A Gravitational Stress Tensor for Asymptotically AdS Spaces from Holography

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfilment of the requirements for the
BS-MS Dual Degree Programme

by

Feroz Mohamed Hatha

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2020

Supervisor: Dr. Chethan Krishnan

Certificate

This is to certify that this dissertation entitled **A Gravitational Stress Tensor for Asymptotically AdS Spaces from Holography** towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Feroz Mohamed Hatha at the Indian Institute of Science, Bangalore under the supervision of Dr. Chethan Krishnan, Associate Professor, Department of Physics, during the academic year 2019-2020.

Chethan K

Dr. Chethan Krishnan

Feroz

Feroz Mohamed Hatha

Committee:

Dr. Chethan Krishnan

Dr. Sachin Jain Sachin Jain

Dedicated to Vappa, Umma and Ithatha.

In memory of Velluppa.

Declaration

I hereby declare that the matter embodied in the report entitled **A Gravitational Stress Tensor for Asymptotically AdS Spaces from Holography** are the results of the work carried out by me at the Department of Physics, Indian Institute of Science, Bangalore under the supervision of Dr. Chethan Krishnan and the same has not been submitted elsewhere for any other degree.

Feroz

Feroz Mohamed Hatha

Acknowledgments

I would like to express my most sincere thanks and appreciation to my thesis advisor Dr. Chethan Krishnan for his constant support and guidance over the past year. Apart from immensely benefitting from the discussions on Physics I had with him, the time spent with Chethan has also helped me shape my thoughts and reflect upon my strengths and weaknesses in a better manner.

I am much indebted to my thesis co-advisor Dr. Sachin Jain for his invaluable inputs that motivated me to work harder during the latter half of the academic year. I would also like to thank Prof. Justin Raj David of CHEP, IISc for letting me attend his general relativity lectures which has helped me further my understanding of the subject.

Thanks to Rithwik, Akhil, Vaishnavi, Zakhiya, Harsha, Kiran and Immanuel, I have thoroughly enjoyed my stay at Bangalore over the past year. I would like to thank Rithwik in particular for being a wonderful roommate and for completely tolerating me during the period of stay! I would like to thank Ajay and Naveen for the stimulating discussions we have had over various topics, both academic and non-academic.

Any number of words would be inadequate to convey my gratitude to my parents and sister for their constant, unparalleled support and encouragement during this period.

Finally, it would be inappropriate to conclude without thanking IISER, Pune for providing me the opportunity to work at IISc, Bangalore and thus to interact with a number of great minds in the field of my interest.

Abstract

This thesis presents a study of the quasilocal stress tensor proposed by Brown and York in 1993 from the perspective of the AdS/CFT correspondence which was carried out by Kraus and Balasubramanian in 1999. The thesis begins with a quick review of the essential ideas of AdS/CFT. The subsequent chapter explains the variational principle for the gravitational action and explains the need for introducing the Gibbons-Hawking-York boundary term and the nondynamical counterterm in the action. The chapter following this elucidates the challenges involved in describing a local stress-energy tensor for the metric of a spacetime and describes the earlier mentioned proposal by Brown and York. Later, we work out in detail and confirm the expressions for quasilocal stress tensor for asymptotically AdS spacetimes in various dimensions and also show that these lead to the right masses and momenta for various spacetimes as shown by Kraus and Balasubramanian. The interpretation of the quasilocal stress tensor from the CFT side is also discussed. We verify the result obtained by Brown and Henneaux and also make a key observation that for metrics that satisfy the fall-offs suggested by them, one may be able to drop the GHY term in the action thus leading to a modified stress tensor that could potentially give the same (correct) results for mass and momenta of various spacetimes.

Contents

Chapter 1

Introduction

This thesis presents an attempt, from the point of view of *holography*, to resolve a long-standing problem in the field of gravity, which is to define a stress-energy tensor for the gravitational field. Holography is the idea that a physical theory in $d+1$ dimensions is essentially equivalent to some other theory living at infinity, in d or fewer dimensions. This is, in fact, great since it enables us to easily compute some (not so easily calculable) quantities in one theory by using the other equivalent description of the theory. The holographic principle was first proposed by Gerard 't Hooft [1] and later made more concrete by the work of Leonard Susskind [2], where he suggested that *string theory* could be a sensible realization of the notion introduced by 't Hooft. A concrete realization of the holographic principle is the *anti de-Sitter/conformal field theory (AdS/CFT) correspondence* also termed as the *gauge/gravity duality*, which was first proposed by Juan Maldacena in 1997 [3]. The AdS/CFT correspondence has been instrumental in scientists' search for a quantum theory of gravity and is widely considered as the best-understood theory of *quantum gravity* till date.

Many have tried and failed in the attempt to define a stress-energy tensor for the gravitational field (or equivalently, a stress-energy momentum for the metric of a spacetime manifold) which is an open problem even now. J. David Brown and James W. York have been somewhat successful in this attempt; in 1992, they gave a definition for such a quantity, which was not exactly defined at a point but localized over the boundary of the spacetime under consideration, thus being partially successful [14]. Since the quantity cannot actually be described as local, it is referred to as a *quasilocal* stress tensor. It is basically defined as the functional derivative of the gravitational action (action for the metric of a spacetime manifold) with respect to the metric on the boundary of the spacetime.

Per Kraus and Vijay Balasubramanian in 1999 realized that this problem is particularly attractive from the point of view of holography [15]. This is because, via the AdS/CFT correspondence, one can assume the equivalence between the partition function on the gravity side and the CFT partition function (subject to the constraint that the partition function on the gravity side is expressed as a functional of a bulk field whose boundary limit couples to the CFT operator with respect to which we express the partition function of the CFT). In the semiclassical limit of the gravitational theory, using saddle point approximation, one can see that the partition function on the gravity side essentially reduces to the bulk gravitational action viewed as a functional of the boundary field(s). Hence, from the point of AdS/CFT correspondence, there is a natural notion of viewing the gravitational action as a function of the boundary metric, thus enabling us to compute functional derivatives of the action with respect to the boundary metric.

A major drawback of the proposal by Brown and York stems from the definition that they propose for the counterterm in the gravitational action which renders the action itself as well as the quasilocal stress tensor finite. They propose an embedding of the boundary hypersurface (which is the region of integration to obtain the counterterm) in a reference spacetime and to consider the resulting gravitational action as the counterterm. The above procedure, commonly referred to as *background subtraction*, is not always feasible since the embedding of a hypersurface with an arbitrary metric in a reference spacetime may not be possible always. Thus, the procedure suggested by Brown and York is ill-defined to some extent. This problem has an effective resolution from the perspective of AdS/CFT correspondence. From the perspective of CFT, the quasilocal stress tensor can be thought of as giving the expectation value of its stress tensor. Therefore, the divergences (that show up in the asymptotic limit) in the quasilocal stress tensor can be thought of as ultraviolet divergences arising in a field theory which can be gotten rid of by considering counterterms that depend solely on the geometry of the boundary hypersurface over which the CFT is defined. One can express the integrand of such a counterterm as a function of the invariants formed from the curvature(s) of the boundary metric, which is always possible as opposed to the method by Brown and York.

The following chapter gives a brief summary of the key details of anti-de Sitter spaces, conformal field thoeries, and the method of computing correlators by making use of the duality. The chapter following the next one explains the Lagrangian formulation of general relativity and explains the need for extra terms in the gravitational action so that we have both a well defined variational principle and also an action that is finite in the limit of large distances. The chapter after that expounds on the proposal by Brown and York and that by Kraus and Balasubramanian. We also explicitly show all the calculations required to arrive at the results observed by Kraus and Balasubramanian for various spacetimes. In addition, we make a key observation which might pave the way for an alternate method to arrive at the results observed by Kraus and Balasubramanian.

Chapter 2

Overview of the AdS/CFT duality

AdS/CFT correspondence is the observation that *any theory of quantum gravity in a family of asymptotically AdS spacetimes in $d + 1$ dimensions is equivalent to a conformal field theory in a d dimensional spacetime* [3, 4]. This relationship between the two theories is an instance of holography since the conformal field theory lives in one smaller dimension compared to the gravitational theory. We know that a hologram is a 2-dimensional version of a 3-dimensional object which completely describes the latter. Similar is the equivalence between the gravitational theory and CFT. The claim is that there is an equivalence between the two theories because one can interpret the calculations of physical quantities in one theory via a 'dictionary' that relates these to calculations in the dual theory. Thus, every physical quantity in one theory has a dual in the other theory. The fact that any physical quantity that can be computed in one theory can also be computed in the dual theory lets us conclude that there is a total equivalence between the two theories. As we will see later, the isometry group of AdS_{d+1}, $SO(d, 2)$, acts on the d dimensional conformal field theory as the conformal group (which happens to be the symmetry group of the CFT) and this therefore lets us conclude that symmetries on either sides of the duality are equivalent as well. Gravity in AdS can essentially be viewed as 'gravity in a box'. We will see why this is the case in one of the coming sections. Any theory that is well approximated by coupling matter to general relativity in AdS may be regarded as a 'quantum gravity' theory. One way of viewing the duality between the two theories is to assume the existence of an isomorphism between the Hilbert spaces of the two theories [5].

2.1 Anti-de Sitter spacetime

Anti-de Sitter spacetime is a maximally symmetric spacetime with a negative curvature. Since it is maximally symmetric, the curvature is guaranteed to be a constant. AdS solves Einstein's equations for an empty universe with a negative cosmological constant. $d+1$ dimensional AdS can be obtained by embedding a hypersurface of the same dimension in Minkowski spacetime in one higher dimension, i.e, $d+2$ dimensions, in $(2,d)$ signature. $d+2$ dimensional Minkowski metric in $(2,d)$ signature is given by

$$ds^2 = -dX_0^2 - dX_{d+1}^2 + dX_1^2 + ... + dX_d^2, \tag{2.1}$$

and the hypersurface to be embedded takes the form

$$X_0^2 + X_{d+1}^2 - \vec{X}^2 = l^2, \tag{2.2}$$

where l is a parameter with units of length, conventionally referred to as the *AdS length scale*. In *global coordinates*, defined by

$$\begin{aligned} X_0 &= \sqrt{l^2 + r^2}\cos(t/l) \\ X_{d+1} &= \sqrt{l^2 + r^2}\sin(t/l) \\ \vec{X}^2 &= r^2, \end{aligned} \tag{2.3}$$

the metric of AdS takes the form

$$ds^2 = -\left(1 + \frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{l^2}\right)} + r^2 d\Omega_{d-1}^2, \tag{2.4}$$

where $r \in [0, \infty), t \in (-\infty, \infty)$, and Ω_{d-1} stands for the round metric on $\mathbb{S}^{d-1}$. Defining $r = \tan\rho$, with $\rho \in (0, \pi/2)$, we see that AdS metric now assumes the form

$$ds^2 = \frac{1}{\cos\rho^2}(-dt^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2), \tag{2.5}$$

which lets us conclude that the causal structure of AdS_{d+1} can be identified to be that of a "solid cylinder",

$$ds^2 = -dt^2 + d\rho^2 + \sin^2\rho d\Omega_{d-1}^2, \tag{2.6}$$

as the conformal factor plays no role in determining the null geodesics of the space time. Therefore, in these coordinates, AdS can be pictured as the interior of a cylinder. It is seen that a signal sent from the centre of the spacetime by an observer at rest hits the asymptotic boundary and reverts back to the centre in a finite proper time, $\Delta t = \pi$. This is the reason why Ads gravity is commonly referred to as 'gravity in a box'. The boundary of AdS at $r = \infty$ or $\rho = \pi/2$, with topology $\mathbb{R} \times \mathbb{S}^{d-1}$, is what is popularly referred to as its *asymptotic boundary*. Now, the metric in (2.4) is the appropriate one for empty universe; the presence of any matter in the spacetime would cause the metric to deviate from (2.4).

Maximal symmetry of AdS implies that it has as many independent symmetries or as many killing vectors as Minkowski spacetime of the same dimension. In $d+1$ dimensional Minkowski spacetime, we have $d+1$ translations, d boosts, and $\frac{1}{2}d(d-1)$ rotations and thus a total of $d+1+d+\frac{1}{2}d(d-1) = \frac{1}{2}(d+1)(d+2)$ independent symmetries. As we saw earlier, AdS$_{d+1}$ can be obtained by embedding the hypersurface

$$X_A X^A \equiv X_0^2 + X_{d+1}^2 - \vec{X}^2 = l^2 \tag{2.7}$$

in $d+2$ dimensional Minkowski spacetime in $(2, d)$ signature. In terms of the coordinates given in (2.5), the parametrization of this hypersurface looks like

$$\begin{aligned} X_0 &= l\frac{\cos t}{\cos\rho} \\ X_{d+1} &= l\frac{\sin t}{\cos\rho} \\ \vec{X}^2 &= l^2\tan^2\rho. \end{aligned} \tag{2.8}$$

The above 2 equations elucidate the fact that isometries of AdS are merely the rotations and boosts of X_A [6]; we have one rotation between the timelike coordinates X_0 and X_{d+1}, d boosts between either of the timelike coordinate and the $X_i s$, and $\binom{d}{2} = \frac{1}{2}d(d-1)$ rotations between the $X_i s$. The general form of the generator of these transformations look like

$$G_{AB} = X_A\partial_B - X_B\partial_A, \tag{2.9}$$

with all of these individual generators together generating the isometry group of AdS, $SO(d, 2)$.

The AdS metric(s) that we saw till now were all in *Lorentzian* signature; the metric had all positive eigenvalues but one. In contrast to this, a metric with all eigenvalues being positive is said to be in *Euclidean* signature. For obtaining AdS metric in Euclidean signature, the hypersurface to be embedded looks like

$$X_0^2 - X_{d+1}^2 - \vec{X}^2 = l^2, \tag{2.10}$$

and the embedding space is $d+2$ dimensional Minkowski spacetime in $(1, d+1)$ signature. In global coordinates, Euclidean AdS looks like

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{l^2}\right)} + r^2 d\Omega_{d-1}^2. \tag{2.11}$$

Compared to the Lorentzian case, we just have a flip in the sign of dt^2 in the Euclidean case.

Another commonly used set of coordinates for AdS are the *Poincare* coordinates, which parametrizes the hypersurface in (2.10) as

$$\begin{aligned} X_0 &= l\left(\frac{1 + \vec{x}^2 + z^2}{2z}\right) \\ X_{d+1} &= l\left(\frac{1 - \vec{x}^2 - z^2}{2z}\right) \\ \vec{X} &= l\frac{\vec{x}}{z}. \end{aligned} \tag{2.12}$$

Here, $\vec{x}$ is a spatial d dimensional vector, i.e., $\vec{x} \in \mathbb{R}^d$, and $z > 0$ (so that we have a consistent positive sign for X_0). In Poincare coordinates, euclidean AdS metric looks like

$$ds^2 = l^2 \left(\frac{dz^2 + \sum_{i=1}^{d} dx_i{}^2}{z^2}\right). \tag{2.13}$$

This makes explicit the fact that AdS is conformal to $\mathbb{R}^+ \times \mathbb{R}^d$ and also the fact that in Poincare coordinates, the asymptotic boundary (situated at $z = 0$) has $\mathbb{R}^d$ geometry [7]. To find the metric of Lorentzian AdS in Poincare patch, we parametrize the hypersurface in (2.2) as

$$\begin{aligned} X_0 &= \frac{z}{2}\left(1 + \frac{l^2 + \vec{x}^2 - t^2}{z^2}\right) & X_{d+1} &= \frac{l}{z}t \\ X_{i<d} &= \frac{l}{z}x_i X_d = \frac{z}{2}\left(1 - \frac{l^2 - \vec{x}^2 + t^2}{z^2}\right) & X_{i<d} &= \frac{l}{z}x_i, \end{aligned} \tag{2.14}$$

where $\vec{x}$ is a $d-1$ dimensional spatial vector. The metric of AdS in these coordinates look like

$$ds^2 = l^2 \left(\frac{-dt^2 + dz^2 + \sum_{i=1}^{d-1} dx_i{}^2}{z^2} \right), \tag{2.15}$$

where $0 < z < \infty$. It can be seen that Poincare coordinates in Euclidean signature cover the whole of Ads while the same set of coordinates in Lorentzian signature covers only a wedge shaped region of the whole spacetime, popularly referred to as the *Poincare patch.*

2.1.1 Classical equations of motion in AdS

For illustrative purposes, consider a free scalar particle in 3 dimensional AdS spacetime. Since we know that free particles traverse along geodesics, the action for the same in an arbitrary spacetime could be expressed as

$$S = m \int d\tau = m \int dt \sqrt{g_{\alpha\beta}(x(t)) \frac{dx^\alpha(t)}{dt} \frac{dx^\beta(t)}{dt}}, \tag{2.16}$$

where τ is the proper time, m is the mass of the particle, and $x^\alpha(t)$ corresponds to the world line of the particle. For AdS_3 in global coordinates, $x^\alpha(t) = (t, \rho(t), \theta(t))$, and therefore the action in (2.16) looks like

$$S = m \int dt \frac{\sqrt{-1 + \dot{\rho}^2 + \dot{\theta}^2 \sin^2 \rho}}{\cos \rho}. \tag{2.17}$$

Since the Euler-Lagrange equations for this action take a complicated form, we can choose an alternate approach to deriving the classical equations of motion. Since we know that Ads is obtained by embedding a hypersurface in Minkowski spacetime of one higher dimension, we could write an action principle for the coordinates X_A in (2.7), considering $X_A = X_A(\tau)$. The action with a Lagrange multiplier takes the form

$$S = \int d\tau \left(\dot{X}_A \dot{X}^A + \lambda(R^2 - X_A X^A) \right). \tag{2.18}$$

From the Euler-Lagrange equation for this action, $\ddot{X}_A = -\lambda X_A$, we see that different values of λ are all equivalent upto a sign (since the equation of motion is a homogenous second order differential equation). Therefore, the possible values that λ can take are -1, 0, or 1. These correspond to spacelike, null, or timelike geodesics in AdS. From our study of the harmonic oscillator, one can easily infer that *massive particles in AdS move along periodic geodesics.* The fact that all trajectories for a free particle in AdS have the same period with respect to the coordinate t implies that discrete energy levels obtained by quantizing the dynamics of an AdS free particle are integer spaced. Stated otherwise, the discrete energy levels must obey

$$E = E_0 + n, \tag{2.19}$$

for some E_0, with n being a non negative integer.

