counting is used to speed-up the solutions to many NP-complete problems.

The performance of the algorithm can be quantified by the number of Grover iterations required to rotate $|\psi\rangle$ to within an angle of $\frac{\theta}{2} \leq \frac{\pi}{4}$ of $|\beta\rangle$ is given by

$$R = \frac{\arccos\sqrt{(\frac{M}{N})}}{\theta} \tag{2.15}$$

However, this requires us to know the number of solutions to the problem beforehand. In some cases, $R$ is too high for the algorithm's performance to be efficient, as $|\psi\rangle$ keeps sweeping around in circles without converging to $|\beta\rangle$. Though $M$ can be evaluated separately by quantum counting, needing to that alongside using Grover's algorithm takes away its advantage of quadratic speed up. In 2005, Grover developed the "fixed-point quantum search" by making a few modifications to the original quantum search [20]. The convergence of $U_d U_w|\psi\rangle$ to $|\beta\rangle$ was ensured in this version, but with a compromise on the quadratic speed up. However, using quantum control techniques to modify the algorithm, the fixed point search can implement the search without losing the quadratic speed up, while also ensuring the success of the search [17].

## 2.3 Dynamical Decoupling

### 2.3.1 Robust operations and sequences

As discussed in Section 1.3, it is important to make quantum operations more robust, so that errors that arise in the process are within the range in which they can be corrected. To make operations more 'robust', the parameters controlling the experimental implementation of the gates should be independent of the operation's performance, or cause only negligible deviations to it.

*Threshold theorem* shows that quantum algorithms and their corresponding gate operations can be as long as needed, given that the noise generated is below a given limit [19]. The usage of pulses to correct errors was first done in NMR. Different rotations can be combined to make the quantum states undergoing processing more resilient to environmental noise, thereby eliminating decoherence.

The addition of the pulse sequences to generate extra rotations in the quantum system during processing may increase the length of operations and the corresponding energy deposited in the system. However, it can be used to increase the fidelity of the entire sequence, and not just the individual operators, making it worthwhile, if used with careful consideration.

There are multiple types of errors that cause decoherence. To correct the deviant behaviour, pulses need to be designed accordingly to be effective. *Optimal control theory* offers the necessary theoretical tools to design them.

### 2.3.2 Robust Dynamical Decoupling

Dynamical Decoupling (DD) is a technique that modulates the system and environmental interactions to suppress decoherence and incoherence. It consists of a sequence of pulses that are temporally spaced in scales much shorter than the environment's evolution. These pulses methodically modulate the environment interactions by performing a series of instantaneous qubit flips. Unlike schemes based on error correcting codes or decoherence free subspaces, DD requires lesser resources - like ancilla qubits.

DD can been incorporated as a part of optimal control procedure [1], and has been realized experimentally, in Grover's Algorithm. It is advantageous to integrate DD pulses into the gates, because there is no need to manually divide gates into segments, or correct for pulse errors like *radiofrequency field inhomogeneity* (RFI) separately.

### 2.3.3 Dynamically protected gates

Consider the total Hamiltonian of the system that evolves with time $H(t)$ with

$$H(t) = H_S(t) + H_C(t) + H_{SE}(t) \tag{2.16}$$

Where $H_S(t)$ refers to the system's Hamiltonian, $H_C(t)$ refers to the Hamiltonian of the control fields, and $H_{SE}(t)$ is the interaction Hamiltonian for the system and the environment.

Ideally, unitaries involved in the evolution of the system are realized in quantum operations as discrete segments in time, and need to be dependent only on $H_S(t)$ and $H_C(t)$. But in realistic experimental scenarios, the system-environment Hamiltonian will also evolve the system-environment state, causing decoherence that manifests in subsequent measurements. However, by applying DD pulses, we can suppress the decoherence caused by $H_{SE}(t)$, by interrupting the evolution of the system-environment state.

The propagators for the DD pulses are given by $P_j = e^{-i\beta I_\alpha}$, where $\beta$ is the DD flip-angle, and $I_\alpha$ can be $I_x$ or $I_y$. Consider the total propagator $U_T$ for the protected sequence that includes the DD pulses:

$$U_T = U_{N+1} \prod_{i=1}^{N} P_i U_i \tag{2.17}$$

In the toggling-frame [25], the total propagator can be rewritten as:

$$U_T = U_{N+1} \prod_{i=1}^{N} \tilde{U}_i \tag{2.18}$$

where $\tilde{U}_i = V_i^\dagger U_i V_i$, and $V_j = P_{i-1} P_{i-2} .... P_2 P_1$ and $V_1 = V_{N+1} = \mathbb{I}$.

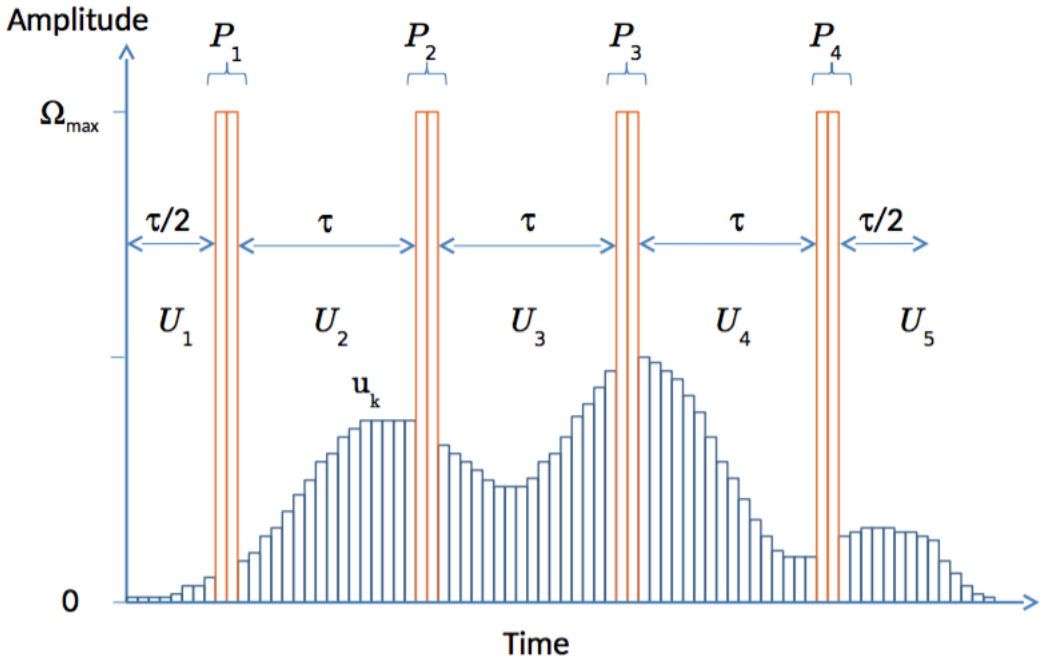


Figure 2.7: Schematic of the protected quantum gate scheme (Source [1])

To implement a protected quantum gate scheme, we reserve certain segments for the full amplitude DD pulses, and compensate for the DD pulse in the other segments, by optimizing them to realize a target unitary that matches the ideal case (Figure 2.7).

# Chapter 3

# Simulations

## 3.1 Evolution of Quantum Discord in a 2-Qubit NMR system

To gain better understanding about the evolution of quantum discord for 2-qubit systems, the results from [16] were simulated.

The system studied is a two qubit NMR system with spins $I_a$ and $I_b$ that evolve under the Hamiltonian

$$H = H_z + H_J \tag{3.1}$$

where $H_z = -\frac{h}{2\pi}(\omega_a I_z^a + \omega_b I_z^b)$ is the *Zeeman Hamiltonian*, and $H_J = 2\pi J I_a \cdot I_b$ is the indirect *spin-spin coupling Hamiltonian*.

The system is initialized by preparing a pseudo-pure state corresponding to the pure state $|00\rangle$ and the density matrix after different durations of free evolution ($\tau$) under $H_J$ can be calculated as a function of the *delay parameter* $\theta$ (where $\theta = \pi J\tau$).

