A proof of The Hodge Decomposition Theorem and some applications

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Certificate

This is to certify that this dissertation entitled A proof of The Hodge Decomposition Theorem and some applications towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rithwik S V at Indian Institute of Science, Bengaluru under the supervision of Dr. Harish Seshadri, Associate Professor, Department of Mathematics, during the academic year 2019-2020.

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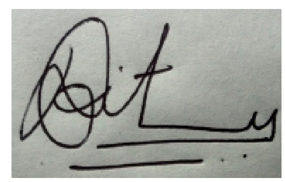
This thesis is dedicated to my family and friends for their unending encouragement.

Declaration

I hereby declare that the matter embodied in the report entitled A proof of The Hodge Decomposition Theorem and some applications are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science, Bengaluru. under the supervision of Dr. Harish Seshadri and the same has not been submitted elsewhere for any other degree.



Rithwik S V

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Abstract

The goal of the project is to prove the Hodge decomposition theorem for compact Riemannian manifolds. This theorem states that any smooth differential form on such a manifold can be expressed in a unique way as a sum of a harmonic form, a closed form and a co closed form. The proof involves a study of elliptic differential operators on manifolds. We will see applications of this theorem can be used to prove some theorems like the Poincaré duality and the Kunneth Formulae.

Contents

Introduction

Let us start with two finite dimensional vector spaces U and V and a linear map T between them. We have the adjoint of this map, T^*. Now what can we tell about the solutions of the equation $Tu = v$. One necessary condition for the existence of the solution would be that $v \perp Ker(T^*)$, since if $x \in ker(T^*)$, then $\langle v, x\rangle = \langle Tu, x\rangle = \langle u, T^*x\rangle = 0$, had a solution existed. In fact, for the finite dimensional case this is in fact a sufficient condition as it can be shown without a lot of trouble that $Im(T) = (Ker(T^*))^{\perp}$. The Hodge Decomposition Theorem is a generalisation of this simple theorem.

Consider the the differential operator $L := \frac{d}{dx^2}$ for smooth function on the real line, this is just the map that sends a function f to its second derivative f''. This is a linear operator. Let us work with the space of functions with period 2π so that we can define an inner product on this space as, $\langle f, g\rangle = \int_0^{2\pi} fgdx$. Then L is a linear operator, which we observe using integration by parts, is in fact self-adjoint. What is the kernel of L? $u'' = 0$ forces u to be linear of the form $u = ax + b$, but it has to be 2π periodic, this constrains $a = 0$. Hence $Ker(L) = constants$.

Suppose we are trying to look for a solution of $Lu = f$. We cannot use the arguments in the previous paragraphs as we are working with infinite dimensional spaces. However integrating both sides from 0 to x, we see $u'(x) = u'(0) + \int_0^x f$. Now suppose we have a solution, then it has to be periodic, and $u'(0) = u'(2\pi) \implies \int_0^x f = 0$. Recall that L is a self adjoint operator $Ker(L^*) = Ker(L) = constants$. Clearly $\int_0^x f = 0 \implies f \perp Ker(L)$. Hence we have a necessary condition, which in fact turns out to be sufficient. Hence, even in this infinite dimensional case, $Lu = f$ has a solution iff $f \perp Ker(L)$.

This is precisely the Hodge Decomposition Theorem where your manifold is $\mathbb{R}$. For a general manifold, L turns out to be what is called the Hodge-Laplacian, Δ which is a self adjoint linear operator on smooth forms. The Hodge-Decomposition Theorem states that $\Delta\omega = \alpha$ has a solution precisely when $\alpha \in (Ker(\Delta))^{\perp}$. However there are a lot of subtle things that need to be done since we don't have a well defined notion of a PDE on a manifold to start with, or an inner product on the space of smooth forms.

The goal of this project is to give a self contained proof of the Hodge Decomposition Theorem. This is an exposition closely following the proof outlined in [1]. The result has far reaching consequences in getting information about the cohomology groups of the manifold which is an algebraic quantity and as such not an easy quantity to calculate. Below is a brief outline of the chapters.

The proof involves a study of PDE on manifolds. But what does it mean to have a PDE on a manifold? Is it independent of the co-ordinate system? Does it match with our usual notions when we consider $\mathbb{R}^n$ as a smooth manifold. All these questions are dealt with in the first chapter. The analysis of PDE involves a inevitable study of Sobolev Spaces. This is a vast theory and studying it in it's generality will sufficiently take us off course. Hence we limit our study to the case of periodic functions which is far easier to deal with. The second chapter involves a brief section on Fourier Analysis, as you would've guessed, as does any theory on periodic functions, then we define Sobolev spaces and see generalisations of some things from calculus.

The third chapter deals with defining PDE and looking at some properties of the solutions. Most of these are a direct consequence A special class of PDE, called the elliptic PDE. In order to work with and study PDE e develop the theory of Sobolev Spaces, not in generality, but for periodic functions, which is much easier due to the techniques of Fourier Analysis. We give a brief exposition on the same(Fourier Analysis. We will see this is sufficient for our purpose for compact manifolds. The final chapter gives us the proof the main analytical theorems that will be used to prove the Hodge-Decomposition Theorem. This is a self contained read, with the reader assumed to have a working understanding of functional analysis and differential geometry along with the elementary notions of a Riemannian metric and the concept of partitions of unity.

Chapter 1

The Hodge Decomposition Theorem

The Hodge decomposition theorem is an important result in differential geometry about certain representatives of the de Rham cohomology class. This theorem tells us that we can always find certain "nice" representatives called Harmonic forms for each class.These are "nice" in the sense that these representatives minimizes some sort of energy function on p-forms. This is very useful in calculating the cohomology groups of manifolds which is not a trivial task. Using something called the Bochner Technique we can conclude results about the cohomology classes given some conditions on the curvature, for instance if the curvature of a compact oriented manifold is positive then the cohomology class is trivial since there exist no harmonic forms.

Now we would like to define the Laplacian on the manifolds, but it is not straight forward how to define PDE's on manifolds, let alone the Laplacian, because the usual notion of partial derivatives seems to change with the choice of co ordinate system on the manifold. However we have an exterior derivative d which is a well-defined operator resembling some sort of derivative which acts on forms, and functions are nothing but zero forms. We will define the δ operator which is the adjoint of d, and then the Laplacian using these two linear operators. We will see that defining the Laplacian like this in fact coincides with the usual definition of the Laplacian on functions in $\mathbb{R}^n$.

1.1 The $*$ operator

Let V be a finite dimensional vector space of dimension n, with an inner product $\langle . , . \rangle$. Since we have an inner product, we can use this to get an isomorphism between V and V^* and as a result $\Lambda^p V \simeq \Lambda^p V^*$. Since $Hom(V, W) = V^* \otimes W$ and we already have an isomorphism between $\Lambda^p V$ and $\Lambda^p V^*$, the inverse gives an element $Hom(\Lambda^p V^*, \Lambda^p V)$, we get an element of $\Lambda^p V^* \otimes \Lambda^p V$. But an element here is nothing but an inner product on $\Lambda^p V$. We have used the fact that $\Lambda^p V^* = (\Lambda^p V)^*$ This inner product on $\Lambda^p(V)$ is follows:

$$\langle w_1 \wedge w_2 \wedge w_p \ldots \wedge_p, v_1 \wedge v_2 \wedge \ldots \wedge v_p \rangle = \det \langle w_i, v_j \rangle$$

where w_i's and v_i's are in V. Extend this bilinearly to whole of $\Lambda^p(V)$.

The inner product defined on $\Lambda^p(V)$ as above is clearly independent of the initial vectors you start with, since we are using the inner product on V and there is no reference to any basis. Now if $\mathfrak{B} = \{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $V, \langle . , . \rangle$, then $\mathfrak{B}_p = \{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} : e_{i_k} \in \mathfrak{B}, i_1 < i_2 < \ldots < i_p\}$ is an orthonormal basis for $\Lambda^p V$ with the inner product defined as above. This follows directly from the definition of the inner product on $\Lambda^p V$

Definition 1.1.1 (The * operator)**.** Let V be an n dimensional vector space with an inner product $\langle . , . \rangle$. Choose an orthonormal basis of V, $\mathfrak{B} = \{e_1, e_1, \ldots, e_n\}$. Now define $*$ as follows:

$$* : \Lambda^p(V) \to \Lambda^{n-p}(V)$$

$$*(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = (-1)^{sgn(I)} e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}}$$

where $\{j_1, j_2, \ldots, j_{n-p}\}$ is the complement of $\{i_1, i_2, \ldots, i_p\}$ in $\{1, 2, \ldots, n\}$, that is,

$$\{1, 2, \ldots, n\} \backslash \{i_1, i_2, \ldots, i_p\} = \{j_1, j_2, \ldots, j_{n-p}\}$$

and $sgn(I)$ is the sign of the permutation I, which takes $\{1, 2, \ldots, n\}$ to $\{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_{n-p}\}$ Another way to look at is just as the complement in the standard volume form.

$$(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) \wedge *(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = (-1)^{sgn(I)} e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p} \wedge (e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}}) = e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

First we need to show that this definition is well-defined. What if we start with another basis $\mathfrak{B}^{'} = \{e_1', e_2', \ldots, e_n'\}$? Note we have to put an extra condition that this basis needs to be the same orientation $\mathfrak{B}$ since we used this to define the sign of the map. We can find a $n \times n$ matrix A such that $Ae_i = e_i'$. A will be an orthogonal matrix since with positive determinant since we have orthogonal basis of the same orientation. Now,

$$*(e_{i_1}' \wedge e_{i_2}' \wedge \ldots \wedge e_{i_p}') = *(Ae_{i_1} \wedge Ae_{i_2} \wedge \ldots \wedge Ae_{i_p}) = det(A) * (e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p})$$

Since $det(A) = 1$, we conclude that the $*$ operator is well-defined.

Lemma 1.1.1. *The $*$ operator satisfies the following property:* $** = (-1)^{(n)(n-p)}$

Proof. Suppose $*(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}}$ or in other words

$$(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) \wedge e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}} = e_1 \wedge e_2 \wedge \ldots \wedge e_n \tag{1.1}$$

Here we have used up the factor of $(-1)^{sgn(I)}$ to reorder the basis elements of the image. Now, clearly $* * (e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = (-1)^{?}(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p})$, i.e it gives back the same element(since complement of a complement is the same thing) upto a sign which we have to determine. This is done by:

$$*(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) \wedge * * (e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

$$e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}} \wedge (-1)^{?}(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = e_1 \wedge e_2 \wedge \ldots \wedge e_n$$

Notice that from 1.1, we just need to show

$$e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}} \wedge (-1)^{?}(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = (e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) \wedge e_{j_1} \wedge e_{j_2} \wedge \ldots \wedge e_{j_{n-p}}$$

Now we need to move each e_{i_k} across $n-p$ terms of e_{j_l}. There are p e_{i_k}'s. Since each step gives us a factor of -1, we see that the the we need to multiply by $(-1)^{p(n-p)}$. Hence $* * (e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p}) = (-1)^{p(n-p)}(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_p})$ □

This operator seems to have popped out of nowhere. But in fact it is closely related to the inner product that we started out with. We already have some hints towards some relation as the definition of $*$ involved choosing an orthonormal basis.

Lemma 1.1.2. $\langle v, w \rangle vol_M = *(v \wedge *w)$

Proof. We just have to verify this for the basis elements, due to linearity it will follow for all vectors. Suppose we start with two distinct basis vectors v and w, then there will be some common e_i between v and $*w$, and due to this $*(v \wedge *w) = 0$. Since v and w are orthogonal $\langle v, w \rangle = 0$. Next when we take the same basis vectors, $*(v \wedge *v) = *(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = 1$. Also $\langle v, v \rangle = 1$ since it's an orthonormal basis. Hence by verification we have proved the lemma. □

We have the inner product as above on $\Lambda^p(T_x M)$ for each $x \in M$. Given two smooth p forms ω, η, we can define

$$\langle \omega, \eta \rangle(x) = \langle \omega(x), \eta(x) \rangle$$

Using this we can define an inner product on $\Omega^p(M)$ as follows:

$$\langle \omega, \eta \rangle = \int_M \langle \omega, \eta \rangle(x) vol_M = \int_M *(\omega(x) \wedge *\eta(x))$$

1.2 The δ operator

The δ operator is the formal adjoint of the exterior derivative, d, where $\Omega^p(M)$ is equipped with the inner product as defined as above.

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle$$

where $\omega \in \Omega^p(M)$ and $\eta \in \Omega^{p+1}M$

Definition 1.2.1 (The δ operator)**.** The δ operator is defined as follow:

$$\delta : \Omega^{p+1}(M) \to \Omega^p(M)$$

$$\delta = (-1)^{n(p+2)+1} * d*$$

and δ satisfies:

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle$$

for all $\omega \in \Omega^p(M)$ and $\eta \in \Omega^{p+1}M$. For zero forms it is just the zero map.

We will show that with the definition of δ as above it is in fact the adjoint of d. Take

$\alpha \in \Omega^{p-1}M$ and $\beta \in \Omega^p M$. Then

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1}\alpha \wedge d * \beta$$

Since $** = (-1)^{(p-1)(n-p+1)}$ acting on $d * \beta$, we have:

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1}(-1)^{(p-1)(n-p+1)}\alpha \wedge * * d * \beta$$

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta - \alpha \wedge *\delta\beta$$

Now integrate both sides and using stokes theorem we obtain $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$

1.3 The Hodge Laplacian

We can use d and δ to define a symmetric linear operator on the space of smooth p-forms.

Definition 1.3.1 (The Hodge Laplacian)**.** The Hodge Laplacian denoted by Δ, is defined as follows:

$$\Delta : \Omega^p(M) \to \Omega^p(M)$$

$$\Delta = d\delta + \delta d$$

Lemma 1.3.1. *Properties of* Δ

1. *The Hodge Laplacian,* Δ *is a symmetric linear form on the space of smooth p-forms.*
2. $\Delta\phi = 0 \iff d\phi = 0$ *and* $\delta\phi = 0$
3. *The Hodge star commutes with the Laplacian,* $*\Delta = \Delta*$

Proof. 1. Let $\alpha, \beta \in \Omega^p(M)$. Clearly Δ is linear as it is the composition of linear operators.

$$\begin{aligned}\langle \Delta\alpha, \beta\rangle &= \langle (d\delta + \delta d)\alpha, \beta\rangle \\ &= \langle d\delta\alpha, \beta\rangle + \langle \delta d\alpha, \beta\rangle \\ &= \langle \alpha, \delta d\beta\rangle + \langle \alpha, d\delta\beta\rangle \\ &= \langle \alpha, (\delta d + d\delta)\beta\rangle \\ &= \langle \alpha, \Delta\beta\rangle\end{aligned}$$

This shows that it is symmetric.

