Geometric Knot Theory

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Geometric Knot Theory towards the partial fulfilment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Adithyan P at Indian Institute of Science Education and Research under the supervision of Dr. Rama Mishra, Professor, Department of Mathematics, during the academic year 2019-2020.

Rama Mishra

Dr. Rama Mishra

Committee:

Dr. Rama Mishra

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This thesis is dedicated to my parents who have always been supportive throughout my life.

Declaration

I hereby declare that the matter embodied in the report entitled Geometric Knot Theory are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rama Mishra and the same has not been submitted elsewhere for any other degree.

Adithyan

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Abstract

This thesis provides an exposition to some geometric aspects of knot theory. We discuss certain numerical invariants of knots which are geometric in nature. Many of them are defined as the minimum value of some quantity over all the directions taken over one particular diagram or configuration and then minimizing this over all possible configurations. Crossing index, unknotting index and bridge index are some of the examples of such invariants. Later some invariants were defined by first taking the maximum value of these quantities over all the directions taken over one particular diagram or configuration and then minimizing this over all possible configurations. They were termed as superinvariants. Superbridge number, supercrossing number and superunknotting number are studied lately. All these invariants are very difficult to compute. Certain parametrizations are used to obtain some bounds for these invariants. Polygonal representation for knots has been instrumental in PL category. Similarly polynomial representation plays an important role in smooth category. In this thesis we also study the topology of the spaces of polygonal knots as well as polynomial knots. We discuss some applications of these spaces at the end.

Contents

Chapter 1

Introduction

This thesis is an exposition on the geometric aspects of knot theory. We discuss certain knot invariants that arise owing to the configurations used to represent the knot. Knot theory has been a rapidly growing field of research for the past few decades. Understanding the depth of the subject gives a flavour of pure and sophisticated mathematics ingrained in it. At the same time, it has tremendous applications in almost all the areas that we can think of. A knot is defined as an embedding of unit circle S^1 in the three-dimensional space $\mathbb{R}^3$. Since the three-dimensional sphere S^3 contains $\mathbb{R}^3$, it is preferred to define a knot as an embedding of S^1 in S^3. One identifies the embedding with the subset of S^3 which is the image of S^1 under the embedding. As there can be so many such embeddings possible and all these images are topologically homeomorphic to S^1, the question arises that how to classify them. This classification problem comes under the placement problem. The equivalence is known as *ambient isotopy* (see Definition 2.1.2). Historically, the subject started in 1867 with the imagination of physicists who were exploring the structure of atoms. They proposed that atoms might consist of knotted vortex tubes of the ether, with different elements corresponding to different knots. Here different means up to the ambient isotopy. P.G.Tait, a mathematician of that era, took this project of classifying all knots. This was a difficult task as there were no tools available. He was a discrete mathematician and hence developed a theory based on his expertise that is Combinatorics!. He assigned a combinatorial data known as *knot diagram* associated to a given knot. A knot diagram is made of two things, a projection of the knot into a suitable plane where the projected image contains only transverse double points as possible singular points and at each such double point the information regarding which point was over/under before projecting. A transverse double point after such information is referred to as *overcrossing/undercrossing*. After obtaining a knot diagram, Tait made several conjectures and used these conjectures to enumerate knots up to ten crossings. Tait's theory was based on the assumption that it is possible to obtain a plane on which the projected image has only transverse double points as singular points. Tait called it a *regular projection*. Question then was raised, if such a plane always exists for every knot. For that purpose, knots were segregated into two types, *tame knots* and *wild knots.* Tame knots are the ones which are either Piecewise Linear or smooth embeddings of S^1 in S^3. Any knot which is not tame is a wild knot. Using a simple algebraic geometry argument one can prove that for every tame knot there exists a regular projection. In fact, the set of all regular projections for a given knot is an open and dense set in the space of all projections of the knot. In this thesis, we confine ourselves to only tame knots. We have discussed both polygonal as well as smooth embeddings.

Once, we have a tame knot, we can find a regular projection and obtain a knot diagram associated to it. As there can be many choices of regular projections, there are many knot diagrams associated to the same knot. Thus, looking at two different knot diagrams, how can one infer if they are associated to the same knot. Even more difficult to predict if two knot diagrams are associated to knots which are ambient isotopic. This is resolved with the help of three combinatorial moves developed by Reidemeister, known as *Reidemeister Moves* (see Definition 2.4.1). Reidemeister's theorem asserts that two knot diagrams represent isotopic knots if and only if one can be transformed to the other using finitely many Reidemister moves. Now, we have a complete kit to work on the classification problem of knots. We choose a knot diagram, define some quantity associated to it and check if it remains invariant under all three Reidemister moves. Any such quantity is a knot invariant and can be utilized to classify knots. All of them may not be very useful though because many non-isotopic knots may share the same value. The knot theory is a study of defining powerful knot invariants which can distinguish as many knots as possible. In the last three decades, several knot invariants are defined. Some of them are numerical, some are polynomials and some are sophisticated algebraic structures. Thus far, there is no invariant which is complete and which can be computed easily.

Let us discuss some numerical invariants. For a given knot diagram of a knot K, using the Reidemister moves we can simplify the crossings and obtain a simpler diagram and call it a *reduced diagram*. The number of crossings in this reduced diagram is called the crossing number of the diagram. If we take the minimum of this crossing number taken over all the diagrams of K, we obtain an invariant, *crossing index* or *minimal crossing number* $c(K)$ (see Subsection 3.1.1) of K. Another interesting invariant is, the *unknotting number* $u(K)$ of K. It is defined again in two steps. Least number of crossing changes (from over to under or from under to over) required to change the diagram into a diagram of an unknot is called the unknotting number of the diagram and when this is minimized over all possible diagrams is an invariant $u(K)$ the unknotting index (see Subsection 3.1.2). Several numerical invariants are defined in this manner. *Bridge index* (see Subsection 3.1.3) is another important invariant which is defined in the same fashion. Thinking of a knot diagram as a space curve, we can study its behaviour along any particular direction, where it will be treated as a function of a single variable. Since we are in tame knot category, there will be a finite number of local extrema. Also, the unit vectors along each direction constitute a compact set, when we count the number of local maxima over all possible directions, it attains its minima as well maxima. The least number of local maxima over all directions in a diagram is defined as the Bridge number of the diagram and when we minimize the bridge number of all the diagrams we get the bridge number $b(K)$ of the knot K which is an invariant. Some numerical invariants such as genus of a knot (see Definition 2.2.3) are defined using other geometric objects associated with a knot. There are some numerical invariants such as *braid index* that arise through the representation of knots as closure of braids (see Subsection 3.1.5).

In 1987, Kuiper [Kui87] came up with another kind of invariant. At the diagram level, he took the maximum number of local maxima taken over all directions. We have pointed out earlier that the maximum will also exist. He calls it the superbridge index of the diagram. When we minimize the superbridge index of all the diagrams we get an invariant called the *superbridge index* $Sb(K)$ (see Subsection 3.1.6). Clearly $b(K) < Sb(K)$. The least value of $Sb(K)$ for any nontrivial knot is 3. With this theme in mind, some more superinvariants like *supercrossing index* (see Subsection 3.1.7), *superunknotting index* (see Subsection 3.1.8) were defined later.

We have noticed that numerical invariants are associated with a representation of knots. In these connections, several parametrizations were studied. One can represent a knot using line segments or sticks in different ways. In one such representation, we can count the number of sticks that are used. Clearly, the least number of sticks required by a particular knot is an invariant called the *stick index* (see Subsection 3.1.4) of a knot. Similarly, in smooth category it has been proved that every knot is isotopic to one point compactification of an embedding of $\mathbb{R}$ in $\mathbb{R}^3$ given by $t \mapsto (f(t), g(t), h(t))$ where $f(t)$, $g(t)$ and $h(t)$ are polynomials of degree d. This is known as a polynomial representation of the knot. For a given knot K the least value of d (positive integer) such that K has a polynomial representation in degree d is an invariant called *polynomial degree* of K (see Definition 6.1.5). Both stick index, as well as the polynomial degree, has been instrumental in drawing inferences of other invariants.

These representations of knots have another side to it. Suppose we consider the set of all knots obtained using d sticks. We can topologize the set and see the knot equivalence here. Putting the restriction on the number of sticks may make the classification more rigid. In other words, we can compare the path components of this space with the ambient isotopy classes of knots represented in this space. Likewise, for a fixed d the space of all polynomial knots of degree d can be studied and the path components of this space can be compared with the equivalence of knots.

This thesis has two parts. In the first part, we have studied various numerical invariants of knots that we mentioned earlier and also their relation with each other. Most invariants are well understood, we have provided the interesting results on them. We have taken the study of *superbridge index* a little more deeply. It has been known that there are infinitely many knots with superbridge index n for each $n \geq 4$. However, there are only finitely many knots that can have superbridge index 3. We have provided a detailed proof of this result in this thesis. In the second part of the thesis we have studied the topology of spaces of polygonal knots with stick number d and the topology of spaces of polynomial knots of degree d for various values of d. Finding specific polynomial representation of different knots and plotting them in Mathematica is really fascinating. We have taken off a few weaving knots and using some known results found their polynomial representation. Our polynomials provide an evidence that these knots are *strongly negative amphicheiral* (see Remark 2.1.3).

This thesis is organized as follows: In chapter 2, we provide the necessary basic definitions that are required for this thesis. In chapter 3, we define certain numerical invariants of knots and discuss how these invariants are related to one another. Choon Bae Jeon and Gyo Taek Jin proved that there are only finitely many 3-superbridge index knots, in chapter 4, the proof of the same is written in a more detailed way. Chapter 5 discusses polygonal knots and the topology of spaces of polygonal knots, specifically polygonal knots which can be realized as a hexagon or a heptagon. In chapter 6, polynomial knots and topology of spaces of polynomial knots are discussed, polynomial knots with minimal degree 5 are described in detail. In chapter 7, we solve a problem of finding polynomial representation for weaving knots motivated from similar results for torus knots. We conclude the thesis by mentioning few problems that were attempted, and the ideas that we have so far in this direction. We also share some problems that can be taken up as a project in the future.

Chapter 2

Basic Definitions in Knot Theory

2.1 Knots and knot equivalences

Definition 2.1.1. *A knot is homeomorphic image of an embedding* $f : S^1 \hookrightarrow \mathbb{R}^3$.

Example: Unknot (0_1), Trefoil knot (3_1), Figure eight knot (4_1)

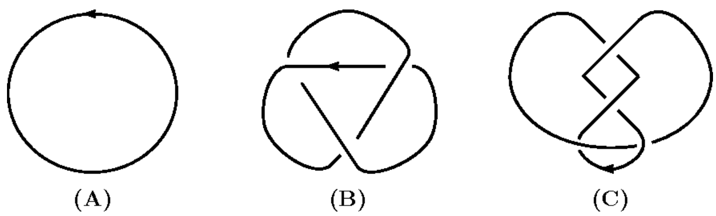


Figure 2.1: A, B, C are unknot, trefoil knot and figure eight knot respectively

Remark 2.1.1. *A link is homeomorphic image of an embedding* $f : S^1 \sqcup S^1 \sqcup ... \sqcup S^1 \hookrightarrow \mathbb{R}^3$.

Two knots, K_1 and K_2 are said to be equivalent if there exists a homeomorphism $h : S^3 \longrightarrow S^3$ such that $h(K_1) = K_2$.

Definition 2.1.2. *An ambient isotopy of* $\mathbb{R}^3$ *is a map* $H_t(s) : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *such that* $H_0(s)$ *is the identity map in* $\mathbb{R}^3$. *Two knots* K_1 *and* K_2 *are said to be isotopic if* $\exists$ *an ambient isotopy* H_t *of* $\mathbb{R}^3$ *such that* $H_1(K_1) = K_2$.

All knots are orientable and knots with a chosen orientation are called oriented knots.

Definition 2.1.3. *An oriented knot, K is said to be invertible if it is equivalent to the same knot with opposite orientation,* $-K$.

Remark 2.1.2. *If the equivalence between K and* $-K$ *is through an involution then K is called strongly invertible.*

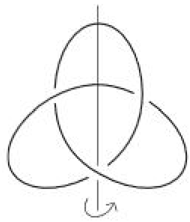


Figure 2.2: Strongly invertible (taken from [Sak20])

A mirror image of a knot is the knot composed with an orientation reversing homeomorphism of $\mathbb{R}^3$. It is essentially the knot with the crossings reversed.

Definition 2.1.4. *A knot, K is said to be amphichiral if it is equivalent to its mirror image K^*. An oriented knot, K is said to be positive-amphichiral if it is equivalent to its mirror image with the same orientation, K^* and is said to be negative-amphichiral if it is equivalent to its mirror image with opposite orientation, $-K^*$.*

Remark 2.1.3. *If the equivalence described between K and K^* is through an involution then K is called strongly positive-amphichiral. Similarly if the equivalence described between K and $-K^*$ is through an involution then K is called strongly negative-amphichiral respectively.*

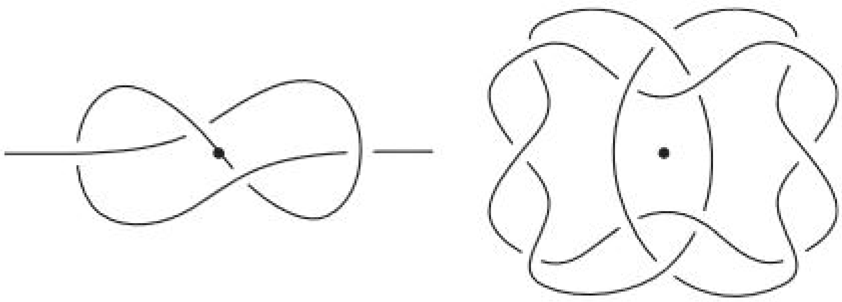


Figure 2.3: Strongly positive-amphichiral and strongly negative-amphichiral respectively (taken from [Sak20])

Definition 2.1.5. *A knot is said to be polygonal if it is made of a finite number of straight lines (sticks or edges). A polygonal knot is said to be generic if no four of its vertices are coplanar and no three of them are collinear.*

Definition 2.1.6. *A projection of a polygonal knot is regular if the projection has only finitely many double points which are transverse and there exist no other multiple points.*

Definition 2.1.7. *A knot is called tame if it is equivalent to a polygonal knot. Knots which are not tame are called wild knots (see Figure 2.4).*

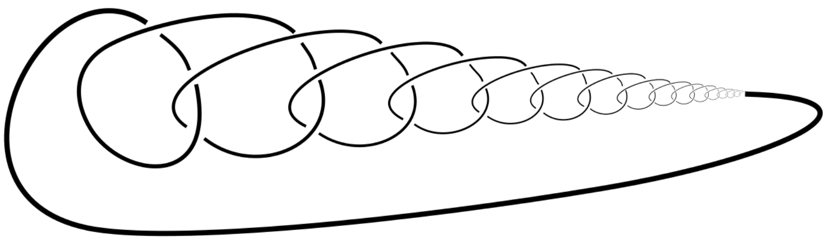


Figure 2.4: Example of a wild knot

We study invariants which comes from the projection of knots and in order to ensure the existence of projection we restrict ourselves into smooth category.

Definition 2.1.8. *A smooth knot is the image of a smooth embedding of S^1 in $\mathbb{R}^3$.*

Remark 2.1.4. *Every smooth knot has a polygonal representation (see Appendix I of [CF77]) and every polygonal knot which is essentially a piece-wise linear knot has a smooth approximation.*

From the above remark tame knots can be defined as smooth knots and the remark ensures that as long as we are dealing with tame knots we can restrict ourselves to the smooth category with no loss of information. A regular projection thus can be defined as,

Definition 2.1.9. *A projection of a knot, $f : S^1 \longrightarrow \mathbb{R}^3$ to a plane is regular if $\pi \circ f$ is an immersion where $\pi : \mathbb{R}^3 \longrightarrow \mathcal{P}$ is projection to a plane $\mathcal{P}$.*

It can be checked that the above definition ensures that the projection has finitely many double points with no other multiple points, there are no cusps and there are no overlapping of strands. A projection of a knot is called reduced if there are no obvious ways of decreasing the crossing number such as removing twists in knots ie, one part of the knot is twisted with respect to the other and removing self crossings in knot strands (see Figure 2.5).

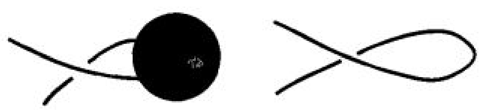


Figure 2.5: The two obvious ways of decreasing crossing number (taken from [ABF09])

The projection of the knot is simply a 4-valent graph and we lose a lot of information about the knot. In order to counter this we look at something called the knot diagram.

A knot diagram is the image of a regular projection with the added over-under information of knot strands at each double points.
Since we are interested in studying invariants that come from knot projections we will be studying only tame knots.

2.2 Connected sum and prime decomposition of knots

Definition 2.2.1. *Connected sum of two knots or Composition of two knots K_1 and K_2 denoted as $K_1\#K_2$ is an operation in which we remove two arcs from the two knots and join the two knots (see Figure 2.6). The arcs removed are chosen such that joining the rest won't produce any extra crossings (see Figure 2.7).*

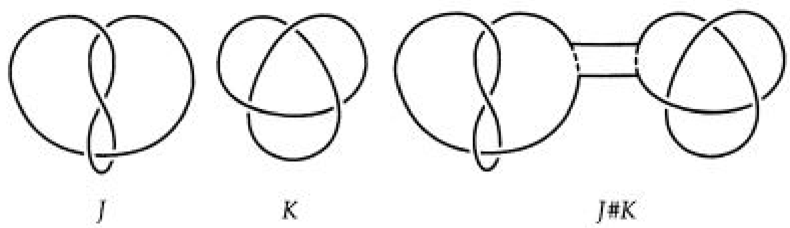


Figure 2.6: Composition of the knots J and K (taken from [Ada94])

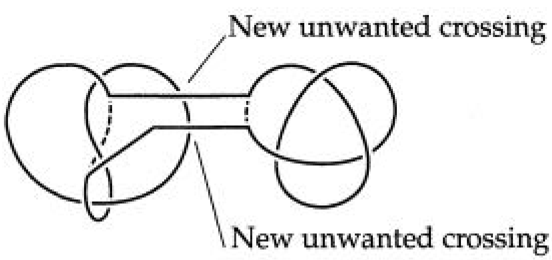


Figure 2.7: Unwanted crossings coming up due to bad choice of arcs removed (taken from [Ada94])

The definition does not guarantee that the operation connected sum is well defined but if we choose the knots to be oriented and force the operation to be such that it respects the orientation of both the knots in consideration then it becomes well defined. A knot is composite if it is equivalent to the composition of two nontrivial knots and a knot that is not composite is called a prime knot. For understanding certain results on the composition of knots we look at a new concept, Seifert surface of knots.

Definition 2.2.2. *A Seifert surface of a knot is a connected compact orientable surface with the knot as the boundary.*

It was proved that every knot has a Seifert surface and later Seifert gave a simpler algorithm to find a Seifert surface of a knot given the knot diagram.

(i) Give an orientation to the knot, start moving through the strand of knot and at each crossing switch between under and over strand preserving the orientation. If this completes a circle then choose some other point on the knot strand and repeat this until you travel through the knot once. Now we have a bunch of disjoint circles called as Seifert circles.

(ii) If two Seifert circles are concentric lift the circle that is inside a little above the plane containing the other circle and if the circles are disjoint keep it in the same plane.

