Elliptic Partial Differential Equations

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by

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Certificate

This is to certify that this dissertation entitled Elliptic Partial Differential Equationstowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Arghya Rakshitat Indian Institute of Science Education and Research under the supervision of Dr. Mousomi Bhakta, Associate Professor, Department of Mathematics , during the academic year 2019-2020.

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Declaration

I hereby declare that the matter embodied in the report entitled Elliptic Partial Differential Equations are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Mousomi Bhakta and the same has not been submitted elsewhere for any other degree.

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I dedicate the thesis to my parents and my sister for their endless love and support throughout the years of my existence.

Abstract

In this dissertation we present a brief introduction to theory of elliptic partial differential equations (PDE). First we review theory of Sobolev spaces. After that we discuss existence, regularity and other qualitative properties of weak solutions to the second order linear elliptic PDE. Afterwards, we discuss various standard variational and non-variational techniques to study nonlinear elliptic pde, mainly existence/nonexistence and various qualitative properties. Finally, in the last two chapters we mention various regularity results for weak solutions to elliptic equations in divergence form, in particular well-known theory of De Giorgi-Nash-Moser.

Contents

Chapter 1

Introduction

The aim of this thesis is to give an introduction to the theory of elliptic partial differential equations(PDEs). To start with, let's define what is a PDE. Throughout the thesis Ω is an open subset of $\mathbb{R}^n$ unless otherwise mentioned.

Definition 1.0.1. An equation

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0 \quad (x \in \Omega) \tag{1.1}$$

where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a known map and $u : \Omega \to \mathbb{R}$ is the unknown, is known as a partial differential equation of order k.

We say u solves the above PDE if u satisfies (1.1) and certain boundary conditions on some part of $\partial\Omega$. By finding solution, we mean obtaining explicit solution, which is very difficult in most of the cases. Therefore, mostly we try to establish existence and various other properties of solution.

Classification of PDE: (i) The PDE (1.1) is said to be linear if it is of the form

$$\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u = f(x)$$

where a_α ($|\alpha| \leq k$), f are given functions. The linear PDE is said to be homogeneous, if $f = 0$.
Example: $\Delta u := \sum_{i=1}^{n} u_{x_i x_i} = \lambda u$.
(ii) The PDE (1.1) is said to be semilinear if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha + a_0(D^{k-1}u......, Du, u, x) = 0.$$

Example: $-\Delta u = \lambda u^p$, $p > 1$.

(iii) The PDE (1.1) is said to be quasilinear if it is of the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, ..., Du, u, x)D^\alpha u + a_0(D^{k-1}u, ..., Du, u, x).$$

Examples: $-\Delta_p u := \nabla \cdot |\nabla u|^{p-2}\nabla u = f(x)$.
(iv) The PDE (1.1) is said to be fully nonlinear if it depends nonlinearly on the highest order derivatives.
Example: $|\Delta u| = 1$.
In the classical sense, the solution of a PDE must be differentiable at least as many times as the order of the equation. However, such a point of view is very restrictive - several interesting equations which model real life physical phenomena fail to possess such solutions.Hence it is necessary to generalize the notion of solutions of PDEs, which in turn motivates the generalization of the notion of differentiable functions. This is the main motivation behind weak derivative.
Throughout this thesis, we do not try to find explicit formulas. The general picture is using "energy estimate" we first try to find solutions of certain class of PDEs. Then we attempt to see if the solution is unique. We also study regularity properties and other qualitative properties of the solution.

In next chapter, we study Sobolev spaces, a proper setting for the study of several linear and nonlinear PDEs. Third chapter deals with second order linear *elliptic PDEs*. The fourth chapter is about calculus of variation: finding critical points of the Euler Lagrange functional associated with the given PDE, which in turn becomes the solution of the PDE. Variational theory is one of the most useful tools in the study of nonlinear PDEs. We study the existence and nonexistence of solutions to various nonlinear PDEs and qualitative properties of the solutions. In fifth chapter using some non-variational techniques like fixed point theorem, sub and super solution method. Last chapter is about regularities and maximum principle of solutions of *nonlinear elliptic PDEs*.

Chapter 2

Sobolev Spaces

The aim of this chapter is to develop the theory of *Sobolev spaces* which turns out to be the proper functional setting to study Partial Differential Equations(PDEs) applying the ideas of functional analysis. To solve a PDE, we look it as $A : X \to Y$, where A carries the structure of the partial differential operator, including possibly boundary condition, etc., acting on elements of space of functions X and gives output in space of functions Y.

2.1 Hölder Spaces

Before we get into Sobolev space, let's define Hölder Spaces.

Definition 2.1.1. Let Ω be an open set, $0 < \gamma \leq 1$ and $u : \Omega \to \mathbb{R}$ be a continuous bounded map. We define

$$\|u\| := \sup_{x \in \Omega} |u(x)|$$

and the γ^{th} Hölder seminorm of u to be

$$[u]_{C^{0,\gamma}(\overline{\Omega})} := \sup_{x,y \in \Omega, x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right\}.$$

γ^{th} Hölder norm is defined to be

$$\|u\|_{C^{0,\gamma}(\overline{\Omega})} = \|u\|_{C(\Omega)} + [u]_{C^{0,\gamma}(\overline{\Omega})}.$$

Definition 2.1.2. The Hólder space $C^{k,\gamma}(\overline{\Omega})$ is the space of functions $u \in C^k(\overline{\Omega})$ such that

$$\|u\|_{C^{k,\gamma}(\overline{\Omega})} = \sum_{|\alpha| \leq k} \|D^{\alpha} u\|_{C(\overline{\Omega})} + \sum_{|\alpha| = k} [D^{\alpha} u]_{C^{0,\gamma}(\overline{\Omega})}$$

is finite, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multi-index.

It's easy to check that Hölder spaces are Banach spaces.

2.2 Weak Dervative and Sobolev Spaces

Unfortunately the Hölder spaces does not provide appropriate platform to study the theory of PDE. Therefore, our plan is to define Sobolev spaces, which contain less smooth functions than that of Hölder spaces.

Definition 2.2.1. Let α be a multi-index and $u, v \in L^1_{loc}(\Omega)$. We say v is the α^{th} weak partial derivate of u, if

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^\alpha \int_\Omega v \, \phi \, dx \quad \forall \phi \in C_c^\infty(\Omega),$$

where $C_c^\infty(\Omega)$ is the space of infinitely differentiable functions with compact support. We write $D^\alpha u = v$ if v is the α^{th} weak partial derivative of u.

It can easily be shown that α^{th} weak partial derivative is unique up to a measure zero set, when exists. Functions that are not differentiable in the usual sense can be weakly differentiable (Example 1). But if there is a jump discontinuity, it is never going to be weakly differentiabe (Example 2).
Example 1: Consider $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = |x|$. This function is not differentiable at 0, but it is weakly differentiable at all points in $\mathbb{R}$.
Example 2: Consider the heaviside function defined by

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 1, & \text{if } x > 0. \end{cases}$$

As there is a jump at 0, the function is neither differentiable nor weakly differentiable in weak sense.
Now we are in a position to define Sobolev spaces. For detailed exposition on Sobolev spaces, refer to [R.75].

Definition 2.2.2. The Sobolev space $W^{k,p}(\Omega)$ consists of all $u : \Omega \to \mathbb{R}$ such that $u \in L^1_{loc}(\Omega)$ and for every multi-index α with $|\alpha| \leq k$, the α^{th} weak partial derivative, $D^\alpha u$, exists and belongs to $L^p(\Omega)$.

For $u \in W^{k,p}(\Omega)$, define,

$$\|u\| = \begin{cases} \left(\sum_{|\alpha|\le k} \int_\Omega |D^\alpha u|^p dx \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty \\ \sum_{|\alpha|\le k} \text{esssup}_\Omega |D^\alpha u|, & \text{if } p = \infty. \end{cases}$$

Notation:

1. $H^k(\Omega) = W^{k,2}(\Omega)$.
2. $u_m \to u$ in $W^{k,p}_{loc}(\Omega)$, we mean $u_m \to u$ in $W^{k,p}(V)$ for all $V \subset\subset \Omega$.
3. $W^{k,p}_0(\Omega)$ means the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

It is easy to check that $W^{k,p}(\Omega)$ is a Banach space $\forall\, k \in \mathbb{N}$ and $\forall\, 1 \le p \le \infty$.

Theorem 2.2.1. (Elementary properties of weak derivative) Let $u, v \in W^{k,p}(\Omega)$ and $|\alpha| \le k$. Then
(i) $D^\alpha(\Omega) \in W^{k-|\alpha|,p}(\Omega)$.
(ii) $D^\alpha(D^\beta u) = D^\beta(D^\alpha u) = D^{\alpha+\beta} u$ where $|\alpha| + |\beta| \le k$.
(iii) $\forall\, \lambda,\, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(\Omega)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$.
(iv) If V is an open subset of Ω then $u \in W^{k,p}(V)$.
(v) If $\zeta \in C_c^\infty(\Omega)$ then $\zeta u \in W^{k,p}(\Omega)$ and

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u, \quad \text{where} \quad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha! - \beta!)}.$$

The above formula is called Leibniz's formula.

Proof. (i), (ii), (iii) and (iv) easily follows using the definition. (v) can be proved using induction on $|\alpha|$. For detailed proof, we refer [L.C86, Theorem 1, Section 5.2]. ∎

2.3 Approximations, Extension and Trace

In this section, we try to approximate functions in Sobolev space using smooth functions. Given any $\epsilon > 0$, define $\Omega_\epsilon := \{x \in \Omega | \text{dist}(x, \partial\Omega) > \epsilon\}$ and $\eta_\epsilon(x) := \frac{1}{\epsilon^n} \eta(\frac{x}{\epsilon})$, where $\eta(x)$ is defined as

$$\eta(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & \text{if } |x| < 1 \\ 0, & \text{if } |x| \ge 1. \end{cases}$$

The constant in the definition of η is chosen such a way that $\int_{\mathbb{R}^n} \eta(x) \mathrm{d}x = 1$. The function η is known as the standard mollifier. Note that $\eta_\epsilon \in C^\infty(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} \eta_\epsilon \, dx = 1, \quad \text{supp}\ (\eta_\epsilon) \subset B(0, \epsilon)$.

Theorem 2.3.1. (Local approximation) Assume $u \in W^{k,p}(\Omega)$, for some $1 \leq p < \infty$. Set, $u^\epsilon := \eta_\epsilon * u$ in Ω_ϵ. Then
(i) $u^\epsilon \in C^\infty(\Omega_\epsilon)$ $\forall\, \epsilon > 0$
(ii) $u^\epsilon \to u$ in $W^{k,p}_{loc}(\Omega)$ as $\epsilon \to 0$.

Proof. As n_ϵ is C^∞, (i) follows from the properties of convolution.
To prove (ii), first note that

$$D^\alpha u^\epsilon = \eta_\epsilon * D^\alpha u \quad \text{in} \quad \Omega_\epsilon. \tag{2.1}$$

Since $D^\alpha u \in L^p_{loc}(\Omega)$, doing a standard computation using (2.1), it can be shown that for any $V \subset\subset \Omega$, $D^\alpha u^\epsilon \to D^\alpha u$ in $L^p(V)$ as $\epsilon \to 0$, for all $|\alpha| \leq k$. Hence,

$$\|u^\epsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha| \leq k} \|D^\alpha u^\epsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\epsilon \to 0$. For detailed proof we refer [L.C86, Theorem 1, Section 5.3]. ■

Theorem 2.3.2. (Global approximation by smooth function) Let Ω be a bounded open subset of $\mathbb{R}^n$ and $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists $u_m \in W^{k,p}(\Omega) \cap C^\infty(\Omega)$ satisfying

$$u_m \to u \quad \text{in} \quad W^{k,p}(\Omega).$$

Proof. For proof, we refer [L.C86, Theorem 2, Section 5.3]. ■

In the next theorem, we approximate the functions in the Sobolev space upto the boundary. Here we will need some 'smoothness' of boundary of Ω.

Definition 2.3.1. The boundary $\partial\Omega$ of a bounded open subset Ω of $\mathbb{R}^n$ is said to be C^k provided $\forall\ x \in \partial\Omega$, there exists $r > 0$ and some C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ so that after relabeling and reorienting of coordinates if needed, we can get

$$\Omega \cap B(x,r) = \{y = (y_1, \ldots, y_n) \in B(x,r) | y_n > \gamma(y_1, \ldots, y_{n-1})\}.$$

We call $\partial\Omega$ to be C^∞ provided $\partial\Omega$ is C^k for any integer k.

Theorem 2.3.3. (Global approximation by functions smooth up to the boundary) Let Ω be a bounded open subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Assume, $u \in W^{k,p}(\Omega)$ for some $1 \leq p < \infty$. Then there exists $u_m \in C^\infty(\overline{\Omega})$ satisfying

$$u_m \to u \quad \text{in} \quad W^{k,p}(\Omega).$$

Proof. For proof, we refer [L.C86, Theorem 3, Section 5.3]. ∎

Our next goal is to extend functions in $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^n)$. Note that we can not extend the function by defining zero outside Ω as then the extended function may not be weakly differentiable. Next theorem tells us about how to extend such a function.

Theorem 2.3.4. (Extension Theorem) Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ be bounded and open such that $\partial\Omega$ be C^1. Let V be any bounded open set such that $\Omega \subset\subset V$. Then there exists a bounded linear operator

$$E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n)$$

such that $\forall\, u \in W^{1,p}(\Omega)$:
(i) $Eu = u$ almost everywhere in Ω
(ii) Supp $(Eu) \subset V$ and
(iii) $\|Eu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)}$, where C depends only on p, V and Ω.

Definition 2.3.2. The function Eu in the above theorem is called an extension of u to $\mathbb{R}^n$.

Proof. For proof, we refer [L.C86, Theorem 1, Section 5.4]. ∎

Next we discuss how to assign "boundary value" along $\partial\Omega$ for functions in $u \in W^{1,p}(\Omega)$. Note that, functions u in $W^{1,p}(\Omega)$ is defined in almost everywhere sense and so it does not make any sense if we talk about the value of such a function on a measure zero set. Since $\partial\Omega$ has measure zero, there is no direct meaning we can give to $u|_{\partial\Omega}$. Using the next theorem we resolve this problem.

Theorem 2.3.5. (Trace Theorem) Assume $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be such that $\partial\Omega$ is C^1. Then there exists a bounded linear operator

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that
(i) $Tu = u_{|\partial\Omega}$ if $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$.
(ii)

$$\|Tu\|_{L^p(\partial\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \quad \forall\, u \in W^{1,p}(\Omega),$$

where C depends only on p and Ω.

Proof. See [L.C86, Theorem 1, Section 5.5]. ∎

Definition 2.3.3. Tu is called the trace of u on $\partial\Omega$.

Theorem 2.3.6. (Trace zero functions) Let $u \in W^{1,p}(\Omega)$ where Ω is a bounded domain with $\partial\Omega$ is C^1. Then

$$u \in W_0^{1,p}(\Omega) \iff Tu = 0 \quad \text{on} \quad \partial\Omega.$$

Proof. See [L.C86, Theorem 2, Section 5.5]. ■

The last theorem justifies why we denoted the closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ as $W_0^{1,p}(\Omega)$.

2.4 Sobolev embedding

Next we study some very important inequalities which will give embedding of some Sobolev spaces into other spaces. We set

$$p^* = \frac{np}{n-p}.$$

Theorem 2.4.1. (Gagliardo-Nirenberg-Sobolev inequality) Let $1 \leq p < \infty$. Then $\forall\, u \in C_c^1(\mathbb{R}^n)$

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Du\|_{L^p(\mathbb{R}^n)},$$

where C depends only on p and n.

Proof. See [L.C86, Theorem 1, Section 5.6]. ■

Theorem 2.4.2. (Estimate for $W^{1,p}(\Omega)$ for $1 \leq p < n$) Let Ω be a bounded and open subset of $\mathbb{R}^n$ with $\partial\Omega$ being C^1. Assume $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Then $u \in L^{p^*}(\Omega)$ with

$$\|u\|_{L^{p^*}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)},$$

where C depends on p, n and Ω.

Proof. See [L.C86, Theorem 2, Section 5.6]. ■

Theorem 2.4.3. (Estimate for $W_0^{1,p}(\Omega)$ for $1 \leq p < n$) Let Ω be a bounded and open subset of $\mathbb{R}^n$ and $u \in W_0^{1,p}(\Omega)$, for some $1 \leq p < n$. Then

$$\|u\|_{L^q(\Omega)} \leq C\|Du\|_{L^p(\Omega)} \quad \forall\, q \in [1, p^*],$$

where $C = C(p, q, n, \Omega)$.

Proof. See [L.C86, Theorem 3, Section 5.6]. ■

Taking $q = p$ in the above theorem yields

Corollary 2.4.3.1. If $u \in W_0^{1,p}(\Omega)$ with $1 \leq p \leq \infty$, then

$$\|u\|_{L^p(\Omega)} \leq C\|Du\|_{L^p(\Omega)}$$

i.e., $(\int_\Omega |Du|^p)^{\frac{1}{p}}$ is an equivalent norm in $W_0^{1,p}(\Omega)$.

The above estimate is also called Poincaré's Inequality. The other version (more general version) of Poincaré's Inequality, will be mentioned later.
In all the previous theorems we have assumed that $p < n$. Next we deal the case $n < p < \infty$, where we will see that if $u \in W^{1,p}(\Omega)$ then u is infact Hölder continuous. To consider the case $p = n$, first we observe that $p^* = \frac{np}{n-p} \to \infty$ as $p \to n$. Therefore, we may expect that $u \in W^{1,n}(\Omega)$ implies $u \in L^\infty(\Omega)$, which is false for $n > 1$. In fact, if $u \in W^{1,n}(\Omega)$ implies $u \in L^q(\Omega)$ for all $q \in [1, \infty)$ but in general $u \notin L^\infty(\Omega)$. For example, consider $\Omega = B(0,1)$ (a unit ball in $\mathbb{R}^n$) and $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$. Then $u \in W^{1,n}(\Omega)$ but $u \notin L^\infty(\Omega)$.

Theorem 2.4.4. (Morrey's Inequality) Assume $n < p \leq \infty$ and $u \in W^{1,p}(\mathbb{R}^n)$. Then

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\mathbb{R}^n)} \quad \forall\, u \in C^1(\mathbb{R}^n),$$

where $\gamma := 1 - \frac{n}{p}$.

Proof. See [L.C86, Theorem 4, Section 5.6]. ■

Definition 2.4.1. We say u^* is a version of u, if $u^* = u$ almost everywhere.

Theorem 2.4.5. (Estimates for $W^{1,p}(\Omega)$, $n < p \leq \infty$) Suppose Ω is a bounded and open subset of $\mathbb{R}^n$ with C^1 boundary. Let $u \in W^{1,p}(\Omega)$ for some $n < p \leq \infty$. Then u has a version $u^* \in C^{0,\gamma}(\overline{\Omega})$, with $\gamma := 1 - \frac{n}{p}$ and satisfies

$$u^*_{C^{0,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{1,p}(\Omega)}.$$

Here $C = C(n, p, \Omega)$.

