Gravitational waves from inspiralling compact binaries:
Partial 3.5 PN modes, Higher order stationary phase approximation

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

Anilkumar Tolamatti

under the guidance of

Prof. Bala Iyer

Raman Research Institute, Bengaluru & International Center for Theoretical Sciences (ICTS) Bengaluru.

Indian Institute of Science Education and Research Pune

Certificate

This is to certify that this thesis entitled *Gravitational waves from inspiralling compact binaries: Partial 3.5 PN modes, Higher order stationary phase approximation* submitted towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by *Anilkumar Tolamatti* at *Raman Research Institute, Bengaluru* and *International Center for Theoretical Sciences (ICTS), Bengaluru*, under the supervision of *Prof. Bala Iyer* during the academic year 2014-2015.

Student
ANILKUMAR
TOLAMATTI

Supervisor
PROF. BALA IYER

Declaration

I hereby declare that the matter embodied in the report entitled *Gravitational Waves from Inspiralling compact binaries: Partial 3.5 PN modes, Higher order stationary phase approximation* are the results of the investigations carried out by me at Raman Research Institute, Bengaluru and International Center for Theoretical Sciences (ICTS), Bengaluru, under the supervision of Prof. Bala Iyer and the same has not been submitted elsewhere for any other degree.

Student
ANILKUMAR
TOLAMATTI

Supervisor
PROF.BALA IYER

Acknowledgements

I would like to express my deepest gratitude to my advisor, Prof. Bala Iyer, for his excellent guidance, caring nature, patience and providing me an excellent opportunity to pursue my interest. I am thankful to him for always being available for discussions. I am forever indebted to him.

Secondly, I wish to thank Prof.P.Ajith and Dr.Chandra Kant Mishra for their valuable inputs during programming and suggestions in general. I would like to thank Dr. Suneeta Vardarajan for being my TAC member and also for introducing me to this field.

I would also like to specially thank the Astrorel Group at ICTS. The environment in the group has been most friendly and supportive and the suggestions I got from the group have always been helpful. I am also thankful for the weekly seminars, paper discussions and journal clubs that take place. They have really fueled my interest in this field and increased the breadth of my knowledge greatly.

I am grateful to RRI and ICTS for the hospitality provided during my stay here in Bengaluru. Finally, I would like to thank my family and friends especially Abhijith,Ashok, Bhavesh, Nagesh, Sainath, Siddharth, Viraj and Utkarsh for their constant support and care.

Abstract

For both data analysis and validation of numerical waveforms we need to know the Post-Newtonian (PN) waveforms as accurately as possible. Comparison between PN waveforms and numerical waveforms is implemented by projecting the waveforms onto spin weighted spherical harmonics and comparing the individual components. I am focusing on the inspiral part of non-spinning coalescing compact binary (CCB) system, as they are the most promising sources for laser interferometric detectors. To compute the spherical harmonic modes (h^{lm}s) for inspiralling compact binaries (ICBs) we need to know the source multipole moments of the system. Earlier investigations have calculated the spherical harmonic modes for 3 PN accurate gravitational waveform and some modes for $l = 2, 3$, at 3.5 PN accuracy. My aim is to go half a PN order further and calculate spherical harmonic modes for 3.5 PN accurate full waveform for non-spinning ICBs. To this end, I discuss the accuracy of the source multipole moments which will contribute to 3.5 PN waveform. By doing literature survey I have checked the availability of these moments and listed the available ones. I show that with the available source moments we can calculate spherical harmonic modes(h^{lm}s) with full 3.5 PN accuracy, only for modes $l = 2, 3$ when $l + m$ is even, for $l = 5$ when $l + m$ is odd and for $l \geq 6$. With the available source multipole moments as inputs, I have written a *mathematica* code, which gives spherical harmonic modes for 3.5 PN accurate waveform as a function of l and m. This new code also reproduces the spherical harmonic modes till 3 PN order consistent with the earlier work. Finally I display the spherical harmonic modes for $l = 5$ when $l + m$ is odd and for $l = 6$ to $l = 9$ which are obtained in this work and are new.

Extreme mass ratio inspirals (EMRIs) are one of the important sources for evolved laser interferometric space antenna (eLISA). Recent works [4] have calculated the 22 PN waveforms for extreme mass ratio binaries (EMRIs) using the black hole perturbation theory. Vijay et al [8] have calculated the Fourier transform of gravitational waveform for EMRIs to 22 PN order using the leading stationary phase approximation (SPA). The SPA can be improved by computing the next order correction terms. In the second part of my project, I find the first order correction terms to the Fourier transform of the gravitational waveform for EMRIs.