To calculate the discord for a given density matrix, we have to perform measurements over extensive sets of orthonormal basis vectors, and then maximize the quantity $J(A:B)$. Since this is a two-qubit system, consider generalized orthonormal basis vectors,

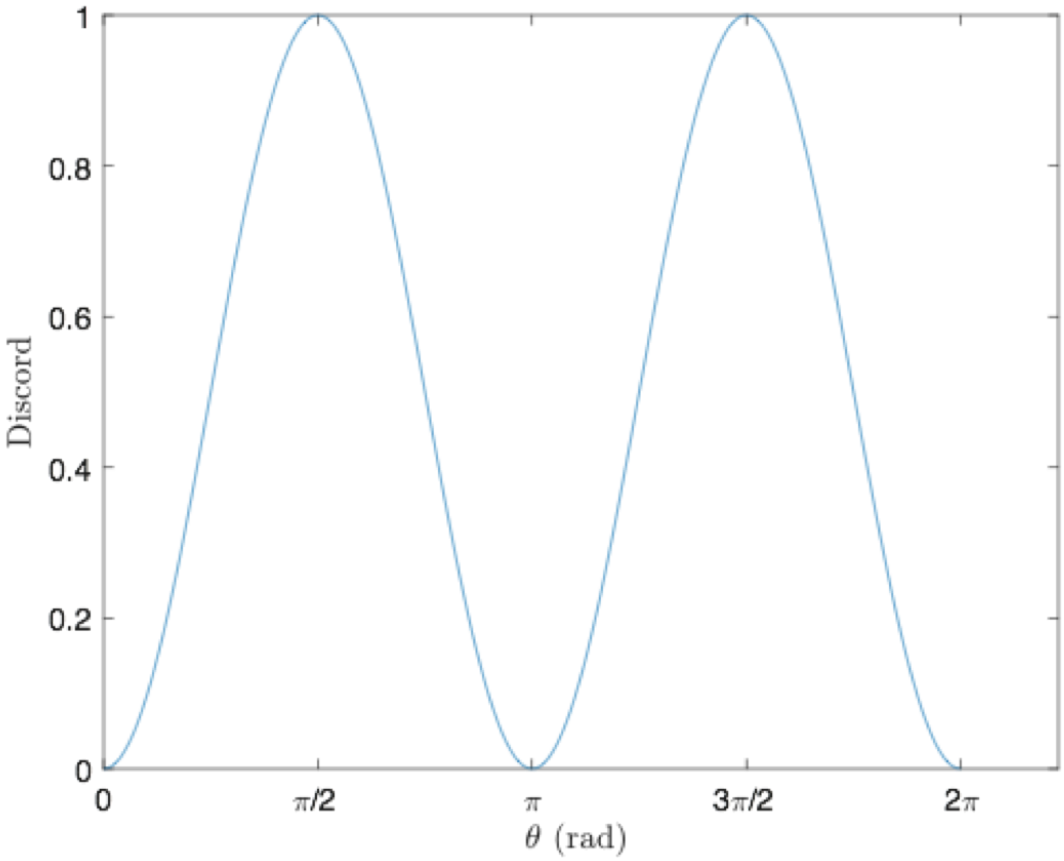


Figure 3.1: Evolution of quantum discord in a two qubit NMR system

$$|u\rangle = \cos\alpha|0\rangle + e^{i\phi}\sin\alpha|1\rangle \tag{3.2}$$

$$|v\rangle = \sin\alpha|0\rangle - e^{i\phi}\cos\alpha|1\rangle \tag{3.3}$$

We vary $\alpha \in [0, \pi]$ and $\phi \in [0, 2\pi]$ in small increments, and evaluate $J(A : B)$ for different $\alpha$ and $\phi$. Then $J(A : B)$ is maximized over all these states, and the discord for a given density matrix is evaluated as discussed in Section 2.1.

The evolution of discord can be plotted as a function of the delay parameter $\theta$. The results match with the simulations in [16].

## 3.2 Evolution of Quantum Discord in Grover's Algorithm for a two qubit system

The quantum search algorithm for the two qubit case is special because the search is 100% successful after just a single Grover iteration. In other words, the state vector of the search rotates itself into alignment with the $|\beta\rangle$ axis after a single Grover iteration, indicating that the search algorithm was successful in finding the solution state.

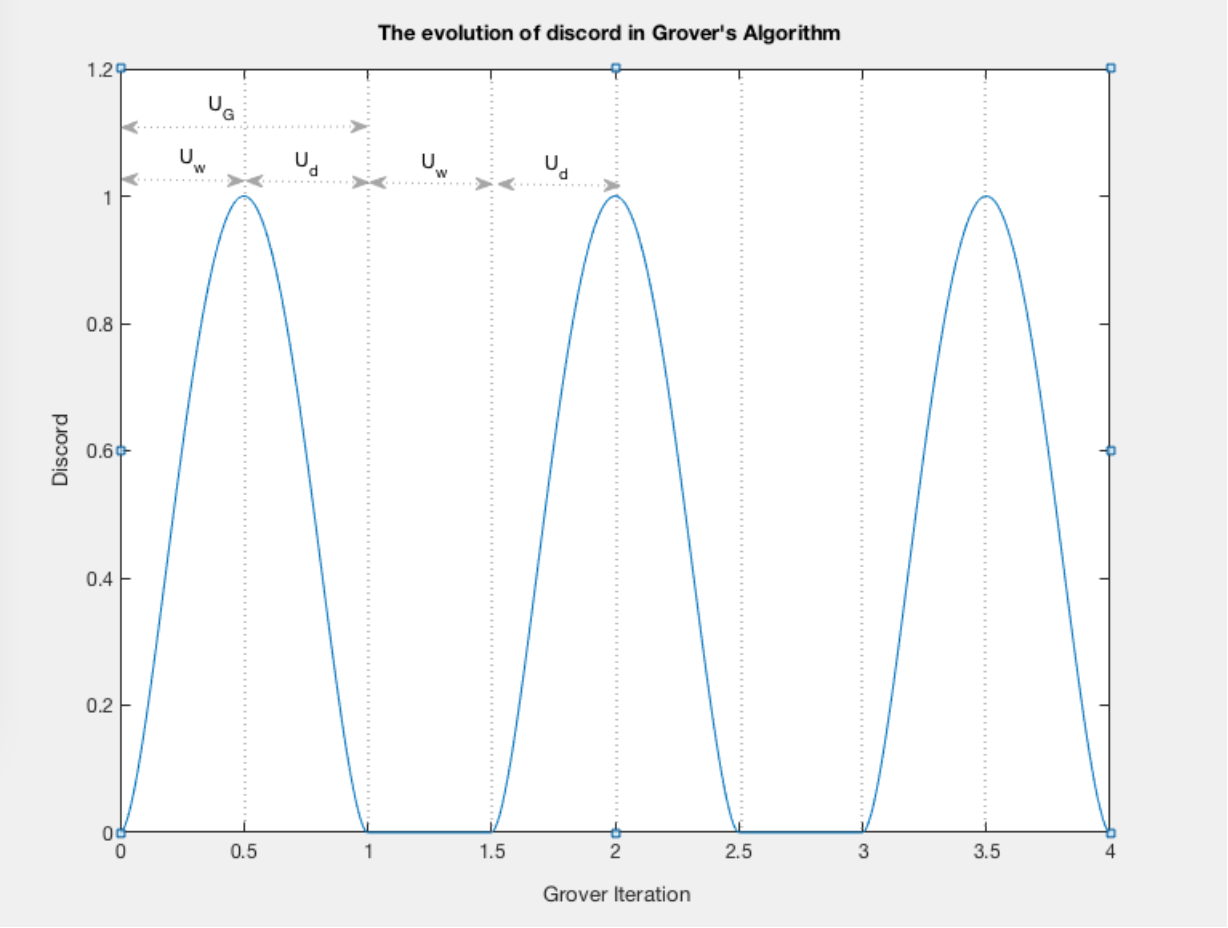


Figure 3.2: Evolution of quantum discord in Grover's algorithm

Since this is a two qubit system, we consider the basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ with $|01\rangle$ as the marked state. The initial superposition state is $|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, with $|\beta\rangle = |01\rangle$.

$$U_w = \mathbb{I} - 2|01\rangle\langle 01| \tag{3.4}$$

$$U_d = 2|\psi\rangle\langle\psi| - \mathbb{I} \tag{3.5}$$

We start with the initial superposition state and evolve it under the Grover iterate $G = U_d U_w$. We break each operator into a $2mn$ steps so that we can sample the intermediate points for discord as the algorithm runs its course. At each point, we can evaluate the evolved density matrix $\rho$, and evaluate their corresponding discord. We then plot the evolution of discord against the Grover iterate.

Figure 3.2 displays the evolution of quantum discord for Grover's algorithm over 4 iterations. The discord vanishes after the first iteration as it reaches the marked state, which is classical. It also vanishes for uncorrelated states. The discord is maximum after the action of the oracle, which flips the phase of the marked state by $\pi$.