(iii) Fill the circles, we will color each of the discs obtained such that the two sides get two different colors. If two discs are in the same plane color them in a similar way and if they are in different plane switch the colors.

(iv) Take colored twisted bands and use them to connect the the discs such that the boundary of the surface obtained is the knot.

Figure 2.8 shows the construction. The surface thus obtained is compact, connected and orientable (the coloring shows that it has two sides) surface such that the boundary is the knot we started with ; i.e, we have a Seifert surface.

Definition 2.2.3. *A genus of a knot is the minimum genera of all the Seifert surfaces of a knot. For a knot K it is denoted by $g(K)$.*

Genus of a knot characterizes unknots. We have K is an unknot if and only if genus of K is 0. The if part is obvious since a disc is a Seifert surface of unknot. For the only if part we need a little more machinery we need to prove that if a knot bounds a disc then it is an unknot which requires Dehn's theorem (see [Hem04]). Schubert in [Sch53] proved the following theorem on the behaviour of knot genus under connected sum.

Theorem 2.2.1. *For two knots K_1 and K_2,*

$$g(K_1 \# K_2) = g(K_1) + g(K_2)$$

Using these we get the composition of two nontrivial knots must yield a nontrivial knot and genus one knots are prime knots. Schubert later proved the existence and uniqueness of prime decomposition of knots.

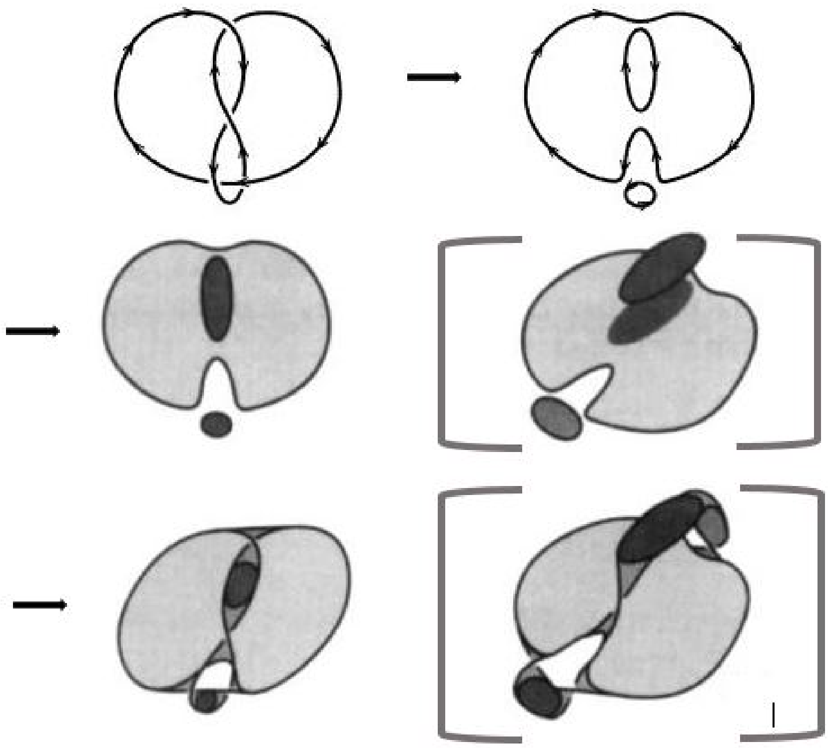


Figure 2.8: Seifert algorithm, the figures in brackets gives a 3D view (taken from [Ada94])

Theorem 2.2.2. *Every nontrivial oriented knot can be written as the connected sum of finitely many oriented prime knots and this decomposition is unique upto ordering and knot equivalence.*

The existence follows from the above theorem using strong induction on the genus of a knot. Since genus one knots are prime the theorem is trivially true. Assume every knot having genus upto n has a prime decomposition. Now any knot of genus $n+1$ is either prime or composite. If it is composite it can be written as the connected sum of two nontrivial knots and since each of these has genus strictly less than $n+1$, each of these nontrivial knots has prime decomposition by our assumption. The uniqueness part follows from the following theorem which can be proved using a constructive argument.

Theorem 2.2.3. *Let K be a knot and suppose $K = P\#Q$ and $K = M\#N$ where P is a prime knot Q, M and N not necessarily prime. Then either*

(i) $M = P\#J$ *for some knot* J *and* $Q = J\#N$ *or*

(ii) $N = P\#J'$ *for some knot* J' *and* $Q = M\#J'$

Now if a knot K has two prime decompositions $K = P_1\#P_2....\#P_n$ and $K = Q_1\#Q_2....\#Q_m$ then using the above theorem $Q_1 = P_i\#J$ for some knot J but since Q_1 is prime and P_i

is nontrivial, J should be unknot. Repeating this argument would give $n = m$ and for each $i \in \{1, 2, ..., n\}$ $P_i = Q_j$ for some $j \in \{1, 2, ..., n\}$ ie, the prime decomposition is unique upto ordering and knot equivalence.

2.3 Certain types of knots

2.3.1 Torus knots

Definition 2.3.1. *A nontrivial knot which can be realized on the surface of an unknotted torus in three-dimesional space is called a torus knot. If it wraps p times around the meridinal direction and q times around the longitudinal direction, then it is called a (p, q) torus knot denoted by $T(p, q)$.*

Example: Trefoil is a (3,2)-torus knot (see Figure 2.9).

A (p, q)-torus knot is also a (q, p)-torus knot. Considering the knot as an embedding of $\mathbb{S}^1$ in the 3-sphere $\mathbb{S}^3$ and $T(p, q)$ on a solid torus, T inside $\mathbb{S}^3$ then $\mathbb{S}^3 - Int(T)$ is a solid torus with $T(q, p)$ on it.

2.3.2 Alternating knots

Definition 2.3.2. *A knot or link K is alternating if it has a projection such that the crossings alternate from over to under as we travel along each of the components of K.*

Example: Trefoil knot(3_1), Figure eight knot(4_1).

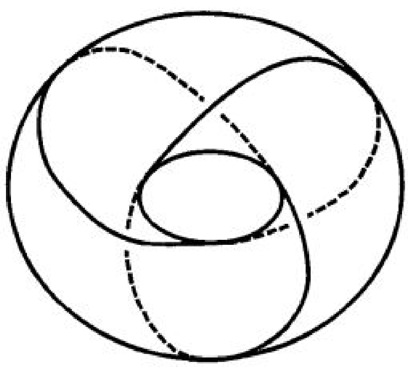


Figure 2.9: Trefoil: (3,2)-torus knot (taken from [ABF09])

2.4 Reidemeister moves

Definition 2.4.1. *A Reidemeister move is one of the following three moves that can be done on the knot diagram.*

- *Reidemeister move I allows us to put in and take out a twist on knot strands as in Figure 2.10, this move can change the crossing number by* 1*.*
- *Reidemeister move II allows us two add or delete* 2 *crossings as in Figure 2.11.*
- *Reidemeister move III keeps the crossing number unchanged but allows us slide a strand of knot from one side of a crossing to the other side of the crossing as in Figure 2.12.*

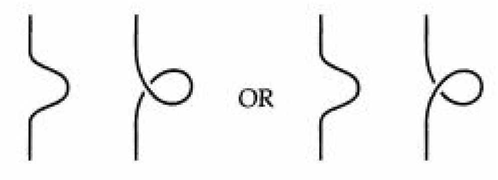


Figure 2.10: Reidemeister move I (taken from [Ada94])

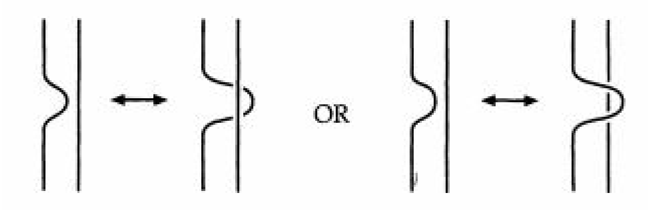


Figure 2.11: Reidemeister move II (taken from [Ada94])

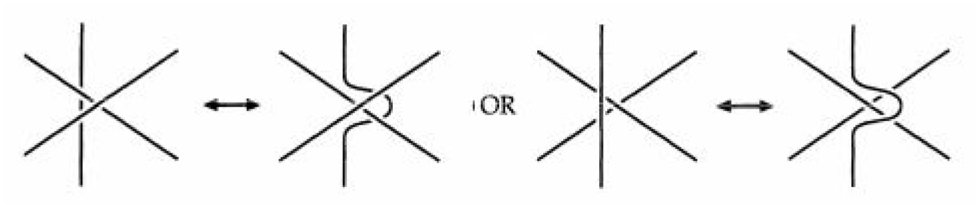


Figure 2.12: Reidemeister move III (taken from [Ada94])

Reidemeister [Rei27] proved the following theorem.

Theorem 2.4.1 (Reidemeister theorem)**.** *Two knot diagrams represent the same isotopy class of knots if they are related by a finite number of Reidemeister moves and planar isotopy.*

Chapter 3

Knot invariants

One of the central problems in knot theory is the classification of knots up to ambient isotopy. A knot invariant is a mathematical object such as a number, a polynomial, a ring, a group, etc that is attached to every knot such that if the knots are ambient isotopic then the mathematical object attached are isomorphic in its category. Over years different invariants have been studied for knots.

In classical knot theory, one studies numerical and polynomial invariants of knots. One example of a polynomial invariant is Jones polynomial (see Definition 3.6 in [Lic12]) which is a very celebrated invariant due to its ease in computation from the knot diagrams. Another example of knot invariant is the knot complement in S^3. Gordon-Luecke theorem (see Theorem 11.9 in [Lic12]) says that knot complement is a complete invariant of knots; i.e, knot complement can distinguish a knot from other knots up to ambient isotopy and mirror image. Also, there are some homological invariants such as Khovanov homology (see [Kau11]) and Heegaard-Floer homology (see [Sah10]) which are of particular interest in recent years. We will be concentrating on the study of numerical invariants throughout the project.

3.1 Numerical invariants

Numerical invariants are knot invariants that attach a number to every knot and the numerical invariant is equal for all the knots in the same isotopy class. In this section, we will look at certain numerical invariants of knots and look at results that compare these invariants. In this chapter, we will be using knot to refer to the equivalence class and conformation to refer to a particular element in the equivalence class. We would use the notation K for knot as well as conformations which will be clear from the contexts.

3.1.1 Crossing index

It is the least number of crossings for any projection of the conformations of that knot type. Let us give a formal definition.

Definition 3.1.1. *Crossing number of a conformation K is defined as $c(K) = \min_{\vec{v} \in S^2}(c_{\vec{v}}(K))$ where $c_{\vec{v}}(K)$ is the number of crossings in the projection of the knot to a plane perpendicular to $\vec{v}$. Crossing index of knot $[K]$ is defined as $c[K] = \min_{K' \in [K]} c(K')$*

$$c[K] = \min_{K' \in [K]} \min_{\vec{v} \in S^2}(c_{\vec{v}}(K'))$$

The natural question to ask is how crossing index behaves under composition of knots. It is straight forward to observe that $c[K_1 \# K_2] \leq c[K_1] + c[K_2]$. M. Lackenby [Lac09] proved the following theorem on the lower bound.

Theorem 3.1.1. $\frac{c[K_1]+c[K_2]}{152} \leq c[K_1 \# K_2] \leq c[K_1] + c[K_2]$

We have a much stronger result for crossing index of composition of two torus knots proved by Y.Diao [Dia04]. He defined something called the deficiency of a knot, $d[K] = c[K] - b[K] - 2g[K] + 1$ where $b[K]$ is the bridge index of knot which will be defined later. He proved a more general theorem that for two knots having deficiency zero, the crossing number of composition is sum of crossing numbers of individual knots. Then the following theorem follows from the fact that all torus knots have deficiency zero.

Theorem 3.1.2. *If K_1 and K_2 are torus knots then $c[K_1 \# K_2] = c[K_1] + c[K_2]$*

We explicitly know the crossing index of torus knots.

Theorem 3.1.3. *For $p < q$, $c[T(p, q)] = (p - 1)q$.*

Proof. Refer [Mur91]. ■

For alternating knots we have the following theorem.

Theorem 3.1.4. *The crossing index of an alternating knot or link is realized in any reduced alternating projection.*

Proof. Refer [Kau87]. ■

From the alternating projection of the individual knots an alternating projection of the composition of these knots can be constructed. Therefore the composition of two alternating knots is an alternating knot. Using this fact and the above theorem we have the following theorem.

Theorem 3.1.5. *If K_1 and K_2 are alternating knots then $c[K_1 \# K_2] = c[K_1] + c[K_2]$*

3.1.2 Unknotting index

It is the least number of crossing changes in any projection of the conformations of that knot type to obtain a projection of an unknot.

Definition 3.1.2. *Unknotting number of a conformation K is defined as $u(K) = \min_{\vec{v} \in S^2}(u_{\vec{v}}(K))$ where $u_{\vec{v}}(K)$ is the number of crossing changes required in the projection of the knot to a plane perpendicular to $\vec{v}$ for making it into an unknot. Unknotting index of knot $[K]$ is defined as $u[K] = \min_{K' \in [K]} u(K')$*

$$u[K] = \min_{K' \in [K]} \min_{\vec{v} \in S^2}(u_{\vec{v}}(K'))$$

It is straight forward to see that $u[K] = 0$ if and only if K is an unknot. In Figure 3.1 if we switch the darkened crossings we get unknots. For trefoil knot and figure-eight knot switching just one crossing gives us unknot and since they are nontrivial the unknotting index should be 1. Proving that the unknotting index of knot 7_4 (the third knot given in Figure 3.1) is 2 is much more difficult, one has to show that switching one crossing does not unknot 7_4.

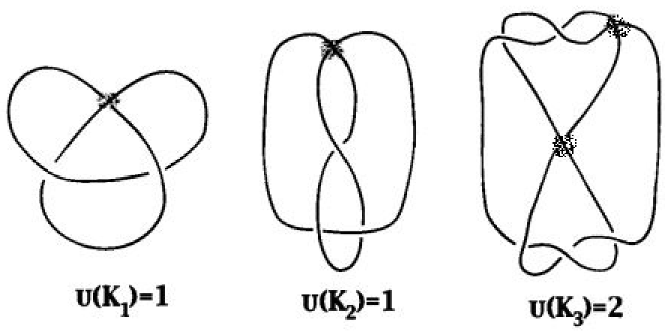


Figure 3.1: The darkened crossings are the candidates for unknotting the given knots (taken from [ABF09])

It was proved by M. Scharlemann in [Sch85] that $u(K_1 \# K_2) \geq 2$ for nontrivial knots K_1 and K_2. In the same paper he proved the following theorem.

Theorem 3.1.6. *Knots of unknotting number one are prime.*

Kronheimer and Mrowka in [KM93] gave an explicit formula for unknotting index of torus knots.

Theorem 3.1.7. *For $p < q$, $u[T(p,q)] = \frac{(p-1)(q-1)}{2}$.*

3.1.3 Bridge index

This invariant was introduced by H. Schubert [Sch54].

Definition 3.1.3. *Bridge number of a conformation K is defined as $b(K) = \min_{\vec{v} \in S^2}(b_{\vec{v}}(K))$ where $b_{\vec{v}}(K)$ called as crookedness of K with respect to $\vec{v}$ is the number of local maxima of the function $\vec{v} \cdot \vec{\gamma(t)}$, $\vec{\gamma(t)}$ being the parameterization of the knot. Bridge index of knot $[K]$ is defined as $b[K] = \min_{K' \in [K]} b(K')$*

$$b[K] = \min_{K' \in [K]} \min_{\vec{v} \in S^2}(b_{\vec{v}}(K'))$$

It is easy to observe that the bridge index of an unknot is 1. There is an alternate definition for bridge index using knot diagrams. It is defined as the minimal number of disjoint bridges over all the knot diagrams and knot conformations where a bridge is a strand of the knot diagram which is an overcrossing strand. From this alternate definition it is easy to see that if $b[K] = 1$ then K is an unknot. Hence, we have $b[K] = 1$ if and only if K is an unknot.
Again we need to know how the bridge index modifies with respect to composition of knots. The following theorem was proved by Schubert in [Sch54] on behaviour of bridge index under connected sum.

Theorem 3.1.8. *For knots K_1 and K_2 we have,*

$$b[K_1 \# K_2] = b[K_1] + b[K_2] - 1$$

Schubert also proved the following theorem on the bridge index of a torus knot.

Theorem 3.1.9. *For $p < q$, $b[T(p,q)] = p$.*

Schubert gave a complete classification of 2-bridge knots, knots with bridge index 2. 2-bridge knots are called rational knots, they are numerator closures of a rational tangles and each rational tangle is associated with a rational number (see [KL03] for details). Then the theorem is stated as,

Theorem 3.1.10. *Suppose that the numerator closures of rational tangles with fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ ($\gcd(p,q) = 1$ and $\gcd(p',q') = 1$) are denoted by $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$, then $K(\frac{p}{q})$ and $K(\frac{p'}{q'})$ are isotopic if and only if*

(i) $p = p$ and

(ii) either $q \equiv q' (\mod p)$ or $qq' \equiv 1 (\mod p)$.

3.1.4 Stick index

Definition 3.1.4. *Stick index or minimal stick number of knot [K] is the smallest number of sticks required to realize the knot as a polygonal knot. It is denoted by* $s[K]$

We have the following theorem regarding the relation between stick index and crossing index.

Theorem 3.1.11. *For any nontrivial knot K,*

$$\frac{5+\sqrt{9+8c[K]}}{2} \leq s[K] \leq 2c[K]$$

Proof. Refer [Cal01]. ■

Corollary 3.1.12. *Any knot with stick index n where* $3 \leq n \leq 5$ *is an unknot.*

Corollary 3.1.13. *The stick index of trefoil knot is 6.*

In Theorem 3.1.11 the lower bound is obtained by considering the total number of crossings possible in the projection along an edge of the polygonal knot. If we instead consider the projection onto a sphere with one of the vertices of the polygonal knot as the centre and look at the possible number of crossings we get a much stronger lower bound. The following theorem gives the new lower bound on stick index.

Theorem 3.1.14. *For a nontrivial knot K,*

$$s[K] \geq \frac{7+\sqrt{1+8c[K]}}{2}$$

Proof. Refer [Cal01]. ■

Corollary 3.1.15. *The stick index of figure eight knot is 7.*

Proof. From the above theorem the stick index is strictly greater than 6. We also know

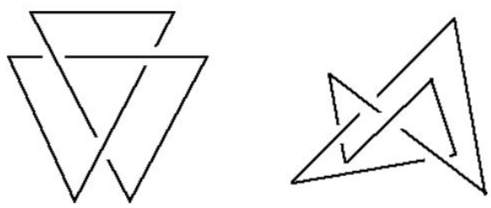


Figure 3.2: Stick representations of Trefoil knot and Figure eight knot respectively (from the [Cal01])

that there is a stick representation of figure eight knot with 7 sticks (see Figure 3.2). So we get $s[4_1] = 7$. ■

For torus knots in general we have

Theorem 3.1.16. *For $p < q$,*

1. $s[T(p,q)] = 2q$ *for* $2 \leq p < q < 2p$
2. $s[T_{p,2p+1}] = 4p$ *for* $p \geq 2$

Proof. Refer [AS09] and [Jin97]. ∎

3.1.5 Braid index

In $\mathbb{R}^3$, let $A = \{(x,0,1) | x \in \mathbb{Z}, 1 \leq x \leq n\}$ and $B = \{(x,0,0) | x \in \mathbb{Z}, 1 \leq x \leq n\}$. An n-strand braid is defined as set of n non-intersecting smooth paths which connects points in A to points in B. For all $x \in \{1,2,...,n\}$ joining $(x,0,1) \in A$ to $(x,0,0) \in B$ through smooth paths outside the 3-dimensional cube $[0,1]^3$ would give us a closure of the braid or closed braid. J.W Alexander in [Ale23] proved that every knot is a closed braid, this is called the Alexander's theorem.