2. Observe the following:

$$\begin{aligned}\langle \Delta\phi, \phi\rangle &= \langle d\delta\phi + \delta d\phi, \phi\rangle \\ &= \langle d\delta\phi, \phi\rangle + \langle \delta d\phi, \phi\rangle \\ &= \langle \delta\phi, \delta\phi\rangle + \langle d\phi, d\phi\rangle \\ &= ||\delta\phi||^2 + ||d\phi||^2\end{aligned}$$

It follows directly from the above equality and the definition of Δ.

3. Straightforward calculation using the definition of δ and that result $** = (-1)^{(n)(n-p)}$.

$\square$

Okay now we have defined the Hodge-Laplacian. The least we can expect is for this operator to be equal to the usual Laplacian on $\mathbb{R}^n$ for functions or zero forms with the euclidean metric. We will show that this is indeed that case. We will also calculate it for smooth p forms and notice that it is just the Laplacian of each of the components. We have a global basis for the tangnet bundle and co-tangent bundle given by $\{\frac{\partial}{\partial dx_1}, \frac{\partial}{\partial dx_2}, \ldots, \frac{\partial}{\partial dx_n}\}$ and $\{dx_1, dx_2, \ldots, dx_n\}$. $\delta f = 0$ since it is a zero form. Next we need to calculate δdf, that is $(-1)^{2n+1} * d * df$ (Since df is a one form). Hence $\delta df = - * d * df$

$$\begin{aligned}
df &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i \\
*df &= *(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i) = \sum_{i=1}^{n} *(\frac{\partial f}{\partial x_i} dx^i) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x_i} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n \\
d * df &= d(\sum_{i=1}^{n} (-1)^{i-1} \frac{\partial f}{\partial x_i} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n) \\
&= \sum_{i=1}^{n} (-1)^{i-1} d(\frac{\partial f}{\partial x_i} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n) \qquad (1.2) \\
&= \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial^2 f}{\partial x_i^2} dx^i \wedge dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \ldots \wedge dx^n) \\
&= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \\
- * d * df &= - * (\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n) \\
&= - \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}
\end{aligned}$$

Hence

$$d\delta f + \delta df = - \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

So we get back our usual Laplacian with a negative sign. The calculations for p forms can be similarly be computed and we will see that for $\omega = \sum_I \omega_I dx^I$ where $I = (i_1, \ldots, i_p)$ is a multi-index and $dx^I = dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_p}$. Then

$$\Delta \omega = (d\delta + \delta d)\omega = \sum_I (d\delta + \delta d)\omega_I dx^I = \sum_I \omega_I' dx^I$$

where

$$\omega_I' = - \sum_{k=1}^{p} \frac{\partial^2 \omega_I}{\partial x_{i_k}^2}$$

for $I = \{i_1, \ldots, i_p\}$

Definition 1.3.2 (Harmonic p-forms)**.** The space of harmonic p-forms in just the kernel of the Hodge-Laplacian. It is denoted by H^p.

$$H^p = \{\omega \in \Omega^p(M) : \Delta\omega = 0\}$$

We have now developed the the various definitions and operators for stating the Hodge theorem.

1.4 The Hodge Decomposition Theorem and Applications

Theorem 1.4.1 (The Hodge Decomposition Theorem)**.** *Given any integer $0 \leq p \leq n$, the space of harmonic forms is finite dimensional and we have the following orthogonal direct sum decomposition of the space $\Omega^p(M)$ of smooth forms on a compact oriented Riemannian manifold, M:*

$$\Omega^p(M) = \Delta(\Omega^p(M)) \oplus H^p(M)$$

Consequently, the equation $\Delta\omega = \alpha$ has a solution $\omega \in \Omega^p(M)$ if and only if the p-form α is orthogonal to the space of harmonic p-forms.

Once we have this theorem we now show that any de-Rham cohomology class has a unique harmonic representative.

Corollary 1.4.1.1. *Every de-Rham cohomology class of a compact oriented manifold contains a unique harmonic representative. Hence the de-Rham cohomology groups are all finite dimensional.*

Proof. Let $[\omega]$ be some p-th de-Rham cohomology class. Take an element in this class, $\alpha \in [\omega]$. Now from Theorem 1.4.1 we see that $\alpha = \Delta\phi + \eta$, where $\phi \in \Omega^p(M)$ and $\eta \in H^p$. Hence

$$\alpha = \Delta\phi + \eta = d\delta\phi + \delta d\phi + \eta$$

We apply d on both sides. Since $d^2 = 0$ and $\Delta\eta = 0 \implies d\eta = 0$, and α is a closed form, we obtain:

$$\begin{aligned}
0 &= d\delta d\phi \\
||\delta d\phi||^2 &= \langle \delta d\phi, \delta d\phi \rangle \\
&= \langle d\delta d\phi, d\phi \rangle \\
&= \langle 0, d\phi \rangle \\
&= 0
\end{aligned}$$

Hence $||\delta d\phi||^2 = 0 \implies \delta d\phi = 0$. Now since

$$\alpha = d\delta\phi + \delta d\phi + \eta = d\delta\phi + \eta$$

Observe that α and η differ by a closed form. Hence η is the desired harmonic representative in $[\omega]$. Now we show that this form is unique. Let η_1, η_2 be two harmonic forms in the same class. Then $\eta_1 - \eta_2$ is a harmonic forms and they differ by a closed form, i.e, $\eta_1 - \eta_2 = d\alpha$. Now,

$$\begin{aligned}
||\eta_1 - \eta_2||^2 &= \langle \eta_1 - \eta_2, \eta_1 - \eta_2 \rangle \\
&= \langle \eta_1 - \eta_2, d\alpha \rangle \\
&= \langle \delta(\eta_1 - \eta_2), \alpha \rangle \\
&= \langle 0, \alpha \rangle \\
&= 0
\end{aligned}$$

Hence $\eta_1 - \eta_2 = 0$ which means that $\eta_1 = \eta_2$. Hence the harmonic representative is unique. The finite dimensionality of the de-Rham cohomology group follows as the space of harmonic forms is finite dimensional and from above we have a unique harmonic representative for each class. □

Corollary 1.4.1.2 (Poincaré Duality)**.** *Let $H^p_{de-Rham}(M)$ denote the p-th de-Rham cohomology of a compact oriented manifold M. Then we have an isomorphsim:*

$$H^{n-p}_{de-Rham}(M) \cong (H^p_{de-Rham}(M))^*$$

Proof. Since we are working with finite dimensional spaces with an inner product, $H^{n-p}_{de-Rham}(M) \cong (H^p_{de-Rham}(M))^*$ is the same as having a non-degenerate bilinear from on $H^{n-p}_{de-Rham}(M) \times H^p_{de-Rham}(M)$. We define the form as follows. Represent any class in the cohomology groups by their harmonic representatives. Then:

$$([\alpha],[\beta]) := \int_M \alpha \wedge *\beta$$

This is well defined since the harmonic representatives are unique.Clearly this is bilinear. Now suppose for a fixed $[\alpha]$, $([\alpha],[\beta]) = 0$ for every β. Since $*$ commutes with Δ, $\Delta(*\alpha) = 0$ and as a result $*\alpha$ is a closed form. Take the class $[*\alpha]$. Now,

$$([\alpha],[*\alpha]) = 0 \implies \int_M \alpha \wedge *\alpha = 0 \implies ||\alpha||^2 = 0$$

Hence $\alpha = 0$. This proves non-degeneracy. $\square$

1.5 Proof of The Hodge Decomposition Theorem

We will give a proof of the Hodge Decomposition Theorem in this section. We will be assuming two theorems, which we will be proving in the subsequent sections.

Theorem 1.5.1 (Regularity Theorem). *Let $\alpha \in \Omega^p(M)$, and l be a weak solution to the equation $\Delta\omega = \alpha$. That is $l : \Omega^p(M) \to \mathbb{R}$ is a linear map with the property that $l(\Delta\omega) = \langle \alpha, \omega \rangle$. Then there exists $\omega_0 \in \Omega^p(M)$ such that*

$$l(\beta) = \langle \omega_0, \beta \rangle$$

for every $\beta \in \Omega^p(M)$, which implies that $\Delta\omega_0 = \alpha$.

In general it is a difficult to say whether a general PDE has a smooth solution. However the Laplacian is a very special PDE,called an elliptic PDE (which we will show later). Roughly it means that all the partial derivatives of the solution are controlled by the a smaller set of partial derivatives for every order. Hence we can conclude the existence of smooth solutions under some additional conditions.

Theorem 1.5.2 (Δ is a compact operator). *Let $\{\alpha_n\}$ be a sequence in $\Omega^p(M)$ such that $||\alpha_n|| \leq c$ and $||\Delta\alpha_n|| \leq c$ for all n and for some fixed $c > 0$. Then there exists a subsequence $\{\alpha_{n_k}\}$ which in Cauchy in $\Omega^p(M)$.*

Again this theorem makes use of the fact that Δ is elliptic. It is not surprising then since we have a bound on both the values of the function and all it's partial derivatives of all order, to expect some sort of convergence. Now we need a technical lemma.

Lemma 1.5.1. *For any $\beta \in (H^p)^\perp$ there exists $c > 0$ such that*

$$||\beta|| \leq c||\Delta\beta||$$

Proof. Assume to the contrary that there exists no such constant c. Then for every $n \in \mathbb{N}$, there exists $\beta_n \in (H^p)^\perp$ such that $||\beta_n|| \geq n||\Delta\beta_n||$. By renormalizing, i.e setting $\psi_n = \frac{\beta_n}{||\beta_n||^{\frac{1}{2}}}$, we obtain a sequence of forms, $\{\psi_n\} \in (H^p)^\perp$. Now $||\psi_n|| = 1$. Since $||\Delta\psi_n|| \leq \frac{1}{n}$ for all n, this means that $||\Delta\psi_n|| \to 0$. Since the series $\{||\Delta\psi_n||\}$ converges, it is bounded, say $||\Delta\psi_n|| \leq c$. Hence we have

$$||\psi_n|| \leq c + 1$$

$$||\Delta\psi_n|| \leq c + 1$$

from Theorem 1.5.2, we obtain a Cauchy subsequence which we denote by the same index $\{\psi_n\}$. We conclude that for a fixed ϕ, $\langle\psi_i, \phi\rangle$ is a Cauchy sequence because

$$|\langle\psi_l, \phi\rangle - \langle\psi_m, \phi\rangle| = |\langle\psi_l - \psi_m, \phi\rangle| \leq ||\psi_l - \psi_m||\,||\phi||$$

Now define a linear functional on $\Omega^p(M)$ as follows:

$$l : \Omega^p(M) \to \mathbb{R}$$

$$l(\phi) = \lim_{n\to\infty} \langle\psi_i, \phi\rangle$$

This is well defined since for a fixed ϕ the sequence converges as it is Cauchy as shown above. Now we observe that:

$$l(\Delta\phi) = \lim_{n\to\infty} \langle\psi_n, \Delta\phi\rangle = \lim_{n\to\infty} \langle\Delta\psi_n, \phi\rangle = 0.$$

Hence l is actually a weak solution of the PDE $\Delta\omega = 0$. From Theorem 1.5.1, we know that there exists $\psi \in \Omega^p(M)$ such that $l(\phi) = \langle\psi, \phi\rangle$. Hence,

$$\lim_{n\to\infty} \langle\psi_i, \phi\rangle = \langle\psi, \phi\rangle.$$

This means that $\psi_n \to \psi$. Since $\psi_i \in (H^p)^\perp$ and $||\psi_i|| = 1$ for all i, we obtain that $\psi \in (H^p)^\perp$, since $(H^p)^\perp$ is closed and $||\psi|| = 1$, since $||.||$ is a continuous function. However since $\Delta\psi = 0$, we have $\psi \in H^p$. Then $\psi \in (H^p)^\perp \cap H^p = \{0\}$. But this is impossible since $||\psi|| = 1$. Hence we have a contradiction. $\square$

Now we proceed to the main proof:

Proof of Theorem 1.4.1(The Hodge Decomposition Theorem). First we will show that H^p is finite dimensional. Assume to the contrary that H^p is not finite-dimensional. Then we can find an infinite sequence of orthonormal elements in H^p, say $\{\alpha_n\}$. Now since $\alpha_n \in H^p, \Delta\alpha_n = 0$. Also since our sequence is orthonormal we have $||\alpha_n|| = 1$. Hence we can use Theorem 1.5.1 to conclude that there exists a Cauchy subsequence. But $||\alpha_n - \alpha_m|| = \sqrt{2}$ for all n, m since the sequence is orthonormal, and consequently there cannot be a Cauchy subsequence. So we have a contradiction. Hence H^p is finite dimensional.

Select an orthonormal basis of H^p, $\{\omega_1, \omega_2, \ldots, \omega_l\}$. Then for any $\alpha \in \Omega^p(M)$ set $\beta = \alpha - \sum_{i=1}^{l} \langle \omega_i, \alpha\rangle \omega_i$. Note that $\langle \beta, \omega_i\rangle = 0$ for each i. Hence $\beta \in (H^p)^\perp$. Since $H^p \cap (H^p)^\perp = \{0\}$, we conclude that $\Omega^p(M) = (H^p)^\perp \oplus H^p$. Hence the proof is reduced to showing that $(H^p)^\perp = \Delta(\Omega^p(M))$.

First we show that $\Delta(\Omega^p(M)) \subseteq (H^p)^\perp$. Suppose $\omega \in \Delta(\Omega^p(M))$ and $\alpha \in H^p$ i.e $\Delta\alpha = 0$. Let $\Delta\omega' = \omega$ Then

$$\langle \omega, \alpha\rangle = \langle \Delta\omega', \alpha\rangle = \langle \omega', \Delta\alpha\rangle = 0.$$

Hence we have one way containment.

Next to show that $(H^p)^\perp \subseteq \Delta(\Omega^p(M))$. So given $\alpha \in (H^P)^\perp$, we need to find $\omega \in \Omega^p(M)$ such that $\Delta\omega = \alpha$. Hence we need a solution to the PDE $\Delta\omega = \alpha$. The procedure is to construct a weak solution and then obtain the required ω by using Theorem 1.5.1. Hence we need

$$l : \Omega^p(M) \to \mathbb{R}$$

which satisfies

$$l(\Delta\omega) = \langle \alpha, \omega\rangle$$

We will start by defining a linear functional with this property on the subspace $\Delta(\Omega^p(M))$.

