Representation Theory of p-adic Groups and the Local Langlands Correspondence for $GL(2)$

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Certificate

This is to certify that this dissertation entitled Representation Theory of p-adic Groups and the Local Langlands Correspondence for $GL(2)$ towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Nazia V. at Indian Institute of Science Education and Research under the supervision of Dr. Chandrasheel Bhagwat, Associate Professor, Department of Mathematics , during the academic year 2019-2020.

चं.रा. भागवत May 22, 2020

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Declaration

I hereby declare that the matter embodied in the report entitled Representation Theory of p-adic Groups and the Local Langlands Correspondence for $GL(2)$ are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Chandrasheel Bhagwat and the same has not been submitted elsewhere for any other degree.

May 22, 2020

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Abstract

One aim of this thesis is to discuss the representation theory of $GL_2(F)$, where F is a non-archimedean local field. Then we look at the decomposition of $L^2(\Gamma \setminus PGL_2(F))$ into irreducible unitary representations and we aim to prove a correspondence between multiplicity of spherical representations in the decomposition and eigenvalue of Hecke operator on the quotient graph of Bruhat-Tits tree. To prove this, we use the theory of spherical functions. We also aim to state the local Langlands correspondence for $GL_2(F)$, by discussing all the required machinery to understand the statement. Along with the representation theory of $GL_2(F)$, this also requires representation theory of Weil group and the theory of L-functions and local constants of these two classes of representations.

Contents

Introduction

In this thesis we study the representation theory of p-adic groups, specifically $GL_2(F)$, where F is a non-archimedean local field. Then we study a problem about representations of $PGL_2(F)$. We also study the local Langlands correspondence for $GL_2(F)$.
Representation theory of p-adic groups was first studied by F. I. Mautner in 1958, in his paper [Mau58] on spherical functions of $PGL_2(F)$. The subject developed due to the work of pioneers in the field like Harish-Chandra, Bruhat, Jacquet, Howe etc. Following the book [BH06] by C.J Bushnell and G. Henniart, we aim to study the classification of irreducible smooth representations of $GL_2(F)$.
In the second part of the thesis, we prove a theorem which lies in the intersection of representation theory of p-adic groups, harmonic analysis and graph theory. The motivation for this problem comes from the harmonic analysis on locally symmetric Riemannian manifolds. Let's look at the Lie group case here. The action of $SL_2(\mathbb{R})$ on the upper half plain gives the hyperbolic plane. Hence the Laplacian operator on this space can be viewed as acting on $SL_2(\mathbb{R})/SO(2)$. For a uniform lattice Γ of $SL_2(\mathbb{R})$, the space of complex smooth functions on the locally symmetric space $\Gamma \setminus SL_2(\mathbb{R})/SO(2)$ decomposes into direct sum of eigenspaces by the action of Laplacian. There is another decomposition that happens here, which is of the right regular representation of the Hilbert space $L^2(\Gamma \setminus SL_2(\mathbb{R}))$ (space of square integrable functions on $\Gamma \setminus SL_2(\mathbb{R})$) into irreducible unitary representations. This duality gives raise to a relation between dimension of eigenspaces of Laplacian and the multiplicity of representations occurring in the decomposition. This duality theorem is proved by Gelfand, Graev and Shapiro in [GGPS68], for a semisimple lie group. In this thesis, we try to prove the local field equivalent of this theorem. The p-adic group $PGL_2(F)$ acts on Bruhat-Tits tree, which is an infinite regular graph and this is our p-adic equivalent of upper half space. There is a Hecke operator acting on Bruhat-Tits tree which results in the decomposition of $\Gamma \setminus PGL_2(F)/PGL_2(\mathfrak{o})$ into eigenspaces $V(\lambda_\pi, \Gamma)$. Similar to the Lie group case, $L^2(\Gamma \setminus PGL_2(F))$ also decomposes. The two decompositions are related by the theory of spherical function, which is well studied for any unimodular group by S. Lang in [Lan75]. The precise statement of the problem which we prove is:

(Duality Theorem) Let Γ be a uniform lattice in $PGL_2(F)$ and π be an irreducible unitary spherical representation of $PGL_2(F)$ with associated eigenvalue λ_π. Consider the representation $(L^2(\Gamma \setminus PGL_2(F)), \mathcal{R}_\Gamma)$ of $PGL_2(F)$. Then

$$\mathrm{m}(\pi, \Gamma) = \dim V(\lambda_\pi, \Gamma),$$

where $\mathrm{m}(\pi, \Gamma)$ is the multiplicity of π in $\mathcal{R}_\Gamma$ and $V(\lambda_\pi, \Gamma)$ is the eigenspace of $\lambda_\pi \in \mathbb{C}$. The connection between spherical functions, Hecke operator and representations of $PGL_2(F)$ can be understood from [Lub10] by A. Lubotzky.

Third part of this thesis is aimed at stating the local Langlands correspondence for $GL_2(F)$. This correspondence is a part of the Langlands program, which is a series of conjectures articulated by Robert Langlands in 1967, in a letter to Andre Weil. Through these conjectures, Langlands envisioned close relations between number theory and representation theory. Local Langlands correspondence unifies the representation theory of p-adic groups, which is harmonic analysis on p-adic groups and the local Galois representation, which is a part of algebraic number theory. A naive version of the statement of local Langlands correspondence goes like: $\mathcal{G}_n(F)$ is the class of complex n-dimensional semisimple Weil-Deligne representations of Weil group of F and $\mathcal{A}_n(F)$ is the class of complex smooth irreducible representations of $GL_n(F)$. Then there exists a unique bijection from $\boldsymbol{\pi} : \mathcal{G}_n(F) \longrightarrow \mathcal{A}_n(F)$ such that the bijection preserves L-functions and local constants associated to representations in both classes. This correspondence for $GL(n)$ was first proved by Guy Henniart in 2000 and M. Harris-Richard Taylor in 2001. In 2010, Peter Scholze gave a new proof of the theorem. The centrality of local Langlands correspondence in modern number theory can be understood by observing that the local class field theory is $n = 1$ case of this theorem.

Our aim in the local Langlands part of the thesis is to state the correspondence for $GL_2(F)$ by discussing all the machinery required to understand the statement. We completely follow [BH06] for this purpose. In addition to the representation theory of $GL_2(F)$ discussed in the first part of thesis, we require theory of L-functions and local constants for both $GL(2)$ and the Galois side. In the above statement of correspondence, we saw that it uses a slight variation of Galois group, called the Weil group, hence we shall discuss its representation theory.

We explain the structure of the thesis: Chapter 1 covers all the required tools from algebraic number theory and representation theory to understand this thesis. Chapter 2 discusses representation theory of $GL_2(F)$, which consists of classification of smooth irreducible representations in four sections and the remaining three sections have a discussion of some other important classes of representations of $GL_2(F)$. In chapter 3 we describe the necessary background to understand the problem and we give a proof for problem. Next two chapters are devoted to discuss the tools to get to the statement of local Langlands correspondence. This consists of chapter 4 which describes the theory of L-functions and local constants for representations of $GL(2)$ and chapter 5 with essential details about the representation theory of Weil group and L-functions and local constants associated to it. We conclude the thesis with chapter 6 which has the statement of local Langlands correspondence and a glimpse to its proof.

Chapter 1

Preliminaries

1.1 Local Field

Definition 1.1.1. An absolute value $|\cdot|$ defined on a field K is said to be *discrete* if the valuation group $|K^\times|$ is discrete in $\mathbb{R}$. If the absolute value satisfies the property $|x+y| \leq \max\{|x|, |y|\}$, it is called a *non-archimedean absolute value*.

Of course any finite field with the trivial absolute value is an example, but we will get to more interesting examples.

Definition 1.1.2. A *local field* is a field with a non-trivial absolute value such that the induced topology is locally compact.

Example 1.1.1. The field of real numbers $\mathbb{R}$ and the field of complex numbers $\mathbb{C}$ with the usual absolute value are examples of local field.
Let's discuss the most important example of a local field. Fix a prime p. Then for any $x \in \mathbb{Q}$, there exists an integer $v_p(x)$ such that $x = p^{v_p(x)}\frac{b}{c}$ where $b, c \in \mathbb{Z}$ and are coprime to p. Define the p-adic norm

$$\begin{aligned} |\cdot|_p : \mathbb{Q} &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto p^{-v_p(x)}. \end{aligned}$$

The field obtained by completing $\mathbb{Q}$ with respect to the p-adic norm is called the *p-adic field*, denoted by $\mathbb{Q}_p$.

Theorem 1.1.1. *(classification theorem)* Let K be a local field. Then

(a) if K is a characteristic zero field, then either $K = \mathbb{R}$ or $\mathbb{C}$ or a finite field extension of $\mathbb{Q}_p$ for some prime p.

(b) if K is a characteristic p field, then K is non-archimedean and it is isomorphic to $F_{p^r}((t))$, which is the field of formal Laurent series in one variable over the finite field F_{p^r}.

Proof. Refer to [RV02] for the proof. ■

1.1.1 Some Properties of the Non-Archimedean Local Fields

Throughout the thesis, we will fix F to be a non-archimedean local field.

Definition 1.1.3. Let

$$\mathfrak{o} = \{x \in F : |x| \leq 1\},$$
$$\mathfrak{p} = \{x \in F : |x| < 1\}.$$

From the non-archimedean property it follows that $\mathfrak{o}$ is a local ring with the unique maximal ideal $\mathfrak{p}$. The ring $\mathfrak{o}$ is called the *discrete valuation ring* of F.

The discrete valuation ring and the maximal ideal has a number of fascinating properties which we will list below:

1. $\mathfrak{o}/\mathfrak{p}$ is a finite field, with cardinality q. We will denote this residue field by $\boldsymbol{k}$.

2. Units in the ring $\mathfrak{o}$ denoted by U_F is the set of all elements in $\mathfrak{o}$ with an absolute value 1 and this forms a group under multiplication.

3. The ideal $\mathfrak{p}$ is principal. A generator of $\mathfrak{p}$, which is called a *uniformizer* is not unique, but we will fix an element $\varpi \in \mathfrak{p}$ as the generator.

4. For any $x \in F^{\times}$, $x = \varpi^n u$ for some $n \in \mathbb{Z}$ and $u \in U_F$.

5. $\mathfrak{o} \cong \varprojlim_n \mathfrak{o}/\mathfrak{p}^n$, which proves that $\mathfrak{o}$ is compact. Consequently $\mathfrak{p}^n = \varpi^n \mathfrak{o}$ is compact for all $n \in \mathbb{Z}$.

6. $U_F \cong \varprojlim_n U_F/\mathfrak{p}^n$, which proves that U_F is compact.

1.2 Locally Profinite Groups

Since the p-adic groups we will discuss here and the Weil group are locally profinite, we will talk about some general results of representation theory of locally profinite groups.

Definition 1.2.1. A topological group G is said to be *locally profinite* if every open neighbourhood of identity in G contains a compact open subgroup of G.

Equivalently, G is said to be locally profinte if it is totally disconnected and locally compact topological group. Hence, every profinite group is locally profinite. Any group with discrete topology is trivial example of locally profinite group.

Remark 1.2.1. Any closed subgroup of a locally profinite group is locally profinite.

Example 1.2.1. The subgroups $\mathfrak{o} \supset \mathfrak{p}^2 \supset \mathfrak{p}^3$...... which are compact and open, give a filtration of 0 in F. Hence F is a locally profinite goup. Similarly, for the multiplicative group $F^\times$, the subgroups

$$U_F = \mathfrak{o}^\times$$
$$U_F^k = 1 + \mathfrak{p}^k \quad \forall \ k \geq 1.$$

form compact and open neighbourhoods around the identity.

Example 1.2.2. Determinant of $M_n(F)$ is a continuous map. Hence, $GL_n(F)$, which is the inverse image of $F^\times$ is open. The subgroups

$$GL_n(\mathfrak{o}) \text{ and } K_j = 1 + \mathfrak{p}^j M_n(\mathfrak{o}), j \geq 1$$

which are compact and open, give a filtration of $1 \in GL_n(F)$. This makes $GL_n(F)$ a locally profinite group.

1.3 Representation Theory of Locally Profinite Groups

Let G be a locally profinite group and V be a complex vector space. We will restrict our attention to smooth representations of $GL_2(F)$.

Definition 1.3.1. We say a representation (π, V) of G is a *smooth representation* if $\pi : G \to \text{Aut}_{\mathbb{C}}(V)$ is a homomorphism such that for all $v \in V$, $v \in V^K$ for some compact open subgroup K of G, where V^K is the space of vectors in V which are fixed by K.

Equivalently, $\pi : G \times V \to V$ has to be a locally constant map, given discrete topology on V. Hence, every smooth representation is continuous.

Proposition 1.3.1. Let $\chi : G \to \mathbb{C}^\times$ be a group homomorphism. Then, χ is continuous if and only if the kernel of χ is open.

We call continuous homomorphism from G to $\mathbb{C}^\times$ as *character* of G. Since the kernel of a character is open, it is a smooth representation $(\chi, \mathbb{C})$ of dimension one.
Consider the additive group of a local field F. The set of all characters of F, denoted by $\hat{F}$ forms a group under multiplication.

Proposition 1.3.2. Let ψ be a non-trivial character of F. Then the map

$$\begin{aligned} F &\longrightarrow \widehat{F} \\ a &\longmapsto a\psi \end{aligned}$$

where $a\psi$ is the character such that, $a\psi(x) = \psi(ax)$.

Notation 1.3.1. For any locally profinite group G, we denote $\widehat{G}$ to be the abelian group of characters of G.
For any character χ of G, $\chi^{-1} : G \longrightarrow \mathbb{C}$ is defined such that $\chi^{-1}(x) = \chi(x^{-1})$ for all $x \in G$.

1.3.1 Induced Representation

Let H be a closed subgroup of G and (σ, W) be a smooth representation of H. Consider the space X of functions $f : G \to W$ which satisfy
(1) $f(hg) = \sigma(h)f(g)$,
(2) There exists a compact open subgroup K of G such that $f(xk) = f(x)$ for all $k \in K$ and $x \in G$.
Consider the action Σ of G on X,

$$\Sigma(g)f(x) = f(xg) \text{ for all } g, x \in G.$$

The second condition forces (X, Σ) to be a smooth representation, denoted as $\mathrm{Ind}_B^G\sigma$, called *smoothly induced representation* of G. This induced representation has the following fundamental property called *Frobenius Reciprocity*,

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_H(\pi, \sigma).$$

As a variation of this theme, we construct another representation of G called *compactly induced representation*. Consider the space

$$X_c = \{f \in X : \text{ support of} f \text{ is compact modulo } H\}$$

and G acting on X_c by right translation. We denote $(X_c, \Sigma) = c\text{-}\mathrm{Ind}_H^G \sigma$. When H is an open subgroup of G, we have another version of Frobenius reciprocity which says,

$$\mathrm{Hom}_G(c\text{-}\mathrm{Ind}_H^G \sigma, \pi) \cong \mathrm{Hom}_H(\sigma, \pi).$$

Note that all open subgroups of a topological group are closed.

Lemma 1.3.1. (*Schur's Lemma*) Assume G to have a compact and open subgroup K, such that the set G/K is atmost countable. If (π, V) is a smooth irreducible representation of G, then $\mathrm{End}_G(V) = \mathbb{C}$.

Proof. See [BH06]. ∎

Corollary 1.3.0.1. Let (π, V) is a smooth irreducible representation of G. For any z in the center of G, $\pi(z)$ commutes with $\pi(g)$ for any $g \in G$. Hence, $\pi(z) \in \mathrm{End}_G(V)$. Schur's Lemma implies that $\pi(z) = \omega_\pi(z)$ for some character ω_π of center.