Proof. See [L.C86, Theorem 5, Section 5.6]. ■

Remark: In view of the above theorem, here onwards we will identify a $W^{1,p}(\Omega)$ ($p > n$) function u with its continuous version u^*.
Next, we state the general Sobolev inequalities for functions in $W^{k,p}(\Omega)$ ($k \geq 1$).

Theorem 2.4.6. (General Sobolev Inequalities) Suppose Ω is a bounded and open subset of $\mathbb{R}^n$ with $\partial\Omega$ is C^1. Let $u \in W^{k,p}(\Omega)$.
(i) If $k < \frac{n}{p}$, then $u \in L^q(\Omega)$ where $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$ and

$$\|u\|_{L^q(\Omega)} \leq C\|u\|_{W^{k,p}(\Omega)},$$

where C depends only on Ω, k, p and n.
(ii) If $k > \frac{n}{p}$, then $u \in C^{k-\left[\frac{n}{p}\right]-1,\gamma}(\overline{\Omega})$, where

$$\gamma = \begin{cases} \left[\frac{n}{p}\right] + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \text{ is not an integer} \\ \text{any positive number less than 1} & \text{otherwise.} \end{cases}$$

Moreover,

$$\|u\|_{C^{k-\frac{n}{p}-1,\gamma}(\overline{\Omega})} \leq C\|u\|_{W^{k,p}(\Omega)},$$

where C depends on $n.\gamma, k, p$ and Ω.

Proof. See [L.C86, Theorem 6, Section 5.6]. ■

The Gagliardo-Nirenberg-Sobolev inequality yields us that for $1 \leq p < n$, $W^{1,p}(\Omega)$ is embeded in $L^{p^*}(\Omega)$, where $p^* = \frac{np}{n-p}$. Next we show that $W^{1,p}(\Omega)$ is in fact compactly embeded in $L^q(\Omega)$, for all $q \in [1, p^*)$.

2.5 Compact embedding

Definition 2.5.1. Let X, Y be Banach be such that $X \subset Y$. X is said to be compactly embeded in Y, denoted by $X \subset\subset Y$ if
(i) $\|u\|_Y \leq C\|u\|_X$, where C does not depend on $u \in X$.
(ii) each bounded sequence in X has a convergent subsequence in Y.

Theorem 2.5.1. (Rellich-Kondrachov Compactness Theorem) Let Ω be an open and bounded subset of $\mathbb{R}^n$ with $\partial\Omega$ be C^1. Then,
(i) $W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \forall, q \in [1, p^*),$ if $1 \leq p < n,$
(ii) $W^{1,p}(\Omega) \subset\subset L^q(\Omega) \quad \forall, q \in [p, \infty),$ if $p = n,$
(iii) $W^{1,p}(\Omega) \subset\subset C(\overline{\Omega})$ if $p > n.$

Proof. See [L.C86, Theorem 1, Section 5.7] and [H.11, Theorem 9.16] . ■

We note that since $p^* > p$ and $p^* \to \infty$ as $p \to n$, the last theorem yields

$$W^{1,p}(\Omega) \subset\subset L^p(\Omega)$$

for all $1 \leq p \leq \infty$.

2.6 Additional topics

Theorem 2.6.1. (Poincaré's inequality) Assume Ω is a bounded, open and connected subset of $\mathbb{R}^n$ and $\partial\Omega$ is C^1. Let $1 \leq p \leq \infty$ and set $(u)_\Omega := \frac{1}{|u|}\int_\Omega u\,dy$. Then $\forall u \in W^{1,p}(\Omega)$,

$$\|u-(u)\|_{L^p(\Omega)} \leq C\|Du\|_{L^p},$$

where the constant C depends only on p, n and Ω.

Proof. See [L.C86, Theorem 1, Section 5.8]. ∎

The next theorem is a special case of the above Poincaré's inequality in ball. For that, we set

$$(u)_{x,r} = \frac{1}{|B(x,r)|}\int_{B(x,r)} u\mathrm{d}y.$$

Theorem 2.6.2. $\big($**Poincaré's inequality for** $B(x,r)\big)$ Assume $u \in W^{1,p}(B^0(x,r))$, for some $1 \leq p \leq \infty$. Then

$$\|u-(u)_{x,r}\|_{L^p(B(x,r))} \leq Cr\|Du\|_{L^p(B(x,r))},$$

where the constant C depends only on p and n.

Proof. See [L.C86, Theorem 2, Section 5.8]. ∎

Next we aim to study about Different quotients which will be helpful when we'll study regularity of solutions of second order elliptic equations.

Definition 2.6.1. Suppose $u : \Omega \to \mathbb{R}$ is locally integrable and $V \subset\subset \Omega$.
(i) For $1 \leq i \leq n$, the i^{th} different quotient of size h is defined to be

$$D_i^h u(x) = \frac{u(x+he_i)-u(x)}{h}$$

where $h \in \mathbb{R}$ be such that $0 < |h| < dist(V, \partial\Omega)$ and $x \in V$.
(ii) $D^h u := (D_1^h u, ..., D_n^h u)$.

Theorem 2.6.3. (Different quotient and weak derivative) (i) Assume $u \in W^{1,p}(\Omega)$, for some $1 \leq p < \infty$. Then, for $V \subset\subset \Omega$

$$\left\| D^h(u) \right\|_{L^p(V)} \leq C \|Du\|_{L^p}$$

for some constant C and all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$.
(ii) Suppose, $u \in L^p(V)$ with $1 < p < \infty$ and there exists a constant C such that

$$\left\| D^h u \right\|_{L^p(V)} \leq C$$

for all $0 < |h| < \frac{1}{2} dist(V, \partial\Omega)$. Then $u \in L^p(V)$ and

$$\|Du\|_{L^p(V)} \leq C.$$

Remark. Assertion (ii) of the previous theorem is false for $p = 1$.

Proof. See [L.C86, Theorem 3, Section 5.8]. ■

Now using Fourier transform method we see the characterization of $H^k(\mathbb{R}^n)$.

Theorem 2.6.4. (Characterization of H^k) Let k be a nonnegative integer.
(i) A function $u \in L^2(\mathbb{R}^n)$ belongs to $H^k(\mathbb{R}^n) \iff (1 + |y|^k)\hat{u} \in L^2(\mathbb{R}^n)$, where $\hat{u}$ denotes the Fourier transformation of u.
(ii) There exists a constant $C > 0$ s.t.

$$\frac{1}{C} \|u\|_{H^k(\mathbb{R}^n)} \leq \left\| (1 + |y|^k)\hat{u} \right\|_{L^2 \mathbb{R}^n} \leq C \|u\|_{H^k(\mathbb{R}^n)} \quad \forall\, u \in H^k(\mathbb{R}^n).$$

Proof. See [L.C86, Theorem 7, Section 5.8]. ■

The last theorem motivates us to define fractional Sobolev spaces in the following way:

Definition 2.6.2. (Fractional Sobolev spaces) Let $u \in L^2(\mathbb{R}^n)$ and $0 < s < \infty$. Then $u \in H^s(\mathbb{R}^n)$ if $(1 + |y|^s)\hat{u} \in L^2(\mathbb{R}^n)$. For s not being an integer, we set

$$\|u\|_{H^s(\mathbb{R}^n)} := \|(1 + |y|^s)\hat{u}\|_{L^2(\mathbb{R}^n)}.$$

Now we define the dual space of the Hilbert space $W_0^{1,p}(\Omega)$.

Definition 2.6.3. (dual space of $W_0^{1,p}(\Omega)$) The dual space of $W_0^{1,p}(\Omega)$ is denoted by $W^{-1,p'}(\Omega)$, where $p' = \frac{p}{p-1}$. If $p = 2$ then dual space of $H_0^1(\Omega)$ is denoted as $H^{-1}(\Omega)$.

The dual of $L^2(\Omega)$ is identified with $L^2(\Omega)$ but we do not identify $H_0^1(\Omega)$ with it's dual space $H^{-1}(\Omega)$. Rather we have the following inclusion

$$H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega).$$

If Ω is bounded then

$$W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega) \quad \text{if} \quad \frac{2n}{n+2} \leq p < \infty.$$

If Ω is unbounded subset of $\mathbb{R}^n$, then the same holds but in the range $\frac{2n}{n+2} \leq p \leq 2$.

Theorem 2.6.5. Characterization of $W^{-1,p'}(\Omega)$:
(i) Let $f \in W^{-1,p'}(\Omega)$. Then $\exists\, f^0,\ f^1, \ldots.., f^n \in L^{p'}(\Omega)$ s.t.

$$\langle f, v \rangle = \int_\Omega f^0 v + \sum_{i=1}^{n} f^i v_{x_i} \mathrm{d}x \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{2.2}$$

and

$$\|f\|_{W^{-1,p'}(\Omega)} = \max_{0 \leq i \leq n} \|f_i\|_{L^{p'}(\Omega)}.$$

If Ω is bounded then we take $f_0 = 0$.

Notation: In the context of previous theorem we write $f = f^0 - \sum_{i=1}^{n} f^i_{x_i}$.

Proof. See [H.11, Proposition 9.20]. ■

Chapter 3

Second Order Elliptic Partial Differential Equations

In this chapter our goal is to study the existence of solution to second order uniformly elliptic boundary value problems via energy method within Sobolev spaces. We will also discuss regularity properties of the solution and maximum principle. For more details on these topics, [N.S98] is a good reference.

3.1 Definitions

Mostly in this Chapter, we discuss the problem of the type

$$\begin{cases} Lu = f & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded and open subset of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ is given function and L is the second order partial differential operator of one of the two forms

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{3.2}$$

or

$$Lu = -\sum_{i,j=1}^{n} a^{ij}(x)u_{x_i}x_j + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{3.3}$$

for given coefficient function a^{ij}, b^i, c $(i,\, j = 1, \cdots, n)$.
If L is of the form (3.2), we say (3.1) is in divergence form and if L is of the form (3.3), we say (3.1) is in nondivergence form. The boundary condition here, that is , $u = 0$ on

$\partial\Omega$ is known as *Dirichlet boundary condition*.
Remark: There is another important kind of boundary value problem, known as Neumann boundary value problem. There we know the value of $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$, where ν is the unit outward normal on the boundary. We'll discuss results about this problem separately in the last section.
Throughout this chapter, we assume $a^{ij} = a^{ji}$.

Definition 3.1.1. The partial differential operator L is said to be uniformly elliptic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$$

for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^n$.

This means L is elliptic if for each $x \in \mathbb{R}^n$, the matrix $\left((a^{ij}(x))\right)_{i,j}^{n}$ is positive definite and the smallest eigenvalue is greater than equal to θ.

3.1.1 Weak Solution

Consider (3.1) where L is in the divergence form. First we try to define the weak solution of (3.1) in the appropriate Sobolev space and show the existence of such solutions and then in later sections we'll see the study the smoothness of such solutions.
In the following part, we'll assume

$$a^{ij}, b^i, c \in L^\infty(\Omega) \quad \forall\ i,\ j = 1, ..., n \tag{3.4}$$

and also

$$f \in L^2(\Omega). \tag{3.5}$$

Definition 3.1.2. (i) For a given elliptic operator L in its divergence form (3.2), the associated bilinear form is defined by

$$B[u,v] := \int_\Omega \Big(\sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuv \Big) \mathrm{d}x \qquad \forall\ u,\ v \in H_0^1(\Omega).$$

(ii) u is said to be a weak solution of the problem (3.1), if it satisfies

$$B[u,v] := (f,v) \qquad \forall\ v \in H_0^1(\Omega).$$

By (f,v), we denote the usual inner product in $L^2(\Omega)$.

We now study a more general problem

$$\begin{cases} Lu = f^0 - \sum_{i=1}^n f^i_{x_i} & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.6}$$

where L has the divergence form and each f^i is in $L^2(\Omega)$. From the characterization of $H^{-1}(\Omega)$, $f^0 - \sum_{i=1}^n f^i_{x_i} \in H^{-1}(\Omega)$.

Definition 3.1.3. $u \in H^1_0(\Omega)$ is said to be a weak solution of the problem (3.6), if it satisfies

$$B[u,v] = \langle f, v\rangle \qquad \forall\, v \in H^1_0(\Omega),$$

where $\langle f, v\rangle = \int_\Omega \left(f^0 v + \sum_{i=1}^n f^i v_{x_i}\right) dx$, the usual dual space action of $H^{-1}(\Omega)$ on $H^1_0(\Omega)$.

3.2 Existence of Weak Solutions

To study the existence of weak solution, first we discuss an abstract theorem, namely Lax-Milgram theorem from functional analysis.

Theorem 3.2.1. (Lax-Milgram theorem) Let $B : H \times H \to \mathbb{R}$ be a bilinear form which satisfies the following
(i) $\exists\, \alpha > 0$ s.t. $|B[u,v]| \le \alpha\|u\|\|v\|$ for all u, $v \in H$
(ii) $\exists\, \beta > 0$ s.t. $\beta\|u\|^2 \le B[u,u]$ for all $u \in H$.
Then for a bounded linear functional $f : H \to \mathbb{R}$, there exists unique $u \in H$ such that $B[u,v] = \langle f, v\rangle\ \forall\, v \in H$.

Proof. See [L.C86, Theorem 1, Section 6.2]. ■

The Lax-Milgram theorem is a generalization of Riesz Representation theorem in the sense it does not assume that the bilinear form is symmetric.
In the context of weak solution of second order elliptic PDEs, take the particular bilinear form $B : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ having the form

$$B[u,v] = \int_\Omega \Big(\sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cuv\Big) dx \qquad \forall\, u,\, v \in H^1_0(\Omega).$$

We try to use Lax Milgram thorem in this set-up.

Theorem 3.2.2. (Energy Estimates) There exist constants α, $\beta > 0$ and $\gamma > 0$ such that
(i) $|B[u,v]| \le \|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}$.

(ii) $\beta\|u\|^2_{H^1_0(\Omega)} \leq B[u,u] + \gamma\|u\|^2_{L^2(\Omega)}$ $\forall$ u, $v \in H^1_0(\Omega)$.

Proof. (i) is straight forward application of traingle inequality via (3.4).
(ii)

$$\begin{aligned}\theta\int_\Omega |Du|^2\mathrm{d}x \leq \int_\Omega \sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j}\mathrm{d}x &= B[u,u] - \int_\Omega\left(\sum_{i=1}^n b^i u_{x_i}u + cu^2\right)\mathrm{d}x \\ &\leq B[u,u] + \sum_{i=1}^n \left\|b^i\right\|_{L^\infty}\int_\Omega |Du||u|\mathrm{d}x + \|c\|_{L^\infty}\int_\Omega u^2\mathrm{d}x \qquad (3.7)\end{aligned}$$

Using the Young inequality with $\epsilon > 0$, we estimate the middle term of the last inequality

$$\int_\Omega |Du||u|\mathrm{d}x \leq \epsilon\int_\Omega |Du|^2\mathrm{d}x + \frac{1}{4\epsilon}\int_\Omega u^2\mathrm{d}x.$$

Now we plug the above expression back into (3.7) and choose ϵ small enough so that $\epsilon\sum_{i=1}^n \left\|b^i\right\|_{L^\infty(\Omega)} < \frac{\theta}{2}$. This yields

$$\frac{\theta}{2}\int_\Omega |Du|^2\mathrm{d}x \leq B[u,u] + C\int_\Omega u^2\mathrm{d}x$$

where $C > 0$ is a constant.
Now using the Poincaré's inequality Corollary 2.4.3.1, we conclude the proof of (ii). ∎

As can we see, if $\gamma > 0$ in previous theorem then $B[\,,\,]$ does not satisfy the conditions of Lax-Milgram theorem. In the the next theorem, we resolve this issue.

Theorem 3.2.3. (First existance theorem for weak solutions) There exists a constant $\gamma \geq 0$ such that forall $\mu \geq \gamma$ and each $f \in L^2(\Omega)$, there exists a unique weak solution u of the boundary value problem

$$\begin{cases} Lu + \mu u = f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

Proof. Let γ be as obtained in Theorem 3.2.2. For $\mu \geq \gamma$, define the bilinear form

$$B_\mu[u,v] := B[u,v] + \mu(u,v) \quad (u,v \in H^1_0(\Omega)).$$

This bilinear problem corresponds to the operator $L_\mu u := Lu + \mu u$. We have defined B_μ in such a way that it satisfies the hypothesis of Lax-Milgram theorem.

Now fix $f \in L^2(\Omega)$ as in the statement of the theorem. Define the dual action of f on $v \in H_1^0(\Omega)$ as $\langle f, v\rangle := (f,v)_{L^2(\Omega)}$. Clearly it is bounded linear functional on $L^2(\Omega)$ and thus on $H_0^1(\Omega)$. Therefore, using Lax-Milgram theorem we find unique $u \in H_0^1(\Omega)$ such that $B_\mu[u,v] = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega)$. This proves the theorem. ■

Using the theory of Fredholm alternative, we can gain some valuable information about solvability of second order elliptic PDE. Before that we define adjoint of elliptic operator, adjoint bilinear form and weak solution to adjoint problem.

Definition 3.2.1. (i) The adjoint of L denoted by L^* is defined by

$$L^*v := -\sum_{i,j=1}^{n}(a^{ij}v_{x_j})_{x_i} - \sum_{i=1}^{n} b^i v_{x_i} + (c - \sum_{i=1}^{n} b^i_{x_i})v$$

provided each $b^i \in C^1(\overline{\Omega})$, $i = 1, \cdots, n$.
(ii) $B^* : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ defined by $B^*[v,u] = B[u,v]$ $\forall\, u,\, v \in H_0^1(\Omega)$ is called the adjoint bilinear form of B.
(iii) $v \in H_0^1(\Omega)$ is a weak solution of the adjoint problem

$$\begin{cases} Lv^* = f & in \quad \Omega \\ v = 0 & on \quad \partial\Omega \end{cases}$$

if $B^*[v,u] = (f,u)\ \forall\, u \in H_0^1(\Omega)$.

Theorem 3.2.4. (Second Existence Theorem for Weak Solutions) (i) Exactly one of the below holds true: either

$$(1) \begin{cases} \text{For every } f \in L^2(\Omega) \text{, there exists unique weak solution } u \text{ of} \\ (A) \quad \begin{cases} Lu = f \text{ in} & \Omega \\ u = 0 \text{ on} & \partial\Omega, \end{cases} \end{cases}$$

or

$$(2) \begin{cases} \text{there exists unique weak solution } u \not\equiv 0 \text{ of} \\ (B) \quad \begin{cases} Lu = 0 \text{ in} & \Omega \\ u = 0 \text{ on} & \partial\Omega \end{cases} \end{cases}$$

(ii) Moreover, if (2) holds true, then the dimension of the subspace $N \subset H_0^1(\Omega)$ of weak solutions of (B) is finite and equals the dimension of subspace $N^* \subset H_0^1(\Omega)$ of weak solutions of

$$\begin{cases} L^*v = 0 & \text{in} \quad \Omega \\ v = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{3.8}$$

(iii) (A) has a weak solution $\iff (f,v) = 0 \quad \forall\, v \in N^*$.