Define:

$$l : \Delta(\Omega^p(M)) \to \mathbb{R}$$

$$l(\Delta\omega) = \langle \alpha, \omega \rangle$$

We want to extend this function to all of $\Omega^p(M)$ is order to use Theorem 1.5.1. This can be achieved by using Hahn-Banach Extension theorem, which requires l to be bounded. We use Lemma 1.5.1 to show this. But first we need to show that l is well-defined, i.e if $\Delta\omega_1 = \Delta\omega_2$, then $l(\Delta\omega_1) = l(\Delta\omega_2)$. But this is clear since $\Delta(\omega_1 - \omega_2) = 0$ which means $l(\Delta(\omega_1 - \omega_2)) = 0$. By linearity we conclude that l is a well-defined function.

Now to show that l is bounded. Note that in order to apply Lemma 1.5.1 we need $\omega \in (H^p)^\perp$. But we already know that

$$\Omega^p(M) = (H^p)^\perp \oplus H^p$$

Hence write $\omega = \omega_1 + \omega_2$ where $\omega_1 \in (H^p)^\perp$ and $\omega_2 \in H^p$. Now,

$$\begin{aligned} ||l(\Delta\omega)|| = ||l(\Delta\omega_1)|| &= |\langle \alpha, \omega_1 \rangle| \\ &\leq ||\alpha||\,||\omega_1|| \leq c||\alpha||\,||\Delta\omega_1|| = c'||\Delta\omega|| \end{aligned}$$

We have shown that l is a bounded linear functional on $\Delta(\Omega^p(M))$. Hence, using the Hahn-Banach Extension theorem we can extend l to obtain,

$$l' : \Omega^p(M) \to \mathbb{R}$$

satisfying the property $l'(\Delta\omega) = l(\Delta\omega) = \langle \alpha, \omega \rangle$. Hence from Theorem 1.5.1, there exists $\omega' \in \Omega^p(M)$ such that $\Delta\omega' = \alpha$.Therefore we have found a solution to the equation $\Delta\omega = \alpha$ and this completes the proof of the theorem. $\square$

Chapter 2

Sobolev Spaces

This section we develop the relevant theory for proving the two theorems used in the proof of the Hodge Decomposition Theorem. The general theory of Sobolev spaces is a huge topic and studying that will take us considerably off course. So we restrict ourselves to space of periodic functions, these are a lot easier to deal with since we can use the Fourier decomposition. This is sufficient in the case of manifolds since we will always prove things locally. We will use charts and partitions of unity, and due to compact support we can extend it periodically to the whole of $\mathbb{R}^n$. These will become more clear in the next chapter where we will actually put to use all the machinery developed here.

First we define some notations:

$\mathcal{P} = \{f : \mathbb{R}^n \to \mathbb{C}^m$ such that f is smooth and periodic with period $2\pi\}$

$C^\infty = \{f : \mathbb{R}^n \to \mathbb{C}^m$ such that f is smooth$\}$

$C_0^\infty = \{f : \mathbb{R}^n \to \mathbb{C}^m$ such that f is smooth and compactly supported$\}$

$C_0^\infty(V) = \{f : \mathbb{R}^n \to \mathbb{C}^m$ such that f is smooth and supp $f \subseteq V\}$

2.1 Fourier Series

In the section we will look at the necessary results from Fourier analysis. In the next section of Sobolev Spaces, we will prove of lot of properties and theorems, for functions that are periodic and smooth, so that we can use some of the results of this chapter. Proving the theorems of general Sobolev spaces, which requires us to do a lot of analysis on L^p functions,

will take us significantly away from the present topic. Hence we prove all the theorems for a certain subset of functions, which are nice, in the sense they have a Fourier series expansion. Some notations: $\alpha = (\alpha_1, \ldots, \alpha_n)$ where α_i's are all non-negative integers. For $\xi \in \mathbb{Z}^n$, we denote:
$\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$
$[\alpha] = \alpha_1 + \ldots + \alpha_n$
$|\xi|^2 = \xi_1^2 + \ldots + \xi_n^2$

We denote

$$D^\alpha = \frac{\partial}{\partial x_n^{\alpha_n}} \cdots \frac{\partial}{\partial x_1^{\alpha_1}}$$

Let $\mathcal{P} = \{f : \mathbb{R}^n \to \mathbb{C}^m$ such that f is smooth and periodic with period $2\pi\}$. And Q be the cube of side 4π centered at zero in $\mathbb{R}^n$, also called the 2π cube centered at 0, i.e $Q = \{p = (p_1, \ldots, p_n) \in \mathbb{R}^n : |p_j| < 2\pi, 1 \leq j \leq n\}$
Take an element $\phi \in \mathcal{P}$. For each $\xi \in \mathbb{Z}^n$, define

$$\phi_\xi = \frac{1}{(2\pi)^n} \int_Q \phi(x) e^{-ix.\xi}$$

Theorem 2.1.1. *Let $\phi \in \mathcal{P}$, and ϕ_ξ defined as above. Then the following series converges uniformly to ϕ.*

$$\sum_{\xi \in \mathbb{Z}^n} \phi_\xi e^{ix.\xi}$$

Proof. By repeatedly integrating by parts we obtain the following for each non-zero $\xi \in \mathbb{Z}^n$:

$$|\phi_\xi| \leq \frac{c_k}{\prod \xi^{2k}}$$

Now we prove a useful inequality that we will be using very frequently.

$$\begin{aligned}
(1 + |\xi|^2)^k &= (1 + \xi_1^2 + \ldots + \xi_n^2)^k \\
&= \sum_{[\alpha]=k} \binom{k}{\alpha} \prod \xi_i^{2\alpha_{i+1}} \\
&\leq c \prod \xi_i^{2k}
\end{aligned}$$

Here $\binom{k}{\alpha} = \frac{k!}{\alpha_1!\ldots\alpha_n!}$ Hence,

$$|\phi_\xi| \leq \frac{c_k}{\prod \xi^{2k}} \leq \frac{c_k'}{(1+|\xi|^2)^k}$$

Hence we reduce the problem to the convergence of the series $\sum \frac{1}{(1+|\xi|^2)^k}$. First define the set

$$A_j = \{\xi = (\xi_1, \ldots, \xi_n) : \max_{0\leq i\leq n} |\xi_i| = j\}$$

We will calculate the cardinality of A_j. First we fix the component which attains the maximum, there are n possibilities for this. It could take the value j or $-j$.The other $n-1$ elements can take values ranging from $-j, \ldots, j$, i.e, any of these $2j+1$ elements. There is a subtle overcounting going on but we can anyways conclude than the cardinality of A_j is less than $2n(2j+1)^{n-1}$. Also observe that for each $\xi \in A_j$ we have $|\xi|^2 \geq j^2$.

$$a_j = \sum_{\xi\in A_j} \frac{1}{(1+|\xi|^2)^k} \leq \sum_{\xi\in A_j} \frac{1}{(1+j^2)^k} \leq 2n(2j+1)^{n-1}\frac{1}{(1+j^2)^k}$$

Now for $j \geq 1$, we have $(2j+1)^{n-1} \leq cj^{n-1}$ and $(1+j^2)^k \geq j^{2k}$. Hence we conclude that,

$$a_j \leq cj^{n-1-2k}$$

Since $\mathbb{Z}^n = \cup_{j\geq 0} A_j$,

$$\begin{aligned}\sum_{\xi\in\mathbb{Z}^n} \frac{1}{(1+|\xi|^2)^k} &= 1 + \sum_{j>0}\sum_{\xi\in A_j} \frac{1}{(1+|\xi|^2)^k} \\ &= 1 + \sum_{j>0} a_j \\ &\leq 1 + \sum_{j>0} cj^{n-1-2k}\end{aligned}$$

We know that the the second term which is a geometric series converges when $n-1-2k < -1$, that is when $k > \lfloor \frac{n}{2} \rfloor + 1$. But since ϕ is smooth, we can take k to be as large as possible. Hence by the comparison test, we conclude that the $\sum_\xi \phi_\xi$ converges absolutely to some continuous function ψ (since the finite sums are all smooth functions and the convergence is uniform). Now we show that in fact $\phi = \psi$. Let $\Phi = \phi - \psi$. Let t be a trigonometric polynomial, i.e, a finite sum of terms of the from $a_\xi e^{ix.\xi}$ where $a_\xi \in \mathbb{C}^m$. Take

$t = \sum_{finite} a_\xi e^{ix.\xi}$ Now notice:

$$\begin{aligned}
\int_Q \Phi.t &= \int_Q (\phi - \psi).t \\
&= \int_Q \phi.t - \int_Q \psi.t \\
&= \int_Q \phi.(\sum_{finite} a_\xi e^{ix.\xi}) - \int_Q \psi.(\sum_{finite} a_\xi e^{ix.\xi}) \\
&= \sum_{finite} \int_Q \phi.a_\xi e^{ix.\xi} - \sum_{finite} \int_Q \psi.a_\xi e^{ix.\xi} \\
&= \sum_{finite} a_\xi.\phi_{-\xi} - \sum_{finite} a_\xi.\phi_{-\xi} \\
&= 0
\end{aligned}$$

□

Now by the Stone-Weierstrass Theorem we can find a trigonometric polynomial t, such that $||\Phi - t||_\infty < \epsilon$.

$$||\Phi||^2 = \frac{1}{(2\pi)^n} \int_Q \Phi.\Phi = \frac{1}{(2\pi)^n} \int_Q - \frac{1}{(2\pi)^n} \int_Q \Phi.t = \frac{1}{(2\pi)^n} \int_Q \Phi.(\Phi - t) \leq ||\Phi - t||_\infty ||\Phi||$$

$$||\Phi|| \leq \epsilon$$

Since ϵ is arbitrary and Φ is a continuous function, we conclude that $\Phi = 0$ and hence $\phi = \psi$. We have proved that the series converges uniformly to ϕ.

$$\phi = \sum_{\xi \in \mathbb{Z}^n} \phi_\xi e^{ix.\xi}$$

We now look at the derivatives of ϕ and how these change the Fourier coefficients. By using integration by parts, we observe:

$$\begin{aligned}
(D^\alpha \phi)_\xi &= \frac{1}{(2\pi)^n} \int_Q D^\alpha \phi . e^{-ix.\xi} \\
&= \frac{(-1)^{[\alpha]}}{(2\pi)^n} \int_Q D^\alpha (e^{-ix.\xi}) . \phi \\
&= \frac{(-1)^{[\alpha]}}{(2\pi)^n} \int_Q (-i)^{[\alpha]} \xi^\alpha e^{-ix.\xi} . \phi \\
&= i^{[\alpha]} \xi^\alpha \phi_\xi
\end{aligned}$$

For ease of notation we will denote D^α to actually mean

$$D^{[\alpha]} = \frac{1}{i^{[\alpha]}} \frac{\partial}{\partial x_n^{\alpha_n}} \cdots \frac{\partial}{\partial x_1^{\alpha_1}}$$

Then we have

$$(D^\alpha \phi)_\xi = \xi^\alpha \phi_\xi$$

2.2 Sobolev Spaces

We define the set $\mathcal{S} :=$ { sequences in $\mathbb{C}^m$ indexed by $\xi \in \mathbb{Z}^n$}. Hence a typical element of $\mathcal{S}$ would be denoted by $u = \{u_\xi\}$ where each $u_\xi \in \mathbb{C}^m$ as ξ varies over $\mathbb{Z}^n$.

Definition 2.2.1 (Sobolev Spaces H_s)**.**

$$H_s = \{u \in \mathcal{S} \, : \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s |u_\xi|^2 < \infty\}$$

If we define an inner product on $\mathcal{S}$ as

$$\langle u, v \rangle_s = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s u_\xi \overline{v_\xi}$$

then H_s is precisely the subset of $\mathcal{S}$ with a finite norm, the norm being induced from $\langle . \, , . \rangle_s$ Notice that any $\phi \in \mathcal{P}$ has a Fourier series and hence we can identify ϕ with the series $\{\phi_\xi\}_\xi \in \mathcal{S}$, and from the results of the previous section we conclude that $\mathcal{P} \subset H_s$ for each s. Now we define the operators D^α. The intuition is as follows. We know that for smooth periodic functions we have the Fourier coefficients, ϕ_ξ, and we have an obvious way for defining $D^\alpha \phi_\xi$ to be $(D^\alpha \phi)_\xi$, i.e, $D^\alpha \phi_\xi = \xi^\alpha \phi_\xi$. We extend this to all the sequences in $\mathcal{S}$. For $\phi \in \mathcal{S}$,

$$(D^\alpha(\phi))_\xi = \xi^\alpha u_\xi$$

Also from previous inequality we can conclude that the more the differentiable the function is, the higher the order of the Sobolev space it belongs to. And the converse is also true. Suppose $u \in H_s$ for large s, then we can show that the formal Fourier series actually converges to a smooth functions with some derivatives.

Theorem 2.2.1 (Properties of Sobolev Spaces)**.** *We have the follow inequalities which will be useful:*
ω is a complex valued smooth periodic function on $\mathbb{R}^n$, $\phi, \psi \in \mathcal{P}$

*(**a**) If $t < s$, then $||u||_t \leq ||u||_s$, hence $H_s \subset H_t$. Denote by $H_{-\infty}$ to be the union of all H_s.*

*(**b**) D^α is a bounded operator from $H_{s+[\alpha]}$ to H_s for each s. We have the inequality,*

$$||D^\alpha u||_s \leq ||u||_{s+[\alpha]}$$

*(**c**) $|\langle \phi, \psi \rangle_s| \leq ||\phi||_{s+t} ||\psi||_{s-t}$*

*(**d**) (Peter Paul Inequality) Given $t' < t < t''$ and any $\epsilon > 0$, we can find another constant which depends only on ϵ, call it $c(\epsilon)$ such that :*

$$||\phi||_t^2 \leq \epsilon ||\phi||_{t''}^2 + c(\epsilon) ||\phi||_{t'}^2$$

Proof. (**a**) $t < s \implies (1 + |\xi|^2)^t < (1 + |\xi|^2)^s$, as a result $||u||_t \leq ||u||_s$, and $H_s \subset H_t$.