1.3.2 Haar Measure

We denote $C_c^\infty(G)$ as the space of all locally constant and compactly supported functions from G to $\mathbb{C}$. A *right Haar measure* on G is a non-zero linear functional I from $C_c^\infty(G)$ to $\mathbb{C}$, which is invariant under right action of G, and $I(f) \geq 0$ for all $f \in C_c^\infty(G)$, $f \geq 0$. For every locally profinite group G, there exist a right Haar measure which is unique upto multiplication by a positive real number. Similarly, existence of left Haar measure is also true. A group G is said to be *unimodular* if any left Haar measure in G is a right Haar measure. We will use the traditional notation

$$I(f) = \int_G f(g)\, d\mu(g), \text{ for all } f \in C_c^\infty(G).$$

If μ_G is a left Haar measure on G, observe that

$$f \longmapsto \int_G f(gx)\, d\mu(x), \text{ for all } f \in C_c^\infty(G)$$

also defines a left Haar measure, but the uniqueness property implies that for every $g \in G$, there exists a positive real number $\delta_G(g)$ such that

$$\delta_G(g) \int_G f(gx)\, d\mu(x) = \int_G f(x)\, d\mu(x), \text{ for all } f \in C_c^\infty(G).$$

The function δ_G is clearly a homomorphism and it is trivial on every compact open subgroup. Consequently, δ_G is a locally constant homomorphism of G and hence a character. We will call δ_G as *modular character*.

Remark 1.3.1. It is easy to see that a group G is unimodular if and only if δ_G is trivial. All compact groups and all abelian groups are unimodular. $GL_2(F)$ is also unimodular. We will shortly see an example of a non-unimodular group.

1.3.3 Dual of a Representation

For a smooth representation (π, V) of G the space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ gives a natural representation π^* of G via the action $g \cdot \langle v^*, v\rangle \longmapsto \langle v^*, \pi(g^{-1})v\rangle$, where $\langle v^*, v\rangle := v^*(v)$ for all $v^* \in V^*$, $v \in V$. But this representation need not be smooth. Thus by 'taking the smooth part' of π, we define the smooth dual $\check{V} = \bigcup_{K \leq G} (V^*)^K$ where K is compact open. The action π^* on $\check{V}$ is certainly smooth which is called *smooth dual* of (π, V) and denoted by $(\check{\pi}, \check{V})$.

1.4 About $GL(2)$

We will begin by fixing the notations, $G = GL_2(F)$, $K = GL_2(\mathfrak{o})$,
$B = \{g \in G : g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), a, d \in F^\times, b \in F\}$, $N = \{g \in G : g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), x \in F\}$,
$T = \{g \in G : g = \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right), s, t \in F^\times\}$, $Z = \{g \in G : g = \left(\begin{smallmatrix} y & 0 \\ 0 & y \end{smallmatrix}\right), y \in F^\times\}$
.
where B is the standard Borel subgroup, N is the unipotent radical of B, T is the standard split maximal torus and Z is the centre of G. It is easy to see that these subgroups are closed, hence locally profinite and $B = N \rtimes T$.

Remark 1.4.1. A left Haar measure defined on B cannot be right invariant, which gives the group a non-trivial modular character δ_B. For any $nt \in B$, where $n \in N$ and $t = \left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_2 \end{smallmatrix}\right)$, the modular character is given by $\delta_B(nt) = \frac{\|t_2\|}{\|t_1\|}$.

Now we look at two important decompositions of G.
(Iwasawa decomposition) $G = BK$.

Corollary 1.4.0.1. $B \setminus G$ is compact. This is because of the continuous sujective map from K to $B \setminus G$, where $k \mapsto Bk$.

(Cartan decomposition) $G = \bigcup_{a,b \in \mathbb{Z}, a \leq b} K\left(\begin{smallmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{smallmatrix}\right)K.$

Corollary 1.4.0.2. The subgroup K is a maximal compact subgroup of G.

Proof. Suppose a compact subgroup H of G contains K. But for every $h \in H$,

$$h = k_1\left(\begin{smallmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{smallmatrix}\right)k_2 \text{ for some } k_1, k_2 \in K \text{ and } a, b \in \mathbb{Z}.$$

This implies that the group $J = \langle\left(\begin{smallmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{smallmatrix}\right)\rangle$ is a subgroup of H. But J is clearly unbounded which is a contradiction to the compactness of H. ∎

Corollary 1.4.0.3. For every compact open subgroup K_0 of G, the set G/K_0 is countable.

Remark 1.4.2. Let (π, V) be an irreducible smooth representation of G. Let v be a non-zero vector fixed by some open compact subgroup K_0. Then the space spanned by $\{\pi(g)v : g \in G/K_0\}$ is G-invariant subspace of V, hence it is V. But G/K_0 is countable which implies (π, V) of G is of at most countable dimension. This also implies that Schur's lemma holds for any irreducible smooth representation of G, since countable dimension was the only hypothesis.

Theorem 1.4.1. *Let (σ, W) be a smooth representation of B which is trivial on N. Then*

$$(c\text{-Ind}_B^G\,\sigma)^\vee \simeq \text{Ind}_B^G\,(\delta_B^{-1} \otimes \check{\sigma})$$

Proof. See [BH06] page no. 56 for the proof. ∎

1.4.1 $GL(2)$ Over Finite Fields

Representation theory of G has close resemblance with the representation theory of general linear group over finite fields. Since the residue field of the local field F is a finite field $\boldsymbol{k}$, we have a natural surjection from G to $GL_2(\boldsymbol{k})$. In future, this map will help us to study a certain class of representations of G using that of $GL_2(\boldsymbol{k})$. We will just state few important facts.
For a finite field $\boldsymbol{k}$, fix the notations $G(\boldsymbol{k}) = GL_2(\boldsymbol{k})$, $\boldsymbol{b}$ is the group of upper triangular matrices in $G(\boldsymbol{k})$, $\boldsymbol{n}$ is the unipotent radical of $\boldsymbol{b}$ and $\boldsymbol{t}$ is the split maximal torus of $G(\boldsymbol{k})$. Parallel to the case of Borel subgroup of G, here we have $\boldsymbol{b} = \boldsymbol{n} \rtimes \boldsymbol{t}$.

(Bruhat Decomposition)

Let $\bar{w} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in G(\boldsymbol{k})$, then $G(\boldsymbol{k}) = \boldsymbol{b} \cup \boldsymbol{b}\bar{w}\boldsymbol{b}$ and this union is disjoint.

Definition 1.4.1. An irreducible representation (π, V) of $G(\boldsymbol{k})$ is defined to be *cuspidal* if π doesn't contain a trivial character of $\boldsymbol{n}$. Otherwise, we call it *non-cuspidal*.

Chapter 2

Representation Theory of $GL(2)$

In this chapter, we will discuss the construction and classification of irreducible smooth representations of $GL_2(F)$. We will also discuss some other important classes of representations of $GL_2(F)$, called admissible, spherical and unitary.

2.1 Jacquet Module

In order to classify irreducible smooth representations of $GL_2(F)$, we introduce the notion of Jacquet module. Let (π, V) be a smooth representations of G and $V(N)$ be the linear span of

$$\{v - \pi(n)v : v \in V, n \in N\}.$$

Since N is normal in B, $\pi(b)(v - \pi(n)v) = \pi(b)v - \pi(bnb^{-1})(b)v$ gives that $V(N)$ is a B-stable subspace of V. Hence we define $V_N = V/V(N)$, which is the maximal quotient space of V such that N acts trivially. Hence, V_N gives a representation of B. But recall that $B = N \rtimes T$. Essentially, (π_N, V_N) is a representation of T. The representation (π_N, V_N) of T is called *Jacquet module* of (π, V) at N. We can also view this as a functor

$$\begin{aligned} \mathrm{Rep}(G) &\longrightarrow \mathrm{Rep}(T) \\ ((\pi, V)) &\longmapsto (\pi_N, V_N), \end{aligned}$$

where $\mathrm{Rep}(G)$ and $\mathrm{Rep}(T)$ are the category of smooth representations of G and that of T respectively. In particular, the Jacquet functor is exact.

Proposition 2.1.1. Let (π, V) be a smooth irreducible representation of G. The following are equivalent:

i) Jacquet module V_N is zero.
ii) $\mathrm{Hom}_G(\pi, \mathrm{Ind}_B^G \chi) = 0$, for all characters χ of T.

If (π, V) is an irreducible smooth representation of G which satisfies any of these conditions, then π is called *cuspidal* or *supercuspidal*. Otherwise, π is called *non-cuspidal.*

Example 2.1.1. For every character ϕ of G, the kernel of ϕ has to contain commutator of G, which is exactly $SL_2(F)$. Essentially, ϕ is a homomorphism from $SL_2(F)\backslash G$. So, ϕ factors through the determinant map, which is surjective and open. Hence, $\phi = \chi \circ \det$ for some $\chi \in \widehat{F^\times}$.
Observe that all the characters of G acts trivially on N, which implies $V_N = V$, hence we have our first example for irreducible non-cuspidal representation of G.

2.2 Principal Series Representations

In this section, we will construct the remaining irreducible non-cuspidals of G. The definition itself gives an idea about the construction.
Suppose χ_1 and χ_2 are two characters of $F^\times$. Define a character χ of T

$$\chi : \left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right) \longmapsto \chi_1(s)\chi_2(t).$$

Inflate $\chi = \chi_1 \otimes \chi_2$ to a character of B which is trivial on N and then induce it to get a representation of G. This method of constructing representations of a reductive group from representations of its parabolic subgroups (Borel subgroup in our case) is called *Parabolic induction* and these induced representations are called *Principal Series Representations*. $\mathrm{Ind}_B^G \chi$ need not be irreducible always.

Proposition 2.2.1. Let $\chi = \chi_1 \otimes \chi_2$ be a character of T, where χ_1, χ_2 are characters of $F^\times$. The representation $\mathrm{Ind}_B^G \chi$ is reducible if and only if it satisfies any of the following conditions.
case 1: $\chi_1\chi_2^{-1}$ is the trivial character.
case 2: $\chi_1\chi_2^{-1}$ maps $x \mapsto \|x\|^2$ of $F^\times$.

This proposition tells us that among all the characters of T, very few of them give reducible representation when induced. We will shortly see that even these reducible representations are 'not very far' from being irreducible. We shall look at those $\mathrm{Ind}_B^G \chi$

which are reducible and try to get irreducible representations out of those.
Consider the most trivial situation in case 1, when $\chi = 1_T$, the trivial character of T. Frobenius reciprocity implies $\mathrm{Hom}_G(1_G, \mathrm{Ind}_B^G 1_T) \neq 0$. The G-quotient of $\mathrm{Ind}_B^G 1_T$ is called *Steinberg representation*, which we will denote by St_G. We have the short exact sequence

$$0 \to 1_G \longrightarrow \mathrm{Ind}_B^G 1_T \longrightarrow St_G \to 0. \tag{2.1}$$

For a character ϕ of $F^\times$ and a representation π of G, the representation $\chi \cdot \phi := \chi \circ \det \otimes \phi$ is called *twist*. We twist the exact sequence 2.1 with a character ϕ of $F^\times$ to get the short exact sequence

$$0 \to \phi \otimes \phi \longrightarrow \mathrm{Ind}_B^G (\phi \otimes \phi) \longrightarrow \phi \cdot \mathrm{St}_G \to 0, \tag{2.2}$$

Which is exactly our case 1. This concludes that the case 1 gives an irreducible subrepresentation, $\phi \circ \det$ which is non-cuspidal. The representations of the type $\phi \cdot \mathrm{St}_G$ are called *special representations*.
Dualizing (2.2) and applying theorem 1.4.1 we obtain

$$0 \to \phi^{-1} \cdot \check{\mathrm{St}}_G \longrightarrow \mathrm{Ind}_B^G(\delta_B^{-1} \otimes (\phi^{-1} \otimes \phi^{-1})) \longrightarrow \phi^{-1} \otimes \phi^{-1} \to 0 \tag{2.3}$$

But

$$\delta_B^{-1} \otimes (\phi^{-1} \otimes \phi^{-1})\left(\begin{smallmatrix} s & 0 \\ 0 & t \end{smallmatrix}\right) = \phi^{-1}(s)\|s\| \cdot \phi^{-1}(t)\|t\|^{-1},$$

which is exactly case 2. Irreducibility, non-cuspidality and self-duality of St_G comes from:

Proposition 2.2.2. Let χ, ξ are two characters of T. The space

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) = \begin{cases} \mathbb{C} \text{ if } \xi = \chi \text{ or } \xi = \chi^w \delta_B^{-1}, \\ 0 \text{ otherwise.} \end{cases}$$

where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the Weyl element.

Gathering up all these results we get:

Theorem 2.2.1. *(Classification theorem)* Any isomorphism class of irreducible, non-cuspidal representations of G has to be of one of the following:

1. the irreducible induced representation $\mathrm{Ind}_B^G \chi$, where $\chi_1\chi_2^{-1} \neq 1$ and $\chi_1\chi_2^{-1}(x) \neq \|x\|^2$;

2. the one-dimensional representations $\phi \circ \det$, for a character ϕ of $F^\times$;

3. the special representations $\phi \cdot \mathrm{St}_G$, for a character ϕ of $F^\times$.

These classes are distinct except that $\mathrm{Ind}_B^G \chi \simeq \mathrm{Ind}_B^G \chi^w \delta_B^{-1}$.

Remark 2.2.1. We should be also familiar with the notation $\boldsymbol{\iota}_B^G \sigma := \mathrm{Ind}_B^G(\delta_B^{-1/2} \otimes \sigma)$, called the *normalised induction* for some future purpose. With this notation, in theorem 2.2.1 it becomes '$\boldsymbol{\iota}_B^G \chi$ with $\chi_1\chi_2^{-1} \neq \|\cdot\|^{\pm}1$' and $\boldsymbol{\iota}_B^G(\chi_1 \otimes \chi_2) \simeq \boldsymbol{\iota}_B^G(\chi_2 \otimes \chi_1)$.

2.3 Supercuspidal Representations

The definition we have for supercuspidal representation gives no clue about its construction. Hence we find an equivalent definition using the concept of 'matrix coefficients'.

Definition 2.3.1. Let (π, V) be a smooth representation. Fix vectors $v \in V$ and $\check{v} \in \check{V}$. We define a function from G to $\mathbb{C}$:

$$\gamma_{v\otimes\check{v}} : g \longmapsto \langle \check{v}, \pi(g)v \rangle$$

Let $\mathcal{C}(\pi)$ be the linear span of $\{\gamma_{v\otimes\check{v}} : \forall\, v \in V, \check{v} \in \check{V}\}$. Any $f \in \mathcal{C}(\pi)$ is called a *matrix coefficient* of π.

Remark 2.3.1. When π is irreducible, it acts as a central character via Z on the matrix coefficient γ which makes the support of γ invariant under translation by Z.

With this tool, we can get to the following important result proved by Harish-Chandra. He proved the theorem for $GL_n(F)$, but a proof for $n = 2$ can be found in [BH06].

Theorem 2.3.1. Let (π, V) be an irreducible smooth representation of G. Then the following are equivalent:

(i) π is supercuspidal;

(ii) At least one non-zero matrix coefficient of π is compactly supported modulo Z;

(iii) All matrix coefficients of π are compactly supported modulo Z.

2.3.1 Intertwining and Hecke Algebra

We require some more tools to get to the construction.

Definition 2.3.2. Suppose K_1, K_2 are compact open subgroup of G and ρ_1, ρ_2 are irreducible smooth representation of these subgroups, respectively. An element $g \in G$ is said to *intertwine* ρ_1 with ρ_2 if

$$Hom_{K_1^g \cap K_2}(\rho_1^g, \rho_2) \neq 0$$

where ρ_1^g denotes the representation $x \mapsto \rho(gxg^{-1})$ of the group $K_1^g = g^{-1}K_1g$. Such a $g \in G$ is called an *intertwiner*.