The division (A) and (B) is known as Fredholm alternative.

Proof. Choose $\mu = \gamma$ as Theorem 3.2.3 and define the bilinear form

$$B_\gamma[u, v] = B[u, v] + \gamma(u, v),$$

corresponding to the operator $L_\gamma u := Lu + \gamma u$. Therefore, for every $g \in L^2(\Omega)$ there exists unique $u \in H_0^1(\Omega)$ satisfying

$$B_\gamma[u, v] := (g, v) \quad \forall\ v \in H_0^1(\Omega). \tag{3.9}$$

When ever (3.9) is satisfied, we denote $u = L_\gamma^{-1} g$, We note that $u \in H_0^1(\Omega)$ is a weak solution of $(A) \iff B_\gamma[u, v] = (\gamma u + f, v) \quad \forall\ v \in H_0^1(\Omega) \iff u = L_\gamma^{-1}(\gamma u + f)$.

Writing the last equality as $u - Ku = h$, where $Ku := \gamma L_\gamma^{-1} u$ and $h := L_\gamma^{-1} f$, it can be easily checked that $K : L^2(\Omega) \to L^2(\Omega)$ is a compact bounded linear operator. Now applying the theory of Fredholm alternative we obtain exactly one of the following holds true:

$$(C) \begin{cases} \forall\ h \in L^2(\Omega), \text{ there exists } u \in L^2(\Omega) \\ \text{satisfying } u - Ku = h \end{cases}$$

or

$$(D) \quad \begin{cases} \text{there exists a nonzero } \ u \in L^2(\Omega) \\ \text{satisfying } \ u - Ku = 0 \end{cases}$$

If (C) holds true, then by the above arguments there exists a unique weak solution u satisfying (A).

If (D) holds true, then definitely $\gamma \neq 0$ and using the theory of Fredholm alternative we get that the space N consists of solutions of (D) is finite dimentional and has same dimention as N^*, the space of solutions of $v - K^* v = 0$.

Now note that (D) holds if and only if u is a weak solution of (B) and v is a weak solution of (3.8) if and only if $v - K^* v = 0$.

Recall, (C) has a weak solution if and only if $(h, v) = 0$ for all v satisfying $v - K^* v = 0$. observe that

$$(h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v).$$

Hence (A) has a weak solution if and only if $(f, v) = 0$ for all v satisfying (3.8). This ends the proof.

■

Theorem 3.2.5. (Third existance theorem for weak solutions) (i) There exists at most a countable set $\Sigma \subset \mathbb{R}$ s.t. the boundary value problem

$$(\alpha) \begin{cases} Lu = \lambda u + f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega \end{cases}$$

has a unique weak solution for each $f \in L^2(\Omega)$ if and only if $\lambda \notin \Sigma$.
(ii) If Σ is infinite, then $\lambda_1 \leq \lambda_2 \leq \cdots$ with $\lambda_k \to \infty$ as $k \to \infty$.

Proof. (i) Let γ be as in Theorem 3.2.2 and $\lambda > -\gamma$. Without loss of generality, we may assume $\gamma > 0$. The theory of Fredholm alternative says that (α) has unique solution in weak sense for each $f \in L^2(\Omega)$ if and only if $u \equiv 0$ is the one and only weak solution of

$$\begin{cases} Lu = \lambda u & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

This will be true if and only if $u \equiv 0$ is the unique weak solution of

$$(\beta) \begin{cases} Lu + \gamma u = (\lambda + \gamma) u & in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases}$$

For (β) to hold true, we exactly need

$$u = L_\gamma^{-1}(\gamma + \lambda) u = \frac{\gamma + \lambda}{\gamma} K u, \tag{3.10}$$

where $K : L^2(\Omega) \to L^2(\Omega)$ defined as $Ku = \gamma L_\gamma^{-1} u$ is the bounded linear compact operator as in the proof of the previous theorem. Now if $u \equiv 0$ is the only solution of the problem (3.10), then

$$\frac{\gamma}{\gamma + \lambda} \text{ is not an eigenvalue of K.} \tag{3.11}$$

Hence $\forall f \in L^2(\Omega)$, (α) has a unique solution in weak sense if and only if (3.11) holds.
(ii) From the spectral theory of compact self adjoint operator, we know K has either a finite set or a sequence of eigenvalue which converges to 0. In the second case, since $\alpha > -\gamma$, from (3.10) we can conclude that for every $f \in L^2(\Omega)$, (α) admits a weak solution except for a sequence $\lambda_k \to \infty$.

∎

Definition 3.2.2. Σ is known as real spectrum of the elliptic operator L.

Theorem 3.2.6. (Boundedness of inverse) For $\lambda \notin \Sigma$, there exists a positive constant

C depending only on λ, coefficient of L and Ω such that

$$\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)},$$

where f is any function in $L^2(\Omega)$ and $u \in H_0^1(\Omega)$ is the unique weak solution of

$$\begin{cases} Lu = \lambda u + f & in \quad \Omega \\ u = 0 & on \quad \partial\Omega. \end{cases}$$

Proof. See [L.C86, Theorem 6, Section 6.2]. ■

3.3 Regularity of Solutions

After the existence theory of linear elliptic equations of second order, now we focus on the question whether these weak solutions are smooth. This is called *r*egularity problem of weak solutions. Throughout this section Ω will be will be open, bounded subset of $\mathbb{R}^n$. Let, $u \in H_0^1(\Omega)$ to be the weak solution of

$$Lu = f \text{ } in \text{ } \Omega \tag{3.12}$$

and L be in the divergence form

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)u_{x_i} + c(x)u \tag{3.13}$$

We also assume L is uniformly elliptic.

Theorem 3.3.1. **(Interior H^2 regularity)** Suppose

$$a^{ij} \in C^1(\Omega), \; b^i, c \in L^\infty(\Omega) \;\; \forall \; i,j = 1, \cdots, n$$

and $f \in L^2(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of

$$Lu = f \text{ } in \;\; \Omega,$$

Then

$$u \in H^2_{loc}(\Omega)$$

and for $V \subset\subset \Omega$,

$$\|u\|_{H^2(V)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})$$

for some C which depends on V, coefficient of L and Ω.

Proof. Take some $V \subset\subset \Omega$. Now select W such that $V \subset\subset W \subset\subset \Omega$. Using locally finite C^∞ partition of unity, we can choose a smooth function ζ such that

$$\begin{cases} 0 \leq \zeta \leq 1, & \zeta = 1 \text{ on } \quad V \\ \zeta = 0 & \text{on} \quad \mathbb{R}^n - W. \end{cases}$$

Clearly distance between $\partial\Omega$ and W is positive.
As u is a weak solution of (3.12), we have $B[u, v] = (f, v) \quad \forall\, v \in H_0^1(\Omega)$. Therefore,

$$\sum_{i,j=1}^{n} \int_\Omega a^{ij} u_{x_i} v_{x_j} \,\mathrm{d}x = \int_\Omega \tilde{f} v \,\mathrm{d}x, \tag{3.14}$$

where $\tilde{f} = f - \sum_{i=1}^{n} b^i u_{x_i} - cu$.
Choose $|h| > 0$ small and $k \in \{1, ..., n\}$. Then define the test function v as $v = -D_k^{-h}(\zeta^2 D_k^h u)$ and we substitute it in (3.14), where D_k^h denotes the difference quotient as defined in Definition 2.6.1. This yields

$$\begin{aligned} \text{LHS of (3.14)} &= -\sum_{i,j=1}^{n} \int_\Omega a^{ij} u_{x_i} \left[D_k^{-h} \left(\zeta^2 D_k^h u \right) \right]_{x_j} dx \\ &= \sum_{i,j=1}^{n} \int_\Omega D_k^h \left(a^{ij} u_{x_i} \right) \left(\zeta^2 D_k^h u \right)_{x_j} dx \\ &= \sum_{i,j=1}^{n} \int_\Omega a^{ij,h} \left(D_k^h u_{x_i} \right) \left(\zeta^2 D_k^h u \right)_{x_j} + \left(D_k^h a^{ij} \right) u_{x_i} \left(\zeta^2 D_k^h u \right)_{x_j} dx. \end{aligned}$$

Here we used the formulas

$$\int_\Omega v D_k^{-h} w \,\mathrm{d}x = -\int_\Omega w D_k^h v \,\mathrm{d}x \quad \text{and} \quad D_k^h(vw) = v^h D_k^h w + w D_k^h v$$

for $v^h(x) := v\,(x + he_k)$. Hence,

$$\begin{aligned} \text{LHS} &= \sum_{i,j=1}^{n} \int_\Omega a^{ij,h} D_k^h u_{x_i} D_k^h u_{x_j} \zeta^2 \,\mathrm{d}x \\ &\quad + \sum_{i,j=1}^{n} \int_\Omega \left[a^{ij,h} D_k^h u_{x_i} D_k^h u 2\zeta\zeta_{x_j} + \left(D_k^h a^{ij} \right) u_{x_i} D_k^h u_{x_j} \zeta^2 + \left(D_k^h a^{ij} \right) u_{x_i} D_k^{h_k} u 2\zeta\zeta_{x_j} \right] \mathrm{d}x \\ &= : A + B. \end{aligned} \tag{3.15}$$

Using uniform ellipticity condition of L,

$$A \geq \theta \int_\Omega \zeta^2 \left| D_k^h Du \right|^2 \, \mathrm{d}x.$$

From the regularity of coefficients

$$|B| \leq C \int_\Omega \left(\zeta \left| D_k^h Du \right| \left| D_k^h u \right| + \zeta \left| D_k^h Du \right| |Du| + \zeta \left| D_k^h u \right| |Du| \right) \mathrm{d}x,$$

where C is a positive constant. Further more, using Young's inequality with $\epsilon > 0$

$$|B| \leq \epsilon \int_\Omega \zeta^2 \left| D_k^h Du \right|^2 \, \mathrm{d}x + \frac{C}{\epsilon} \int_W \left| D_k^h u \right|^2 + |Du|^2 \, \mathrm{d}x.$$

Now, taking $\epsilon = \frac{\theta}{2}$ and applying Theorem 6.3 in Chapter 2, we obtain

$$|B| \leq \frac{\theta}{2} \int_\Omega \zeta^2 \left| D_k^h Du \right|^2 \, \mathrm{d}x + C \int_\Omega |Du|^2 \, \mathrm{d}x$$

Hence plugging the above estimates back into (3.15) yields

$$\text{LHS of } (3.14) \quad \geq \frac{\theta}{2} \int_\Omega \zeta^2 \left| D_k^h Du \right|^2 \, \mathrm{d}x - C \int_\Omega |Du|^2 \, \mathrm{d}x.$$

From the definition of $\tilde{f}$ and v, we also have

$$|\text{RHS of } (3.14)| \leq C \int_\Omega (|f| + |Du| + |u|)|v| \, \mathrm{d}x. \tag{3.16}$$

Applying Theorem 2.6.3 from Chapter 2,

$$\begin{aligned} \int_\Omega |v|^2 \mathrm{d}x \leq C \int_\Omega \left| D\left(\zeta^2 D_k^h u \right) \right|^2 \mathrm{d}x &\leq C \int_W \left| D_k^h u \right|^2 + \zeta^2 \left| D_k^h Du \right|^2 \mathrm{d}x \\ &\leq C \int_\Omega |Du|^2 \mathrm{d}x + \zeta^2 \left| D_k^h Du \right|^2 \mathrm{d}x. \end{aligned} \tag{3.17}$$

Therefore, using Hölder inequality in (3.16) and followed by (3.17), we obtain

$$|\text{RHS of (3.14)}| \leq \epsilon \int_\Omega \zeta^2 \left| D_k^h Du \right|^2 \mathrm{d}x + \frac{C}{\epsilon} \int_\Omega f^2 + u^2 \mathrm{d}x + \frac{C}{\epsilon} \int_\Omega |Du|^2 \mathrm{d}x.$$

Next, set $\epsilon = \frac{\theta}{4}$. This reduces

$$|\text{RHS of (3.14)}| \leq \frac{\theta}{4} \int_\Omega \zeta^2 \left| D_k^h Du \right|^2 \mathrm{d}x + C \int_\Omega f^2 + u^2 + |Du|^2 \mathrm{d}x.$$

Hence comparing the final estimations for LHS and RHS of (3.14), we write

$$\int_V \left|D_k^h Du\right|^2 \mathrm{d}x \leq \int_\Omega \zeta^2 \left|D_k^h Du\right|^2 \mathrm{d}x \leq C \int_\Omega f^2 + u^2 + |Du|^2 \mathrm{d}x,$$

where $|h| > 0$ is small enough and $k \in \{1, ..., n\}$. In view of Theorem 2.6.3(ii) of Chapter 2, we can deduce from the above estimate that $Du \in H^1_{loc}(\Omega)$ and thus $u \in H^2_{loc}(\Omega)$ with

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}\right).$$

We can refine the previous estimate by observing that if $V \subset\subset W \subset\subset \Omega$, then

$$\|u\|_{H^2(V)} \leq C\left(\|f\|_{L^2(W)} + \|u\|_{H^1(W)}\right), \tag{3.18}$$

where C is a constant which depends on V, W, etc. Now define a new map ζ such that

$$\zeta \equiv 1 \text{ on } W, \quad \text{supp}\, \zeta \subset U, \quad 0 \leq \zeta \leq 1.$$

Put $v = \zeta^2 u$ in equation (3.14) to find

$$\int_\Omega \zeta^2 |Du|^2 dx \leq C \int_\Omega (f^2 + u^2) \mathrm{d}x.$$

Hence,

$$\|u\|_{H^1(W)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

The last inequality and (3.18) completes the proof of the theorem. ■

Theorem 3.3.2. (Higher interior regularity) Let m be a nonnegative integer and

$$a^{ij}, b^i, c \in C^{m+1}(\Omega) \quad \forall\, i, j = 1, \cdots, n.$$

Also $f \in H^m(\Omega)$ and $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω. Then

$$u \in H^{m+2}_{loc}(\Omega)$$

and if $V \subset\subset \Omega$, then

$$\|u\|_{H^{2+m}(V)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2()\Omega}),$$

where C depends on m, V, coefficient of L and Ω.

Proof. This theorem can be proved by induction on m. We have proved the base case in Theorem 3.3.1. For the detailed proof, we refer [L.C86, Theorem 2, Section 6.3]. ■

Theorem 3.3.3. (Infinite interior differentiability) Let $a^{ij}, b^i, c \in C^\infty(\Omega)$ for all $i, j = 1, \cdots, n$ and $f \in C^\infty(\Omega)$. If $u \in H^1(\Omega)$ is a weak solution of $Lu = f$ in Ω, then $u \in C^\infty(\Omega)$.

Proof. By Theorem 3.3.2, $u \in H^m_{loc}(\Omega)$, where m is any positive integer. Therefore, applying Theorem 2.4.6 of Chapter 2, we have $u \in C^k(\Omega)$ for each $k = 1, 2, \cdots$. Hence, $u \in C^\infty(\Omega)$. ■

In the next theorem, we aim to extend the above regularity unto the boundary of Ω. To obtain this, we need to assume some "smoothness" property of $\partial\Omega$. Note that in Theorem 3.3.1 and Theorem 3.3.2, we have not assumed any condition on boundary of Ω and that is why the improved regularity of the solution which we have obtained is local.

Theorem 3.3.4. (Higher boundary regularity) Let m be a nonnegative integer, Ω be a bounded domain of class C^{m+2} and a^{ij}, b^i, $c \in C^{m+1}(\overline{\Omega})$ $\forall\ i, j = 1, \cdots, n$. Also $f \in H^m(\Omega)$ and $u \in H^1_0(\Omega)$ is a weak solution of the elliptic PDE

$$\begin{cases} Lu = f & in\ \Omega \\ u = 0 & on\ \partial\Omega. \end{cases}$$

Then

$$u \in H^{m+2}(\Omega)$$

and

$$\|u\|_{H^{2+m}(\Omega)} \leq C(\|f\|_{H^m(\Omega)} + \|u\|_{L^2(\Omega)})$$

where C depends on m, coefficient of L and Ω.
In particular, if $u \in H^m(\Omega)$ with $m > N/2$ then $u \in C^2(\bar{\Omega})$. Finally if $\partial\Omega$ is C^∞ and a^{ij}, b^i, $c \in C^\infty(\Omega)$ $\forall\ i, j = 1, \cdots, n$ and $f \in C^\infty(\Omega)$ then $u \in C^\infty(\overline{\Omega})$.

Proof. See [L.C86, Theorem 3, Section 6.3] and [H.11, Theorem 9.25] . ■

3.4 Maximum Principles

In this section we assume Ω is a bounded and open subset of $\mathbb{R}^n$ and L be given in the divergence form (3.2).

Definition 3.4.1. If $Lu \leq 0$ in Ω, then u is called a subsolution of

$$Lu = 0 \tag{3.19}$$

and if $Lu \geq 0$ in Ω, then u is called a supersolution of (3.19).

Theorem 3.4.1. (Weak maximum principle for $c = 0$) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \equiv 0$ in Ω.
(i) $Lu \leq 0$ in $\Omega \implies \max_{\overline{\Omega}} u = \max_{\partial\Omega} u$.
(ii) $Lu \geq 0$ in $\Omega \implies \min_{\overline{\Omega}} u = \min_{\partial\Omega} u$.

Proof. First take $Lu < 0$ and on the contrary, let's assume there exists $x_0 \in \Omega$ such that $u(x_0) = \max_{\overline{\Omega}} u$. As it is a minima, we have $Du(x_0) = 0$ and $D^2u(x_0) \leq 0$.
Consider the matrix $A := (a^{ij}(x_0))$, which is symmetric and positive definite. Hence, from basic linear algebra we know there exists an orthogonal matrix $O = (o^{ij})$ which satisfies

$$OAO^T = \operatorname{diag}(d_1, ..., d_n),$$

where each $d_i > 0$. Take $y = x_0 + O(x - x_0)$. Then $x - x_0 = O^T(y - x_0)$. So we have

$$u_{x_i} = \sum_{k=1}^{n} u_{y_k} o_{ki}, \quad u_{x_i x_j} = \sum_{k,l=1}^{n} u_{y_k y_l} o_{ki} o_{lj} \quad (i, j = 1, \ldots, n)$$

Hence at x_0, we have (since O is orthogonal)

$$\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} = \sum_{k,l=1}^{n} \sum_{i,j=1}^{n} a^{ij} u_{y_k y_l} o_{ki} o_{lj} = \sum_{k=1}^{n} d_k u_{y_k y_k} < 0$$

as $d_k > 0$ and $u_{y_k y_k}(x_0) \leq 0 \ \ \forall 1 \leq k \leq n$. Hence,

$$Lu = -\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} \geq 0$$

at x_0 by our previous calculation.
For $Lu \leq 0$ in Ω, define $u^\epsilon(x) := u(x) + \epsilon e^{\lambda x_1}$ where λ is a particular constant to be selected below and ϵ be greater than 0. Using uniform ellipticity we now show that $Lu^\epsilon < 0$ where we choose λ sufficiently large. Thus, by the previous calculations we already have $\max_{\overline{\Omega}} u^\epsilon = \max_{\partial\Omega} u^\epsilon$. Taking $\epsilon \to 0$, we prove (i).
To prove (ii), just note that if u is a supersolution of (3.19), then $-u$ is a subsolution of (3.19). This proves the theorem. ∎

Theorem 3.4.2. (Weak maximum principle for $c \geq 0$) Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and $c \geq 0$ in Ω.
(i) $Lu \leq 0$ in $\Omega \implies \max_{\overline{\Omega}} u = \max_{\partial\Omega} u^+$.
(ii) $Lu \geq 0$ in $\Omega \implies \min_{\overline{\Omega}} u = -\min_{\partial\Omega} u^-$.