(**b**) $(D^\alpha u)_\xi = \xi^\alpha u_\xi$
So

$$\begin{aligned} ||D^\alpha u||_s^2 &= \sum_\xi (1 + |\xi|^2)^s |(D^\alpha u)_\xi|^2 \\ &= \sum_\xi (1 + |\xi|^2)^s |\xi^\alpha u_\xi|^2 = \sum_\xi (1 + |\xi|^2)^s |\xi^\alpha u_\xi|^2 \\ &\leq \sum_\xi (1 + |\xi|^2)^s (1 + |\xi|^2)^{[\alpha]} |u_\xi|^2 = ||u||_{s+[\alpha]}^2 \end{aligned}$$

(**c**) We will use the Cauchy Schwartz inequality for the complex numbers to prove this.

$$\begin{aligned}
|\langle\phi,\psi\rangle_s| &= \sum_\xi (1+|\xi|^2)^s \phi_\xi \overline{\psi_\xi} \\
&= \sum_\xi (1+|\xi|^2)^{\frac{s+t}{2}} \phi_\xi . (1+|\xi|^2)^{\frac{s-t}{2}} \overline{\psi_\xi} \\
&\leq (\sum_\xi (1+|\xi|^2)^{s+t} |\phi_\xi|^2)^{\frac{1}{2}} (\sum_\xi (1+|\xi|^2)^{s-t} |\overline{\psi_\xi}|^2)^{\frac{1}{2}} \\
&= ||\phi||_{s+t} ||\psi||_{s-t}
\end{aligned}$$

(**d**) We have the following observation: $t''-t>0$ and $t-t'>0$. Now if $y>0$, then either y or $\frac{1}{y}$ has to be greater than 1. Hence we have:

$$1 \leq y^{t''-t} + \left(\frac{1}{y}\right)^{t-t'}$$

Now given ϵ, take $y = \epsilon^{\frac{1}{t''-t}}(1+|\xi|^2)$ to obtain:

$$(1+|\xi|^2)^t \leq \epsilon(1+|\xi|^2)t'' + c(\epsilon)(1+|\xi|)^{t'}$$

where $c(\epsilon) = \epsilon^{\frac{t'-t}{t''-t}}$ The required inequality follows from the above by multiplying throughout by $|\phi_\xi|^2$ and summing over ξ.

□

2.3 Sobolev and Rellich theorem

In this section we prove two useful theorems. Sobolev theorem makes our previous intuition, of a formal Fourier series converging to a function that is more differentiable, the higher the order of the Sobolev space it belongs to, more rigorous. Rellich theorem is concerned with whether we have a convergent sequence or in particular a Cauchy sequence. This is a hard question in general, and whether there is exists a Cauchy subsequence is easier to answer. This too is not easy to do using elementary analysis. The way we get around this is, we impose conditions on the sequence in higher order Sobolev spaces, and this is sufficient to give us some sort of convergence in a lower order Sobolev space.

Theorem 2.3.1 (Sobolev Theorem)**.** *If $s > \lfloor \frac{n}{2} \rfloor + 1$, then the series $\sum_\xi u_\xi e^{\iota x.\xi}$ converges uniformly. Thus each $s > \lfloor \frac{n}{2} \rfloor + 1$ corresponds to a continuous function.*

Proof. We will show that $\sum_\xi u_\xi e^{\iota x.\xi}$ converges absolutely. Since for finite sums this is a continuous function, that means the limit is also a continuous function.

$$\begin{aligned}
|| \sum_{|\xi| \leq N} u_\xi e^{\iota x.\xi} || &\leq \sum_{|\xi| \leq N} |u_\xi| \\
&= \sum_{|\xi| \leq N} (1 + |\xi|^2)^{\frac{-s}{2}} . (1 + |\xi|^2)^{\frac{s}{2}} |u_\xi| \\
&\leq (\sum_{\xi \leq N} (1 + |\xi|^2)^{-s})^{\frac{1}{2}} ((1 + |\xi|^2)^s |u_\xi|^2)^{\frac{1}{2}} \\
&= (\sum_{\xi \leq N} (1 + |\xi|^2)^{-s})^{\frac{1}{2}} ||u||_s
\end{aligned}$$

But since $s > \lfloor \frac{n}{2} \rfloor + 1$, $\sum_{\xi \leq N} (1 + |\xi|^2)^{-s}$ converges. Hence by using the Weierstrass M-test, we conclude that the series $\sum_\xi u_\xi e^{\iota x.\xi}$ is uniformly convergent. $\square$

Corollary 2.3.1.1. *If $u \in H_s$, $t > \lfloor \frac{n}{2} \rfloor + 1 + m$. Then if $[\alpha] \leq m$, $D^\alpha u \in H_{t-[\alpha]}$ from (a) of 2.2.1. Now $t - [\alpha] > \lfloor \frac{n}{2} \rfloor + 1$. Hence from the Sobolev Lemma we conclude that $D^\alpha u$ is continuous. We conclude that $u \in C^m$*

Theorem 2.3.2 (Rellich Theorem)**.** *Let $\{u^i\}$ be a bounded sequence in H_t, that is $||u^i||_t \leq c$ for some $c > 0$. Suppose $s < t$. Then there exists a subsequence $\{u^{i_k}\}$ which converges in H_s.*

Proof. We have $\sum_\xi (1 + |\xi|^2)^t |u^i_\xi|^2 \leq c$ for each i. Now fix $\xi \in \mathbb{Z}^n$. The sequence $\{(1 + |\xi|^2)^{\frac{t}{2}} |u^i_\xi|\}$ is a bounded sequence in $\mathbb{R}$. Hence we have a convergent subsequence which we denote by the same index. Now we use the diagonal argument to obtain a sequence $\{u^{j_k}\}$, such that $\{(1 + |\xi|^2)^{\frac{t}{2}} |u^{j_k}_\xi|\}$ converges for all ξ. $\{u^{i_k}\}$ is the required Cauchy subsequence in H_s

$$\begin{aligned}
||u^{j_k} - u^{j_l}||_s^2 &= \sum_\xi (1 + |\xi|^2)^s |u^{j_k}_\xi - u^{j_l}_\xi|^2 \\
&= \sum_{\xi < N} (1 + |\xi|^2)^{s-t} (1 + |\xi|^2)^t |u^{j_k}_\xi - u^{j_l}_\xi|^2 + \sum_{\xi \geq N} (1 + |\xi|^2)^{s-t} (1 + |\xi|^2)^t |u^{j_k}_\xi - u^{j_l}_\xi|^2 \\
&=: A + B
\end{aligned}$$

$$
\begin{aligned}
B &= \sum_{\xi \geq N} (1+|\xi|^2)^{s-t}(1+|\xi|^2)^t |u_\xi^{j_k} - u_\xi^{j_l}|^2 \\
&\leq (1+N^2)^{s-t} \sum_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^t (|u_\xi^{j_k}|^2 + |u_\xi^{j_l}|^2 + 2|u_\xi^{j_k}||u_\xi^{j_l}|) \\
&= (1+N^2)^{s-t} \left(\sum_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^t |u_\xi^{j_k}|^2 + \sum_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^t ||u_\xi^{j_l}|^2 + 2 \sum_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^{\frac{t}{2}} |u_\xi^{j_k}| (1+|\xi|^2)^{\frac{t}{2}} |u_\xi^{j_l}| \right) \\
&= (1+N^2)^{s-t} (||u^{j_k}||_t + ||u^{j_l}||_t + \langle u^{j_k}, u^{j_l} \rangle_s) \\
&\leq (1+N^2)^{s-t} (c + c + ||u^{j_k}||_s \, ||u^{j_l}||_s) \\
&\leq (1+N^2)^{s-t} (c + c + c^2) \\
&< \frac{\epsilon}{2}
\end{aligned}
$$

by choosing N large enough since $s - t < 0$. Now after choosing such an $N = N_o$,

$$
\begin{aligned}
A &= \sum_{\xi < N_o} (1+|\xi|^2)^{s-t}(1+|\xi|^2)^t |u_\xi^{j_k} - u_\xi^{j_l}|^2 \\
&\leq \sum_{\xi < N_o} (1+|\xi|^2)^t |u_\xi^{j_k} - u_\xi^{j_l}|^2
\end{aligned}
$$

Since for a fixed ξ, $\{(1+|\xi|^2)^{\frac{t}{2}} |u_\xi^{j_i}|\}$ converges, we can find J_ξ such that, for all $j_k, j_l > J_\xi$,

$$
(1+|\xi|^2)^{\frac{t}{2}} (|u_\xi^{j_k}| - |u_\xi^{j_l}|) \leq \sqrt{\frac{\epsilon}{2\sigma}}
$$

where $\sigma = \#\{\xi \in \mathbb{Z}^n : |\xi| < N_0\}$. Since we have only a finite number of $|\xi| < N_o$, we choose:

$$
J = max\{J_\xi : |\xi| < N_o\}
$$

Now if $j_k, j_l > J$, then

$$
\begin{aligned}
A &\leq \sum_{\xi < N_o} (1+|\xi|^2)^t |u_\xi^{j_k} - u_\xi^{j_l}|^2 \\
&\leq \sum_{\xi < N_o} \frac{\epsilon}{\sigma} \\
&= \frac{\epsilon}{2}
\end{aligned}
$$

Hence combining the estimates for A and B, we conclude that, for large enough j_k, j_l,

$$\begin{aligned} ||u^{j_k} - u^{j_l}||_s^2 &= A + B \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Hence $\{u^{j_i}\}$ is our required Cauchy subsequence in H_s. $\square$

2.4 Difference Quotients

What does it mean for a function to be differentiable. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Then we say that f is differentiable at x_0 if the following limit exists and the limit is called the differential of f at x_o, denoted by $f'(x_0)$.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Note that if this limit exists then $\frac{f(x_0+h)-f(x_0)}{h}$ is necessarily bounded. And conversely if there is a uniform bound on this term for all h, then in fact the limit exists and f is differentiable. Keeping this example and intuition in mind we proceed to define difference quotients.
Suppose $\phi \in \mathcal{P}$, then we want to look at the term $\phi(x+h)$. Since we already have a Fourier decomposition, we observe the foll wing:

$$\phi(x+h) = \sum_\xi \phi_\xi e^{i(x+h).\xi} = \sum_\xi \phi_\xi e^{i(x.\xi + h.\xi)} = \sum_\xi \phi_\xi e^{ix.\xi} e^{ih.\xi} = \sum_\xi e^{ih.\xi} \phi_\xi e^{ix.\xi}$$

Hence we see that a translation by h corresponds to the ξth Fourier coefficients being scaled by $e^{ih.\xi}$. We extend this to the space $\mathcal{S}$. We define the translation map, T_h, as follows:

$$(T_h(u))_\xi = e^{ih.\xi} u_\xi$$

Now we can define the difference quotients for $u \in \mathcal{S}$. We denote it by u^h, defined as:

$$u^h = \frac{T_h(u) - u}{|h|}$$

$$(u^h)_\xi = \frac{(T_h(u))_\xi - u_\xi}{|h|} = \frac{e^{ih.\xi}u_\xi - u_\xi}{|h|}$$

Similar to the real one dimensional case, we have analogous theorem for Sobolev spaces.

Lemma 2.4.1. *If $u \in H_{s+1}$, then $||u^h||_s \leq ||u||_{s+1}$ for all non-zero $h \in \mathbb{R}^n$.*

Proof.

$$\begin{aligned}
||u^h||_s &= \sum_\xi (1+|\xi|^2)^s |u^h_\xi|^2 \\
&= \sum_\xi (1+|\xi|^2)^s \left|\left|\frac{e^{ih.\xi}u_\xi - u_\xi}{|h|}\right|\right|^2 \\
&= \sum_\xi (1+|\xi|^2)^s \left|\left|\frac{e^{ih.\xi} - 1}{|h|}\right|\right|^2 ||u_\xi||^2
\end{aligned}$$

By using the series expansion of $e^{ihx.\xi}$ and the Cauchy-Shwarz inequality we can that

$$\left|\left|\frac{e^{ih.\xi} - 1}{|h|}\right|\right|^2 \leq 1 + |\xi|^2$$

Hence, we have:

$$\begin{aligned}
||u^h||_s^2 &= \sum_\xi (1+|\xi|^2)^s \left|\left|\frac{e^{ih.\xi} - 1}{|h|}\right|\right|^2 ||u_\xi||^2 \\
&\leq \sum_\xi (1+|\xi|^2)^s (1+|\xi|^2) ||u_\xi||^2 \\
&= ||u||_{s+1}
\end{aligned}$$

□

Now for the converse,

Lemma 2.4.2. *If $u \in H_s$, and there exists a constant c, such that $||u^h||_s \leq c$ for all non-zero $h \in \mathbb{R}^n$, i.e, $||u^h||_s$ is uniformly bounded, then $u \in H_{s+1}$.*

Proof. We will give a sequence in H_{s+1} which converges to u. The obvious candidate for this sequence which agrees with u in a finite number of places and is zero for all the other indices, i.e,$\{u_N\}$ is defined as follows:

$$(u_N)_\xi = \begin{cases} u_\xi & \text{if } |\xi| < N \\ 0 & \text{otherwise} \end{cases}$$

Clearly this sequence converges to u. We will show that this sequence is uniformly bounded in H_{s+1}. First an observation, if we let $h = te_i$ where $\{e_1, \ldots, e_n\}$ form an orthonornmal basis for $\mathbb{R}^n$, then:

$$\lim_{t\to 0} \frac{e^{i\xi.te_i} - 1}{|te_i|} = \lim_{t\to 0} \frac{e^{it\xi_i} - 1}{|t|} = i\xi_i e^{i\xi_i}$$

Now,

$$||u^h||_s < k \implies \sum_{|\xi|<N} (1 + |\xi|^2)^s |u_\xi|^2 \left|\left| \frac{e^{ih.\xi} - 1}{|h|} \right|\right|^2 \leq k^2$$

We take $h = te_i$, then take limit as $t \to 0$ to obtain

$$\sum_{\xi<N} (1 + |\xi|^2)^s |u_\xi|^2 |\xi_i|^2 < k^2 \text{ for } 0 \leq i \leq n$$

Adding up all these inequalities and the fact that $u \in H_s$ we conclude:

$$||u_N||_{s+1}^2 = \sum_{\xi<N} (1 + |\xi|^2)^s |u_\xi|^2 (1 + \sum_{i=1}^{n} |\xi_i|^2) < nk^2 + ||u||_s^2$$

We see that $\{u_N\}$ is bounded in H_{s+1} uniformly. Therefore $u \in H_{s+1}$. $\square$

2.5 Some Technical Results

We prove some technical lemmas in this section which are required to prove the various theorems and inequalities in the upcoming sections the most important of which is the elliptic regularity and the fundamental inequality for periodic elliptic partial differential operators. First we will prove some lemmas for H_s, which will then be used to prove required lemmas for partial differential operators.