Now we define a Hecke algebra that contains all the information of a smooth representation. For an open subgroup of K of G, containing and compact modulo Z and an irreducible smooth representation (ρ, W) of K, consider the space $\mathcal{H}(G, \rho)$ of compactly supported modulo Z functions f from G to $\mathrm{End}_{\mathbb{C}}(W)$ which satisfy

$$f(k_1gk_2) = \rho(k_1)f(g)\rho(k_2) \;\; k_i \in K, \, g \in G.$$

From the definition it is clear that support of any $f \in \mathcal{H}(G, \rho)$ is a finite union of double cosets KgK. Let $\dot{\mu}$ be a Haar measure on G/Z. Then we define convolution as multiplication in $\mathcal{H}(G, \rho)$ and this makes it an associative $\mathbb{C}$-algebra with unit. The algebra $\mathcal{H}(G, \rho)$ is called the *ρ-spherical Hecke algebra* of G. Next lemma tells us how the intertwining and ρ-spherical Hecke algebra are related.

Lemma 2.3.1. Let $g \in G$, there exists $\phi \in \mathcal{H}(G, \rho)$ with support KgK if and only if g intertwines ρ (g intertwines ρ with itself).

Let's get to the central theorem of construction of supercuspidals.

Theorem 2.3.2. Suppose J is an open subgroup of G contains Z and compact modulo Z and (Λ, W) is an irreducible smooth representation of J such that, an element $g \in G$ intertwines Λ if and only if $g \in J$. Then, $c\text{-Ind}_J^G \Lambda$ is irreducible and supercuspidal.

Theorem above implies the importance of intertwining. We have to find the pairs (J, Λ) of open subgroups J of G such that it contains and compact modulo center, and its irreducible representations Λ such that the intertwiners in G of Λ is J.

2.3.2 Maximal Compact Subgroup and Iwahori Subgroup

Construction of supercuspidal is involved, so it demands a good amount of motivation before beginning the theory. In chapter 1, we introduced the subgroup K=$GL_2(\mathfrak{o})$ and its filtrations. $GL_2(\mathfrak{o})$ is the unique maximal compact subgroup of G upto conjugation. Define the *standard Iwahori subgroup* of K as

$$I = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, d \in U_F,\ b \in \mathfrak{o},\ c \in \mathfrak{p} \}.$$

It is easy to see that I also has a filtration of compact open normal subgroups given by

$$I_k = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) : a, d \in U_F^k,\ b \in \mathfrak{o},\ c \in \mathfrak{p} \} \ \ \forall \ \ k \geq 1.$$

For a given smooth representation (π, V), we look for the largest subgroup in the filtration which has a fixed vector in V. Hence, action of next largest subgroup in the filtration has interesting information about π. We will demonstrate this using the simplest example.

Theorem 2.3.3. Suppose (π, V) is an irreducible smooth representation of G such that π contains the trivial character of K_1 and π doesn't contain the trivial character of I_1. Then π is supercuspidal and there exists an irreducible representation λ of K such that

$$\pi \simeq \operatorname{Ind}_{KZ}^{G} \Lambda,$$

where $\Lambda|_K \simeq \lambda$.

Proof. Let Ψ be the natural surjective homomorphism from K to $GL_2(\boldsymbol{k})$, which gives $K/K_1 \cong GL_2(\boldsymbol{k})$. Also observe that $I_1 = \Psi^{-1}(\boldsymbol{n})$, the upper triangular unipotent matrices in $GL_2(\boldsymbol{k})$. For the representation (π, V) it is given that V^{K_1} is non empty. But

$$\pi(k_1)\pi(k)v = \pi(k)(k^{-1}k_1k)v = \pi(k)v \quad \text{for all} \quad k \in K,\, k_1 \in K_1,\, v \in V^{K_1}.$$

So $(\pi|_K,\, V^{K_1})$ is a representation which is trivial on K_1, hence is a representation of $GL_2(\boldsymbol{k})$. Since this will be a semisimple representation of $GL_2(\boldsymbol{k})$, choose one irreducible component $\bar{\lambda}$ from the decomposition. Inflate $\bar{\lambda}$ to get an irreducible representation λ of K. Either $\bar{\lambda}$ is cuspidal, or not. If $\bar{\lambda}$ is non-cuspidal, it contains the trivial character of $\boldsymbol{n}$. This will imply that λ contains the trivial character of I_1, which is a contradiction. We need a lemma to show that the set of intertwiners of π is KZ.

Lemma 2.3.2. For i=1, 2 suppose $\bar{\rho_i}$ is an irreducible representation of $GL_2(\boldsymbol{k})$ and ρ_i is the inflation of $\bar{\rho_i}$ to a representation of K. If $\bar{\rho_i}$ is cuspidal, then

(1) The representations ρ_i intertwine in G if and only if $\bar{\rho_1} \simeq \bar{\rho_2}$.

(2) Any $g \in G$ intertwines ρ_1 if and only if $g \in ZK$.

Proof. In (1), $\Longleftarrow$ is trivial. To prove $\Longrightarrow$, suppose $g \in G$ intertwines ρ_i. If g is trivial, then $\bar{\rho_1} \simeq \bar{\rho_2}$.So assume it is not. Since intertwining is the property of coset $KgZK$, we will assume that $g = \left(\begin{smallmatrix} \varpi^k & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for some $k \geq 1$. But

$$\begin{pmatrix} 1 & \mathfrak{p} \\ 0 & 1 \end{pmatrix}^g = \begin{pmatrix} 1 & \mathfrak{p}^{1-k} \\ 0 & 1 \end{pmatrix} \supset \begin{pmatrix} 1 & \mathfrak{o} \\ 0 & 1 \end{pmatrix} = N_0.$$

Hence $N_0 \leq K_1^g \cap K_1$. This implies that ρ_2^g is trivial on N_0 since it is inflated from $GL_2(\boldsymbol{k})$. We know that $\mathrm{Hom}_{N_0}(1, \rho_1) = 0$ because $\bar{\rho_1}$ is cuspidal. Hence $\mathrm{Hom}_{N_0}(\rho_2^g, \rho_1) = 0$ which is a contradiction to the intertwining of ρ_i. So $g = e$ and this proves (1). Proof of (2) easily follows from (1). If $g \in G$ intertwines ρ_1, then $\bar{\rho_1} \simeq \overline{\rho_1^g}$ which implies the equivalence of ρ_1 and ρ_1^g. So,

$$\rho_1(gxg^{-1}) = \rho_1(x) \quad \forall x \in K.$$

This happens if and only if $g \in KZ$. ■

Lemma proves that the intertwinings of λ is exactly K. Now extend λ to a representation Λ of KZ by defining $\Lambda(kz) = \lambda(k)\omega_\pi(z)$. Observe that

$$\mathrm{Hom}_{KZ}^G(\Lambda, \pi) \cong \mathrm{Hom}_{KZ}^G(c\text{-}\mathrm{Ind}_{KZ}^G \Lambda, \pi) \neq 0$$

From theorem 2.3.2 and the lemma, we can see that $c\text{-}\mathrm{Ind}_{KZ}^G \Lambda$ is irreducible and supercuspidal. Together with the irreducibility of π, we conclude that $\pi \simeq \mathrm{Ind}_K^G Z\Lambda$. ■

One important observation is that the representation $\Lambda|K = \lambda$ is an inflation of a cuspidal represetation of $GL_2(\boldsymbol{k})$. Moreover, the theorem gives a pathway for the construction of supercuspidals: For a given π,

step I. find the maximal open subgroup in the filtration such that the restriction contains the trivial character,

step II. choose the next largest subgroup $\mathcal{J}$ in the filtration and its irreducible representation λ such that $\pi|\mathcal{J}$ contains λ and the set of intertwiners of λ is $\mathcal{J}$,

step III. extend λ to get a representation Λ of $\mathcal{J}Z$ such that it agrees with π.

Let's look at the intricacies we will face on our way: In step I, we will have to choose a filtration from that of K, I and conjugates of these two. It is evident that Step II will require a lot of work with all the choices we have to make, which makes the construction highly technical. Instead of going through the technicalities of proofs, we will state the important theorems. We use the ring-theoretic interpretation of K, I and its filtrations for developing the theory. This kind of treatment to supercuspidals is mainly due to the works of C. Bushnell, P. Kutzko, R. Howe, H. Carayol and A. Moy.

2.3.3 Theory of Fundamental Strata

The objects of study in this section are not at all new to us, but we have to develop a new perspective for further study. Consider the F-vector space $V = F \oplus F$, then $G = \mathrm{Aut}_G(V)$ and $A = M_2(F) = \mathrm{End}_F(V)$.

Definition 2.3.3. An $\mathfrak{o}$-*lattice* in V is a finitely generated $\mathfrak{o}$-submodule L such that the F-linear span of L is V. An $\mathfrak{o}$-*lattice chain* in V is a non-empty set $\mathcal{L}$ of $\mathfrak{o}$-lattices in V such that,

$$\mathcal{L} = \{L_i : i \in \mathbb{Z}\}, \ \ L_i \supsetneq L_{i+1}$$

and there exists $e_{\mathcal{L}} \in \mathbb{Z}$ such that, $xL_i = L_{i+ev_F(x)}$, for all $x \in F^\times$ and $i \in \mathbb{Z}$. This property is called stability under translation by $F^\times$.

For any $\mathfrak{o}$-lattice chain $\mathcal{L}$ in V, $e_{\mathcal{L}} = 1$ or 2. Refer to [BH06] for the proof. Define

$$\mathfrak{U}_{\mathcal{L}} = \bigcap_{i\in\mathbb{Z}} \mathrm{End}_{\mathfrak{o}}(L_i) = \{x \in A : xL_i \subset L_i, \, i \in \mathbb{Z}\} = \bigcap_{0\leq i\leq e-1} \mathrm{End}_{\mathfrak{o}}(L_i).$$

Last equality is obtained because of stability under translation property of $\mathcal{L}$. Thus, $\mathfrak{U}_{\mathcal{L}}$ is a ring with identity.

Definition 2.3.4. A *chain order* in A is a ring of the form $\mathfrak{U}_{\mathcal{L}}$ for some $\mathfrak{o}$-lattice chain $\mathcal{L}$ in V.

For every chain order $\mathfrak{U}_{\mathcal{L}}$ of a lattice chain $\mathcal{L}$, there exists a $g \in G$ such that

$$g\mathfrak{U}g^{-1} = \begin{cases} \mathfrak{M} = \left(\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} \end{smallmatrix}\right) & \text{if } e_{\mathcal{L}} = 1, \\ \mathfrak{J} = \left(\begin{smallmatrix} \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p} & \mathfrak{o} \end{smallmatrix}\right) & \text{if } e_{\mathcal{L}} = 2. \end{cases}$$

Thus, instead of studying all the chain orders, we just have to study $\mathfrak{M}$ and $\mathfrak{J}$. Importance of these representatives will be evident soon:
Let $\mathfrak{P}$ be the Jacobson radical of the chain order $\mathfrak{U}_{\mathcal{L}} = \mathfrak{U}$. Then $\mathfrak{P}$ is a two sided ideal and there exists $\Pi \in G$ such that

$$\mathfrak{P} = \Pi\mathfrak{U} = \mathfrak{U}\Pi$$

which gives rise to a filtration of compact, open and normal subgroups

$$U^0_{\mathfrak{U}} = U_{\mathfrak{U}} = \mathfrak{U}^\times$$
$$U^n_{\mathfrak{U}} = 1 + \mathfrak{P}^n, \ \ n \geq 1.$$

Remark 2.3.2. $U_{\mathfrak{M}} = K$ and $U_{\mathfrak{J}} = I$. But we will stick to the ring-theoretic point of view throughout the section.

Define $\psi_A \in \hat{A}$ by

$$\psi_A(x) = \psi(\mathrm{tr}_A\ x),$$

where tr_A is the trace map $A \to F$. For a $\psi \in \widehat{F}$, $\psi \neq 1$, *level* of ψ is defined as the least integer d such that $\mathfrak{p}^d \subset Ker\, \psi$.

Proposition 2.3.1. Let $\mathfrak{U}$ be a chain order in A with Jacobson radical $\mathfrak{P}$, and let $\psi \in \widehat{F}$ has level one. Let $m, n \in \mathbb{Z}$ such that $2m + 1 \geqslant n \geqslant m \geqslant 0$. Let $(U_{\mathfrak{U}}^{m+1}/U_{\mathfrak{U}}^{n+1})\hat{}$ be the group of characters of finite abelian group $U_{\mathfrak{U}}^{m+1}/U_{\mathfrak{U}}^{n+1}$. Then the map

$$\begin{aligned} \mathfrak{P}^{-n}/\mathfrak{P}^{-m} &\longrightarrow (U_{\mathfrak{U}}^{m+1}/U_{\mathfrak{U}}^{n+1})\hat{} \\ a + \mathfrak{P}^{-m} &\mapsto \psi_a | U_{\mathfrak{U}}^{m+1} \end{aligned}$$

is an isomorphism, where ψ_a is the function $x \mapsto \psi_A(a(x-1))$.

Now onwards we fix a choice of character $\psi \in \hat{F}$ of level one.

Definition 2.3.5. A *stratum* in A is a triple $(\mathfrak{U}, n, a)$, where $\mathfrak{U}$ is a chain order in A with the radical $\mathfrak{P}$, an integer n and $a \in \mathfrak{P}^{-n}$.

We say two strata $(\mathfrak{U}, n, a_1)$ and $(\mathfrak{U}, n, a_2)$ are equivalent if $a_1 \equiv a_2 \bmod \mathfrak{P}^{1-n}$. From 2.3.1, we can see that if $n \geq 1$, a stratum $(\mathfrak{U}, n, a)$ corresponds to a character ψ_a of $U_{\mathfrak{U}}^n$, which is trivial on $U_{\mathfrak{U}}^{n+1}$. In short, strata give a family of characters of certain compact open subgroups.

Definition 2.3.6. Let $\mathcal{S}(\pi)$ be the set of all pairs $(\mathfrak{U}, n)$ where $\mathfrak{U}$ is a chain order in A and $n \geq 0$ is the smallest integer such that π contains the trivial representation of $U_{\mathfrak{U}}^{n+1}$. Define the *normalized level* of π denoted by $\ell(\pi)$ as:

$$\ell(\pi) = \min\{n/e_{\mathfrak{U}} :\ (\mathfrak{U}, n) \in \mathcal{S}(\pi)\}.$$

Given any irreducible smooth representation (π, V) of G, we say that π contains the stratum $(\mathfrak{U}$, n, a$)$ if $n \geq 1$ and π contains the character ψ_a of $U_{\mathfrak{U}}^n$.

Definition 2.3.7. A stratum $(\mathfrak{U},\ n,\ a)$ is called *fundamental* if the coset $a + \mathfrak{P}^{1-n}$ contains no nilpotent element of A.

Remark: Fundamental stratum also appears in the name *minimal K-types* and *fundamental G-stratum* in literature. The importance of fundamental stratum will be evident from next theorem, which is a fundamental result proved by Bushnell in [Bus87] for $GL_n(F)$.

Theorem 2.3.4. Suppose π is an irreducible smooth representation of G. Then π contains a fundamental stratum if and only if $\ell(\pi) > 0$.

Note that $I_1 = U^1_{\mathfrak{J}}$ contains $K_1 = U^1_{\mathfrak{M}}$. So the case $\ell(\pi) = 0$ means that π contains the trivial character of K_1, which we discussed earlier partially. The case which π contains the trivial character of I_1 was remaining. It is proved in [BH06], page no.102, that in this case π is non-cuspidal. So we just have to focus on $\ell(\pi) > 0$ case.

Classification of Fundamental Strata

We begin the classification of fundamental strata so that it will be easy to identify what kind of strata are contained by supercuspidals. First we make some observations. Let ($\mathfrak{U}$, n, a) be a stratum in A. If $e_{\mathfrak{U}} = 1$, then $a \in \mathfrak{P}^{-n} = \Pi^{-n} g^{-1}\mathfrak{M}g$. So we have $a \in \varpi^{-n} g^{-1}\mathfrak{M}g$ and this implies $a = \varpi^{-n}g^{-1}\alpha g$ for some $\alpha \in \mathfrak{M}$. Let $f_\alpha(x)$ be the characteristic polynomial of α. Let $\overline{f_\alpha(x)} \in \boldsymbol{k}[x]$ be the reduction modulo $\mathfrak{p}$ of $f_\alpha(x)$. A fundamental stratum ($\mathfrak{U}$, n, a) in A is defined to be:

1. unramified simple if $\overline{f_\alpha(x)}$ is irreducible in $\boldsymbol{k}[x]$,
2. split if $\overline{f_\alpha(x)}$ has distinct roots in $\boldsymbol{k}$,
3. essentially scalar if $\overline{f_\alpha(x)}$ has a repeated root in $\boldsymbol{k}^\times$,
4. ramified simple if $e_{\mathfrak{U}} = 2$ and n is odd.