2.1.2 Quantum mechanics and field theory in AdS

For illustrative puropses, we'll work with a free particle in AdS_2. The action in (2.16) for AdS_2 looks like

$$S = m \int dt \frac{\sqrt{-1+\dot{\rho}^2}}{\cos\rho}. \tag{2.20}$$

The momentum conjugate to the coordinate ρ is

$$P_\rho = \frac{m\dot{\rho}}{\cos\rho\sqrt{\dot{\rho}^2-1}}, \tag{2.21}$$

and therefore, the Hamiltonian is given by

$$H = P_\rho\dot{\rho} - L = \sqrt{{P_\rho}^2 - \frac{m^2}{\cos^2\rho}}. \tag{2.22}$$

Quantization can be imposed by demanding $[\rho, P_\rho] = i$, which is satisfied if we have $P_\rho = -i\partial_\rho$. We could then solve for the wavefunction of a free particle by expressing the Schrodinger equation using the above information; in fact, solving the squared form of Schrodinger equation is less laborious.

Although we could solve for the free particle wavefunction by writing down the Schrodinger equation as above, an easier method to obtain the same would be to use the isometries of AdS. From (2.9), we see that the three generators of AdS_2 isometries are G_{01}, G_{02}, and G_{12}. Now, G_{01} basically mixes X_0 and X_1 between themselves and therefore its action could be expressed as

$$X_0 \to X_0 + \epsilon X_1, \text{ and } X_1 \to X_1 + \epsilon X_0. \tag{2.23}$$

If we are working with global coordinates, using the parametrization given in (2.8), we can express the above transformation as a transformation of t and ρ as

$$\begin{aligned}\frac{\cos(t+\epsilon g_t)}{\cos(\rho+\epsilon g_\rho)} &\approx \frac{\cos t}{\cos\rho} + \epsilon\tan\rho \\ \tan(\rho+\epsilon g_\rho) &\approx \tan\rho + \epsilon\frac{\cos t}{\cos\rho},\end{aligned} \tag{2.24}$$

whose solutions are

$$\begin{aligned}g_t &= -\sin t\sin\rho \\ g_\rho &= \cos t\cos\rho.\end{aligned} \tag{2.25}$$

From this, we can conclude that

$$G_{01} = -\sin t\sin\rho\partial_t + \cos t\cos\rho\partial_\rho. \tag{2.26}$$

By following the same approach, we can arrive at the expressions for the other two generators and see that they satisfy the commutation relations

$$[G_{01}, G_{02}] = G_{12}, \quad [G_{02}, G_{12}] = G_{01}, \quad \text{and} \;\; [G_{01}, G_{12}] = G_{02}, \tag{2.27}$$

which we identify with the $SO(2,1)$ algebra. The above equation tells us that defining $D = G_{02}$, $P = \frac{1}{2}(G_{01} + iG_{12})$, and $K = \frac{1}{2}(G_{01} - iG_{12})$ helps us recast them in the form

$$[D, P] = iP, \quad [D, K] = -iK, \quad \text{and} \;\; [K, P] = iD. \tag{2.28}$$

The above equations elucidate the fact that the states can be labelled as eigenstates of the operator D, and therefore satisfy

$$D\left|\psi\right\rangle = \Delta\left|\psi\right\rangle, \tag{2.29}$$

where Δ is an eigenvalue of D. With respect to these eigenvalues, P and K act respectively as raising and lowering operators. Since K is the lowering operator, if $|\psi_0\rangle$ is the ground state, it should satisfy

$$K\left|\psi_0\right\rangle = 0. \tag{2.30}$$

Any arbitrary state can be constructed by the (repeated) action of the raising operator on $|\psi_0\rangle$. Using (2.26) and the similar expression for G_{12}, we see that K is given by

$$\frac{1}{2}e^{-it}(-i\sin\rho\partial_t + \cos\rho\partial_\rho) \tag{2.31}$$

for ψ expressed in the (t, ρ) coordinate basis. We see that (2.30) when expressed in the coordinate basis takes the form

$$-i\sin\rho\partial_t\psi_0(t,\rho) + \cos\rho\partial_\rho\psi_0(t,\rho) = 0. \tag{2.32}$$

Assuming a separable solution $\psi_0 = e^{i\Delta t}\chi(\rho)$, we see that the above equation becomes

$$\Delta\chi = -\cot\rho(\partial_\rho\chi), \tag{2.33}$$

which is solved by $\chi(\rho) = \cos^\Delta\rho$, and thus the full ground state solution is given by

$$\psi_0(t,\rho) = e^{i\Delta t}\cos^\Delta\rho. \tag{2.34}$$

Now, any two successive eigenstates obtained by the repeated action of the raising operator on the ground state differ in energy by the same amount. That is, energy increases in equal steps of a finite quantity.

Now, we only had 3 isometry generators for the case we considered previously since there were only 3 independent isometries in the simple case of AdS_2. However, in arbitrary dimensions (say $d+1$), as discussed before, there are $\frac{1}{2}(d+1)(d+2)$ number of independent isometries. The conformal algebra $SO(d,2)$ pertinent to that case is given by

$$\begin{gathered}
[D, P_\mu] = iP_\mu \qquad [D, K_\mu] = -iK_\mu \qquad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \qquad [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}),
\end{gathered} \tag{2.35}$$

where the index μ ranges from 1 to d. Comparing with (2.9), we see that the operator definitions that are consistent with the above equation are given by $D \equiv -iG_{0,d+1}$ (dilation generator), $P_\mu \equiv iG_{d+1,\mu} + G_{0,\mu}$ (translation generators), $K_\mu \equiv iG_{d+1,\mu} - G_{0,\mu}$ (SCT generators), and $M_{\mu\nu} \equiv -iL_{\mu\nu}$ (rotation generators). The second line of (2.35) describes the transformation of P_μ and K_μ as vectors under rotations, while the vanishing commutator of D with $L_{\mu\nu}$ is a reflection of the fact that it transforms as a scalar under rotations. The latter fact enables us to label quantum states by their energy and angular momentum which are the eigenvalues of D and $L_{\mu\nu}$ respectively. The first 2 equations in the first line of (2.35) lets us conclude that P_μ acts as a 'raising operator' while K_μ acts as a 'lowering operator' with respect to the generator of dilations, which explains the integer spacing between the energy levels of single particle quantum states. If we denote the lowest energy state by $|\psi_0\rangle$, it should satisfy

$$K_\mu \left|\psi_0\right\rangle = 0, \tag{2.36}$$

since the lowering operator annihilates the lowest energy state. We call this operator as the *primary* operator, which, from the perspective of quantum mechanics in AdS, is the lowest energy state of a particle. A general one particle quantum mechanical state in AdS can be constructed from the primary operator by the repeated application of raising operator(s) as

$$\left|\psi_{n,l}\right\rangle = \left(P_\mu^2\right)^n P_{\mu_1} P_{\mu_2} P_{\mu_3} \cdots P_{\mu_l} \left|\psi_0\right\rangle, \tag{2.37}$$

which (by construction) has energy given by $E_{n,l} = \Delta + 2n + l$, where we assume that the energy of $|\psi_0\rangle$ is given by Δ.

The position space wavefunctions of these quantum states can be obtained by expressing the lowering operator in coordinate basis using equations (2.8) and (2.9) as well as the definition of the lowering operator in terms of (2.9). Once we obtain this, one can impose the constraint that the lowering operator annihilates the ground state to obtain the position space wavefuntion of the ground state. Applying the raising operator repeatedly on the so obtained ground state wavefunction gives us position space wavefunction of the state given in (2.37), whose completely general form is expressed in terms of a hypergeometric function.

Multiparticle states (states in the Fock space) have quantum numbers that are sums of quantum numbers of individual single particle states. Since this applies to the energy eigenvalue as well, one can construct the eigenstates using harmonic oscillator creation and annihilation operators. The Hamiltonian operator for AdS maybe expressed using these operators as

$$D = \sum_{n,l,J} \left(\Delta + 2n + l\right) a^\dagger_{nlJ} a_{nlJ}, \tag{2.38}$$

where n, l and J are individual eigenvaluess of different single particle states and $a^\dagger_{nlJ} a_{nlJ}$ is the number operator familiar from the harmonic oscillator analysis. One can arrive at the same expression for energy by quantizing the AdS free scalar field action, given by

$$S = \int_{AdS} d^{d+1}x \sqrt{-g} \left(\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2\right). \tag{2.39}$$

2.2 Conformal field theory

A conformal field theory is one that has 2 additional symmetries compared to ordinary field theories with Poincare symmetry; *dilations* and *special conformal transformations*. The complete symmetry group of a CFT is what is commonly referred to as the *conformal group*.

2.2.1 Conformal transformations in d dimensions

Any symmetry transformation of a CFT is what we refer to as a conformal transformation. Let $\mathcal{M} = \mathbb{R}^{p,q}$ be a manifold $\mathbb{R}^d$, with $d = p+q$ such that the metric is diagonal with p of the entries equal to +1 and q of them equal to -1. We have a Euclidean metric when $p = d$, and a Lorentzian metric when $p = 1$ and $q = d-1$. For AdS/CFT, we usual work with $p = 1$ and $q = 2$ when the space is 3 dimensional. Consider a smooth change of coordinates from x to x' (both belonging to $\mathcal{M}$), $x \to x'(x)$, such that the transformed metric is a scalar multiple of the old one:

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x), \tag{2.40}$$

with

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x), \text{ and } \Omega(x) > 0. \tag{2.41}$$

Such a transformation is called *conformal*. The group of all such transformations is called the conformal group, denoted by Conf($\mathbb{R}^{p,q}$).

Let us consider an infinitesimal conformal transformation

$$x'^\mu = x^\mu + \epsilon^\mu(x), \tag{2.42}$$

so that upto first order in ϵ^μ, the metric changes as

$$g'_{\mu\nu}(x') = \eta_{\mu\nu}(x) + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{2.43}$$

We use $\eta_{\mu\nu}$ in place of $g_{\mu\nu}$ since we are working with flat space metric. For the above equation to satisfy (2.41), we require

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu \propto \eta_{\mu\nu}, \tag{2.44}$$

so that

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial.\epsilon)\eta_{\mu\nu}, \tag{2.45}$$

where the proportionality constant can be fixed by tracing (2.44). Therefore, the overall factor by which the metric scales is given by

$$\Omega(x) = 1 + \frac{2}{d}(\partial_\mu \epsilon^\mu). \tag{2.46}$$

It follows that

$$\eta_{\mu\nu}\Box + (d-2)\partial_\mu\partial_\nu(\partial.\epsilon) = 0, \tag{2.47}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ [8]. (2.45) and (2.47) are highly constraining; for $d > 2$, they demand that derivates of ϵ of order 3 or greater must vanish. That is, ϵ can at most be quadratic in x. Therefore, the possible dependences of ϵ on x, and thus the possible infinitesimal conformal transformations are given by

$$\begin{aligned}
&\epsilon^\mu(x) = a^\mu,\ \text{a constant} && (\textit{translations})\\
&\epsilon^\mu(x) = \omega^\mu_\nu x^\nu,\ \omega\ \text{antisymmetric} && (\textit{Lorentz transformations})\\
&\epsilon^\mu(x) = \lambda x^\mu,\ \lambda > 0 && (\textit{dilations})\\
&\epsilon^\mu(x) = b^\mu x^2 - 2x^\mu(b.x) && (\textit{special conformal transformations}),
\end{aligned} \tag{2.48}$$

where the first expression is zeroth order in x, second and third are first order in x, and the final expression is second order in x. Integrating these expressions, we get finite conformal transformations. The most general conformal transformation is a combination of these 4. Special conformal transformations bring infinity to a finite point and vice versa. If one adds infinity in (compactification), these transformations form a group under composition. The corresponding generators of the above infinitesimal transformations look like [8]

$$\begin{aligned}
P_\mu &= -i\partial_\mu && (\textit{translations})\\
L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) && (\textit{Lorentz transformations})\\
D &= -ix^\mu\partial_\mu && (\textit{dilations})\\
K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) && (\textit{special conformal transformations}).
\end{aligned} \tag{2.49}$$

2.2.2 Classical conformal field theory

A conformal theory, at its most basic, gives some kind of representation of the conformal group on the degrees of freedom of the theory. Classical CFTs essentially describe classical field representations of conformal symmetry. In the non relativistic case, we impose $[\mathcal{H}, G_s] = 0$ as a requirement for a symmetry, where $\mathcal{H}$ is the Hamiltonian of the theory and G_s is the generator of a symmetry. This definiton does not hold in the relativistic case; we need our theory to be *local*, which is defined by having a set of local observables $\phi_a(x)$, where $x \in \mathbb{R}^{p,q}$ and $a \in I$, some index set. Not all representations may give rise to local theories.

Working with fields is perhaps the only way we can furnish a local set of observables, since by definition, a field $\phi(x)$ is defined at a point. We all know that a general classical field theory is defined by an action given by

$$S = \int d^4x \mathscr{L}(\phi_a, \partial_\mu \phi_a). \tag{2.50}$$

We say that our theory is conformally invariant if the Euler-Lagrange equations of motion (obtained from a variational principle of the above action) are invariant under conformal transformations; the action as such need not be invariant. This means that $\mathscr{L}$ is allowed to change by a total derivative under a conformal transformation. The corresponding transformations of fields can be studied by analyzing representations of the infinitesimal generators of conformal group.

Under an (active) symmetry transformation, the general transformation of a field can be expressed as

$$\phi'_a(x) = \pi(O)_{ab} \phi_b(O^{-1}x), \tag{2.51}$$

where O is the symmetry transformation matrix, and $\pi(O)$ is a matrix referred to as the *representation*, which acts (possibly) non trivially on the field configurations at $O^{-1}x$. Here, a repeated index implies a summation over the same (Einstein summation convention). The representation is called *trivial* when $\pi(O)_{ab} = \delta_{ab}$, and *fundamental* when $\pi(O)_{ab} = O_{ab}$.

In the case of conformal group, we see that its generators satisfy the commutation relations

$$\begin{gathered}
[D, P_\mu] = iP_\mu \qquad [D, K_\mu] = -iK_\mu \qquad [K_\mu, P_\nu] = 2i(\eta_{\mu\nu} D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu) \qquad [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu) \\
[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}).
\end{gathered} \tag{2.52}$$

We will look at the kind of fields that transform under representations of Conf($\mathbb{R}^{p,q}$) and also how they transform. This task is greatly simplified if we know the Lie algebra of Lorentz group, which is a sub-algebra of the Lie algebra of Conf($\mathbb{R}^{p,q}$). From the above equation, we see that the Lie algebra of Poincare group is given by

$$\begin{aligned}[][P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\ [L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}).\end{aligned} \tag{2.53}$$

If we have a representation of Conf($\mathbb{R}^{p,q}$), then by restriction, we have a Poincare Lie algebra. A field transforming under Conf($\mathbb{R}^{p,q}$) transforms under the Poincare subgroup (generated by P_μ and $L_{\mu\nu}$) as

$$\phi_a(x) \to \pi(\Lambda)_{ab}\phi_b(\Lambda^{-1}x), \tag{2.54}$$

by restricting to just the Poincare subalgebra. We will focus on a subgroup of Conf($\mathbb{R}^{p,q}$) that leaves the origin invariant. Except translations, all elements of Conf($\mathbb{R}^{p,q}$) fall into this category. Such a transformation (infinitesimal) could be expressed as

$$\Lambda = e^{i\omega^\alpha G_\alpha}, \tag{2.55}$$

where $G_\alpha \in \{K_\mu, D, L_{\mu\nu}\}$. Since the origin is left invariant, the corresponding transformation of the field is given by

$$\phi_a(0) \to \pi(e^{i\omega^\alpha G_\alpha})_{ab}\phi_b(0). \tag{2.56}$$

Since ω^α is infinitesimal, we could approximate $\pi(e^{i\omega^\alpha G_\alpha})$ to first order in ω^α as

$$\pi(e^{i\omega^\alpha G_\alpha}) \approx \pi(\mathbb{1}) + i\omega^\alpha \pi(G_\alpha). \tag{2.57}$$

Now, if we define $\pi(D) \equiv \widetilde{\Delta}$, $\pi(K_\mu) \equiv \mathcal{K}_\mu$, and $\pi(L_{\mu\nu}) \equiv \mathcal{S}_{\mu\nu}$, then near the origin, we have

$$[\widetilde{\Delta}, \mathcal{S}_{\mu\nu}] = 0, \qquad [\widetilde{\Delta}, \mathcal{K}_\mu] = -i\mathcal{K}_\mu, \text{ and } \qquad [\mathcal{K}_\mu, \mathcal{K}_\nu] = 0. \tag{2.58}$$

Suppose $\mathcal{S}_{\mu\nu}$ are irreps of Lorentz group. Then, since $[\widetilde{\Delta}, \mathcal{S}_{\mu\nu}] = 0$, by Schur's lemma, we have $\widetilde{\Delta} \propto \mathbb{1}$, and therefore, from (2.58), we have $\mathcal{K}_\mu = 0$. For the purpose of convenience, let us assume that

$$\widetilde{\Delta} = i\Delta\mathbb{1}. \tag{2.59}$$

At the origin, under a dilation $x \to \lambda x$, a field transforms as

$$\phi_a(0) \to (\mathbb{1} + i\epsilon\widetilde{\Delta})(\mathbb{1} + i\epsilon\widetilde{\Delta}) \cdots\cdots \phi_a(0), \tag{2.60}$$

where we have constructed a dilation by composing a large number of infinitesimal dilations. The RHS of the above equation can be expressed [9] as $\lambda^{i\widetilde{\Delta}}\phi_a(0)$, which is in turn equal to $\lambda^{-\Delta}\phi_a(0)$ by (2.59). This formula can be extended to the transformation of fields at arbitrary points by the Baker-Campbell-Hausdorff formula [9]. Therefore, under a dilation $x \to \lambda x$ a field transforms as

$$\phi'_a(x) = \lambda^{-\Delta_a}\phi_a(\lambda^{-1}x), \tag{2.61}$$

where Δ_a is referred to as the *scaling dimension* of the field ϕ_a. Since the Jacobian for a dilation is $\left|\frac{\partial x'}{\partial x}\right| = \lambda^d$, (2.61) can also be expressed as

$$\phi'_a(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_a}{d}} \phi_a(x). \tag{2.62}$$

2.2.3 Constraints of conformal invariance on Quantum CFTs

The only way a quantum theory can be conformally invariant is if we have a projective unitary representation, $U(g)$ with $g \in \mathrm{Conf}(\mathbb{R}^{\mathbb{p},\mathbb{q}})$, to describe the symmetry transformations of the theory, i.e., conformal transformations. In a quantum CFT, we demand that the vacuum, $|0\rangle$, is invariant under global conformal transformations; at max, we allow the transformed state to pick up a phase: $U(g)\,|0\rangle = e^{i\phi(g)}\,|0\rangle$. Also, we demand that the theory be *local*, which means that $[A_j(x), A_j(y)] = 0$ when $(x-y)$ is spacelike. Here $A_j(x)$ is an observable labelled by x and j, where j belongs to an index set J and $x \in \mathbb{R}^{\mathbb{p},\mathbb{q}}$. There exists a subset of $\{A_j(x) | j \in J, x \in \mathbb{R}^{\mathbb{p},\mathbb{q}}\}$ called *quasi primaries*, denoted by $\{\hat{\phi}_k(x) | k \in K\}$, such that, under a conformal transformation $U(g)$, they transform as

$$\hat{\phi}_k(x) \to \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_k}{d}} \hat{\phi}_k(x) \qquad (= U^\dagger(g)\phi_k(x)U(g)), \tag{2.63}$$

where $x' = gx$ and $g \in \mathrm{Conf}(\mathbb{R}^{\mathbb{p},\mathbb{q}})$. Local operators in a CFT are of types; *primary* and *descendant*. A primary operator is defined as one that satisfies (2.61) in addition to the requirement that, at the origin, it is left invariant under the action of special conformal transformations. Therefore, a primary operator may be defined as one that satisfies

$$[K_\mu, \phi(0)] = 0, \quad [D, \phi(0)] = \Delta\phi(0) \tag{2.64}$$

where Δ is the scaling dimension of the operator. Each primary operator is a quasi primary operator, but not all quasi primaries are primaries. A descendant operator maybe expressed in terms linear combinations of derivatives of primary operators, while this is not the case with a primary operator.