To prove the following two lemmas, we define a new operator called K^t. This will be an isometry between Sobolev space of different another orders. The reason behind doing this is, we can prove these lemmas rather easily for when $s \geq 0$. Then the case for $s < 0$ is proved by using K^t to reduce it to the case of $s \geq 0$.
Define for $u \in \mathcal{S}$, $K^t u$, where the sequence is given by,

$$(K^t u)_\xi = (1 + |\xi|^2)^t u_\xi$$

Lemma 2.5.1. *Properties of K^t*

1. *$K^t : H_s \to H_{s-2t}$ is an isometry*

2. *The inverse of K^t is K^{-t}*

3. *If $\phi \in \mathcal{P}$ and $t \geq 0$, then*
$$K^t \phi = \left(1 - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}\right)^t \phi$$

4. *For every s and t,$u, v \in H_s$ we have the following:*
$$\langle u, v \rangle_s = \langle u, K^t v \rangle_{s-t} = \langle K^t u, v \rangle_{s-t}$$

Now we have defined K^t, we proceed to prove the following technical lemmas.

Lemma 2.5.2. *Let ω be a smooth real valued periodic function on $\mathbb{R}^n$. Then given an integer s, we can find constants c which depends only on s and n, and c' which depends on s,n and ω and it's derivatives such that:*

$$||\omega\phi||_s \leq c||\omega||_\infty ||\phi||_s + c'||\phi||_{s-1}$$

In particular, due to Theorem 2.2.1 we can say, $||\omega\phi||_s \leq c''||\phi||_s$.

Proof. We first prove for the case $s \geq 0$, follows from the fact that: $||\phi||_s \leq C \sum_{[\alpha]=0}^{s} ||D^\alpha \phi||$.

Then

$$
\begin{aligned}
||\omega\phi||_s &\leq C\sum_{[\alpha]=0}^{s}||D^\alpha\omega\phi|| = C\sum_{[\alpha]=0}^{s}||D^\alpha\omega\phi - \omega D^\alpha\phi + \omega D^\alpha\phi|| \\
&\leq C\sum_{[\alpha]=0}^{s}||(D^\alpha\omega - \omega D^\alpha)\phi|| + C\sum_{[\alpha]=0}^{s}||\omega D^\alpha\phi|| \\
&\leq c'\sum_{i=0}^{s}||\phi||_{i-1} + c||\omega||_\infty\sum_{i=0}^{s}||\phi||_i \\
&\leq c'(s-1)||\phi||_{s-1} + cs||\omega||_\infty||\phi||_s = C'||\phi||_{s-1} + C''||\omega||_\infty||\phi||_s
\end{aligned}
$$

Now when $s < 0$, we hit it by K^s to reduce it to the $s = 0$ case.

$$
\begin{aligned}
||\omega\phi||_s = \langle\omega\phi, \omega\phi\rangle_s &= \langle\omega K^{-s}K^s\phi, K^s\omega\phi\rangle_0 \\
&= \langle\omega K^{-s}K^s\phi + K^{-s}\omega K^s\phi - K^{-s}\omega K^s\phi, K^s\omega\phi\rangle_0 \\
&= \langle K^{-s}\omega K^s\phi, K^s\omega\phi\rangle_0 + \langle(\omega K^{-s} - K^{-s}\omega)K^s\phi, K^s\omega\phi\rangle_0
\end{aligned}
$$

Using the case of $s \geq 0$(since $-s > 0$), Cauchy Schwartz and properties of K^s, we see that for the first term:

$$
\begin{aligned}
|\langle K^{-s}\omega K^s\phi, K^s\omega\phi\rangle_0| &= |\langle\omega K^s\phi, K^s\omega\phi\rangle_{-s}| \\
&\leq ||\omega K^s\phi||_{-s}||K^s\omega\phi||_{-s} \\
&\leq (c||\omega||_\infty||K^s\phi||_{-s} + k'||K^s\phi||_{-s-1})||\omega\phi||_s \\
&\leq (c||\omega||_\infty||\phi||_s + k'||\phi||_{s-1})||\omega\phi||_s
\end{aligned}
$$

□

For the second term, we use the definition of K^s on smooth functions, i.e 3) of Lemma 2.5.1 to obtain that:

$$
\omega K^{-s} - K^{-s}\omega = \sum_{[\alpha]=0}^{-2s-1} a_\alpha D^\alpha
$$

where a_α are continuous functions of derivatives of ω. Then by using elementary analysis, Cauchy Schwartz and the properties of D^α and K^s we obtain:

$$
|\langle(\omega K^{-s} - K^{-s}\omega)K^s\phi, K^s\omega\phi\rangle_0| \leq const||\phi||_{s-1}||\omega||_\infty||\phi||_s
$$

From the bounds on the two terms we get our desired inequality.

Lemma 2.5.3. *Let ω be a complex-valued smooth periodic function on $\mathbb{R}^n$. Then for any integer s,*

$$|\langle \omega\phi, \psi\rangle_s - \langle \phi, \overline{\omega}\psi\rangle_s| \leq c(||\phi||_s||\psi||_{s-1} + ||\phi||_{s-1}||\psi||_s)$$

And we have a special condition when $s = 0$, $\langle \omega\phi, \psi\rangle_0 = \langle \phi, \overline{\omega}\psi\rangle_0$

Proof. The case for $s = 0$ when $\phi, \psi \in \mathcal{P}$ is nothing but a direct consequence of the fact that $\langle . , . \rangle_0$ is the same as the L^2 norm and as $\mathcal{P}$ is dense in H_0. Take $s < 0$ and $\phi, \psi \in \mathcal{P}$. Then we have:

$$\begin{aligned}
\langle \omega\phi, \psi\rangle_s &= \langle \omega\phi, K^s\psi\rangle_0 = \langle \omega K^{-s}K^s\phi, \psi\rangle_0 \\
&= \langle K^{-s}K^s\phi, \overline{\omega}K^s\psi\rangle_0 = \langle K^s\phi, K^{-s}\overline{\omega}K^s\psi\rangle_0 \\
&= \langle \phi, \overline{\omega}\psi\rangle_s - \langle \phi, \overline{\omega}\psi\rangle_s + \langle K^s\phi, K^{-s}\overline{\omega}K^s\psi\rangle_0 \\
&= \langle \phi, \overline{\omega}\psi\rangle_s - \langle K^s\phi, \overline{\omega}\psi\rangle_0 + \langle K^s\phi, K^{-s}\overline{\omega}K^s\psi\rangle_0 \\
&= \langle \phi, \overline{\omega}\psi\rangle_s + \langle K^s\phi, (K^{-s}\overline{\omega} - \overline{\omega}K^{-s})K^s\psi\rangle_0
\end{aligned}$$

We already did an approximation in the previous lemma for a term similar to the second term. After re-arrangement and observing the symmetry we obtain the desired inequality. The case for $s > 0$ is proved exactly as above but you use K^{-s} first. $\square$

Chapter 3

PDE Theory

In this chapter we define partial differential operators and some analytic results on them. These are pretty straightforward from the bounds that we proved the D^α operator and after all a partial differential operator will be made up of these, so we can easily extend those results. More importantly e deal with a very special kind of operator called an elliptic operator, the prime example being the Laplacian. These operators have positive eigen values and that allows us to prove more stronger results and have better control over the solutions.

3.1 Partial Differential Operators

We prove some lemmas on partial differential operators which are a direct consequence of the lemmas of the previous section.

Definition 3.1.1 (Partial Differential Operators)**.** An partial differential operator of order l is,

$$L = P_l(D) + ... + P_0(D),$$

where each $P_l(D)$ is a $m \times m$ matrix, and each element of the matrix is a partial differential operator on complex-valued functions of a constant order, that is, $(P_l(D))_{ij} = \sum_{[\alpha]=l} a_{ij}^{\alpha} D^{\alpha}$, where α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and a_{ij}^{α} are complex valued smooth functions on $\mathbb{R}^n$. The partial differential operator is called periodic when the a_{ij}^{α}'s are periodic functions and we say that it is an operator on $\mathcal{P}$

The simplest examples partial differential operators are the usual constant coefficients partial derivatives of functions on $\mathbb{R}^n$, $\frac{\partial^k}{\partial x_i^k}$. Also linear combinations of these and varying the orders, along with multiplying them with smooth functions give us the class of linear partial differential operators for single valued functions. These are the building blocks for the differential operators on multi-valued functions, which are nothing but matrices composed of elements, which look like the partial differential operators on single valued functions.

Lemma 3.1.1. *Let L be a partial differential operator on $\mathcal{P}$ of order l. Given $\phi \in \mathcal{P}$, we can find constants:*
c which depends on n, m, l and s;
k which is a bound on the coefficients of the highest order term of L;
c' which depends on n, m, l, s and all the coefficients of L and it's derivatives;
such that:

$$||L\phi||_s \leq ck||\phi||_{s+l} + c'||\phi||_{s+l-1}$$

In particular $||L\phi||_s \leq c''||\phi||_{s+l}$.

Proof. The proof for $m = 1$, is nothing but a direct consequence of the inequalities in b0 of 2.2.1 and 2.5.2. The case for an arbitrary m, follows from seeing that

$$\begin{aligned}
||L\phi||_s^2 &= \sum_\xi (1+|\xi|^2)^s ||(L\phi)_\xi||^2 \\
&= \sum_\xi \left((1+|\xi|^2)^s \sum_{i=1}^n |((L\phi)_\xi)_i|^2 \right) \\
&= \sum_{i=1}^n \left(\sum_\xi (1+|\xi|^2)^s |((L\phi)_\xi)_i|^2 \right) \\
&= \sum_{i=1}^n ||(L\phi)_i||_s^2
\end{aligned}$$

Now we have: $||L\phi||_s^2 = \sum_{i=1}^n ||(L\phi)_i||_s^2 = \sum_{i=1}^n \sum_{j=1}^n ||L_{ij}\phi_j||_s^2$. Hence we can find a constant which depends only on m, to get $||L\phi||_s \leq \sum_{i,j} ||L_{ij}\phi_j||_s$. Now the general inequality follows from the $m = 1$ case. □

Remark. 1.Notice that looking at $\mathcal{P}$ as a subset of H_s we can look at L as a bounded linear map from a dense subset of H_{s+l} to H_s. Hence by Hahn-Banach extension theorem we can extend this map to obtain, $L : H_{s+l} \to H_s$, satisfying the above inequalities.

Remark. 2. If L is a partial differential operator of order l and ω is a smooth function, then $L\omega - \omega L$ is a partial differential operator of order $l-1$. Hence we have $(L\omega - \omega L)(\phi) \leq c||\phi||_{s+l-1}$

Lemma 3.1.2. *Let ω be a smooth-real valued periodic function on $\mathbb{R}^n$ and L be a periodic partial differential operator of order l. Then for all $u \in H_{s+l}$,there exists $c > 0$, such that:*

$$|\langle L(\omega^2 u), Lu\rangle_s - \langle L(\omega u), L(\omega u)\rangle_s| \leq c||u||_{s+l}||u||_{s+l-1}$$

Proof.

$$\begin{aligned}\langle L(\omega^2 u), Lu\rangle_s - \langle L(\omega u), L(\omega u)\rangle_s = {}& \langle \omega L(\omega u), Lu\rangle_s - \langle L(\omega u), \omega Lu\rangle_s + \\ & \langle L(\omega u), \omega Lu\rangle_s - \langle L(\omega u), L(\omega u)\rangle_s + \\ & \langle L(\omega . \omega u), Lu\rangle_s - \langle \omega L(\omega u), Lu\rangle_s\end{aligned}$$

$$\begin{aligned}|\langle L(\omega^2 u), Lu\rangle_s - \langle L(\omega u), L(\omega u)\rangle_s| \leq {}& |\langle \omega L(\omega u), Lu\rangle_s - \langle L(\omega u), \omega Lu\rangle_s| + \\ & |\langle L(\omega u), (\omega L - L\omega)u\rangle_s| + \\ & |\langle (L\omega - \omega L)(\omega u), Lu\rangle_s|\end{aligned}$$

The first term is reduced using Lemma 2.5.3 and use the Cauchy Schwartz on the last two terms. Then repeated use of Lemma 3.1.1 and remarks, and Lemma 2.5.2 gives us the desired inequality. □

3.2 Elliptic PDE

Definition 3.2.1. Elliptic Partial Differential Operator Given a partial differential operator of order l ,

$$L = P_l(D) + ... + P_0(D),$$

where , $(P_l(D))_{ij} = \sum_{[\alpha=l]} a^{\alpha}_{ij} D^{\alpha}$, where α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and a^{α}_{ij} are complex valued smooth functions on $\mathbb{R}^n$. For each $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ construct a new matrix $P_l(\xi)$, where you substitute for each D^{α} that element $\xi^{\alpha} = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$. We call the partial differential operator L elliptic at x if $P_l(\xi)(x)$ is invertible for each non-zero ξ.

The prototypical example for an elliptic PDE is the Laplacian on $\mathbb{R}^n$, which is a second order operator. The Laplacian, L, of a function f on $\mathbb{R}^n$ is defined as

$$Lf = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Writing this in the form of our definition, since the range of the function is in $\mathbb{R}$, the $P_k(D)$ are 1×1 matrices or just numbers. Also only there are only terms with order 2, i.e,

$$L = P_2(D)$$

$$P_2(D) = \sum_{i=1}^{n} D^{e_i}$$

where $e_i = (0, \ldots, 2, \ldots, 0)$, with 2 at the i-th place. Then for any non-zero $\xi \in \mathbb{R}^n$, $P_2(\xi) = \sum_{i=1}^{n} (\xi_i)^2$. Since ξ is non-zero we see that $P_2(\xi)$ is non-zero and hence invertible. Therefore the Laplacian on $\mathbb{R}^n$ is an elliptic operator.

Remark. Notice that the condition is only for the matrix consisting of the highest order derivatives.