Chapter 1 gives a broad overview of the field, I will discuss the importance of the present project. Chapter 2 deals with the generation of gravitational waves under different approximations, summarizing the standard material [10, 9, 2]. In chapter 3, I discuss the dynamics of binary system under gravitational radiation reaction. Chapter 4 deals with the present work and lists the new results. Chapter 5 discusses the the Fourier transform of gravitational wave signal using SPA. I compute the the correction terms to the phase of the Fourier transform for GWs from EMRIs and display the results.

Contents

1 Introduction 4

2 Theory of Gravitational Waves 7

2.1 Linearized Theory . . . 7

2.1.1 Transverse and Traceless (TT) Guage and Lambda Tensor (Λ) . . 8

2.2 Generation of Gravitational Waves in Linearized Theory . . . 9

2.2.1 Low Velocity Expansion . . . 10

2.2.2 Quadrupolar Radiation . . . 11

2.3 Generation of GWs from Post Newtonian Sources . . . 12

2.3.1 The relaxed Einstein equations . . . 12

2.3.2 Post-Newtonian Expansion in near zone . . . 13

2.3.3 Post-Minkowskian expansion outside the source . . . 14

2.3.4 Matching of the solutions . . . 15

3 Inspiralling compact binaries 16

3.1 Adiabatic inspiral quasi circular orbits . . . 17

3.2 Source mass multipole and source current multipole moments for ICBs . 19

4 Gravitational Waveform and Computing spherical harmonic modes for 3.5 PN accurate gravitational waveform for non-spinning ICBs 21

4.1 Radiative multipole moments . . . 23

4.2 Accuracy of radiative multipole moments required for 3.5 PN waveform. . 23

4.2.1 Instantaneous terms . . . 24

4.2.2 Tail terms . . . 25

4.2.3 Memory terms . . . 25

4.3 Computing 3.5 PN spherical harmonic modes for non-spinning ICBs . . . 25

4.3.1 Availability of source moments . . . 26

4.3.2 Mode separation for non-spinning binaries . . . 26

4.3.3 Available radiative moments for ICBs. . . . 27

4.3.4 Procedure . . . 27

4.3.5 Results and Discussion . . . 28

5 Fourier Transform of GW signal and Stationary phase approximation 34
5.1 Time domain phasing formulas 34
5.2 Fourier transform of GW signal and usual SPA 35
5.3 Extreme mass ratio inspirals (EMRIs) 37
5.4 Adding corrections to the uSPA phase for EMRIs 37
5.4.1 Procedure 38
5.5 Results and Discussion 39

Notations 1

References 2

Chapter 1

Introduction

Existence of gravitational waves (GWs) is one of the most important predictions of the General Theory of Relativity. In General relativity, gravity is described as the curvature of spacetime produced due to the presence of mass and energy in spacetime. When massive objects accelerate they produce the distortions or ripples in spacetime. We call these ripples as gravitational waves and they travel with speed of light. According to the Newtonian theory the orbital period of the bound binary system should be a constant. However General relativity predicts that, two masses moving around each other emit gravitational waves. Since the GWs carry away energy and angular momentum of the system, orbital energy of the system reduces and binaries spiral towards each other resulting in decrease of the orbital period. This predicted decrease in orbital period was first observed in the binary pulsar PSR $1913 + 16$ by Hulse and Taylor[11] which won them a Nobel prize for physics in 1993.

Gravitational waves carry some information about astrophysical objects that can not be obtained by any other means. For example,

- Electromagnetic waves from astrophysical objects are emitted from atoms or electrons and hence they are microscopic origin. GWs are emitted due to coherent, bulk movement of the sources and are macroscopic. Therefore GWs give a completely different kind information about the universe complementing the electromagnetic observations.

- Unlike electromagnetic waves, GWs couple with matter very weakly, so they travel through spacetime almost un-scattered or unabsorbed. Thus they can bring information about regions like interior of supernova explosion or Big Bang which have never been probed before.

GWs produced due to most of the astrophysical events in the universe are extremely small and the fact that the GWs couple with matter very weakly make it very challenging to detect them. Currently many laser interferometric detectors such as advanced LIGO,

advanced VIRGO are operational to hunt down the first gravitational wave signal.
One may wonder the need for high post-Newtonian waveforms. Let me give two main reasons for why we should calculate the gravitational waveforms with high precision.