## 3.3 Evolution of Quantum Discord with the effect of noise

Quantum correlations are lost due to effects like decoherence and incoherence. To realistically simulate the evolution of quantum discord, it is necessary to account for noise. For studying the effects of local noise on quantum discord, we apply a noise operator with a given amplitude ($n_{amp}$) generated randomly. We average the discord obtained from several repetitions, to obtain the effect of noise on discord. Figure 3.3 shows the effect of a random dephasing noise on the discord.

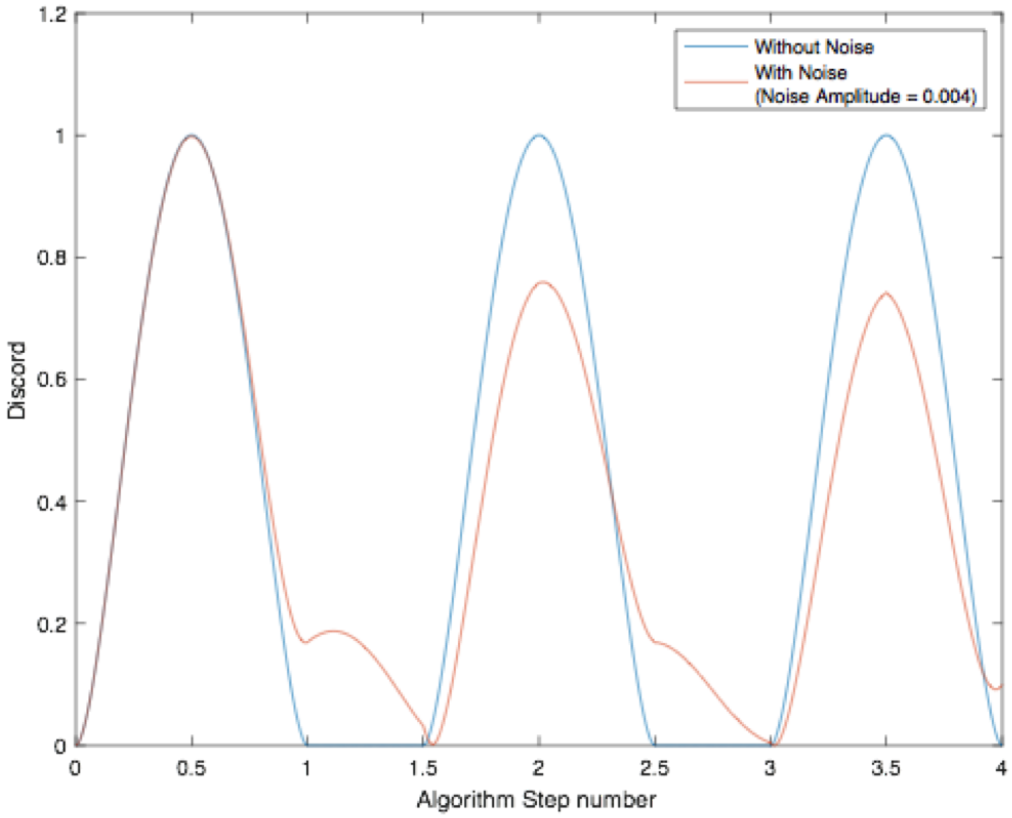


Figure 3.3: Evolution of quantum discord in Grover's algorithm in the presence of a dephasing noise of amplitude $n_{amp} = 0.004$

The randomly generated dephasing noise applied is of the form

$$U_N = e^{-i\eta n_{amp}(I_z^a + I_z^b)} \tag{3.6}$$

where $\eta$ is random, $n_{amp}$ is the noise amplitude and $I_z^a$ and $I_z^b$ are the z-components of the spin angular momentum operators. The noise unitary now acts on the constituent states in unison with the oracle $U_w$ and the diffusion operator $U_d$.

Figure 3.3 represents the evolution of discord in Grover's algorithm in the presence and absence of noise. On gradually increasing the noise amplitude, we find that the evolution of discord gets correspondingly aberrant (Figure 3.4).

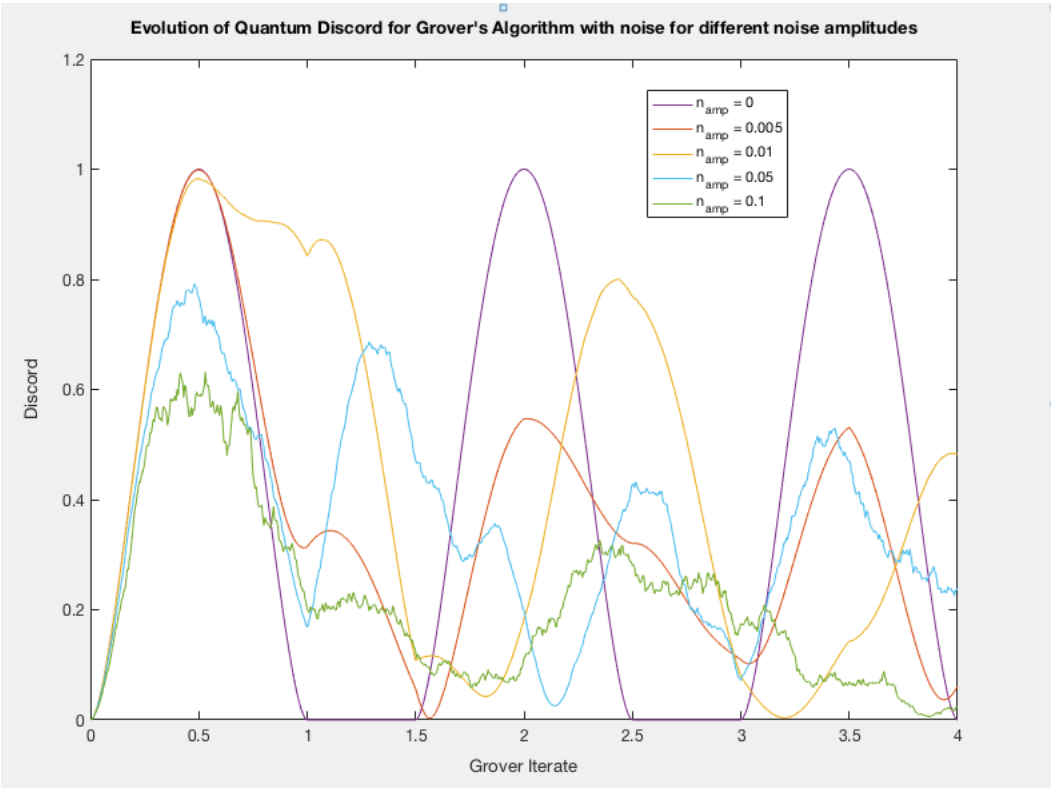


Figure 3.4: The evolution of quantum discord in Grover's algorithm for various noise amplitudes ($n_{amp}$)

We observe that the effect of noise on the evolution of discord during the first half ($U_w$) of the Grover Iterate is much more than the second half ($U_d$). This is illustrated in Figure 3.5.

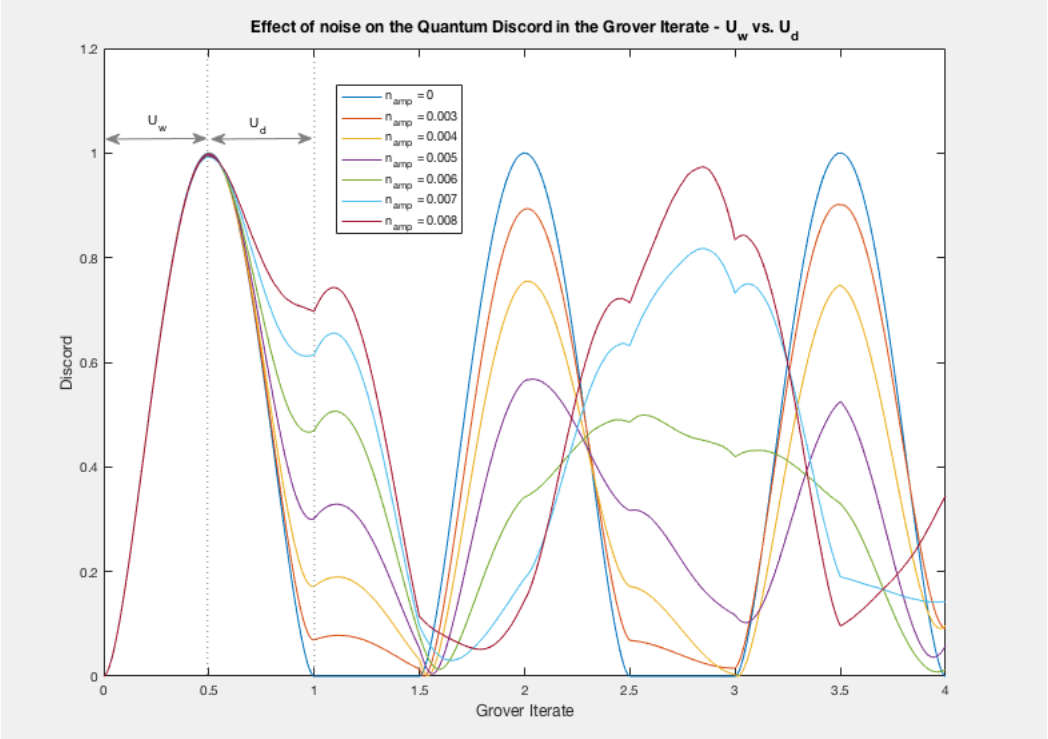


Figure 3.5: Effect of noise on $U_w$ vs $U_d$ for different noise amplitudes ($n_{amp}$)