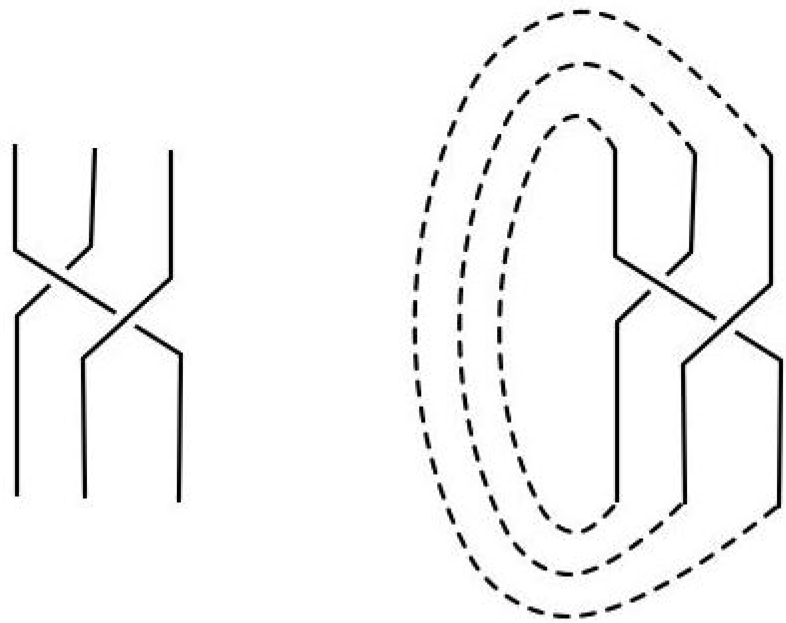


Figure 3.3: The figure in the left is a braid and the one in the left is its braid closure

Definition 3.1.5. *Braid index of knot isotopy class $[K]$ is the smallest n such that K is the closure of n-strand braid. It is denoted by $\beta[K]$.*

There is an alternate definition for braid index which will be of our interest, The equivalence of these definitions can be found in [Yam87].

Definition 3.1.6. *Braid number of a conformation K is defined as* $\beta(K) = \min_{\vec{v}\in S^2}(\beta_{\vec{v}}(K))$ *where* $\beta_{\vec{v}}(K)$ *is the number of Seifert circles in the knot diagram when K is projected to a plane perpendicular to* $\vec{v}$. *Braid index of knot* $[K]$ *is defined as* $\beta[K] = \min_{K'\in[K]} \beta(K')$

$$\beta[K] = \min_{K'\in[K]} \min_{\vec{v}\in S^2}(\beta_{\vec{v}}(K'))$$

Birman and Menasco in [BM90] proved the following theorem on how braid index behaves under connected sum.

Theorem 3.1.17. *For knots* K_1 *and* K_2 *we have,*

$$\beta[K_1\#K_2] = \beta[K_1] + \beta[K_2] - 1.$$

We have the following relation between braid index and bridge index.

Theorem 3.1.18. *For a knot* K, $b[K] \leq \beta[K]$.

Proof. Let Figure 3.4 represent the minimal braid representation of the knot, K. It can be clearly observed there are $\beta[K]$ no. of local maxima in the direction given by the arrow. Therefore $b[K] \leq \beta[K]$.

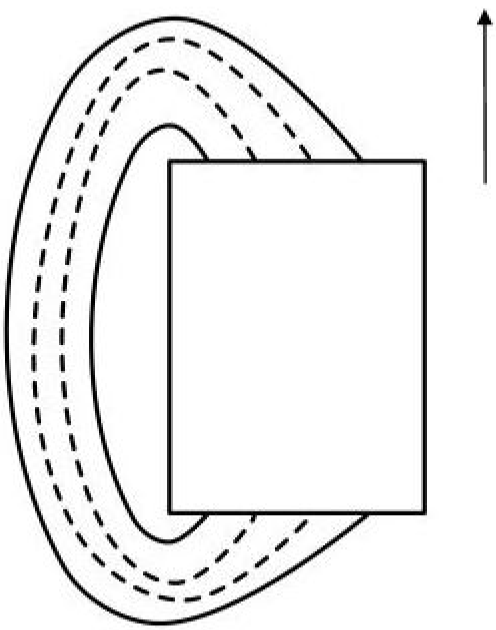


Figure 3.4: There are $\beta[K]$ no.of local maxima in the direction given by the arrow.

■

It can be observed that all the invariants bridge index, crossing index, braid index, and unknotting index were defined as a min-min invariant; i.e, these invariants are defined as taking a minimum of a function over S^2 and then taking the minimum over all the knot conformations in a knot isotopy type. Any continuous function from a compact set has an absolute maximum and an absolute minimum hence, it is possible to define define min-max invariants; i.e, take the maximum of the function over S^2 and then minimize it over all the knot conformations in a knot isotopy type. These min-max invariants are called the superinvariants, for example, superbridge index, supercrossing index, superbraid index, and superunknotting index. We will study some of these in detail.

3.1.6 Superbridge index

N.H.Kuiper introduced this new knot invariant in [Kui87].

Definition 3.1.7. $Sb[K] = \min_{K' \in [K]} \max_{\vec{v} \in S^2}(b_{\vec{v}}(K'))$.

From the definition it is clear that for a knot K, $b[K] \leq Sb[K]$ but Kuiper infact proved that this inequality is actually strict ie, $b[K] < Sb[K]$ making use of the concepts of total curvature of knots introduced by Milnor in [Mil50].

Theorem 3.1.19. *For any knot K, $Sb[K] \leq 5b[K] + 3$.*

Proof. Refer [WS03]. ∎

Kuiper in [Kui87] proved that the superbridge index is bounded above by twice the braid index and explicitly calculated the superbridge index for torus knots.

Theorem 3.1.20. *For a knot K, $Sb[K] \leq 2\beta[K]$.*

The proof of the above theorem follows by giving a parameterization for knots of braid index $\beta[K]$ and then calculating the number of local maxima making use of this parameterization.

Theorem 3.1.21. *For coprime integers p and q such that $2 \leq p < q$, superbridge index of the torus knot $T(p,q)$ is given by $Sb[T(p,q)] = \min\{2p, q\}$.*

The proof follows by looking at a general parameterization for torus knots $T(p,q)$ and then computing the number of local maxima from this parameterization.

The above theorem guarantees the existence of infinitely many knots of even superbridge index. For torus knots of type $(n, nk+1)$ where $n \geq 2$ and $k \geq 2$, the superbridge index is $2n$. Hence, for a fixed n varying the value of k over integers greater than 2 will give us infinitely many knots of superbridge index $2n$. It is infact proved that there are infinitely many knots of superbridge index greater than 4. What about knots of superbridge index 3? Choon Bae Jeon and Gyo Taek Jin proved that there are only finitely many 3-superbridge index knots. We will be giving a detailed proof of this theorem in the next chapter.
The following theorem discusses how bridge index and superbridge index of a knot changes when acted upon by an isomorphism of the ambient space $\mathbb{R}^3$.

Theorem 3.1.22. *Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be an isomorphism then for every knot K,*

$$Sb[K] = Sb[T(K)],$$

$$b[K] = b[T(K)].$$

.

Proof. Refer [WS03]. ∎

Jin noted that for a knot in stick conformation projected along a particular vector, the local maxima must occur at the vertices of sticks or along the sticks perpendicular to the vector. Also, the local maxima will not occur at an edge and its vertices simultaneously. Therefore the number of local maxima obtained is atmost half the number of vertices or edges; i.e,

Theorem 3.1.23. *For a knot K, $s[K] \geq 2Sb[K]$.*

Proof. Refer [Jin01]. ∎

3.1.7 Supercrossing index

Definition 3.1.8. $Sc[K] = \min\limits_{K' \in [K]} \max\limits_{\vec{v} \in S^2} (c_{\vec{v}}(K'))$.

Theorem 3.1.24. *For any knot K, $Sc[K] \geq c[K] + 3$*

Proof. It is sufficient to prove these for polynomial knots. We will prove that for a given knot there will be distinct projections that differ in crossing number by 3. The idea is to find a projection such that a slight change of perspective would mimic the Reidmeister moves I and II.

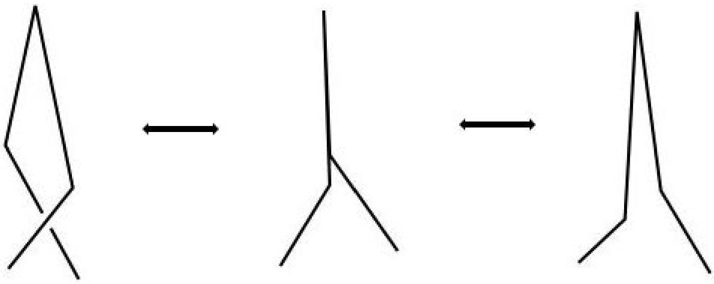


Figure 3.5: Slight change of perspective which induces Reidemeister move I.

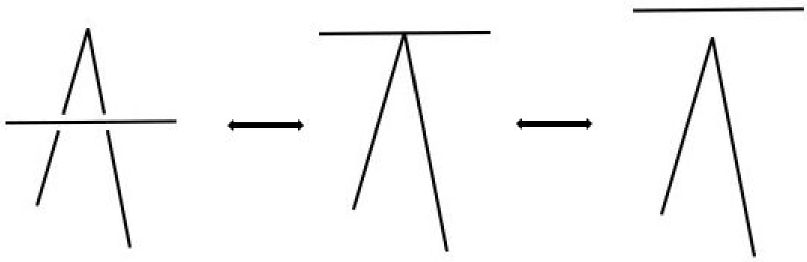


Figure 3.6: Slight change of perspective which induces Reidemeister move II.

We fix a vertex p and choose a tangent at p which intersects the knot at q a point not on the edges containing p. Now if we project along this tangent then the two edges containing p will be projected onto one line segment. Looking only at these adjacent edges, a slight change of perspective (see Figure 3.5) will induce Reidemeister move I. Now the projection takes p to q a slight change of perspective (see Figure 3.6) in looking at the edge containing q and the two edges containing p would induce a Reidemeister move II. Together they give two distinct projections with a crossing number difference of 3. ■

Corollary 3.1.25. *For every nontrivial knot K, $Sc[K] \geq 6$*

Theorem 3.1.26. *If the stick index of a knot is s then,*

$$Sc[K] \leq \frac{s(s-3)}{2} \; for\ odd\ s,\ and$$

$$Sc[K] \leq \frac{s(s-4)}{2} + 1 \; for\ even\ s.$$

Proof. Refer [Ada+02]. ■

For a knot with stick index s every edge can intersect $(s-3)$ other non-adjacent edges hence $Sc[K] \leq \frac{s(s-3)}{2}$ is true for all s. $\frac{s(s-3)}{2}$ is the total number of crossings that are possible in general. But when s is even we have restrictions on the number of edges which can intersect all other edges which are non-adjacent to it. Hence, when s is even we have a much stronger inequality. Using this inequality and Theorem 3.1.11 we get the following relation between crossing index and supercrossing index.

Theorem 3.1.27. $c[K] + 3 \leq Sc[K] \leq 2(c[K])^2 - 3c[K]$.

Again if we already know that the stick index is even, we have a stronger upper bound $Sc[K] \leq 2(c[K])^2 - 4c[K] + 1$. From this new bound we have the following.

Corollary 3.1.28. *For a trefoil knot* (3_1), $6 \leq Sc[3_1] \leq 7$.

The problem of whether the supercrossing index of 3_1 is actually 6 or 7 still remains unproven.
The following theorem discusses how crossing index and supercrossing index of a knot changes when acted upon by an isomorphism of the ambient space $\mathbb{R}^3$.

Theorem 3.1.29. *Let* $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *be an isomorphism then for every knot* K,

$$Sc[K] = Sc[T(K)]$$

$$c[K] = c[T(K)]$$

.

Proof. Refer [Dia18]. ∎

Similarly, we can define the super counterparts of the min-min invariants: unknotting index and braid index. No active study has been done on these, we will be defining these.

3.1.8 Superunknotting index

Definition 3.1.9. $Su[K] = \min\limits_{K' \in [K]} \max\limits_{\vec{v} \in S^2} (u_{\vec{v}}(K'))$.

3.1.9 Superbraid index

Definition 3.1.10. $S\beta[K] = \min\limits_{K' \in [K]} \max\limits_{\vec{v} \in S^2} (\beta_{\vec{v}}(K'))$.

Chapter 4

3-Superbridge knots

In this chapter we will focus on 3-superbridge knots, knots with superbridge index 3. It was proved by Choon Bae Jeon and Gyo Taek Jin in [JJ01] that there are only finitely many 3-superbridge knots. We will be giving the proof in detail.

Lemma 4.0.1 (Local straightening)**.** *Given a knot K, let K' be a knot obtained by replacing a subarc of K with a straight line segment joining the end points of the subarc. Then $Sb(K) \geqslant Sb(K')$.*

Proof. Refer [JJ01]. ■

Remark 4.0.1. *For 3-superbridge knots, the superbridge number remains unchanged under local straightening.*

This is because for nontrivial knots, superbridge index is always strictly greater than the bridge index. It is therefore strictly greater than 2. Hence, local straightening cannot reduce the superbridge number.

Definition 4.0.1. *Quadrisecant is a straight line intersecting a knot at four distinct points.*

The following theorem was proved by E. Pannwitz in [Pan33].

Theorem 4.0.2. *Every nontrivial knot has a quadrisecant.*

Definition 4.0.2. *The quadrisecant Q of a nontrivial knot K is said to be topologically nontrivial if, for any two points X and Y of $K \cap Q$ which are adjacent along Q, any disk bounded by the line segment XY and the arc of $K - Q$ that has end points X and Y meets the interior of the knot K.*

G. Kuperberg [Kup97] proved the following theorem.

Theorem 4.0.3. *Every nontrivial knot has a topologically nontrivial quadrisecant.*

Let K be a 3-superbridge knot with superbridge number 3 ie, $Sb[K] = Sb(K) = 3$ and let Q be a quadrisecant of K. Then $K - Q$ consists of four disjoint open arcs l_1, l_2, l_3 and l_4. Suppose $\pi : \mathbb{R}^3 \longrightarrow Q^{\perp}$ is the orthogonal projection of $\mathbb{R}^3$ onto a plane $Q^{\perp}$ perpendicular to the quadrisecant, denote $\pi(l_i)$ and $\pi(Q \cup l_i)$ by $\overline{l_i}$ and $\widetilde{l_i}$ respectively. Using local straightening we assume that the singular points of $\pi(K)$ are the the quadruple point $\pi(Q)$ and finitely many transversal double points. For every open subarc l of K, let $b_{\vec{v}}(K|l)$ be the number of local maxima of K along the direction $\vec{v}$ on l. Since each $\widetilde{l_i}$ is closed loop in $Q^{\perp}$, we must have

$$b_{\vec{v}}(K|l_i) \geqslant 1 \text{ or } b_{-\vec{v}}(K|l_i) \geqslant 1 \tag{4.1}$$

for every $\vec{v} \in Q^{\perp}$.
For every straight line ρ in $Q^{\perp}$, denote a unit vector perpendicular to ρ by $\vec{v}_\rho$.

Sublemma 4.0.1. *We may assume that $\overline{l_i}$ has no self crossings $\forall\ i \in \{1, 2, 3, 4\}$.*

Proof. Suppose $\overline{l_i}$ has self crossings. Then choose a loop λ of $\overline{l_i}$ which is minimal (no subarc of λ is a loop). Then λ bounds an open disc δ.

Case 1: $\pi(K) \cap \delta = \phi$
We can eliminate this arc by local straightening.

Case 2: $\overline{l_i} \cap \delta \neq \phi$
$\exists\ \rho$ a half line starting from $\pi(Q)$ which meets $\overline{l_i}$ atleast 3 times. Since λ is a closed loop $\exists$ one local maxima along $\vec{v}_\rho$ and $-\vec{v}_\rho$. Since ρ crosses $\overline{l_i}$ and $\overline{l_i}$ should complete a loop at $\pi(Q)$ there should be atleast one additional local maxima in the direction $\vec{v}_\rho$ and $-\vec{v}_\rho$. Therefore,

$$b_{\vec{v}_\rho}(K|l_i) \geqslant 2 \text{ and } b_{-\vec{v}_\rho}(K|l_i) \geqslant 2 \tag{4.2}$$

Fixing this i and a direction $\vec{v}_\rho$ or $-\vec{v}_\rho$, two among the other three arcs would have atleast one local maxima in that direction, which implies

$$b_{\vec{v}_\rho}(K) \geqslant \sum_{j=1}^{4} b_{\vec{v}_\rho}(K|l_j) \geqslant 4 \text{ or } b_{-\vec{v}_\rho}(K) \geqslant \sum_{j=1}^{4} b_{-\vec{v}_\rho}(K|l_j) \geqslant 4 \tag{4.3}$$

contradicting $Sb(K) = 3$

Case 3: $\overline{l_j} \cap \delta \neq \phi$ where $j \neq i$
Among the half lines starting from $\pi(Q)$ through δ, $\exists\ \rho$ which crosses $\overline{l_j}$ such that for

$\vec{w} = \vec{v}_\rho$ or $\vec{w} = -\vec{v}_\rho$ we have

$$b_{\vec{w}}(K|l_i) \geqslant 2,\ b_{-\vec{w}}(K|l_i) \geqslant 1,\ b_{\vec{w}}(K|l_j) \geqslant 1,\ b_{-\vec{w}}(K|l_j) \geqslant 1 \tag{4.4}$$

Fixing this i and j, one among the other two arcs should have a local maxima in the direction $\vec{w}$, which implies

$$b_{\vec{w}}(K) \geqslant \sum_{j=1}^{4} b_{\vec{w}}(K|l_j) \geqslant 4 \tag{4.5}$$

contradicting $Sb(K) = 3$ ■

Sublemma 4.0.2. *We may assume that $\widetilde{l_i}$ bounds an open disk δ_i in $Q^\perp$ which is star-shaped with respect to $\pi(Q)$.*

Proof. Since $\widetilde{l_i}$ is a loop, it bounds an open disc δ_i in $Q^\perp$.
Suppose δ_i is not star-shaped with respect to $\pi(Q) \implies \exists$ a half line ρ in $Q^\perp$ starting from $\pi(Q)$ such that it meets $\overline{l_i}$ more than once.

Case 1: If ρ meets $\overline{l_i}$ at three or more points. Then condition 4.2 happens and we get the contradiction 4.4.