- One of the most promising sources of gravitational radiation for laser interferometric detectors are coalescing compact binary (CCB) systems of neutron stars or black holes. In GW detectors, signal is buried in the noise of detectors which makes detection even more challenging. In order to unearth the the weak GW signal from the strong noise, we make use of the technique called *matched filtering technique.* This data analysis technique relies on cross correlation between noisy detector output and theoretical template banks. In order for matched filtering to be successful we need templates as accurate as possible.

- The evolution of the binary system can be divided into 3 major phases viz. *inspiral phase, merger phase* and *ringdown phase.* In the inspiral phase the component objects of binary system are well away from each other and over the time they spiral towards each other emitting GWs. As the system lose energy in the form of GWs the companions come close and plunge towards each other resulting in merger. This phase is called the merger phase. In the ringdown phase the resulting compact object emits energy in excited quasi-normal modes.

  Well before the merger phase, velocities involved are much less than the speed of light(c) and gravitational fields are weak. Hence we can apply post-Newtonian (PN) approximation to general relativity to describe the GWs from binaries in the inspiral stage. However merger phase and ringdown phases being highly relativistic it is difficult model them. So we switch to numerical relativity to calculate the numerical waveforms in these phases. In order to validate, interpret and to assess accuracy of the numerical waveforms, we need to compare them with the high accuracy post-Newtonian (PN) waveforms in an overlapping region. Currently such comparisons have proved to be very successful, showing the need to compute the accurate PN waveforms in terms of both phase and amplitude.

In the current project, I am focusing on inspiralling compact binaries (ICBs) with spin of individual binaries being zero as they are the most prominent sources for detection. I compute the gravitational waveform produced by such systems in quasi-circular orbits till three and half a post-Newtonian (3.5 PN) order[1]. The full waveform (FWF) at 3.5 PN order we mean, the waveform with amplitude till 3.5 PN order and all the higher order harmonics contributing to 3.5 PN waveform. This in contrast with restricted waveform (RWF) which focuses only on the dominant harmonic, twice the orbital frequency of the

[1] We usually refer terms of the order$\sim$ $(v/c)^{2n}$ as n PN terms. Here v is typical velocity of binary system and c is the velocity of light.

source. Calculation of FWF is more difficult compared to computation of RWF, since we need source multipoles with higher multipolarity and with higher accuracy. BFIS[16] provided the results to 3 PN and listed down all the harmonics in the 3PN accurate FWF.

To calculate FWF we need to use the multipolar post-Minkowskian (MPM) formalism, wherein the radiation field in the region where our detectors are located is parametrized using two sets of radiative moments. These moments are next expressed in terms of canonical moments which are in turn functions of source moments. Recent studies [17, 13, 18, 19] have calculated the required source moments for binaries. In the present work I have done a literature survey and identified the availability of the source moments. With the help of these available moments, I compute the spherical harmonic modes for 3.5 PN FWF.

Chapter 2

Theory of Gravitational Waves

The essence of general relativity can be summarized by Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}. \tag{2.0.1}$$

General relativity is invariant under huge group of coordinate transformations of the form

$$x^\mu \to x'^\mu(x), \tag{2.0.2}$$

where, x'^μ is an invertible, differentiable function of x^μ and having differentiable inverse. This property is referred as *gauge symmetry* of general theory of relativity.

2.1 Linearized Theory

Movements of the mass and energy densities (sources) in the spacetime causes perturbations in the usual flat spacetime metric $\eta_{\mu\nu}$. We choose a reference frame, in which the metric is slightly perturbed from the flat space metric. *i.e*

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \text{with,} \quad |h_{\mu\nu}| << 1. \tag{2.1.1}$$

We study the Einstein's equations (2.0.1) till linear order in $h_{\mu\nu}$. For convenience we define the new perturbations related to old ones as

$$\overline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h. \tag{2.1.2}$$

Using the gauge symmetry of general relativity, in the *Lorentz gauge*

$$\partial^\nu \overline{h}_{\mu\nu} = 0, \tag{2.1.3}$$

it can be shown that, the linearized Einstein's equations become,

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{2.1.4}$$

where $\Box$ is defined as flat space d'Alembertian given by,

$$\Box \equiv \eta_{\mu\nu}\partial^\mu \partial^\nu = \partial_\mu \partial^\mu. \tag{2.1.5}$$

Since, the perturbations $h_{\mu\nu}$ follow the wave equation with the source $T_{\mu\nu}$ we call them as gravitational waves. Conceptually in linearized theory, we use Newtonian gravity to describe the dynamics of the sources of GWs. However, while describing the motion of test masses in the presence of GWs $h_{\mu\nu}$, generated by these sources, we use the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. And again we keep only linear order terms in $h_{\mu\nu}$ while calculating their equations of motion (EOM). At the distances, far away from the source where GW detectors are located, $T_{\mu\nu} = 0$, and the linearized Einstein equations in Lorentz gauge (2.1.3), take the form,

$$\Box \bar{h}_{\mu\nu} = \eta^{\alpha\beta}\partial_\alpha \partial_\beta \bar{h}_{\mu\nu} = 0. \tag{2.1.6}$$

This is the three dimensional wave equation. $\Box = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2$, suggests that GWs propagate with the speed c equal to that of light.