Proof. The idea is to define a new operator K defined as $Ku := Lu - cu$ and apply the previous theorem. Detailed proof can be found in [L.C86, Theorem 2, Section 6.4]. ■

Lemma 3.4.1. (Hopf's Lemma) Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and in the expression of L, $c \equiv 0$ in Ω. Suppose $Lu \leq 0$ in Ω and there exists $x^0 \in \partial\Omega$ for which $u(x^0) > u(x)$, for every $x \in \Omega$. Also assume there exists an open ball $B \subset \Omega$ with $x^0 \in \partial B$ (interior ball condition). Then
(i)

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where ν is the unit outward normal to B at x^0.
(ii) If $c \geq 0$ in Ω, the same conclusion holds provided $u(x^0) \geq 0$.

Note: As $u(x_0)$ is a maxima, we already know $\frac{\partial u}{\partial \nu}(x_0) \geq 0$. The strength of Hopf's lemma is that it makes the inequality strict.

Proof. We will just prove the second part, that is for $c \geq 0$. First part follows if we put $c = 0$.
Take B to be $B(0,r)$, where $r > 0$. For $x \in \overline{B(0,r)}$, define $v(x) := \exp\{-\lambda|x|^2\} - \exp\{\lambda r^2\}$ where $\lambda > 0$ will be chosen below. Thus,

$$\begin{aligned} Lv = -\sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} + \sum_{i=1}^{n} b^i v_{x_i} + cv =& e^{-\lambda|x|^2} \sum_{i,j=1}^{n} a^{ij}\left(-4\lambda^2 x_i x_j + 2\lambda\delta_{ij}\right) \\ & - e^{-\lambda|x|^2} \sum_{i=1}^{n} b^i 2\lambda x_i + c\left(e^{-\lambda|x|^2} - e^{-\lambda r^2}\right) \\ \leq& e^{-\lambda|x|^2}\left(-4\theta\lambda^2|x|^2 + 2\lambda \operatorname{tr} \mathbf{A} + 2\lambda|\mathbf{b}||x| + c\right) \end{aligned}$$

for $\mathbf{A} = \left(\left(a^{ij}\right)\right), \mathbf{b} = \left(b^1, \ldots, b^n\right)$. Define $A_r := B(0,r) - \overline{B(0,r/2)}$.

$$Lv \leq e^{-\lambda|x|^2}\left(-\theta\lambda^2 r^2 + 2\lambda \operatorname{tr} \mathbf{A} + 2\lambda|\mathbf{b}|r + c\right) \leq 0 \quad \text{in} \quad A_r$$

provided $\lambda > 0$ is chosen large enough. As $u(x^0) > u(x)$, there exists $\epsilon > 0$ small enough such that

$$u(x_0) \geq u(x) + \epsilon v(x) \quad \forall\, x \in \partial B\left(0, \frac{r}{2}\right). \tag{3.20}$$

Since, $v \equiv 0$ in $\partial B(0,r)$, clearly

$$u(x^0) \geq u(x) + \epsilon v(x) \quad \text{on} \quad \partial B(0,r). \tag{3.21}$$

Moreover, $Lv \leq 0$ implies

$$L(u + \epsilon v - u(x^0)) \leq -cu(x^0) \leq 0 \quad \text{in} \quad A_r.$$

From (3.20) and (3.21), it follows that $u + \epsilon v - u()x^0 \leq 0$ on ∂A_r.
Therefore, applying weak maximum principle, we have $u + \epsilon v - v(x^0) \leq 0$ in A_r. As $u(x^0) + \epsilon v(x^0) - u(x^0) = 0$, we find

$$\frac{\partial u}{\partial \nu}(x^0) + \epsilon \frac{\partial v}{\partial \nu}(x^0) \geq 0.$$

This implies

$$\frac{\partial u}{\partial \nu}(x^0) \geq -\epsilon \frac{\partial v}{\partial \nu}(x^0) = -\frac{\epsilon}{r} Dv(x^0).x^0 = -2\lambda \epsilon r e^{-\lambda r^2} > 0$$

and we are done. ■

Theorem 3.4.3. (Strong maximum principle) Let Ω be a connected subset of $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Suppose $c \geq 0$ in the expression of L.
(i) If $Lu \leq 0$ in Ω and u attains a nonnegative maximum over $\overline{\Omega}$ at an interior point, then u is cosntant in Ω.
(ii) If $Lu \geq 0$ in Ω and u attains a nonpositive minimum over $\overline{\Omega}$ at an interior point, then u is constant in Ω.

Proof. Define $M := \max_{\overline{\Omega}} u$ and $S := \{x \in \Omega | u(x) = M\}$. If $u \equiv M$, there is nothing left to prove. Else, define $T := \{x \in \Omega | u(x) < M\}$. Now take $y \in T$ such that $\text{dist}(y, S) < \text{dist}(y, \partial\Omega)$. Assume B is the biggest ball centred at y and completely contained in T. Then obviously, $\exists\, z \in S$ along with $z \in \partial\Omega$.
As T satisfies the interior ball condition at the point z, using Hopf's lemma we have $\frac{\partial u}{\partial \nu}(z) > 0$. This leads to a contradiction as u has maxima at z and $Du(z) = 0$. ■

Theorem 3.4.4. (Harnack's inequality) Let $u \geq 0$ be a c^2 function satisfying $Lu = 0$ in Ω and $V \subset\subset \Omega$ be connected. Then there exists a positive constant C depending only on V and coefficients of L such that

$$\sup_V u \leq C \inf_V u.$$

Note: For simplicity we'll assume a^{ij}'s are given smooth functions and $b^i = c = 0$.

Proof. See [L.C86, Theorem 5, Section 6.4]. ■

3.5 Eigenvalues and Eigenfunctions

Let Ω be a connected bounded open subset of $\mathbb{R}^n$ and w be a nontrivial solution of

$$\begin{cases} Lw = \lambda w & \text{in} \quad \Omega \\ w = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x) u_{x_i}\right)_{x_j},$$

with the assumption that $a^{ij} \in C^\infty(\overline{\Omega})$, $\forall\ i,\ j = 1, \cdots, n$, L is uniformly elliptic and $a^{ij} = a^{ji}$ $(i, j = 1, \cdots, n)$.
The set of all eigenvalues, denoted by Σ can be at most countable (from the theory developed in Section 2 of this chapter).

Theorem 3.5.1. (Eigenvalues of symmetric elliptic operator) (i) Eigenvalues of L are real.
(ii) If we repeat each eigenvalues according to it's finite multiplicity then $0 < \lambda_1 \leq \lambda_2 \leq$ ....., and $\lambda_k \to \infty$ as $k \to \infty$.
(iii) There exists an orthonormal basis $\{w_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $w_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k

$$\begin{cases} Lw = \lambda w & in\ \Omega \\ w = 0 & on\ \partial\Omega. \end{cases}$$

for $k \in \mathbb{N}$.

Remark: λ_1 is known as the principle eigenvalue of L.

Proof. See [L.C86, Theorem 1, Section 6.5]. ■

Theorem 3.5.2. (Variational principle for the principle eigenvalue)
(i)

$$\lambda_1 = min\{B[u,u] \ :\ u \in H_0^1(\Omega),\ \|u\|_{L^2(\Omega)} = 1\}.$$

(ii) The above minimum is being attained by a function w_1 which does not change sign in Ω and solves

$$\begin{cases} Lw_1 = \lambda_1 w_1 & in\ \Omega \\ w = 0 & on\ \partial\Omega. \end{cases}$$

(iii) If $u \in H_0^1(\Omega)$ is weak solution of

$$\begin{cases} Lu = \lambda_1 u & in\ \Omega \\ u = 0 & on\ \partial\Omega, \end{cases}$$

then u is a multiple of w_1.

Remark: (i) From Theorem 3.5.2(iii) , we conclude that $\lambda_2 > \lambda_1$ i.e. λ_1 is simple eigenvalue.
(ii) The value of λ_1 given in the above theorem is known as Rayleigh's formula and can equivalently be expressed as

$$\lambda_1 = min_{u \in H_0^1(\Omega), u \neq 0} \frac{B[u,u]}{\|u\|^2_{L^2(\Omega)}}.$$

Proof. See [L.C86, Theorem 2, Section 6.5]. ■

3.6 Neumann Boundary Value Problem

Let Ω be a bounded domain in $\mathbb{R}^n$ of the class C^1. For a given $f : \Omega \to \mathbb{R}$, we want to solve

$$\begin{cases} -\Delta u + u = f & in\ \Omega \\ \frac{\partial u}{\partial \nu} = 0 & on\ \partial\Omega, \end{cases} \tag{3.22}$$

that is, we want to find $u : \overline{\Omega} \to \mathbb{R}$ satisfying the above problem. Here ν is the unit outward normal vector to $\partial\Omega$ and this means $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

Definition 3.6.1. A function $u \in H^1(\Omega)$ is said to be a weak solution of (3.22) provided u satisfies

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Omega uv \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad \forall\ v \in H^1(\Omega).$$

Using Green's formula we can show that every classical solution is also a weak solution.

Theorem 3.6.1. (Existance and uniqueness of weak solution) For any $f \in L^2(\Omega)$, there exists unique $u \in H^1(\Omega)$ satisfying (3.22). Moreover, the unique solution can be expressed as

$$u = \min_{v \in H^1(\Omega)} \left\{ \frac{1}{2} \int_\Omega (|\nabla v|^2 + v^2) \, \mathrm{d}x - \int_\Omega fv \, \mathrm{d}x \right\}.$$

Proof. Just like the proof of existence and uniqueness of Dirichlet boundary value problem, we use Lax-Milgram theorem on $H^1(\Omega)$. ■

Theorem 3.6.2. (Regularity of solutions) Assume Ω is an open subset of $\mathbb{R}^n$ of the class C^2 and $\partial\Omega$ is a bounded set. Suppose also $f \in L^2(\Omega)$ and $u \in H_0^1(\Omega)$ satisfy the following:

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega u\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in H_0^1(\Omega).$$

Then $u \in H^2(\Omega)$ with $\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ for some constant C depending only upon the set Ω.
If Ω is C^{m+2} and $f \in H^m(\Omega)$, then we have $v \in H^{m+2}(\Omega)$ with $\|u\|_{H^{m+2}(\Omega)} \leq C\|f\|_{H^m(\Omega)}$, where C depends only on the set Ω.
From this we can conclude, if Ω is C^∞ and $f \in C^\infty(\overline{\Omega})$, then $u \in C^\infty(\overline{\Omega})$.

Proof. The proof is not very different from the proof of regularity results for Dirichlet boundary value problem. For detailed proof see [H.11, Theorem 9.26, Section 9.6]. ■

Theorem 3.6.3. (Maximum principle) Assume $f \in L^2(\Omega)$ and $u \in H^1(\Omega)$ satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi \, dx + \int_\Omega u\varphi \, dx = \int_\Omega f\varphi \, dx \quad \forall \varphi \in H_0^1(\Omega).$$

Then we have

$$\inf_\Omega f \leq u(x) \leq \sup_\Omega f \quad \text{for almost every } x \in \Omega.$$

Proof. Similar to the proof of Maximum principle for the Dirichlet boundary value problem. ■

Note: There are other boundary conditions for second order elliptic PDEs like Robin condition and oblique derivative problem. In Robin condition, the boundary value is given as

$$\frac{\partial u}{\partial \nu} + \alpha u = 0 \quad \text{and} \quad \alpha > 0 \quad \text{on} \partial\Omega$$

where ν is same as in Neumann problem. In oblique derivative problem the boundary value is given by

$$\alpha\frac{\partial u}{\partial \nu} + \beta\frac{\partial u}{\partial \tau} = 0 \quad \text{on} \quad \partial\Omega$$

where τ is the tangent vector. Refer Chapter 3 of [S.03] for more details on these types of boundary value problems.

Chapter 4

Variational Techniques

Calculus of variation is about finding critical points. Suppose we want to solve $A[u] = 0$, where A is any partial differential operator (can even be nonlinear). There is no general theory to address this problem, but using calculus of variation we can solve a particular class of problems(called 'variational problem') in the following way:
Suppose the operator $A[\cdot]$ is *derivative* of an *energy* functional I i.e.

$$A[\cdot] = I'[\cdot].$$

Then to solve $A[u] = 0$, it is enough to find the critical points of I and here comes the role of calculus of variation.
For more detailed exposition on calculus of variation, refer [A.07] and [M.96].

4.1 Eular-Lagrange equation, First variation and Second Variation

Definition 4.1.1. Let Ω be an open and bounded subset of $\mathbb{R}^n$ with smooth boundary. A smooth function

$$L : \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}$$

is called the Lagrangian.

Notation We write

$$L = L(p, z, x) = L(p_1, ..., p_n, z, x_1, ..., x_n)$$

where $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in \Omega$. Below, we substitute $Dw(x)$ for the variable p and

$w(x)$ for the variable z. Set

$$\begin{cases} D_pL = (L_{p_1}, ..., L_{p_n}) \\ D_zL = L_z \\ D_xL = (L_{x_1}, ..., L_{x_n}) \end{cases}$$

Let $I[\cdot]$ have an explicit form

$$I[w] = \int_\Omega L(Dw(x), w(x), x)\mathrm{d}x$$

where $w : \overline{\Omega} \to \mathbb{R}$ be smooth and satisfies the boundary condition

$$w = g \quad \text{on } \partial\Omega.$$

Moreover, assume there exists a smooth function u satisfying the boundary condition and is also a minimizer of I among all functions which satisfies the boundary condition. We now see that u is automatically a solution for some PDE.
Choose $v \in C_c^\infty(\Omega)$ and define

$$\gamma(\tau) := I[u + \tau v].$$

Since u is a minimizer of I, $u + \tau v = u$ when $\tau = 0$ and thus $u + \tau v = u = g$ on $\partial\Omega$, we conclude that $\gamma(\cdot)$ has a minimum at $\tau = 0$. Hence, $\gamma'(0) = 0$. As we know the explicit formula for $\gamma(\tau)$, we calculate $\gamma'(0)$ and equate it to 0 to get

$$\int_\Omega \Big[-\sum_{i=1}^n (Lp_i(Du, u, x))_{x_i} + L_z(Du, u, x) \Big] v \,\mathrm{d}x = 0.$$

The above equation holds for any test function v. So, u solves

$$-\sum_{i=1}^n (Lp_i(Du, u, x))_{x_i} + L_z(Du, u, x)) = 0.$$

The above equation is known as *Eular-Lagrange equation* associated with the energy functional $I[\cdot]$ defined above.
Just now we have seen that any smooth minimizer of $I[\cdot]$ is a solution for the Eular-Lagrange equation. Now we try to find solution of Eular-Lagrange equation by searching for the appropriate energy functional.

Since u was the minimum of $I[\cdot]$, we can conclude

$$\gamma''(0) \geq 0.$$

From the explicit formula for γ, we get

$$\sum_{i,j=1}^{n} L_{p_i p_j}(Du, u, x)\xi_i\xi_j \geq 0 \quad \forall\ \xi \in \mathbb{R}^n,\ x \in \Omega.$$

Later we will see that this inequality is necessary for the existence theory.

4.2 Existence of Minimizers

Within an appropriate Sobolev space, we try to put some conditions on L so that $I[\cdot]$ has a minimizer. To be precise, consider the functional

$$I[w] := \int_\Omega L(Dw(x), w(x), x)\mathrm{d}x$$

defined for appropriate functions $w : \Omega \to \mathbb{R}$ satisfying

$$w = g \ \ on\ \partial\Omega.$$

What assumption(s) should we make to ensure the existence of a minimizer of I?

Coercivity: Consider $f(x) = e^{-x^2}$. It is a function from $\mathbb{R}$ to $\mathbb{R}$, which is smooth and bounded. Still it does not attend it's infimum. This example suggests us that we should have some control over $I[w]$ when w is large. The most intuitive way is to assume that $I[w]$ grows rapidly when $\|w\| \to \infty$.

To be precise, for $1 < q < \infty$ assume

$$\begin{cases} \exists\ \alpha > 0 \text{ and } \beta \geq 0 \text{ such that} \\ L(p, z, x) \geq \alpha|p|^q - \beta \\ \forall\ p \in \mathbb{R}^n,\ z \in \mathbb{R},\ x \in \Omega. \end{cases} \tag{4.1}$$

From this assumption, we can easily conclude

$$I[w] \geq \delta\|Dw\|^q_{L^q(\Omega)} - \gamma \tag{4.2}$$

where $\gamma := \beta|\Omega|$ and $\delta > 0$ be some constant.

Hence, when $\|Dw\|_{L^q} \to \infty$, we have $I[w] \to \infty$. The equation (4.2) is known as a

coercivity condition on $I[\cdot]$.
From the coercivity condition (4.2), we see that it is a good idea to define $I[w]$ for $W^{1,q}(\Omega)$ functions rather than for smooth functions. Obviously, w also has to satisfy boundary condition $w = g$ on $\partial\Omega$. So, we define

$$\mathcal{A} := \{w \in W^{1,q}(\Omega) \mid w = g \text{ on } \partial\Omega \text{ in trace sense}\}$$

to denote the class of *admissible* functions w.
Weak Lower Semicontinuity: Given coersivity, a function from $\mathbb{R}$ to $\mathbb{R}$ attains minimum, but in general $I[w]$ will not. The main problem here is the space $W^{1,q}(\Omega)$ is infinite dimensional and it is difficult to get compactness. For our purpose, we need one more assumption.

Definition 4.2.1. A functional $I[\cdot]$ is called weakly lower semicontinuous (w.l.s.c, in short) on $W^{1,q}(\Omega)$ if for every sequence $\{u_k\}$ in $W^{1,q}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W^{1,q}(\Omega)$ implies

$$I[u] \leq \liminf_{k\to\infty} I[u_k].$$

Lemma 4.2.1. Let X be a reflexive Banach space and $I : X \to \mathbb{R}$ be coercive and w.l.s.c. Then I is bounded from below on X. Moreover, there exists $u \in X$ such that $I(u) = \min\{I(v) \mid v \in X\}$. If I is differentiable, then $I'(u) = 0$.