As a consequence of the above requirements and observations, we see that the transformation of n-point functions of quasi primaries under global conformal transformations is given by

$$\langle 0|\, \hat{\phi}'_{k_1}(x'_1) \cdots \hat{\phi}'_{k_n}(x'_n)\, |0\rangle = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_{k_n}}{d}} \cdots \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_{k_1}}{d}} \langle 0|\, \hat{\phi}_{k_1}(x_1) \cdots \hat{\phi}_{k_n}(x_n)\, |0\rangle\,. \tag{2.65}$$

The above equation constrains the form of correlation functions in a conformal field theory. For $x_j, x_k \in \mathbb{R}^{\mathbb{p},\mathbb{q}}$, we see that $(x_j - x_k)$ is left invariant under translations. For the case of rotations, the quantity left invariant would be $r_{jk} \equiv |x_j - x_k|$, while for the case of dilations, the corresponding invariant quantity would be $\frac{r_{jk}}{r_{lm}}$. Although not as straightforward to see as the other cases, one can work out that the quantity left invariant under special conformal transformations is $\frac{r_{jk} r_{lm}}{r_{jl} r_{km}}$, commonly referred to as *cross ratio*. Therefore, for correlation functions to be invariant in the sense of (2.65), we require that they be functions of cross ratios.

For the case of 2-point functions, (2.65) tells us that

$$\langle \hat{\phi}'_1(x'_1)\hat{\phi}'_2(x'_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_1}{d}} \left|\frac{\partial x'}{\partial x}\right|^{-\frac{\Delta_2}{d}} \langle \hat{\phi}_1(x_1)\hat{\phi}_2(x_2)\rangle. \tag{2.66}$$

Translational and rotational invariance together impose the 2 point function to be of the form $f(r_{12})$, i.e., to have dependence only on the magnitude of separation between the two points, r_{12}. Furthermore, invariance under dilations enforce $f(r_{12}) = \lambda^{\Delta_1+\Delta_2} f(\lambda r_{12})$. To deduce the sort of functions that satisfy this property, we expand $f(r_{12})$ in a Taylor series as: $f(r_{12}) = \sum_a f_a r_{12}^a$. We see that for all $a \neq -(\Delta_1 + \Delta_2)$, f_a vanishes, and thus

$$f(r_{12}) = \frac{f_{-(\Delta_1+\Delta_2)}}{r_{12}^{\Delta_1+\Delta_2}} = \frac{c_{12}}{r_{12}^{\Delta_1+\Delta_2}}, \tag{2.67}$$

where c_{12} is a constant that can be fixed by normalizing the fields. Following this, if we demand invariance under special conformal transformations as well, we see that the 2-point function vanishes whenever $\Delta_1 \neq \Delta_2$. Therefore, we see that the general form of a CFT 2-point function is given by

$$\langle \hat{\phi}_1(x_1)\hat{\phi}_2(x_2)\rangle = \frac{c_{12}}{r_{12}^{2\Delta}}, \tag{2.68}$$

where $\Delta_1 = \Delta_2 = \Delta$. Similar calculations show that the general form of the 3-point function in a CFT looks like

$$\langle \hat{\phi}_1(x_1)\hat{\phi}_2(x_2)\hat{\phi}_3(x_3)\rangle = \frac{c_{123}}{r_{12}^{\Delta_1+\Delta_2-\Delta_3} r_{23}^{\Delta_2+\Delta_3-\Delta_1} r_{13}^{\Delta_1+\Delta_3-\Delta_2}}. \tag{2.69}$$

For n ≥ 4, constraining the general form of n-point functions are not as easy as the previous two cases and thus we do not mention it here.

2.3 Correlators from the duality

A massive particle at rest in AdS is the dual of a primary operator in CFT. Hence, as we saw before, a primary state in Ads is the ground state of a massive particle with energy given by Δ, which is the scaling dimension of the primary operator in CFT. Similarly, descendant operators in a CFT correspond to moving particles in the AdS dual. The process of taking derivatives of primary operators in a CFT to get descendants essentially corresponds to going from free particle ground states to excited states in the AdS dual. Also, the conformal group $SO(d,2)$, which is the symmetry group of a CFT, corresponds to the isometry group $SO(d,2)$ of AdS. Furthermore, local operators in a CFT correspond to bulk fields in AdS; scalar operators correspond to scalar fields in the bulk, vector operators correspond to vector fields in the bulk, and finally tensor operators correspond to tensor fields in the bulk.

It is a well known fact [10] that if one has a bulk field $\phi(x)$, with a boundary value equal to $\phi_0(x)$, which is dual to a boundary (CFT) operator $\mathcal{O}(x)$, it is equivalent to deforming the CFT by adding a source term for $\mathcal{O}(x)$ as

$$\int d^d x \phi_0(x)\mathcal{O}(x). \tag{2.70}$$

This is a very important result as it helps us in proving a maxim in AdS/CFT; *a global symmetry in the boundary theory corresponds to a gauge symmetry on the bulk side.* To prove this, consider deforming the boundary theory by adding the term

$$\int d^d x a_\mu(x) J^\mu(x), \tag{2.71}$$

where $a_\mu(x) = A_\mu(x,z)|_{z=0}$ (in Poincare coordinates) is the boundary limit of some bulk field $A_\mu(x)$, which is dual to the conserved boundary current $J_\mu(x)$ (conserved since it corresponds to a global symmetry in the boundary). Now, for some arbitrary differentiable $\Lambda(x)$, the transformation

$$a_\mu(x) \to a_\mu(x) + \partial_\mu \Lambda(x), \tag{2.72}$$

is an invariance of (2.71) if we assume that a_μ vanishes reasonably fast at the boundary. Now, since (2.72) is a sub class of the gauge transformations of $A_\mu(x)$ (which leaves the boundary dynamics invariant), we see that a global symmetry in the boundary theory corresponds to a gauge symmetry in the dual bulk theory.

We will explore the computation of CFT 2-point function from the bulk side as done in [4]. To start with, we consider a massive scalar field ϕ in AdS$_{d+1}$ with its boundary value equal to ϕ_0. As seen previously, we assume that ϕ_0 couples to some boundary operator (which is the CFT dual of ϕ) $\mathcal{O}$ via the coupling $\int_{\partial AdS} \phi_0 \mathcal{O}$. Let $Z_s(\phi_0)$ be the partition on the bulk side with the constraint that ϕ at the boundary takes the value ϕ_0. In the semiclassical limit, using the saddle point approximation, we see that

$$Z_s(\phi_0) \simeq e^{-I_s(\phi_0)}, \tag{2.73}$$

where I_s is the action corresponding to the bulk theory.

The interpretation given in [4] of the duality between the two theories is that

$$\langle \exp \int_{\partial AdS} \phi_0 \mathcal{O} \rangle = Z_s(\phi_0), \tag{2.74}$$

where on the LHS, we have the generating functional of CFT correlation functions. As a result, one can obtain CFT correlation functions by taking functional derivatives with respect to the bulk partition function as opposed to taking them with respect to the generating functional of CFT correlation functions. For this, we will first need to solve for the equations of motion of a massive scalar field in the bulk and see its asymptotic behaviour.

The action for a free massive scalar field in AdS_{d+1} is given by

$$S=\frac{1}{2}\int d^{d+1}x\sqrt{-g}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+m^2\phi^2), \tag{2.75}$$

where $g^{\mu\nu}dx_\mu dx_\nu = \frac{R^2}{z^2}(-dt^2+d\vec{x}^2+dz^2)$, and thus corresponds to the metric of AdS_{d+1} in Lorentzian Poincare coordinates. The equations of motion that follow from a variational principle for the above action is given by

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)=m^2\phi. \tag{2.76}$$

We make the following ansatz [10] for the solution to this equation:

$$\phi(z,x^i)=\int\frac{d^dk}{(2\pi)^4}e^{ik.x}\phi(z;k), \tag{2.77}$$

where $x^i\equiv(t,\vec{x})$ and $k.x=\eta_{ij}k^ix^j$. Substitution of the above expression into (2.76) gives

$$z^{d+1}\partial_z(z^{1-d}\partial_z\phi)-k^2z^2\phi-m^2R^2\phi=0, \tag{2.78}$$

where $k^i=(\omega,\vec{k})$ and $k^2=-\omega^2+\vec{k}^2$. In the asymptotic limit $(z\to 0)$, the above equation reduces to

$$z^2{\partial_z}^2\phi+(1-d)z\partial_z\phi-m^2R^2\phi=0. \tag{2.79}$$

This homogeneous partial differential equation could be solved by assuming a power series solution for ϕ. If we let $\phi\sim z^\Delta$, (2.79) reduces to

$$\Delta(\Delta-1)+(1-d)\Delta-m^2R^2=0, \tag{2.80}$$

whose solutions are given by

$$\Delta=\frac{d}{2}\pm\sqrt{\frac{d^2}{4}+m^2R^2}. \tag{2.81}$$

If we define $v\equiv\sqrt{\frac{d^2}{4}+m^2R^2}$, $\Delta\equiv\frac{d}{2}+v$, and $\Delta_-\equiv\frac{d}{2}-v=d-\Delta$, then the general form of asymptotic $(z\to 0)$ solution can be expressed as

$$\phi(k^i,z)=\alpha(k^i)z^{d-\Delta}+\beta(k^i)z^\Delta, \tag{2.82}$$

or equivalently as

$$\phi(x^i, z) = \alpha(x^i) z^{d-\Delta} + \beta(x^i) z^{\Delta}. \tag{2.83}$$

Now, we see that $\Delta \geq \frac{d}{2}$, and therefore z^{Δ} always approaches 0 as $z \to 0$. Hence, we may conclude that

$$\phi(z, x^i)\big|_{z=0} = \lim_{z \to 0} z^{d-\Delta} \phi_0(x^i), \tag{2.84}$$

$\phi_0(x^i)$ is a function of the boundary coordinates.

Now that we have obtained the asymptotic behaviour of the free scalar field in the bulk, we may express the general solution of bulk wave equation (in terms of the solution at the boundary) as

$$\phi(z, x) = \int d^{d+1}x' K(x, x'; z) \phi_0(x'), \tag{2.85}$$

where $\phi_0(x')$ is the boundary value of ϕ as in (2.84), and $K(x, x'; z)$ is the bulk to boundary propagator, which gives us the bulk solution in response to a source localized on the boundary (which is the boundary value of the bulk field). As one can easily see, the bulk-boundary propagator satisfies the equation

$$(\Box - m^2) K(x, x'; z) = z^{d-\Delta} \delta(x - x'), \tag{2.86}$$

since ϕ satisfies the free massive scalar field equation (Klein-Gordon equation). One can solve for the bulk-boundary propagator [11–13] to obtain

$$K(x, x'; z) = C_{\Delta} \left(\frac{z}{z^2 + (x - x')^2} \right)^{\Delta}, \tag{2.87}$$

where C_{Δ} is a constant that depends solely on Δ. Now that we have obtained (2.85), we can plug this expression into the bulk on-shell action and take functional derivatives of the same with respect to ϕ_0 (and evaluate the resultant at $\phi_0 = 0$) to obtain CFT correlators. One can verify that the CFT 2-point function so obtained is given by

$$\langle \mathcal{O}(x) \mathcal{O}(x') \rangle = \frac{\delta^2}{\delta\phi_0(x) \delta\phi_0(x')} Z_s \phi_0 \bigg|_{\phi_0 = 0} = \frac{1}{|x - x'|^{2\Delta}}. \tag{2.88}$$

Chapter 3

Gravitational action and an action principle for general relativity

In this chapter, we look at a variational principle for gravitational action starting with the simplest form which it can assume. We then introduce term(s) in the action so that we have a well defined variational principle (by which we mean that setting the variation of the action with respect to the metric equal to zero yields the Einstein's equations) as well as a non-divergent action when we take the asymptotic limit (large r). We conclude the chapter by giving the most general form of gravitational action from which one may obtain the Einstein's equations.

3.1 Einstein-Hilbert action

By the principle of general covariance, one can deduce that the action for an arbitrary metric $g_{\mu\nu}$ should assume the form

$$S = \frac{1}{2\kappa}\int \sqrt{-g}\, d^4x f(g_{\mu\nu}), \tag{3.1}$$

where f is a scalar function of the metric, g is the determinant of the metric and $\kappa = 8\pi G_N$, where G_N is the gravitational constant. The simplest scalar that can be constructed out of the metric is the Ricci scalar R, which is also the unique choice when we demand metric derivatives that are not higher than order 2 [16]. This gives rise to the *Einstein-Hilbert* *action*, given by

$$S_{EH} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R. \tag{3.2}$$

With the aim of obtaining vacuum Einstein's equations, we vary this action with respect to the metric, which gives us

$$\delta S_{EH} = \frac{1}{2\kappa}\int d^4x(R\,\delta\sqrt{-g} + \sqrt{-g}\,\delta R). \tag{3.3}$$

Now, we know that $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$. Also, since, $R = g_{\mu\nu}R^{\mu\nu} = g^{\mu\nu}R_{\mu\nu}$, we obtain

$$\delta S_{EH} = \frac{1}{2\kappa}\int \sqrt{-g}d^4x(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \frac{1}{2\kappa}\int \sqrt{-g}d^4x\, g^{\mu\nu}\delta R_{\mu\nu}. \tag{3.4}$$

Since the first term on the RHS alone gives us the Einstein tensor, we expect the second term on the RHS to be a boundary term that would arise from a total derivative. From the formula for Ricci scalar obtained by the contraction of the first and 3rd indices of Riemann tensor, we see that

$$\delta R_{\mu\nu} = \partial_\lambda\delta\Gamma^\lambda_{\mu\nu} - \partial_\nu\delta\Gamma^\lambda_{\mu\lambda} + \delta\Gamma^\lambda_{\lambda\rho}\Gamma^\rho_{\nu\mu} + \Gamma^\lambda_{\lambda\rho}\delta\Gamma^\rho_{\nu\mu} - \delta\Gamma^\lambda_{\nu\rho}\Gamma^\rho_{\lambda\mu} - \Gamma^\lambda_{\nu\rho}\delta\Gamma^\rho_{\lambda\mu}. \tag{3.5}$$

Although Christoffel symbols themselves are not tensors, their variations with respect to the metric transform like tensors. This is because of the fact that the (second derivative, inhomogenous)term obtained under coordinate transformations that makes christoffel symbols non tensorial is independent of the metric. Thus, metric variations of Christoffel symbols transform like tensors. In fact, these metric variations are given by [16]

$$\delta\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}(\nabla_\nu\delta g_{\rho\lambda} + \nabla_\lambda\delta g_{\rho\nu} - \nabla_\rho\delta g_{\nu\lambda}). \tag{3.6}$$

Also, one can see that using covariant derivatives, (3.5) can be expressed in a much simpler looking form as

$$\delta R_{\mu\nu} = \nabla_\lambda\delta\Gamma^\lambda_{\mu\nu} - \nabla_\nu\delta\Gamma^\lambda_{\mu\lambda}. \tag{3.7}$$

Using the above formula, we obtain

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= \nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}) \\ &= \nabla_\lambda(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^\nu_{\nu\mu}), \end{aligned} \tag{3.8}$$

which establishes our claim that $g^{\mu\nu}\delta R_{\mu\nu}$ must be a total derivative. Using (3.6), the above formula can be equivalently expressed as