Note that the case $\overline{f_\alpha(x)} = x^2$ will not occur since a can't be nilpotent. From the theorem 2.3.3, 2.3.4 and using the classification, we can characterize the irreducible supercuspial representations of G.

Theorem 2.3.5. Let (π, V) be a smooth irreducible representation of G such that $\ell(\pi) \leq \ell(\chi \cdot \pi)$ for all characters χ of $F^\times$. Then π is supercuspidal if and only if

(a) either $\ell(\pi) = 0$ and π is an inflation of an irreducible cuspidal representation of $GL_2(\boldsymbol{k})$;

(b) or $\ell(\pi) > 0$ and π contains a simple stratum.

2.3.4 Classification of Supercuspidal Representaions

We will refine the theorem 2.3.5 to arrive at a classification theorem for supercuspidals. We have to choose the right open subgroup $\mathcal{J}$ of G and all its irreducible representations such that it contains the character ψ_α, and the set of intertwiners are exactly $\mathcal{J}$. Recall that any quadratic extension of F can be embedded in G. Suppose $(\mathfrak{U}, n, \alpha)$ is a simple stratum in A such that $E = F(\alpha)$ is a quadratic extension of F and $n \geq 1$. Then

$$\mathcal{J}_\alpha = E^\times U_{\mathfrak{U}}^{\lfloor (n+1)/2 \rfloor}$$

is an open subgroup of G which contains Z and compact modulo Z. It can be proved that an element $g \in G$ intertwines the character ψ_α of $U_{\mathfrak{U}}^{\lfloor n/2 \rfloor+1}$ if and only if $g \in \mathcal{J}_\alpha$.

Theorem 2.3.6. Let Λ be an irreducible representation of $\mathcal{J}_\alpha$ which contains the character ψ_α of $U_{\mathfrak{U}}^{\lfloor n/2 \rfloor+1}$. Then the representation $\pi_\Lambda = c\text{-Ind}_{\mathcal{J}_\alpha}^G \Lambda$ is irreducible and supercuspidal.

Proof. The restriction of Λ to $U_{\mathfrak{U}}^{\lfloor n/2 \rfloor+1}$ is a multiple of ψ_α (Refer to [BH06] page no. 106). So if g intertwines Λ, it has to intertwine ψ_α, which implies $g \in \mathcal{J}_\alpha$. Now the theorem 2.3.2 tells us that π_Λ is irreducible and supercuspidal. ∎

Finally we get to the classification. We introduce a new triple for convenience.

Definition 2.3.8. A cuspidal type in G is a triple $(\mathfrak{U}, \mathcal{J}, \Lambda)$ which is one of the following forms:

I. $\mathfrak{U} \cong \mathfrak{M}$, $\mathcal{J} = ZU_{\mathfrak{U}} \cong ZK$ and the restriction of Λ to $U_{\mathfrak{U}}$ is the inflation of a supercuspidal representation of $U_{\mathfrak{U}}/U_{\mathfrak{U}}^1 \cong K/K_1 = GL_2(\boldsymbol{k})$,

II. there is a simple stratum $(\mathfrak{U}, n, \alpha)$ such that $n \geq 1$, $\mathcal{J} = \mathcal{J}_\alpha$ and an irreducible representation Λ of $\mathcal{J}_\alpha$ such that the restriction of Λ to the subgroup $U_{\mathfrak{U}}^{\lfloor n/2 \rfloor+1}$ contains the character ψ_α,

III. There exists a triple $(\mathfrak{U}, \mathcal{J}, \Lambda)$ which satisfies (1) or (2) and a character χ of $F^\times$ such that $\Lambda \simeq \Lambda_\alpha \otimes \chi \circ \det$.

Theorem 2.3.7. *(induction theorem)* Let π be an irreducible supercuspidal representation of G. Then there exists a cuspidal type $(\mathfrak{U}, \mathcal{J}, \Lambda)$ such that $\pi_\Lambda \simeq c\text{-Ind}_{\mathcal{J}_\alpha}^G \Lambda$.

Corollary 2.3.7.1. *(classification theorem)* There is a bijection between the set of all conjugacy classes of cuspidal types and the set of all equivalence classes of irreducible supercuspidal representations of G, which is given by the map

$$(\mathfrak{U}, \mathcal{J}, \Lambda) \longmapsto \pi_\Lambda = \text{Ind}_{\mathcal{J}}^G \Lambda$$

Proof. If a cuspidal type $(\mathfrak{U}, \mathcal{J}, \Lambda)$ is of type I or II, the theorem 2.3.3 and 2.3.6 proves that there exists a π_Λ which is irreducible and supercuspidal. Type III overlaps with the types I and II, hence follows trivially. So it is easy to see that the map is one-one. Surjectivity follows from the induction theorem. ∎

2.4 Admissible Representations

Before we end the chapter, we discuss few more important themes in the theory of smooth representations.

Definition 2.4.1. A smooth representation (π, V) of a locally profinite group is said to be *admissible* if for every open compact subgroup K' of the group, the space of fixed vectors $V^{K'}$ is finite dimensional.

Admissible representations were introduced by Harish-Chandra and this class of representations were widely studied since the finiteness of fixed space allowed a character theory of infinite dimensional representations.

Theorem 2.4.1. Every irreducible smooth representations of G are admissible.

Proof. We will prove for supercuspidal and non-cuspidal case separately. Suppose (π, V) is non-cuspidal irreducible representation of G. Then either π is equivalent to a subrepresentation of $\operatorname{Ind}_B^G \chi$ for some $\chi \in \widehat{T}$, or it is equivalent to $\operatorname{Ind}_B^G \chi$. Anyway we just have to prove $\operatorname{Ind}_B^G \chi = (\Sigma, X)$ is irreducible for every χ. Let K' be any compact open subgroup of G.
It is enough to prove that V^{K_0} is finite dimensional, where $K_0 = K' \cap K$. For any $f \in X^{K_0}$, $f(bgk) = \chi(b)f(g)$ for all $b \in B$, $g \in G$ and $k \in K_0$. So X^{K_0} is spanned by characteristic functions of the double cosets BgK_0. From Iwasawa decomposition, we have $B \setminus G$ is compact. But

$$B \setminus G = \bigcup_{g \in B \setminus G / K_0} gK_0$$

and gK_0 is open since K_0 is open. Hence $B \setminus G/K_0$ is finite, which yields X^{K_0} is finite dimensional.
Suppose (π, V) is supercuspidal. Assume to the contrary that there exists a compact open subgroup K_0 of G such that V^{K_0} is infinite dimensional. But π is irreducible, hence has dimension atmost countable. Assume V^{K_0} is of countable dimension. That will imply that $\check{V}^K \cong \operatorname{Hom}_{\mathbb{C}}(V^{K_0}, \mathbb{C})$ has uncountable dimension.
Now consider the map $\mathcal{F}_v : \check{V}^{K_0} \longrightarrow \mathcal{C}(\pi)$ where $\check{v} \longmapsto \gamma_{\check{v} \otimes v}$ for all $\check{v} \in \check{V}_0$. This map

is clearly injective. Observe that $\mathcal{F}_v(\check{v})(zk_1gk_2) = \omega_\pi(z)\mathcal{F}_v(\check{v})(g)$, that is $\mathcal{F}_v(\check{V_0}^K)$ is spanned by the cosets $ZKgK$. But recall that if π is supercuspidal, every $f \in \mathcal{C}(\pi)$ is compactly supported modulo Z. Which implies $\mathcal{F}_v(\check{V_0}^K)$ is finite dimensional and therefore the injective preimage $\check{V_0}^K$. This is a contradiction for $\check{V_0}^K$ being uncountable dimension, which proves that V^{K_0} is finite dimensional. ∎

As a consequence of this theorem, we have an equivalence of the category of irreducible smooth representations of G and the category of irreducible admissible representations of G.

2.5 Spherical Representations

Definitions 2.5.1. A smooth irreducible representation (π, V) of G is said to be *spherical* if $V^K \neq 0$, where V^K is the space of vectors fixed by $K = GL_2(\mathfrak{o})$. A non-zero element of V^K is called a *spherical vector*.
A character χ of $F^\times$ is said to be *unramified* if χ is trivial on U_F.

Example 2.5.1. Suppose a character ϕ of G is spherical. But $\phi = \chi \circ \det$ for some character χ of $F^\times$. Since $\det(k) \in U_F$ for all $k \in K$, we need χ to be unramified for it to be spherical.

Theorem 2.5.1. Let (π, V) be a smooth irreducible spherical representation of G. Then either $\pi = \chi \circ \det$ for some unramified character χ of $F^\times$, or $\pi = \mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ with both χ_i unramified.

Proof. Refer to [Bum97] p. 496, Theorem 4.6.4 for the proof. ∎

2.6 Unitary Representations

This discussion is from [Bum97], but the reader is warned that [Bum97] uses the normalised induced representation for the discussion and this gives a slightly different parameter in theorems. We will begin with few definitions.

Definition 2.6.1. A real bilinear pairing $\langle\, , \rangle$ on the complex vector space V is called *sesquilinear* if it is complex linear in the first variable and antilinear in the second variable. A sesquilinear form is called *Hermitian* if it satisfies

$$\langle x, y\rangle = \overline{\langle y, x\rangle}.$$

Definition 2.6.2. A representation (π, V) of a group G, where V is a complex Hilbert space is said to be *unitary* if $\pi(g)$ is a unitary operator for all $g \in G$. A representation (π, V) of G is said to be *unitarizable* if there exists a positive definite G-invariant Hermitian pairing on V.

Our goal is to find out all unitarizable principal series representations of G.

Proposition 2.6.1. Let χ_1 and χ_2 are unitary characters of $F^\times$. Then the smooth representation $\mathrm{Ind}_B^G\,(\bar{\delta}_B^{-1/2}(\chi_1 \otimes \chi_2))$ is unitarizable.

Proof. Define a bilinear pairing

$$\langle f, g\rangle = \int_K f(k)\overline{g(k)}\, d\mu_K(k).$$

This pairing is clearly Hermitian and positive definite. Proof of G-invariance is given in [Bum97], Lemma 2.6.1. ■

Next proposition gives a partial converse of this.

Proposition 2.6.2. Suppose that $\mathrm{Ind}_B^G\,(\chi_1 \otimes \chi_2)$ admits a G-invariant nondegenerate Hermitian pairing. Then either $\bar{\delta}_B^{1/2}\chi_1 \otimes \chi_2$ is unitary, or $\chi_1 = \overline{\chi_2}^{-1}$.

Proof. Consider the G-invariant map

$$\begin{aligned} \mathrm{Ind}_B^G\,(\chi_1 \otimes \chi_2) &\longrightarrow \mathrm{Ind}_B^G\,(\overline{\chi_1} \otimes \overline{\chi_2}) \\ f &\longmapsto \bar{f}. \end{aligned}$$

This gives a G-invariant nondegenerate bilinear pairing

$$\begin{aligned} \mathrm{Ind}_B^G\,(\chi_1 \otimes \chi_2) \times \mathrm{Ind}_B^G\,(\overline{\chi_1} \otimes \overline{\chi_2}) &\longrightarrow \mathbb{C} \\ (f_1, f_2) &\longmapsto \langle f_1, \overline{f_2}\rangle, \end{aligned}$$

which gives a G-invariant linear map

$$\begin{aligned} \mathrm{Ind}_B^G\,(\overline{\chi_1} \otimes \overline{\chi_2}) &\longrightarrow (\mathrm{Ind}_B^G\,(\chi_1 \otimes \chi_2))^\vee \\ f &\longmapsto \check{f}, \end{aligned}$$

where $\check{f} : h \mapsto (h, f)$. This map is injective since $(\ ,\ )$ is non-degenerate. Irreducibility of $(\mathrm{Ind}_B^G\,(\chi_1 \otimes \chi_2))^\vee$ implies that it is surjective. Thus

$$\mathrm{Ind}_B^G\,(\overline{\chi_1} \otimes \overline{\chi_2}) \simeq (\mathrm{Ind}_B^G\,(\chi_1 \otimes \chi_2))^\vee \simeq \mathrm{Ind}_B^G\,(\bar{\delta}_B^{-1}(\chi_1^{-1} \otimes \chi_2^{-1})),$$

where the second equivalence is obtained from theorem 1.4.1. From 2.2, observe that either $|\chi_1 \otimes \chi_2|^2 = \bar{\delta}_B^{-1}$ is unitary, or $\chi_1 = \overline{\chi_2}^{-1}$. ■

We have another useful lemma which says:

Lemma 2.6.1. For any character χ of $F^\times$, there exists characters $\dot{\chi}$ and χ_s of $F^\times$ such that $\chi = \dot{\chi}\chi_s$, where $\dot{\chi}$ is a unitary chracter and $\chi_s(x) = \|x\|^s,\ s \in \mathbb{R}$ for any x in $F^\times$.

Proof. The proof immediately follows from the fact that $F^\times \cong \mathbb{Z} \times U_F$, any character of $\mathbb{Z}$ of the form $n \mapsto e^{ins}$ for some $s \in \mathbb{R}$, for all $n \in \mathbb{Z}$. All characters of U_F are unitary as it is a compact group. ■

Notice that any representation π is unitarizable if and only if $\chi \otimes \pi$ is unitarizable for all unitary characters χ of $F^\times$. Consider the representation $\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2)$ such that $\chi_1 = \overline{\chi_2}^{-1}$. But 2.6.1 implies $\chi_1 = \dot{\chi}\chi_s$ and $\chi_2 = \dot{\chi}\chi_s^{-1}$ for some unitary character $\dot{\chi}$ of $F^\times$ and a real number s. Then

$$\mathrm{Ind}_B^G(\chi_1 \otimes \chi_2) = \mathrm{Ind}_B^G(\dot{\chi}\chi_s \otimes \dot{\chi}\chi_s^{-1}) \simeq \dot{\chi} \cdot \mathrm{Ind}_B^G(\chi_s \otimes \chi_s^{-1}),$$

which helps us to conclude that determining all the unitarizable representations of the type $\chi_s \otimes \chi_s^{-1}$ is enough.

Proposition 2.6.3. Suppose that s is a real number, $s \neq= 0$ and $s \neq 1$ (so that $\mathrm{Ind}_B^G(\chi_s \otimes \chi_s^{-1})$ irreducible). Then $\mathrm{Ind}_B^G(\chi_s \otimes \chi_s^{-1})$ is unitarizable if and only if $-1 < s < 0$.

Proof. Refer to [Bum97], Proposition 4.6.13 for the proof. ■

Chapter 3

Bruhat-Tits Tree and Spectral Decomposition

This chapter is devoted to the discussion of a duality theorem which we discussed in the introduction. Two important concepts in the theory of p-adic groups, Bruhat-Tits tree and spherical functions are discussed initially. We mentioned earlier that the study of representation theory of p-adic group was originated from theory of spherical functions. In this chapter, we will also get a glimpse of this connection. We study the action of a certain Hecke operator on Bruhat-Tits tree and the right regular representation on the space of L^2 functions on $PGL_2(F)$. In the end we prove a duality theorem which relates these two actions, where the interconnections of Bruhat-Tits theory, spherical functions and representation theory become explicit.
Throughout this chapter, we denote $\mathcal{G} = PGL_2(F)$. This is a quotient by closed subgroup of a locally profinite group, hence locally profinite, and $\mathcal{K}$, $\mathcal{I}$, $\mathcal{I}_1$ are the image of K, I and I_1 respectively in the quotient map from G to $\mathcal{G}$, hence open and compact. Since Z is abelian, it is unimodular and hence there exists a G-invariant Haar measure $d\mu_{\mathcal{G}}$ on the quotient space $\mathcal{G}$. A representation of $\mathcal{G}$ which has a non-zero vector fixed by $\mathcal{K}$ is called spherical representation of $\mathcal{G}$ and recall the definition of unitary representation from chapter 3.