Proof. Suppose not, that is we assume there exists a sequence $\{u_n\} \subset X$ such that $I(u_n) \to -\infty$. Since I is coercive, there exists $M > 0$ such that $\|u_n\|_E \leq M$. This in turn implies there exists $u \in X$ such that $u_n \rightharpoonup u$, since X is reflexive. Therefore, as I is w.l.s.c, we obtain $I[u] \leq \liminf_{n\to\infty} I[u_n] = -\infty$, which is a contradiction. Hence I is bounded from below. Consequently, $m := \inf\{I(v) \mid v \in X\}$ is finite. Let w_n be a minimizing sequence for m, i.e., $I(w_n) \to m$. Again by coercivity, $\{w_n\}$ is bounded in X and there exists $w \in X$ such that $w_n \rightharpoonup w$, for some $w \in X$. Combining this with w.l.s.c. of I,

$$I[w] \leq \liminf_{n\to\infty} I[w_n] = m.$$

Clearly $I(w)$ can not be strictly smaller than m and hence $I(w) = m$. This proves the lemma. ∎

Applications

Theorem 4.2.1. Assume f is locally Hölder continuous and there exists $a_1 \in L^2(\Omega)$, $a_2 > 0$ and $p \in (0,1)$ such that

$$|f(x, u(x))| \leq a_1(x) + a_2|u|^p \quad \forall\, x \in \Omega,\ t \in \mathbb{R}. \tag{4.3}$$

Then the following Dirichlet boundary value problem admits a solution in $H_0^1(\Omega)$.

$$\begin{cases} -\Delta u &= f(x, u(x)) \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.4}$$

Proof. Define $I : H_0^1(\Omega) \to \mathbb{R}$ by

$$I(u) = \|u\|^2_{H_0^1(\Omega)} - \Phi(u),$$

where

$$\Phi(u) = \int_\Omega F(x, u(x))dx, \quad \text{and} \quad F(x, u) = \int_0^u f(x, s)ds.$$

Clearly, $\Phi \in C^1(H_0^1(\Omega), \mathbb{R})$ and weakly continuous (i.e., $u_n \rightharpoonup u$ implies $\Phi(u_n) \to \Phi(u)$). Also it is easy to see that critical points of I are solutions of (4.4) and vice versa. From (4.3), we have

$$|\Phi(u)| \leq \|a_1\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \frac{a_2}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}.$$

Consequently, $I(u) \geq \|u\|^2_{H_0^1(\Omega)} - \|a_1\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} - \frac{a_2}{p+1}\|u\|^{p+1}_{L^{p+1}(\Omega)}$.. Therefore, I is coercive as $p < 1$. Moreover, as $u \mapsto \|u\|^2$ is w.l.s.c. and Φ is weakly continuous, then I is w.l.s.c. Hence the theorem follows from Lemma 4.2.1.

■

Our next aim is to find suitable assumptions on L so that weak lower semicontinuity is satisfied.
The second variation suggests us to assume L is convex in its first variable.

Theorem 4.2.2. (Weak lower semicontinuity) If L is bounded below, smooth and $p \to L(p, z, x)$ is convex for all $z \in \mathbb{R}$, $x \in \Omega$, then $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(\Omega)$.

Proof. See [L.C86, Theorem 1, Section 8.2]. ■

Now we show that $I[\cdot]$ has a minimizer among all $w \in \mathcal{A}$.

Theorem 4.2.3. (Existence) Assume $\mathcal{A} \neq \emptyset$ and L satisfies coersivity inequality and is convex in the variable p. Then $\exists\ u \in \mathcal{A}$ satisfying

$$I[u] = min_{w \in \mathcal{A}} I[w].$$

Proof. Define $m := \inf_{w\in\mathcal{A}} I[w]$. For $m = \infty$, there is nothing to prove. Therefore, we suppose $m < \infty$. Let $\{u_k\} \subset \mathcal{A}$ be a minimizing sequence, that is $I[u_k] \to m$. Without loss of generality, take $\beta = 0$ in the definition of coercivity. Hence, coercivity says $L \geq \alpha|p|^q$. Hence $I[w] \geq \alpha \int_\Omega |Dw|^q \mathrm{d}x$.
As $m < \infty$, we see

$$\sup_k \|Du_k\|_{L^q(\Omega)} < \infty. \tag{4.5}$$

Fix any $w \in \mathcal{A}$. Since, $u_k \in \mathcal{A}$, therefore we have $u_k - w \in W_0^{1,q}(\Omega)$. Now using Poincaré's inequality and (4.5) we have,

$$\|u_k\|_{L^q(\Omega)} \leq \|u_k - w\|_{L^q(\Omega)} + \|w\|_{L^q(\Omega)} \leq C\|Du_k - Dw\|_{L^q(\Omega)} + C \leq C.$$

So, $\sup_k \|u_k\|_{L^q(\Omega)} < \infty$. This result and (4.5) proves that $\{u_k\}_{k=1}^\infty$ is bounded in $W^{1,q}(\Omega)$. As a consequence, there exists $u \in W^{1,q}(\Omega)$ such that up to a subsequence $u_{k_j} \rightharpoonup u$ in $W^{1,q}(\Omega)$. Next, we claim that $u \in \mathcal{A}$. Indeed, for any $w \in \mathcal{A}$, $u_k - w \in W_0^{1,q}(\Omega)$ and $W_0^{1,q}(\Omega)$ is closed linear subspace of $W^{1,q}(\Omega)$ and thus by Mazur's theorem weakly closed. consequently, $u - w \in W_0^{1,q}(\Omega)$. Therefore, trace of u is g. Hence the claim follows. Therefore, by Theorem 4.2.2, $I[u] \leq \liminf_{j\to\infty} I[u_{k_j}] = m$. Using the fact $u \in \mathcal{A}$, $I[u] = m = \min_{w\in\mathcal{A}} I[w]$. ■

The next obvious question is uniqueness of minimizers. To guarantee uniqueness we need some more assumptions.

Theorem 4.2.4. (Uniqueness) Suppose $L = L(p, x)$ doesn't depend on z and

$$\exists\, \theta > 0 \text{ such that } \sum_{i,j=1}^{n} L_{p_ip_j}(p,x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{where } \ p,\, \xi \in \mathbf{R}^n,\ x \in \Omega \quad \text{(uniform convexity)}$$

Then the minimizer in $\mathcal{A}$ is unique.

Proof. Suppose $u \neq \tilde{u}$ in almost everywhere sense and both of them are minimizers. Define $v := \frac{u+\tilde{u}}{2}$. We'll prove that

$$I[v] < \frac{I[u] + I[\tilde{u}]}{2}.$$

Note that this is sufficient to prove as it contradicts the minimality of u and $\tilde{u}$. From uniform convexity, we deduce,

$$L(p,x) \geq L(q,x) + D_pL(q,x)\dot{(}p-q) + \frac{\theta}{2}|p-q|^2 \tag{4.6}$$

where $x \in \Omega$ and $p, q \in \mathbb{R}^n$.
Now put $p = Du$ and $q = \frac{Du+D\tilde{u}}{2}$ and integrate on the set Ω to get

$$I[v] + \int_\Omega D_p L\Big(\frac{Du + D\tilde{u}}{2}, x\Big) \cdot \Big(\frac{Du - D\tilde{u}}{2}, x\Big) dx + \frac{\theta}{8} \int_\Omega |Du + D\tilde{u}|^2 dx \leq I[u].$$

In a similar way, if we put $p = D\tilde{u}$ and $q = \frac{Du+D\tilde{u}}{2}$ and integrate on the set Ω, we get

$$I[v] + \int_\Omega D_p L\Big(\frac{Du + D\tilde{u}}{2}, x\Big) \cdot \Big(\frac{D\tilde{u} - Du}{2}, x\Big) dx + \frac{\theta}{8} \int_\Omega |Du + D\tilde{u}|^2 dx \leq I[\tilde{u}].$$

If we add the last two inequality, we find

$$I[v] + \frac{\theta}{8} \int_\Omega |Du + D\tilde{u}|^2 \mathrm{d}x \leq \frac{I[u] + I[\tilde{u}]}{2}.$$

From this we conclude

$$I[v] \leq \frac{I[u] + I[\tilde{U}]}{2}.$$

If the above inequality is not strict we will get $Du = D\tilde{u}$ almost everywhere, which in turn will say $u = \tilde{u}$ almost everywhere. This implies the minimizer is unique. ■

Recall that our aim was to solve the Eular-Lagrange equation. Now we show that any minimizer $u \in \mathcal{A}$ of $I[\cdot]$ solves Eular-Lagrange equation under certain assumptions. We have to assume some growth conditions on L and its derivatives:

$$|L(p, z, x)| \leq C(|p|^q + ||z|^q + 1) \tag{4.7}$$

and

$$\begin{cases} |D_p L(p, z, x)| \leq C(|p|^{q-1} + ||z|^{q-1} + 1) \\ |D_z L(p, z, x)| \leq C(|p|^{q-1} + ||z|^{q-1} + 1) \end{cases} \tag{4.8}$$

for some C and all $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in \Omega$.

Motivation behind the definition of weak solutions: Consider

$$\begin{cases} -\sum_{i=1}^n (Lp_i(Du, u, x))_{x_i} + L_z(Du, u, x) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{4.9}$$

Multiply (4.9) by $v \in C_c^\infty(\Omega)$ and integrate by parts to get

$$\int_\Omega \sum_{i=1}^n L_{p_i}(DU, u, x) v_{x_i} + L_z(Du, u, x) v \mathrm{d}x = 0 \tag{4.10}$$

For $u \in W^{1,q}(\Omega)$ using (4.8) we see

$$|D_p L(Du, u, x)| \in L^{q'}(\Omega)$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. In a similar way,

$$|D_z L(Du, u, x)| \in L^{q'}(\Omega).$$

Now use the standard approximation argument to see (4.10) is valid $\forall\ v \in W_0^{1,q}(\Omega)$. This calculation motivates us to define the following:

Definition 4.2.2. $u \in \mathcal{A}$ is said to a weak solution of the boundary value problem (4.9) if

$$\int_\Omega \sum_{i=1}^{n} L_{p_i}(DU, u, x)v_{x_i} + L_z(Du, u, x)v\mathrm{d}x = 0$$

$\forall\ v \in W_0^{1,q}(\Omega)$.

Theorem 4.2.5. (Solution of Eular-Lagrange equation) If L follows the two growth conditions (4.7) and (4.8) and $u \in \mathcal{A}$ satisfies

$$I[u] = min_{w \in \mathcal{A}} I[w].$$

Then u is a weak solution of (4.9).

Proof. This proof is very similar to the motivation part we did in the beginning of the chapter. Only thing is that we can't assume u is smooth.
Fix $v \in W_0^{1,q}(\Omega)$. Define $\gamma(\tau) := I[u + \tau v]$. From the growth condition on L, we always have $\gamma(\tau)$ is finite.
For $\tau \neq 0$,

$$\frac{\gamma(\tau) - \gamma(0)}{\tau} = \int_\Omega L^\tau(x)\mathrm{d}x \tag{4.11}$$

where

$$L^\tau(x) := \frac{1}{\tau}[L(Du(x) + \tau Dv(x), u(x) + \tau v(x), x) - L(Du(x), u(x), x)]$$

for a.e. $x \in \Omega$. Obviously, $L^\tau(x) \to \sum_{i=1}^{n} L_{p_i}(Du, u, x)v_{x_i} + L_z(Du, u, x)v$ a.e. as

$\tau \to 0$. We also have

$$\begin{aligned} L^\tau(x) =& \frac{1}{\tau}\int_0^\tau \frac{d}{ds}L(Du+sDv,u+sv,x)\mathrm{d}s \\ =& \frac{1}{\tau}\int_0^\tau \Big(\sum_{i=1}^n L_{p_i}(Du+sDv,u+sv,x)v_{x_i} \\ & + L_z(Du+sDv,u+sv,x)v\Big)\mathrm{d}s. \end{aligned}$$

Use Young's inequality and (4.8), to conclude

$$|L^\tau(x)| \le C(|Du|^q + |u|^q + |Dv|^q + |v|^q + 1) \in L^1(\Omega)$$

$\forall\, \tau \neq 0$. Now using Lebesgue dominated convergence theorem and previous computation to prove that $\gamma'(0)$ exists and

$$\gamma'(0) = \int_\Omega \sum_{i=1}^n \Big(L_{p_i}(Du,u,x)v_{x_i} + L_z(Du,u,x)v\Big)\mathrm{d}x.$$

As γ has a minima at $\tau = 0$, we have $\gamma'(0) = 0$ and this proves our claim.

■

4.3 Regularity of solution

Let I has the particular form

$$I[w] := \int_\Omega \big(L(Dw) - wf\big)\mathrm{d}x \tag{4.12}$$

where $f \in L^2(\Omega)$. Take $q = 2$ and

$$|D_pL(p)| \le C(|p|+1) \ \ (p \in \mathbb{R}^n).$$

Then as before, any minimizer $u \in \mathcal{A}$ is a weak solution of

$$-\sum_{i=1}^n (L_{p_i}(Du))_{x_i} = f \ \ in \ \ \Omega. \tag{4.13}$$

We'll now show that under the assumptions

$$|D^2L(p)| \le C \ \text{ where } \ p \in \mathbb{R}^n \tag{4.14}$$

and

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p)\xi_i\xi_j \geq \theta|\xi|^2 \quad \forall\ p,\ \xi \in \mathbb{R}^n \quad \text{and some constant } \theta > 0 \tag{4.15}$$

if $u \in H^1(\Omega)$ is a weak solution of (4.13) then $u \in H^2_{loc}(\Omega)$.

Theorem 4.3.1. (Second derivative for minimizers) (i) If $u \in H^1(\Omega)$ be a weak solution of (4.13) where L satisfies (4.14) and (4.15), then $u \in H^2_{\text{loc}}(\Omega)$.
(ii) Moreover if $u \in H^1_0(\Omega)$ and $\partial\Omega$ is C^2, then $u \in H^2(\Omega)$ with

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$$

for some C.

Proof. Proof is very similar to the proof of regularity of weak solutions of second order elliptic PDEs. For detailed proof we refer [L.C86, Theorem 1, Section 8.3]. ■

4.4 Constraints

In this section we use Lagrange multiplier method to solve Eular-Lagrange equation. Let's say we want to minimize

$$I[w] := \frac{1}{2}\int_{\Omega} |Dw|^2 \mathrm{d}x$$

such that $w = 0$ on $\partial\Omega$ and also w satisfies

$$J[w] := \int_{\Omega} G(w)\mathrm{d}x = 0$$

where $G : \mathbb{R} \to \mathbb{R}$ be a given smooth function. Denote by g, the derivative of G. We'll also assume some growth conditions on g and G:

$$|g(z)| \leq C(|z| + 1)$$

and

$$|G(z)| \leq C(|z|^2 + 1)$$

for some appropriate constant C.
The appropriate admissible class here is

$$\mathcal{A} := \{w \in H^1_0(\Omega) \mid J[w] = 0\}.$$

We take Ω to be a bounded, open and connected subset of $\mathbb{R}^n$ with a smooth boundary.

Theorem 4.4.1. (Existence of minimizer in the constrained case) If $\mathcal{A} \neq \phi$, then $\exists\, u \in \mathcal{A}$ for which

$$I[u] = \min_{w \in \mathcal{A}} I[w].$$

Proof. Refer [L.C86, Theorem 1, Section 8.4]. . ■

Theorem 4.4.2. If $u \in \mathcal{A}$ be such that

$$I[u] = min_{w \in \mathcal{A}} I[w],$$

then $\exists\, \lambda \in \mathbb{R}$ satisfying

$$\int_\Omega Du \cdot Dv = \lambda \int_\Omega g(u) v \, \mathrm{d}x \quad \forall\, v \in H_0^1(\Omega).$$

Proof. See [L.C86, Theorem 2, Section 8.4]. ■

Now let's try to minimize the energy functional

$$I[w] := \int_\Omega \left(\frac{1}{2} |dw|^2 - fw \right) \mathrm{d}w$$

over all w in the admissible set

$$\mathcal{A} := \{ w \in H_0^1(\Omega) | w \geq h \quad \text{almost everywhere} \quad in \quad \Omega \}.$$

Here $h : \overline{\Omega} \to \mathbb{R}$ be a given smooth function, known as the obstacle. Moreover assume f to be smooth.

Theorem 4.4.3. (Existane of minimizer) If $\mathcal{A} \neq \phi$, then $\exists\, u \in \mathcal{A}$ for which

$$I[u] = min_{w \in \mathcal{A}} I[w].$$

Proof. See [L.C86, Theorem 3, Section 8.4]. ■

Theorem 4.4.4. (Variational characterization of minimizer) If $u \in \mathcal{A}$ be the unique solution of

$$I[u] = min_{w \in \mathcal{A}} I[w],$$

then

$$\int_\Omega Du \cdot D(w - u) \mathrm{d}x \geq \int_\Omega f(w - u) \mathrm{d}x$$

for all $w \in \mathcal{A}$.

Proof. See [L.C86, Theorem 4, Section 8.4]. . ■

4.5 Critical Points

Till now we were looking for minimizers of functional associated to the given nonlinear PDE. Now we see some additional method to find other critical points. These critical points need not in general be minimizers, rather mostly they will be *saddle points*.

Definition 4.5.1. Let X and Y be two sets and $f : X \times Y \to \mathbb{R}$. A point $(x_0, y_0) \in X \times Y$ is called a saddle point of f over $X \times Y$ if

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0) \quad \forall (x, y) \in X \times Y.$$

Example: Let $X = Y = \mathbb{R}$ and $f(x, y) = x^2 - y^2$. Then $(0, 0)$ is a saddle point of f.

Lemma 4.5.1. Let X and Y be two sets and $f : X \times Y \to \mathbb{R}$. Then f attains saddle point on $X \times Y$ if and only if

$$\max_{y \in Y} \inf_{x \in X} f(x, y) = \min_{x \in X} \sup_{y \in Y} f(x, y).$$

Proof. See [H.79, Lemma 2.2] (also see [S.11, Proposition 5.2.1]). ■

Theorem 4.5.1. (Ky Fan – von Neumann) Let H_1 and H_2 be two Hilbert spaces and $K_i \subset H_i$ $(i = 1, 2)$ be bounded and closed convex subsets. Let $f : K_1 \times K_2 \to \mathbb{R}$ satisfy the following:
(i) For every $x \in K_1$, the map $y \mapsto f(x, y)$ is convex and upper semi continuous on K_2.
(ii) For every $y \in K_2$, the map $x \mapsto f(x, y)$ is convex and lower semi continuous on K_1.
Then f admits at least one saddle point over $K_1 \times K_2$.