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= (\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box)\delta g_{\mu\nu} \\ &= (g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta})\nabla_\mu\nabla_\nu\delta g_{\alpha\beta} \\ &= \nabla_\lambda\left((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta}\right), \end{aligned} \tag{3.9}$$

which indeed is consistent with our hunch that it must be a total derivative. Thus, using the above formula, the variation of Einstein-Hilbert action could be expressed as

$$\begin{aligned} \delta S_{EH} &= \frac{1}{2\kappa}\int\sqrt{-g}d^4x(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)\delta g^{\mu\nu} + \frac{1}{2\kappa}\int\sqrt{-g}d^4x\nabla_\lambda\left((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta}\right) \\ &= \frac{1}{2\kappa}\int\sqrt{-g}d^4xG_{\mu\nu} + \frac{\epsilon}{2\kappa}\oint_\Sigma\sqrt{-h}d^3y\ n_\lambda\left((g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta}\right), \end{aligned} \tag{3.10}$$

where $G_{\mu\nu}$ is the Einstein tensor for the metric, n_λ is the normal vector to the boundary hypersurface Σ of the manifold, h is the determinant of the induced metric on Σ, and $n_\mu n^\mu \equiv \epsilon = +1$ or -1 depending on whether the boundary hypersurface is timelike or spacelike respectively. For writing the second equation in (3.10), we use the Gauss integral theorem in general relativity, which is given by

$$\int_{\mathcal{M}}\sqrt{-g}d^nx\nabla_\mu V^\mu = \int_\Sigma\sqrt{-h}d^{n-1}x\ n_\mu V^\mu, \tag{3.11}$$

where $\mathcal{M}$ is the n dimensional spacetime manifold with boundary Σ, g and h are the determinants of the metric on $\mathcal{M}$ and Σ respectively, and n_μ is the normal vector to Σ. Now, the integrand of the boundary integral in (3.10) can be expressed using the decomposition of the metric on Σ, given by

$$g^{\mu\nu} = h^{\mu\nu} + \epsilon n^\mu n^\nu, \tag{3.12}$$

as

$$\begin{aligned} (g^{\lambda\alpha}g^{\nu\beta} - g^{\lambda\nu}g^{\alpha\beta})\nabla_\nu\delta g_{\alpha\beta} &= (n^\rho g^{\mu\nu} - n^\mu g^{\rho\nu})\nabla_\mu\delta g_{\rho\nu} \\ &= n^\rho h^{\mu\nu}\nabla_\mu\delta g_{\rho\nu} - n^\mu h^{\rho\nu}\nabla_\mu\delta g_{\rho\nu}. \end{aligned} \tag{3.13}$$

Now, if we impose the condition that the variation of the metric vanishes at the boundary (Dirichlet boundary conditions), i.e.,

$$\delta g_{\rho\nu}|_\Sigma = 0, \tag{3.14}$$

then, since the first term in the RHS of the second equation in (3.13) depends only on $\delta g_{\rho\nu}$ and its tangential derivatives $h^{\mu\nu}\nabla_\mu \delta g_{\rho\nu}$, it would vanish if we impose Dirichlet boundary conditions as in (3.14). On the other hand, the second term in the RHS of the second equation in (3.13) depends on $\delta g_{\rho\nu}$ and their normal derivatives $n^\mu \nabla_\mu \delta g_{\rho\nu}$, and is therefore non-zero. Thus, we see that with Dirichlet boundary conditions, the variation of Einstein-Hilbert action gives rise to a term that gives the Einstein tensor and a non vanishing boundary term:

$$\delta S_{EH} = \frac{1}{2\kappa}\int \sqrt{-g} d^4x G_{\mu\nu} + \frac{\epsilon}{2\kappa}\oint_\Sigma \sqrt{-h} d^3y(-h^{\rho\nu}n^\mu \partial_\mu \delta g_{\rho\nu}). \tag{3.15}$$

Hence, we see that the variation of Einstein-Hilbert action is not exactly equal to the Einstein tensor; another term needs to be added to the action for the metric so that its variation cancels with the boundary term arising from the variation of Einstein-Hilbert action such that the net variation gives us just the Einstein tensor. This term is the *Gibbons-Hawking-York* boundary term.

3.2 Gibbons-Hawking-York boundary term

In the previous section, we saw that apart from just the Einstein-Hilbert action, another piece needs to be added to the gravitational action in order for the net variation to be exactly equal to the Einstein tensor. We choose this term to be a boundary term so that the bulk variation is unaffected, and for Dirichlet boundary conditions, we could choose this term in such a way that it exactly cancels the boundary term in the variation of Einstein-Hilbert action.

The Gibbons-Hawking-York boundary term is given by

$$S_{GHY} = \frac{\epsilon}{\kappa}\oint_\Sigma \sqrt{-h}\ d^3y K, \tag{3.16}$$

where K is the trace $h^{ab}K_{ab}$ of the *extrinsic curvature* K_{ab} of Σ, which is defined as

$$K_{ab} \equiv \nabla_\beta n_\alpha e^\alpha_a e^\beta_b, \tag{3.17}$$

where $e^\alpha_a \equiv \frac{\partial x^\alpha}{\partial y^a}$ are tangent vectors on the hypersurface (since we specify the hypersurface using the parametrization $x^\alpha(y^a)$). Here, x^α are coordinates on the spacetime manifold while

y^a are coordinates on the hypersurface. The trace of the extrinsic curvature of the boundary hypersurface Σ can be expressed as

$$\begin{aligned} K &= h^{ab} K_{ab} = h^{ab} \nabla_\beta n_\alpha e^\alpha_a e^\beta_b \\ &= \nabla_\beta n_\alpha (h^{ab} e^\alpha_a e^\beta_b + \epsilon n^\alpha n^\beta) \\ &= \nabla_\beta n_\alpha g^{\alpha\beta} \\ &= \nabla_\alpha n^\alpha, \end{aligned} \tag{3.18}$$

where in the second equation, we have used the fact that $n_\alpha n^\alpha = \epsilon = \pm 1$. Also, from the second equation, we see that

$$\begin{aligned} K &= h^{\alpha\beta} \nabla_\beta n_\alpha \\ &= h^{\alpha\beta} (\partial_\beta n_\alpha - \Gamma^\gamma_{\alpha\beta} n_\gamma), \end{aligned} \tag{3.19}$$

so that the variation of the scalar extrinsic curvature (under Dirichlet boundary conditions) is given by

$$\begin{aligned} \delta K &= -h^{\alpha\beta} n_\gamma \delta \Gamma^\gamma_{\alpha\beta} \\ &= -\frac{1}{2} h^{\alpha\beta} n^\mu (\partial_\beta \delta g_{\mu\alpha} + \partial_\alpha \delta g_{\mu\beta} - \partial_\mu \delta g_{\alpha\beta}) \\ &= \frac{1}{2} h^{\alpha\beta} n^\mu \partial_\mu \delta g_{\alpha\beta}, \end{aligned} \tag{3.20}$$

where we have used the fact that the tangential derivatives of the metric, $h^{\mu\nu} \nabla_\mu \delta g_{\rho\nu}$, vanish on Σ. Also, it is because we impose Dirichlet boundary conditions that we do not have any variation of the normal vector with respect to the metric. Therefore, the variation of the Gibbons-Hawking-York boundary term with respect to the metric is observed to be

$$\delta S_{GHY} = \frac{\epsilon}{2\kappa} \oint_\Sigma \sqrt{-h} d^3 y h^{\rho\nu} n^\mu \partial_\mu \delta g_{\rho\nu}. \tag{3.21}$$

We see that this exactly cancels with the boundary term in the variation of the Einstein-Hilbert action, and thus we conclude that the sensible gravitational action from which we can obtain Einstein's equations is given by

$$S = S_{EH} + S_{GHY} = \frac{1}{2\kappa} \int d^4 x \sqrt{-g} R + \frac{\epsilon}{\kappa} \oint_\Sigma \sqrt{-h}\ d^3 y K. \tag{3.22}$$

However, it might be the case that the above action might lead to divergences when we integrate over the whole spacetime manifold (which is non compact). In order to fix this problem, one can introduce a counterterm to the above action which is independent of the metric with respect to which we vary, and thus renders the variational problem as well as equations of motion invariant. We achieve this by a technique commonly known as *background subtraction* as will be explained in the subsequent section.

3.3 Nondynamical counterterm

The counterterm that cancels out the divergences arising from the dynamical pieces (Einstein-Hilbert term and Gibbons-Hawking-York term), as mentioned in the last section, is given by

$$S_{ct} = \frac{\epsilon}{\kappa} \oint_{\Sigma} \sqrt{-h}\, d^3y K_0, \tag{3.23}$$

where K_0 is the scalar extrinsic curvature of Σ embedded in a reference background spacetime, which is responsible for the 'background subtraction' terminology [17].

As an example, consider the 4 dimensional flat space metric in spherical polar coordinates with Σ defined by two constant time slices, $t = t_1$ and $t = t_2$, and a constant r hypersurface defined by $r = R$. One can easily verify that the scalar extrinsic curvature vanishes on the constant time slices. The induced metric on the $r = R$ hypersurface is simply $ds^2 = -dt^2 = R^2 d\Omega_2{}^2$, for which $\sqrt{-h} = R^2 sin\theta$. Also, for this hypersurface, $n_\mu = \partial_\mu r = \delta^r_\mu$, and $\epsilon = 1$ since it is timelike. One can also check that the scalar extrinsic curvature on this hypersurface is given by $K = \nabla_\alpha n^\alpha = \frac{2}{R}$. Therefore, for the 4 dimensional flat space metric, the action in (3.22) looks like

$$\begin{aligned} S = S_{GHY} &= \frac{\epsilon}{\kappa} \oint_{\Sigma} \sqrt{-h}\; d^3yK \\ &= \frac{8\pi}{\kappa} R(t_2 - t_1), \end{aligned} \tag{3.24}$$

where the contribution from Einstein-Hilbert action vanishes since Ricci scalar vanishes for flat spacetime. We see that when the constant r hypersurface is pushed to infinity , i.e., when $R \to \infty$, the action in (3.2) diverges. Thus, we see that, even with (finite) constant time slices, the gravitational action for Minkowski (flat) spacetime diverges. This problem would persist even for curved spacetimes, and thus, the gravitational action for asymptotically flat spacetimes diverge (provided the spacetime manifold is non-compact). The remedy for this is to consider the nondynamical counterterm in gravitatonal action, which would cancel off the divergence. Thus, when the spacetime is asymptotically flat, we embed the boundary in flat spacetime and compute the counterterm using the scalar extrinsic curvature of the boundary as viewed as en embedded hypersurface in flat spacetime.

3.4 Most general gravitational action

From the discussions we have had till now in this chapter, we see that the most general form of gravitational action is given by

$$\begin{aligned} S_g[g_{\mu\nu}] &= S_{EH}[g_{\mu\nu}] + S_{GHY}[g_{\mu\nu}] - S_{ct} \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + \frac{\epsilon}{\kappa}\oint_\Sigma \sqrt{-h}\ d^3yK - \frac{\epsilon}{\kappa}\oint_\Sigma \sqrt{-h}\ d^3yK_0, \end{aligned} \tag{3.25}$$

where κ is the gravitational constant, g and h are the determinants of the metric $g_{\mu\nu}$ and the induced metric h_{ab} on the boundary hypersurface Σ, R is the Ricci scalar of $g_{\mu\nu}$, K is the trace of the extrinsic curvature K_{ab} of Σ, K_0 is the scalar extrinsic curvature of Σ embedded in a reference background spacetime, and $\epsilon = +1$ or -1 depending on whether the boundary hypersurface is timelike or spacelike respectively.

Chapter 4

Quasilocal stress tensor for asymptotically AdS spaces

In this chapter, we look at a definition of the quasilocal stress tensor proposed by Brown and York [14] in 1993. We review the interpretation of the same quantity when applied to asymptotically AdS spaces as proposed by Kraus and Balasubramanian [15] in 1999, and also make a key observation which may pave the way for obtaining the results obtained by Kraus and Balasubramanian starting with a slightly different definition of the quasilocal stress tensor.

We have seen multiple definitions of stress-energy tensor such as the Hilbert stress-energy tensor (defined as the functional derivative of matter action with respect to the metric: $T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g_{\mu\nu}} = \frac{-2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_{matter})}{\delta g_{\mu\nu}}$), the canonical stress-energy tensor (defined as the stress tensor associated with a conserved Noether current resulting from invariance under translations in spacetime), etc. Nevertheless, defining a stress tensor for the gravitational field (metric) itself is a non-trivial problem and has been a subject of active research for quite some time now.

There are a few reasons why such a definition is hard. First of all, a definition analogous to that of the covariant matter energy-momentum tensor (Hilbert stress-energy tensor) for either the metric or for matter coupled to the metric does not work (where the metric satisfies Einstein's equations), since by Einstein's equations, we identically obtain zero:

$$\begin{aligned} T^{tot}_{\mu\nu} &= \frac{-2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} \left(S_{EH}[g_{\alpha\beta}] + S_M[\phi, g_{\alpha\beta}] \right) \\ &= -\frac{1}{\kappa} G_{\mu\nu} + T_{\mu\nu} = 0, \end{aligned} \tag{4.1}$$

where S_{EH} is the Einstein-Hilbert action, S_M is the matter action, $T_{\mu\nu}$ is the matter energy-momentum tensor, and $g_{\alpha\beta}$ is assumed to be a solution of Einstein's equations.

Furthermore, a local definition of the stress tensor also seems pointless from the perspective of the equivalence principle [16], by which one can always go to an inertial frame where gravitational effects vanish. This implies that the gravitational energy-momentum tensor should vanish in a local inertial frame, and since the object is tensorial in nature, it must vanish in any arbitrary frame as well. Thus, it seems as if the only way a non vanishing momentum-energy stress 'tensor' for the gravitational field could exist would be if it were non-tensorial in nature. However, since a local inertial frame is defined by the metric being Minkowski along with vanishing first derivatives of the metric, it might be possible to come up with a notion of 'non-local' gravitational energy-momentum stress tensor (constructed out of metric derivatives of order 2 or greater) which need not vanish.

From the ADM formalism of general relativity, it has however been observed that it makes sense to define the 'total' energy, momentum, and angular momentum of an isolated spacetime, and that these can be obtained from boundary surface integrals over a 2-sphere placed at infinity. In addition, one can also make sense of a notion of a somewhat more localised form of energy (possibly attributed to a localised finite region of spacetime) [16]. This is the idea behind the *quasilocal* energy-momentum stress tensor, which we will make precise in the coming section.

4.1 Brown and York's definition of the stress tensor

In 1993, J.D. Brown and J.W. York, Jr suggested a definition of the quasilocal stress tensor which was primarily motivated by the Hamilton-Jacobi theory [14]. In Hamilton-Jacobi theory for a mechanical system, we start out by expressing the action in the usual canonical form given by

$$S = \int dt \left(p\frac{dx}{dt} - H(x,p,t) \right). \tag{4.2}$$

The above equation can be expressed equivalently as

$$S = \int_{\eta_1}^{\eta_2} d\eta (p\dot{x} - \dot{t}H(x,p,t)), \tag{4.3}$$

where η parametrizes the dynamics of the system in phase space, and $\dot{x} \equiv \frac{dx}{d\eta}$ and $\dot{t} \equiv \frac{dt}{d\eta}$. Variation of this action looks like

$$\delta S = (\text{terms that give classical EOM}) + p\delta x|_{\eta_1}^{\eta_2} - H\delta t|_{\eta_1}^{\eta_2}. \tag{4.4}$$

We observe that fixing the variations δx and δt at the boundary points η_1 and η_2 makes the boundary terms vanish and directly gives us the equations of motion (just like the case with Dirichlet boundary conditions in the variation of gravitational action). Now, if we restrict the variations to happen between classical paths, we see that

$$\delta S_{cl} = p_{cl}\delta x|_{\eta_1}^{\eta_2} - H_{cl}\delta t|_{\eta_1}^{\eta_2}., \tag{4.5}$$

where "cl" implies that the corresponding quantity is evaluated at a classical solution. From the above equation, we observe that

$$\begin{aligned} p_{cl}|_{\eta_2} &= \frac{\partial S_{cl}}{\partial x_2} \\ H_{cl}|_{\eta_2} &= -\frac{\partial S_{cl}}{\partial t_2}, \end{aligned} \tag{4.6}$$

where $t(\eta_2) = t_2$ and $x(\eta_2) = x_2$, and $p_{cl}|_{\eta_2}$ and $H_{cl}|_{\eta_2}$ correspond to the classical momentum and energy at the boundary η_2.

A rough sketch of the proof given by Brown and York is as follows. Consider the action for general relativity as given in [14]:

$$S = \frac{1}{2\kappa}\int_{\mathcal{M}} R\sqrt{-g}d^4x + \frac{1}{\kappa}\int_{t_1}^{t_2} K\sqrt{-h}d^3x - \frac{1}{\kappa}\int_{{}^3B} \Theta\sqrt{-\gamma}d^3x, \tag{4.7}$$

where t_1 and t_2 are constant time (spacelike) hypersurfaces in the manifold $\mathcal{M}$, ${}^3B$ is the 3 dimensional timelike hypersurface of $\mathcal{M}$ which when put together with t_1 and t_2 constitute the boundary $\partial\mathcal{M}$ of the manifold $\mathcal{M}$. Varying the above action gives

$$\delta S = (\text{terms that give EOM}) + \int_{t_1}^{t_2} d^3x\ P^{ij}\delta h_{ij} + \int_{{}^3B} d^3x\ \pi^{ij}\delta\gamma_{ij}, \tag{4.8}$$

where P^{ij} is the momentum conjugate to the metric h_{ij} on the spacelike hypersufaces embedded in $\mathcal{M}$, while π^{ij} is the momentum conjugate to the metric γ_{ij} on the boundary hypersurface ${}^3B$. In order for us to obtain classical equations of motion from this variation of the action, we impose boundary conditions that fix the metric h_{ij} on constant time hypersurfaces and the metric γ_{ij} on the boundary hypersurface ${}^3B$. Now, as before, if we consider variations just between classical solutions so that we can determine the dependence of classical action on the fixed boundary data γ_{ij}, $h^1_{ij} \equiv h_{ij}(t_1)$, and $h^2_{ij} \equiv h_{ij}(t_2)$, we obtain

$$\delta S_{cl} = \int_{t_1}^{t_2} d^3x\ P^{ij}_{cl}\delta h_{ij} + \int_{{}^3B} d^3x\ \pi^{ij}_{cl}\delta\gamma_{ij}. \tag{4.9}$$

From this, we see that the analogue of the first equation in (4.6) is

$$P^{ij}_{cl}\big|_{t_2} = \frac{\delta S_{cl}}{\delta h^2_{ij}} \tag{4.10}$$

for the gravitational momentum conjugate to h_{ij}, evaluated at t_2. Now, as we saw before in the case of gravitational action, we can add a nondynamical boundary counterterm to the action which does not disturb the bulk equations of motion. Including this counterterm, we see that the analogue of the second equation of (4.6) is given by

$$\tau^{ij} \equiv \frac{2}{\sqrt{-\gamma}}(\pi^{ij}_{cl} - \pi^{ij}_0) = \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{cl}}{\delta\gamma_{ij}}, \tag{4.11}$$

where $\pi^{ij}_0 = \frac{\delta S_0}{\delta\gamma_{ij}}$ is the functional derivative of the counterterm S_0 with respect to the boundary metric γ_{ij}. The quantity that functions as the gravitational analogue of the fixed boundary quantity t_2 from the earlier problem of nonrelativistic mechanics is the metric γ_{ij}, which determines the time interval between constant time (spacelike) hypersurfaces. However, since the metric γ_{ij} also gives us the spacetime intervals on the boundary ${}^3B$, the quantity in (4.11) has an interpretation as a surface stress energy-momentum tensor, rather than just energy, as was the case previously.