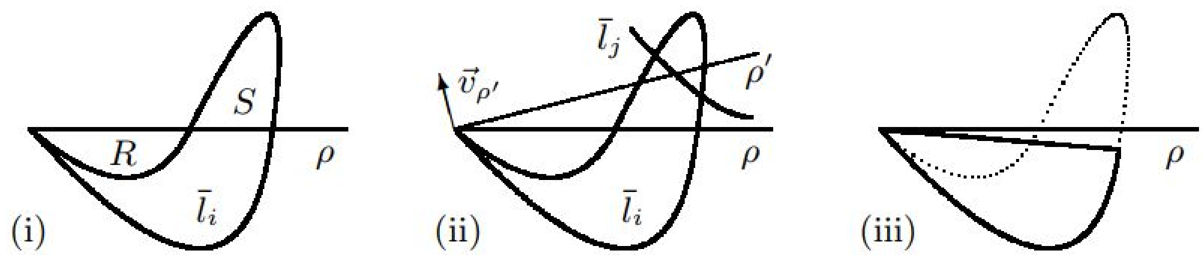


Figure 4.1: Deformations that makes the disc bounded star-shaped (taken from [JJ01]).

Case 2: If ρ meets $\overline{l_i}$ at two points. There are two discs R and S bounded by $\overline{l_i}$ and ρ as in Figure 4.1. There are two subcases.
Subcase 1: If $\overline{l_j}$ meets $R \cup S$ then $\exists\ \rho'$ starting from $\pi(Q)$ crossing $\overline{l_j}$ at a point in $R \cup S$. Then condition 4.3 happens and we get the contradiction 4.5.
Subcase 2: $\pi(K) \cap (R \cup S) = \phi$. We can straighten out the part of l_i without changing the knot type. δ_i can be made star-shaped with finitely many modifications.

■

Sublemma 4.0.3. *None of the following conditions hold when h, i, j, k are distinct elements of $\{1,2,3,4\}$.*

$$\delta_i \cap \delta_j \cap \delta_k \neq \phi \tag{4.6}$$

$$\delta_i \cap \delta_j \neq \phi \text{ and } \delta_k \cap \delta_h \neq \phi \tag{4.7}$$

$$\delta_i \cap \delta_j \neq \phi,\ \delta_i \cap \delta_k \neq \phi \text{ and } \delta_i \cap \delta_h \neq \phi \tag{4.8}$$

Proof. Assuming these conditions are true, we will find a line ρ in $Q^{\perp}$ such that it intersects the knot atleast 8 times which will inturn show that $b_{\vec{v}_\rho}(K) \geqslant 4$ contradicting $Sb(K) = 3$

Case 1: If condition 4.6 is true choose points $P_1 \in \delta_i \cap \delta_j \cap \delta_k$ and $P_2 \in \delta_h$ such that the straight line ρ joining the points does not pass through $\pi(Q)$. Then for $a \in \{i, j, k, h\}$, $\overline{l_a}$ crosses ρ at least twice.

Case 2: If condition 4.7 is true then choose points $P_1 \in \delta_i \cap \delta_j$ and $P_2 \in \delta_k \cap \delta_h$ such that the straight line ρ joining the points does not pass through $\pi(Q)$. Then for $a \in \{i, j, k, h\}$, $\overline{l_a}$ crosses ρ at least twice.

Case 3: If condition 4.8 is true, we can choose three points $P_a \in \delta_i \cap \delta_a$ for $a \in \{j, k, h\}$ such that the three line straight lines determined by pairs of P_a's do not pass through $\pi(Q)$. Since condition 4.6 cannot happen all edges of $\Delta P_j P_k P_h$ intersects $\pi(K)$ even number of times. There are two subcases to consider:
Subcase 1: If $\pi(Q)$ is contained inside $\Delta P_j P_k P_h$, then the boundary of the triangle intersects $\pi(K)$ atleast eight times and since all the edges intersects even number of times, one among the three edge should intersect $\pi(K)$ atleast four times. Extent this edge and choose this as ρ.
Subcase 2: If $\pi(Q)$ is contained outside $\Delta P_j P_k P_h$, then $\exists$ a vertex say P_h such that it intersects the opposite side $P_j P_k$ here. Since condition 4.6 cannot occur, the edge $P_j P_k$ meets $\pi(K)$ atleast four times. Take ρ to be the extension of $P_j P_k$. ∎

Sublemma 4.0.4. *We may assume that the only crossings in $\overline{l_i} \cup \overline{l_j}$ are a set of finitely many consecutive half twists.*

Proof. Sublemma 4.0.1 prevents self crossings and condition 4.6 of Sublemma 4.0.3 implies $\overline{l_i} \cup \overline{l_j}$ cannot intersect any of the other to $\overline{l_a}$'s. From Sublemma 4.0.2 we can assume that δ_i's are star shaped discs with respect to $\pi(Q)$.
Therefore if a ray in $Q^{\perp}$ starting from $\pi(Q)$ meets $\overline{l_i}$ and $\overline{l_j}$, then it meets each of them exactly once and no other $\overline{l_a}$'s. This inturn implies that the only crossing in $\overline{l_i} \cup \overline{l_j}$ if it exists is a set of finitely many half twists. ∎

Sublemma 4.0.5. *We may assume that $\pi(K)$ is as in Figure 4.2 upto planar isotopies of $Q^{\perp}$, where each rectangle contains a pair of arcs with finitely many half twists.*

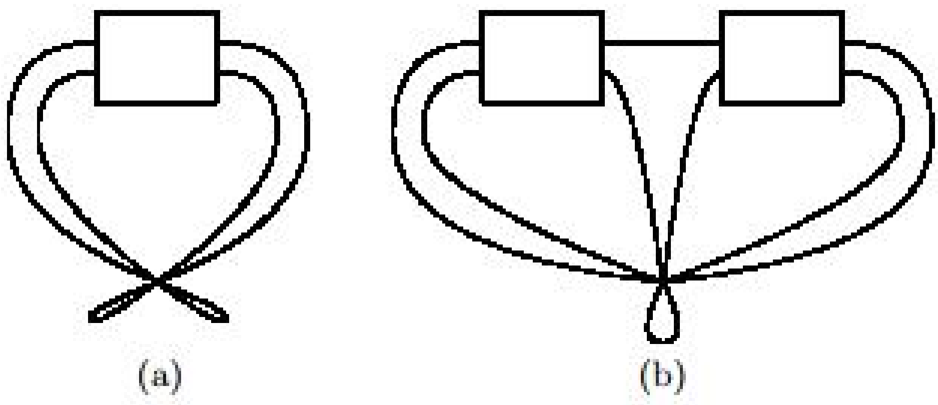


Figure 4.2: The distinct projections along quadrisecant possible upto planar isotopies (taken from [KSY01]).

Proof. This follows directly from the previous sublemmas. ∎

Suppose $\delta_i \cap \delta_j \neq \phi$ and that none of δ_i and δ_j contains the other completely. The connected component of $\delta_i - (\delta_j \cup l_j)$ which does not meet $\pi(K)$. Such a region is referred to as a crescent. The boundary of a crescent has only two singular points of $\pi(K)$ called as the ends of the crescent. The crescent can have the quadruple point $\pi(Q)$ as one of it's ends.
Now suppose δ_i is completely inside δ_j, then we have a loop which passes through $\pi(Q)$ which is the projection of one subarc of K. Then the disc bounded is called a loop crescent.
Crescents are said to be alternating if one of the arcs on the boundary of the crescent passes over the other arc at one end then it passes under the other arc at the other end.

Sublemma 4.0.6. *We may assume that every crescent which is not a loop crescent is alternating. We may also assume that no crescent is a loop crescent.*

Proof. If the crescent is non-alternating then we can remove the two crossings which makes it non-alternating by a local straightening in the two arcs as shown in the Figure 4.3. By Remark 4.0.1 the superbridge number remains unchanged.

If there is a loop crescent, we may straighten a small arc of the loop near $\pi(Q)$ without changing the knot type and the superbridge number. Then Q becomes a trisecant. We look at distinct configurations of cylindrical neighbourhoods of trisecant upto small

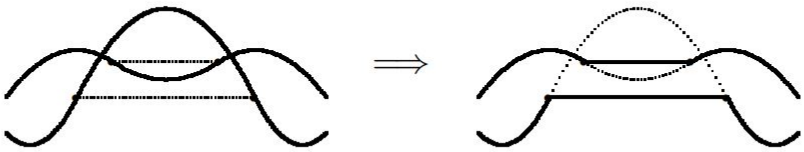


Figure 4.3: Removing non-alternating crescents (taken from [KSY01]).

perturbations and planar isotopies. For the number of such distinct configurations we look at the choice of pairs of 6 points on a circle ($\binom{6}{2}\binom{4}{2}\binom{2}{2}$) and to remove repetition divide it by number of order in which the pairs could be chosen (3!) ie, $\frac{\binom{6}{2}\binom{4}{2}\binom{2}{2}}{3!} = 15$. Figure 4.4 gives distinct configurations upto rotations.
Upon rotation each of these configurations give rise to 2, 3, 6, 3, and 1 distinct configurations respectively. Therefore there are 15 possible configurations proving that upto rotations these five are the distinct configurations.

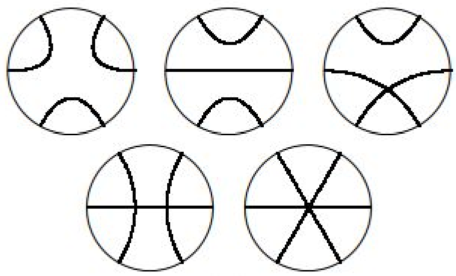


Figure 4.4: The 5 distinct configurations possible at the cylindrical neighbourhood around the trisecant upto rotations, small perturbations and planar isotopy (taken from [KSY01]).

From Table 4.1 it can be observed that if loop crescents exist then it is an unknot, a link or a torus knot. 3-superbridge knots are nontrivial and the only 3-superbridge torus knot is trefoil. Since presence of loop crescents leads to only one knot type: trefoil, it can be assumed that there exists no loop crescents. ■

Sublemma 4.0.7. *We may assume that no two arcs $\overline{l_i}$ and $\overline{l_j}$ can meet more than three times.*

Proof. Suppose the two arcs $\overline{l_i}$ and $\overline{l_j}$ meet atleast 4 times. For any crossing point Z of $\overline{l_i} \cup \overline{l_j}$ let Z_0 be the mid point of the line segment joining $Z_i = \pi^{-1}(Z) \cap l_i$ and $Z_j = \pi^{-1}(Z) \cap l_j$. Let O be a point on Q which separates the four points of $K \cap Q$ two by two. There are two cases to consider,

Case 1: Suppose, the closure of $l_i \cup l_j$ is not connected. Let us denote the starting and ending points of l_i by A and B respectively and that of l_j by C and D respectively.

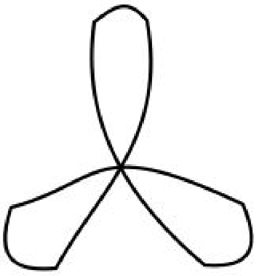
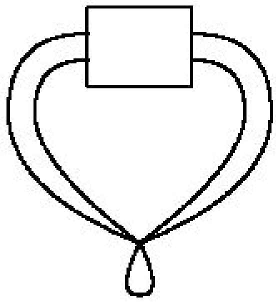
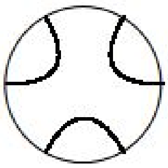
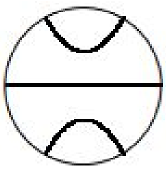
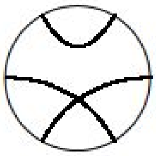
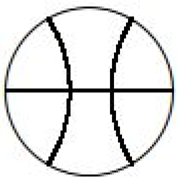
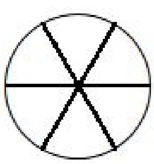
Configurations		
	either an unknot or a link	torus knot
	link	unknot
	either an unknot or a link	either an unknot or a torus knot
	unknot	torus knot
	unknot	torus knot

Table 4.1: The knots or links possible for a given projection along a trisecant and for each of the 5 distinct configurations possible at the cylindrical neighbourhood around the trisecant upto rotations, small perturbations and planar isotopy.

Subcase 1.1:
Suppose O separates A and D, then, B and C is also separated by O. Let X and Y be the first and second crossing of $\overline{l_i} \cup \overline{l_j}$ along $\overline{l_i}$, respectively. Then Q, $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ cuts the knot K into 8 disjoint arcs. Take $\mathcal{P}$ to be the plane determined by O, X_0 and Y_0. Then it can be observed that each of the six arcs, two from $K - (l_i \cup l_j)$, two between Q and $\pi^{-1}(X)$, and two between $\pi^{-1}(X)$ and $\pi^{-1}(Y)$ crosses $\mathcal{P}$.
If O separates A and B, then, C and D is also separated by O. Then, it can be observed that the remaining two arcs between $\pi^{-1}(Y)$ and Q crosses $\mathcal{P}$. So we have a plane $\mathcal{P}$ such that the knot crosses the plane 8 times. Therefore $Sb(K) \geqslant 4$, a contradiction.
If O does not separate A and B, then it does not separate C and D. Then again the remaining two arcs between $\pi^{-1}(Y)$ and Q crosses $\mathcal{P}$ (look at the crossing next to Y). Therefore, we have the same contradiction here.
Subcase 1.2:
Suppose O does not separate A and D, then, it does not separate B and C either. Then, it can be observed that there exists a disc such that it does not intersect K. Such a case can be avoided by choosing a topologically nontrivial quadrisecant to begin with.

Case 2: Suppose the closure of $l_i \cup l_j$ is connected. We assume that the starting point point and the ending point of l_i is A and B respectively. We also assume that the starting point of l_j is the ending point of l_i. Let C be the end point of l_j and D be the remaining point in $K \cap Q$.
Subcase 2.1:
Suppose O separates B from A and C. Then B and D are on the same side of O in Q. Let X and Y be the first and second crossing points of $\overline{l_i} \cup \overline{l_j}$ along $\overline{l_i}$. Choose points $X_+ \in \pi^{-1}(X)$ and $Y_- \in \pi^{-1}(Y)$ such that

$$\overrightarrow{OX}_+ = \overrightarrow{OX}_0 + \frac{||\overrightarrow{X_iX_j}||}{||\overrightarrow{OA}||}\overrightarrow{OA} \text{ and } \overrightarrow{OY}_- = \overrightarrow{OY}_0 + \frac{||\overrightarrow{Y_iY_j}||}{||\overrightarrow{OB}||}\overrightarrow{OB}$$

Choosing $\mathcal{P}$ to be the plane determined by the points O, X_+ and Y_- we have all the 8 arcs in $K - (\pi^{-1}(X) \cup \pi^{-1}(Y) \cup Q)$ crossing $\mathcal{P}$ arriving at the contadiction, $Sb(K) \geqslant 4$.
Subcase 2.2:
Suppose O separates A from B and C. Then, A and D is in the same side of O in Q. Let l_a be the arc with end points A and D and let l_b be the arc with end points C and D. By assuming that Q is topologically nontrivial, we know that $\overline{l_b}$ is the only arc corresponding to the simple loop. Therefore there is a crossing point between $\overline{l_a}$ and $\overline{l_i} \cup \overline{l_j}$, say Y. Consider the simple loop in $\widetilde{l_i} \cup \widetilde{l_j}$ created by the last crossing point of $\overline{l_i} \cup \overline{l_j}$

along $\overline{l_i}$. By Sublemma 4.0.6, this loop must have a crossing with $\overline{l_a}$. Again, by the assumption of Q being topologically nontrivial, Y can be chosen such that $\overrightarrow{Y_0Y_a}$ and $\overrightarrow{OA}$ are in opposite directions. Also, let X be the first crossing point of $\overline{l_i} \cup \overline{l_j}$ along $\overline{l_i}$. Take $\mathcal{P}$ to be the plane determined by the points O, X_0 and Y_0. Then the following five arcs, two from $l_a - \pi^{-1}(Y)$, the arc l_b and the two between Q and $\pi^{-1}(X)$, crosses the plane.
If $\overline{l_a}$ crosses $\overline{l_i}$ at Y, then, the three arcs joins X_i, Y_i, B and X_j, successively. Consequently, K crosses $\mathcal{P}$ 8 times and results in the contradiction, $Sb(K) \geqslant 4$.
If $\overline{l_a}$ crosses $\overline{l_j}$ at Y, then, the three arcs joins X_i, B, Y_j and X_j, successively. In this case, the arc joining the points B and Y_j crosses $\mathcal{P}$. Also, since there is a crossing point in $\overline{l_i} \cup \overline{l_j}$ other than X, the union of the two remaining arcs should cross $\mathcal{P}$ atleast twice. Then again, K crosses $\mathcal{P}$ atleast 8 times and results in the contradiction, $Sb(K) \geqslant 4$.
Subcase 2.3:
Suppose O seperates C from the two points A and B. This can be handled in a similar way as the above subcase. ■

Sublemma 4.0.8. *There are finitely many possible diagrams for K obtained from $\pi(K)$ by perturbing near the quadrisecant.*

Proof. We look at the distinct configurations possible at the cylindrical neighbourhood around the quadrisecant upto rotations, small perturbations and planar isotopy. There are 18 such configurations as shown in the Figure 4.5.

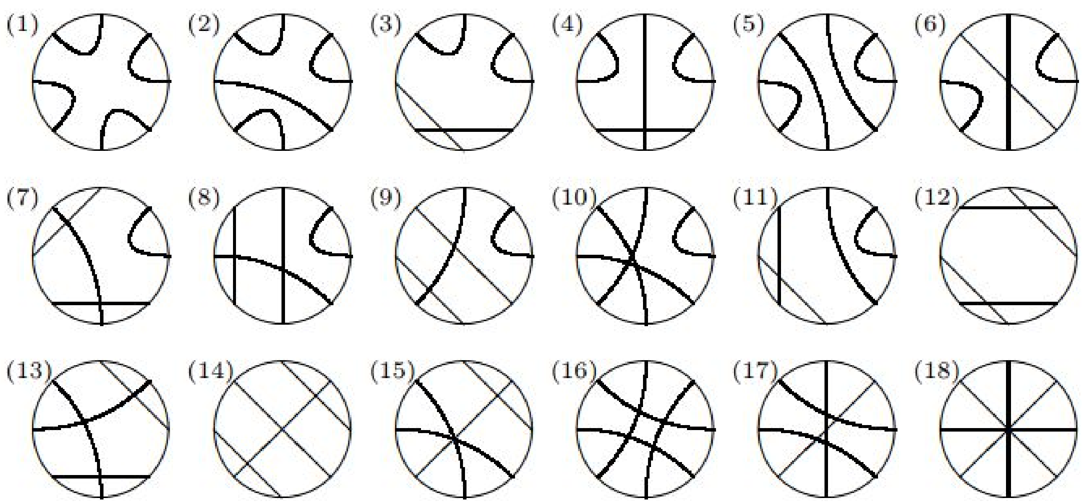


Figure 4.5: The 18 distinct configurations possible at the cylindrical neighbourhood around the quadrisecant upto rotations, small perturbations and planar isotopy (taken from [KSY01]).

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The number of distinct configurations possible ie, $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}\binom{2}{2}}{4!} = 105$ coincides with the number of distinct configurations obtained from the 18 configurations upon rotation. Therefore, these are all the configurations possible upto rotations, small perturbations and planar isotopies. ■

Theorem 4.0.4. *There are only finitely many 3-superbridge index knots.*

Proof. From all the sublemmas described above it can be noted that there are only finitely many knot diagrams that are possible. Therefore, there are only finitely many possible knot types. ■

It is conjectured that trefoil and figure-eight knot are the only knots with superbridge index 3.