3.1 Bruhat-Tits Tree for $GL(2)$

Bruhat-Tits theory gives a geometric parametrization of reductive groups over a non-archimedean local field. One gets more insight about the inner structure of these groups via this construction, hence has wide applications especially in the study of smooth representations of reductive groups over a non-archimedean local field. But

we are here to study about a certain operator on the Bruhat-Tits tree and eigenvalues of this operator.
Recall the definition of $\mathfrak{o}$- lattice from chapter 2. Two $\mathfrak{o}$- lattices L_1 and L_2 are said to be *similar* if there exists $k \in F^{\times}$ such that $L_1 = kL_2$.

Definition 3.1.1. Define a graph X with vertex set, similarity classes of lattices, which we denote by $\mathfrak{X}$. We set M, $L \in \mathfrak{X}$ have an edge between them if

$$\varpi L \subset M \subset L.$$

Remark 3.1.1. The graph X is a $q+1$-regular tree ([Ser02], page no.70) and it is called the *Bruhat-Tits tree for* $GL_2(F)$.

Suppose L is an $\mathfrak{o}$-lattice, it is easy to see that L is a rank two free $\mathfrak{o}$-module and $L = \mathfrak{o}u \oplus \mathfrak{o}v$ for some u and v in F^2. Then, for any other lattice M, there exists m and n in $\mathbb{Z}$ such that $M = \mathfrak{o}\varpi^m u \oplus \mathfrak{o}\varpi^n v$. We define the *distance* between the two lattices L and M to be $|m-n|$. Observe that L and M have an edge between them if and only if the distance is 1. In that case, we say L and M are *neighbours*.
There is a natural action of G on the tree. For any $g \in G$ and a vertex $L \in \mathfrak{X}$, where L is the $\mathfrak{o}$-span of two vectors u, $v \in F^2$, define the action $g \cdot L = \text{span}_{\mathfrak{o}}\{gu, gv\}$. It is easy to prove that this action is transitive. Due to the similarity, center of G acts trivially. Essentially, this is a transitive action of $\mathcal{G}$ on the tree. Let L_0 be the lattice spanned by the vectors $u_0 = (1,0)$ and $v_0 = (0,1)$ in F^2. This lattice is invariant under the action of $\mathcal{K}$. Hence we have the bijection

$$\begin{aligned} \mathcal{G}/\mathcal{K} &\longrightarrow \mathfrak{X} \\ x\mathcal{K} &\longmapsto x \cdot L_0. \end{aligned}$$

The bijection gives us a better perspective in many instances, for example we will view the complex functions on $\mathcal{G}/\mathcal{K}$ as the complex functions on vertices of the tree. We will find the neighbours of e, the identity element in $\mathcal{G}$. Let $\mathbb{P}^1(\boldsymbol{k})$ be the projective space of $\boldsymbol{k}$. It consists of q lines each passing through origin and points $(a, 1)$ in $\boldsymbol{k}^2$, and a line which passes through $(1, 0)$.

Proposition 3.1.1. Let $\mathfrak{X}_1$ be the neighbours of L_0. The map

$$\begin{aligned} \mathbb{P}^1(\boldsymbol{k}) &\longrightarrow \mathfrak{X}_1 \\ \lambda = (x,y) &\longmapsto L_\lambda \end{aligned}$$

is a bijection, where $L_\lambda = \varpi L_0 + \mathfrak{o}(x,y)$ and (x,y) is the point on $\boldsymbol{k}^2$ which represents an element in $\mathbb{P}^1(\boldsymbol{k})$.

Hence, the neighbours of $L_0 = \text{span}_{\mathfrak{o}}\{(1,0),\ (0,\ 1)\}$ at distance 1 are

$$\begin{aligned}&\{\varpi L_0 + \mathfrak{o}(a,1), \forall a \in \boldsymbol{k}\} \cup \{\varpi L_0 + \mathfrak{o}(0,1)\}\\ &= \text{span}_{\mathfrak{o}}\{(\varpi,0),(0,\varpi),(a,1)\} \cup \text{span}_{\mathfrak{o}}\{(1,0),(0,\varpi)\}\\ &= \text{span}_{\mathfrak{o}}\{(\varpi,0),(a,1)\} \cup \text{span}_{\mathfrak{o}}\{(1,0),(0,\varpi)\}\end{aligned}$$

Hence, the neighbours of e are

$$\{\left(\begin{smallmatrix}\varpi & a\\ 0 & 1\end{smallmatrix}\right), \forall a \in \boldsymbol{k}\} \cup \{\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)\}. \tag{3.1}$$

3.1.1 Hecke Operator

Let $\mathcal{C}_c(\mathcal{G}/\mathcal{K})$ be the space of all continuous complex valued functions on $\mathcal{G}/\mathcal{K}$ with compact support. Define an operator acting on $f \in \mathcal{C}_c(\mathcal{G}/\mathcal{K})$,

$$T_m(f)(x) = \sum_{|x:y|=m} f(y)$$

where, $|x : y|$ is the distance between x and y, and m is any positive integer. Denote $T_1 = \mathcal{A}$, which we call the *adjacency operator* or *Hecke operator*.
We define a function $f \in \mathcal{C}_c(\mathcal{G}/\mathcal{K})$ to be *K-bi-invariant* if $f(k_1 g k_2) = f(g)$ for all k_1, $k_2, \in \mathcal{K}$ and $g \in \mathcal{G}$. Denote $\mathcal{C}_c(\mathcal{G}//\mathcal{K})$ to be the space of all $\mathcal{K}$-bi-invariant functions in $\mathcal{C}_c(\mathcal{G}/\mathcal{K})$. The convolution operation makes this an algebra. The characteristic function of $\mathcal{K}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{K}$, denoted by $\bar{\delta}$ is clearly an element in $\mathcal{C}_c(\mathcal{G}//\mathcal{K})$.

Proposition 3.1.2.

$$\mathcal{K}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{K} = \sum_{a\in\boldsymbol{k}}\left(\begin{smallmatrix}\varpi & a\\ 0 & 1\end{smallmatrix}\right)\mathcal{K} \cup \left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)\mathcal{K}. \tag{3.2}$$

We use $\cup$ to denote disjoint union throughout the chapter.

Proof. Recall the surjective map from G to $GL_2(\boldsymbol{k})$. Observe that the inverse image of upper triangular matrices in $GL_2(\boldsymbol{k})$ under this map is the Iwahori subgroup I of G. From Bruhat decomposition of $GL_2(\boldsymbol{k})$, we get $K = I \cup IwI$ where w is the Weyl element which implies that $\mathcal{K} = \mathcal{I} \cup \mathcal{I}w\mathcal{I}$. Then (3.2) becomes

$$\mathcal{K}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{K} = \mathcal{I}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{K} \cup \mathcal{I}w\mathcal{I}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{K}. \tag{3.3}$$

Now writing the cosets $\mathcal{I}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\mathcal{I}w\mathcal{I}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)$ as sum of left $\mathcal{I}_1$ cosets yields the desired result. ■

Using this we prove a very useful proposition.

Proposition 3.1.3. For any $f \in \mathcal{C}_c(\mathcal{G}/\mathcal{K})$, $\mathcal{A}(f) = f * \bar{\delta}$, where $*$ is the convolution.

Proof. We have

$$f * \bar{\delta}(x) = \int_{\mathcal{G}} f(xg^{-1})\bar{\delta}(g)\, d\mu_{\mathcal{G}}(g) = \int_{\mathcal{K}\left(\begin{smallmatrix}\varpi & 0\\ 0 & 1\end{smallmatrix}\right)\mathcal{K}} f(xg^{-1})\, d\mu_{\mathcal{G}}(g). \tag{3.4}$$

But by Proposition 3.2.1, Equation (3.4) becomes

$$f * \bar{\delta}(x) = \sum_{a\in \boldsymbol{k}} \int_{\left(\begin{smallmatrix}\varpi & a\\ 0 & 1\end{smallmatrix}\right)\mathcal{K}} f(xg^{-1})\, d\mu_{\mathcal{G}}(g) + \int_{\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right)\mathcal{K}} f(xg^{-1})\, d\mu_{\mathcal{G}}(g) \tag{3.5}$$

$$= \sum_{a\in \boldsymbol{k}} \int_{\mathcal{K}} f(gk^{-1}\left(\begin{smallmatrix}\varpi & a\\ 0 & 1\end{smallmatrix}\right))\, d\mu_{\mathcal{K}}(k) + \int_{\mathcal{K}} f(gk^{-1}\left(\begin{smallmatrix}1 & 0\\ 0 & \varpi\end{smallmatrix}\right))\, d\mu_{\mathcal{K}}(k) \tag{3.6}$$

and the left $\mathcal{K}$-invariance of f gives the desired result. ■

Suppose $f \in \mathcal{C}_c(\mathcal{G}/\mathcal{K})$ is an eigenfunction of $\mathcal{A}$ with eigenvalue λ then f is also an eigenfunction for all operators T in $\mathbb{C}[\mathcal{A}]$, which is the ring of polynomials in Hecke operator. We will denote eigenvalue of T by $\lambda(T)$. In fact we have an algebra homomorphism,

$$\begin{aligned}\mathbb{C}[\mathcal{A}] &\longrightarrow \mathbb{C}\\ T &\longmapsto \lambda(T).\end{aligned}$$

We will call this map *eigencharacter*.

3.2 Spherical Functions of $PGL(2)$

Definition 3.2.1. A function $\phi \in \mathcal{C}_c(\mathcal{G}/\mathcal{K})$ is called *spherical function* if:

1. $\phi(e) = 1$,
2. ϕ is $\mathcal{K}$-bi-invariant,
3. $T(\phi) = \lambda(T)\phi$ for all $T \in \mathbb{C}[\mathcal{A}]$.

Before getting into some representation theory of $\mathcal{G}$, recall the classes of spherical and unitary representations of G. Choose the representations of theses classes for which the action is trivial on center. This gives most of the spherical and unitary representations for $\mathcal{G}$. For example, consider the one-dimensional spherical representation of G, that is $\chi \circ \det$ for an unramified character χ of $F^{\times}$. For it to be spherical, suppose $z = \begin{pmatrix} \varpi^n u & 0 \\ 0 & \varpi^n u \end{pmatrix} \in Z$ then we need $\chi \circ \det(z) = \chi(\varpi u)^2 = \chi(\varpi^2) = 1$. This gives a non-trivial one-dimesional spherical representation of $\mathcal{G}$, and indeed, is unitary.

Theorem 3.2.1. Let (ρ, V) be an irreducible unitary spherical representation of $\mathcal{G}$ and let $v \in V^{\mathcal{K}}$ be of unit norm. Then the function $\phi(g) = \langle \rho(g)v, v \rangle$ is spherical.

Proof. We use the following:

Proposition 3.2.1. ([Lub10], page no. 63) Let (π, W) be a unitary representation of $\mathcal{G}$. Then $\dim W^{\mathcal{K}} = 1$ or 0.

Theorem 3.2.2. ([Lan75], page no.61) Let (π, W) is an irreducible unitary representation of $\mathcal{G}$. Then $\dim W^{\mathcal{K}} = 1$ if and only if the function $\phi(g) = \langle \rho(g)v, v \rangle$ is spherical, where $v \in W^{\mathcal{K}}$.

Since the unitary irreducible representation (ρ, V) is also spherical, $V^{\mathcal{K}}$ can't be 0. Then Proposition 3.2.1 implies $\dim V^{\mathcal{K}} = 1$. Now the Theorem 3.2.2 implies that ϕ is spherical function. ■

Remark 3.2.1. The theorem discussed in [Lan75] is with much more generality, for any unimodular group G and compact subgroup K. He defines a function $f \in \mathcal{C}_c(G/K)$ is spherical, if it takes 1 at identity, K-bi invariant and $f * F = \lambda(F)f$, for all $F \in \mathcal{C}_c(G//K)$ and some complex number $\lambda(F)$.

The Theorem 3.2.1 enables us to denote every irreducible unitary spherical representation of $\mathcal{G}$ by ρ_λ, with eigencharacter λ and associated spherical function ϕ_λ.
Now the connection between spherical functions and representation theory is apparent. Anyway, [Lub10] classifies all irreducible spherical representations of $\mathcal{G}$ using eigenvalues of associated spherical functions. We won't get to this classification, since the explicit form of these classes are not important in our discussions.

3.3 A Duality Theorem

In order to discuss the duality theorem, we need some background on the L^2 space and its spectral decomposition.
A *uniform lattice* Γ of $\mathcal{G}$ is a discrete subgroup of $\mathcal{G}$ such that the quotient space $\Gamma \setminus \mathcal{G}$ is compact. The double coset $\Gamma \setminus \mathcal{G}/\mathcal{K}$ can be viewed as a quotient graph of the Bruhat-Tits tree, also as a quotient space of $\Gamma \setminus \mathcal{G}$. Since Γ is discrete, it is unimodular and

hence there exists a Haar measure $d\mu_{\Gamma\backslash\mathcal{G}}$ on $\Gamma \backslash \mathcal{G}$. So we consider $L^2(\Gamma \backslash \mathcal{G})$ to be the space of all square integrable functions from $\Gamma \backslash \mathcal{G}$ to $\mathbb{C}$. The action $\mathcal{R}_\Gamma$ on this Hilbert space given by

$$\mathcal{R}_\Gamma(g)f(x) = f(xg) \quad \text{for all } g,\, x \in \mathcal{G},\, f \in L^2(\Gamma \backslash \mathcal{G})$$

gives a unitary representation. Note that the space $\mathcal{C}_c(\Gamma \backslash \mathcal{G})$ is dense in $L^2(\Gamma \backslash \mathcal{G})$. Moreover,

Theorem 3.3.1. ([Bum97], page no. 174) The space $(L^2(\Gamma \backslash \mathcal{G})$ decomposes into Hilbert space direct sum of its subspaces. These subspaces are G-invariant and irreducible under the action $\mathcal{R}_\Gamma$.

This yields,

$$L^2(\Gamma \backslash \mathcal{G}) = \widehat{\bigoplus_{\pi \in \widehat{\mathcal{G}}_u}} \mathrm{m}(\pi, \Gamma)\pi$$

where $\widehat{\mathcal{G}}_u$ is the set of isomorphism classes of unitary representations of $\mathcal{G}$ and $\mathrm{m}(\pi, \Gamma)$ is the multiplicity with which $\pi \in \widehat{\mathcal{G}}_u$ occurs in the decomposition.
What we stated in the beginning of this section is more clear now. There are two decompositions happening; the space $\mathcal{C}_c(\Gamma \backslash \mathcal{G}/\mathcal{K})$ into direct sum of eigenspaces of $\mathcal{C}[\mathcal{A}]$ and the decomposition of $L^2(\Gamma \backslash \mathcal{G})$ into irreducible unitary representations by the action $\mathcal{R}_\Gamma$. This phenomenon gives rise to a duality theorem. For the proof of duality theorem, we use the following central theorem from [Lub10] and an important observation from its proof. Now onwards we say a representation π of $\mathcal{G}$ 'occurs' in $L^2(\Gamma \backslash \mathcal{G})$ if $\mathrm{m}(\pi, \Gamma)$ is non-zero.

Theorem 3.3.2. ([Lub10], page no. 72) Suppose Γ is a uniform lattice in $\mathcal{G}$. Then λ is an eigencharacter of $\mathcal{C}[\mathcal{A}]$ acting on $\Gamma \backslash \mathcal{G}/\mathcal{K}$ if and only if ρ_λ occurs as a subrepresentation of $L^2(\Gamma \backslash \mathcal{G})$.