This can be explained by the fact that $U_d$ is more sensitive to noise than $U_w$. The oracle $U_w$ only acts on the solution states in the superposition, which it has to "mark" by flipping their phase. But, the diffusion operator $U_d$ flips all the states about their mean, hence acting on all the states in the superposition. As a result, the noise from $U_d$ disturbs the correlations more, compared to the oracle $U_w$.

## 3.4 Evolution of quantum discord with the effect of noise with DD ($\pi$ pulse)

The effect of noise on the evolution of discord was shown in the previous section. Now, we apply DD to suppress decoherence, using the methods described in Section 2.7.

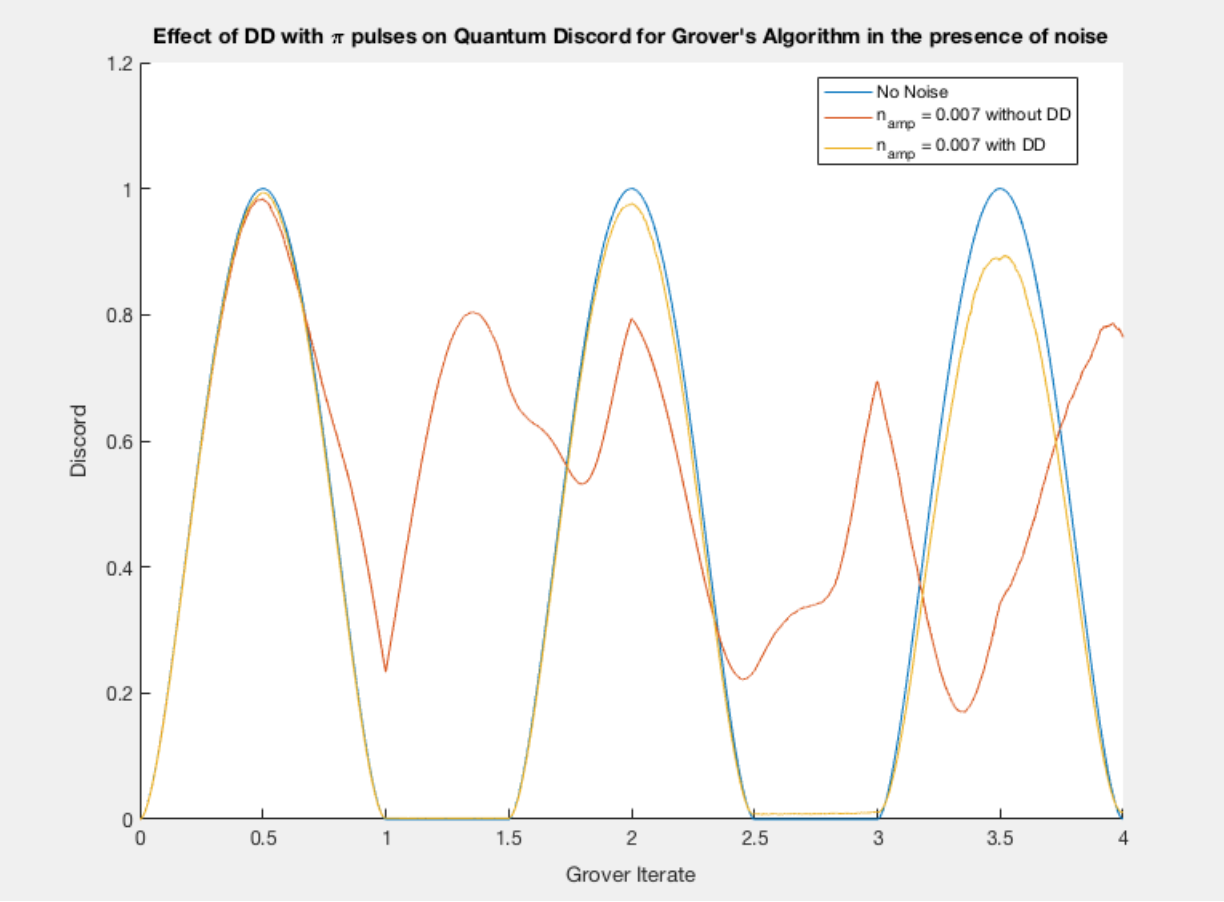


Figure 3.6: Effect of DD on the evolution of discord in Grover's algorithm in the presence of noise

We apply the unitary for $DD_y$ to the system as it evolves, which is of the form:

$$V = e^{-i\beta I_\alpha} \tag{3.7}$$

where $\beta = \pi$ and $I_\alpha = I_y$. We divide the Grover iterate into $4mn$ segments each, for both the oracle and the diffusion. Suppose we apply DD after every $n$th segment, then the next $n$

unitaries that come after the DD pulse are modified to adjust to hit the target unitary after $2n$ operations:

$$U_w = \{\ UU....{}_{n\ times}....UU\ V\ U_pU_p....{}_{2n\ times}....U_pU_p\ V'\ UU....{}_{n\ times}....UU\ \}_{m\ times}$$

This is the protected gate scheme that integrates DD into the quantum gates. It decouples the system-environment interactions to successfully suppress the randomly generated noise and protects correlations during the evolution of Grover's algorithm.