In summary, Brown and York define the quasilocal energy-momentum stress tensor associated with a spacetime region as

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_g}{\delta \gamma_{\mu\nu}}, \tag{4.12}$$

where S_g is the gravitational action as defined in (3.25) thought of as a functional of the metric on the boundary, $\gamma_{\mu\nu}$.

4.2 Kraus and Balasubramanian's prescription for asymptotically AdS spaces

The counterterm present in the action S_g to render the stress tensor finite (or equivalently the action S_g itself finite) is obtained by embedding the boundary hypersurface in a reference spacetime as we have seen previously in section 3.3. However, embedding an arbitrary boundary metric in a reference spacetime may not always be feasible, and thus the prescription by Brown and York is ill-defined to an extent. Kraus and Balasubramanian in 1999 suggested an effective remedy for this drawback in the case of asymptotically AdS spacetimes [15]. By the AdS/CFT correspondence, one can view the bulk gravitational action as a functional of the data on the boundary, as we had seen before (in the last section of the first chapter). Therefore, by the application of this duality, one can view the quasilocal stress tensor for asymptotically AdS spacetimes as giving the expectation value of the stress tensor of the conformal field theory living on the boundary:

$$\langle T^{\mu\nu} \rangle_{CFT} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{eff}}{\delta \gamma_{\mu\nu}}. \tag{4.13}$$

The divergences observed in the asymptotic limit in the quasilocal stress tensor can now be given the interpretation of UV divergences in the CFT, which maybe taken care of by adding counterterms to the action which depend solely on the boundary geometry. This therefore helps us get around the ill-defined prescription of embedding the boundary hypersurface in a reference spacetime to get the counterterm, which may not always be possible as discussed before.

Consider the foliation of a $d+1$ dimensional spacetime manifold $\mathcal{M}$ by a set of d dimensional timelike hypersurfaces that maybe obtained by continuously deforming the boundary hypersurface $\partial\mathcal{M}$. A decomposition analogous to the ADM decomposition (where we foliate the spacetime using spacelike hypersurfaces) for this spacetime maybe expressed as

$$ds^2 = N^2 dr^2 + \gamma_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr), \tag{4.14}$$

where $g^{rr} = -\frac{1}{N^2}$, and $g_{\nu r} = \gamma_{\mu\nu}N^\mu \equiv N_\nu$. Here $\gamma_{\mu\nu}$ is the induced metric on a constant r (timelike) hypersurface $\partial\mathcal{M}_r$ which maybe thought of as bounding the region $\mathcal{M}_r$. Also, the coordinates x^μ are assumed to span the constant r hypersurfaces.
Varying the gravitational action, with respect to the boundary metric, among the solutions to Einstein's equations yields

$$\delta S = \int_{\partial\mathcal{M}_r} d^d x \pi^{\mu\nu}\delta\gamma_{\mu\nu} + \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}_r} d^d x \frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\delta\gamma_{\mu\nu}, \tag{4.15}$$

where $\pi^{\mu\nu}$ is the momentum conjugate to the boundary metric $\gamma_{\mu\nu}$ and is explicitly given by

$$\pi^{\mu\nu} = \frac{1}{16\pi G_N}\sqrt{-\gamma}(\theta^{\mu\nu} - \theta\gamma^{\mu\nu}), \tag{4.16}$$

where $\theta^{\mu\nu}$ is the extrinsic curvature of the hypersurface $\partial\mathcal{M}_r$, and is given by

$$\theta^{\mu\nu} = -\frac{1}{2}(\nabla^\mu n^\nu + \nabla^\nu n^\mu), \tag{4.17}$$

where n^μ is the (normalized) normal vector to the hypersurface $\partial\mathcal{M}_r$. Also, θ is the trace of the extrinsic curvature (a.k.a scalar extrinsic curvature) given by $\theta = h^{\mu\nu}\theta_{\mu\nu}$. Using (4.16), we can express (4.15) as

$$\delta S = \frac{1}{16\pi G_N}\int_{\partial\mathcal{M}_r} \sqrt{-\gamma}\; d^d x(\theta^{\mu\nu} - \theta\gamma^{\mu\nu})\delta\gamma_{\mu\nu} + \frac{1}{8\pi G_N}\int_{\partial\mathcal{M}_r} d^d x\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\delta\gamma_{\mu\nu}. \tag{4.18}$$

From the above equation, we see that the quasilocal stress tensor as given in (4.12) may be expressed as

$$T^{\mu\nu} = \frac{1}{8\pi G_N}\left(\theta^{\mu\nu} - \theta\gamma^{\mu\nu} + \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma_{\mu\nu}}\right). \tag{4.19}$$

The above equation is general and it applies to Brown and York's prescription too. The difference arises in the case of asymptotically AdS spaces where (as suggested by Kraus and

Balasubramanian) we construct the (local, covariant) counterterm out of boundary curvature invariants (R, R^2, etc., where R stands for the Ricci scalar of the boundary metric), since it may be assumed to be a local functional of the geometry of the boundary hypersurface $\partial\mathcal{M}_r$. In general, higher order curvature invariants like R^2, R^3, ...,etc., do not contribute to the counterterm since in the asymptotic limit, their contribution is negligible (i.e., they vanish rapidly at large r). Thus, in most cases, we are allowed to express the counterterm as

$$S_{ct} = \int_{\partial\mathcal{M}_r} \sqrt{-\gamma}\; d^d x(a + bR), \tag{4.20}$$

where, as mentioned before, R is the Ricci scalar of the boundary metric $\gamma_{\mu\nu}$. The counterterm should be chosen in such a way that at large r, the quasilocal stress does not exhibit any divergences. Brown and York propose to embed the hypersurface $\partial\mathcal{M}_r$ in an AdS background and consider the gravitational action of the resulting spacetime region as the counterterm. As mentioned before, such an embedding may not be possible always and that is why their prescription is, strictly speaking, ill-defined.

Performing an ADM decomposition of the boundary metric lets us define expressions for the mass and momentum of asymptotically AdS spacetimes [14]. The ADM decomposition of the boundary metric $\gamma_{\mu\nu}$ may be expressed as

$$\gamma_{\mu\nu} dx^\mu dx^\nu = -N_\Sigma^2 dt^2 + \sigma_{ab}(dx^a + N_\Sigma^a dt)(dx^b + N_\Sigma^b dt), \tag{4.21}$$

where Σ is a constant time spacelike hypersurface in $\partial\mathcal{M}$ (the boundary hypersurface of $\mathcal{M}$), with induced metric σ_{ab}. Here, the function N_Σ is referred to as the *lapse function*, and N_Σ^a is referrred to as the *shift vector field*. If u^μ is the (normalized) normal vector to the hypersurface Σ, then it describes the direction of time flow in $\partial\mathcal{M}$. The conserved charge associated with a boundary killing vector ξ^μ (generator of an isometry of the boundary metric) has been defined by Brown and York [14] to be

$$Q_\xi = \int_\Sigma d^{d-1}x\sqrt{\sigma}(u^\mu T_{\mu\nu}\xi^\nu) \tag{4.22}$$

The conserved charge associated with the symmetry under time translations is the mass of the spacetime, which maybe expressed as

$$M = \int_\Sigma d^{d-1}x\sqrt{\sigma}N_\Sigma\epsilon, \tag{4.23}$$

where we define a (proper) energy density

$$\epsilon \equiv u^\mu u^\nu T_{\mu\nu}. \tag{4.24}$$

We observe that if we choose $\xi^\mu = N_\Sigma u^\mu$ in (4.22), we obtain the the expression (4.23) for the mass of the spacetime. A similar definition holds for the momentum of a spacetime, and is given by

$$P_a = \int_\Sigma d^{d-1}x\sqrt{\sigma} j_a, \tag{4.25}$$

where we define $j_a \equiv \sigma_{ab} u_\mu T^{b\mu}$.

In addition, we also observe that with the fall-offs suggested by Brown and Henneaux, for leaving the asymptotic form of the metric of an asymptotically AdS space space invariant, the variation of the Gibbons-Hawking-York boundary term vanishes, and so does the boundary piece in the variation of the Einstein-Hilbert action. Therefore, we assume that the Gibbons-Hawking-York boundary term might not be needed in the action, and one can possibly make sense of a quasilocal stress tensor derived from the action obtained by dropping the GHY term.

4.3 Quasilocal stress tensor for asymptotically AdS$_3$ spaces

Consider the metric of AdS$_3$ in Poincare coordinates:

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}(-dt^2 + dx^2). \tag{4.26}$$

For a constant r hypersurface, the (normalized) normal vector is given by

$$\begin{aligned} n_\mu &= \frac{\partial_\mu r}{\sqrt{g^{\mu\nu}\partial_\mu r \partial_\nu r}} \\ &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} = \frac{l}{r}\delta_{\mu,r}. \end{aligned} \tag{4.27}$$

The extrinsic curvature of this hypersurface may be expressed as

$$\theta_{ab} = -\nabla_\beta n_\alpha e^\alpha_a e^\beta_b, \tag{4.28}$$

where $e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}$. Here, $x^\alpha = (t, r, x)$ are the coordinates that span the spacetime manifold while $y^a = (t, x)$ are the coordinates that span the constant r hypersurface.
The component θ_{tt} of the extrinsic curvature maybe expressed as

$$\begin{aligned}
\theta_{tt} &= -\nabla_\beta n_\alpha e^\alpha_t e^\beta_t = -\nabla_t n_t \\
&= -\partial_t n_t + \Gamma^\sigma_{tt} n_\sigma \\
&= \Gamma^r_{tt} n_r = -\frac{1}{2} n_r g^{rr} \partial_r g_{tt} \\
&= \frac{r^2}{l^3}.
\end{aligned} \tag{4.29}$$

Similarly, the other components of the extrinsic curvature are given by

$$\theta_{xx} = -\frac{1}{2} n_r g^{rr} \partial_r g_{xx} = -\frac{r^2}{l^3}, \tag{4.30}$$

and

$$\theta_{tx} = -\frac{1}{2} n_r g^{rr} \partial_r g_{tx} = 0. \tag{4.31}$$

The metric on the timelike hypersurface (denote it by Σ) is given by

$$ds^2_\Sigma = \frac{r^2}{l^2}(-dt^2 + dx^2), \tag{4.32}$$

where r is the constant value of the radial coordinate on the hypersurface. The scalar extrinsic curvature is therefore given by

$$\begin{aligned}
\theta = \gamma^{ab}\theta_{ab} &= \left(-\frac{l^2}{r^2}\right)\left(\frac{r^2}{l^3}\right) + \left(\frac{l^2}{r^2}\right)\left(-\frac{r^2}{l^3}\right) \\
&= -\frac{2}{l},
\end{aligned} \tag{4.33}$$

where γ^{ab} are the components of the induced metric on the hypersurface. Applying the above results in (4.19), we find that the components of the quasilocal stress tensor for AdS_3 are given by

$$\begin{aligned}
8\pi G_N T_{tt} &= \theta_{tt} - \theta\gamma_{tt} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta \gamma^{tt}} \\
&= \frac{r^2}{l^3} - \left(-\frac{2}{l}\right)\left(-\frac{r^2}{l^2}\right) - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta \gamma^{tt}}
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{r^2}{l^3} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\
8\pi G_N T_{xx} &= \theta_{xx} - \theta\gamma_{xx} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{xx}} \\
&= -\frac{r^2}{l^3} - \left(-\frac{2}{l}\right)\left(\frac{r^2}{l^2}\right) - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{xx}} \\
&= \frac{r^2}{l^3} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{xx}} \\
8\pi G_N T_{tx} &= \theta_{tx} - \theta\gamma_{tx} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tx}} \\
&= -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tx}}. \qquad (4.34)
\end{aligned}
$$

4.3.1 Determining the counterterm

In (4.34), S_{ct} is the yet to be determined counterterm for the gravitational action which should make the action itself and the stress tensor finite in the large r limit. Without the counterterm, we see that, for instance, the stress tensor components T_{tt} and T_{xx} diverge. Therefore, it is necessary to have the counterterm. We assume the form as given in (4.20) for the counterterm:

$$
S_{ct} = \int_{\partial M_r} \sqrt{-\gamma}\, dtdx(a + b^{(2)}R), \qquad (4.35)
$$

where ${}^{(2)}R$ is the Ricci scalar of the metric on the boundary hypersurface, $\gamma_{\mu\nu}$, and the coefficients a and b have to be chosen so as to cancel the divergences. The variation of S_{ct} can be expressed as

$$
\begin{aligned}
\delta S_{ct} &= \int_{\partial M_r} dtdx\Big(\delta\sqrt{-\gamma}(a + b^{(2)}R) + b\sqrt{-\gamma}\delta^{(2)}R\Big) \\
&= \int_{\partial M_r} dtdx\Big(-\frac{1}{2}\sqrt{-\gamma}\gamma_{\mu\nu}\delta\gamma^{\mu\nu}(a + b^{(2)}R) + b\sqrt{-\gamma}\left({}^{(2)}R_{\mu\nu}\delta\gamma^{\mu\nu} + \gamma^{\mu\nu}\delta^{(2)}R_{\mu\nu}\right)\Big) \\
&= \int_{\partial M_r} \sqrt{-\gamma} dtdx\Big(\delta\gamma^{\mu\nu}\left(b\left({}^{(2)}R_{\mu\nu} - \frac{1}{2}\gamma^{(2)}_{\mu\nu}R\right) - \frac{1}{2}a\gamma_{\mu\nu}\right) + b\gamma^{\mu\nu}\delta^{(2)}R_{\mu\nu}\Big) \\
&= \int_{\partial M_r} \sqrt{-\gamma} dtdx\Big(\delta\gamma^{\mu\nu}\left(b^{(2)}G_{\mu\nu} - \frac{1}{2}a\gamma_{\mu\nu}\right) + b\gamma^{\mu\nu}\delta^{(2)}R_{\mu\nu}\Big), \qquad (4.36)
\end{aligned}
$$

where ${}^{(2)}R_{\mu\nu}$ and ${}^{(2)}G_{\mu\nu}$ are the Ricci tensor and the Einstein tensor of $\gamma_{\mu\nu}$ respectively. Since the Einstein tensor vanishes identically for a 2 dimensional metric, we have ${}^{(2)}G_{\mu\nu} = 0$. Also, since the last term in the paranthesis is independent of $\delta\gamma^{\mu\nu}$ (cf. (3.9)), we see that

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{\mu\nu}} = -a\gamma_{\mu\nu}. \tag{4.37}$$

To fix the coefficient a, we apply the above result to make T_{tt} in (4.34) finite (zero in this case) in the large r limit:

$$8\pi G_N T_{tt} = -\frac{r^2}{l^3} + a\left(-\frac{r^2}{l^2}\right) = 0, \tag{4.38}$$

which is satisfied only when we have $a = -\frac{1}{l}$. Also, since the Einstein tensor identically vanishes in 2 dimensions, we see that the coefficient b maybe arbitrary; for convenience, we set it to zero.

Thus, the counterterm in the action for asymptotically AdS_3 spaces is given by

$$S_{ct} = \left(-\frac{1}{l}\right)\int_{\partial M_r}\sqrt{-\gamma}\, dtdx. \tag{4.39}$$

We also see that, with the above definition, the components T_{xx} and T_{tx} also vanish, and thus we see that the quasilocal stress tensor for AdS_3 vanishes. Also, the stress tensor is free of divergences since it vanishes everywhere.

4.3.2 General formula for asymptotically AdS_3 spaces and an application

Since the counterterm is the same for all asymptotically AdS_3 spacetimes, we apply the definition of quasilocal stress tensor given in (4.19) to compute and verify the masses and momenta of some of these spacetimes. We assume these spacetimes to have the form

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}(-dt^2 + dx^2) + \delta g_{ab}dx^a dx^b, \tag{4.40}$$

where δg_{ab} are functions of $r,\ t,$ and x, and $x^a = r,\ t$ or x. We see that the induced metric at a constant r hypersurface $\partial\mathcal{M}_r$ (denote in by Σ) is given by

$$ds_\Sigma^2 = \frac{r^2}{l^2}(-dt^2 + dx^2) + \delta g_{MN} dx^M dx^N, \tag{4.41}$$

where $x^M = t$ or x.