Chapter 5

Polygonal knots

Parameterizing knots has been useful in defining certain knot invariants such as stick index, polynomial degree, etc. and estimating certain knot invariants such as bridge index, superbridge index, etc. Two of the parameterizations that we will be interested in are polygonal knots and polynomial knots. While polygonal knots give us classical compact knots, polynomial knots give us non-compact long knots. We will be studying about the space of these parameterizations, its topology and see how knot isotopy problems get translated to path equivalence problems in this space.

5.1 The space of polygonal knots

Every knot has a polygonal representation and the stick index gives the minimum number of edges required to give a polygonal representation of the knot. A knot in its polygonal representation is called a polygonal knot.

Definition 5.1.1. *A polygonal isotopy is an ambient isotopy of $\mathbb{R}^3$ which keeps the number of edges of the polygon in $\mathbb{R}^3$ fixed.*

Two polygonal knots are said to be equivalent if there exists a polygonal isotopy between them. We will see later that polygonal isotopy is in fact stronger than isotopy. Any embedded n-sided polygon (one which has no self-intersections), $\mathcal{P}$ in $\mathbb{R}^3$ with a choice of distinguished vertex and an orientation is called a rooted oriented n-polygon. Any rooted oriented n-polygon, $\mathcal{P}$ can be viewed as a point in $\mathbb{R}^{3n}$ listing the coordinates in the order in which the vertices appear with respect to the orientation taking the distinguished vertex as the first vertex. If $v_i = (x_i, y_i, z_i)$ for $1 \leq i \leq n$ are the vertices of the n-polygon with v_1 as the first vertex and the vertices numbered according to the orientation chosen, then $\mathcal{P} = \langle v_1, v_2, ...v_n \rangle$ can be represented by $(x_1, y_1, z_1, x_2, y_2, z_2, ..., x_n, y_n, z_n) \in \mathbb{R}^{3n}$.

But not every point in $\mathbb{R}^{3n}$ would give you an embedded n-sided polygon since there are possibilities for self intersections. The set of such points is called the discriminant denoted by $\Sigma^{(n)}$. For polygons $\mathcal{P} = \langle v_1, v_2, ...v_n\rangle$ with v_1v_2 intersecting v_3v_4 we have,

$$(v_2 - v_1) \times (v_3 - v_1) \cdot (v_4 - v_1) = 0$$

$$(v_2 - v_1) \times (v_3 - v_1) \cdot (v_2 - v_1) \times (v_4 - v_1) < 0$$

$$(v_4 - v_3) \times (v_1 - v_3) \cdot (v_4 - v_3) \times (v_2 - v_3) < 0$$

So the set of polygons with v_1v_2 intersecting v_3v_4 is the closure of the locus described by the above system, hence forming a real semi-algebraic cubic variety. Discriminant can also be defined as the union of closure of $\frac{n(n-1)}{3}$ of such real semi-algebraic varieties each consisting of n-polygons having a pair of intersecting edges.

Definition 5.1.2. *The space of geometric knots or geometric knot space is defined to be the compliment of determinant,*

$$\mathfrak{Geo}^{(n)} = \mathbb{R}^{3n} - \Sigma^{(n)}$$

$\mathfrak{Geo}^{(n)}$ is an open subset of $\mathbb{R}^n$ since $\Sigma^{(n)}$ being a finite union of closed sets is closed. Hence $\mathfrak{Geo}^{(n)}$ is an open submanifold of $\mathbb{R}^n$, it is infact an open dense submanifold since any element in $\Sigma^{(n)}$ can be perturbed by a small amount to get an embedded n-polygon. Using the bijection from the set of embedded n-polygons to $\mathfrak{Geo}^{(n)}$ we can give a topology on the set of embedded n-polygons inducing from the subspace topology on $\mathfrak{Geo}^{(n)}$.
It can be observed that a path $h : [0, 1] \longrightarrow \mathfrak{Geo}^{(n)}$ gives a polygonal isotopy in the space of polygonal knots. So, path components in $\mathfrak{Geo}^{(n)}$ gives the equivalence classes of polygonal knots.

Theorem 5.1.1. *The spaces $\mathfrak{Geo}^{(3)}$, $\mathfrak{Geo}^{(4)}$ and $\mathfrak{Geo}^{(5)}$ are path connected and consists only of unknot.*

Proof. Fix a plane and choose a triangle in it. Any element in $\mathfrak{Geo}^{(3)}$ is a triangle and we can find an polygonal isotopy from this element to the chosen triangle. Hence, $\mathfrak{Geo}^{(3)}$ is path connected.
Any element in $\mathfrak{Geo}^{(4)}$ is a quadrilateral and each quadrilateral has two triangles attached along a side (a hinge). We can rotate one of the triangles along the hinge to get a planar quadrilateral. Hence, $\mathfrak{Geo}^{(4)}$ is path connected.
Suppose $\mathcal{P} = \langle v_1, v_2, v_3, v_4, v_5\rangle$ is an element of $\mathfrak{Geo}^{(5)}$ then we have two cases. If $\Delta v_1v_2v_3$ is not intersected by $\mathcal{P}$ then you can just straighten the sides v_1v_2 and v_2v_3 into one long line segment making it a quadrilateral. If $\Delta v_1v_2v_3$ is intersected by $\mathcal{P}$

then v_4v_5 intersects the triangle and in this case $\Delta v_4v_5v_1$ is not intersected hence can be straightened into a quadrilateral. Then path connectedness of $\mathfrak{Geo}^{(4)}$ guarantees the path connectedness of $\mathfrak{Geo}^{(5)}$. ■

To study the topology of $\mathfrak{Geo}^{(n)}$ we look at a stratification of the knot space. Consider the map $g : \mathfrak{Geo}^{(n)} \longrightarrow \mathfrak{Geo}^{(n-1)}$ defined by

$$\mathcal{P} = \langle v_1, v_2, ..., v_n \rangle \mapsto g(\mathcal{P}) = \langle v_1, v_2, ..., v_{n-1} \rangle$$

This map is not necessarily well defined. There can be cases in which $\mathfrak{Geo}^{(n)}$ has polygons where the line segment joining v_1v_{n-1} is intersected. But in all these cases small perturbations can be made so that the image of every element is in $\mathfrak{Geo}^{(n-1)}$. Fixing $\mathcal{Q} \in \mathfrak{Geo}^{(n-1)}$ it can be observed that $g^{-1}(\mathcal{Q})$ is a 3-manifold of $\mathfrak{Geo}^{(n)}$. Considering $g^{-1}(\mathcal{Q})$ for all $\mathcal{Q} \in \mathfrak{Geo}^{(n-1)}$ we get a stratification of $\mathfrak{Geo}^{(n)}$ into 3-manifolds.
The geometric knot spaces $\mathfrak{Geo}^{(6)}$ and $\mathfrak{Geo}^{(7)}$ were studied in detail. It was proved that $\mathfrak{Geo}^{(6)}$ contains a component of unknot and two components each of right-handed trefoil and left-handed trefoil. In the case of $\mathfrak{Geo}^{(7)}$ unknot, right-handed trefoil and left-handed trefoil has one component each and there are two components of figure-eight knot. We will be studying these theorem and will be studying about invariants for geometric hexagons and geometric heptagons.

5.2 Geometric hexagons

5.2.1 The topology of $\mathfrak{Geo}^{(6)}$

We will study the knot types that can be realized as hexagons.

Theorem 5.2.1. $\mathfrak{Geo}^{(6)}$ *contains five path components, one component of unknots, two components of right-handed trefoil and two components of left-handed trefoil.*

Proof. Let $H = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle \in \mathfrak{Geo}^{(6)}$, through a series of linear transformations in $\mathbb{R}^3$ we can take $v_1 = (0,0,0)$ and $v_5 = (a,0,0)$ where $a > 0$. We can also perturb H such that H intersects the x-axis only at the points v_1 and v_5. Consider half planes with x-axis as the boundary, let $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ denote the half planes containing v_2, v_3 and v_4. There is nothing special about the x-axis, we could've taken the line containing v_1 and v_5 and all the half planes with this line as boundary.
Consider the order in which the these half planes occur as we rotate around the x-axis in the clockwise direction starting from a half plane which does not intersect H. This divides $\mathfrak{Geo}^{(6)}$ into six regions which intersect along 2-dimensional regions where

1. two $\mathcal{P}_i$'s are the same
2. edge of H intersects the x-axis

We have the following lemma which describes all the possible knots that can occur in each of these regions. If the half planes occur in the order $\mathcal{P}_2 - \mathcal{P}_3 - \mathcal{P}_4$ we denote this configuration by $2-3-4$.

Lemma 5.2.2 (Structure lemma for $\mathfrak{Geo}^{(6)}$). *The given table gives the number of connected components of different knot types in the six regions in $\mathfrak{Geo}^{(6)}$.*

Configurations of the half planes	0_1	3_1	-3_1
$2-3-4$	1	0	0
$2-4-3$	1	1	0
$3-2-4$	1	1	0
$3-4-2$	1	0	1
$4-2-3$	1	0	1
$4-3-2$	1	0	0

Table 5.1: The number of components of unknot(0_1), right-handed trefoil(3_1) and left-handed trefoil(-3_1) in each of the regions.

The proof of this lemma is just brute force analysis of possibility of knot types in $g^{-1}(g(H))$ if H is in any of these regions. The proof can be found in J.A Calvo's PhD thesis, [Cal98].

It can be noted that the regions $2-3-4$ and $2-4-3$ meet along the two dimensional subsets where $\mathcal{P}_3 = \mathcal{P}_4$ and regions $4-3-2$ and $3-2-4$ meet along the two dimensional subsets which contains hexagons for which v_2v_4 intersects x-axis. We call these codimension 1 connections between different regions. All the connections between the six regions can be summarised in the following figure.

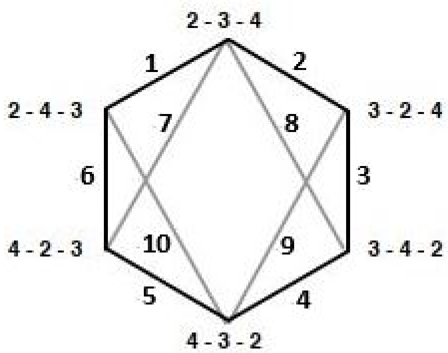

Figure 5.1: Codimension 1 connections between different regions.

The codimension 1 connections are set of hexagons where

1. $\mathcal{P}_3 = \mathcal{P}_4$
2. $\mathcal{P}_2 = \mathcal{P}_3$
3. $\mathcal{P}_2 = \mathcal{P}_4$
4. $\mathcal{P}_3 = \mathcal{P}_4$
5. $\mathcal{P}_2 = \mathcal{P}_3$
6. $\mathcal{P}_2 = \mathcal{P}_4$
7. v_3v_4 intersects x-axis
8. v_2v_4 intersects x-axis
9. v_2v_4 intersects x-axis
10. v_2v_3 intersects x-axis

It can be observed that adjacent regions have unknot common and the set of unknots of all the regions form one component of $\mathfrak{Geo}^{(6)}$. Table 5.2 shows that $2-4-3$ and $3-2-4$ has right-handed trefoils. If these trefoils are in the same component we should get a path from elements in $2-4-3$ to $3-2-4$ since these two are connected through region $4-3-2$, the path should contain a point which represents unknot. But then this would imply that unknot and right handed trefoil are equivalent which is a contradiction. Hence, trefoils in $2-4-3$ and $3-2-4$ forms two separate components of $\mathfrak{Geo}^{(6)}$. Similarly, we can show that the left-handed trefoils in $4-2-3$ and $3-4-2$ forms the other two separate components of $\mathfrak{Geo}^{(6)}$. ∎

5.2.2 Generating invariants for geometric hexagons

Definition 5.2.1. *The curl of a hexagon $H = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle$ is defined as*

$$\mathfrak{curl}(H) = \text{sign}((v_3 - v_1) \times (v_5 - v_1) \cdot (v_2 - v_1))$$

The curl of a hexagon in some sense measures whether it twists up or down.

Note 5.2.1. *It is an easy calculation to see that*

1. *Every trefoils of type* $3-2-4$ *and* $3-4-2$ *have negative curl*
2. *Every trefoils of type* $2-4-3$ *and* $4-2-3$ *have positive curl*

Corollary 5.2.3. *The curl of a rooted oriented hexagonal trefoil is invariant under geometric deformations.*

Proof. Let us assume that this is not the case, then there exists a path $h : [0,1] \longrightarrow \mathfrak{Geo}^{(6)}$ such that $\mathfrak{curl}(h(0)) = 1$ and $\mathfrak{curl}(h(1)) = -1$. By intermediate value theorem $\exists H \in \mathfrak{Geo}^{(6)}$ such that $\mathfrak{curl}(H) = 0$ which implies that v_1, v_2, v_3 and v_5 lies in the same plane. Such an H would be isotopic to Figure 5.2. It is easy to see that H is clearly an unknot. But then we got that geometric trefoils are isotopic to an unknot, a contradiction. ∎

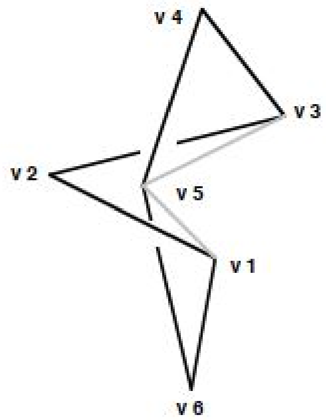

Figure 5.2: Hexagon of curl zero (taken from [Cal98]).

Remark 5.2.1. *It can be observed that for any geometric trefoils the plane containing the even vertices lie either above or below the plane containing the odd vertices.*

Let us define automorphisms r and s in $\mathfrak{Geo}^{(6)}$, for $H = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle \in \mathfrak{Geo}^{(6)}$

$$r\langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle = \langle v_1, v_6, v_5, v_4, v_3, v_2 \rangle$$
$$s\langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle = \langle v_2, v_3, v_4, v_5, v_6, v_1 \rangle$$

To begin with $\mathfrak{Geo}^{(6)}$ is the space of rooted oriented hexagons, r basically reverses the orientation and s changes the root vertex by one cyclic permutation.
Let H be a geometric trefoil in $\mathfrak{Geo}^{(6)}$ such that all the even vertices lie above the plane of all the odd vertices, then

$$\begin{aligned} \mathfrak{curl}(H) &= \operatorname{sign}((v_3 - v_1) \times (v_5 - v_1) \cdot (v_2 - v_1)) \\ &= +1 \end{aligned} \tag{5.1}$$

while for H with opposite orientation,

$$\begin{aligned} \mathfrak{curl}(rH) &= \operatorname{sign}((v_5 - v_1) \times (v_3 - v_1) \cdot (v_6 - v_1)) \\ &= -\operatorname{sign}((v_3 - v_1) \times (v_5 - v_1) \cdot (v_6 - v_1)) \\ &= -1 \end{aligned} \tag{5.2}$$

From equations 5.1 and 5.2 we have,

$$\mathfrak{curl}(rH) = -\mathfrak{curl}(H)$$

.

This proves the following corollary.

Corollary 5.2.4. *Geometric trefoils in $\mathfrak{Geo}^{(6)}$ are not reversible.*

But trefoils are topologically reversible. There are restrictions in the deformations that one can perform on geometric hexagons due to the presence of rigid edges and this indeed is why the topological isotopy can't be translated into a geometric isotopy.
Let us inspect more and see if the presence of two types of left and right handed trefoil has something to do with the choice of the rooted vertex. We will see how curl behaves under the automorphism s. Lets start with the same H we took before and it can be observed that in the case of sH the even vertices lie below the plane containing all the odd vertices. Therefore we have,

$$\mathfrak{curl}(sH) = -\mathfrak{curl}(H)$$

Which implies that curl clearly depends on the choice of rooted vertex. So we can conclude that the irreducubility of trefoils in $\mathfrak{Geo}^{(6)}$ is a consequence of fixing a root.

Remark 5.2.2. *The space $\mathfrak{Geo}^{(6)}/\langle s\rangle$ of non-rooted oriented embedded hexagons, and the space $\mathfrak{Geo}^{(6)}/\langle r,s\rangle$ of non-rooted non-oriented embedded hexagons, each consist of exactly three path-components.*

If we know already that the trefoil is either right-handed or left-handed then curl can distinguish between the two geometrically different trefoils in each of these. This is the same as saying curl cannot completely distinguish all the five different geometric hexagons. We will be introducing something called chirality which along with curl will be able to distinguish between all the five hexagonal knot types.
For a rooted oriented hexagon $H = \langle v_1, v_2, v_3, v_4, v_5, v_6\rangle$ a triangle based at v_2, Δ_{123} is triangle with vertices v_1, v_2, and v_3 oriented by the right hand rule. If H intersects this triangle, it is either intersected by v_4v_5, v_5v_6 or both. The algebraic intersection number of Δ_{123}, Δ_2 is taken to be 1 if one edge intersects in the direction of the orientation and -1 if the intersection is in the opposite direction. If both the edges intersect then they intersect in opposite directions hence Δ_2 is taken to be 0. In a similar way algebraic intersection number of Δ_{345}, Δ_4 and algebraic intersection number of Δ_{561}, Δ_6 are defined. The following lemma shows that Δ_2, Δ_4 and Δ_6 completely distinguishes different topological knots in $\mathfrak{Geo}^{(6)}$.

Lemma 5.2.5. *Let H be a hexagon. Then*

(i) *H is a right-handed trefoil* $\iff \Delta_2 = \Delta_4 = \Delta_6 = 1$

(ii) *H is a left-handed trefoil* $\iff \Delta_2 = \Delta_4 = \Delta_6 = -1$

(iii) *H is an unknot* $\iff \Delta_i = 0$ *for some* $i \in \{2, 4, 6\}$.

Proof. Refer [Cal98]. ∎

Definition 5.2.2. *The chirality of a hexagon H is defined as* $\Delta(H) = \Delta_2\Delta_4\Delta_6$.

Note 5.2.2. *From the above lemma we have,*

- $\Delta(H) = 0 \iff$ *H is an unknot*
- $\Delta(H) = 1 \iff$ *H is an right-handed trefoil*
- $\Delta(H) = 1 \iff$ *H is an left-handed trefoil*

Now we will use both the tools, curl and chirality we just introduced to define joint chirality-curl which will be a complete invariant for geometric hexagons.

Definition 5.2.3. *The joint chirality-curl of a hexagon H is defined as*

$$\mathcal{J}(H) = (\Delta(H), \Delta^2(H)\mathfrak{curl}(H))$$

Theorem 5.2.6. *The joint chirality-curl is an invariant of hexagons under geometric deformation. In fact, the geometric knot type of a hexagon H is completely determined by the value of its joint chirality-curl, since*

$$\mathcal{J}(H) = \begin{cases} (0,0) & \iff \text{ H is an unknot} \\ (+1, c) & \iff \text{ H is a right-handed trefoil with } \mathfrak{curl}(H) = c \\ (-1, c) & \iff \text{ H is a left-handed trefoil with } \mathfrak{curl}(H) = c \end{cases}$$

Proof. Theorem 5.2.1 shows that there are 5 path components and it can be observed that for right-handed trefoil and left-handed trefoil the two different path components corresponds to hexagons with curl $+1$ and -1 respectively. From Lemma 5.2.5 we know that $\Delta(H)$ clearly distinguishes topological knots and for different topological trefoils, curl distinguishes the geometrically different ones. ∎

5.3 Geometric heptagons

5.3.1 The topology of $\mathfrak{Geo}^{(7)}$

We will study the knot types that can be realized as heptagons.