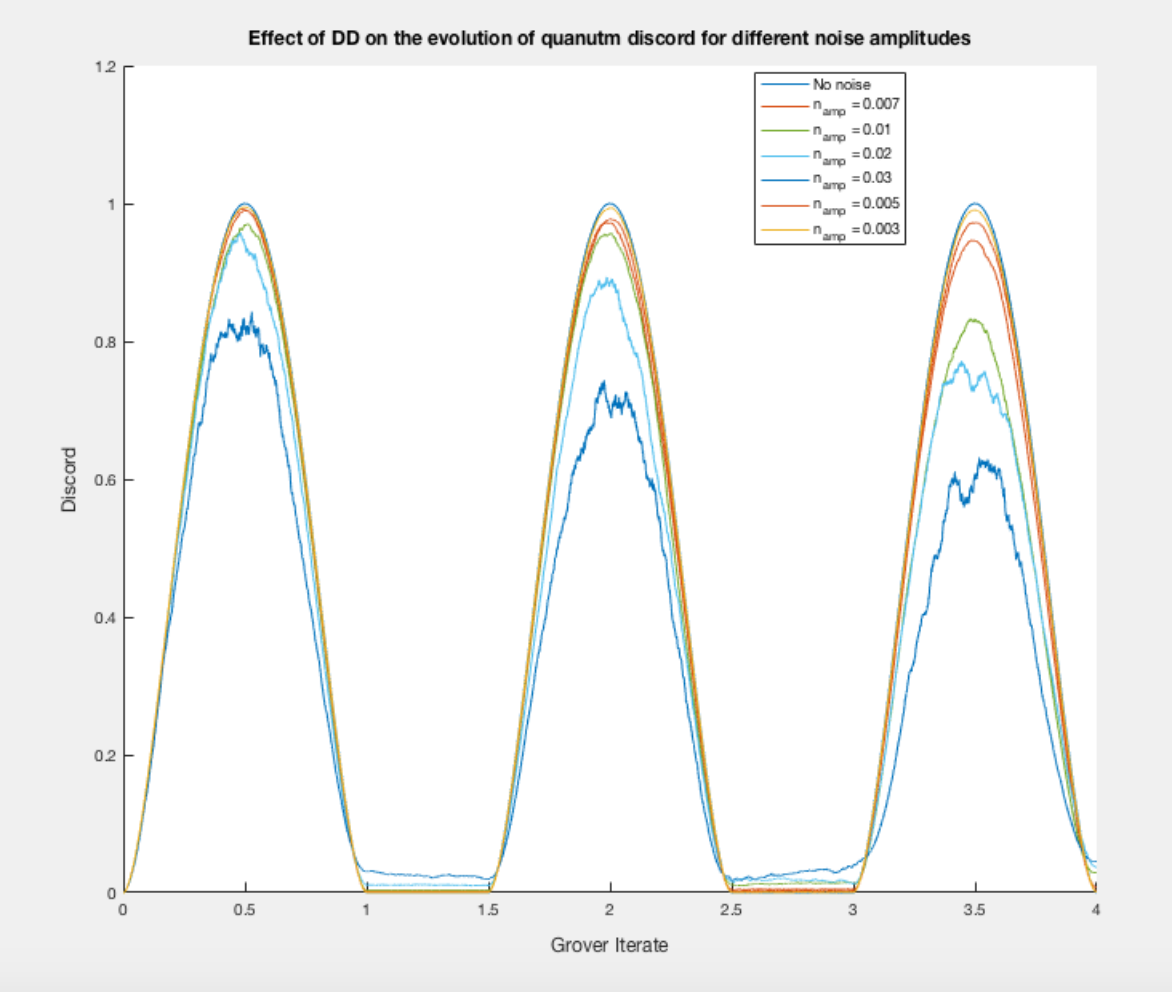


Figure 3.7: Effect of DD on the evolution of discord in Grover's algorithm for various $n_{amp}$

Figure 3.6 compares the evolution of quantum discord in Grover with noise $n_a mp = 0.07$ with and without DD and compares them with the ideal evolution with no noise. We observe that the robust-DD scheme is quite successful in noise suppression.

Figure 3.7 shows that this scheme is pretty effective at suppressing decoherence even for higher amplitudes of noise, which exhibit extremely aberrant behaviour from the ideal evolution (Figure 3.4).

## 3.5 Evolution of quantum discord with the effect of DD based on different pulse angles ($\beta$)

It was experimentally shown that DD sequences based on $\pi/2$ pulses performed just as well as, if not better than $\pi$ pulses in protecting the quantum discord in Grover's algorithm [1]. Hence it is of interest to further explore the performance of DD sequences based on non-$\pi$ pulses to find out their potential in suppressing noise.

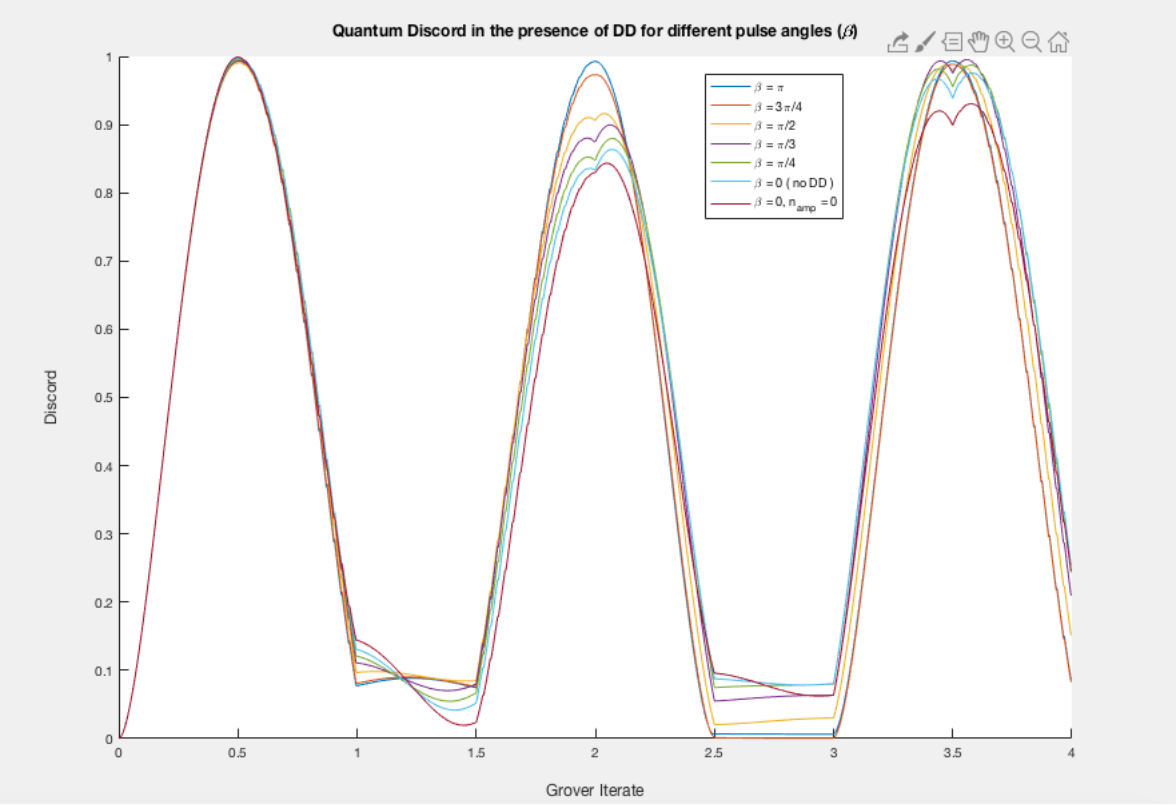


Figure 3.8: Effect of DD on the evolution of discord for different pulse angles ($\beta$)

Figure 3.8 shows the evolution of quantum discord in the presence of noise ($n_{amp} = 0.003$), for DD sequences based on various pulse angles ($\beta$). We observe that $\pi$ pulse based DD outperforms the DD based on other pulse angles. Their performances of different $\beta$ can be ranked as:

$$\pi > \frac{3\pi}{4} > \frac{\pi}{2} > \frac{\pi}{3} > \frac{\pi}{4} > \frac{\pi}{6} > \ no\ DD$$

Though not as effective as $\pi$ pulse based DD, the DD sequences based on other pulse

angles, still successfully suppress the noise in the system and correct the noise to some extent. It is of interest to further explore upon this, and compare the simulated results with experimental results from [1].

## 3.6 Evolution of quantum discord with DD in the presence of RFI

An experimentally applied $\pi$ pulse is imperfect, and not purely a $\pi$ pulse but a distribution of pulses $[\pi - \delta, \pi + \delta]$ whose probability peaks for the value $\pi$. This can have a significant effect on the effectiveness of DD in preserving the correlations and decoupling the system from the environment.

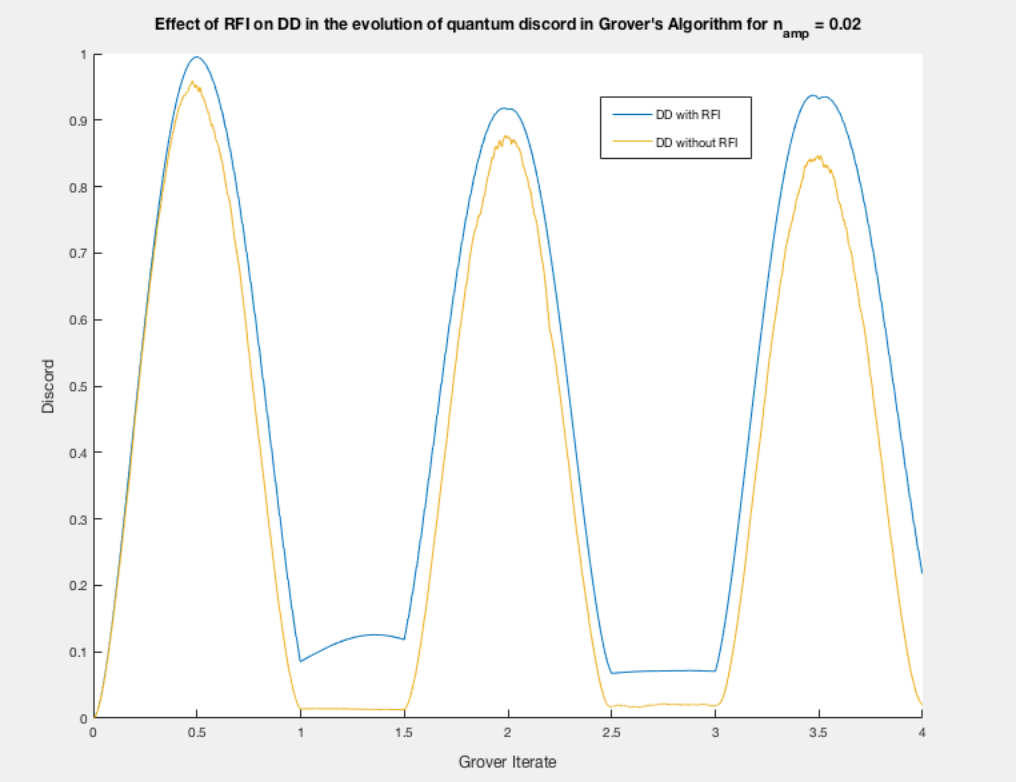


Figure 3.9: Effect of RFI on evolution of quantum discord with $\pi$-pulse based DD for $n_{amp}$ = 0.02