Proof. $\Longrightarrow$ Suppose $f \in \mathcal{C}_c(\Gamma \backslash \mathcal{G}/\mathcal{K})$ is of eigencharacter λ, assume f to be of unit norm. By Theorem 3.3.1, f is contained in a subspace V of $L^2(\Gamma \backslash \mathcal{G})$, and the representation (π, V) is irreducible, where π is the right action of $\mathcal{G}$. We also have ρ_λ, which is the irreducible spherical unitary representation of $\mathcal{G}$ with eigencharacter λ and associated spherical function ϕ_λ. We use the lemma:

Lemma 3.3.1.

$$\int_{\mathcal{K}} f(xky)\, d\mu_{\mathcal{K}}(k) = \phi_\lambda(y) f(x) \quad \text{for all } x,\, y \in \mathcal{G}.$$

Now we look at the associated spherical function of π,

$$\langle \pi(g)f, f\rangle = \int_{\Gamma\backslash\mathcal{G}} f(xg)\overline{f(x)}\, d\mu_{\Gamma\backslash\mathcal{G}}(x) = \int_{\Gamma\backslash\mathcal{G}}\int_{\mathcal{K}} f(xkg)\overline{f(xk)}\, d\mu_{\mathcal{K}}(k)\, d\mu_{\Gamma\backslash\mathcal{G}}(x) \quad (3.7)$$

$$= \phi_\lambda(g)\int_{\Gamma\backslash\mathcal{G}} f(x)\overline{f(x)}\, d\mu_{\Gamma\backslash\mathcal{G}}(x) = \phi_\pi(g)\|f\|^2 = \phi_\lambda(g), \quad \text{for all } g\in\mathcal{G}. \quad (3.8)$$

which implies $\pi \simeq \rho_\lambda$ and hence ρ_λ occurs in $L^2(\Gamma \setminus \mathcal{G})$.
$\Longleftarrow$ Suppose (ρ_λ, W) occurs as a subrepresentation of $L^2(\Gamma \setminus \mathcal{G})$. Then there exists a subspace V of $L^2(\Gamma \setminus \mathcal{G})$ such that the right action π on V gives the representation (π, V), and $\pi \simeq \rho_\lambda$. Now for a spherical vector $f \in V$, since the Hecke operator commutes with right translation and proposition 3.2.1, $\rho_\lambda \mathcal{A}(f) = \mathcal{A}(\rho_\lambda f) = \mathcal{A}(f) = cf$, for some complex number c. Reader can refer [Lub10], page no. 74 for a computation which proves $c = \lambda$. ■

Remark 3.3.1. From the above proof we observe: Let Γ be a uniform lattice in $\mathcal{G}$, $(\rho_\lambda, W_\lambda)$ is an irreducible unitary spherical representation of $\mathcal{G}$ and ρ_λ occurs as a subrepresentation of $L^2(\Gamma \setminus \mathcal{G})$. Then a function $f \in \mathcal{C}_c(\Gamma \setminus \mathcal{G}/\mathcal{K})$ of unit norm is an eigenfunction with an eigencharacter λ if and only if f is in a subspace V of $L^2(\Gamma \setminus \mathcal{G})$, such that the representation $(\pi, V) \simeq (\rho_\lambda, W_\lambda)$, where π is the right action on $L^2(\Gamma \setminus \mathcal{G})$.

We denote the eigenspace

$$V(\lambda, \Gamma) = \{f \in \mathcal{C}_c(\Gamma \setminus \mathcal{G}/\mathcal{K}) : T(f) = \lambda(T)f \quad \text{for all } T \in \mathbb{C}[\mathcal{A}]\}$$

for any eigencharacter λ and a uniform lattice Γ. Now we are ready to prove the theorem.

Theorem 3.3.3. ***(Duality Theorem)*** Let Γ be a uniform lattice in $\mathcal{G}$ and π be an irreducible unitary spherical representation of $\mathcal{G}$ with associated eigencharacter λ_π. Consider the representation $(L^2(\Gamma \setminus \mathcal{G}), \mathcal{R}_\Gamma)$ of $\mathcal{G}$. Then

$$\mathrm{m}(\pi, \Gamma) = \dim V(\lambda_\pi, \Gamma),$$

where $\mathrm{m}(\pi, \Gamma)$ is the multiplicity of π in $\mathcal{R}_\Gamma$ and $V(\lambda_\pi, \Gamma)$ is the eigenspace of eigencharacter λ_π.

Proof. If $\mathrm{m}(\pi, \Gamma) = 0$, Remark 3.3 implies that $V(\lambda_\pi, \Gamma) = 0$. So consider a spherical representation (π, V) of $\mathcal{G}$ such that, $\mathrm{m}(\pi, \Gamma) \neq 0$. Choose an $f \in V^{\mathcal{K}}$ of unit norm, then from remark 3.3, $f \in V(\lambda_\pi, \Gamma)$. But there are $\mathrm{m}(\pi, \Gamma)$ many copies of V, which are mutually orthogonal. Hence we have that many linearly independent vectors such that all of them are in $V(\lambda_\pi, \Gamma)$. Hence

$$\dim V(\lambda_\pi, \Gamma) \leq \mathrm{m}(\pi, \Gamma).$$

$\Longleftarrow$ Let $f \in V(\lambda, \Gamma)$ for some eigencharacter λ of $\mathbb{C}[\mathcal{A}]$. Then $f \in L^2(\Gamma/\mathcal{G})$ which implies

$$f = \sum_{\pi \in \widehat{\mathcal{G}}_u} \sum_{j=1}^{\mathrm{m}(\pi,\Gamma)} a_{\pi,j} f_{\pi,j}, \quad a_{\pi,j} \neq 0. \tag{3.9}$$

But

$$\mathcal{R}_\Gamma(k) f = \sum_{\pi \in \widehat{\mathcal{G}}_u} \sum_{j=1}^{\mathrm{m}(\pi,\Gamma)} a_{\pi,j} \pi(k) f_{\pi,j}. \tag{3.10}$$

Invariance of f under right K-translation gives,

$$\sum_{\pi \in \widehat{\mathcal{G}}_u} \sum_{j=1}^{\mathrm{m}(\pi,\Gamma)} a_{\pi,j} \pi(k) f_{\pi,j} = \sum_{\pi \in \widehat{\mathcal{G}}_u} \sum_{j=1}^{\mathrm{m}(\pi,\Gamma)} a_{\pi,j} f_{\pi,j}. \tag{3.11}$$

But note that the expression 3.9 for f is unique since the $\mathrm{m}(\pi, \Gamma)$ copies of π are mutually orthogonal. Therefore,

$$\pi(k) f_{\pi,j} = f_{\pi,j}. \tag{3.12}$$

That is, all π occurring in $L^2(\Gamma/G)$ with non-zero multiplicity are spherical, and $f_{\pi,j}$ are spherical unit vectors for all $0 \leq j \leq \mathrm{m}(\pi, \Gamma)$. But by Theorem 3.3.2, for all such π, there exists a $\lambda_\pi \in \mathbb{R}$ such that $\pi \simeq \rho_{\lambda_\pi}$. Moreover, by Remark 3.3, $f_{\pi,j}$ is an eigenfunction with eigencharacter λ_π for all $0 \leq j \leq \mathrm{m}(\pi, \Gamma)$. Hence operating with $T \in \mathbb{C}[\mathcal{A}]$ on both sides of 3.9 gives,

$$\lambda(T) f = \sum_{\pi \in \widehat{\mathcal{G}}_u} \lambda_\pi(T) \sum_{j=1}^{m(\pi,\Gamma)} a_{\pi,j} f_{\pi,j}. \tag{3.13}$$

Argument similar to 3.10 gives

$$\lambda(T) = \lambda_\pi(T) \quad \text{for all } T \in \mathbb{C}[A]. \tag{3.14}$$

But inequivalent representations give distinct eigencharacter. Thus, $a_{\pi,j} = 0$ for all $\pi \in \widehat{\mathcal{G}}_u$ except one π such that $\lambda = \lambda_\pi$. Thus $a_{\pi,j} = 0$ for all π in the summation, except the one with $\lambda = \lambda_\pi$. This gives

$$f = \sum_{j=1}^{m(\pi,\Gamma)} a_{\pi,j} f_{\pi,j}. \tag{3.15}$$

From this we conclude

$$\dim V(\lambda_\pi, \Gamma) \leq m(\pi, \Gamma).$$

■

Indeed, above proof shows that $V(\lambda_\pi, \Gamma)$ is exactly equal to the direct sum of $m(\pi, \Gamma)$ many copies of π.

Chapter 4

L-functions and Local Constants

This chapter aims to discuss the L-functions and local constants associated to the smooth representations of G. L-functions are complex functions which can be associated to many mathematical objects including modular forms, elliptic curves, representations and so on. L-functions and local constants which we will also refer to as *L- and ε- factors* come in our story, because for every smooth irreducible representations of G and that of the Weil group (which is yet to be discussed), we will attach these complex functions and the Langlands correspondence will preserve these functions and this makes the correspondence more natural.

4.1 L- and ε- Factors for $GL(1)$

Before discussing the L- and ε-factors for G, we shall discuss the invariants for F, since the irreducible representations of F play an important role in the representations of G. Fix a non-trivial character $\psi \in \hat{F}$ and a Haar measure μ on F. Let $C_c^\infty(F)$ be the space of all locally constant and compactly supported functions from F to $\mathbb{C}$. For any $\Phi \in C_c^\infty(F)$, define the *Fourier transform* of Φ to be

$$\hat{\Phi}(x) := \int_F \Phi(y)\psi(xy)\, d\mu(y) \text{ for all } x \in F.$$

The function $\hat{\Phi} \in C_c^\infty(F)$.

Proposition 4.1.1. For every $\psi \in \widehat{F}$, there exists a Haar measure μ_ψ of F such that

$$\hat{\hat{\Phi}}(x) = \Phi(-x)$$

for all $\Phi \in C_c^\infty(F)$ and $x \in F$ where Fourier transform is computed using the measure μ_ψ.

The above equality is called the *Fourier inversion formula* and the measure μ_ψ called *self-dual Haar measure on F*.

We require few more definitions.

Definition 4.1.1. Suppose $\Phi \in C_c^\infty(F)$, $\chi \in \widehat{F^\times}$ and $d\mu^*$ is a Haar measure on $F^\times$. Let

$$z_m(\Phi, \chi) = \int_{\varpi^m U_F} \phi(x)\chi(x)\, d\mu^*(x).$$

Since the support of Φ is compact, $z_m(\Phi, \chi)$ has to vanish for every $m \leq M$ form some integer M. Thus we can define a formal Laurent series,

$$Z(\Phi, \chi, X) = \sum_{m\in\mathbb{Z}} z_m X^m$$

and $Z(\Phi, \chi, X) \in \mathbb{C}((X))$. Also define the space

$$Z(\chi, X) = \{Z(\Phi, \chi, X) : \Phi \in C_c^\infty(F)\}.$$

Proposition 4.1.2. For any character χ of $F^\times$,

$$Z(\chi, X) = P_\chi(X)^{-1}\mathbb{C}[X, X^{-1}]$$

where

$$P_\chi(X) = \begin{cases} 1 - \chi(\varpi)X \text{ if } \chi \text{ is unramified,} \\ 1 \text{ if } \chi \text{ is ramified.} \end{cases}$$

Proof. There is an embedding of $C_c^\infty(F^\times)$ in $C_c^\infty(F)$ in which $\Phi \in C_c^\infty(F^\times) \mapsto \Phi'$, where $\Phi'(0) = 0$ and $\Phi'(x) = \Phi(x)$ for all the non-zero $x \in F$. This yields that $C_c^\infty(F)$ is spanned by $C_c^\infty(F^\times)$ and the characteristic function $\Gamma_\mathfrak{o}$ of $\mathfrak{o}$. Let's analyse what happens at $\Gamma_\mathfrak{o}$.

$$Z(\Gamma_\mathfrak{o}, \chi, X) = \sum_{m\geq 0} \int_{\varpi^m U_F} \chi(x)\, d\mu^*(x)\, X^m.$$

Observe that $\chi|_{\varpi^m U_F}$ is a character of the compact subgroup $\varpi^m U_F$ of $F^\times$. It is any easy exercise to prove that for any compact subgroup K, and its non-trivial character χ,

$\int_K \chi(x)\, d\mu^*(x) = 0$. Which implies $Z(\Gamma_\mathfrak{o}, \chi, X) = 0$ if χ is unramified.

When χ is unramified,

$$Z(\Gamma_\mathfrak{o}, \chi, X) = \sum_{m\geq 0} \int_{\varpi^m U_F} \chi(\varpi)^m\, d\mu^*(x)\, X^m = \sum_{m\geq 0} \chi(\varpi)^m\, \mu^*(\varpi^m U_F)\, X^m.$$

But μ^* is left invariant. Hence,

$$(\mu^*(U_F))^{-1}Z(\Gamma_{\mathfrak{o}}, \chi, X) = \begin{cases} (1-\chi(\varpi)X)^{-1} \text{ if } \chi \text{ is unramified,} \\ 0 \text{ if } \chi \text{ is ramified.} \end{cases} \tag{4.1}$$

For any $\Phi \in C_c^\infty(F^\times)$, $z_m = 0$ for $m >> 0$ as $\varpi^m U_F$ is approaching zero for larger m. Hence

$$\{Z(\Phi, \chi, X) : \Phi \in C_c^\infty(F^\times)\} = \mathbb{C}[X, X^{-1}]. \tag{4.2}$$

Observe that for some integer $m < 0$, $\Gamma_{\varpi^m U_F} \in C_c^\infty(F^\times)$ and $Z(\Gamma_{\varpi^m U_F}, \chi, X) = \mu(U_F)$ which shows that $1 \in Z(\chi, X)$ for all χ. This obsevation, together with 4.1 and 4.2 gives the desired result. ■

Theorem 4.1.1. Suppose $\chi \in F^\times$, then there exists a unique rational function $c(\chi, \psi, X) \in \mathbb{C}(X)$ such that

$$Z(\hat{\Phi}, \check{\chi}, q^{-1}X^{-1}) = c(\chi, \psi, X)Z(\hat{\Phi}, \chi, X) \text{ for all } \Phi \in C_c^\infty(F).$$

Now we replace the complex variable X with q^{-s} where s is a complex variable.

Definitions 4.1.1.

$$\zeta(\Phi, \chi, s) := Z(\Phi, \chi, q^{-s}) = \int_{F^\times} \Phi(x)\chi(x)\|x\|^s \, d\mu^*(x)$$

which is called as the *local zeta function*. In the Ph.D. thesis of J. Tate [CF10], he developed the theory of local zeta functions and proved that the zeta function has meromorphic continuation for all $s \in \mathbb{C}$ and it has no poles in the region $\mathrm{re}(s) > 0$.

$$L(\chi, s) := P_\chi(q^{-s}) = \begin{cases} (1-\chi(\varpi)q^{-s})^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

The complex rational function $L(\chi, s)$ is called *local L-function*.

$$\gamma(\chi, s, \psi) := c(\chi, \psi, X).$$

As soon as we have the L-functions defined, we can easily define the next important invariant such that it satisfies a functional equation.

Definition 4.1.2. The local constant for any character χ of $F^\times$ is defined to be

$$\varepsilon(\chi, s, \psi) = \gamma(\chi, s, \psi)\frac{L(\chi, s)}{L(\check{\chi}, 1-s)}.$$

The function $\varepsilon(\chi, s, \psi)$ is generally referred to as *Tate's local constant*.