Lemma 5.3.1 (Structure lemma for $\mathfrak{Geo}^{(7)}$)**.** *The given table gives the number of connected components of possible knot types in different regions of* $\mathfrak{Geo}^{(7)}$.

Configurations of the half planes	0_1	3_1	-3_1	4_1
$2-3-4-5$	1	0	0	0
$2-3-5-4$	1	1	1	0
$2-4-3-5$	3	3	3	4
$2-4-5-3$	5	4	4	6
$2-5-3-4$	1	1	1	2
$2-5-4-3$	1	1	1	2
$3-2-4-5$	1	1	1	0
$3-2-5-4$	1	2	2	0
$3-4-2-5$	1	1	1	2
$3-4-5-2$	1	1	1	2
$3-5-2-4$	10	5	4	8
$3-5-4-2$	5	4	4	6
$4-2-3-5$	5	4	4	6
$4-2-5-3$	10	5	4	8
$4-3-2-5$	1	1	1	2
$4-3-5-2$	1	1	1	2
$4-5-2-3$	1	2	2	0
$4-5-3-2$	1	1	1	0
$5-2-3-4$	1	1	1	2
$5-2-4-3$	1	1	1	2
$5-3-2-4$	5	4	4	6
$5-3-4-2$	3	3	3	4
$5-4-2-3$	1	1	1	0
$5-4-3-2$	1	0	0	0
$Loop^+$	3	1	1	2
$Loop^-$	3	1	1	2

Table 5.2: The number of components of unknot(0_1), right-handed trefoil(3_1), left-handed trefoil(-3_1) and figure-eight knot(4_1) in each of the regions.

Theorem 5.3.2. $\mathfrak{Geo}^{(7)}$ *contains five path components, one component of unknots, one of right-handed trefoil, one of left-handed trefoil and two components of figure-eight knots.*

Proof. The proof is similar to the geometric hexagons case. We choose a generic heptagon such that v_1 and v_6 lie in x-axis with $v_6 = (a, 0, 0)$ where $a > 0$. The 24 regions in $\mathfrak{Geo}^{(7)}$ are based on order in which the half planes $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$ containing v_2, v_3, v_4 and v_5 respectively exists when we go about clockwise around the x-axis starting from a half plane which does not intersect the heptagon at all. But there are geometric heptagons which intersects all the half planes and so in addition to the 24 regions we have two additional regions $Loop^+$ and $Loop^-$ in which the vertices v_2, v_3, v_4 and v_5 loop around the x-axis in right-handed and left-handed fashion respectively. The structure lemma for geometric heptagon is given above. It is not easy as the hexagonal case since each region has a lot of path components. We won't be going into the details there are multiple connections possible between components of trefoil and unknot hence there are one component each of unknot, right-handed trefoil and left-handed trefoil. It can be noted that there are two path components of figure-eight knot. Suppose there exists a path between the figure-eight knots in the two components then the path should contain a point in the boundary of one of the 26 regions. By small perturbations this point can be assumed to be such that the heptagon on this path intersects the x-axis more than twice. But this is impossible for a figure-eight knot (one have to take cases by case to see this). ■

5.3.2 Generating invariants for geometric heptagons

For a geometric heptagon $H = \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7 \rangle$ let us define the the following functions,

$$\Theta_3(H) = \mathrm{sign}((v_7 - v_1) \times (v_2 - v_1) \cdot (v_3 - v_1)),$$

$$\Theta_6(H) = \mathrm{sign}((v_6 - v_1) \times (v_7 - v_1) \cdot (v_2 - v_1)).$$

Note 5.3.1. *It can be noted that,*

- *if v_3 and v_6 lie on the same side of the plane containing v_7, v_1 and v_2 then $\Theta_3 = \Theta_6$,*
- *if v_3 and v_6 lie on different sides of the plane containing v_7, v_1 and v_2 then $\Theta_3 = -\Theta_6$.*

Therefore, for a generic heptagon in $\mathfrak{Geo}^{(7)}$ the functions $\frac{1}{2}(\Theta_3 + \Theta_6)$ and $\frac{1}{2}(\Theta_3 - \Theta_6)$ are $\mathbb{Z}_2$-complementary; i.e, exactly one of them is zero while the other is ± 1.
For $(i, j) \in \{(3, 4), (4, 5), (5, 6)\}$ define I_{ij} to be the algebraic intersection number of the edge $v_i v_j$ with Δ_{712}, giving orientation from that of H.

Lemma 5.3.3. *If H is a heptagonal figure-eight knot with $\Theta_3 = \Theta_6$, then exactly one of the intersection numbers I_{34}, I_{45} or I_{56} is non-zero. In particular, $I_{34} + I_{45} + I_{56} = \pm 1$*

Proof. Refer [Cal98]. ■

Lemma 5.3.4. *If H is a heptagonal figure-eight knot with $\Theta_3 = -\Theta_6$, then exactly one of the intersection numbers I_{34} or I_{56} is non-zero. In particular, $I_{34} - I_{56} = \pm 1$*

Proof. Refer [Cal98]. ∎

Theorem 5.3.5. *The function*

$$\Xi(H) = \frac{1}{2}(\Theta_3 + \Theta_6)(I_{34} + I_{45} + I_{56}) + \frac{1}{2}(\Theta_3 - \Theta_6)(I_{34} - I_{56})$$

is an invariant of heptagonal figure-eight knots under geometric deformations.

Proof. From the previous two lemmas it can be noted that Ξ takes values $+1$ or -1. Therefore, if the values of Θ_3 and Θ_6 remains unchanged throughout the isotopy, Ξ must also remain unchanged. We will consider a geometric deformation of figure-eight knot which changes Θ_3 and show that Ξ remains unchanged under this deformation.
Let H_0 be a heptagonal figure-eight knot with $\Theta_3 = 0$ which means v_3 is in the plane containing v_7, v_1 and v_2. Pushing v_3 away from the plane towards v_6, would give us a heptagon H_0^+ with $\Theta_3 = \Theta_6$, similarly pushing v_3 away from the plane and away from v_6, would give us a heptagon H_0^- with $\Theta_3 = -\Theta_6$. It can be noted that v_3v_4 is the edge which is being moved in these transformations, we will denote algebraic intersection number of v_3v_4 in H_0^+ and H_0^- by I_{34}^+ and I_{34}^- respectively.
Suppose that $I_{34}^- = 0$ then by Lemma 5.3.3, $I_{56} = \pm 1$ and hence by Lemma 5.3.4, $I_{34}^+ = I_{45} = 0$. Therefore we have,

$$\left(I_{34}^+ + I_{45} + I_{56}\right) = I_{56} = -\left(I_{34}^- - I_{56}\right).$$

So, whatever value Θ_3 takes Ξ remains unchanged.
Suppose that $I_{34}^- = \pm 1$ then by Lemma 5.3.3, $I_{56} = 0$. We also have $I_{34}^+ = 0$ hence by Lemma 5.3.4, $I_{45} = \pm 1$. Since edges v_3v_4 and v_4v_5 are adjacent if they intersect Δ_{712} they should do so in opposite directions, hence $I_{34}^- = -I_{45}$. Making use of all these we have,

$$\left(I_{34}^+ + I_{45} + I_{56}\right) = I_{45} = -I_{34}^- = -\left(I_{34}^- - I_{56}\right).$$

So whatever value Θ_3 takes, Ξ again remains unchanged. Therefore Ξ remains unchanged under geometric deformations.

∎

Topological knots being achiral does not guarantee that the geometric knots should also be achiral. But in the case of figure-eight knots this is infact the case. It can be observed that heptagonal figure-eight knots are also achiral. Figure 5.3 shows the deformations.

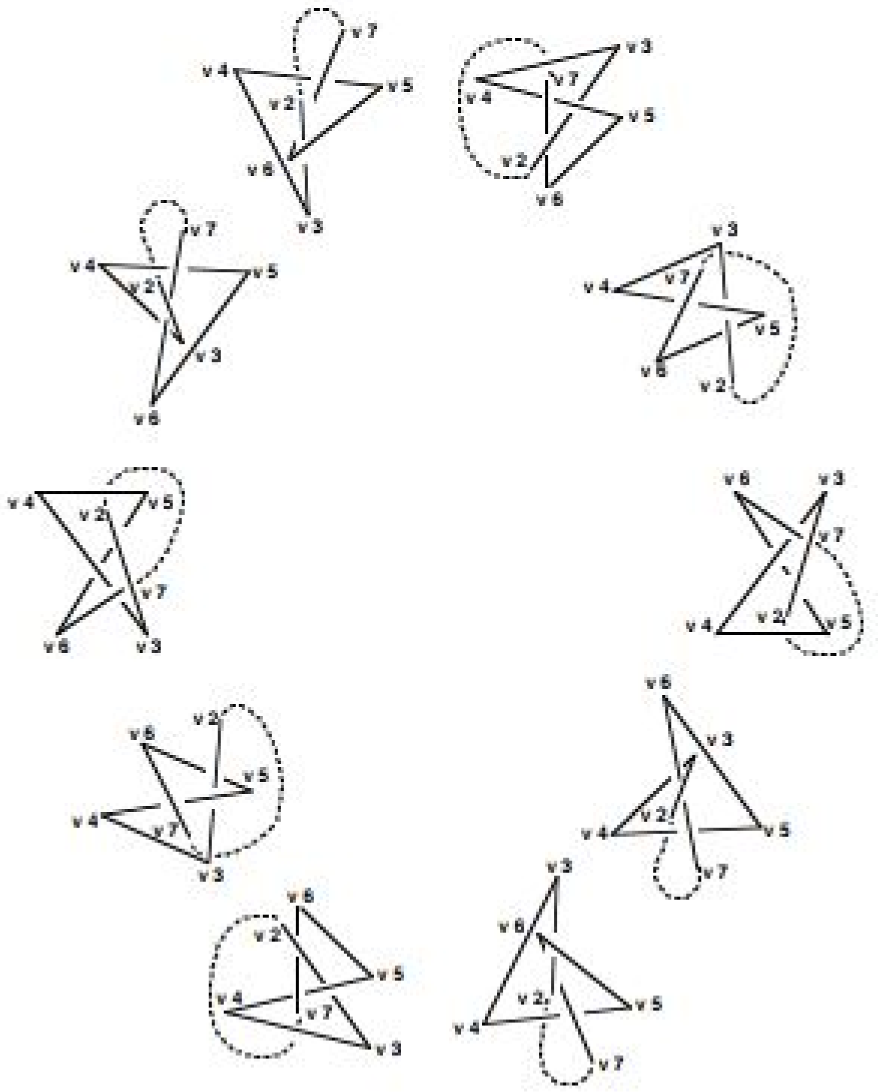

Figure 5.3: Heptagonal figure-eight knots are achiral (taken from [Cal98]).

For $H = \langle v_1, v_2, v_3, v_4, v_5, v_6, v_7 \rangle \in \mathfrak{Geo}^{(7)}$ we will define the automorphisms r and s of $\mathfrak{Geo}^{(7)}$ as follows,

$$r\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7 \rangle = \langle v_1, v_7, v_6, v_5, v_4, v_3, v_2 \rangle$$

$$s\langle v_1, v_2, v_3, v_4, v_5, v_6, v_7 \rangle = \langle v_2, v_3, v_4, v_5, v_6, v_7, v_1 \rangle$$

Action of r reverses the orientation of H, $I_{34}(rH)$ is the algebraic intersection number of v_6v_5 with Δ_{217} which is the same as the algebraic intersection number of v_5v_6 with Δ_{712}. Similarly $I_{45}(rH)$ and $I_{56}(rH)$ can be calculated. Then we have the following,

$$\begin{aligned} I_{34}(rH) &= I_{56}(H) && (5.3)\\ I_{45}(rH) &= I_{45}(H) && (5.4)\\ I_{56}(rH) &= I_{34}(H) && (5.5)\end{aligned}$$

On the other hand we have,

$$\begin{aligned} \Theta_3(rH) &= \operatorname{sign}((v_2-v_1)\times(v_7-v_1)\cdot(v_6-v_1)) \\ &= -\operatorname{sign}((v_6-v_1)\times(v_7-v_1)\cdot(v_2-v_1)) \\ &= -\Theta_6(H), && (5.6)\end{aligned}$$

$$\begin{aligned} \Theta_6(rH) &= \operatorname{sign}((v_3-v_1)\times(v_2-v_1)\cdot(v_7-v_1)) \\ &= -\operatorname{sign}((v_7-v_1)\times(v_2-v_1)\cdot(v_3-v_1)) \\ &= -\Theta_3(H). && (5.7)\end{aligned}$$

Therefore we have,

$$\begin{aligned} \Xi(rH) &= \frac{1}{2}\Big(\Theta_3(rH)+\Theta_6(rH)\Big)\Big(I_{34}(rH)+I_{45}(rH)+I_{56}(rH)\Big) \\ &\quad+\frac{1}{2}\Big(\Theta_3(rH)-\Theta_6(rH)\Big)\Big(I_{34}(rH)-I_{56}(rH)\Big) \\ &= \frac{1}{2}\Big(-\Theta_6(H)-\Theta_3(H)\Big)\Big(I_{56}(H)+I_{45}(H)+I_{34}(H)) \\ &\quad+\frac{1}{2}\Big(-\Theta_6(H)+\Theta_3(rH)\Big)\Big(I_{56}(H)-I_{34}(H)\Big) \\ &= -\Xi(H). && (5.8)\end{aligned}$$

Equation 5.8 shows that figure-eight knot in $\mathfrak{Geo}^{(7)}$ is irreversible as opposed to topological figure-eight knots which are reversible.

Corollary 5.3.6. *Figure-eight knots in $\mathfrak{Geo}^{(7)}$ are achiral but not reversible.*

Let us look at how choice of root affects the invariant. Suppose,

$$\Xi(sH) = -\Xi(H) \implies \Xi(s^7H) = -\Xi(H)$$

which is a contradiction. Therefore,

$$\Xi(sH) = \Xi(H)$$

i.e, unlike in $\mathfrak{Geo}^{(6)}$ the geometric knot type of heptagons does not depend on the choice of the root.

Remark 5.3.1. *The space $\mathfrak{Geo}^{(7)} / \langle s \rangle$ of non-rooted oriented embedded heptagons consists of five path-components. On the other hand, the space $\mathfrak{Geo}^{(7)} / \langle r, s \rangle$ of non-rooted non-oriented embedded heptagons consists of four path-components.*

Chapter 6

Polynomial knots

This chapter is based on the study of these two papers by Rama Mishra and Hitesh Raundal; [MR15] and [RM17].

6.1 Polynomial knots and equivalences

Definition 6.1.1. *A long knot is a proper smooth embedding $\phi : \mathbb{R} \longrightarrow \mathbb{R}^3$ such that the map $t \mapsto \|\phi(t)\|$ is strictly monotone outside some closed interval of $\mathbb{R}$ and $\|\phi(t)\| \to \infty$ as $|t| \to \infty$.*

Definition 6.1.2. *Two long knots $\phi, \psi : \mathbb{R} \longrightarrow \mathbb{R}^3$ are said to be topologically equivalent if there exists orientation preserving diffeomorphisms $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that $\psi = h \circ \phi \circ f$.*

A diffeotopy of $\mathbb{R}^3$ is a continuous map $H : [0,1] \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that for all $t \in [0,1]$, $H_t = H(t, \cdot)$ is a diffeomorphism of $\mathbb{R}^3$ and H_0 is the identity map of $\mathbb{R}^3$.

Definition 6.1.3. *Two long knots ϕ and ψ are said to be ambient isotopic if there exists a diffeotopy $H : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}^3$ such that $\phi = H \circ \psi$. It is denoted by $\phi \simeq \psi$.*

There is a one-one correspondence between the ambient isotopy classes of long knots and the ambient isotopy classes of tame knots. For every long knot $\phi : \mathbb{R} \longrightarrow \mathbb{R}^3$ there is a unique embedding $\widetilde{\phi} : S^1 \longrightarrow S^3$ extended from ϕ through the inverse stereographic projection from the north pole of S^3. Every ambient isotopy class of tame knots contains a smooth knot $\psi : S^1 \longrightarrow S^3$ which fixes the north pole, the restriction of which, $\widehat{\psi} : \mathbb{R} \longrightarrow \mathbb{R}^3$ is a long knot. It can be observed that $\widehat{\widetilde{\phi}} \simeq \phi$ and $\widetilde{\widehat{\psi}} \simeq \psi$ giving the bijective correspondence.

Definition 6.1.4. *A polynomial map is a map $\phi : \mathbb{R} \longrightarrow \mathbb{R}^3$ whose component functions are univariate real polynomials. A polynomial knot is a polynomial map which is an embedding.*

A polynomial knot is a long knot. A. Shastri in [Sha92] proved that every long knot is topologically equivalent to some polynomial knot. Thus all tame knots $\mathcal{K} : S^1 \longrightarrow S^3$ are ambient isotopic to $\widetilde{\phi} : S^1 \longrightarrow S^3$ which is an extension of some polynomial knot $\phi : \mathbb{R} \longrightarrow \mathbb{R}^3$;i.e, every knot has a polynomial representation. For example, $t \mapsto (t^3 - 3t, t^4 - 4t^2, t^5 - 10t)$ and $t \mapsto (t^3 - 3t, t^5 - t^3 + 4t, t^7 - 42t)$ are the polynomial representations of a trefoil (see figure 6.1) and figure-eight knots (see Figure 6.2) respectively.

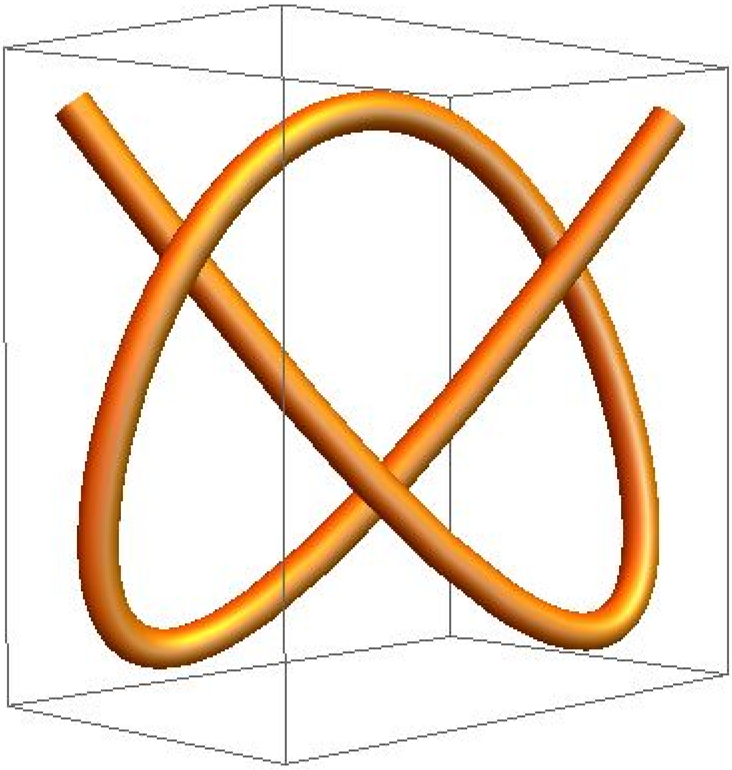

Figure 6.1: Polynomial representation of a trefoil knot

Definition 6.1.5. *A polynomial map $\phi = (f, g, h)$ is said to have a degree sequence (d_1, d_2, d_3) if deg$(f) = d_1$, deg$(g) = d_2$ and deg$(h) = d_3$. A polynomial degree of ϕ is maximum of the degrees of its component functions.*

An ambient isotopy class $[\mathcal{K}]$ of a tame knot $\mathcal{K} : S^1 \longrightarrow S^3$ is said to have a polynomial representation if there exists a polynomial knot ϕ such that $\widetilde{\phi} \simeq \mathcal{K}$ and ϕ is called a polynomial representation of the knot isotopy class $[\mathcal{K}]$. A knot isotopy class $[\mathcal{K}]$ is said to have polynomial degree d if d is the least degree of all the polynomial representations of $\mathcal{K}$ and any polynomial representation ϕ of degree d is called the minimal polynomial representation of $[\mathcal{K}]$.