Let us calculate the components of the extrinsic curvature of the hypersurface $\partial\mathcal{M}_r$. The normal vector to the hypersurface is given by

$$\begin{aligned} n_\mu &= \frac{\partial_\mu r}{\sqrt{g^{\mu\nu}\partial_\mu r \partial_\nu r}} \\ &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}}, \end{aligned} \tag{4.42}$$

where $g^{rr} = \frac{1}{g_{rr}}$. Therefore, $\frac{1}{\sqrt{g^{rr}}} = \sqrt{g_{rr}}$. Since $g_{rr} = \frac{l^2}{r^2} + \delta g_{rr}$, by taylor expansion to first order in δg_{rr}, we see that

$$n_\mu = \left(\frac{l}{r} + \frac{r}{2l}\delta g_{rr}\right)\delta_{\mu,r}. \tag{4.43}$$

Analogous to (4.29), we see that

$$\begin{aligned} \theta_{tt} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{tt} \\ &= -\frac{1}{2}\left(\frac{l}{r} + \frac{r}{2l}\delta g_{rr}\right)\left(\frac{r^2}{l^2} - \frac{r^4}{l^4}\delta g_{rr}\right)\partial_r\left(-\frac{r^2}{l^2} + \delta g_{tt}\right) \\ &= \frac{r^2}{l^3} - \frac{r^4}{2l^5}\delta g_{rr} - \frac{r}{2l}\partial_r \delta g_{tt}, \end{aligned} \tag{4.44}$$

where the RHS is upto terms first order in δg_{ab}. Also, $g^{rr} = \frac{1}{g_{rr}} = \left(\frac{l^2}{r^2} + \delta g_{rr}\right)^{-1}$ is obtained by Taylor expansion to first order in δg_{rr} as

$$g^{rr} = \frac{r^2}{l^2} - \frac{r^4}{l^4}\delta g_{rr}, \tag{4.45}$$

which is made use of in (4.44).

Similarly, the other components of the extrinsic curvature are given by

$$\begin{aligned}
\theta_{xx} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{xx} \\
&= -\frac{1}{2}\left(\frac{l}{r} + \frac{r}{2l}\delta g_{rr}\right)\left(\frac{r^2}{l^2} - \frac{r^4}{l^4}\delta g_{rr}\right)\partial_r\left(\frac{r^2}{l^2} + \delta g_{xx}\right) \\
&= -\frac{r^2}{l^3} + \frac{r^4}{2l^5}\delta g_{rr} - \frac{r}{2l}\partial_r \delta g_{xx} \\
\theta_{tx} = \theta_{xt} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{tx} \\
&= -\frac{1}{2}\left(\frac{l}{r} + \frac{r}{2l}\delta g_{rr}\right)\left(\frac{r^2}{l^2} - \frac{r^4}{l^4}\delta g_{rr}\right)\partial_r \delta g_{tx} \\
&= -\frac{r}{2l}\partial_r \delta g_{tx}.
\end{aligned} \tag{4.46}$$

The scalar extrinsic curvature (to first order in δg_{ab}) is given by

$$\begin{aligned}
\theta &= \gamma^{ab}\theta_{ab} = \gamma^{tt}\theta_{tt} + \gamma^{xx}\theta_{xx} + 2\gamma^{tx}\theta_{tx} \\
&= \frac{1}{\gamma}\left(\left(\frac{r^2}{l^2} + \delta g_{xx}\right)\left(\frac{r^2}{l^3} - \frac{r^4}{2l^5}\delta g_{rr} - \frac{r}{2l}\partial_r \delta g_{tt}\right) + \left(-\frac{r^2}{l^2} + \delta g_{tt}\right)\left(-\frac{r^2}{l^3} + \frac{r^4}{2l^5}\delta g_{rr} - \frac{r}{2l}\partial_r \delta g_{xx}\right)\right) \\
&= \frac{1}{\gamma}\left(\frac{2r^4}{l^5} - \frac{r^6}{l^7}\delta g_{rr} + \frac{r^3}{2l^3}\partial_r(\delta g_{xx} - \delta g_{tt}) + \frac{r^2}{l^3}(\delta g_{xx} - \delta g_{tt})\right),
\end{aligned} \tag{4.47}$$

where γ^{ab} is the inverse of the induced metric on $\partial\mathcal{M}_r$ and γ is the determinant of the induced metric γ_{ab}.

The determinant, upto first order in δg_{ab}, is given by

$$\gamma = -\frac{r^4}{l^4} + \frac{r^2}{l^2}(\delta g_{tt} - \delta g_{xx}), \tag{4.48}$$

which is a function of $\delta g_{tt} - \delta g_{xx}$. Therefore, one can express the inverse of the determinant as well as a Taylor series in $\delta g_{tt} - \delta g_{xx}$, which to first order in the same is given by

$$\frac{1}{\gamma} = -\frac{l^4}{r^4} - \frac{l^6}{r^6}(\delta g_{tt} - \delta g_{xx}). \tag{4.49}$$

Plugging the above equation in (4.47), we find the scalar extrinsic curvature (to first order in δg_{ab}) to be

$$\theta = -\frac{2}{l} + \frac{r^2}{l^3}\delta g_{rr} + \frac{l}{2r}\partial_r(\delta g_{xx} - \delta g_{tt}) + \frac{l}{r^2}(\delta g_{xx} - \delta g_{tt}). \tag{4.50}$$

Using the above results, we see that the components of the quasilocal stress tensor are given by

$$\begin{aligned}
8\pi G_N T_{tt} &= \theta_{tt} - \left(\theta + \frac{1}{l}\right)\gamma_{tt} \\
&= \frac{r^4}{2l^5}\delta g_{rr} + \frac{\delta g_{xx}}{l} - \frac{r}{2l}\partial_r \delta g_{xx} \\
8\pi G_N T_{xx} &= \theta_{xx} - \left(\theta + \frac{1}{l}\right)\gamma_{xx} \\
&= \frac{\delta g_{tt}}{l} - \frac{r}{2l}\partial_r \delta g_{tt} - \frac{r^4}{2l^5}\delta g_{rr} \\
8\pi G_N T_{tx} &= \theta_{tx} - \left(\theta + \frac{1}{l}\right)\gamma_{tx} \\
&= \frac{\delta g_{tx}}{l} - \frac{r}{2l}\partial_r \delta g_{tx}.
\end{aligned} \tag{4.51}$$

As is obvious from (4.23) and (4.24), the expression for the mass of the spacetime with metric described by (4.40) is given by

$$\begin{aligned}
M &= \int dx \sqrt{\sigma_{xx}} N_\Sigma \, u^\mu u^\nu T_{\mu\nu} \\
&= \int dx \sqrt{g_{xx}} N_\Sigma \, u^t u^t T_{tt}.
\end{aligned} \tag{4.52}$$

In the above repression, the lapse function N_Σ is given by

$$N_\Sigma = (-\gamma^{tt})^{-\frac{1}{2}} = \sqrt{-\frac{\gamma}{g_{xx}}}, \tag{4.53}$$

where γ is the determinant of the induced metric on $\partial\mathcal{M}_r$. Also, the timelike normal to Σ is given by

$$u_\mu = \frac{\delta_{\mu,t}}{\sqrt{\gamma^{tt}}} = \sqrt{\frac{\gamma}{g_{xx}}}\delta_{\mu,t}. \tag{4.54}$$

Plugging the above two equations in (4.52), we see that the mass of the spacetime is given by

$$M = \int dx\, T_{tt}, \tag{4.55}$$

which clearly diverges in the absence of the counterterm. For a spacetime with metric of the form in (4.40), we see that its mass is given by

$$M = \frac{1}{8\pi G_N} \int dx \left[\frac{r^4}{2l^5}\delta g_{rr} + \frac{\delta g_{xx}}{l} - \frac{r}{2l}\partial_r \delta g_{xx} \right]. \tag{4.56}$$

A similar calculation reveals that the momentum of such a spacetime is given by

$$\begin{aligned} P_x &= -\int dx\, T_{tx} \\ &= -\frac{1}{8\pi G_N} \int dx \left[\frac{\delta g_{tx}}{l} - \frac{r}{2l} - \partial_r \delta g_{tx} \right]. \end{aligned} \tag{4.57}$$

As an application of the above results, we consider the BTZ metric [18, 19] given by

$$ds^2 = -N^2 dt^2 + \rho^2 (d\phi + N^\phi dt)^2 + \frac{r^2}{N^2 \rho^2} dr^2, \tag{4.58}$$

where

$$\begin{aligned} \rho^2 &= r^2 + 4GMl^2 - \frac{1}{2} r_+^2, & r_+^2 &= 8Gl\sqrt{M^2 l^2 - J^2}, \\ N^2 &= \frac{r^2 (r^2 - r_+^2)}{l^2 \rho^2}, & N^\phi &= -\frac{4GJ}{\rho^2}, \end{aligned} \tag{4.59}$$

and ϕ has a period of 2π. The above metric can be rearranged in the form

$$ds^2 = \left(\frac{l^2}{r^2 - r_+^2} \right) dr^2 + \frac{dt^2}{\rho^2} \left(\frac{r^2}{l^2}(r_+^2 - r^2) + (4GJ)^2 \right) + \left(\frac{\rho^2}{l^2} \right) dx^2 - \left(\frac{8GJ}{l} \right) dt dx. \tag{4.60}$$

Now, in the limit of large r, one can approximate

$$\frac{1}{\rho^2} \approx \frac{1}{r^2} - \frac{1}{r^4} \left(4GMl^2 + \frac{1}{2} r_+^2 \right) \tag{4.61}$$

$$\frac{1}{r^2 - r_+^2} \approx \frac{1}{r^2} + \frac{r_+^2}{r^4}. \tag{4.62}$$

Plugging the above 2 expressions in (4.60), we see that the BTZ metric looks like

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}(-dt^2 + dx^2) + \frac{8GMl^4}{r^4}dr^2 + 8GMdt^2 - \frac{8GJ}{l}dtdx \tag{4.63}$$

in the limit $J \ll Ml$. Therefore, when $J \ll Ml$, in the large r limit, we see that the metric looks like the one given in (4.40), where

$$\delta g_{rr} = \frac{8GMl^4}{r^4}, \qquad \delta g_{tt} = 8GM, \qquad \delta g_{tx} = -\frac{4GJ}{l}. \tag{4.64}$$

Substituting $x = l\phi$ with $\int dx \to l\int_0^{2\pi} d\phi$, we see from (4.56) that the spacetime mass is given by

$$M = \frac{l}{8\pi G}\int_0^{2\pi} d\phi \left[\frac{r^4}{2l^5}\frac{8GMl^4}{r^4}\right] = M, \tag{4.64}$$

which is consistent with the observed result. Similarly, from (4.57), we see that the momentum of the spacetime is given by

$$\begin{aligned} P_\phi &= -\frac{l}{8\pi G}\int_0^{2\pi} d\phi \left[-\frac{4GJ}{l}\right] \\ &= J, \end{aligned} \tag{4.65}$$

which again is in agreement with the expected result.

4.3.3 Reproducing Brown and Henneaux's result

Brown and Henneaux in 1993 have shown that the 2 dimensional conformal field theory dual to gravity in asymptotically AdS_3 spacetime has central charge equal to $c = \frac{3l}{2G}$ [20]. Since we interpret the quasilocal stress tensor to be equivalent to the expectation value of the stress tensor of the CFT, we can verify the above fact by studying the transformation of the stress tensor under diffeomorphisms which preserve the asymptotic form of the metric of AdS_3.

Consider the metric of Poincare AdS expressed in the form

$$ds^2 = \frac{l^2}{r^2}dr^2 - r^2 dx^+ dx^-, \tag{4.66}$$

where

$$x^+ = \frac{t+x}{l} \text{ and } x^- = \frac{t-x}{l}. \tag{4.67}$$

The conformal field theory can be thought as living on the boundary of AdS with metric $ds^2 = -r^2 dx^+ dx^-$, with r eventually taken to infinity. Diffeomorphisms of the form

$$x^+ \longrightarrow x^+ - \xi^+(x^+), \qquad x^- \longrightarrow x^- - \xi^-(x^-) \tag{4.68}$$

transform the stress tensor on the boundary as

$$\begin{aligned} T_{++} &\longrightarrow T_{++} + 2\partial_+\xi^+ T_{++} + \xi^+\partial_+ T_{++} - \frac{c}{24\pi}\partial_+^3\xi^+ \\ T_{--} &\longrightarrow T_{--} + 2\partial_-\xi^- T_{--} + \xi^-\partial_- T_{--} - \frac{c}{24\pi}\partial_-^3\xi^-, \end{aligned} \tag{4.69}$$

where in either of the equations, the last term arises from an effect that is of quantum origin while the other terms just follow from the transformation rules for a tensor under coordinate transformations. Although, (4.68) is a translation and thus classically a symmetry of the CFT, at the quantum mechanical level, it fails to be one since they introduce a conformal factor in the metric [15]. Brown and Henneaux observed that only those diffeomorphisms that leave the asymptotic form of the boundary metric of AdS invariant correspond to symmetries of the CFT (at the quantum level) for which the metric of AdS is bound to satisfy the following fall-offs [20]:

$$\begin{aligned} g_{rr} &= \frac{l^2}{r^2} + \mathcal{O}(\frac{1}{r^4}), \qquad g_{++} = \mathcal{O}(1), \qquad g_{--} = \mathcal{O}(1), \\ g_{+-} &= -\frac{r^2}{2} + \mathcal{O}(1), \qquad g_{+r} = \mathcal{O}(\frac{1}{r^3}), \qquad g_{-r} = \mathcal{O}(\frac{1}{r^3}). \end{aligned} \tag{4.70}$$

The coordinate transformations that preserve these conditions are given by

$$\begin{aligned} x^+ &\longrightarrow x^+ - \xi^+ - \frac{l^2}{2r^2}\partial_-^2\xi^- \\ x^- &\longrightarrow x^- - \xi^- - \frac{l^2}{2r^2}\partial_+^2\xi^+ \\ r &\longrightarrow r + \frac{r}{2}(\partial_+\xi^+ + \partial_-\xi^-). \end{aligned} \tag{4.71}$$

For large r, the transformations of x^+ and x^- look like those in (4.68) since the extra corrections in the above equations are negligilble. Under the above diffeomorphisms, the metric of AdS_3 transforms as

$$ds^2 \longrightarrow \frac{l^2}{r^2}dr^2 - r^2 dx^+ dx^- - \frac{l^2}{2}(\partial_+^3 \xi^+)(dx^+)^2 - \frac{l^2}{2}(\partial_-^3 \xi^-)(dx^-)^2. \tag{4.72}$$

As this transformation leaves the metric invariant in the asymptotic limit, it is a symmetry. For the above form (4.72) of the metric, we compute the quasilocal stress tensor as follows.

The metric in (4.72) could be equivalently expressed in terms of the usual Poincare coordinates of AdS as

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}(-dt^2 + dx^2) - \frac{1}{2}(\partial_+^3 \xi^+ + \partial_-^3 \xi^-)(dt^2 + dx^2) + (\partial_-^3 \xi^- - \partial_+^3 \xi^+)dtdx, \tag{4.73}$$

from which we identify that

$$\delta g_{tt} = \delta g_{xx} = -\frac{1}{2}(\partial_+^3 \xi^+ + \partial_-^3 \xi^-), \qquad \delta g_{tx} = \frac{1}{2}(\partial_-^3 \xi^- - \partial_+^3 \xi^+). \tag{4.74}$$

Therefore, from (4.51), we see that the components of the quasilocal stress tensor are given by

$$\begin{aligned} 8\pi G T_{tt} &= -\frac{1}{2l}(\partial_+^3 \xi^+ + \partial_-^3 \xi^-) \\ 8\pi G T_{xx} &= -\frac{1}{2l}(\partial_+^3 \xi^+ + \partial_-^3 \xi^-) \\ 8\pi G T_{tx} &= \frac{1}{2l}(\partial_-^3 \xi^- - \partial_+^3 \xi^+). \end{aligned} \tag{4.75}$$

Now, from the tensor transformation rules, on going from (r, t, x) coordinates to (r, x^+, x^-) coordinates, we see that

$$\begin{aligned} T_{++} &= (\partial_+ t)^2 \, T_{tt} + (\partial_+ x)^2 \, T_{xx} + 2(\partial_+ t)(\partial_+ x) T_{tx} \\ T_{--} &= (\partial_- t)^2 \, T_{tt} + (\partial_- x)^2 \, T_{xx} + 2(\partial_- t)(\partial_- x) T_{tx}. \end{aligned} \tag{4.76}$$

From (4.67), we see that

$$\partial_+ t = \partial_+ x = \partial_- t = \frac{l}{2}, \qquad \partial_- x = -\frac{l}{2}. \tag{4.77}$$

Therefore, using the above 3 equations, we have

$$\begin{aligned} T_{++} &= \frac{l^2}{32\pi G}\left[-\frac{1}{l}(\partial_+^3\xi^+ + \partial_-^3\xi^-)\right] + \frac{l^2}{16\pi G}\left[\frac{1}{2l}(\partial_-^3\xi^- - \partial_+^3\xi^+)\right] \\ &= -\frac{l}{16\pi G}\partial_+^3\xi^+, \end{aligned} \tag{4.78}$$

and

$$\begin{aligned} T_{--} &= \frac{l^2}{32\pi G}\left[-\frac{1}{l}(\partial_+^3\xi^+ + \partial_-^3\xi^-)\right] - \frac{l^2}{16\pi G}\left[\frac{1}{2l}(\partial_-^3\xi^- - \partial_+^3\xi^+)\right] \\ &= -\frac{l}{16\pi G}\partial_-^3\xi^-. \end{aligned} \tag{4.79}$$

Since these transformation rules should match with those given in (4.69), we see that

$$-\frac{c}{24\pi} = -\frac{l}{16\pi G} \iff c = \frac{3l}{2G}, \tag{4.80}$$

which matches perfectly with the result obtained by Brown and Henneaux.