Remark 4.1.1. Applying the definition of zeta function twice and the definition of local constant yields the functional equation

$$\varepsilon(\chi, s, \psi)\varepsilon(\check{\chi}, 1-s, \psi) = \chi(-1).$$

4.2 L- and ε- factors for $GL(2)$

The definitions for L- and ε- factors for G are analogous to that of F, but the proof of existence is more involved. We start by defining the Fourier transform of elements in the space $C_c^\infty(A)$. We fix a non-trivial character ψ of F and define $\psi_A = \psi \circ \mathrm{tr}_A$. Similar to the local field case, we define the *Fourier transform* of $\Phi \in C_c^\infty(A)$ relative to the Haar measure $d\mu$ od A as:

$$\hat{\Phi} = \int_A \Phi(y)\psi_A(xy)\, d\mu(y).$$

Remark 4.2.1. For any character ψ of F, there exists a Haar measure $d\mu_\psi^A$ of A such that

$$\hat{\hat{\Phi}}(x) = \Phi(-x)$$

for all $\Phi \in C_c^\infty(A)$ and $x \in A$.

Definition 4.2.1. For any $\Phi \in C_c^\infty(A)$ and $f \in \mathcal{C}(\pi)$ where $\mathcal{C}(\pi)$ is the space of matrix coefficients, define the zeta function to be:

$$\zeta(\Phi, f, s) = \int_G \Phi(x)f(x)\|\det x\|^s d\mu^*(x),$$

where s is a complex variable and $d\mu^*$ is a Haar measure on G.

Next three theorems give all the information we need to know about L- and ε- factors of representations of G.

Theorem 4.2.1. For an irreducible smooth representation (π, V) of G,

1. There exists $s_0 \in \mathbb{R}$ such that the zeta function converges absolutely and uniformly in the region $\mathrm{Re}(s) > s_0$, for every Φ and f.

Let $Z(\pi) := \{\zeta(\Phi, f, s+\frac{1}{2}) : \Phi \in C_c^\infty(A), f \in \mathcal{C}(\pi)\}$. Then there exists a unique polynomial $P_\pi(X) \in \mathbb{C}[X]$ such that $P_\pi(0) = 1$ and

$$Z(\pi) = P_\pi(q^{-s})^{-1}\mathbb{C}[q^s, q^{-s}].$$

Also define the L-function associated to the representation π to be:

$$L(\pi, s) = P_\pi(q^{-s})^{-1}.$$

Proposition 4.2.1. Suppose (π, V) is an irreducible supercuspidal representation of G, then $L(\pi, s) = 1$.

Let π be a representation of G. A representation π_i of G is said to be G-composition factor of π, if π_i belongs to a sequence of representations such that

$$0 = \pi_0 \subset \pi_1 \subset ... \subset \pi_{m-1} \subset \pi_m = \pi$$

and π_j/π_{j-1} is irreducible for all $1 \leq j \leq m$. Observe from chapter 2 that the only G-composition factors of a reducible $\boldsymbol{\iota}_B^G \chi$ is itself, a character and a special representation. Consequently, the following theorem gives L-and ε-factor for all irreducible non-cuspidal representations of G.

Theorem 4.2.2. Let (π, V) is an irreducible non-cuspidal representation of G. Then if π is a G-composition factor of $\boldsymbol{\iota}_B^G \chi$ where $\chi = \chi_1 \otimes \chi_2$ for some $\chi_1, \chi_2 \in \widehat{F^\times}$, then

$$L(\pi, s) = L(\chi_1, s)L(\chi_2, s),$$
$$\varepsilon(\pi, s, \psi) = \varepsilon(\chi_1, s, \psi)\varepsilon(\chi_2, s, \psi),$$

except if $\pi = \phi \cdot \mathrm{St}_G$ for some unramified character ϕ of $F^\times$, then

$$L(\pi, s) = L(\phi, s + \tfrac{1}{2}) \text{ and } \varepsilon(\pi, s, \psi) = -\varepsilon(\phi, s, \psi).$$

Chapter 5

Weil Groups

Let's gather knowledge to understand the Galois side of the correspondence. The central object here is a locally profinite group called the Weil group. In order to define this group, we need some background on Galois theory.

5.1 Infinite Galois Theory

Let E be any field and K/E is a Galois extension, need not be finite. Let $\Omega = \mathrm{Gal}(K/E)$ and $\mathcal{N}$ be the set of all normal subgroups of G with finite index. When $M, N \in \mathcal{N}$ the projection map $G/M \longrightarrow G/N$ gives an inverse system. Then Ω is isomorphic to $\varprojlim_{N \in \mathcal{N}} G/N$ ([RV02], Pr. 1.19). We topologize Ω by giving the profinite topology and this topology is called *Krull topology*. Hence Ω is compact and Hausdorff. Now we can extend the fundamental theorem of Galois theory to infinite extensions.

Theorem 5.1.1. Suppose E is a field and K/E is a Galois extension and $\Omega = \mathrm{Gal}(K/E)$. The map sending all closed subgroups of Ω to the fixed fields is a bijection between the set of all closed subgroups of Ω and the set of all subfields of K cotaining E.

5.2 Weil Group

Let $\overline{F}$ be a separable algebraic closure of F and $\Omega_F = \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group. In order to state the Langlands correspondence, we shall slightly modify the absolute Galois group.

We know that for every positive integer m, adjoining $q^m - 1^{\text{th}}$ root of unity to $\boldsymbol{k}$ gives the unique extension of $\boldsymbol{k}$ of degree m, which we will denote as $\boldsymbol{k}_m$. Lifting the polynomial $x^{p^m-1} - 1 \in \boldsymbol{k}[x]$ to $F[x]$ and adjoining its roots to F gives the unique unramified extension of F and we will denote it as F_m. The Frobenius automorphism

$\mathrm{Frob}_m : x \mapsto x^q$ is an element in $\mathrm{Gal}(\boldsymbol{k}_m/\boldsymbol{k})$ and it gives a canonical isomorphism $\mathrm{Gal}(\boldsymbol{k}_m/\boldsymbol{k}) \to \mathbb{Z}/m\mathbb{Z}$. Let $\Phi_m \in \mathrm{Gal}(F_m/F)$ be the element which act as Frob_m on the residue field $\boldsymbol{k}_m$. The map which sends Φ_m to Frob_m is an isomorphism. Define

$$F^{\mathrm{nr}} = \varinjlim_{m \in \mathbb{Z}_{\geq 1}} F_m = \bigcup_{m \in \mathbb{Z}_{\geq 1}} F_m,$$

where F^{nr} is called the *maximal unramified* extension of F. It is clear that F^{nr}/F is Galois. Then

$$\mathrm{Gal}(F^{\mathrm{nr}}/F) \cong \varprojlim_{m \in \mathbb{Z}_{\geq 1}} \mathrm{Gal}(F_m/F) \cong \varprojlim_{m \in \mathbb{Z}_{\geq 1}} \mathrm{Gal}(\boldsymbol{k}_m/\boldsymbol{k}) \cong \varprojlim_{m \in \mathbb{Z}_{\geq 1}} \mathbb{Z}/m\mathbb{Z} = \widehat{\mathbb{Z}}.$$

There is a unique element $\Phi_F \in \mathrm{Gal}(F^{\mathrm{nr}}/F)$ which acts as Φ_m on F_m for all m. It is clear that under the canonical map, Φ_F is mapped to $(1, 1, 1.....) \in \widehat{\mathbb{Z}}$. The profinite group $\mathcal{I}_F = \mathrm{Gal}(\overline{F}/F^{\mathrm{nr}})$ has an inclusion map to Ω_F and there is a restriction map res from Ω_F to $\mathrm{Gal}(F^{\mathrm{nr}}/F)$. This yields the short exact sequence

$$1 \to \mathcal{I}_F \longrightarrow \Omega_F \longrightarrow \widehat{\mathbb{Z}} \to 0. \tag{5.1}$$

It is clear that this exact sequence splits. Moreover, we get a canonical valuation map, $\upsilon : \Omega_F \to \widehat{\mathbb{Z}}$. Note that $\mathbb{Z}$ is a dense subgroup of $\widehat{\mathbb{Z}}$.

Definition 5.2.1. Consider the infinite cyclic subgroup of $\mathrm{Gal}(F^{\mathrm{nr}}/F)$ generated by Φ_F. We define the Weil group of F, $\mathcal{W}_F$ is a subgroup of Ω_F such that, $\mathcal{W}_F = \mathrm{res}^{-1}(\langle \Phi_F \rangle)$. In other words, $\mathcal{W}_F = \{\sigma \in \Omega_F : \upsilon(\sigma) \in \mathbb{Z}\}$.

This gives the short exact sequence

$$1 \to \mathcal{I}_F \longrightarrow \mathcal{W}_F \longrightarrow \mathbb{Z} \to 0. \tag{5.2}$$

Let Φ be any element in Ω_F such that $\Phi|_{F^{\mathrm{nr}}} = \Phi_F \in \mathrm{Gal}(F^{\mathrm{nr}}/F)$. Then Φ is called a *Frobenius element* of Ω_F. Now fix a Frobenius element in $\mathcal{W}_F$, and define a map $\iota : \mathbb{Z} \to \mathcal{W}_F$ such that $1 \mapsto \Phi$. Then $\iota \circ \mathrm{res} = 1_{\mathcal{W}_F}$ and $\mathrm{res} \circ \iota = 1_{\mathbb{Z}}$, which results in the right splitting of 5.2 and hence $\mathcal{W}_F = \mathcal{I}_F \rtimes \mathbb{Z}$.
We topologize $\mathcal{W}_F$ not with the subspace topology, but with product topology, where $\mathcal{I}_F$ is given the natural profinite topology and $\mathbb{Z}$ with discrete topology. This makes the image of $\mathcal{I}_F$ open in $\mathcal{W}_F$. As a result, the Weil group of a local field is locally profinite.

Remark 5.2.1. Fix a Frobenius element Φ of $\mathcal{W}_F$. From the short exact sequence 5.2, observe that for any Weil group element, there exists an integer d and $\sigma \in \mathcal{I}_F$ such that the element can be uniquely written as $\Phi^d\sigma$. Thus, for any smooth irreducible representation ρ of $\mathcal{W}_F$, there exists a smooth irreducible representation τ of $\mathcal{I}_F$, and a character θ of $\mathbb{Z}$ such that $\rho(\Phi^d\sigma) = \theta(d)\tau(\sigma)$, for all $\Phi^d\sigma \in \mathcal{W}_F$. But any irreducible smooth representation of a compact group is finite dimensional. This yields that ρ is finite dimensional.

5.3 Results from Local Class Field Theory

In chapter 4, we witnessed the theory of L-functions and local constants attached to irreducible smooth representations of G. To develop a theory of L-functions and local constants attached to representations of Weil group, we use statements from local class field theory. Local class field theory classifies the finite abelian extensions of a local field. We will begin with the Artin reciprocity map which yields the classification.

Theorem 5.3.1. There is a canonical continuous group homomorphism

$$\boldsymbol{a}_F : \mathcal{W}_F \longrightarrow F^\times$$

with the following properties:

(1) The map $\boldsymbol{a}_F$ induces an isomorphism $\mathcal{W}_F^{\mathrm{ab}} \cong F^\times$.

(2) $\boldsymbol{a}_F(x)$ is a uniformizer of F if and only if x is Frobenius element of $\mathcal{W}_F$.

(3) $\boldsymbol{a}_F(\mathcal{I}_F) = U_F$.

(4) If E/F is a finite separable extension, the diagram

$$\begin{array}{ccc} \mathcal{W}_E & \xrightarrow{\boldsymbol{a}_E} & E^\times \\ {\scriptstyle \boldsymbol{a}_F}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{N}_{E/F}} \\ \mathcal{W}_F & \xrightarrow{\boldsymbol{a}_F} & F^\times \end{array}$$

commutes.

The map $\boldsymbol{a}_F$ is the *Artin reciprocity map*. As a consequence of (2), we have the properties:

(i) The map from the set of all finite abelian extensions of F inside $\overline{F}$ to the set of open subgroups of $F^\times$ of finite index, where $E/F \mapsto \mathrm{N}_{\mathrm{E/F}}(E^\times)$ is a bijection.

(ii) For any finite abelian extension E/F, Artin map induces an isomorphism between $\mathrm{Gal}(E/F)$ and $F^\times / \mathrm{N}_{\mathrm{E/F}}(E^\times)$.

(iii) Let E/F be a finite separable extension and let E^{ab} be its maximal abelian subextension. Then $\mathrm{N}_{\mathrm{E/F}}(E^\times) = F^\times/\mathrm{N}_{E^{\mathrm{ab}}}((E^{\mathrm{ab}})^\times)$.

Rather than the abelian extensions, for now we are more interested in statement (1) of the theorem.

5.4 Artin L-function

For every group, the character group is isomorphic to that of its abelianization. From the statement of local class field theory we get

$$\widehat{\mathcal{W}_F} \cong \widehat{\mathcal{W}_F^{\mathrm{ab}}} \cong \widehat{F^\times}.$$

Thus any character ϕ of $\mathcal{W}_F$ is of the form $\chi \circ \boldsymbol{a}_F$ for some character χ of $F^\times$. We say a character ϕ of $\mathcal{W}_F$ is *unramified* if it is trivial on $\mathcal{I}_F$. Thus 5.3.1 implies that every unramified character of $F^\times$ gives an unramified character of $\mathcal{W}_F$ when composed with the Artin reciprocity map.

$$L(\chi \circ \boldsymbol{a}_F, s) := L(\chi, s),$$
$$\varepsilon(\chi \circ \boldsymbol{a}_F, s, \psi) := \varepsilon(\chi, s, \psi)$$

Define $\mathcal{G}^{ss}(F)$ to be the set of isomorphism classes of semisimple smooth representations of $\mathcal{W}_F$. Let's extend the definition to all $\sigma \in \mathcal{G}^{ss}(F)$.

Definition 5.4.1. Let (σ, V) be a semisimple smooth finite-dimensional representation of $\mathcal{W}_F$, let $V^{\mathcal{I}}$ be the space of vectors in V which are fixed by $\mathcal{I}_F$. Fix a Frobenius element Φ of $\mathcal{W}_F$. Let $\tau', \tau \in \mathcal{I}$ then $\Phi^a\tau \in \mathcal{W}_F$ for some integer a. Since $\mathcal{I}_F$ is normal in $\mathcal{W}_F$, we have

$$\sigma(\tau')\sigma(\Phi^a\tau)v = \sigma(\Phi^a)\sigma(\Phi^{-a}\tau'\Phi^a)v = \sigma(\Phi^a)v = \sigma(\Phi^a\tau)v, \text{for all } v \in V^{\mathcal{I}}.$$

This implies $V^{\mathcal{I}}$ is a $\mathcal{W}_F$-stable subspace of V. Let $\sigma_{\mathcal{I}}$ denote this action of $\mathcal{W}_F$ on $V^{\mathcal{I}}$.

$$L(\sigma, s) := \det(1 - \sigma_{\mathcal{I}}(\Phi)q^{-s})^{-1}.$$

The definition is independent of choice of Φ because any two Frobenius elements differ by an element in $\mathcal{I}_F$ and $\sigma_{\mathcal{I}}|_{\mathcal{I}_F}$ is anyway trivial. Let's check if this definition is consistent with the earlier definition of L-function for characters:

Suppose a character $\theta = \chi \circ \boldsymbol{a}_F$ of $\mathcal{W}_F$ is unramified, $L(\theta, s) = L(\chi, s)$ since $\boldsymbol{a}_F(\Phi) = \varpi$. If θ is ramified, $V^{\mathcal{I}} = 0$ which gives $L(\theta, s) = 1$ and hence consistent. It is easy to see that

$$L(\sigma_1 \oplus \sigma_2, s) = L(\sigma_1 s)L(\sigma_2, s)\ .$$

L-function associated to semisimple smooth representations of Weil group is called *Artin L-function*.

5.5 Langlands-Deligne Local Constant

The Langlands-Deligne local constants are the local constants attached to semisimple representations of Weil group of a local field. Proving existence of such functions is not as easy as that of L-functions.
Before stating the theorem, recall the trace map Tr from any finite field extension E/F to F. For any character ψ of F, $\psi_E :- \mathrm{Tr} \circ \psi$ is a character of E.