Consider a distribution of pulse angles about $\beta$

$$[0.9\ \beta,\ 0.95\ \beta,\ \beta,\ 1.05\ \beta,\ 1.1\ \beta\ ]$$

with the probability distribution (A)

$$[\ 0.1,\ 0.25,\ 0.3,\ 0.25,\ 0.1\ ]$$

We obtain the discord for each of $\beta$ values in the distribution and then obtain the weighted means of the discord values with the probability from the distribution as the weight. We then plot the time evolution of discord for a given pulse angle $\beta$ with RFI for a particular $n_{amp}$.

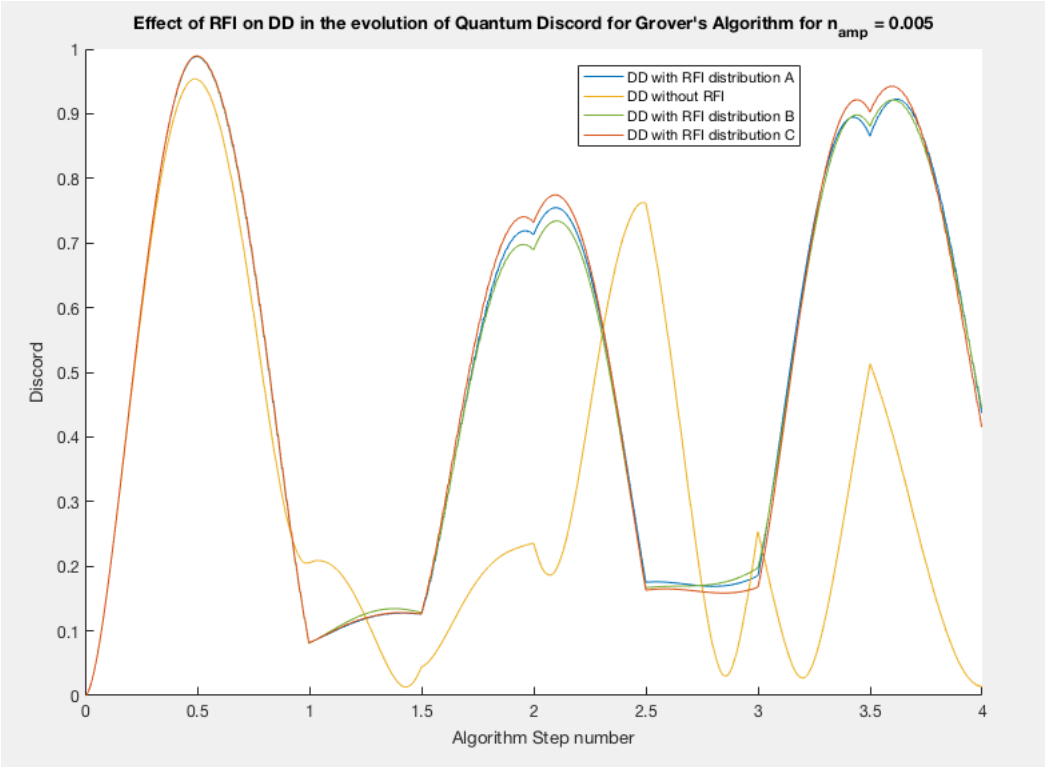


Figure 3.10: Evolution of quantum discord with $\pi/2$-pulse based DD for $n_{amp} = 0.005$, in the presence and absence of RFI

From Figure 3.9, we observe that RFI reduces the effectiveness of DD for $\pi$-pulse based DD, but not significantly. This because $\pi$-pulse based DD is seen to be effective for even high noise amplitudes, unlike non-$\pi$ pulse based DD.

However, it is observed that $\pi/2$-pulse based DD may perform better in the presence of RFI than when it is absent. In Figure 3.10, the evolution of discord with $\pi/2$ based DD has been plotted in the absence of RFI and the presence of RFI for three probability distributions A, B and C. The imperfect pulse angle $\beta$ constitutes the pulses $[0.9\,\beta,\ 0.95\,\beta,\ \beta,\ 1.05\,\beta,\ 1.1\,\beta\,]$, where $\beta = \pi/2$, and each pulse angle occurs as an impurity with a corresponding probability

as given in one of the normalized probability distributions below:

$$Distribution\ A = [\ 0.1,\ 0.25,\ 0.3,\ 0.25,\ 0.1\ ]$$

$$Distribution\ B = [\ 0.05,\ 0.1,\ 0.7,\ 0.1,\ 0.05\ ]$$

$$Distribution\ C = [\ 0.05,\ 0.1,\ 0.55,\ 0.2,\ 0.1\ ]$$

The better performance of DD in the presence of RFI rather than its absence can be attributed to the mixture of slightly higher pulse angles for an imperfect pulse. Since the performance of DD increases with the increase in pulse angle $\beta$, the presence of RFI means that the imperfect pulse may constitute some pulses that perform slightly better than a pure $\pi/2$ pulse. This is illustrated by the performance of DD for an imperfect $\pi/2$-pulse with different probability distributions for the impurities. The distribution with more probability for higher pulse angles perform better:

$$Distribution\ C > Distribution\ A > Distribution\ B$$

However, this feature of $\pi/2$ pulse based DD, where it performs better in the presence of RFI than its absence, does not take away from the fact that $\pi$ pulse based DD is better at noise suppression than the $\pi/2$ pulse based DD. These results, in fact, further attest to the better performance of DD for higher values of $\beta$.

## 3.7 Evolution of quantum discord for various DD schemes ($DD_x$, $DD_y$, $DD_{xy}$) in the presence of RFI for different pulse angles ($\beta$)

Non-$\pi$-pulse based DD sequences are found to be weaker for higher levels of noise amplitudes. We plot the evolution of discord for $DD_x$, $DD_y$ and $DD_{xy}$ sequences for different pulse angles $\beta$ for a low $n_{amp}$ to compare their decoherence suppression capacity, in the presence of RFI.

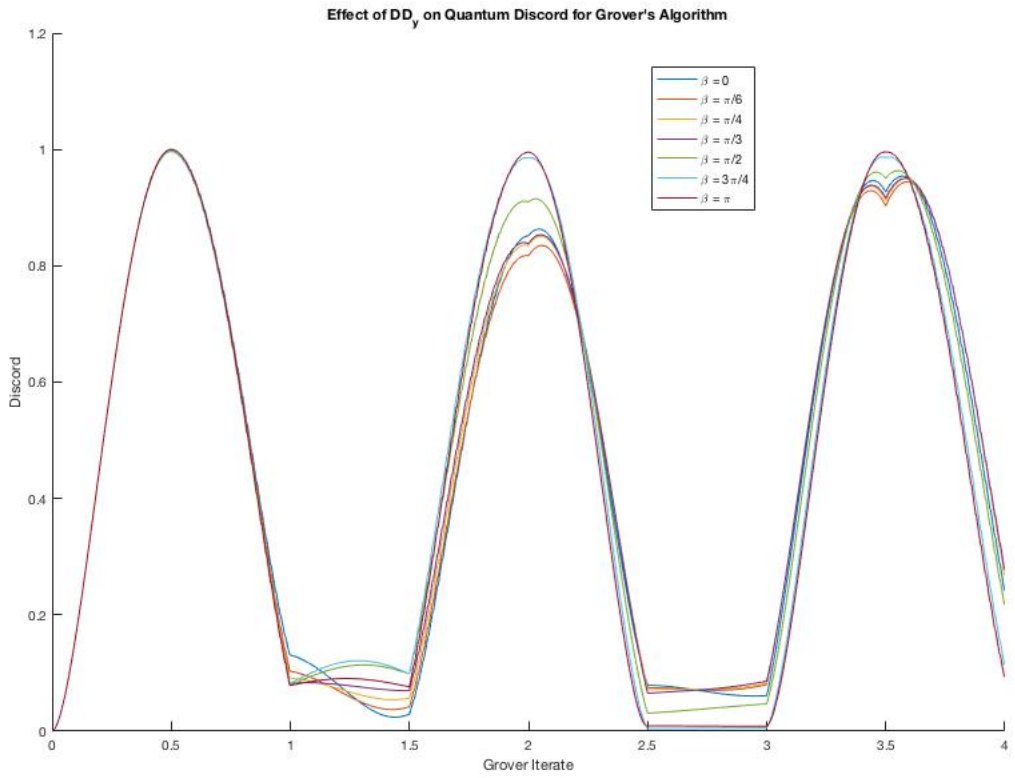


Figure 3.11: Evolution of quantum discord for Grover's algorithm with $DD_x$ in the presence of RFI for different pulse angles ($\beta$) for $n_{amp} = 0.002$