For any polynomial knot ϕ of degree d, we can appropriately choose an orientation

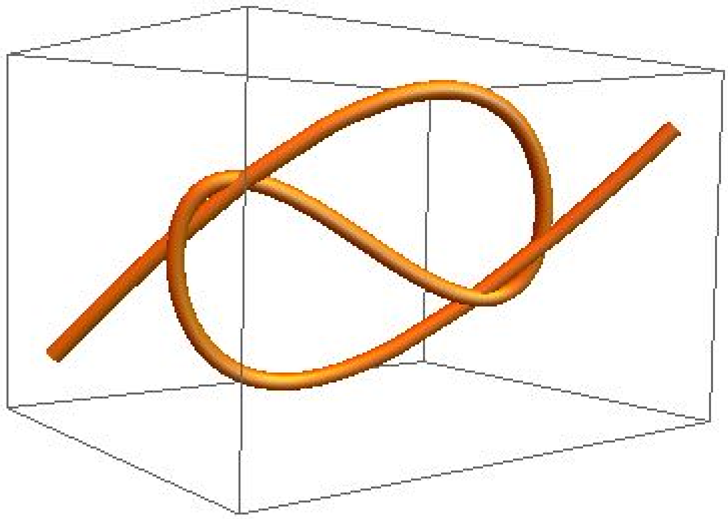

Figure 6.2: Polynomial representation of a figure-eight knot

preserving automorphism of $\mathbb{R}^3$ such that ϕ under this automorphism gives $\sigma = (f, g, h)$ such that $\deg(f) < \deg(g) < \deg(h) \leq d$. Also by adding δt^{d-2}, δt^{d-1} and δt^d for sufficiently small $\delta > 0$ in the respective components, one change the degree sequence of the polynomial knot into consecutive numbers without changing the knot type. In other words, every tame knot has a polynomial representation $\tau = (u, v, w)$ such that $\deg(u) < \deg(v) < \deg(w) \leq d$.

Remark 6.1.1. *The polynomial degree of a knot isotopy class is a knot invariant.*

But polynomial degree cannot detect chirality. If $\phi = (f,\ g,\ h)$ is a minimal polynomial representation of knot isotopy class $[\mathcal{K}]$ then $(-f,\ g,\ h)$, $(f,\ -g,\ h)$, $(f,\ g,\ -h)$ and $(-f,\ -g,\ -h)$ are minimal polynomial representations of knot isotopy class $[\mathcal{K}^*]$ of the mirror image of $\mathcal{K}$ which shows that a knot and its mirror image have the same polynomial degree.

Proposition 6.1.1. *For a classical tame knot $\mathcal{K} : S^1 \longrightarrow S^3$ with polynomial degree d we have the following*

1. $c[\mathcal{K}] \leq \frac{(d-2)(d-3)}{2}$
2. $b[\mathcal{K}] \leq \frac{(d-1)}{2}$
3. $Sb[\mathcal{K}] \leq \frac{(d+1)}{2}$

Proof. Refer [DO06]. ■

Corollary 6.1.2. *The unknot is the only knot that can be represented as a polynomial knot of degree $d \leq 4$.*

Corollary 6.1.3. *The unknot, the left-handed trefoil and the right-handed trefoil are the only knots those can be represented as polynomial knots of degree* 5.

6.2 Spaces of polynomial knots

Let $\mathcal{A}$ be the set of all polynomial maps and $\mathcal{P}$ be the set of all the polynomial knots. For an integer $d \geq 2$, we define the following sets:

$$\mathcal{A}_d = \{(f,\ g,\ h) \in \mathcal{A} |\deg(f) \leq d-2,\ \deg(g) \leq d-1,\ \deg(h) \leq d\},$$

$$\mathcal{B}_d = \{(f,\ g,\ h) \in \mathcal{A} |\deg(f) < \deg(g) < \deg(h) \leq d\},$$

$$\mathcal{C}_d = \{(f,\ g,\ h) \in \mathcal{A} |\deg(f) = d-2,\ \deg(g) = d-1,\ \deg(h) = d\}.$$

Also, let $\mathcal{O}_d$, $\mathcal{P}_d$ and $\mathcal{Q}_d$ be the set of polynomial knots in $\mathcal{A}_d$, $\mathcal{B}_d$ and $\mathcal{C}_d$ respectively; i.e, $\mathcal{O}_d = \mathcal{A}_d \cap \mathcal{P}$, $\mathcal{P}_d = \mathcal{B}_d \cap \mathcal{P}$ and $\mathcal{Q}_d = \mathcal{C}_d \cap \mathcal{P}$. Writing the components of $\phi = (f,\ g,\ h) \in \mathcal{A}_d$ as

$$f(t) = a_0 + a_1t + a_2t^2 + ... + a_{d-2}t^{d-2},$$

$$g(t) = b_0 + b_1t + b_2t^2 + ... + b_{d-1}t^{d-1},$$

$$h(t) = c_0 + c_1t + c_2t^2 + ... + c_dt^d$$

we can define a natural bijection η from $\mathcal{A}_d$ to the euclidean space $\mathbb{R}^{3d}$ given by

$$\phi \overset{\eta}{\mapsto} (a_0, a_1, ..., a_{d-1}, b_0, b_1, ..., b_{d-1}, c_0, c_1, ...c_d)$$

We can define a natural metric ρ on $\mathcal{A}_d$ given by

$$\rho(\phi, \psi) = \xi(\eta(\phi), \eta(\psi))$$

for $\phi,\ \psi \in \mathcal{A}_d$, where ξ denotes the Euclidean metric in $\mathbb{R}^{3d}$. This induces a topology on $\mathcal{A}_d$ and with this induced topology on $\mathcal{A}_d$ the map η becomes a diffeomorphism. Then the sets $\mathcal{B}_d,\ \mathcal{C}_d,\ \mathcal{O}_d,\ \mathcal{P}_d$ and $\mathcal{Q}_d$ are given the subspace topology, making these into subspaces of $\mathcal{A}_d$ which are homeomorphic to some subspaces of $\mathbb{R}^{3d}$.

It can be noted that the set of all polynomial knots $\mathcal{P} = \bigcup_{d\geq 2} \mathcal{O}_d$ can be given the inductive limit topology; i.e, $U \subseteq \mathcal{P}$ is open if and only if $U \cap \mathcal{O}_d$ is open in $\mathcal{O}_d$ for all $d \geq 2$. Similarly the set of all polynomial maps $\mathcal{A} = \bigcup_{d\geq 2} \mathcal{A}_d$ can also be given inductive limit topology.

Note 6.2.1. *It is obvious to note that:*

- $\mathcal{B}_2 = \mathcal{C}_2 = \mathcal{P}_2 = \mathcal{Q}_2$
- $\mathcal{C}_3 = \mathcal{Q}_3$
- $\mathcal{C}_d \nsubseteq \mathcal{C}_{d+1}$ *and* $\mathcal{Q}_d \nsubseteq \mathcal{Q}_{d+1}$

Proposition 6.2.1. *If two polynomial knots are path equivalent in $\mathcal{K}_d$, then they are topologically equivalent.*

Corollary 6.2.2. *If two polynomial knots are path equivalent in $\mathcal{P}$, then they are topologically equivalent.*

Corollary 6.2.3. *If two polynomial knots are path equivalent in $\mathcal{Q}_d$, then they are topologically equivalent.*

Theorem 6.2.4. *Suppose $\phi = (f,\ g,\ h) \in \mathcal{Q}_d$ be a polynomial knot, then ϕ and its mirror image $\psi = (f,\ g,\ -h)$ belong to different path components of the space $\mathcal{Q}_d$.*

Proof. Refer [RM17]. ■

Corollary 6.2.5. *Let $\phi \in \mathcal{Q}_d$, for $d \geq 3$, be a polynomial representation of a classical tame knot $\mathcal{K} : S^1 \longrightarrow S^3$. Then we have the following:*

1. *If $\mathcal{K}$ is amphichiral, then there are at least eight path components corresponding to $\mathcal{K}$ in $\mathcal{Q}_d$.*
2. *If $\mathcal{K}$ is chiral, then there are at least four path components corresponding to $\mathcal{K}$ and $\mathcal{K}^*$ each in $\mathcal{Q}_d$.*

6.3 The spaces $\mathcal{P}_d$ and $\mathcal{Q}_d$ for $d \leq 4$

We have the following theorems and propositions for spaces of polynomial knots with $d \leq 4$. The proofs can be found in [MR15].

Proposition 6.3.1. *The space $\mathcal{P}_2$ is open in $\mathcal{A}_2$ and it has exactly four path components.*

Theorem 6.3.2. *The space $\mathcal{P}_3$ is path connected.*

Proposition 6.3.3. *The space $\mathcal{Q}_3$ has exactly eight path components.*

Theorem 6.3.4. *The space $\mathcal{P}_4$ is path connected.*

Theorem 6.3.5. *The space $\mathcal{Q}_4$ has exactly eight path components.*

6.4 Polynomial knots of degree five

We will be giving topology to the set of polynomial knots of degree five in a different way in contrast to what was introduced earlier. This section is based on the paper [KSY01].

For fixed positive numbers p, q and r, $\mathcal{K}_{p,q,r}$ is the set of polynomial knots $\phi : \mathbb{R} \longrightarrow \mathbb{R}^3$ such that $\phi(t) = (f(t),\ g(t),\ h(t))$ where f, g and h are given by,

$$\begin{aligned} f(t) &= t^p + a_{p-1}t^{p-1} + ... + a_1 t \\ g(t) &= t^q + b_{q-1}t^{q-1} + ... + b_1 t \\ h(t) &= t^r + c_{r-1}t^{r-1} + ... + c_1 t \end{aligned} \tag{6.1}$$

for a_i, b_i and $c_i \in \mathbb{R}$.

Similar to Section 6.2 we will be able to give this set a topology by inducing a metric on $\mathcal{K}_{p,q,r}$ using the natural bijection to a subspace of $\mathbb{R}^{p+q+r-3}$. If $p = q = r$ then we will denote the space by $\mathcal{K}_p$.

Lemma 6.4.1. *For a knot K with polynomial representation having $f(t)$ and $g(t)$ as given in the equation 6.1, the crossing index $c[K] \leq \frac{(p-1)(q-1)}{2}$.*

Proof. We have a crossing when there exists a pair (s,t), $s \neq t$ such that

$$f(s) - f(t) = 0 \text{ and } g(s) - g(t) = 0 \tag{6.2}$$

We have an $(s-t)$ common in both the equations in 6.2. Since $s \neq t$, the pair that satisfies the equations in 6.2 also satisfies the following equations

$$\frac{f(s) - f(t)}{s - t} = 0 \text{ and } \frac{g(s) - g(t)}{s - t} = 0 \tag{6.3}$$

with degrees $p-1$ and $q-1$ respectively. Then by Bezout's theorem (see page no. 97 of [CLO05]), the number of common roots is at most $(p-1)(q-1)$. Since roots, (s,t) and (t,s) yields the same crossing, the number of crossings is at most $\frac{(p-1)(q-1)}{2}$. ■

Remark 6.4.1. *Part 1 of Proposition 6.1.1 follows from the above lemma taking $p = d-2$ and $q = d-1$.*

Remark 6.4.2. *It follows from Corollary 6.1.3 that all the knots in $\mathcal{K}_{3,4,5}$ are unknots or trefoils.*

6.4.1 The space $\mathcal{K}_{3,4,5}$

Proposition 6.4.2. *There exists no right-handed trefoils in $\mathcal{K}_{3,4,5}$ with coordinate functions given by:*

$$\begin{aligned} f(t) &= t^3 + a_1 t, \\ g(t) &= t^4 + b_2 t^2, \\ h(t) &= t^5 + c_4 t^4 + c_3 t^3 + c_2 t^2 + c_1 t. \end{aligned}$$

Proof. Refer [KSY01] ∎

In order to study trefoils in $\mathcal{K}_{3,4,5}$ we will consider a much simpler space $\mathcal{K}^s_{3,4,5}$ with similar topology. $\mathcal{K}^s_{3,4,5}$ is a subspace of $\mathcal{K}_{3,4,5}$ which contains all the polynomial knots $\phi = (f(t),\ g(t),\ h(t))$ with its coordinate functions of the form:

$$\begin{aligned} f(t) &= t^3 + a_1 t, \\ g(t) &= t^4 + b_2 t^2 + b_1 t, \\ h(t) &= t^5 + c_2 t^2 + c_1 t. \end{aligned} \tag{6.4}$$

Lemma 6.4.3. *If $\phi \in \mathcal{K}_{3,4,5}$, then ϕ is isotopic to a map in $\mathcal{K}^s_{3,4,5}$. Furthermore, the space $\mathcal{K}^s_{3,4,5}$ is homotopy equivalent to $\mathcal{K}_{3,4,5}$.*

Proof. For any $\phi \in \mathcal{K}_{3,4,5}$ the parametric substitution $t \mapsto (t - a_2/3)$ eliminates the quadratic term in the first-coordinate function. Let $\mathcal{K}'$ be the subset of $\mathcal{K}_{3,4,5}$ such that the quadratic term of the first coordinate is zero. Then it can be checked that the map $H : \mathcal{K}_{3,4,5} \times [0,1] \longrightarrow \mathcal{K}_{3,4,5}$ defined by

$$F(\phi, s) = \phi(t - sa_2/3) - \phi(-sa_2/3)$$

is a deformation retraction from $\mathcal{K}_{3,4,5}$ to $\mathcal{K}'$. Now using the coordinate transformation

$$(x,\ y,\ z) \mapsto (x,\ y + q_1 x,\ z + q_2 x + q_3 y)$$

taking $q_1 = -b_3$, $q_2 = -c_4$ and $q_3 = -c_3$ we can eliminate the cubic term in the second coordinate function and the cubic and quadratic terms in third coordinate function to obtain $\phi' \in \mathcal{K}^s_{3,4,5}$. It can be checked that $H' : \mathcal{K}' \times [0,1] \longrightarrow \mathcal{K}'$ given by

$$H'((f(t),\ g(t),\ h(t)), s) = (f(t),\ g(t) - sb_3 f(t),\ h(t) - s(c_4 g(t) + c_3 f(t)))$$

is a deformation retract from $\mathcal{K}'$ to $\mathcal{K}^s_{3,4,5}$. Hence $\mathcal{K}_{3,4,5}$ deformation retracts to $\mathcal{K}^s_{3,4,5}$. So they are homotopy equivalent. ∎

Let T^L and T^R be the set of points $(a_1, b_2, b_1, c_2, c_1) \in \mathcal{K}^s_{3,4,5}$ corresponding to left-handed trefoil and right-handed trefoil respectively.

Proposition 6.4.4. *The sets T^L and T^R are contractible.*

Proof. Refer [KSY01]. ∎

Lemma 6.4.5. *The polynomial knots associated with the coefficients $(a_1, b_2, b_1, c_2, c_1)$ and $(a_1, b_2, -b_1, -c_2, c_1)$ are isotopic.*

Proof. From a polynomial knot with coefficients $(a_1, b_2, b_1, c_2, c_1)$ we get one with coefficients $(a_1, b_2, -b_1, -c_2, c_1)$ if we make a parameter substitution $t \mapsto -t$ followed by the coordinate transformation $(x, y, z) \mapsto (-x, y, -z)$. Since these are orientation preserving linear transformations the composition of these map can be extended to an isotopy to identity, hence the knots are isotopic. ∎

Proposition 6.4.6. *There are no right-handed trefoils in $\mathcal{K}^s_{3,4,5}$.*

Proof. By Proposition 6.4.4, T^R is contractible, hence it is path connected. Suppose $(a_1, b_2, b_1, c_2, c_1) \in T^R$ then by the above lemma $(a_1, b_2, -b_1, -c_2, c_1) \in T^R$ and since T^R is path connected there is a path between them. It can be noted that the third coordinate changes sign so by intermediate value theorem there exists a point in the path for which the third coordinate is 0. But then by Lemma 6.4.2 such a right-handed trefoil cannot exist. This is a contradiction, so there are no right-handed trefoils in $\mathcal{K}^s_{3,4,5}$. ∎

Proposition 6.4.7. *The space of $\mathcal{K}_{3,4,5}$ contains a contractible region of left-handed trefoils but does not contain any right-handed trefoils.*

Proof. By Lemma 6.4.3, the space $\mathcal{K}^s_{3,4,5}$ is homotopy equivalent to $\mathcal{K}_{3,4,5}$. Therefore, from Proposition 6.4.4 and Proposition 6.4.6 the result follows. ∎

6.4.2 The space $\mathcal{K}_5$

Let us express every element ϕ in $\mathcal{K}_5$ in the form of a coefficient matrix.

$$M(\phi) = \begin{bmatrix} 1 & a_4 & a_3 & a_2 & a_1 \\ 1 & b_4 & b_3 & b_2 & b_1 \\ 1 & c_4 & c_3 & c_2 & c_1 \end{bmatrix}$$

It can be noted that, the coefficient matrix can be simplified by left multiplying by appropriate matrices of the following form:

$$\begin{aligned}
\tau_p &= \begin{bmatrix} 1 & p_1 & p_2 \\ 0 & 1 & p_3 \\ 0 & 0 & 1 \end{bmatrix} \text{ for } p = (p_1, p_2, p_3) \in \mathbb{R}^3, \\
\lambda_r &= \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \text{ for } r = (r_1, r_2, r_3) \text{ and } r_i > 0, \text{ and} \\
\rho_\theta &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ for } 0 \le \theta \le 2\pi.
\end{aligned}$$

Let us denote the groups $\{\tau_p\}$, $\{\lambda_r\}$ and $\{\rho_\theta\}$ by T, Λ and S^1 respectively. Also, given a set G of 3×3 matrices and a set A of 3×5 matrices, let us define $G(A)$ to be the set $\{ga | g \in G,\ a \in A\}$.