Proof. See [S.11, Theorem 5.2.1]. ■

Let H be any real Hilbert space and with $\|\cdot\|$ be the norm and $(\cdot, \cdot)$ be the inner product. Let $I : H \to \mathbb{R}$ be a nonlinear functional.
Notations: (i) Let

$$\mathcal{C} := \{I \in C^1(H, \mathbb{R}) \mid I' : H \to H \text{ be Lipschitz continuous on bounded subset of } H\}.$$

(ii) For $c \in \mathbb{R}$,

$$A_c := \{u \in H \mid I[u] \leq c\}$$

and

$$K_c = \{u \in H \mid I[u] = c,\ I'[u] = 0\}.$$

Definition 4.5.2. (i) An element $u \in H$ is said to be a critical point of I if $I'[u] = 0$.
(ii) If $K_c \neq \emptyset$, we call c a critical value of the functional I.

Our aim now is to prove if c is not a critical value, then we can deform $A_{c+\epsilon}$ to $A_{c-\epsilon}$ in a nice way for some $\epsilon > 0$. To prove this we'll need some kind of compactness.

Definition 4.5.3. Let X be a Banach space and $I : X \to \mathbb{R}$ be a C^1 functional. $\{u_n\} \subset X$ is called a Palais-Smale sequence (PS sequence, in short) of I on X at level c if $I(u_n) \to c$ and $I'(u_n) \to 0$ in the dual space of X.
We say I satisfies (PS) condition, respectively $(PS)_c$ condition on X if every PS sequence, respectively $(PS)_c$ sequence has a converging subsequence in X.

Lemma 4.5.2. (Deformation Lemma) Let $I \in \mathcal{C}$ satisfies (PS) condition and $K_c \neq \phi$. Then $\forall\ \epsilon > 0$, sufficiently small, $\exists\ \delta \in (0, \epsilon)$ and $\exists\ \eta \in C([0,1] \times H; H)$ such that $\eta_t(u) := \eta(t, u)$ (where $t \in [0,1]$ and $u \in H$) satisfies
(i) $\eta_o(u) = u$ for all $u \in H$,
(ii) $\eta_1(u) = u$ when $u \notin I^{-1}[c - \epsilon, c + \epsilon]$,
(iii) $I[n_t(u)] \leq I[u]$ for all $0 \leq t \leq 1$ and $u \in H$
(iv) $\eta_1(A_{c+\delta}) \subset A_{c-\delta}$.

Proof. Claim : $\exists\ \sigma, \epsilon \in (0,1)$ satisfying $\|I'[u]\| \geq \sigma \quad \forall\ u \in A_{c+\epsilon} - A_{C-\epsilon}$.
We prove the above claim by method of contradiction. That is, if the claim is not true, we can find sequences σ_k, ϵ_k both converges to 0 and $u_k \in A_{c+\epsilon_k} - A_{c-\epsilon_k}$ satisfying $\|I'[u_k]\| \leq \sigma_k$.
Palais-Smale condition tells us there exists a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ such that $u_{k_j} \to u$, where $u \in H$. As $I \in C^1(H; \mathbf{R})$, we conclude $I[u] = c$ with $I'[u] = 0$. This is a contradiction to our assumption $K_c = \phi$.
Fix a δ such that $0 < \delta < \min\{\epsilon, \sigma/2, \sigma^2/2\}$.
Define

$$A := \{u \in H | I[u] \notin (c - \epsilon, c + \epsilon)\}$$

$$B := \{u \in H | I[u] \in [c - \delta, c + \delta]\}.$$

Because I' is Lipschitz continuous in bounded set, we have that I' is bounded in bounded set. This implies $u \to \operatorname{dist}(u, A) + \operatorname{dist}(u, B)$ is bounded by some positive constant on bounded subset of H. Now define for $u \in H$

$$g(u) := \frac{\operatorname{dist}(u, A)}{\operatorname{dist}(u, A) + \operatorname{dist}(u,)}.$$

Note that

- g is Lipschitz continuous on any bounded set
- $0 \leq g \leq 1$
- $g = 0$ on A and $g = 1$ on B.

Define $h : \mathbb{R} \to \mathbb{R}$ and $V : H \to H$ by

$$h(t) := \begin{cases} 1, & \text{if } 0 \leq t < 1 \\ 1/t & \text{if } t \geq 1 \end{cases}$$

and

$$V(u) := -g(u)h(I'[u])I'[u].$$

Now consider

$$\begin{cases} \frac{d\eta}{dt} = V(\eta(t)) \quad t > 0 \\ \eta(0) = 0 \end{cases}$$

where $u \in H$. Keeping in mind that V is Lipschitz continuous on bounded set and bounded, the above ODE has has a unique solution for each $u \in H(t > 0)$. Denote the solution by $\eta = \eta_t(u) = \eta(t, u)$ for all $t \geq 0$ and $u \in H$. Restrict the solution for $0 \leq t \leq 1$. Then $\eta \in C([0, 1] \times H \to H)$ clearly satisfies (i) and (ii).
A little calculation will show that $\frac{d}{dt}I[\eta_t(u)] \leq 0$ for all $u \in H$ and $0 \leq t \leq 1$. Hence (iii) is proved.
To prove (iv), fix any $u \in A_{c+\delta}$. To prove, $\eta_1(u) \in A_{c-\delta}$.
If we assume $\eta_t(u) \notin B$ for some t, then trivially we are done.
For $\eta_t(u) \in B$, we have $g(\eta_t(u)) = 1$. In this case we can show $\frac{d}{dt}I[\eta_t(u)] \leq \min\{-\sigma, -\sigma^2\}$.
Using these we can show $I[\eta_1(u)] \leq c - \delta$, which in turn proves (iv).

■

Using the deformation lemma we prove mountain pass theorem which is intuitive for the finite dimensional cases.

Theorem 4.5.2. Mountain Pass Theorem (Ambrosetti-Rabinowitz): Let $I \in \mathcal{C}$ satisfies (PS) condition and
(i) I[0]=0
(ii) $\exists\, r,\ a > 0$ such that $I[u] \geq a$ on $\|u\| = r$
(iii) $\exists\, v \in H$ such that $\|v\| > r$ and $I[v] \leq 0$.
Define

$$\Gamma := \{\gamma \in C([0, 1]; H) \mid \gamma(0) = 0,\ \gamma(1) = v\}.$$

Then

$$c := \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I[\gamma(t)]$$

is a critical point for I.

The above mountain pass theorem is a classical theorem by Ambrosetti and Rabinowitz [H.73].

Proof. First note that $c \geq a$. Assume to the contrary that $K_c = \phi$. Choose ϵ sufficiently small such that $0 < \epsilon < a/2$.
Using deformation lemma we can say that $\exists\ 0 < \delta < \epsilon$ and a homeomorphism $\eta_1 : H \to H$ satisfying

$$\eta_1(A_{c+\delta}) \subset \eta_1(A_{c-\delta}) \tag{4.16}$$

and

$$\eta_1(u) = u \qquad \text{if } u \notin I^{-1}[c-\epsilon, c+\epsilon]. \tag{4.17}$$

From the definition of c and infimum, we select $\gamma \in \Gamma$ such that

$$\max_{0\le t\le 1} I[\gamma(t)] \le c + \delta. \tag{4.18}$$

Then $\hat{\gamma} := \eta_1(\gamma) \in \Gamma$. Also $\hat{\gamma}$ is continuous as composition of continuous function is continuous. But $\eta_1(A_{c+\delta}) \subset A_{c-\delta}$.
Now equation (4.18) says $\max_{0\le t\le 1} I[\hat{\gamma}(t)] \le c - \delta$. So,

$$c = \inf_{\gamma\in\Gamma}\max_{0\le t\le 1} I[\gamma] \le \max_{0\le t\le 1} I[\hat{\gamma}(t)] \le c - \delta$$

which gives us the contradiction. ∎

Now we apply this theorem to solve some nonlinear PDE.
Application of Mountain Pass Theorem: Consider

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.19}$$

where f is smooth and satisfies the following growth conditions:

$$|f(z)| \le C(1 + |z|^p)$$

and

$$|f'(z)| \le C(1 + |z|^{p-1})$$

where C is some constant. Also. $1 < p < \frac{n+2}{n-2}$.
Define $F(z) := \int_0^z f(s)ds$ and assume $0 \leq F(z) \leq \gamma f(z)z$ for some $\gamma < \frac{1}{2}$. We finally assume that $\exists\ 0 < a \leq A$ such that

$$a|z|^{p+1} \leq |F(z)| \leq A|z|^{p+1}.$$

Clearly $f(0) = 0$ and $u \equiv 0$ is a solution to the above problem. The next theorem guarantees existence of a nontrivial solution.

Theorem 4.5.3. Under the assumptions mentioned above (4.19) has at least one weak solution other than $u \equiv 0$.

Proof. See [L.C86, Theorem 3, Section 8.5]. ∎

4.6 Nonexistence of Solutions

Consider the problem

$$\begin{cases} -\Delta u = |u|^{p-1}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.20}$$

Under certain geometric condition on Ω, we will show that there exists no nontrivial solution of (4.20) when $p > \frac{n+2}{n-2}$.

Definition 4.6.1. Let Ω be an open subset of R^n. We say Ω is star-shaped with respect to 0 if for all $x \in \overline{\Omega}$, the line segment $\{\lambda x |\ 0 \leq \lambda \leq 1\}$ lies within $\overline{\Omega}$.

Clearly, any convex set containing 0 is star shaped but converse need not be true.

Lemma 4.6.1. (Normals to a star shaped domain) If Ω is star shapped with respect to 0 and $\partial\Omega$ is C^1, then

$$x \cdot \nu(x) \geq 0 \ \ \forall\ x \in \partial\Omega,$$

where ν is the unit outward normal.

Proof. See [L.C86, Lemma , Section 9.4]. ∎

4.6.1 Bounded domain

First, we consider the more general boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u \in H_0^1(\Omega), \end{cases} \tag{4.21}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 3$. Define, $F(u) := \int_0^u f(s)\, ds$.

Lemma 4.6.2. (Pohozaev Identity, 1965) Let $u \in H^2_{loc}(\bar{\Omega})$ be a solution of (4.21) such that $F(u) \in L^1(\Omega)$. Then the following identity holds

$$n \int_\Omega F(u) \mathrm{d}x - \frac{n-2}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x = \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (x \cdot \nu) \mathrm{d}S,$$

where ν is the unit outward normal to $\partial\Omega$ and $\mathrm{d}S$ is the surface measure on $\partial\Omega$.

The above celebrated identity due to S. Pohozaev [S.65]. We also refer [J.86] for remarkable extension by Pucci and Serrin.

Proof. Since $u \in H^2_{loc}(\Omega)$, (4.21) is satisfied pointwise almost everywhere. Therefore, multiplying the equation by $x \cdot \nabla u$ we have

$$\big(- \Delta u - f(u) \big) x \cdot \nabla u = 0. \tag{4.22}$$

Note that

$$f(u)\big(x \cdot \nabla u\big) = \mathrm{div}(xF(u)) - nF(u)$$

and

$$\begin{aligned} \Delta u \big(x \cdot \nabla u\big) &= \mathrm{div}\big(\nabla u (x \cdot \nabla u)\big) - |\nabla u|^2 - x \cdot \left(\frac{|\nabla u|^2}{2} \right) \\ &= \mathrm{div}\left(\nabla u (x \cdot \nabla u) - x \Big(\frac{|\nabla u|^2}{2}\Big) \right) + \frac{N-2}{2} |\nabla u|^2. \end{aligned}$$

Therefore, integrating (4.22) in Ω and using integration by parts we obtain

$$- \int_{\partial\Omega} \left(\nabla u (x \cdot \nabla u) - x \Big(\frac{|\nabla u|^2}{2}\Big) + xF(u) \right) \nu \, \mathrm{d}S(x) = \int_\Omega \left(\frac{N-2}{2} |\nabla u|^2 - nF(u) \right) \mathrm{d}x. \tag{4.23}$$

Moreover, as $u = 0$ on $\partial\Omega$, it is easy to see that $F(u) = 0$ on $\partial\Omega$ and

$$x \cdot \nabla u = x \cdot \nu (\nabla u \cdot \nu) \quad \text{on} \quad \partial\Omega.$$

Therefore, $(x \cdot \nabla u) \nabla u \cdot \nu = (x \cdot \nu) |\nabla u|^2$ on $\partial\Omega$. Hence substituting the above relations into (4.23) proves the lemma. ∎

Corollary 4.6.0.1. Suppose Ω is a bounded domain which is star-shaped with respect to 0 and $\partial\Omega$ is C^1. If u is any smooth nontrivial solution of (4.20), then $p \leq \frac{n+2}{n-2}$.

Proof. Assume (4.20) admits a nontrivial smooth solution u. Since Ω is star shaped and bounded, using Lemma 4.3.1 and Lemma 4.6.2, we obtain

$$\frac{n}{p+1}\int_\Omega |u|^{p+1}\mathrm{d}x - \frac{n-2}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x \geq 0.$$

On the other hand, taking u as the test function for (4.20) yields $\int_\Omega |\nabla u|^2 dx = \int_\Omega |u|^{p+1}dx$. Substituting this to the previous expression yields $p \leq \frac{n+2}{n-2}$. Hence the corollary follows. ∎

The previous corollary highlights that the exponent $\frac{n+2}{n-2}$ is critical not only from the point of view of Sobolev embedding, but also from that of the existence of nontrivial solutions to (4.20).

4.6.2 Unbounded domain

Next, we consider the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u \in D^{1,2}(\Omega), \end{cases} \tag{4.24}$$

where $f \in C^1(\mathbb{R},\mathbb{R})$, $f(0) = 0$ and Ω is a smooth unbounded domain in $\mathbb{R}^n$, $n \geq 3$. Here

$$D^{1,2}(\Omega) := \{u \in L^{\frac{2n}{n-2}}(\Omega) : \int_\Omega |\nabla u|^2 \mathrm{d}x < \infty\}.$$

It's easy to check that $D^{1,2}(\Omega)$ is a Hilbert space with the $\|u\|_{D^{1,2}(\Omega)} := \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}}$.

Theorem 4.6.1. Let $u \in H^2_{loc}(\bar{\Omega})$ be a solution of (4.24) such that $F(u) \in L^1(\Omega)$. Then the Pohozaev identity is valid

Proof. In order to prove the Pohozaev identity in unbounded domain, one can use truncation argument. For that let $\psi \in C_c^\infty(\mathbb{R})$ such that $\psi = 0$ if $|x| \geq 2$, $\psi \equiv 1$ if $|x| \leq 1$, $0 \leq \psi \leq 1$. We define, $\psi_R : \mathbb{R}^n \to \mathbb{R}$ as $\psi_R(x) = \psi(\frac{|x|}{R})$. Note that, $(x \cdot \nabla u)\psi_R \in C_c^2(\mathbb{R}^n)$, therefore multiplying (4.24) by $(x \cdot \nabla u)\psi_R$ and integrating by parts, we have

$$\int_\Omega \nabla u \cdot \nabla\left((x \cdot \nabla u)\psi_R\right)\mathrm{d}x - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(x \cdot \nabla u)\psi_R \mathrm{d}S(x) = \int_\Omega f(u)\,(x \cdot \nabla u)\,\psi_R \mathrm{d}x. \tag{4.25}$$

Now RHS of (4.25) can be simplified as

$$\int_\Omega f(u)\,(x\cdot\nabla u)\,\psi_R \mathrm{d}x = \int_\Omega (\nabla F(u)\cdot x)\psi_R \mathrm{d}x = -n\int_\Omega F(u)\psi_R \mathrm{d}x - \int_\Omega F(u)(x\cdot\nabla\psi_R)\mathrm{d}x.$$

Note that $|x\cdot\nabla\psi_R)| \le C$ and hence using the dominated convergence theorem we get

$$\lim_{R\to\infty} RHS = -n\int_\Omega F(u)\mathrm{d}x. \tag{4.26}$$

By direct calculation and integration by parts, LHS of (4.25) simplifies as

$$LHS = -\frac{n-2}{2}\int_\Omega |\nabla u|^2\psi_R \mathrm{d}x - \frac{1}{2}\int_{\partial\Omega}(\frac{\partial u}{\partial\nu})^2(x\cdot\nu)\psi_R \mathrm{d}S(x)$$

$$-\frac{1}{2}\int_\Omega |\nabla u|^2(x\cdot\nabla\psi_R)\mathrm{d}x + \int_\Omega (x\cdot\nabla u)(\nabla u\cdot\nabla\psi_R)\mathrm{d}x.$$

Here we have used the fact $x\cdot\nabla u = x\cdot\nu\frac{\partial u}{\partial\nu}$ on $\partial\Omega$, since $u=0$ on $\partial\Omega$. Now

$$\lim_{R\to\infty} |\int_\Omega (x\cdot\nabla u)(\nabla u\cdot\nabla\psi_R)|\mathrm{d}x \le C\lim_{R\to\infty}\int_{R\le|x|\le 2R}|\nabla u|^2\mathrm{d}x = 0.$$

Using the above estimate and taking the limit using Lebesgue dominated convergence theorem and using the fact $|x\cdot\nabla\psi_R)| \le C$, we get

$$\lim_{R\to\infty} LHS = -\frac{n-2}{2}\int_\Omega |\nabla u|^2\mathrm{d}x - \frac{1}{2}\int_{\partial\Omega}(\frac{\partial u}{\partial\nu})^2(x\cdot\nu)\mathrm{d}S(x). \tag{4.27}$$

Substituting (4.26) and (4.27) in (4.25) yields

$$n\int_\Omega F(u)\mathrm{d}x - \frac{n-2}{2}\int_\Omega |\nabla u|^2\mathrm{d}x = \frac{1}{2}\int_{\partial\Omega}(\frac{\partial u}{\partial\nu})^2(x\cdot\nu)\mathrm{d}S(x) = \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2(x\cdot\nu)\mathrm{d}S(x).$$

■

Chapter 5

Nonvariational Techniques

In this chapter we discuss some non-variational techniques to study the existence/nonexistence of solutions to various nonlinear PDEs and qualitative properties of the solutions.

5.1 Fixed Point Method

For this section, X denotes a Banach space.

Theorem 5.1.1. (Schauder's Fixed Point Theorem)
Let K be a compact and convex subset of X and the map $A : K \to K$ is continuous. Then A has a fixed point in K.

Proof. See [L.C86, Theorem 3, Section 9.2]. ■

Schauder's fixed point theorem is not that easy to use. There is an alternative form which is useful for our purpose. Before seeing the alternative form, let's recall the definition of a compact map.

Definition 5.1.1. A nonlinear map $A : X \to X$ is said to be compact if for each bounded $\{u_k\}_{k=1}^{\infty}$, $\{A[u_k]\}$ has a convergent subsequence.

Theorem 5.1.2. (Schaefer's Fixed Point Theorem) Let the map $A : X \to X$ be compact and continuous. Moreover

$$\{u \in X | u = \lambda A[u] \text{ for some } 0 \leq \lambda \leq 1\}$$

is a bounded set. Then A has a fixed point.

Proof. See [L.C86, Theorem 4, Section 9.2]. ■

Application to a quasilinear elliptic PDE: Consider the semilinear problem

$$\begin{cases} -\Delta u + b(Du) + \mu u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $b : \mathbb{R}^n \to \mathbb{R}$ be smooth, Lipschitz continuous. The domain Ω is bounded and $\partial\Omega$ is smooth.