Lemma 3.2.1. *The partial differential operator L of order l is elliptic at x if and only if*

$$L(\phi^l u)(x) \neq 0$$

for any u, a $\mathbb{C}^m$-valued functions smooth functions on $\mathbb{R}^n$ such that $u(x) \neq 0$ and ϕ, a smooth function on $\mathbb{R}^n$ such that $\phi(x) = 0$ and $d\phi(x) \neq 0$

Proof. Fix $x = (x_1, \ldots, x_n)$. We need to show that this condition is equivalent to the fact that $P_l(\xi)(x)$ is invertible for each non-zero ξ. Another equivalent way of telling this is $P_l(\xi)(x) \neq 0$ when $x \neq 0$. The condition $\phi(x) = 0$ forces only the highest order derivatives to survive in $L(\phi^l u)(x)$ as seen below:

Suppose $\alpha = (\alpha_1, \dots, \alpha_n)$. Then:

$$\begin{aligned}
D^\alpha(\phi^l) &= \frac{1}{i^{[\alpha]}} \frac{\partial}{\partial x_n^{\alpha_n}} \cdots \frac{\partial \phi^l}{\partial x_1^{\alpha_1}} \\
&= \frac{1}{i^{[\alpha]}} \frac{\partial}{\partial x_n^{\alpha_n}} \cdots \frac{\partial}{\partial x_1^{\alpha_1 - 1}} \Big(l\phi^{l-1} \frac{\partial \phi}{\partial x_1} \Big) \\
&= \frac{1}{i^{[\alpha]}} \frac{\partial}{\partial x_n^{\alpha_n}} \cdots \frac{\partial}{\partial x_1^{\alpha_1 - 2}} \Big(l(l-1)\phi^{l-2} \frac{\partial \phi}{\partial x_1} \frac{\partial \phi}{\partial x_1} + l\phi^{l-1} \frac{\partial^2 \phi}{\partial x_1^2} \Big)
\end{aligned}$$

Now since $\phi(x) = 0$, only the terms independent of ϕ survive. This would require us to differentiate the remaining powers of ϕ again and again to get terms of the form (for example in the case above):

$$\frac{l(l-1)\dots(l-[\alpha])}{i^{[\alpha]}} \phi^{l-[\alpha]} \left(\frac{\partial \phi}{\partial x_n} \right)^{\alpha_n} \cdots \left(\frac{\partial \phi}{\partial x_1} \right)^{\alpha_1}$$

Hence only for $[\alpha] = l$, ϕ disappears and the terms survive. Hence,

$$L(\phi^l u)(x) = P_l(d\phi|_x)(u)(x)$$

where $d\phi|_x = \left(\frac{\partial \phi}{\partial x_n}\Big|_x, \dots, \frac{\partial \phi}{\partial x_1}\Big|_x \right)$. Now for each $\xi \in \mathbb{R}^n$ non-zero, set

$$\phi(t) = (\xi_1(t - x_1), \dots, \xi_n(t - x_n))$$

Observe that ϕ is smooth, $\phi(x) = 0$ and $d\phi|_x = (\xi_n, \dots, \xi_1)$ which is non-zero. Now $P_l(d\phi|_x) = P_l(\xi)$. Since this is a constant matrix,

$$P_l(d\phi|_x)(u)(x) = P_l(\xi)(u(x))$$

Now for any non-zero $v \in \mathbb{R}^n$ we define $u(t) = v$, the constant function. This is clearly non-zero. For this particular choice of ϕ and u we see:

$$L(\phi^l u)(x) \neq 0 \iff P_l(d\phi|_x)(u)(x) \neq 0 \iff P_l(\xi)(v) \neq 0$$

Since this is true for all $u(t) = v$ where v is non-zero, the last terms is the same as the matrix being invertible. Hence the two statements are equivalent. $\square$

Now we proceed to prove two of the main theorems on elliptic PDE, the fundamental inequality and the regularity theorem.

Theorem 3.2.1 (Fundamental Inequality)**.** *Let L be an elliptic operator on of order l, and let s be an integer. Then there exists a constant $c > 0$ such that*

$$||u||_{s+l} \leq c(||Lu||_s + ||u||_s)$$

for all $u \in H_{s+l}$

Proof. We will first prove it for the special case where L_0 is an elliptic operator of order l, consisting of only l-order terms and the coefficients being constants. Also since $\mathcal{P}$ is dense in H_s, it is sufficient to prove the theorem for $\phi \in \mathcal{P}$. Since the operator is elliptic, $P_l(\xi)$ is invertible for all non-zero ξ. This means that for any non zero $u \in \mathbb{R}^n$, $P_l(\xi)u \neq 0$, that means $|P_l(\xi)u|^2 > 0$. Now since the order of each term is l, and is the same throughout, we can just look at the map:

$$S^{n-1} \times S^{n-1} \to \mathbb{R}$$

$$(\xi, u) \mapsto |P_l(\xi)u|^2$$

Since $S^{n-1}XS^{n-1}$ is compact, and the map is easily seen to be continuous(since it's a composition of continuous map),we deduce that $|P_l(\xi)u|^2 > c$ for some $c > 0$. Now given any non-zero $\xi, u \in \mathbb{R}^n \backslash 0$, $\frac{\xi}{|\xi|}, \frac{u}{|u|} \in S^{n-1}$. Hence we obtain that:

$$|P_l(\xi)u|^2 > c|\xi|^{2l}|u|^2$$

Thus for $\phi \in \mathcal{P}$,

$$\begin{aligned}
||L_0(\phi)||_s^2 &= ||P_l(\phi)||_s^2 \\
&= \sum_\xi (1 + |\xi|^2)^s |(P_l(\phi))_\xi|^2 \\
&= \sum_\xi (1 + |\xi|^2)^s |P_l(\xi)\phi_\xi|^2 \\
&\geq \sum_\xi (1 + |\xi|^2)^s c|\xi|^{2l}|\phi_\xi|^2 \\
&= c\sum_\xi |\xi|^{2l}|\phi_\xi|^2(1 + |\xi|^2)^s
\end{aligned}$$

Proceeding,

$$
\begin{aligned}
(||L_0(\phi)||_s^2 + ||\phi||_s^2)^2 &\geq ||L_0(\phi)||_s^2 + ||\phi||_s^2 \\
&\geq c\sum_\xi |\xi|^{2l}|\phi_\xi|^2(1+|\xi|^2)^s + \sum_\xi (1+|\xi|^2)^s|\phi_\xi|^2 \\
&\geq c'\sum_\xi |\phi_\xi|^2(1+|\xi|^2)^s(|\xi|^{2l}+1) \\
&\geq c''\sum_\xi |\phi_\xi|^2(1+|\xi|^2)^s(1+|\xi|^2)^l \\
&= c''\sum_\xi |\phi_\xi|^2(1+|\xi|^2)^{s+l} \\
&= c''||\phi||_{s+l}^2
\end{aligned}
$$

Hence we have proved it for elliptic operators with constant coefficients which consists of only the highest order terms. To prove it for general periodic elliptic operators we will show that for any point $p \in \mathbb{R}^n$, we can find a neighbourhood of p,U_p such that for all $\phi \in \mathcal{P}$ with support in U and 2π translates of U, the theorem holds. Then we will use a partition of unity argument to prove it for global periodic smooth functions. Let L_0 denote the highest order coefficient matrix at the point p. We already know that for such operators,

$$||\phi||_{s+l} \leq c(||L_0(\phi)||_s + ||\phi||_s)$$

Writing $L_0 = L_0 - L + L$ and using the triangle inequality(keep this general procedure in mind, we will be frequently using it):

$$||\phi||_{s+l} \leq c'(||L\phi||_s + ||(L_0 - L)\phi||_s + ||\phi||_s)$$

Now we know that for a PDE L on $\mathcal{P}$, we have the following inequality, $\phi \in \mathcal{P}$:

$$||L\phi||_s \leq ck||\phi||_{s+l} + \tilde{c}||\phi||_{s+l-1}$$

Choose $\epsilon < \frac{1}{2cc'}$. Now choose a neighbourhood of of p such that the coefficients of the highest order terms of $\widetilde{L} = L_0 - L$ are bounded by ϵ. This can be done since the coefficients are smooth functions on $\mathbb{R}^n$. Notice that $\widetilde{L}$ is no longer periodic. Hence we choose a smaller neighbourhood U_p on which $\widetilde{L}$ agrees with a periodic elliptic PDE on that set. Abusing notation we denote this too by $\widetilde{L}$. Now for $\phi \in \mathcal{P}$ with support in U_p(and it's 2π translates), we observe that:

$$\begin{aligned}
||\phi||_{s+l} &\leq c'(||L\phi||_s + ||(L_0 - L)\phi||_s + ||\phi||_s) \\
&= c'(||L\phi||_s + ||\widetilde{L}\phi||_s + ||\phi||_s) \\
&\leq c'(||L\phi||_s + ck||\phi||_{s+l} + \tilde{c}||\phi||_{s+l-1} + ||\phi||_s) \\
&\leq c'(||L\phi||_s + c\epsilon||\phi||_{s+l} + \tilde{c}||\phi||_{s+l-1} + ||\phi||_s) \\
&\leq c'||L\phi||_s + \frac{1}{2}||\phi||_{s+l} + c'\tilde{c}||\phi||_{s+l-1} + c'||\phi||_s)
\end{aligned}$$

Now applying Peter-Paul Inequality to $||\phi||_{s+l-1}$ with $s < s+l-1 < s+l$, and $\epsilon = \frac{1}{4c'\tilde{c}}$, we obtain that:

$$||\phi||_{s+l-1} \leq \epsilon||\phi||_{s+l} + const||\phi||_s$$

Hence,

$$\begin{aligned}
||\phi||_{s+l} &\leq c'||L\phi||_s + \frac{1}{2}||\phi||_{s+l} + c'\tilde{c}\epsilon||\phi||_{s+l} + \epsilon\, const||\phi||_s + c'||\phi||_s \\
&= c'||L\phi||_s + \frac{1}{2}||\phi||_{s+l} + \frac{1}{4}||\phi||_{s+l} + const||\phi||_s \\
&\leq c'||L\phi||_s + \frac{3}{4}||\phi||_{s+l} + const||\phi||_s
\end{aligned}$$

$$\begin{aligned}
\frac{1}{4}||\phi||_{s+l} &\leq c'||L\phi||_s + const||\phi||_s \\
||\phi||_{s+l} &\leq C(||L\phi||_s + ||\phi||_s)
\end{aligned}$$

Now we have some sort of local result. We proceed for the global case as follows. First cover $\mathbb{R}^n$ by $\{U_p\}_{p\in\mathbb{R}^n}$, where the U_p are as above.Consider the torus T^n obtained by quotienting out $\mathbb{R}^n$ by translations by 2π. Cover T^n by the projections of these open sets onto the torus. This is an open cover for T^n and since T^n is compact we have a finite subcover, $\{U_1, \ldots, U_k\}$. Now we consider a partition of unity ω_i of sub-ordinate to this finite cover with the extra condition that $\sum_{i=1}^k \omega_i^2 = 1$. This can be done. There are some subtle points to note here. We construct the finite cover and the partitions of unity on the torus. But notice that pulling it back to $\mathbb{R}^n$ we get a cover of $\mathbb{R}^n$ with 2π translates of these open sets and a partition of unity which are real 2π periodic smooth functions on $\mathbb{R}^n$. Now,

$$\begin{aligned}
||\phi||_{s+l}^2 = \langle \phi, \phi \rangle_{s+l} = \langle \sum_{i=1}^{k} \omega_i^2 \phi, \phi \rangle_{s+l} \\
&= \sum_{i=1}^{k} \langle \omega_i^2 \phi, \phi \rangle_{s+l} \\
&\leq \sum_{i=1}^{k} \langle \omega_i \phi, \omega_i \phi \rangle_{s+l} + const||\phi||_{s+l}||\phi||_{s+l-1} \\
&= \sum_{i=1}^{k} ||\omega_i \phi||_{s+l}^2 + const||\phi||_{s+l}||\phi||_{s+l-1}
\end{aligned}$$

From c),d) of Theorem 2.2.1 and the fact that ω_i's are real. Now since $supp\omega_i \subset U_i$, and there are only a finite number of terms, we can take the biggest constant among the terms from the inequality we proved locally.Hence,

$$\begin{aligned}
||\phi||_{s+l}^2 &\leq c_1 \sum_{i=1}^{k} ||L\omega_i \phi||_s^2 + c_2||\phi||_s^2 + c_3||\phi||_{s+l}||\phi||_{s+l-1} \\
&= c_1 \sum_{i=1}^{k} \langle L\omega_i \phi, L\omega_i \phi \rangle_s + c_2||\phi||_s^2 + c_3||\phi||_{s+l}||\phi||_{s+l-1} \\
&\leq c_1' \sum_{i=1}^{k} \langle L(\omega_i^2 \phi), L\phi \rangle_s + c_2||\phi||_s^2 + c_3'||\phi||_{s+l}||\phi||_{s+l-1} \\
&= c_1' \langle \sum_{i=1}^{k} L(\omega_i^2 \phi), L\phi \rangle_s + c_2||\phi||_s^2 + c_3'||\phi||_{s+l}||\phi||_{s+l-1} \\
&\leq c_1' \langle L(\sum_{i=1}^{k} \omega_i^2 \phi), L\phi \rangle_s + c_2||\phi||_s^2 + c_3'(\frac{1}{2c_3'}\frac{||\phi||_{s+l}^2}{2} + 2c_3'\frac{||\phi||_{s+l-1}^2}{2}) \\
&= c_1' \langle L\phi, L\phi \rangle_s + c_2||\phi||_s^2 + \frac{1}{2}||\phi||_{s+l}^2 + c_4||\phi||_{s+l-1}^2
\end{aligned}$$

Using the same trick as before, applying the Peter-Paul Inequality to $||\phi||_{s+l-1}$, with $s <$

$s+l-1 < s+l$, and $\epsilon = \frac{1}{4c_4}$, we obtain:

$$
\begin{aligned}
||\phi||^2_{s+l} &\le c_1'||L\phi||^2_s + c_2||\phi||^2_s + \frac{1}{2}||\phi||^2_{s+l} + \frac{1}{4}||\phi||^2_{s+l} + const||\phi||^2_s \\
||\phi||^2_{s+l} &\le c_1'||L\phi||^2_s + c_2'||\phi||^2_s + \frac{3}{4}||\phi||^2_{s+l} \\
\frac{1}{4}||\phi||^2_{s+l} &\le c_1'||L\phi||^2_s + c_2||\phi||^2_s
\end{aligned}
$$

Hence we conclude that:

$$||\phi||_{s+l} \le c(||L\phi||_s + ||\phi||_s)$$

□

Theorem 3.2.2 (Regularity for Periodic Elliptic Operators)**.** *Let L be a periodic elliptic operator of order l. Let $u \in H_{-\infty}$ and $v \in H_t$ such that*

$$Lu = v$$

Then $u \in H_{t+l}$

Proof. It is sufficient to prove that for $u \in H_s$ and $v = Lu \in H_{s-l+1}$, $u \in H_{s+1}$. We will prove this by showing that the difference quotient of u, is bounded in the $||.||_s$ norm. Before that we denote by L^h the partial differential operator where the coefficients functions are replaced by their difference quotients. Now we observe that,

$$Lu^h + L^h(T_h u) = (Lu)^h$$

Since L is linear and the equality is component wise, it is sufficient to check the condition for a single term, and the above formulae will hold for L due to linearity. Taking a single term, it is of the form: $a^\alpha(x)D^\alpha u(x) = v(x)$.