Theorem 5.5.1. For every non-trivial character ψ of F and finite extension E of F, there is a unique family of functions

$$\mathcal{G}^{ss}(E) \longrightarrow \mathbb{C}[q^s, q^{-s}],$$
$$\rho \longmapsto \varepsilon(\rho, s, \psi_E)$$

with the properties:

(i) For any character χ of $E^\times$,

$$\varepsilon(\chi \circ \boldsymbol{a}_E, s, \psi_E) = \varepsilon(\chi, s, \psi_E).$$

(ii) For any $\rho_1, \rho_2 \in \mathcal{G}^{ss}(E)$

$$\varepsilon(\rho_1 \oplus \rho_2, s, \psi_E) = \varepsilon(\rho_1, s, \psi_E)\varepsilon(\rho_2, s, \psi_E).$$

(iii) For any non-trivial character ψ of E, the functional equation

$$\varepsilon(\rho, s, \psi)\varepsilon(\check{\rho}, 1 - s, \psi) = \det \rho(-1)$$

is satisfied.

Proof requires some abstract machinery and theory of global L- and ε-factors. Refer to [BH06] for the proof.

5.6 Weil-Deligne Representations

The initial form of local Langlands correspondence for GL_n was stated as a bijection between the irreducible supercuspidal representations of GL_n and n-dimensional irreducible smooth representations of Weil group. Weil-Deligne representations, which are families of representations of $\mathcal{W}_F$, were introduced by Deligne in order to extend the bijection to all irreducible smooth representations of GL_n. Before we begin the definition, we obtain a canonical norm on $\mathcal{W}_F$, from the valuation υ on Ω_F that we defined in 5.1. The norm is given by

$$\|x\| = q^{-\upsilon(x)} \text{ for all } x \in \mathcal{W}_F.$$

Definition 5.6.1. A *Weil-Deligne Representation* $(\rho, V, \boldsymbol{n})$ is a triple where (ρ, V) is a smooth finite dimensional representation of $\mathcal{W}_F$ and $\boldsymbol{n} \in \mathrm{End}_{\mathbb{C}}(V)$ which satisfies:

$$\rho(x)\, \boldsymbol{n}\, \rho(x)^{-1} = \|x\|\, \boldsymbol{n} \text{ for all } x \in \mathcal{W}_F. \tag{5.3}$$

Two Weil-Deligne representations $(\rho_1, V1, \boldsymbol{n}_1)$ and $(\rho_2, V_2, \boldsymbol{n}_2)$ of $\mathcal{W}_F$ are said to be *isomorphic* if $\rho_1 \simeq \rho_2$ and $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ are similar.

Remark 5.6.1. In the property 5.6.1, put $x = \Phi$, which gives $\mathrm{tr}(\boldsymbol{n}^m) = 0$ for all $m \in \mathbb{N}$ and this forces $\boldsymbol{n}$ to be nilpotent. Applying Jordan decomposition theorem yields that every smooth representation of $\mathcal{W}_F$ gives a finite number of distinct Weil-Deligne representations.

If the representation (ρ, V) is semisimple, we call a Weil-Deligne representation $(\rho, V, \boldsymbol{n})$ semisimple. An irreducible Weil-Deligne representation is defined analogously. We denote the set of isomorphism classes of semisimple, n-dimensional Weil-Deligne representation of $\mathcal{W}_F$ by $\mathcal{G}_n(F)$.

Remark 5.6.2. We shall identify any Weil-Deligne representation $(\rho, V, 0) \in \mathcal{G}_n(F)$ with $(\rho, V) \in \mathcal{G}_n^{ss}(F)$.

We shall look at an example.

Example 5.6.1. Let's begin with a two-dimensional Weil-Deligne representation. Consider the triple $(\rho, V, \boldsymbol{n})$ in which:

$$\boldsymbol{n} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right),\ V = \mathbb{C}^2 \text{ and } \rho(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \|x\| \end{smallmatrix}\right)$$

But

$$\begin{pmatrix} 1 & 0 \\ 0 & \|x\| \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \|x\| \end{pmatrix}^{-1} = \|x\|\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

This example can be easily extended to get an n-dimensional Weil-Deligne representation $(\rho, V, \boldsymbol{n})$ if we choose $\boldsymbol{n}$ to be the standard jordan block of rank $n-1$ and

$$\rho(x)e_i = \|x\|^{i-1}e_i.$$

where $\{e_i\}$ is the standard basis for $\mathbb{C}^n$. Moreover,

$$\rho = \bigoplus_{i=1}^{n} \| \cdot \|^{i-1}$$

says that $(\rho, V, \boldsymbol{n})$ is semisimple and we denote this representation as $\mathrm{Sp}(n)$, where n is the dimension of ρ.

What remains is to extend the theory of L-functions and local constants to Weil-Deligne representations. For any Weil-Deligne representation $(\rho, V, \boldsymbol{n})$ of $\mathcal{W}_F$,

$$\boldsymbol{n}\,\rho(x)v = \|x\|^{-1}\rho(x)\boldsymbol{n}v$$

which implies that the action ρ on the kernel of $\boldsymbol{n}$ is $\mathcal{W}_F$-stable. We denote this representation of $\mathcal{W}_F$ by $(\rho_{\boldsymbol{n}}, V_{\boldsymbol{n}})$.

Remark 5.6.3. Suppose $(\rho, V, \boldsymbol{n})$ is an irreducible Weil-Deligne representation, $(\rho_{\boldsymbol{n}}, V_{\boldsymbol{n}})$ is a subrepresentation, which implies $\boldsymbol{n} = 0$.

We define:

$$L((\rho, V, \boldsymbol{n}), s) = L(\rho_{\boldsymbol{n}}, s).$$

We set the contragredient of $(\rho, V, \boldsymbol{n})$ to be the Weil-Deligne representation given by $(\check{\rho}, \check{V}, \check{\boldsymbol{n}})$, where $\check{\boldsymbol{n}} = -\boldsymbol{n}^T \in \mathrm{End}_{\mathbb{C}}(\check{V})$. Thus the local constant is defined to be:

$$\varepsilon((\rho, V, \boldsymbol{n}), s, \psi) = \varepsilon(\rho, s, \psi)\tfrac{L(\check{\rho}, 1-s)}{L(\rho, s)}\tfrac{L(\rho_{\boldsymbol{n}}, s)}{L(\check{\rho}_{\check{\boldsymbol{n}}}, 1-s)}.$$

When $\boldsymbol{n} = 0$, $V_{\boldsymbol{n}} = V$ and this says that above definitions agree with the definitions of L-function and local constant for semisimple representations of $\mathcal{W}_F$.

Chapter 6

The Local Langlands Correspondence

Finally we get to the precise statement of the correspondence. Continuing with the notation, $\mathcal{G}_2(F)$ is the set of isomorphism classes of semisimple, 2-dimensional Weil-Deligne representations of $\mathcal{W}_F$. The set of isomorphism classes of irreducible smooth representations of G is denoted by $\mathcal{A}_2(F)$.

Theorem 6.0.1. ***(Local Langlands Correspondence)*** There exists a unique bijection

$$\boldsymbol{\pi} : \mathcal{G}_2(F) \longrightarrow \mathcal{A}_2(F)$$

such that

$$L(\chi \cdot \boldsymbol{\pi}(\rho), s) = L(\chi \otimes \rho, s),$$
$$\varepsilon(\chi \cdot \boldsymbol{\pi}(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi),$$

for all $\rho \in \mathcal{G}_2(F)$, $\chi \in \widehat{F^\times}$ and any non-trivial character ψ of F.

Proof of the theorem is involved and we haven't yet discussed many themes as deep enough as it requires to understand the proof. We could rather explore a very small part.

We begin by partitioning the set of isomorphism classes of representation, so that we can split the theorem into two separate theorems.

The of isomorphism classes of irreducible Weil-Deligne representations of dimension 2 is denoted by $\mathcal{G}_2^0(F)$. From 5.6.2, recall that this is exactly the set of isomorphism classes of 2-dimensional irreducible representation of $\mathcal{W}_F$. We define $\mathcal{G}_2^1(F)$ to be the set isomorphism classes of semisimple Weil-Deligne representations of $\mathcal{W}_F$ of dimension 2. This yields a partition:

$$\mathcal{G}_2(F) = \mathcal{G}_2^0(F) \bigcup \mathcal{G}_2^1(F).$$

We partition the set $\mathcal{A}_2(F)$ as

$$\mathcal{A}_2(F) = \mathcal{A}_2^0(F) \bigcup \mathcal{A}_2^1(F),$$

where $\mathcal{A}_2^0(F)$ is set of isomorphism classes of the irreducible supercuspidal represenations of G and $\mathcal{A}_2^1(F)$ is that of the irreducible non-cuspidal representations.

Proposition 6.0.1.

1. Suppose $\pi \in \mathcal{A}_2(F)$. Then $(\pi, V) \in \mathcal{A}_2^0(F)$ if and only if $L(\chi \cdot \pi, s) = 1$ for any character χ of $F^\times$.

2. Suppose $\rho \in \mathcal{G}_2(F)$. Then $\rho \in \mathcal{G}_2^0(F)$ if and only if $L(\chi \otimes \rho, s) = 1$ for any character χ of $F^\times$.

Proof. (1): Recall that corresponding to any smooth representation π of G, we have the space of matrix coefficients $\mathcal{C}(\pi)$. But π is supercuspidal implies $\chi \cdot \pi$ is supercuspidal for all characters χ of $F^\times$ since $\mathcal{C}(\chi \cdot \pi) = \chi \circ \det(g)\mathcal{C}(\pi)$. Then $L(\chi \cdot \pi, s) = 1$. Conversely, if π is non-cuspidal, then π is a composition factor of $\boldsymbol{\iota}_B^G\,\phi$, where $\phi = \chi_1 \otimes \chi_2 \in \widehat{T}$. Take $\chi = \chi_2^{-1}$. Then $\chi \cdot \pi$ is a composition factor of $\boldsymbol{\iota}_B^G\,\chi \cdot \phi = \boldsymbol{\iota}_B^G\,(\chi_1\chi_2^{-1} \otimes 1)$. But 4.2.2 implies $L(\chi \cdot \pi, s) = \frac{L(\chi \cdot \pi, s)}{1-q^{-s}}$ is clearly not equal to 1.
(2): Observe that the representation ρ is irreducible if and only if $\chi \otimes \pi$ is irreducible for all $\chi \in \widehat{F}^\times$, which proves (2). ∎

It is clear from above proposition that if the map $\boldsymbol{\pi}$ exists, its restriction to $\mathcal{G}_2^i(F)$ is a bijection with the image $\mathcal{A}_2^i(F)$, for $i = 1, 2$. We will prove the existence of the Langlands correspondence for non-cuspidals. Uniqueness of the map easily follows from *converse theorem* ([BH06], page no: 170) which we will skip.

Theorem 6.0.2. There exists a bijection $\boldsymbol{\pi}^1 : \mathcal{G}_2^1(F) \longrightarrow \mathcal{A}_2^1(F)$ such that

$$L(\chi \cdot \boldsymbol{\pi}^1(\rho), s) = L(\chi \otimes \rho, s) \text{ and } \varepsilon(\chi \cdot \boldsymbol{\pi}^1(\rho), s, \psi) = \varepsilon(\chi \otimes \rho, s, \psi), \tag{6.1}$$

for all $\rho \in \mathcal{G}_2^1(F)$, $\chi \in \widehat{F^\times}$ and any non-trivial character ψ of F.

Proof. Let $(\rho, V, \boldsymbol{n}) \in \mathcal{G}_2^1(F)$. Since ρ is semisimple and of dimension 2, $\rho = \chi_1 \oplus \chi_2$ for some characters χ_1 and χ_2 of $F^\times$. We have the representation $\pi = \boldsymbol{\iota}_B^G\,\vartheta$ of G, where $\vartheta = \chi_1 \otimes \chi_2$. If π is irreducible, we define $\boldsymbol{\pi}^1((\rho, V, \boldsymbol{n})) = \pi$. To check that L and ε-factors agrees, recall from Jordan decomposition theorem that when $n = 2$, either $\boldsymbol{n} = 0$ or $\boldsymbol{n} = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. It is an easy check that $\boldsymbol{n}$ being non-zero contradicts the irreducibility of $\boldsymbol{\iota}_B^G\,\vartheta$. From 4.2.2 and definition for L and ε factors for $(\rho, V, 0)$ it is clear that 6.1 is satisfied.
Suppose π is reducible. Then there exists some $\phi \in \widehat{F^\times}$ such that $\chi_1(x) = \phi(x)\|x\|^{-1/2}$

and $\chi_2(x) = \phi(x)\|x\|^{1/2}$. Now we have two candidates for the image of $\boldsymbol{\pi}$, those two G-composition factors, but we also have exactly two choices of nilpotent matrices. Therefore we set

$$\boldsymbol{\pi}^1((\rho, V, \boldsymbol{n})) = \begin{cases} \boldsymbol{\iota}_B^G\,\vartheta & \text{if } \boldsymbol{\iota}_B^G\,\vartheta \text{ is irreducible,} \\ \phi \circ \det & \text{if } \boldsymbol{\iota}_B^G\,\vartheta \text{ is reducible and } \boldsymbol{n} = 0, \\ \phi \cdot \mathrm{St}_G & \text{if } \boldsymbol{\iota}_B^G\,\vartheta \text{ is reducible and } \boldsymbol{n} \neq 0. \end{cases}$$

The property 6.1 is an easy check in all cases except when $\boldsymbol{\pi}^1((\rho, V, \boldsymbol{n})) = \phi \cdot \mathrm{St}_G$ for some unramified character ϕ of $F^\times$. In that case we have $(\rho, V, \boldsymbol{n})) = (\chi_1 \oplus \chi_2, \mathbb{C}^2, \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right))$. An easy calculation gives $(\rho_{\boldsymbol{n}}, V_{\boldsymbol{n}})$ is the character χ_2. On the GL_2 side, $L(\phi \cdot \mathrm{St}_G, s) = L(\phi, s + \frac{1}{2}) = \frac{1}{1-\chi_2(\varpi)\|\varpi\|^{-1/2}q^{s+\frac{1}{2}}} = L(\chi_2, s)$. Once we have the invariance of L and ε factors for $\chi = 1$, the relation $\chi \cdot \iota_B^G \vartheta \simeq \iota_B^G(\chi \cdot \vartheta)$ proves the invariance for all $\chi \in \widehat{F^\times}$.

■

It is apparent from the proof that defining Weil-Deligne representation was the key in extending the correspondence into whole class of semisimple smooth representations of Weil group. From the Jordan decomposition theorem we know that the similarity classes of nilpotent matrix in $M_n(\mathbb{C})$ is in bijection with partitions of n. At the same time there is a relation between partitions of n and the number of $GL_n(F)$-composition factors of $\mathrm{Ind}_B^{GL_n(F)}(\chi)$ when it is reducible, which is very clear to us in the $n = 2$ case.

Conclusion

We have studied the classification of irreducible smooth representations of $GL_2(F)$ and some other important classes of representations of $GL_2(F)$. By exploiting the theory of spherical functions from [Lan75] and [Lub10], we proved the duality theorem in the case of $PGL_2(F)$. In future, we hope to study about the spectral decomposition of $L^2(\Gamma \setminus GL_2(F))$. Here one can ask the question if spectrum of $L^2(\Gamma \setminus GL_2(F))$ is completely determined by the spectrum of spherical representations occuring in the decomposition. That is,
conjecture: Suppose Γ_1 and Γ_2 are two uniform lattices in $GL_2(F)$ such that

$$\mathrm{m}(\pi, \Gamma_1) = \mathrm{m}(\pi, \Gamma_2)$$

for all but finitely many irreducible spherical representations π of $GL_2(F)$. Then

$$\mathrm{m}(\pi, \Gamma_1) = \mathrm{m}(\pi, \Gamma_2)$$

for all irreducible spherical representations π of $GL_2(F)$.
We also stated the local Langlands correspondence for $GL_2(F)$ with a study of all the necessary tools for the statement. Then we divided the correspondence into two parts; correspondence for the class of irreducible non-cuspidal representations of $GL_2(F)$ and that for class of irreducible supercuspidal representations and we discussed the proof of non-cuspidal part.

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