2.1.1 Transverse and Traceless (TT) Guage and Lambda Tensor (Λ)

Out side the source, we define TT gauge by the conditions,

$$h_{0\mu} = 0, \quad h^i_i = 0, \quad \partial^j h_{ij} = 0. \tag{2.1.7}$$

In this gauge we explicitly have the two physically relevant degrees of freedom for $h_{\mu\nu}$. The plane wave solution to linearized vacuum Einstein equation (2.1.6) in the TT gauge looks like,

$$h^{TT}_{\mu\nu}(t,z) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.1.8}$$

with,

$$h_+ = h_{+0} e^{i\omega t} \quad \text{and,} \quad h_\times = h_{\times 0} e^{i\omega t}. \tag{2.1.9}$$

The two independent components h_+ and $h_\times$ are called as "plus" and "cross" polarizations respectively. GWs carry away energy and angular momentum from the source. It can be shown that, energy contained in gravitational wave h_{ij} per unit time, per unit solid angle is given by,

$$\frac{dE}{dtd\Omega} = \frac{c^3r^2}{32\pi G} < \dot{h}_{ij}^{TT}\dot{h}_{ij}^{TT} >, \tag{2.1.10}$$

where $< \cdots >$ denotes the temporal average over time period of GW and r is distance from source.

Lambda tensor(Λ) is a tool to convert the given solution $h_{\mu\nu}(x)$ into a TT gauge solution. Outside the source, If a gravitational wave h_{ij} is traveling in the direction $\hat{\mathbf{n}}$ and is already in Lorentz gauge (2.1.3), then in TT gauge it looks like

$$h_{ij}^{TT} = \Lambda_{ij,kl}(\hat{\mathbf{n}})h_{kl}, \tag{2.1.11}$$

where the Lambda tensor(Λ) is defined as,

$$\Lambda_{ij,kl}(\hat{\mathbf{n}}) = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \quad \text{where} \quad P_{ij}(\hat{\mathbf{n}}) = \delta_{ij} - n_in_j. \tag{2.1.12}$$

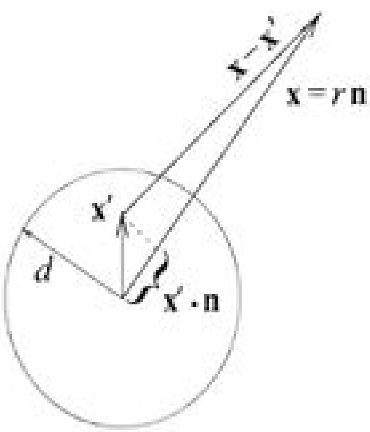
2.2 Generation of Gravitational Waves in Linearized Theory

We assume that gravitational field produced by the sources is sufficiently weak so that we can apply linearized theory to the systems. The system of mass m and having the spatial extent d, produces the weak gravitational field, if it satisfies the conditions, $R_s/d << 1$ and $v << c$, where $R_s = 2Gm/c^2$ is the Schwartzchild's radius and v is the typical internal velocity. For a time being we assume them as independent parameters.

For such systems the solution to linearized Einstein equation (2.1.4) is given as,

$$\bar{h}_{\mu\nu}(\mathbf{x},t) = \frac{4G}{c^4}\int d^3x' \frac{1}{|\mathbf{x}-\mathbf{x}'|}T_{\mu\nu}\big(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}'\big). \tag{2.2.1}$$

The various notations are explained in the following figure.



Let $|\mathbf{x}| = r$, then at $r >> d$ we can write,

$$|\mathbf{x} - \mathbf{x}'| = r - \mathbf{x}' \cdot \hat{\mathbf{n}} + O\Big(\frac{d^2}{r}\Big). \tag{2.2.2}$$

In order to calculate the value of h_{ij}^{TT} at detector location, we take limit $r \to \infty$, at fixed t and making use of Lambda tensor, we will get,

$$h_{ij}^{TT}(t, \mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}}) \int d^3x' T_{kl}\Big(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\Big) + O\big(\frac{1}{r^2}\big). \tag{2.2.3}$$

.