$$
\begin{aligned}
1.\ a^\alpha(x)D^\alpha u^h(x) &= a^\alpha(x)D^\alpha \frac{u(x+h)-u(x)}{|h|} \\
2.\ (a^\alpha(x)D^\alpha)^h(T_h u) &= (a^\alpha)^h(x)D^\alpha(u(x+h)) \\
&= \frac{a^\alpha(x+h)-a^\alpha(x)}{|h|}D^\alpha(u(x+h))
\end{aligned}
$$

Adding 1 and 2, we obtain:

$$
\begin{aligned}
&a^{\alpha}(x)D^{\alpha}u^{h}(x) + (a^{\alpha}(x)D^{\alpha})^{h}(T_{h}u) \\
&= a^{\alpha}(x)D^{\alpha}\frac{u(x+h)-u(x)}{|h|} + \frac{a^{\alpha}(x+h)-a^{\alpha}(x)}{|h|}D^{\alpha}(u(x+h)) \\
&= \frac{a^{\alpha}(x)D^{\alpha}u(x+h) - a^{\alpha}(x)D^{\alpha}u(x) + a^{\alpha}(x+h)D^{\alpha}(u(x+h) - a^{\alpha}(x)D^{\alpha}(u(x+h)}{|h|} \\
&= \frac{a^{\alpha}(x+h)D^{\alpha}(u(x+h) - a^{\alpha}(x)D^{\alpha}u(x)}{|h|} \\
&= \frac{T_{h}(a^{\alpha}D^{\alpha}u)(x) - (a^{\alpha}D^{\alpha}u)(x)}{|h|} \\
&= (a^{\alpha}D^{\alpha}u)^{h}(x)
\end{aligned}
$$

Hence we have showed

$$(a^{\alpha}D^{\alpha})(u^{h}) + (a^{\alpha}D^{\alpha})^{h}(T_{h}u) = (a^{\alpha}D^{\alpha}(u))^{h}$$

Now using the fundamental inequality and the above result,

$$
\begin{aligned}
||u^{h}||_{s} &\leq c(||Lu^{h}||_{s-l} + ||u^{h}||_{s-l}) \\
&\leq c(||(Lu)^{h} - L^{h}(T_{h}u)||_{s-l} + ||u^{h}||_{s-l}) \\
&\leq c(||(Lu)^{h}||_{s-l} + ||L^{h}(T_{h}u)||_{s-l} + ||u^{h}||_{s-l})
\end{aligned}
$$

Since $u \in H_{s}$ and l is at least 1, we see $||u^{h}||_{s-l} \leq ||u^{h}||_{s-1} \leq ||u||_{s}$.
Similarly, $Lu \in H_{s-l+1}$, hence $||(Lu)^{h}||_{s-l} \leq ||(Lu)||_{s-l+1}$.
Now since the coefficients of the differential operator are all smooth, this means that the difference quotients are all uniformly bounded. Hence, $||L^{h}(T_{h}u)||_{s-l} \leq c||T_{h}u||_{s}$. But we know that T_{h} is an isometry, therefore $||T_{h}u||_{s} = ||u||_{s}$.
From all the arguments above we conclude :

$$||u^{h}||_{s} \leq c(||Lu||_{s-l+1} + ||u||_{s})$$

Hence $u \in H_{s+1}$ $\square$

3.3 Reducing to Periodic case

Till now we have been working with and in fact all the theorems we proved are for periodic functions and periodic elliptic partial differential operators. But in general this need not be periodic, for example the Laplacian on the manifold. How do we get around this problem. We will show that given any elliptic differential operator of $\mathbb{R}^n$, for each $p \in \mathbb{R}^n$, we can find a neighbourhood V, on which it agrees with a periodic elliptic partial differential operator. Let L_0 denote the value of this of the elliptic operator at p. Since ellipticity is a continuous condition on the coefficients(the determinant being a continuous map),we can find $\epsilon > 0$, such that the operator is still elliptic if the coefficients differ by at most ϵ from that of L_0. Choose a neighbourhood, U of p, which is contained in the 2π cube centered at p, and on which the coefficients of L differ from that of L_0 by at most ϵ(This is possible since the coefficients are all smooth functions and there are only a finite number of them). Now choose $V \subset \overline{V} \subset U$, and construct a smooth function ω such that $0 \leq \omega \leq 1$, $\omega = 1$ on V and $\operatorname{supp} \omega \subset U$. Define:

$$\widetilde{L} = \omega L + (1 - \omega) L_0$$

Observe that the coefficients of $\widetilde{L}$ differ from that of L_0 by at most ϵ inside U. And outside U it is just L_0. Hence we have a elliptic partial differential operator defined all over $\mathbb{R}^n$. Also, $\widetilde{L} = L$ on V. However there is a subtle point to see that this is not periodic. That can easily be taken care of by observing that it uniformly takes the value of L_0 at the sides of the 2π cube(since $U \subset 2\pi$ cube), which is the same as the value at the center. Hence we can extend it periodically to whole of $\mathbb{R}^n$ to obtain the required periodic elliptic operator $\widetilde{L}$.

3.4 $\triangle$ is an elliptic PDE

We show that the Laplacian is an elliptic PDE. We need some results from vector spaces.

Lemma 3.4.1. *Let U, V, W be finite dimensional inner product spaces. Suppose the following sequence is exact:*

$$U \xrightarrow{A} V \xrightarrow{B} W$$

Suppose the adjoints of A and B are A^* and B^* respectively. Then the map:

$$AA^* + B^*B : V \to V$$

is an isomorphism.

Proof. Since the domanin and co-domain have the same dimension, by the rank-nullity theorem we need to just so that the map is injective. First we wil show, $A^* : V \to U$ is injective on the image of A,$Im(A)$. Suppose $v \in Im(A)$, i.e, $v = Au$ for some $u \in U$ and $v \in Ker(A^*)$, then

$$\begin{aligned} A^*(v) = 0 &\implies A^*Au = 0 \\ &\implies \langle A^*Au, u\rangle = 0 \\ &\implies \langle Au, Au\rangle = 0 \\ &\implies Au = v = 0 \end{aligned}$$

Hence A^* is injective on the $Im(A)$. Now we will show $AA^* + B^*B$ is injective. Let $v \neq 0$.

$$\begin{aligned} (AA^* + B^*B)v = 0 &\implies \langle (AA^* + B^*B)v, v\rangle = 0 \\ &\implies \langle AA^*v, v\rangle + \langle B^*Bv, v\rangle = 0 \\ &\implies \langle A^*v, A^*v\rangle + \langle Bv, Bv\rangle = 0 \end{aligned}$$

If $Bv \neq 0$, then $(AA^*+B^*B)v \neq 0$. Suppose $Bv = 0$, i.e, $v \in Ker(B)$ then since the sequence is exact, $v \in Im(A)$. Now we know that A^* is injective on $Im(A)$. Hence $v \neq 0 \implies A^*v \neq 0$ and consequently $(AA^* + B^*B)v \neq 0$. Hence $v \neq 0 \implies (AA^* + B^*B)v \neq 0$ □

Lemma 3.4.2. *Suppose $\xi \in V$, a finite dimensional inner product space. Then the following sequence is exact:*

$$\Lambda^{p-1}(V) \xrightarrow{\xi} \Lambda^p(V) \xrightarrow{\xi} \Lambda^{p+1}(V)$$

where $\xi(\omega) = \xi \wedge \omega$

Proof. □

Theorem 3.4.1 (Δ is Elliptic)**.**

Proof. Let $\xi \in \Omega^1(M)$. We have the following exact sequence for each $m \in M$:

$$\Lambda^{p-1}(T_m^*M) \xrightarrow{\xi_m} \Lambda^p(T_m^*M) \xrightarrow{\xi_m} \Lambda^{p+1}(T_m^*M)$$

where $\xi_m(\omega(m)) = \xi(m) \wedge \omega(m)$, where ω is an alternating smooth form. We know that the adjoint of ξ_m is

$$\xi_m^* = (-1)^{np} * \xi_m *$$

Now, we have

$$\xi_m \xi_m^* + \xi_m^* \xi_m : \Lambda^p(T_m^* M) \to \Lambda^p(T_m^* M)$$

By using the expression for ξ_m^* we obtain that:

$$\xi_m \xi_m^* + \xi_m^* \xi_m = (-1)^{np} \xi_m * \xi_m * + (-1)^{n(p-1)} * \xi_m * \xi_m \tag{3.1}$$

Now to prove that Δ is elliptic. We will show that for any point m on the manifold, $\Delta(\phi^2 \alpha)(m) \neq 0$ for all smooth p-forms α, $\alpha(m) \neq 0$ and smooth functions ϕ, satisfying $\phi(m) = 0$ and $d\phi(m) \neq 0$.

$$\begin{aligned}
\Delta &= d\delta + \delta d : \Lambda^p(T_m^* M) \to \Lambda^p(T_m^* M) \\
&= (-1)^{n(p+1)+1} d * d * + (-1)^{n(p+1+1)+1} * d * d \\
&= (-1)^{n(p+1)+1} d * d * + (-1)^{np+1} * d * d
\end{aligned}$$

next

$$\begin{aligned}
\Delta(\phi^2 \alpha)(m) &= ((-1)^{n(p+1)+1} d * d * + (-1)^{np+1} * d * d)(\phi^2 \alpha)(m) \\
&= (-1)^{n(p+1)+1} d * d * (\phi^2 \alpha)(m) + (-1)^{np+1} * d * d(\phi^2 \alpha)(m)
\end{aligned}$$

Let us calculate $d * d * (\phi^2 \alpha)(m)$ and $*d * d(\phi^2 \alpha)(m)$.

$$\begin{aligned}
d * d * (\phi^2 \alpha) &= d * d(\phi^2 * \alpha) \\
&= d * (2\phi d\phi \wedge *\alpha + \phi^2 d * \alpha) \\
&= d(2\phi * (d\phi \wedge *\alpha) + \phi^2 * d * \alpha) \\
&= 2(d\phi \wedge *(d\phi \wedge *\alpha) + 2\phi d(*(d\phi \wedge *\alpha))) + 2\phi d\phi \wedge (*d * \alpha) + \phi^2 d * d * \alpha
\end{aligned}$$

Observe that since $\phi(m) = 0$, we obtain that $d * d * (\phi^2 \alpha)(m) = 2(d\phi \wedge *(d\phi \wedge *\alpha)(m)$. Now if we let $d\phi(m) = \xi_m$, we see that

$$d * d * (\phi^2 \alpha)(m) = 2\xi_m * \xi_m * \alpha(m)$$

where we the same symbol ξ_m to denote the map from

Similarly,

$$*d*d(\phi^2\alpha)(m) = 2*\xi_m*\xi_m\alpha(m)$$

Therefore,

$$\begin{aligned}
\Delta(\phi^2\alpha)(m) &= (-1)^{n(p+1)+1}d*d*(\phi^2\alpha)(m) + (-1)^{np+1}*d*d(\phi^2\alpha)(m) \\
&= (-1)^{n(p+1)+1}2\xi_m*\xi_m*\alpha(m) + (-1)^{np+1}2*\xi_m*\xi_m\alpha(m) \\
&= -2[(-1)^{np}\xi_m*\xi_m*+(-1)^{n(p-1)}*\xi_m*\xi_m](\alpha(m))
\end{aligned}$$

From 3.1,we know that this is an isomorphism and since $\alpha(m) \neq 0$, we conclude that $\Delta(\phi^2\alpha)(m) \neq 0$. This shows that Δ is an elliptic operator. $\square$

Chapter 4

Proofs

In this chapter we will prove the two main theorems we used in the proof of the Hodge Decomposition Theorem, namely theorems on regularity and compactness. The procedure for both these proofs is to prove them locally, in a small enough chart so that we can reduce it to working with periodic elliptic PDE which is always possible in a small enough neighbourhood. Choose such neighbourhoods then use a partition of unity to patch the results of each neighbourhood to get a sort of global result. Also using the compactly supported functions lets us extend the functions periodically to whole of $\mathbb{R}^n$.

4.1 Proof of Regularity

We prove the regularity theorem.

Proof of 1.5.1. We will denote the inner product and norms in $\Omega^p(M)$ as $\langle . , . \rangle'$ and $|| \, . \, ||'$ while the standard inner product and norms on the euclidean space as $\langle . , . \rangle$ and $|| \, . \, ||$. So given $f \in \Omega^p(M)$ and a bounded linear map

$$l' : \Omega^p(M) \to \mathbb{R}$$

such that $l'(\Delta\phi) = \langle f, \phi\rangle'$ for all $\phi \in \Omega^p(M)$, then there exists $u \in \Omega^p(M)$ such that

$$l'(\phi) = \langle u, \phi\rangle'$$

and $\Delta u = f$. We first reduce the problem to a local problem. Let (U, γ) be a chart around $m \in M$ such that $\gamma(U) = \mathbb{R}^n$. Now we observe that smooth p-forms correspond to smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m \subseteq \mathbb{C}^m$ where $m = \binom{n}{p}$.
Also functions in C_0^∞ can be extended by zero to obtain complex valued smooth forms on M with support inside U. We would like to extend $\langle .\,, .\rangle'$ to complex valued smooth forms. This is done by defining:

$$\langle u_1 + iu_2, v_1 + iv_2\rangle' = \langle u_1, u_2\rangle' + \langle v_1, v_2\rangle' + i(\langle u_2, v_1\rangle' - \langle u_1, v_2\rangle')$$

We note that this is a hermitian inner product on complex-valued p-forms which agrees with our initial inner product on real valued p-forms. Now we have an the usual L_2 inner product on C_0^∞, given by

$$\langle \phi, \psi\rangle = \int_{\mathbb{R}^n} \phi.\psi$$

where $\phi.\psi = \phi_1.\overline{\psi_1} + \phi_2.\overline{\psi_2} + \ldots + \phi_m.\overline{\psi_m}$.
we can define another inner product on C_0^∞, by first looking at it as complex valued p-forms on the manifold M. Then

$$\langle \phi, \psi\rangle' = \int_{\mathbb{R}^n} \tilde{\phi}.\tilde{\psi}$$

where $\tilde{\phi}, \tilde{\psi}$ are p-forms on M by extension by zero.
Since both $\langle .\,, .\rangle'$ and $\langle .\,, .\rangle$, are defined as integrals of point wise inner product there exists an invertible matrix A such that

$$\langle u, v\rangle' = \langle u, Av\rangle$$

Δ induces a elliptic partial differential operator L of order 2. Let L^* be the adjoint of L w.r.t $\langle .\,, .\rangle$. Note that the adjoint of L w.r.t $\langle .\,, .\rangle'$ is nothing but the adjoint of Δ. Take $\phi, \psi \in C_0^\infty$ and observe that:

$$\begin{aligned}
\langle L^*\phi, \psi\rangle &= \langle \phi, L\psi\rangle \\
&= \langle A^{-1}, L\psi\rangle' \\
&= \langle \Delta^* A^{-1}\phi, \psi\rangle' = \langle \Delta A^{-1}\phi, \psi\rangle' \\
&= \langle A\Delta A^{-1}\phi, \psi\rangle
\end{aligned}$$

Hence $L^* = A\Delta A^{-1}$ Define

$$l : C_0^\infty \to \mathbb{C}$$

$$l(\phi) = l'(A^{-1}\phi)$$

Claim: l is locally represented by a smooth function. That means given any $p \in \mathbb{R}^n$, there exists a neighbourhood W_p of p, and a smooth function $u_p \in \mathcal{P}$, such that, for any $\phi \in C_0^\infty(W_p)$,

$$l(\phi) = \langle u_p, \phi\rangle$$

Claim $\implies$ the proof.