4.3.4 An important observation and its consequences

We see that with the Brown-Henneaux fall-offs, the variation of the Gibbons-Hawking-York term for asymptotically AdS spacetimes vanishes. The expression for the variation of the GHY term is given by (3.21). We consider the metric of AdS_3 as given in (4.66) with the fall-offs given by Brown and Henneaux as in (4.70). We consider a constant r timelike hypersurface for which $\epsilon = 1$. For ease of calculation, we consider the inverse metric whose components are comprised solely of the leading order behaviour in r. This would mean that we are considering the inverse of the exact AdS_3 metric in (4.66). As seen before, the (normalized) normal vector to the boundary hypersurface is given by

$$\begin{aligned} n^\mu &= g^{rr} n_r \delta^{\mu,r} \\ &= \frac{r}{l}\delta^{\mu,r}. \end{aligned} \tag{4.81}$$

With the above ingredients, the variation of the GHY term is given by

$$\delta S_{GHY} = \int_{\partial M_r} \sqrt{-\gamma}\, dx^+ dx^- \gamma^{\alpha\beta} n^\mu \partial_\mu \delta g_{\alpha\beta}. \tag{4.82}$$

The above expression evaluates to

$$\begin{aligned}\delta S_{GHY} &= 2\int_{\partial M_r} \sqrt{-\gamma}\, dx^+ dx^- h^{+-}\, n^r \partial_r \delta g_{+-} \\ &= 2\int_{\partial M_r} dx^+ dx^- \left(-\frac{2}{r^2}\right) \mathcal{O}(\frac{1}{r^2}) \left(\frac{r}{l}\right) \left(\frac{r^2}{2}\right), \\ &\sim \mathcal{O}(\frac{1}{r}), \end{aligned} \tag{4.83}$$

which vanishes in the limit of large r. Here, we have only considered the leading order dependence of r while calculating the determinant of the metric which goes like r^2. Also, since $g_{+-} = -\frac{r^2}{2} + \mathcal{O}(1)$, we see that $\delta g_{+-} \sim \mathcal{O}(\frac{1}{r^2})$. Therefore, we see that the Gibbons-Hawking-York term is not really a mandatory component of the gravitational action for asymptotically AdS_3 spaces; we might as well drop it since the only role it plays is in making the variational problem well defined by cancelling of the boundary term in the variation of the Einstein-Hilbert action which we anyways observe to be zero. This means that we might be able to obtain standard results even from a quasilocal stress tensor derived from the action obtained by dropping the Gibbons-Hawking-York term.

4.4 Quasilocal stress tensor for asymptotically AdS_4 spaces

4.4.1 AdS_4 in Poincare coordinates

Consider the AdS_4 metric in Poincare coordinates:

$$ds^2 = \frac{l^2}{r^2} dr^2 + \frac{r^2}{l^2}(-dt^2 + dx_1^2 + dx_2^2). \tag{4.84}$$

The induced metric on a timelike hypersurface $\partial\mathcal{M}_r$ (denote it by Σ) defined by a constant value of r is given by

$$ds_\Sigma^2 = \frac{r^2}{l^2}(-dt^2 + dx_1^2 + dx_2^2), \tag{4.85}$$

where r is the constant value of the radial coordinate.

From the ADM decomposition of the induced metric, we see that the lapse function N and the shift vector field N^a are given by

$$N = \sqrt{-\gamma_{tt}} = \frac{r}{l}, \qquad\qquad N^a = \frac{\gamma^{at}}{N^2} = 0. \tag{4.86}$$

From (4.23), the mass of the spacetime can be observed to be

$$\begin{aligned} M &= \int_{\partial\Sigma} dx_1 dx_2 \left(\frac{r^2}{l^2}\right)\left(\frac{r}{l}\right)(u^t)^2 T_{tt} \\ &= \int_{\partial\Sigma} dx_1 dx_2 \,\frac{r}{l} T_{tt}, \end{aligned} \tag{4.87}$$

where, in going from the first to the second equation, we use

$$\sqrt{-\sigma} = \frac{r^2}{l^2}, \qquad (u^t)^2 = -\frac{1}{g_{tt}} = -g^{tt} = \frac{l^2}{r^2}. \tag{4.88}$$

Also, the region of integration $\partial\Sigma$ is a spacelike surface in the boundary hypersurface $\partial\mathcal{M}_r$ of AdS_4 with metric σ_{ab}. Therefore, from (4.87), we expect $T_{tt} \sim \frac{1}{r}$ for the mass to not diverge at large r. The (normalized) normal vector to the hypersurface $\partial\mathcal{M}_r$ is given by

$$\begin{aligned} n_\mu &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} \\ &= \frac{l}{r}\delta_{\mu,r}. \end{aligned} \tag{4.89}$$

Analogous to those in (4.29), the components of the extrinsic curvature of the hypersurface $\partial\mathcal{M}_r$ are given by

$$\begin{aligned} \theta_{tt} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{tt} \\ &= \frac{r^2}{l^3} \\ \theta_{x_i x_j} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{x_i x_j} \\ &= -\frac{r^2}{l^3}\delta_{ij} \\ \theta_{t x_i} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{t x_i} \\ &= 0. \end{aligned} \tag{4.90}$$

The scalar extrinsic curvature is given by

$$\begin{aligned}
\theta &= \gamma^{tt}\theta_{tt} + \gamma^{x_ix_j}\theta_{x_ix_j} + \gamma^{tx_i}\theta_{tx_i} \\
&= \left(-\frac{l^2}{r^2}\right)\left(\frac{r^2}{l^3}\right) + 2\left(\frac{l^2}{r^2}\right)\left(-\frac{r^2}{l^3}\right) \\
&= -\frac{3}{l}. \qquad (4.91)
\end{aligned}$$

Therefore, the components of the quasilocal stress tensor are given by

$$\begin{aligned}
8\pi G_N T_{tt} &= \theta_{tt} - \theta\gamma_{tt} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\
&= \frac{r^2}{l^3} - \left(-\frac{3}{l}\right)\left(-\frac{r^2}{l^2}\right) - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\
&= -2\frac{r^2}{l^3} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\
8\pi G_N T_{x_ix_j} &= \theta_{x_ix_j} - \theta\gamma_{x_ix_j} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{x_ix_j}} \\
&= -\frac{r^2}{l^3}\delta_{ij} - \left(-\frac{3}{l}\right)\left(\frac{r^2}{l^2}\right)\delta_{ij} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{x_ix_j}} \\
&= 2\frac{r^2}{l^3}\delta_{ij} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{x_ix_j}} \\
8\pi G_N T_{tx_i} &= \theta_{tx_i} - \theta\gamma_{tx_i} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tx_i}} \\
&= -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tx_i}}. \qquad (4.92)
\end{aligned}$$

In the above expressions, $\gamma_{\mu\nu}$ is the induced metric on the hypersurface $\partial\mathcal{M}_r$. It is easily seen that the counterterm that cancels off the divergences in the stress tensor components is given by

$$S_{ct} = \left(-\frac{2}{l}\right)\int_{\partial\mathcal{M}_r}\sqrt{-\gamma}\,dt dx_1 dx_2, \qquad (4.93)$$

and we see that incorporating the above counterterm in the action for AdS_4 renders $T_{\mu\nu} = 0$.

4.4.2 AdS_4 in global coordinates

Consider the metric of AdS_4 in global coordinates:

$$ds^2 = -\left(1+\frac{r^2}{l^2}\right)dt^2 + \frac{dr^2}{\left(1+\frac{r^2}{l^2}\right)} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.94}$$

The induced metric on a constant r timelike hypersurface $\partial\mathcal{M}_r$ (denote it by Σ) is given by

$$ds_\Sigma^2 = -\left(1+\frac{r^2}{l^2}\right)dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.95}$$

In global coordinates, after doing an ADM decomposition of the above boundary metric, we see that the lapse function N and the shift vector field N^a are given by

$$N = \sqrt{-\gamma_{tt}} = \sqrt{1+\frac{r^2}{l^2}}, \qquad N^a = \frac{\gamma^{at}}{N^2} = 0. \tag{4.96}$$

Also, the other ingredients that make up the RHS of (4.23) are given by

$$\sqrt{-\sigma} = r^2\sin\theta, \qquad (u^t)^2 = -\frac{1}{g_{tt}} = -g^{tt} = \left(1+\frac{r^2}{l^2}\right)^{-1}. \tag{4.97}$$

Using the above information, we see that the mass of the spacetime, in the limit of large r, looks like

$$\begin{aligned} M &= \int_{\partial\Sigma} d\theta d\phi\, r^2\sin\theta\left(\frac{r^2}{l^2}\right)^{-\frac{1}{2}} T_{tt} \\ &= \int_{\partial\Sigma} \sin\theta\, d\theta d\phi\,(lr)T_{tt}. \end{aligned} \tag{4.98}$$

The same expression can be used for determining the mass of asymptotically AdS spacetimes in the limit of large r. The (normalized) normal vector to the hypersurface $\partial\mathcal{M}_r$ is given by

$$\begin{aligned} n_\mu &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} \\ &= \frac{\delta_{\mu,r}}{\sqrt{1+\frac{r^2}{l^2}}}. \end{aligned} \tag{4.99}$$

The components of the extrinsic curvature of the hypersurface $\partial\mathcal{M}_r$ are given by

$$\begin{aligned}
\theta_{tt} &= -\frac{1}{2}n_r g^{rr}\partial_r g_{tt} \\
&= \frac{r}{l^2}\sqrt{1+\frac{r^2}{l^2}} \\
\theta_{\theta\theta} &= -\frac{1}{2}n_r g^{rr}\partial_r g_{\theta\theta} \\
&= -r\sqrt{1+\frac{r^2}{l^2}} \\
\theta_{\phi\phi} &= -\frac{1}{2}n_r g^{rr}\partial_r g_{\phi\phi} \\
&= -r\sin^2\theta\sqrt{1+\frac{r^2}{l^2}}.
\end{aligned} \tag{4.100}$$

The trace of the extrinsic curvature is given by

$$\begin{aligned}
\theta &= \gamma^{tt}\theta_{tt} + \gamma^{\theta\theta}\theta_{\theta\theta} + \gamma^{\phi\phi}\theta_{\phi\phi} \\
&= -\frac{r}{l^2}\left(1+\frac{r^2}{l^2}\right)^{-\frac{1}{2}} - \frac{2}{r}\sqrt{1+\frac{r^2}{l^2}},
\end{aligned} \tag{4.101}$$

where $\gamma_{\mu\nu}$ is the metric on the hypersurface $\partial\mathcal{M}_r$. Now, the difference between the 2 coordinate systems shows up in the counterterms that needs to be added in either case; as opposed to the counterterm in (4.93) for the case of Poincare coordinates, we see that the corresponding counterterm in the action when the metric is expressed in global coordinates is given by

$$S_{ct} = \int_{\partial\mathcal{M}_r}\sqrt{-\gamma}\left(-\frac{2}{l}+\frac{l\,{}^{(3)}R}{2}\right), \tag{4.102}$$

where ${}^{(3)}R$ is the Ricci scalar of the boundary metric $\gamma_{\mu\nu}$. From (4.36), we see that the contribution to the quasilocal stress tensor arising from this counterterm is given by

$$\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{\mu\nu}} = \frac{2}{l}\gamma_{\mu\nu} + lG_{\mu\nu}. \tag{4.103}$$

Finally, we see that the components of the quasilocal stress tensor are given by

$$8\pi G_N T_{tt} = \theta_{tt} - \theta\gamma_{tt} - \frac{2}{l}\gamma_{tt} - lG_{tt}$$

$$
\begin{aligned}
&= \left(1+\frac{r^2}{l^2}\right)\left[-\frac{2}{r}\sqrt{1+\frac{r^2}{l^2}}+\frac{2}{l}+\frac{l}{r^2}\right] \\
&= \frac{l}{4r^2}+\frac{l^3}{8r^4}+\cdots \\
8\pi G_N T_{\theta\theta} &= \theta_{\theta\theta}-\theta\gamma_{\theta\theta}-\frac{2}{l}\gamma_{\theta\theta}-lG_{\theta\theta} \\
&= r\sqrt{1+\frac{r^2}{l^2}}+\frac{r^3}{l^2}\left(1+\frac{r^2}{l^2}\right)^{-\frac{1}{2}}-\frac{2r^2}{l} \\
&= \frac{l^3}{4r^2}-\frac{l^5}{4r^4}+\cdots \\
8\pi G_N T_{\phi\phi} &= \theta_{\phi\phi}-\theta\gamma_{\phi\phi}-\frac{2}{l}\gamma_{\phi\phi}-lG_{\phi\phi} \\
&= 8\pi G_N T_{\theta\theta}\sin^2\theta \\
&= \frac{l^3}{4r^2}\sin^2\theta-\frac{l^5}{4r^4}\sin^2\theta+\cdots . \qquad (4.104)
\end{aligned}
$$

In evaluating the above expressions, we expand the quantities within the square root around $r=\infty$. These expansions look like

$$
\begin{aligned}
\sqrt{1+\frac{r^2}{l^2}} &= \frac{r}{l}+\frac{l}{2r}-\frac{l^3}{8r^3}+\frac{l^5}{16r^5}+\mathcal{O}\left(\left(\frac{1}{r}\right)^7\right) \\
\left(1+\frac{r^2}{l^2}\right)^{-\frac{1}{2}} &= \frac{l}{r}-\frac{l^3}{2r^3}+\frac{3l^5}{8r^5}-\frac{5l^7}{16r^7}+\mathcal{O}\left(\left(\frac{1}{r}\right)^9\right). \qquad (4.105)
\end{aligned}
$$

4.4.3 An example calculation: AdS_4 Schwarzschild metric

We consider the metric of AdS_4 Schwarzschild in global coordinates:

$$
ds^2 = -\left(1+\frac{r^2}{l^2}-\frac{r_0}{r}\right)dt^2+\left(1+\frac{r^2}{l^2}-\frac{r_0}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \qquad (4.106)
$$

The induced metric on a constant r timelike hypersurface $\partial\mathcal{M}_r$ (denote it by Σ) is given by

$$
ds^2_\Sigma = -\left(1+\frac{r^2}{l^2}-\frac{r_0}{r}\right)dt^2+r^2(d\theta^2+\sin^2\theta d\phi^2). \qquad (4.107)
$$

The (normalized) normal vector to this hypersurface is given by

$$\begin{aligned} n_\mu &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} \\ &= \frac{\delta_{\mu,r}}{\sqrt{1+\frac{r^2}{l^2}-\frac{r_0}{r}}}. \end{aligned} \tag{4.108}$$

The components of the extrinsic curvature of this hypersurface are given by

$$\begin{aligned} \theta_{tt} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{tt} \\ &= \left(\frac{r}{l^2}+\frac{r_0}{2r^2}\right)\sqrt{1+\frac{r^2}{l^2}-\frac{r_0}{r}} \\ \theta_{\theta\theta} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{\theta\theta} \\ &= -r\sqrt{1+\frac{r^2}{l^2}-\frac{r_0}{r}} \\ \theta_{\phi\phi} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{\phi\phi} \\ &= -r\sin^2\theta\sqrt{1+\frac{r^2}{l^2}-\frac{r_0}{r}}. \end{aligned} \tag{4.109}$$

The scalar extrinsic curvature is given by

$$\begin{aligned} \theta &= \gamma^{tt}\theta_{tt} + \gamma^{\theta\theta}\theta_{\theta\theta} + \gamma^{\phi\phi}\theta_{\phi\phi} \\ &= -\left(\frac{r}{l^2}+\frac{r_0}{2r^2}\right)\left(1+\frac{r^2}{l^2}-\frac{r_0}{r}\right)^{-\frac{1}{2}} - \frac{2}{r}\sqrt{1+\frac{r^2}{l^2}-\frac{r_0}{r}}, \end{aligned} \tag{4.110}$$

where $\gamma_{\mu\nu}$ is the metric on the hypersurface $\partial\mathcal{M}_r$.

The tt component of the quasilocal stress tensor is given by

$$\begin{aligned} 8\pi G_N T_{tt} &= \theta_{tt} - \theta\gamma_{tt} - \frac{2}{l}\gamma_{tt} - lG_{tt} \\ &= \left(1+\frac{r^2}{l^2}-\frac{r_0}{r}\right)\left[-\frac{2}{r}\sqrt{1+\frac{r^2}{l^2}-\frac{r_0}{r}}+\frac{2}{l}\right] - l\left(\frac{1}{r^2}+\frac{1}{l^2}\right) \\ &= \frac{r_0}{lr} + \cdots , \end{aligned} \tag{4.111}$$

where, in writing the finl expression on the RHS, we use the following expansion around $\frac{1}{r} = 0$:

$$\sqrt{1 + \frac{r^2}{l^2} - \frac{r_0}{r}} = \frac{r}{l} + \frac{l}{2r} - \frac{r_0 l}{2r^2} - \frac{l^3}{8r^3} + \frac{r_0 l^3}{4r^4} + \mathcal{O}\left(\left(\frac{1}{r}\right)^6\right). \tag{4.112}$$

Applying the above result in (4.98), we calculate the mass of the spacetime as

$$\begin{aligned} M &= \frac{1}{8\pi G_N}\int_{\partial\Sigma} \sin\theta \, d\theta d\phi \, (lr)\frac{r_0}{lr} \\ &= \frac{4\pi r_0}{8\pi G_N} \\ &= \frac{r_0}{2G_N}. \end{aligned} \tag{4.113}$$

Since r_0 is the radius of the black hole horizon (or in other words, the Schwarzschild radius), we see that the result we have obtained is consistent with what one would expect for the mass of a black hole.

4.5 Quasilocal stress tensor for asymptotically AdS_5 spaces

Consider the metric of AdS_5 in Poincare coordinates:

$$ds^2 = \frac{l^2}{r^2}dr^2 + \frac{r^2}{l^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2). \tag{4.114}$$

The metric induced on a constant r hypersurface $\partial\mathcal{M}_r$ (denote it by Σ) is given by

$$ds_\Sigma^2 = \frac{r^2}{l^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2). \tag{4.115}$$

From the ADM decomposition of the metric on Σ, we see that the lapse function N and the shift vector field N^a are given by

$$N = \sqrt{-\gamma_{tt}} = \frac{r}{l}, \qquad N^a = \frac{\gamma^{at}}{N^2} = 0, \tag{4.116}$$

where $\gamma_{\mu\nu}$ is the induced metric on the hypersurface $\partial\mathcal{M}_r$.