Lemma 6.4.8. *Given a set A of 3×5 matrices, there is a deformation retraction of $T(A)$ and $\Lambda(A)$ onto A.*

Proof. Let $0 = (0,0,0)$ and $e = (1,1,1)$, note that $\tau_0 = \lambda_e = \text{id}$. It can be checked that $H_T : T(A) \times I \longrightarrow T(A)$ given by $H_T(\tau_p(a), s) = \tau_{p(1-s)}(a)$ is a deformation retract from $T(A)$ to A and $H_\Lambda : \Lambda(A) \times I \longrightarrow \Lambda(A)$ given by $H_\Lambda(\lambda_r(a), s) = \lambda_{r(1-s)+es}(a)$ is a deformation retract from $\Lambda(A)$ to A. ■

Note 6.4.1. *Any action in T, Λ and S^1 are orientation preserving homeomorphisms in $\mathbb{R}^3$, so they preserve knot isotopy types.*

We can choose $\tau_p \in T$ and $\rho_\theta \in S^1$ appropriately such that,

$$\rho_\theta(\tau_p(M(\phi))) = \begin{bmatrix} 0 & 0 & a_3' & a_2' & a_1' \\ 0 & b_4' & b_3' & b_2' & b_1' \\ 1 & c_4 & c_3 & c_2 & c_1 \end{bmatrix}.$$

If both a_3' and b_4' are non-zero, the polynomial knot corresponding to this coefficient matrix can be further reduced by an element in Λ into a polynomial knot in $\mathcal{K}_{3,4,5}$ or to its mirror image $r(\mathcal{K}_{3,4,5})$, where $r : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is the reflection operation along the yz-plane defined by $r : (x, y, z) \mapsto (-x, y, z)$. Otherwise, they reduce to a knot in $\mathcal{K}_{p,q,5}$ or $r(\mathcal{K}_{p,q,5})$ where $p < q$ and $(p, q) < (3, 4)$.

Theorem 6.4.9. *The space $\mathcal{K}_5$ contains a component of left-handed trefoils and a component of right-handed trefoils, which are each homotopy equivalent to S^1.*

Proof. From the previous lemma, components of the left-handed trefoils in $\mathcal{K}_{3,4,5}$ are contractible, denote by T_L. We will apply the reduction operations we described earlier in the opposite order; i.e, $T(S^1(\Lambda(T_L)))$ gives the component of left-handed trefoil in $\mathcal{K}_5$. Since T and Λ preserve homotopy type, $T(S^1(\Lambda(T_L)))$ has the same homotopy type as S^1. Similarly starting with $r(T_L)$ in $r(\mathcal{K}_{3,4,5})$ would give the result for right-handed trefoil. ∎

Chapter 7

Polynomial representations of weaving knots

7.1 Motivation

Over the years various results on polynomial representations of torus knots are proved. A. Ranjan and Rama Mishra in [RS96] proved the following theorem on the polynomial representations for torus knots of type $(3, q)$.

Theorem 7.1.1. *A torus knot of type $(3, q)$ can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ with $\deg(f(t)) = 5$, $\deg(g(t)) = 2q - 1$ and $\deg(h(t)) = 2q$.*

And the same paper proved the following theorem on polynomial representations of knots of type $(2, 2n + 1)$.

Theorem 7.1.2. *A torus knot of type $(2, 2n + 1)$ can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ with $\deg(f(t)) = 3$, $\deg(g(t)) = 4n$ and $\deg(h(t)) = 4n + 1$.*

Later the following much more general theorem was proved in [Mis99].

Theorem 7.1.3. *A torus knot of type (p, q) for $p \geq 3$ and $q > p$ coprime to p can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ with $\deg(f(t)) = 2p - 1$, $\deg(g(t)) = 2q - 1$ and $\deg(h(t)) = 2q$.*

The following theorems regarding the parity of components of the polynomial representations of strongly invertible knots and a restricted class of knots called faithfully strongly negative amphichiral knot where proved by Rama Mishra in [Mis06].

Theorem 7.1.4. *Every strongly invertible (open) knot can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ from $\mathbb{R}$ to $\mathbb{R}^3$ where $f(t)$ and $h(t)$ are odd polynomials and $g(t)$ is an even polynomial.*

Theorem 7.1.5. *Every faithfully strongly negative amphichiral (open) knot can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ from $\mathbb{R}$ to $\mathbb{R}^3$ where $f(t)$, $g(t)$ and $h(t)$ are odd polynomials.*

There is a class of knots that have the same projection of torus knots called weaving knots, we will give a formal definition later. In the case of torus knots, the above theorems give many results on polynomial representation- possible degree sequences and parity of component polynomials. Since weaving knots have the same projections as torus knots all the above theorems would give us partial data, the polynomial maps of two components to be precise for weaving knots. This essentially is the motivation to study polynomial representations of weaving knots. We have proved certain results on the polynomial representations of weaving knots, it is basically an algorithm to find the third coordinate given the other two coordinates. Using this algorithm polynomial representations of certain weaving knots are explicitly calculated. It was also interesting to see how the properties of chirality and invertibility were related to the parity of polynomial representations. We will try to see these relations for weaving knots as well.

7.2 Polynomial representations of weaving knots

Let us formally define what a weaving knot is,

Definition 7.2.1. *Weaving knots are alternating knots which has the same projection as torus knots. Weaving knot having the projection of $T(p,q)$ are denoted by $W(p,q)$.*

It is straight forward to see that for weaving knot $W(p,q)$ with $p < q$, crossing number $c[W(p,q)] = c[T(p,q)] = q(p-1)$.

7.2.1 Main results

Theorem 7.2.1. *A weaving knot of type $(3,q)$, $W(3,q)$ can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ with $\deg(f(t)) = 5$, $\deg(g(t)) = 2q-1$ and $\deg(h(t)) = 4q-1$.*

Proof. Let us assume that the weaving knot has a regular projection to the $x-y$ plane. Now Theorem 7.1.1 gives the existence of polynomial representation with x and y coordinate polynomials having degree 5 and $2q-1$ respectively. Since weaving knots $W(3,q)$ have the same projection as torus knots $T(p,q)$, take f and g to be the same as the one obtained from Theorem 7.1.1.

Let us find all pairs $(s,t) \in \mathbb{R}^2$ such that $f(s) = f(t)$ and $g(s) = g(t)$ simultaneously. Since the knot has $2q$ crossings there will be $2q$ such pairs (s_i, t_i) $1 \leq i \leq 2q$. Write s_is and t_is in increasing order and relabel them as a_i such that $a_i \leq a_{i+1}$ for $1 \leq i \leq 4q-1$. Choose $b_i \in [a_i, a_{i+1}]$ and define $h(t)$ in the following way.

$$h(t) = \prod_{i=1}^{4q-1} (t - b_i)$$

By definition h changes sign at each b_i forcing the property of being alternating; i.e, the polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ is the polynomial representation of weaving knot $W(3,n)$. ■

For weaving knots of type $(3,q)$, Theorem 7.2.1 guarantees the existence of a polynomial representation $t \mapsto (f(t),\ g(t),\ h(t))$ with all the component functions having odd degree. The reparametrization $t \mapsto -t$ is an orientation reversing involution which takes the knot to its mirror image.

Remark 7.2.1. *Weaving knots of type $(3,q)$ are strongly negative-amphichiral.*

The proof of Theorem 7.2.1, in fact, works for weaving knots of type (p,q) for $q > p$ and q coprime to p. There is nothing special about the weaving knot being type $(3,q)$. Starting with the polynomial representations that we get from Theorem 7.1.3 and proceeding in a similar way, it can be proved that,

Theorem 7.2.2. *A weaving knot of type (p,q) for $p \geq 3$ and $q > p$ coprime to p can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ with $\deg(f(t)) = 2p-1$, $\deg(g(t)) = 2q-1$ and $\deg(h(t)) = 2q(p-1)-1$.*

Similarly using Theorem 7.1.2 it can be proved that,

Theorem 7.2.3. *A weaving knot of type $(2, 2n+1)$ can be represented by a polynomial embedding $t \mapsto (f(t),\ g(t),\ h(t))$ with $\deg(f(t)) = 3$, $\deg(g(t)) = 4n$ and $\deg(h(t)) = 4n+1$.*

7.3 Computing polynomial representations of weaving knots using Mathematica

Given $t \mapsto (f(t),\ g(t))$ the polynomial representation for a regular projection of the knot. Define

$$F(s,t) = \frac{(f(s) - f(t))}{(s-t)}$$

$$G(s,t) = \frac{(g(s) - g(t))}{(s - t)}$$

We compute the resultant of $F(s,t)$ and $G(s,t)$ with respect to s. This gives a polynomial in t say $r(t)$ which is zero if and only if the polynomials $F(s,t)$ and $G(s,t)$ have a common root; i.e, for t_*, $r(t_*)$ is zero if and only if $F(s,t_*) = G(s,t_*)$ for some s. Thus, finding all the roots of $r(t)$ will give all the points where the knot projection has a crossing.

The rest of it follows from the proof. Since these roots represent crossings they should occur in pairs. We write them in increasing order and relabel them as a_i such that $a_i \leq a_{i+1}$ for $1 \leq i \leq 4q-1$. Choose $b_i \in [a_i, a_{i+1}]$ and define $h(t)$ in the following way.

$$h(t) = \prod_{i=1}^{4q-1} (t - b_i)$$

7.3.1 Example 1: W(3,4)

The following polynomial map $t \mapsto (f(t),\ g(t),\ h(t))$ gives a polynomial representation of weaving knot $W(3,4)$.

$$f(t) = t(t^2 - 20)(t^2 - 45)$$

$$g(t) = t(t^2 - 5)(t^2 - 36)(t^2 - 49)$$

$$h(t) = t(t^2 - 1)(t^2 - 4)(t^2 - 20)(t^2 - 35)(t^2 - 40)(t^2 - 48)(t^2 - 50)$$

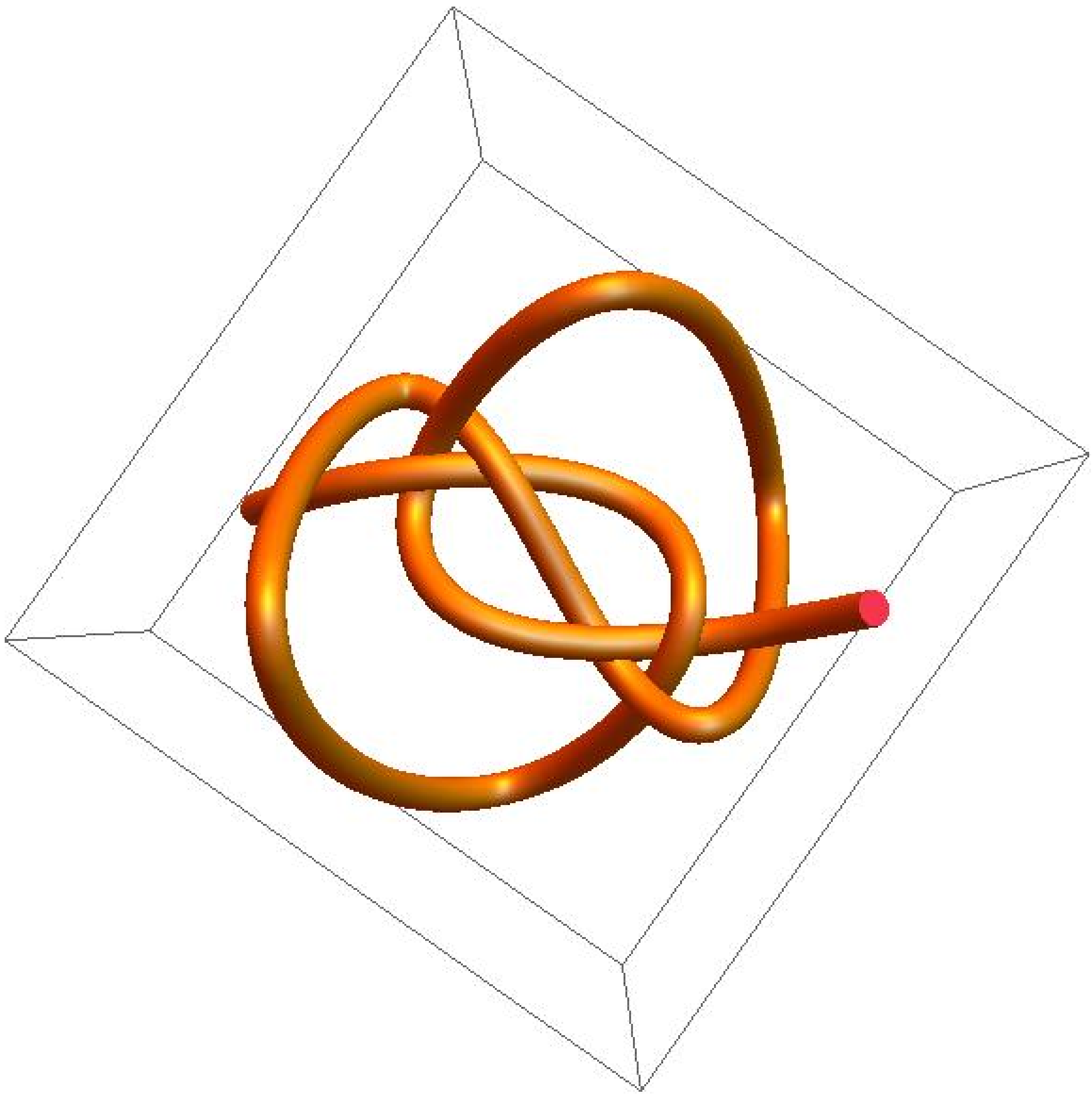

Figure 7.1: Polynomial representation of a W(3,4)

7.3.2 Example 2: W(3,5)

The following polynomial map $t \mapsto (f(t),\ g(t),\ h(t))$ gives a polynomial representation of weaving knot $W(3,5)$.

$$f(t) = t(t^2-10)(t^2-45)$$
$$g(t) = t(t^2-5)(t^2-25)(t^2-40)(t^2-50)$$
$$h(t) = t(t^2-2)(t^2-5.4)(t^2-16)(t^2-31.5)(t^2-40.6)(t^2-43.4)(t^2-45)(t^2-47.7)(t^2-49.7)$$

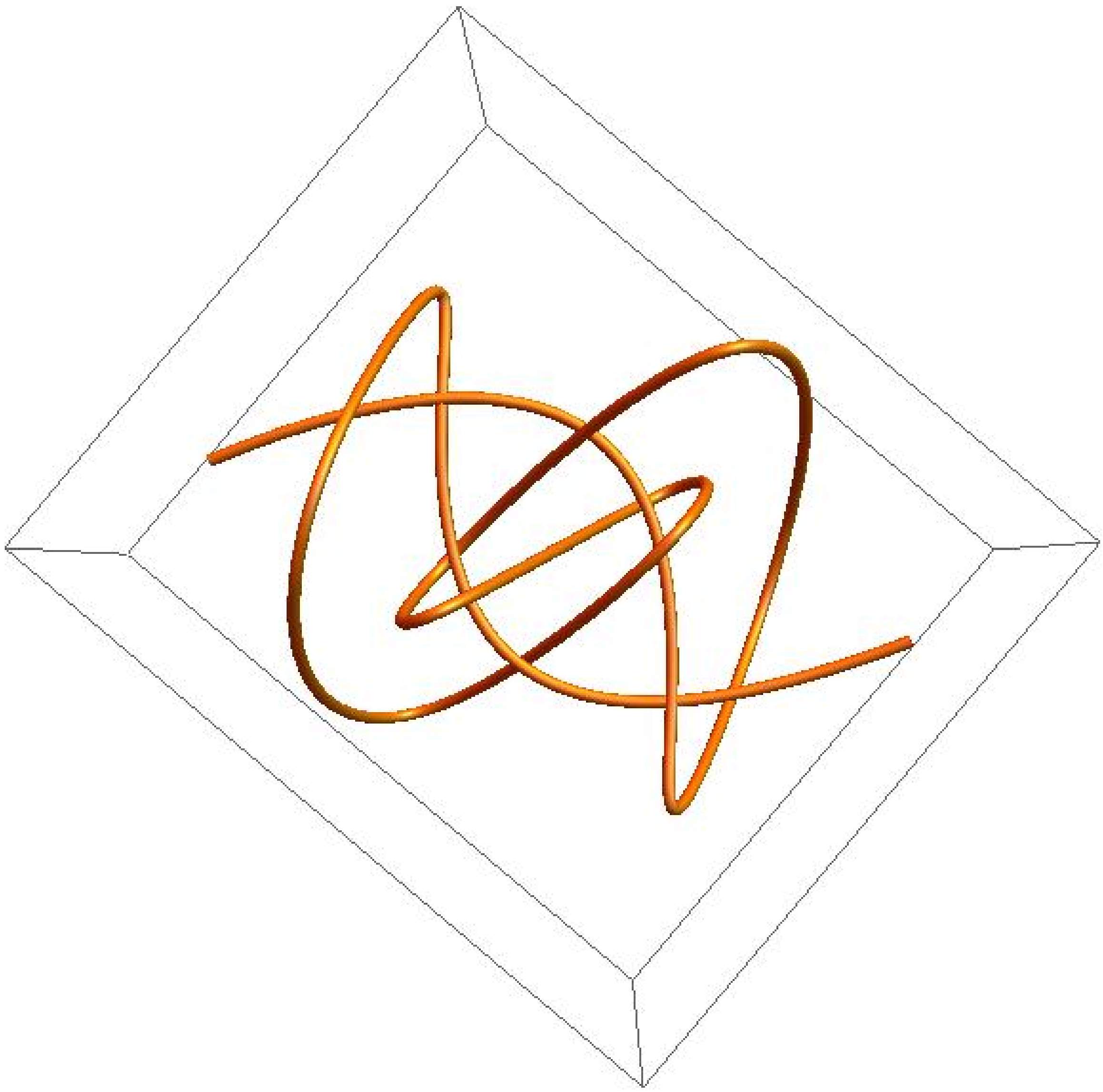

Figure 7.2: Polynomial representation of a W(3,5)

Chapter 8

Conclusion

We studied the geometric aspects of knot theory, it was more of a combinatorial approach. In the initial phase, we studied various numerical invariants and studied 3-superbridge knots in detail. We tried proving the conjecture that trefoil knot and figure-eight knot are the only 3-superbridge knots. I tried finding a combination of numerical invariants which could potentially serve as an upper bound for the superbridge index. Unfortunately, I haven't been able to prove this as of now. Another idea to approach the problem is using the proof of the Theorem 4.0.4. This is a brute force method, we will have to look at all the possibilities of knots, starting with the finite choices of projection along a quadrisecant.

The second part of the project aimed at studying polygonal and polynomial representations of knots and also the topology of spaces of polygonal knots and polynomial knots. It is conjectured that the minimal polynomial degree is one less than the stick index. We studied the spaces of knots which could be expressed as a polygon with 6 edges and the spaces of knots which have polynomial representations of degree 5. It can be noted that both of these spaces contain only unknot, left-handed trefoil, and right-handed trefoil. It somehow sheds light towards the relation between minimal polynomial degree and stick index that is put forth by the conjecture. I tried to see if a bound can be found on the degree of polynomial representations that a knot of a particular stick index can take. I am still working on this problem. If we look at the crossing points of a knot of stick index k and use Lagrange interpolation we get a polynomial of degree $k-1$, now I am trying to see how this polynomial could be perturbed such that the degree remains unchanged and the polynomial obtained in fact becomes a polynomial representation of the knot that we started with. Another problem we tried solving was to find the polynomial representations of weaving knots. This was motivated from similar theorems in the case of torus knots. In the last chapter, we proved that weaving knots of type $(3, n)$ has degree sequence $(5, 2n-1, 4n-1)$

and as a consequence, we get that weaving knots of type $(3, n)$ are strongly negative-amphichiral. We have in fact found an algorithm to find the polynomial representation of a weaving knot of type (p, q) for $p \leq q$ and p co-prime to q given the polynomial representation of its torus counterpart.
For my future studies, I find myself more inclined to homological invariants of knots. One of the very less understood knot invariants is the unknotting number. Recently using a homological invariant, knot Floer homology Akram Alishahi and Eaman Eftekhary in [AE18] gave a much better lower bound on the unknotting number for composite knots. I plan on studying these homological invariants in the future.

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