First we collect all such u_p's on $\mathbb{R}^n$. Now if $W_p \cap W_q \neq \phi$, then $u_p|_{W_p\cap W_q} = u_q|_{W_p\cap W_q}$ since they agree on all $\phi \in C_0^\infty(W_p \cap W_q)$. Hence we can patch up these to define a function $u \in C^\infty$ which satisfies $u|_{W_p} = u_p|_{W_p}$. Now take a partition of unity $\{\phi_i\}$ subordinate to the cover $\{W_p\}$. Now for any $t \in C_0^\infty$,

$$\begin{aligned}
l(t) &= l(\sum_i \phi_i t) \\
&= \sum_i l(\phi_i t) \\
&= \sum_i \langle u, \phi_i t\rangle \\
&= \langle u, \sum_i \phi_i t\rangle \\
&= \langle u, t\rangle
\end{aligned}$$

Now suppose ϕ is a smooth form on M with support in U. Then

$$l'(\phi) = l(A\phi) = \langle u, A\phi\rangle = \langle u, \phi\rangle'$$

Using similar arguments as above we can patch up the various u's and we get that for any $\phi \in \Omega^p(M)$,

$$l(\phi)' = \langle u, \phi\rangle'$$

$\square$

<u>Proof of claim:</u> Fix $p \in \mathbb{R}^n$. Construct a 2π cube with p as the center, $Q = \{y \in \mathbb{R}^n : |x_i(p) - x_i(y)| < 2\pi, \forall\, 0 \leq i \leq n\}$. Choose an open set $V \subset \overline{V} \subset Q$. Denote $\tilde{l} = l|_{C_0^\infty(V)}$.

1. l is a bounded linear functional.
   Since $\overline{V}$ is compact and $||\,.\,||$ is continuous, $||A_x^{-1}||$ attains a maximum value as x varies over $\overline{V}$.

$$\begin{aligned}
|\tilde{l}(\phi)| = |l(\phi)| &= |l'(A^{-1}\phi)| \\
&\leq c||A^{-1}\phi||' \\
&= c|\langle A^{-1}\phi, A^{-1}\phi\rangle'|^{\frac{1}{2}} \\
&= c|\langle \phi, A^{-1}\phi\rangle|^{\frac{1}{2}} \\
&\leq c||\phi||^{\frac{1}{2}}||A^{-1}\phi||^{\frac{1}{2}} \\
&\leq c||\phi||^{\frac{1}{2}} \sup_{x\in\overline{V}}(||A_x^{-1}||^{\frac{1}{2}})||\phi||^{\frac{1}{2}} \\
&\leq c||\phi||
\end{aligned}$$

2.

$$\begin{aligned}
\tilde{l}(L^*\phi) &= l(A^{-1}L^*\phi) \\
&= l'(A^{-1}(A\Delta A^{-1})\phi) \\
&= l'(\Delta(A^{-1}\phi)) \\
&= \langle f, A^{-a}\phi\rangle' = \langle f, \phi\rangle
\end{aligned}$$

Recall $\langle\, ,\, \rangle = \langle\, ,\, \rangle_0$. From 1) we see that $\tilde{l}$ is bounded linear functional, by the Hahn Banach extension theorem, we can extend it to the closure of $C_0^\infty(V) \approx \mathcal{P}$ w.r.t the norm induced by $\langle\, ,\, \rangle_0$ to obtain a Hilbert space. Using the Riez's representation theorem, we can find $\tilde{u}$, such that $\tilde{l}(t) = \langle \tilde{u}, t\rangle_0$. Now we want to show that on a small enough neighbourhood, $\tilde{u}$ represents a periodic smooth function. That is we can find $u \in \mathcal{P}$ and a neighbourhood W_p such that $u|_{W_p} = \tilde{u}|_{W_p}$. Choose a neighbourhood of p, $O_0 \subset \overline{O_0} \subset V$ such that we can find a periodic elliptic partial differential operator,$\widetilde{L}$ which agrees with L on O_0. Choose another neighbourhood of p, O which is contained in in O_0. Now construct a series of open sets, $O_i, i > 0$ satisfying, $\overline{O} \subset O_i$ and $\overline{O}_i \subset O_{i-1}$. For each $i > 0$, define real valued smooth functions ω_i as follows:

$$\omega_i = 1 \text{ on } O_i$$

$$\operatorname{supp} \omega_i \subseteq O_{i-1}$$

Take $v_1 = \omega_1 \tilde{u} \in H_0$

$$\begin{aligned}
\widetilde{L}v_1 &= \widetilde{L}\omega_1\tilde{u} \\
&= \omega_1\widetilde{L}\tilde{u} + \widetilde{L}\omega_1\tilde{u} - \omega_1\widetilde{L}\tilde{u} \\
&= \omega_1\widetilde{L}\tilde{u} + (\widetilde{L}\omega_1 - \omega_1\widetilde{L})\tilde{u}
\end{aligned}$$

Notice that since $\widetilde{L}$ is of order 2, $M_1 = \widetilde{L}\omega_1 - \omega_1\widetilde{L}$ is H_0, we see that $M_1\tilde{u} \in H_{-1}$. The first term $\omega_1\widetilde{L}\tilde{u}$, we claim is equal to $\omega_1 f$.In fact for any $i > 0$, we show $\omega_i\widetilde{L}\tilde{u} = \omega_i f$.

$$\begin{aligned}
\langle \omega_i\widetilde{L}\tilde{u} - \omega_i f, \phi\rangle_0 &= \langle \omega_i\widetilde{L}\tilde{u}\rangle_0 - \langle \omega_i f, \phi\rangle_0 \\
&= \langle \widetilde{L}\tilde{u}, \omega_i\phi\rangle_0 - \langle f, \omega_i\phi\rangle_0 \\
&= \langle \tilde{u}, L * \omega_i\phi\rangle_0 - \tilde{l}(L^*\omega_i\phi) \\
&= \tilde{l}(L^*\omega_i\phi) - \tilde{l}(L^*\omega_i\phi) \\
&= 0
\end{aligned}$$

This means that $\omega_i\widetilde{L}\tilde{u} \in C_0^\infty(O_{i-1})$. In particular, $\omega_1\widetilde{L}\tilde{u} \in C_0^\infty(O_0) \approx \mathcal{P} \subseteq H_s$ for all s. Hence $\widetilde{L}v_1 \in H_{-1}$. Now since $\widetilde{L}$ is a periodic elliptic operator of order 2, and $v \in H_{-1}$, we conclude from the Theorem 3.2.2, that $u \in H_1$.

Next let $v_2 = \omega_2\tilde{u} = \omega_2\omega_1\tilde{u} = \omega_2 v_1$. Proceeding as above:

$$\widetilde{L}v_2 = \omega_2\widetilde{L}\tilde{u} + (\widetilde{L}\omega_2 - \omega_2\widetilde{L})v_1$$

$\omega_2\widetilde{L}\tilde{u} \in C_0^\infty(O_1) \approx \mathcal{P} \subseteq H_s$ for each s. $M_2 = \widetilde{L}\omega_2 - \omega_2\widetilde{L}$ has order 1 so $M_2 v_1 \in H_0$. Hence $v_2 \in H_2$ from Theorem 3.2.2.

Similarly we can prove that $v_i = \omega_i\tilde{u} \in H_i$. Choose an open set containing p, W_p such that $\overline{W_p} \subseteq O_0$. Construct a real valued smooth function ω such that $\omega = 1$ on W_p and $\operatorname{supp}\omega \subset O$. Define $u = \omega\tilde{u} = \omega\omega_i\tilde{u}$ for all $i > 0$. Hence we see that $u \in C_0^\infty(O)) \approx \mathcal{P}$. Take

$\phi \in C_0^\infty(W_p)$:

$$\begin{aligned} l(\phi) &= \tilde{l}(\phi) \\ &= \langle \tilde{u}, \phi \rangle_0 \\ &= \langle \tilde{u}, \omega\phi \rangle_0 \\ &= \langle \omega\tilde{u}, \phi \rangle_0 \\ &= \langle u, \phi \rangle_0 = \langle u, \phi \rangle \end{aligned}$$

■

4.2 Proof of Compactness

This will be a proof of Theorem 1.5.2. Hence given a sequence $\{\alpha_n\} \in \Omega^p(M)$, such that, $||\alpha_n|| \leq c$ and $||\Delta\alpha_n|| \leq c$, for some $c > 0$, then there exists a Cauchy subsequence. Again we will prove the theorem locally. First we show:

Lemma 4.2.1. *Given a point p on the manifold M, there exists a neighbourhood W_p, such that for all functions ϕ, with* $\operatorname{supp}\phi \subseteq V$*, the sequence $\{\phi\alpha_n\}$ has a Cauchy subsequence.*

Proof of 1.5.2(Δ is a compact operator). By using Lemma 4.2.1, cover the manifold M by neighbourhoods W_p such that, for each for each function ϕ, with $\operatorname{supp}\phi \subseteq V$, the sequence $\{\phi\alpha_n\}$ has a Cauchy subsequence. Now since M is compact we have a finite subcover, say $\{W_1, W_2, \ldots, W_k\}$. Now choose a partition of unity $\{\phi_i\}$ subordinate to this cover. Start with W_1. We have a Cauchy subsequence of $\{\phi_1\alpha_n\}$ since $\operatorname{supp}\phi_1\alpha_n = \operatorname{supp}\phi_1 \subseteq W_1$. Call it $\{\phi_1\alpha_{n_j}\}$. Now consider the sequence $\{\phi_2\alpha_{n_j}\}$. Again $\operatorname{supp}\phi_2\alpha_{n_j} = \operatorname{supp}\phi_2 \subseteq W_2$. Hence we get another subsequence of, say $\{\phi_2\alpha_{n'_j}\}$. Now look at $\{\phi_3\alpha_{n'_j}\}$.Continuing this process, as we have to do this only a finite number of times, due to a finite cover, we finally obtain a sequence, $\{a_{n'}\}$ such that for each ϕ_i, the sequence $\{\phi_i\alpha_{n'}\}$ is Cauchy due to our construction.

$\{a_{n'}\}$ is the desired Cauchy subsequence.

$$\begin{aligned}||\alpha_{n'} - \alpha_{m'}|| &= ||1.(\alpha_{n'} - \alpha_{m'})|| \\ &= ||\sum_{i=1}^{k} \phi_i(\alpha_{n'} - \alpha_{m'})|| \\ &= ||\sum_{i=1}^{k} (\phi_i\alpha_{n'} - \phi_i\alpha_{m'})|| \\ &\leq \sum_{i=1}^{k} ||\phi_i\alpha_{n'} - \phi_i\alpha_{m'}||\end{aligned}$$

Since for each i, we have a Cauchy sequence, we can make each of the terms in the last sum as small as we want. Hence $\{a_{n'}\}$ is a Cauchy subsequence. □

Proof of Lemma 4.2.1. Choose a co-ordinate neighbourhood around m, (U, γ) such that $\gamma(m) = p$ and $\gamma(U) = \mathbb{R}^n$. Let us stick to the same notations as in the proof of regularity. Suppose ϕ is a function with support in O_0, we would like to show that $\{\phi\alpha_n\}$ has Cauchy subsequence in the $||\,.\,||'$ norm. But we know that since $\overline{O_0}$ is compact, the $||\,.\,||'$ norm is equivalent to the $||\,.\,||_2$ norm on $C^0_\infty(O_0)$

But we know that on $C^0_\infty(O_0)$ the $||\,.\,||_2$ norm is the same as the $||\,.\,||_0$ norm. Hence our problem is reduced to showing that there is a Cauchy subsequence is the $||\,.\,||_0$ norm. This can be shown by proving that that the sequence is bounded in $||\,.\,||_1$ norm and using Rellich

Theorem 2.3.2.

$$\begin{aligned} ||\phi\alpha_n||_1 &\leq c(||\widetilde{L}\phi\alpha_n||_{-1} + ||\phi\alpha_n||_{-1}) \\ &= c(A + B) \end{aligned}$$

$$\begin{aligned} A &= ||\widetilde{L}\phi\alpha_n||_{-1} = ||L\phi\alpha_n||_{-1} \\ &= ||\phi L\alpha_n + (L\phi - \phi L)\alpha_n||_{-1} \\ &\leq ||\phi L\alpha_n||_{-1} + ||(L\phi - \phi L)\alpha_n||_{-1} \end{aligned}$$

$$\begin{aligned} ||(L\phi - \phi L)\tau\alpha_n||_{-1} &\leq ||\tau\alpha_n||_{-1} \\ &\leq c||\tau\alpha_n||_0 = c||\tau\alpha_n|| \\ &\leq c||\tau\alpha_n||' \\ &\leq c||\alpha_n||' \end{aligned}$$

$$\begin{aligned} ||\phi L\alpha_n||_{-1} &\leq c||\phi L\alpha_n||_0 = c||\phi L\alpha_n||_2 \\ &\leq c||\phi L\alpha_n||' \\ &\leq c||\Delta\alpha_n||' \end{aligned}$$

$$\begin{aligned} B &= ||\phi\alpha_n||_{-1} \leq c||\phi\alpha_n||_0 \\ &\leq c||\phi\alpha_n||' \\ &= c||\alpha_n||' \end{aligned}$$

Since $||\alpha_n||'$ and $||\Delta\alpha_n||'$ are bounded we see that $||\phi\alpha_n||_1$ is bounded for all n. Hence we have a Cauchy subsequence in H_0, which implies we have a Cauchy subsequence in the $||\,.\,||'$ norm. $\square$

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