Figure 3.11 shows the evolution of quantum discord in Grover's algorithm with $DD_x$ ($I_\alpha = I_x$) in the presence of RFI, for different pulse angles ($\beta$) at noise amplitude $n_{amp} = 0.002$

Figure 3.12 shows the evolution of quantum discord in Grover's algorithm with $DD_y$

($I_\alpha = I_y$) in the presence of RFI, for different pulse angles ($\beta$) at noise amplitude $n_{amp} = 0.002$

Figure 3.13 shows the evolution of quantum discord in Grover's algorithm with $DD_{xy}$ (which uses specific alternating pulses of X and Y [27]) in the presence of RFI, for different pulse angles ($\beta$) at noise amplitude $n_{amp} = 0.003$

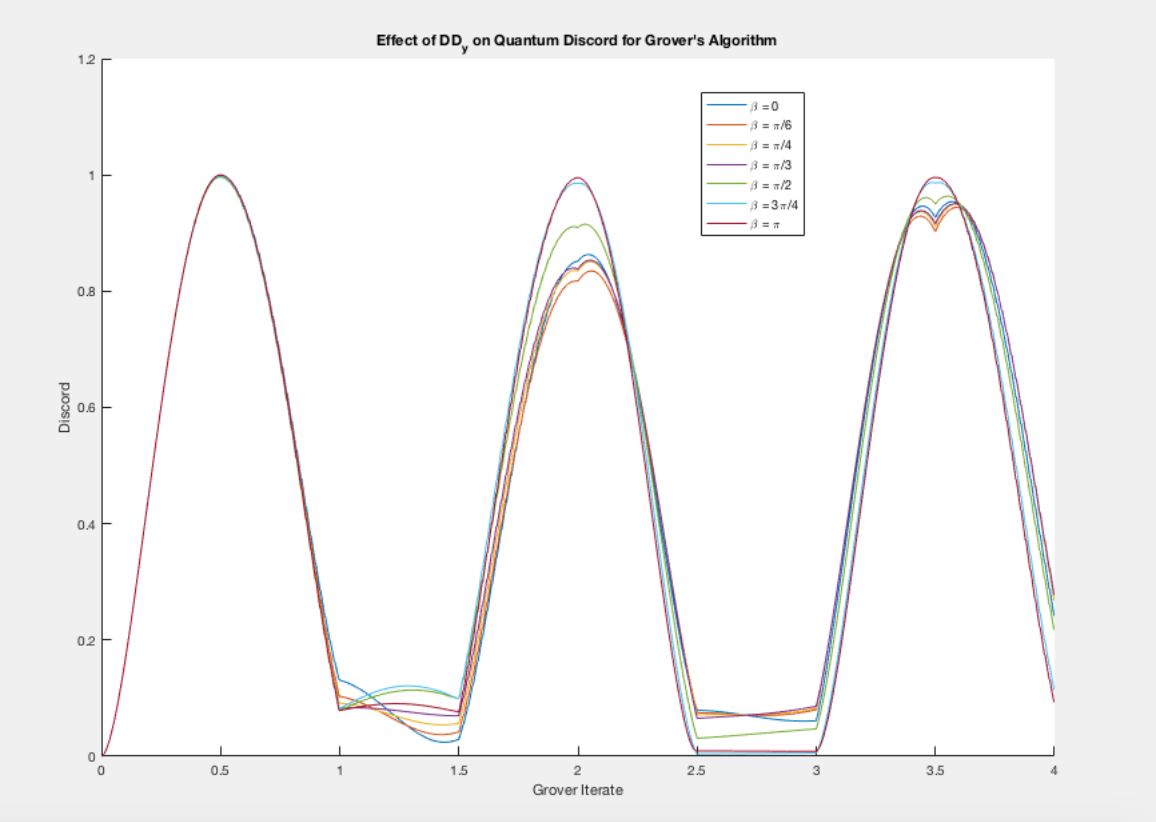


Figure 3.12: Evolution of quantum discord for Grover's algorithm with $DD_y$ in the presence of RFI for different pulse angles ($\beta$) for $n_{amp} = 0.002$

We observe that even in the presence of RFI, DD based on $\pi$ pulses outperform all other pulses. The order of performance in the presence of RFI still remains as:

$$\pi > 3\frac{\pi}{4} > \frac{\pi}{2} > \frac{\pi}{3} > \frac{\pi}{4} > \frac{\pi}{6} > no\ DD$$

This order holds true for all the three DD sequences simuluated - $DD_x$, $DD_y$ and $DD_{xy}$ for different pulse angles.

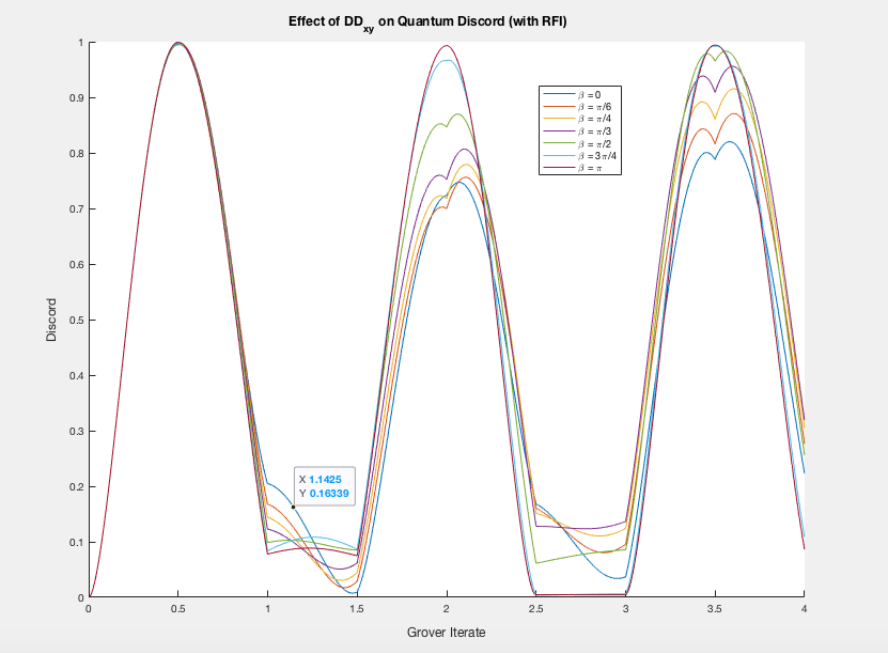


Figure 3.13: Evolution of quantum discord for Grover's algorithm with $DD_{xy}$ in the presence of RFI for different pulse angles ($\beta$)for $n_{amp} = 0.003$

But we also observe that even though non-$\pi$ pulses are not as efficient at decoherence suppression as $\pi$ pulses, they can still significantly decouple the system and environment interactions, especially at low noise amplitudes.

Hence, unlike the conventional notion that dynamical decoupling is achieved only by $\pi$ pulses, we have shown by numerical analysis that if properly incorporated into control sequences, even non-$\pi$ pulses can bring about efficient dynamical decoupling.

Thus the current work supports the earlier experimental results regarding the effectiveness of $\pi/2$-pulse based DD [1], displayed in Figure 3.14.