Theorem 5.1.3. For $\mu > 0$ sufficiently large, $\exists\ u \in H^2(\Omega) \cap H_0^1(\Omega)$ which solves (5.1).

Proof. See [L.C86, Theorem 5, Section 9.2]. ■

5.2 Method of Subsolution and Supersolution

In this section we show that under suitable assumption the existence of subsolution $\underline{u}$ and supersolution $\overline{u}$ ensure the existance of solution provided $\underline{u} \leq \overline{u}$. In this case the solution u satisfies

$$\underline{u} \leq u \leq \overline{u}.$$

Consider the nonlinear Poisson equation

$$\begin{cases} -\Delta u = f(u) & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases} \tag{5.2}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth and $|f'| \leq C$ for some constant C.

Definition 5.2.1. (i) The function $\overline{u} \in H^1(\Omega)$ is said to be a weak supersolution of (5.2) provided

$$\int_\Omega D\overline{u} \cdot Dv \,\mathrm{d}x \geq \int_\Omega f(\overline{u}) v \,\mathrm{d}x$$

for every $v \in H_0^1(\Omega)$ such that $v \geq 0$ almost everywhere.
(ii) The function $\underline{u} \in H^1(\Omega)$ is said to be a weak subsolution of (5.2) provided

$$\int_\Omega D\underline{u} \cdot Dv \,\mathrm{d}x \leq \int_\Omega f(\underline{u}) v \,\mathrm{d}x$$

for every $v \in H_0^1(\Omega)$ such that $v \geq 0$ almost everywhere.
(iii) The function $u \in H_0^1(\Omega)$ is said to be a weak solution of (5.2) provided

$$\int_\Omega Du \cdot Dv \,\mathrm{d}x = \int_\Omega f(u) v \,\mathrm{d}x \quad \forall\ v \in H_0^1(\Omega).$$

Theorem 5.2.1. (Existance of solution between subsolution and supersolution) Let $\overline{u}$ and $\underline{u}$ be a weak supersolution and a weak subsolution of (5.2) such that $\underline{u} \leq \overline{u}$ almost everywhere in Ω and $\overline{u} \geq 0$, $\underline{u} \leq 0$ on $\partial\Omega$ in trace sense. Then $\exists\ u$, a weak solution of (5.2), satisfying $\underline{u} \leq u \leq \overline{u}$ almost everwhere in Ω.

Proof. See [L.C86, Theorem 1, Section 9.3]. ∎

5.3 Radial Symmetry via Moving plane method

Take Ω to be the unit open ball in $\mathbb{R}^n$. Now consider the semilinear Dirichlet problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. Using *moving plane method*, we aim to show that if u is any positive solution of (5.3) then $u(x)$ depends only on $|x|$. This result is due to Gidas, Ni and Nirenberg [L.79].
For this purpose, we need two lemmas.

Lemma 5.3.1. (Refinement of Hopf's Lemma) Let V be an open subset of $\mathbb{R}^n$, $v \in C^2(\overline{V})$ and $c \in L^\infty(V)$. Moreover take

$$\begin{cases} -\Delta v + cv \geq 0 & \text{in } V \\ v \geq 0 & \text{in } V \end{cases} \tag{5.4}$$

where $v \not\equiv 0$.
(i) Now let $x^0 \in V$ be such that $v(x^0) = 0$ and also V satisfies interior ball condition at x^0. Then

$$\frac{\partial v}{\partial \nu}(x^0) < 0.$$

(ii) Moreover $v > 0$ in V.

Proof. See [L.C86, Lemma 1, Section 9.5]. ∎

Lemma 5.3.2. (Boundary estimates) If $u \in C^2(\overline{\Omega})$, $u > 0$ in Ω satisfies (5.3), then for each $x^0 \in \partial\Omega \cap \{x_n > 0\}$ either

$$u_{x_n}(x^0) < 0$$

or

$$u_{x_n}(x^0) = 0,\ u_{x_n x_n}(x^0) > 0.$$

For both cases, near x^0, u is strictly decreasing as a function of x_n.

Proof. See [L.C86, Lemma 2, Section 9.5]. ■

Notation: (i) For $\lambda \in [0,1]$, define the plane

$$P_\lambda := \{x \in \mathbb{R}^n | x_n = \lambda\}.$$

(ii) The reflextion of x in the plane $P_\lambda = x_\lambda := (x_1, ..., x_{n-1}, 2\lambda - x_n)$.
(iii) $E_\lambda := \{x \in \Omega | \lambda < x_n < 1\}$.

Theorem 5.3.1. If $u \in C^2(\overline{\Omega})$, $u > 0$ solves (5.3), then $u(x) = v(|x|)$ where $v : [0,1] \to [0, \infty)$ be a strictly decreasing function.

Proof. First consider the statement S_λ

$$u(x) < u(x_\lambda) \quad \forall\, x \in E_\lambda.$$

By the last lemma we can say that the statement is valid for all $\lambda < 1$ and λ is close enough to 1. Define

$$\lambda_0 := \inf\{0 \leq \lambda < 1 | S_\mu \text{ holds for all } \lambda \leq \mu < 1\}.$$

We claim that $\lambda_0 = 0$.
If not, then $\lambda > 0$. Define for $x \in E_{\lambda_0}$,

$$w(x) := u(x_{\lambda_0}) - u(x)$$

Then we have

$$-\Delta w = f(u(x_{\lambda_0})) - f(u(x)) = -cw \quad \text{in } E_{\lambda_0}$$

where $c(x) := -\int_0^1 (f'(u(x_{\lambda_0})) + (1-s)u(x))ds$.
As $w \geq 0$ in E_{λ_0}, from refined Hopf's lemma we have $w > 0$ in E_{λ_0} and $w_{x_n} > 0$ on $P_{\lambda_0} \cap \Omega$. Hence

$$u(x) < u(x_{\lambda_0}) \text{ in } E_{\lambda_0} \quad \text{and} \quad u_{x_n} < 0 \text{ on } P_{\lambda_0} \cap \Omega.$$

Now from the previous lemma we have

$$u(x) < u(x_{\lambda_0 - \epsilon}) \text{ in } E_{\lambda_0 - \epsilon} \quad \forall\ 0 \leq \epsilon \leq \epsilon_0$$

where ϵ_o is chosen to be small enough. This contradicts the definition of λ_0. Hence, $\lambda_0 = 0$.
$\lambda_0 = 0$ implies $u(x_1, \ldots, x_{n-1}, -x_n) \geq u(x_1, \ldots, x_n)$.

Similarly in $\Omega \cap \{x_n > 0\}$, we have $u(x_1, \ldots, x_{n-1}, -x_n) \leq u(x_1, \ldots, x_n)$. So, u has symmetry around P_0 and $u_{x_n} = 0$ on P_0.
The above logic is valid even after rotation of axes. So, the solution is symmetric about any plane through origin and the proof is done.

■

Chapter 6

Regularity Theory for Nonlinear Equations

In this chapter we discuss various regularity properties of weak solutions to elliptic equations of divergence form and with zeroth order term. More, precisely we consider the equation of the form

$$-D_j(a_{ij}(x)D_iu) + c(x)u = f(x) \text{ in } \Omega$$

where $\Omega \subset \mathbb{R}^n$ is a domain. We recall that $u \in H^1(\Omega)$ is said a weak solution of the above equation provided

$$\int_\Omega (a_{ij}D_iuD_j\phi + cu\phi)\ \mathrm{d}x = \int_\Omega f\phi\ \ \mathrm{d}x\ \forall\ \phi \in H_0^1(\Omega).$$

6.1 Growth of Local Integrals

Let Ω be a bounded, open and connected subset of $\mathbb{R}^n$ and $u \in L^1(\Omega)$. Define

$$u_{x_0,r} := \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u\ \ \mathrm{d}x.$$

Theorem 6.1.1. Suppose $u \in L^2(\Omega)$ be such that

$$\int_{B_r(x)} |u(y) - u_{x,r}(y)|^2\mathrm{d}y \le M^2 r^{n+2\alpha}$$

for all $B_r(x_0) \subset \Omega$, where $0 < \alpha < 1$. Then $u \in C^\alpha(\Omega)$. Moreover for any $\Omega^{'} \subset\subset \Omega$, we have

$$\sup_{\Omega'} |u| + \sup_{x,y\in\Omega',\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^{\alpha}} \leq c(M + \|u\|_{L^2(\Omega)})$$

where c depends on n, α, Ω and Ω'.

Proof. See [Q.00, Theorem 3.1]. ∎

Before stating next theorem, we introduce an important decomposition technique called Calderon-Zygmand decomposition. To describe that we need some terminology.
By Q_0, we denote the unit cube in $\mathbb{R}^n$. Cut Q_0 into 2^n equal pieces and call each of the cubes as first generation. Repeat the same process to the new cubes to get the second generation of cubes. Continue this process. We call these cubes from all generation dyadic cubes. So, any $(k+1)^{th}$ generation cube Q comes from some k^{th} generation cube. In this case, we call the k^{th} generation cube the predecessor of Q.

Lemma 6.1.1. (Calderon-Zygmand decomposition) Let $f \in L^1(Q_0)$ be such that $f \geq 0$. $\alpha > \frac{1}{|Q_0|}\int_{|Q_0|} f \ \mathrm{d}x$ is fixed. Then there exists sequence of dyadic cubes $\{Q_j\}$ in Q_0 which are non overlapping and

$$f(x) \leq \alpha \ \text{ almost everywhere in } \ Q_0 \setminus \cup_j Q_j,$$

$$\alpha \leq \frac{1}{Q_j}\int_{Q_j} f \ \mathrm{d}x < 2^n\alpha.$$

Proof. See [Q.00, Lemma 3.7]. ∎

Theorem 6.1.2. (John-Nirenberg Lemma) If $u \in L^1(\Omega)$ satisfies

$$\int_{B_r(x_0)} |u(y) - u_{x,r}(y)|^2 \mathrm{d}y \leq M^2 r^n$$

for all $B_r(x_0) \subset \Omega$, then for any $B_r(x_0) \subset \Omega$, we have

$$\int_{B_r(x_0)} \exp^{\frac{p_0}{M}|u-u_{x,r}|} \mathrm{d}y \leq Cr^n,$$

where p_0, C are positive constants depend only on n.

Proof. See [Q.00, Theorem 3.5]. ∎

Remark: The functions which satisfy the hypothesis of the previous theorem are known as functions with *b*ounded mean oscilation(BMO). The following relation hold:

$$L^{\infty}(\Omega) \subset \text{BMO}.$$

But the converse need not be true, for example: $u(x) = \log x, \quad x \in (0,1)$.

6.2 Hölder Continuity of Solutions and gradient of solutions

Let $a_{ij} \in L^\infty(B_1)$ be uniformly elliptic in $B_1 := B_1(0)$, i.e.,

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \;\; \forall\, x \in B_1, \xi \in \mathbb{R}^n. \tag{6.1}$$

Throughout this section we assume a_{ij} is at least continuous and $u \in H^1(B_1)$ satisfies the equation

$$\int_{B_1} (a_{ij}D_iuD_j\phi + cu\phi)\, \mathrm{d}x = \int_{B_1} f\phi \;\, \mathrm{d}x \;\; \forall\, \phi \in H_0^1(B_1). \tag{6.2}$$

First we state two lemma which are very important in proving the main theorem of this section.

Lemma 6.2.1. Let $\{a_{ij}\}$ is constant positive definite matrix such that

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \;\; \forall\, \xi \in \mathbb{R}^n$$

where $0 < \lambda \leq \Lambda$. Assume also $w \in H^1(B_r(x_0))$ satisfies

$$a_{ij}D_{ij}w = 0 \quad in \;\; B_r(x_0)$$

in weak sense. Then $\forall\, \rho \in (0, r]$

$$\int_{B_\rho(x_0)} |Dw|^2 \, \mathrm{d}x \leq c\left(\frac{\rho}{r}\right)^n \int_{B_r(x_0)} |Dw|^2 \, \mathrm{d}x$$

and

$$\int_{B_\rho(x_0)} |Dw - (Dw)_{x_0,r}|^2 \, \mathrm{d}x \leq c\left(\frac{\rho}{r}\right)^{n+2} \int_{B_r(x_0)} |Dw - (Dw)_{x_0,r}|^2 \, \mathrm{d}x,$$

where c depends only upon λ and Λ.

Proof. See [Q.00, Lemma 3.10]. ■

From the above lemma we can get the following corollary

Corollary 6.2.0.1. Suppose w be as in the previous lemma. Then $\forall\ u \in H^1(B_r(x_0))$ and $\forall\ \rho \in (0,r]$, we have

$$\int_{B_\rho(x_0)} |Du|^2\, dx \le c\Big[\Big(\frac{\rho}{r}\Big)^n \int_{B_r(x_0)} |Du|^2\, dx + \int_{B_r(x_0)} |D(u-w)|^2\, dx\Big]$$

where c depends on λ and Λ.

Proof. See [Q.00, Corollary 3.11]. ■

Now we state the main theorem of this section.

Theorem 6.2.1. Let $u \in H^1(B_1)$ satisfies (6.2), where $a_{ij} \in C^0(\overline{B_1})$, $c \in L^n(B_1)$ and $f \in L^q(B_1)$ for some $q \in (\frac{n}{2}, n)$. Then $u \in C^\alpha(B_1)$ with $\alpha = 2 - \frac{n}{q}$. Also, $\exists\ R_0$ depending on $\lambda, \Lambda, \tau, \|c\|_{L^n}$ such that for $x \in B_{\frac{1}{2}}$ and $r \le R_0$, we have

$$\int_\Omega |Du|^2\, dx \le Cr^{n-2+2\alpha}(\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}),$$

where C depending on $\lambda, \Lambda, \tau, \|c\|_{L^n}$ with $|a_{ij}(x) - a_{ij}(y)| \le \tau(|x-y|)\ \forall\ x, y \in B_1$.

Remark: If we take $C \equiv 0$ in the elliptic PDE, we can replace $\|u\|_{H^1(B_1)}$ by $\|Du\|_{L^2(B_1)}$.

Proof. Using Corollary 6.2.0.1, one can prove the theorem as in [Q.00, Theorem 3.8]. ■

Theorem 6.2.2. Assume $u \in H^1(B_1)$ satisfies (6.2), where $a_{ij} \in C^\alpha(\bar{B}_1)$ and (6.1) holds. Also assume, $c \in L^q(B_1)$ and $f \in L^q(B_1)$ for some $q > n$ and $\alpha = 1 - n/q \in (0,1)$. Then $Du \in C^\alpha(B_1)$. Moreover, there exists an R_0 depending upon λ, $|a_{ij}|_{C^\alpha}$ and $|c|_{L^q}$ so that $\forall\ x \in B_{\frac{1}{2}}$ and $r \le R_0$ the following holds

$$\int_{B_r(x)} |Du(y) - (Du)_{x,r}(y)|^2\, dy \le Cr^{n+2\alpha}\left\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\right\}$$

where $C > 0$ is a constant depending upon λ, $|a_{ij}|_{C^\alpha}$ and $|c|_{L^q}$.

Proof. We refer [Q.00, Theorem 3.13]. ■

6.3 Local Boundedness

In this section we discuss celebrated De Giorgi-Nash-Moser regularity theory for elliptic equations.

Theorem 6.3.1. Assume $a_{ij} \in L^\infty(B_1)$ and $c \in L^q(B_1)$ for some $q > n/2$ satisfy the following assumptions

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \forall\, x \in B_1, \xi \in \mathbb{R}^n \quad \text{and} \quad |a_{ij}|_{L^\infty} + \|c\|_{L^q} \leq \Lambda$$

where $\lambda, \Lambda > 0$. Let $u \in H^1(B_1)$ is a subsolution, that is,

$$\int_{B_1} (a_{ij}D_iuD_j\varphi + cu\varphi)\,\mathrm{d}x \leq \int_{B_1} f\varphi\,\mathrm{d}x \quad \forall\, \varphi \in H_0^1(B_1), \varphi \geq 0 \quad \text{in} \quad B_1. \tag{6.3}$$

If $f \in L^q(B_1)$, then $u^+ \in L^\infty_{\text{loc}}(B_1)$. Moreover, for any $0 < \theta < 1$ and any $p > 0$ the following holds:

$$\sup_{B_\theta} u^+ \leq C\left\{\frac{1}{(1-\theta)^{n/p}}\|u^+\|_{L^p(B_1)} + \|f\|_{L^q(B_1)}\right\}$$

where C is a positive constant depends upon n, p, q, λ and Λ.