Applying the above equation in (4.23), we see that the mass of the spacetime is given by

$$
\begin{aligned}
M &= \int_{\partial\Sigma} dx_1 dx_2 dx_3 \left(\frac{r^3}{l^3}\right)\left(\frac{r}{l}\right)(u^t)^2 T_{tt} \\
&= \int_{\partial\Sigma} dx_1 dx_2 dx_3 \, \frac{r^2}{l^2} T_{tt},
\end{aligned} \tag{4.117}
$$

where we use $\sqrt{-\sigma} = \frac{r^3}{l^3}$ and $(u^t)^2 = -\gamma^{tt} = -g^{tt}$ in going from the first to the second equation. Therefore, for the mass to not to diverge at large r, we expect $T_{tt} \sim r^{-2}$ in the limit of large r.

We see that that the (normalized) normal vector to the hypersurface $\partial\mathcal{M}_r$ is given by

$$
\begin{aligned}
n_\mu &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} \\
&= \frac{l}{r}\delta_{\mu,r}.
\end{aligned} \tag{4.118}
$$

We see that the components of the extrinsic curvature of the hypersurface $\partial\mathcal{M}_r$ are given by

$$
\begin{aligned}
\theta_{tt} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{tt} \\
&= \frac{r^2}{l^3} \\
\theta_{x_i x_j} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{x_i x_j} \\
&= -\frac{r^2}{l^3}\delta_{ij} \\
\theta_{t x_i} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{t x_i} \\
&= 0,
\end{aligned} \tag{4.119}
$$

which are exactly the same as those for AdS_4 in Poincare coordinates. However, difference arises in the scalar extrinsic curvatures, which for the case of AdS_5 is given by

$$
\begin{aligned}
\theta &= \gamma^{tt}\theta_{tt} + \gamma^{x_i x_j}\theta_{x_i x_j} + \gamma^{t x_i}\theta_{t x_i} \\
&= \left(-\frac{l^2}{r^2}\right)\left(\frac{r^2}{l^3}\right) + 3\left(\frac{l^2}{r^2}\right)\left(-\frac{r^2}{l^3}\right) \\
&= -\frac{4}{l}.
\end{aligned} \tag{4.120}
$$

As expected, this is different from that for AdS_4 given in (4.91). The components of the quasilocal stress tensor are given by

$$\begin{aligned} 8\pi G_N T_{tt} &= \theta_{tt} - \theta\gamma_{tt} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\ &= -\frac{3r^2}{l^3} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{tt}} \\ 8\pi G_N T_{x_i x_j} &= \theta_{x_i x_j} - \theta\gamma_{x_i x_j} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{x_i x_j}} \\ &= \frac{3r^2}{l^3} - \frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{x_i x_j}} \\ 8\pi G_N T_{t x_i} &= -\frac{2}{\sqrt{-\gamma}}\frac{\delta S_{ct}}{\delta\gamma^{t x_i}}. \end{aligned} \tag{4.121}$$

From the above equation, we see that a finite stress tensor for AdS_5 in Poincare coordinates is obtained by picking the counterterm

$$S_{ct} = \left(-\frac{3}{l}\right)\int_{\partial\mathcal{M}_r}\sqrt{-\gamma}\,dtd^3x. \tag{4.122}$$

On the other hand, analogous to the case of AdS_4, we see that in global coordinates, an extra term is required in the counterterm to cancel off the divergences in the quasilocal stress tensor components. We see that the counterterm in the case of AdS_5 in global coordinates is given by

$$S_{ct} = \int_{\partial\mathcal{M}_r}\sqrt{-\gamma}\,dtd^3x\left(-\frac{3}{l} + \frac{l}{4}\,{}^{(4)}R\right), \tag{4.123}$$

where ${}^{(4)}R$ is the Ricci scalar of the induced metric $\gamma_{\mu\nu}$ on the boundary hypersurface $\partial\mathcal{M}_r$. We verify that this is indeed the case with the following examples.

4.5.1 Example calculations

As the first example, we consider the metric that arises in the near-horizon limit of the D3-brane:

$$ds^2 = \frac{r^2}{l^2}\left[-\left(1-\frac{{r_0}^4}{r^4}\right)dt^2 + \sum_{i=1}^{3}(dx_i)^2\right] + \left(1-\frac{{r_0}^4}{r^4}\right)^{-1}\frac{l^2}{r^2}dr^2. \tag{4.124}$$

The induced metric on a constant r hypersurface $\partial\mathcal{M}_r$ (denote it by Σ) is given by

$$ds_{\Sigma}^2 = \frac{r^2}{l^2}\left[-\left(1-\frac{{r_0}^4}{r^4}\right)dt^2 + \sum_{i=1}^{3}(dx_i)^2\right]. \tag{4.125}$$

The (normalized) normal vector to this hypersurface is given by

$$\begin{aligned} n_\mu &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} \\ &= \frac{l}{r}\left(1-\frac{{r_0}^4}{r^4}\right)^{-1}\delta_{\mu,r}. \end{aligned} \tag{4.126}$$

The components of the extrinsic curvature of this hypersurface can be calculated as

$$\begin{aligned} \theta_{tt} &= -\frac{1}{2}n_r g^{rr}\partial_r g_{tt} \\ &= -\frac{1}{2}\left(\frac{l}{r}\right)\left(1-\frac{{r_0}^4}{r^4}\right)^{-1}\left(\frac{r^2}{l^2}\right)\left(1-\frac{{r_0}^4}{r^4}\right)\partial_r\left[\left(-\frac{r^2}{l^2}\right)\left(1-\frac{{r_0}^4}{r^4}\right)\right] \\ &= \left(\frac{r^4+{r_0}^4}{r^2 l^3}\right)\sqrt{1-\frac{{r_0}^4}{r^4}} \\ \theta_{x_i x_i} &= -\frac{1}{2}n_r g^{rr}\partial_r g_{x_i x_i} \\ &= -\frac{1}{2}\left(\frac{l}{r}\right)\left(1-\frac{{r_0}^4}{r^4}\right)^{-1}\left(\frac{r^2}{l^2}\right)\left(1-\frac{{r_0}^4}{r^4}\right)\partial_r\left[\frac{r^2}{l^2}\right] \\ &= -\frac{r^2}{l^3}\sqrt{1-\frac{{r_0}^4}{r^4}}. \end{aligned} \tag{4.127}$$

The scalar extrinsic curvature is given by

$$\begin{aligned} \theta &= \gamma^{tt}\theta_{tt} + \gamma^{x_i x_i}\theta_{x_i x_i} \\ &= \left(-\frac{3}{l}-\frac{1}{l}\left(\frac{r^4+{r_0}^4}{r^4-{r_0}^4}\right)\right)\sqrt{1-\frac{{r_0}^4}{r^4}}. \end{aligned} \tag{4.128}$$

Finally, we see that the components of the quasilocal stress tensor are given by

$$\begin{aligned} 8\pi G_N T_{tt} &= \theta_{tt} - \theta\gamma_{tt} - \frac{3}{l}\gamma_{tt} \\ &= \frac{3(r^4-{r_0}^4)}{l^3 r^3}\left[r - \sqrt{\frac{r^4-{r_0}^4}{r^2}}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{3{r_0}^4}{2l^3r^2} + \cdots \\
8\pi G_N T_{x_ix_i} &= \theta_{x_ix_i} - \theta\gamma_{x_ix_i} - \frac{3}{l}\gamma_{x_ix_i} \\
&= \frac{1}{\sqrt{r^4 - {r_0}^4}}\left(\frac{3r^4}{l^3} - \frac{{r_0}^4}{l^3}\right) - \frac{3r^2}{l^3} \\
&= \frac{{r_0}^4}{2l^3r^2} + \cdots , \qquad (4.129)
\end{aligned}
$$

where to arrive at the final expression on the RHS for the first and the second equations, we use the following expansions (around $r = \infty$ or equivalently around $\frac{1}{r} = 0$) respectively:

$$
\begin{aligned}
\sqrt{\frac{r^4 - {r_0}^4}{r^2}} &= r - \frac{{r_0}^4}{2r^3} - \frac{{r_0}^8}{8r^7} - \frac{{r_0}^{12}}{16r^{11}} + \mathcal{O}\left(\left(\frac{1}{r}\right)^{13}\right) \\
\frac{1}{\sqrt{r^4 - {r_0}^4}} &= \frac{1}{r^2} + \frac{{r_0}^4}{2r^6} + \frac{3{r_0}^8}{8r^{10}} + \mathcal{O}\left(\left(\frac{1}{r}\right)^{13}\right). \qquad (4.130)
\end{aligned}
$$

Using the expression for T_{tt}, we see that the mass of the spacetime is given by

$$
M = \frac{3{r_0}^4}{16\pi G_N l^5}\int_{\partial\Sigma} d^3x, \qquad (4.131)
$$

which is in agreement with the standard formula [21].

As the next example, we consider the AdS$_5$-Schwarzschild metric:

$$
ds^2 = -\left[1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2\right]dt^2 + \frac{dr^2}{\left[1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2\right]} + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2). \qquad (4.132)
$$

The induced metric on a constant r timelike hypersurface $\partial\mathcal{M}_r$ (denote it by Σ) is given by

$$
ds^2 = -\left[1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2\right]dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2). \qquad (4.133)
$$

The (normalized) normal vector to this hypersurface is given by

$$
\begin{aligned}
n_\mu &= \frac{\delta_{\mu,r}}{\sqrt{g^{rr}}} \\
&= \frac{\delta_{\mu,r}}{\sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2}}. \qquad (4.134)
\end{aligned}
$$

The extrinsic curvature components of Σ are given by

$$
\begin{aligned}
\theta_{tt} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{tt} \\
&= \left(\frac{r}{l^2} + \frac{{r_0}^2}{r^3} \right) \sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2} \\
\theta_{\theta\theta} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{\theta\theta} \\
&= -r \sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2} \\
\theta_{\phi\phi} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{\phi\phi} \\
&= -r \sin^2\theta \sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2} \\
\theta_{\psi\psi} &= -\frac{1}{2} n_r g^{rr} \partial_r g_{\psi\psi} \\
&= -r \cos^2\theta \sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2}.
\end{aligned} \tag{4.135}
$$

The trace of the extrinsic curvature of Σ is given by

$$
\begin{aligned}
\theta &= \gamma^{tt}\theta_{tt} + \gamma^{\theta\theta}\theta_{\theta\theta} + \gamma^{\phi\phi}\theta_{\phi\phi} + \gamma^{\psi\psi}\theta_{\psi\psi} \\
&= -\left(\frac{r}{l^2} + \frac{{r_0}^2}{r^3} \right) \frac{1}{\sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2}} - \frac{3}{r} \sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2},
\end{aligned} \tag{4.136}
$$

where $\gamma_{\mu\nu}$ is the metric induced on the hypersurface $\partial\mathcal{M}_r$.

The tt component of the quasilocal stress tensor can be calculated as

$$
\begin{aligned}
8\pi G_N T_{tt} &= \theta_{tt} - \theta\gamma_{tt} - \frac{3}{l}\gamma_{tt} - \frac{l}{2} G_{tt} \\
&= \frac{3(r^4 - {r_0}^4)}{l^3 r^3} \left[r - \sqrt{\frac{r^4 - {r_0}^4}{r^2}} \right] \\
&= \frac{3r^2}{l^3} - \left(\frac{3r}{l^2} + \frac{3}{r} - \frac{3r_0^2}{r^3} \right) \sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r} \right)^2} + \frac{3lr_0^2}{2r^4} - \frac{3r_0^2}{r^2} - \frac{3l}{2r^2} + \frac{3}{2l} \\
&= \frac{3l}{8r^2} + \frac{3r_0^2}{2lr^2} + \cdots ,
\end{aligned} \tag{4.137}
$$

where $G_{\mu\nu}$ is the Einstein tensor of $\gamma_{\mu\nu}$ and is given by $G_{tt} = \frac{3}{r^2}\left(1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2\right)$. In writing the final expression on the RHS, we make use of the following expansion around $\frac{1}{r} = 0$:

$$\sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2} = \frac{r}{l} + \frac{l}{2r} - \frac{l^4 + 4l^2 r_0^2}{8lr^3} + \mathcal{O}\left(\left(\frac{1}{r}\right)^5\right). \tag{4.138}$$

Applying the expression for T_{tt} in (4.23), in the limit of large r, we calculate the mass of the spacetime as

$$\begin{aligned} M &= \int_{\partial\Sigma} d\theta d\phi d\psi\, r^3 \sin\theta \cos\theta \frac{1}{\sqrt{1 + \frac{r^2}{l^2} - \left(\frac{r_0}{r}\right)^2}} \left(\frac{3l}{8r^2} + \frac{3r_0^2}{2lr^2}\right) \\ &= \frac{3\pi l^2}{32 G_N} + \frac{3\pi r_0^2}{8 G_N}, \end{aligned} \tag{4.139}$$

where we use

$$\sqrt{-\sigma} = r^3 \sin\theta\cos\theta, \qquad N = \sqrt{-\gamma_{tt}}, \qquad (u^t)^2 = -\gamma^{tt} = -\frac{1}{\gamma_{tt}}. \tag{4.140}$$

We see that the mass we have obtained has a term in addition to the standard observed value of $3\pi r_0^2/8G_N$ [21]. The additional term corresponds to the mass of empty AdS_5 when $r_0 = 0$.

Chapter 5

Conclusion

The thesis started out by presenting a brief review of the essential aspects about AdS/CFT. We saw that the isometry group of AdS and the symmetry group of AdS are the same thing. We also looked at particle dynamics in AdS, both classical and quantum, and understood the interpretation of CFT primary and descendant operators in AdS. Following this, we saw how conformal invariance constrains the form of CFT correlation functions. In particular, we employed the definition of AdS/CFT dictionary to explicitly compute CFT 2-point correlators.

In the third chapter, we reviewed the Lagrangian formalism of general relativity. We understood that the Einstein-Hilbert action alone does not give rise to a well defined variational principle. Also, neither is it finite in the limit of large distances. Hence the need for the Gibbons-Hawking-York boundary term and the nondynamical counterterm. We concluded the chapter by giving the most explicit form of the action for a spacetime metric.

In the last chapter, we understood why defining a stress-energy tensor for the metric of a spacetime manifold is hard. We studied in detail the proposal of quasilocal stress tensor for a spacetime by Brown and York and how it applies to asymptotically AdS spaces as investigated by Kraus and Balasubramanian. We computed the masses and momenta of various spacetimes using the stress tensor and saw that we indeed were getting the right results. We studied the interpretation of this quantity from the point of view of CFT. We explicitly showed the result obtained by Brown and Henneaux and saw that for metrics that satisfy these fall-offs, one may drop the GHY term in their gravitational action, thus leading to a slightly different definition of the quasilocal stress tensor. This may pave the way for obtaining the standard results obtained before, nevertheless starting with a slightly different definition of the stress tensor.

Bibliography

[1] G. 't Hooft, (1993), "Dimensional reduction in quantum gravity", arXiv:gr-qc/9310026.

[2] L. Susskind, (1995), "The world as a hologram", Journal of Mathematical Physics, 36(11), 6377-6396 [arXiv:hep-th/9409089].

[3] J. Maldacena, (1999), "The large-N limit of superconformal field theories and supergravity", International journal of theoretical physics, 38(4), 1113-1133 [arXiv:hep-th/9711200].

[4] E. Witten, (1998), "Anti-de Sitter space and holography", Adv. Theor. Math. Phys.2:253 [arXiv:hep-th/9802150].

[5] D. Harlow, (2018), "TASI Lectures on the Emergence of Bulk Physics in AdS/CFT", PoS TASI 2017 002 [arXiv:1802.01040 [hep-th]].

[6] J. Kaplan, (2013), "Lectures on AdS/CFT from the bottom up", Notes for Fall.

[7] J. Penedones, (2017), "TASI lectures on AdS/CFT", in New Frontiers in Fields and Strings: TASI 2015 Proceedings of the 2015 Theoretical Advanced Study Institute in Elementary Particle Physics (pp. 75-136) [arXiv:1608.04948 [hep-th]].

[8] P. Ginsparg, (1988), "Applied conformal field theory", arXiv:hep-th/9108028.

[9] T. Osborne, (2018), Video lectues on conformal field theory.

[10] H. Liu, (2014), "String Theory and Holographic Duality", Massachusetts Institute of Technology: MIT OpenCourseWare, Lecture Notes.

[11] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, (2000), "Large N field theories, string theory and gravity", Physics Reports, 323(3-4), 183-386 [arXiv:hep-th/9905111].

[12] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, (1998), "Gauge theory correlators from non-critical string theory", Physics Letters B, 428(1-2), 105-114 [arXiv:hep-th/9802109].

[13] V. Balasubramanian, P. Kraus and A. Lawrence, (1999), "Bulk versus boundary dynamics in anti–de Sitter spacetime", Physical Review D, 59(4), 046003 [arXiv:hep-th/9805171].

[14] J. D. Brown and J. W. York Jr, (1993), "Quasilocal energy and conserved charges derived from the gravitational action", Physical Review D, 47(4), 1407, DOI: 10.1103/PhysRevD.47.1407.

[15] V. Balasubramanian and P. Kraus, (1999), "A stress tensor for anti-de Sitter gravity", Communications in Mathematical Physics, 208(2), 413-428 [arXiv:hep-th/9902121].

[16] M. Blau, (2020), Lecture Notes on General Relativity.

[17] E. Poisson, "A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics", (Cambridge, Cambridge University Press, 2004).

[18] M. Banados, C. Teitelboim and J. Zanelli, (1992), "Black hole in three-dimensional spacetime", Physical Review Letters, 69(13), 1849 [arXiv:hep-th/9204099].

[19] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, (1993), "Geometry of the 2+1 black hole", Physical Review D, 48(4), 1506 [arXiv:gr-qc/9302012].

[20] J. D. Brown and M. Henneaux, (1986), "Central charges in the canonical realization of asymptotic symmetries: an example from three dimensional gravity", Communications in Mathematical Physics, 104(2), 207-226, DOI: 10.1007/BF01211590.

[21] G. T. Horowitz and R. C. Myers, (1998), "AdS-CFT correspondence and a new positive energy conjecture for general relativity", Physical Review D, 59(2), 026005 [arXiv:hep-th/9808079].