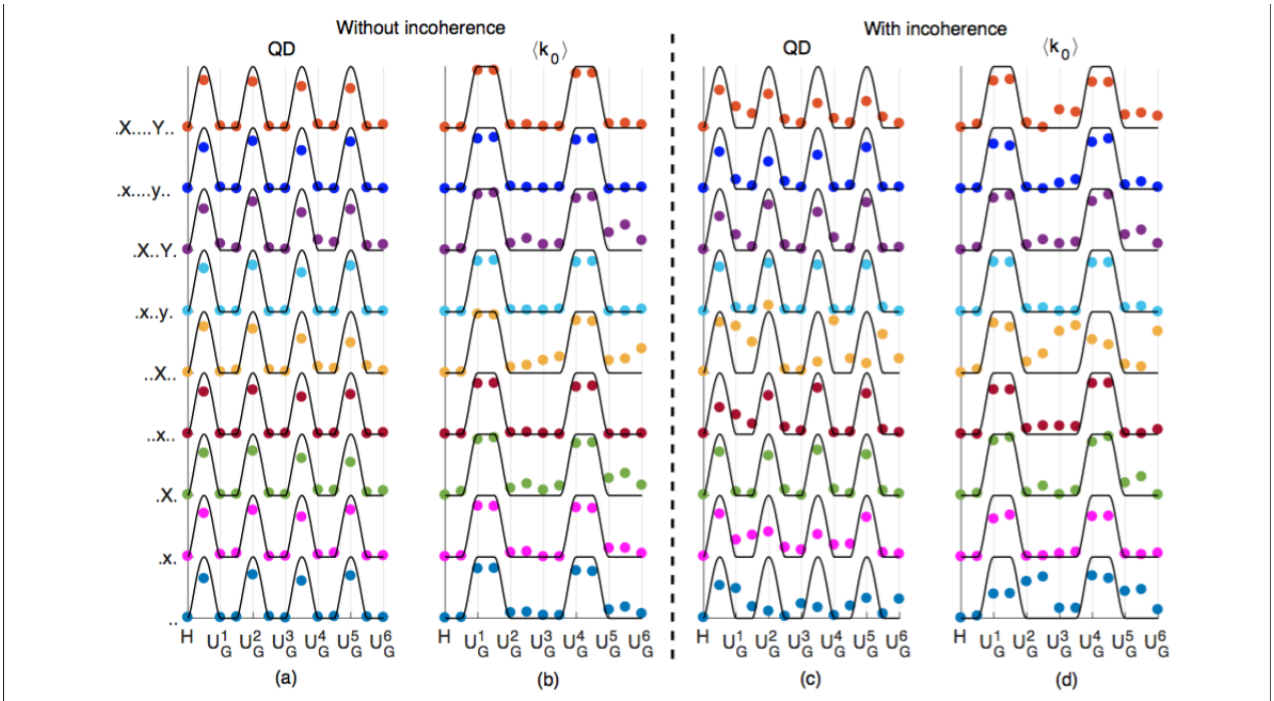


FIG. 4. Idealized estimations (solid lines - ranging from 0 to 1) and experimental (filled circles - in units of $\epsilon^2/\log_e 2$) QD (a, c) and probability $\langle k_0 \rangle$ of the marked state (b, d) under Grover iterates with various DD-protections without (a, b) and with (c, d) additional incoherence. The bottom trace (indicated by ..) corresponds to unprotected Grover iterates. In other traces, lower-case letters x, y indicate $(\pi/2)_x$ and $(\pi/2)_y$ respectively, while the upper-case letters X, Y indicate $\pi_x$ and $\pi_y$ respectively. The spacing between DD-pulses is indicated by one dot for a set of 500 GRAPE segments; eg. "..x.." indicates a $(\pi/2)_x$ pulse in the middle of 2000 GRAPE segments.

Figure 3.14: Experimental results from [1] demonstrating different DD schemes for $\pi/2$ pulses and $\pi$ pulses

# Chapter 4

# Summary and Conclusions

- Quantum Correlations are precious resources in quantum information processing and need to be preserved. It was found that not just entanglement but a broader class of non-classical correlations, was the key to quantum mechanically enhancing classical tasks. Quantum entanglement, discord, contextuality, etc. are some of the well studied quantum correlations today, that have applications in many areas including teleportation, quantum cryptography, etc.

- Quantum algorithms are an essential part of quantum computing and information processing. They comprehensively take advantage of the quantum features of systems that make a quantum computer, to speed up processing (*quantum parallelism*) or simulate naturally occurring quantum systems that are hard to describe using classical means. The study of quantum algorithms garnered further interest when it was found that they can offer exponential or quadratic speed ups to classically hard problems.

- Quantum discord is a measure of the difference between mutual information measured in two classically equivalent but quantum mechanically distinct ways.

- Grover's algorithm or the quantum search algorithm runs a search on an unstructured database to find solutions, and offers a quadratic speed up over all its classical counterparts. A Grover iterate $G$ is composed of two constituent operator steps - the oracle $U_w$ and the diffusion operator $U_d$. The oracle flips the solution states and marks them out by inverting their phase, and the diffusion operator amplifies the amplitude of these marked states by inverting all the states about their mean. The search is successful

when the amplitudes of the marked states can be measured with a high probability.

- In this project, we have studied the evolution of quantum discord in Grover's algorithm for a two qubit system. For the two qubit case of Grover's algorithm, the search is successful after just one iteration, when the system reaches the marked/solution state. Since this is a classical state, the discord vanishes, as it is a quantum correlation. The evolution of quantum discord in Grover's algorithm is plotted in Figure 3.2.

- To take decoherence into account, noise is introduced into the system. The evolution of quantum discord for Grover's algorithm in the presence of a dephasing noise is plotted in Figure 3.3 and Figure 3.4. It is observed that the diffusion operator is more sensitive to noise than the oracle (Figure 3.5)

- Dynamical Decoupling is a well known decoherence suppression scheme. It decouples the interactions between the environment and system by performing a series of rapid qubit flips. Integrating DD into quantum gate operations to make them more robust is advantageous, as there is no need to separately account for pulse errors that come from RFI. [1]. Figure 3.6 and Figure 3.7 show that the protected quantum gate schemes with $\pi$ pulse based DD are effective in suppressing decoherence, even for higher values of noise amplitudes.

- It was experimentally found that $\pi/2$-pulse based DD sequences perform as well as, if not outperform $\pi$-pulse based DD schemes [1]. The effect of DD for different pulse angles ($\beta$) has been studied (Figure 3.8). It is observed that $\pi$-pulse based DD outperforms non-$\pi$ pulse based DD in the order:

  $$\pi > 3\frac{\pi}{4} > \frac{\pi}{2} > \frac{\pi}{3} > \frac{\pi}{4} > \frac{\pi}{6} > no\ DD$$

  However, for low noise amplitudes, non-$\pi$ pulse based DD sequences can also successfully suppress decoherence, even if they are not as effective as the $\pi$-pulse based ones.

- Experimentally applied pulse sequences are not perfect. To account for pulse errors, the effect of RFI on DD is taken into account. We assume a spread of the pulse angle about $\beta$ as the mean, and a corresponding probability distribution that peaks at $\beta$. We see that $\pi$- pulse based DD is more effective at suppressing decoherence than non-$\pi$ pulse based DD. $\pi$-pulse based DD is also effective and stable at higher noise

amplitudes, unlike $\pi/2$-pulse based DD (Figure 3.9). However, non-$\pi$ pulse based DD may sometimes perform better in the presence of RFI than its absence, due to the mixture of higher pulse angles in the distribution of the imperfect pulse (Figure 3.10).

- The evolution of quantum discord for various DD schemes in the presence of RFI for different pulse angles ($\beta$) is simulated. In general, it is observed that $\pi$-pulse based DD outperforms other non-$\pi$ pulse based DD, even in the presence of RFI (Figure 3.11, Figure 3.12, Figure 3.13), in the order:

$$\pi > 3\frac{\pi}{4} > \frac{\pi}{2} > \frac{\pi}{3} > \frac{\pi}{4} > \frac{\pi}{6} > no\ DD$$

  However, for very low noise amplitudes, we see that even the non-$\pi$ pulse based DD sequences are also pretty efficient at suppressing decoherence, even if not as ideally as $\pi$ pulse based DD. This is especially true for the pulse angles $\pi/2$ and $3\pi/4$.

- Contrary to the conventional notion that dynamical decoupling is achieved only by $\pi$ pulses, we show by numerical analysis that if properly incorporated into control sequences, even non-$\pi$ pulses can bring about efficient dynamical decoupling. Thus the current work supports earlier experimental demonstration of dynamical decoupling with non-$\pi$ pulses [1].

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