Proof. There are two famous approaches for the proof. One of them is by **De Giorgi** and the other by **Moser**. We'll discuss the one by **De Giorgi**.
We'll assume $\theta = \frac{1}{2}$ and $p = 2$.
Define $v := (u-k)^+$ where $k \geq 0$ and assume $\zeta \in C_0^1(B_1)$. Take $\varphi = v\zeta^2$ to be the test function. When $\{u > k\}$, we have $v = u - k$ and $Du = Dv$ almost everywhere. When $\{u \leq k\}$, we have $v = 0$ and $Dv = 0$ almost everywhere. Therefore, if we substitute φ in (6.3), all terms will have v or Dv and the integration will be 0 in $\{u \leq k\}$. So, It is enough to carry out the integration in $\{u > k\}$.
Hence, using Hölder inequality we have

$$\begin{aligned}\int a_{ij}D_iuD_j\varphi\,\mathrm{d}x &= \int \left(a_{ij}D_iuD_jv\zeta^2 + 2a_{ij}D_iuD_j\zeta v\zeta\right)\mathrm{d}x \\ &\geq \lambda\int |Dv|^2\zeta^2\,\mathrm{d}x - 2\Lambda\int |Dv||D\zeta|v\zeta\,\mathrm{d}x \\ &\geq \frac{\lambda}{2}\int |Dv|^2\zeta^2\,\mathrm{d}x - \frac{2\Lambda^2}{\lambda}\int |D\zeta|^2v^2\,\mathrm{d}x.\end{aligned}$$

So, one gets

$$\int |Dv|^2\zeta^2\,\mathrm{d}x \leq C\left\{\int v^2|D\zeta|^2\,\mathrm{d}x + \int |c|v^2\zeta^2\,\mathrm{d}x + k^2\int |c|\zeta^2\,\mathrm{d}x + \int |f|v\zeta^2\,\mathrm{d}x\right\},$$

from which we have

$$\int |D(v\zeta)|^2\,\mathrm{d}x \leq C\left\{\int v^2|D\zeta|^2\,\mathrm{d}x + \int |c|v^2\zeta^2\,\mathrm{d}x + k^2\int |c|\zeta^2\,\mathrm{d}x + \int |f|v\zeta^2\,\mathrm{d}x\right\}.$$

Now using Sobolev inequality and Hölder inequality, for small $\delta > 0$ and $\zeta \leq 1$, we have

$$\begin{aligned}\int |f| v\zeta^2 \,\mathrm{d}x &\leq \left(\int |f|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int |v\zeta|^{2^*} \,\mathrm{d}x\right)^{\frac{1}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{1}{2^*}-\frac{1}{q}} \\ &\leq c(n)\|f\|_{L^q} \left(\int |D(v\zeta)|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} |\{v\zeta \neq 0\}|^{\frac{1}{2}+\frac{1}{n}-\frac{1}{q}} \\ &\leq \delta \int |D(v\zeta)|^2 \,\mathrm{d}x + c(n,\delta)\|f\|^2_{L^q} |\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{2}{q}}.\end{aligned}$$

As $1 + \frac{2}{n} - \frac{2}{q} > 1 - \frac{1}{q}$ for $q > n/2$, we get

$$\int |D(v\zeta)|^2 \,\mathrm{d}x \leq C\left\{\int v^2|D\zeta|^2 \,\mathrm{d}x + \int |c|v^2\zeta^2 \,\mathrm{d}x + k^2 \int |c|\zeta^2 \,\mathrm{d}x + F^2|\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\right\},$$

where $F := \|f\|_{L^q(B_1)}$.
Our claim now is

$$\int |D(v\zeta)|^2 \,\mathrm{d}x \leq C\left\{\int v^2|D\zeta|^2 \,\mathrm{d}x + \left(k^2 + F^2\right) |\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\right\} \tag{6.4}$$

for $|\{v\zeta \neq 0\}|$ small enough.
There is not much to prove if $c \equiv 0$. For the general situation, using Hölder inequality we have

$$\begin{aligned}\int |c|v^2\zeta^2 \,\mathrm{d}x &\leq \left(\int |c|^q \,\mathrm{d}x\right)^{\frac{1}{q}} \left(\int (v\zeta)^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}-\frac{1}{q}} \\ &\leq c(n) \int |D(v\zeta)|^2 \,\mathrm{d}x \left(\int |c|^q \,\mathrm{d}x\right)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}}\end{aligned}$$

and

$$\int |c|\zeta^2 \,\mathrm{d}x \leq \left(\int |c|^q \,\mathrm{d}x\right)^{\frac{1}{q}} |\{v\zeta \neq 0\}|^{1-\frac{1}{q}}.$$

Hence we get

$$\int |D(v\zeta)|^2 \,\mathrm{d}x \leq \quad C\left\{\int v^2|D\zeta|^2 \,\mathrm{d}x + \int |D(v\zeta)|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}-\frac{1}{q}} + \left(k^2 + F^2\right) |\{v\zeta \neq 0\}|^{1-\frac{1}{q}}\right\} \tag{6.5}$$

Thus, we have proved our claim. Now, using Sobolev inequality,

$$\int (v\zeta)^2 \,\mathrm{d}x \leq \left(\int (v\zeta)^{2^*} \,\mathrm{d}x\right)^{\frac{2}{2^*}} |\{v\zeta \neq 0\}|^{1-\frac{2}{2^*}} \leq c(n) \int |D(v\zeta)|^2 |\{v\zeta \neq 0\}|^{\frac{2}{n}}.$$

Hence

$$\int (v\zeta)^2 \,\mathrm{d}x \le C\left\{\int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\frac{2}{n}} + (k+F)^2|\{v\zeta \neq 0\}|^{1+\frac{2}{n}-\frac{1}{q}}\right\}.$$

if $|\{v\zeta \neq 0\}|$ is taken to be small. So, $\exists\, \varepsilon > 0$ such that

$$\int (v\zeta)^2 \,\mathrm{d}x \le C\left\{\int v^2|D\zeta|^2|\{v\zeta \neq 0\}|^{\varepsilon} + (k+F)^2|\{v\zeta \neq 0\}|^{1+\varepsilon}\right\}$$

for $\{v\zeta \neq 0\}$ small enough.
To choose the cutoff function ζ, fix $0 < r < R \le 1$. Select $\zeta \in C_0^\infty(B_R)$ so that $\zeta = 1$ in B_r, $0 \le \zeta \le 1$ and $|D\zeta| \le \frac{2}{R-r}$ in B_1.
Define $A(k,r) := \{x \in B_r | u \ge k\}$. Note for $0 < r < R \le 1$ and $k > 0$, we have

$$\int_{A(k,r)} (u-k)^2 \,\mathrm{d}x \le C\left\{\frac{1}{(R-r)^2}|A(k,R)|^{\varepsilon}\int_{A(k,R)} (u-k)^2 \,\mathrm{d}x + (k+F)^2|A(k,R)|^{1+\varepsilon}\right\} \tag{6.6}$$

for $|A(k,R)|$ sufficiently small. We also have

$$|A(k,R)| \le \frac{1}{k}\int_{A(k,R)} u^+ \,\mathrm{d}x \le \frac{1}{k}\left\|u^+\right\|_{L^2}.$$

Thus (6.6) holds true if we have $k \ge k_0 = c\|u^+\|_{L^2}$ for some large $C = C(\lambda,\Lambda)$.
Now we prove that $\exists\, k = C(k_0 + F)$ satisfying $\int_{A(k,1/2)} (u-k)^2 \,\mathrm{d}x = 0$.
Take $h > k \ge k_0$ and $0 < r < 1$. Trivially, $A(h,r) \subset A(k,r)$. This implies, $\int_{A(h,r)} (u-h)^2 \,\mathrm{d}x \le \int_{A(k,r)} (u-k)^2 \,\mathrm{d}x$.
and

$$|A(h,r)| = |B_r \cap \{u-k > h-k\}| \le \frac{1}{(h-k)^2}\int_{A(k,r)} (u-k)^2.$$

Hence using (6.6), for any $h > k \ge k_0$ and $1/2 \le r < R \le 1$,

$$\begin{aligned}\int_{A(h,r)} (u-h)^2 \,\mathrm{d}x &\le C\left\{\frac{1}{(R-r)^2}\int_{A(h,R)} (u-h)^2 \,\mathrm{d}x + (h+F)^2|A(h,R)|\right\}|A(h,R)|^{\varepsilon}\\ &\le C\left\{\frac{1}{(R-r)^2} + \frac{(h+F)^2}{(h-k)^2}\right\}\frac{1}{(h-k)^{2\varepsilon}}\left(\int_{A(k,R)} (u-k)^2 \,\mathrm{d}x\right)^{1+\varepsilon}\end{aligned}$$

or

$$\left\|(u-h)^+\right\|_{L^2(B_r)} \le C\left\{\frac{1}{R-r} + \frac{h+F}{h-k}\right\}\frac{1}{(h-k)^{\varepsilon}}\left\|(u-k)^+\right\|_{L^2(B_R)}^{1+\varepsilon}. \tag{6.7}$$

Now just iterate. Define $\varphi(k,r) := \|(u-k)^+\|_{L^2(B_r)}$ and set $\tau = 1/2$. $k > 0$ will be determined later. For $\ell = 0, 1, \ldots,$ define

$$k_\ell = k_0 + k\left(1 - \frac{1}{2^\ell}\right) \quad (\le k_0 + k)$$
$$r_\ell = \tau + \frac{1}{2^\ell}(1-\tau).$$

A straight forward computation yields

$$k_\ell - k_{\ell-1} = \frac{k}{2^\ell}, \quad r_{\ell-1} - r_\ell = \frac{1}{2^\ell}(1-\tau).$$

Hence,

$$\begin{aligned}\varphi(k_\ell, r_\ell) &\le C\left\{\frac{2^\ell}{1-\tau} + \frac{2^\ell(k_0+F+k)}{k}\right\}\frac{2^{\varepsilon\ell}}{k^\varepsilon}\left[\varphi(k_{\ell-1}, r_{\ell-1})\right]^{1+\varepsilon} \\ &\le \frac{C}{1-\tau}\frac{k_0+F+k}{k^{1+\varepsilon}}2^{(1+\varepsilon)\ell}\left[\varphi(k_{\ell-1}, r_{\ell-1})\right]^{1+\varepsilon}.\end{aligned}$$

Using mathematical induction, it is not difficult to prove for any $\ell = 0, 1, \ldots$

$$\varphi(k_\ell, r_\ell) \le \frac{\varphi(k_0, r_0)}{\gamma^\ell} \quad \text{for some } \gamma > 1,$$

provided we choose k large enough. Choosing $k = C_*(k_0 + F + \varphi(k_0, r_0))$ will do the job provided C_* is sufficiently large.
Take $\ell \to \infty$ in the above equation to get $\varphi(k_0 + k, \tau) = 0$. So, we get

$$\sup_{B_{1/2}} u^+ \le (C_* + 1)\left\{k_0 + F + \varphi(k_0, r_0)\right\}.$$

As $k_0 = C\|u^+\|_{L^2(B_1)}$ and $\varphi(k_0, r_0) \le \|u^+\|_{L^2(B_1)}$, we are done. ∎

Theorem 6.3.2. Assume $a_{ij} \in L^\infty(B_1)$ and $c \in L^{n/2}(B_1)$ satisfy

$$\lambda|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2 \quad \forall\, x \in B_1, \quad \xi \in \mathbb{R}^n$$

for $\lambda, \Lambda > 0$. Suppose that $u \in H^1(B_1)$ is a subsolution, that is,

$$\int_{B_1}(a_{ij}D_iuD_j\varphi + cu\varphi)\,\mathrm{d}x \le \int_{B_1} f\varphi \;\, \mathrm{d}x \quad \forall\, \varphi \in H_0^1(B_1) \text{ and } \varphi \ge 0 \text{ in } B_1$$

If $f \in L^q(B_1)$ for some $\frac{2n}{n+2} \le q < \frac{n}{2}$, then $u^+ \in L^{q^*}_{\mathrm{loc}}(B_1)$ where $\frac{1}{q^*} = \frac{1}{q} - \frac{2}{n}$. Moreover,

$$\|u^+\|_{L^{q^*}(B_{1/2})} \le C\left\{\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)}\right\},$$

where C is a positive constant depends upon n, λ, Λ, q and $\varepsilon(K)$ where

$$\varepsilon(K) = \left(\int_{|c|>K\}} |c|^{\frac{n}{2}} \, \mathrm{d}x \right)^{\frac{2}{n}}.$$

Proof. See [Q.00, Theorem 4.4]. ■

6.4 Hölder Continuity

First let's discuss homogeneous equations without lower-order terms. Take

$$Lu \equiv -D_i \left(a_{ij}(x) D_j u \right) \quad \text{in } B_1(0) \subset \mathbb{R}^n,$$

where we presume $a_{ij} \in L^\infty(B_1)$ satisfies

$$\lambda |\xi|^2 \leq a_{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \quad \forall \, x \in B_1(0), \; \xi \in \mathbb{R}^n$$

for some $\lambda, \Lambda > 0$.

Definition 6.4.1. The function $u \in H^1_{\text{loc}}(B_1)$ is said to be a subsolution (or supersolution) of $Lu = 0$ if u sattisfies

$$\int_{B_1} a_{ij} D_i u D_j \varphi \, \mathrm{d}x \leq 0 (\geq 0) \quad \forall \, \varphi \in H^1_0(B_1), \quad \varphi \geq 0.$$

Lemma 6.4.1. Assume $\Phi \in C^{0.1}_{\text{loc}}(\mathbb{R})$ is convex. Then
(i) if u is a subsolution and $\Phi' \geq 0$, then $v = \Phi(u)$ is also a subsolution presuming $v \in H^1_{\text{loc}}(B_1)$
(ii) if u is a supersolution and $\Phi' \leq 0$, then $v = \Phi(u)$ is a subsolution presuming $v \in H^1_{\text{loc}}(B_1)$.

Proof. See [Q.00, Lemma 4.6]. ■

Lemma 6.4.2. For any given $\varepsilon > 0 \; \exists \; C$ depending upon ε and n such that $\forall \; u \in H^1(B_1)$ with

$$|\{x \in B_1 | u = 0\}| \geq \varepsilon |B_1|,$$

there holds

$$\int_{B_1} u^2 \, \mathrm{d}x \leq C \int_{B_1} |Du|^2 \, \mathrm{d}x.$$

Proof. See [Q.00, Lemma 4.8]. ■

Theorem 6.4.1. (Density Theorem) Let $u > 0$ be a supersolution in B_2 with

$$|\{x \in B_1 | u \geq 1\}| \geq \varepsilon |B_1| .$$

Then there $\exists$ C depending upon ε, n, and Λ/λ such that

$$\inf_{B_{1/2}} u \geq C.$$

Proof. See [Q.00, Theorem 4.9]. ■

Theorem 6.4.2. Let u be a bounded solution of of $Lu = 0$ in B_2. Then $\exists$ $\gamma \in (0,1)$ depending on n and $\frac{\Lambda}{\lambda}$ so that

$$\operatorname{osc}_{B_{\frac{1}{2}}} u \leq \gamma \operatorname{osc}_{B_1} u.$$

Proof. See [Q.00, Theorem 4.10]. ■

Combining the above four results, it is easy to see that the following De Giorgi's theorem follows.

Theorem 6.4.3. (De Giorgi) Assume $Lu = 0$ weakly in B_1. Then

$$\sup_{B_{1/2}} |u(x)| + \sup_{x.y \in B_{1/2}} \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \leq c\left(n, \frac{\Lambda}{\lambda}\right) \|u\|_{L^2(B_1)},$$

where $\alpha \in (0,1)$ depending on n, and Λ/λ.

Lemma 6.4.3. Assume $a_{ij} \in L^{\infty}(B_r)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall\, x \in B_r, \xi \in \mathbb{R}^n$$

for some $0 < \lambda \leq \Lambda < +\infty$. Let $u \in H^1(B_r)$ satisfies

$$\int_{B_r} a_{ij} D_i u D_j \varphi \, \mathrm{d}x = 0 \quad \forall\, \varphi \in H_0^1(B_r).$$

Then there exists $0 < \alpha < 1$ such that $\forall\, \rho < r$ there holds

$$\int_{B_\rho} |Du|^2 \, \mathrm{d}x \leq C\left(\frac{\rho}{r}\right)^{n-2+2\alpha} \int_{B_r} |Du|^2 \, \mathrm{d}x,$$

where C and α are constants depending on n and Λ/λ

Proof. See [Q.00, Lemma 4.12]. ■

Theorem 6.4.4. Suppose $a_{ij} \in L^\infty(B_1)$ and $c \in L^n(B_1)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall\, x \in B_1, \quad \xi \in \mathbb{R}^n$$

for some $0 < \lambda \leq \Lambda < +\infty$. Assume $u \in H^1(B_1)$

$$\int_{B_1} (a_{ij}D_juD_i\varphi + cu\varphi)\,\mathrm{d}x = \int_{B_1} f\varphi\,\mathrm{d}x \quad \forall\, \varphi \in H_0^1(B_1).$$

If $f \in L^q(B_1)$ for some $q > n/2$, then $u \in C^\alpha(B_1)$ for some $0 < \alpha < 1$ depending upon $n, q, \lambda, \Lambda\ \|c\|_{L^n}$. Moreover, $\exists\ R_0 = R_0(q, \lambda, \Lambda, \|c\|_{L^n})$ such that $\forall\ x \in B_{\frac{1}{2}}$ and $r \leq R_0$ the following holds

$$\int_{B_r(x)} |Du|^2\,\mathrm{d}x \leq Cr^{n-2+2\alpha}\left\{\|f\|^2_{L^q(B_1)} + \|u\|^2_{H^1(B_1)}\right\},$$

where $C = C(n, q, \lambda, \Lambda, \|c\|_{L^n})$ is a positive constant.

Proof. See [Q.00, Theorem 4.13]. ■

6.5 Moser Harnack Inequality

Let's assume there is no lower-order terms. Suppose Ω is a domain in $\mathbb{R}^n$. Also take $a_{ij} \in L^\infty(\Omega)$ satisfies

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2 \quad \forall\, x \in \Omega,\ \xi \in \mathbb{R}^n$$

for some $\lambda, \Lambda > 0$.

Theorem 6.5.1. (Local Boundedness) Assume $u \in H^1(\Omega)$ is a nonnegative subsolution in Ω :

$$\int_\Omega a_{ij}D_iuD_j\varphi\,\mathrm{d}x \leq \int_\Omega f\varphi\,\mathrm{d}x \quad \forall\, \varphi \in H_0^1(\Omega),\ \phi \geq 0 \text{ in } \Omega.$$

Let $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$, any $0 < r < R$ and any $p > 0$

$$\sup_{B_r} u \leq C\left\{\frac{1}{(R-r)^{n/p}}\left\|u^+\right\|_{L^p(B_R)} + R^{2-\frac{n}{q}}\|f\|_{L^q(B_R)}\right\},$$

where $C > 0$ is a constant depending on $n, \lambda, \Lambda, p, q$.

Proof. See [Q.00, Theorem 4.14]. ■

Theorem 6.5.2. (Weak Harnack Inequality) Assume $u \in H^1(\Omega)$ is a nonnegative supersolution in Ω, that is,

$$\int_\Omega a_{ij} D_i u D_j \varphi \, \mathrm{d}x \geq \int_\Omega f \varphi \, \mathrm{d}x \quad \forall \, \varphi \in H_0^1(\Omega), \varphi \geq 0 \text{ in } \Omega.$$

Let $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$ for any $0 < p < n/(n-2)$ and any $0 < \theta < \tau < 1$ the following holds

$$\inf_{B_{\theta R}} u + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \geq C \left(\frac{1}{R^n} \int_{B_{\tau R}} u^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

Proof. See [Q.00, Theorem 4.15]. ■

Theorem 6.5.3. (Moser's Harnack Inequality) Assume $u \in H^1(\Omega)$ is a nonnegative solution in Ω, that is,

$$\int_\Omega a_{ij} D_i u D_j \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \quad \forall \, \varphi \in H_0^1(\Omega).$$

Assume also $f \in L^q(\Omega)$ for some $q > n/2$. Then for any $B_R \subset \Omega$ the following holds

$$\max_{B_R} u \leq C \left\{ \min_{B_{R/2}} u + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\},$$

where $C > 0$ is a constant depends on n, λ, Λ and q.

Proof. See [Q.00, Theorem 4.17]. ■

Corollary 6.5.3.1. (Hölder Continuity) Assume $u \in H^1(\Omega)$ is a solution in Ω, that is,

$$\int_\Omega a_{ij} D_i u D_j \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \quad \forall \, \varphi \in H_0^1(\Omega).$$

Assume also $f \in L^q(\Omega)$ for some $q > n/2$. Then $u \in C^\alpha(\Omega)$ for some $\alpha \in (0,1)$ which depends on n, q, λ and Λ. Moreover, for any $B_R \subset \Omega$ the following holds

$$|u(x) - u(y)| \leq C \left(\tfrac{|x-y|}{R} \right)^\alpha \left\{ \left(\tfrac{1}{R^n} \int_{B_R} u^2 \, \mathrm{d}x \right)^{\frac{1}{2}} + R^{2-\frac{n}{q}} \|f\|_{L^q(B_R)} \right\} \quad \forall \, x, y \in B_{\frac{R}{2}},$$

where $C > 0$ is a constant which depends on n, λ, Λ and q.

Proof. See [Q.00, Corollary 4.18]. ■

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