2.2.1 Low Velocity Expansion

Typical velocities inside the source (v) will be comparable to $\omega_s d$, where ω_s is typical frequency of internal motion of the source. The gravitational wave frequency ω will be of the order of ω_s, i.e $\omega \sim \omega_s \sim \frac{v}{d}$. Using reduced wavelength

$$\bar{\lambda} = c/\omega \tag{2.2.4}$$

we can write,

$$\bar{\lambda} \sim \frac{c}{v} d. \tag{2.2.5}$$

It shows that, for non relativistic sources, reduced wavelength generated is much bigger than the size of the source. It clearly suggests that, the macroscopic knowledge of source is good enough to calculate the gravitational radiation at far distances and we do not need to know the internal dynamics in detail.
Taylor expanding $T_{kl}\Big(t - \frac{r}{c} + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\Big)$ around *retarded time*, $t_r = t - \frac{r}{c}$,

$$T_{kl}\Big(t_r + \frac{\mathbf{x}' \cdot \hat{\mathbf{n}}}{c}, \mathbf{x}'\Big) \simeq T_{kl}(t_r, \mathbf{x}') + \frac{x'^i n^i}{c}\partial_0 T_{kl} + \frac{1}{2c^2} x'^i n^i x'^j n^j \partial_0^2 T_{kl} + \cdots, \tag{2.2.6}$$

derivatives are being evaluated at the point $(t_r, \mathbf{x}')$.
Let us define the momenta of the stress energy tensor of matter T^{ij} as,

$$S^{ij}(t) = \int d^3x T^{ij}(t, \mathbf{x}), \tag{2.2.7}$$

$$S^{ij,k}(t) = \int d^3x T^{ij}(t, \mathbf{x}) x^k, \tag{2.2.8}$$

$$S^{ij,kl}(t) = \int d^3x T^{ij}(t, \mathbf{x}) x^k x^l, \tag{2.2.9}$$

$$\cdots .$$

Inserting the expansion (2.2.6) into Eq.(2.2.3), and making use of above moments, we will get,

$$h_{ij}^{TT}(t,\mathbf{x}) = \frac{1}{r}\frac{4G}{c^4}\Big[S^{kl} + \frac{1}{c}\dot{S}^{kl,m} + \frac{1}{c^2}n_m n_p \ddot{S}^{kl,mp} + \cdots\Big]_{ret}, \tag{2.2.10}$$

where the subscript "ret" indicates, bracketed expression is to be evaluated at t_r. This is sometimes referred as multipole expansion. Let us look at S^{kl} and $\dot{S}^{kl,m}$ carefully. $S^{kl,m}$ has an additional factor of $x^m \sim O(d)$ compared to S^{kl}. Time derivative of $S^{kl,m}$ brings an extra factor ω_s to it. So the tensor $\dot{S}^{kl,m}$ gets a factor $O(\omega_s d) \sim O(v)$ with respect to S^{kl}. So in the expansion (2.2.10) 2^{nd} term is correction of the order of v/c, 3^{rd} term is correction of the order of $(v/c)^2$ and so on.

2.2.2 Quadrupolar Radiation

At lowest order in the multipole expansion (2.2.10) only S^{ij} contributes, which can be written as

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij}, \tag{2.2.11}$$

where,

$$M^{ij} = \frac{1}{c^2}\int d^3x T^{00}(t,\mathbf{x})x^i x^j, \tag{2.2.12}$$

known as *quadrupole term*. So the quadrupolar radiation expression becomes,

$$[h_{ij}^{TT}(t,\mathbf{x})]_{quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{M}^{kl}(t_r). \tag{2.2.13}$$

Also in terms of quadrupole moment Q_{ij}, defined by,

$$Q^{ij} \equiv M^{ij} - \frac{1}{3}M_{kk}, \tag{2.2.14}$$

$$= \int d^3x\rho(t,\mathbf{x})(x^i x^j - \frac{1}{3}r^2\delta^{ij}), \tag{2.2.15}$$

the quadrupolar radiation emitted by the source is given by,

$$[h_{ij}^{TT}(t,\mathbf{x})]_{quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}(\hat{\mathbf{n}})\ddot{Q}^{kl}(t_r), \tag{2.2.16}$$

$$= \frac{1}{r}\frac{2G}{c^4}\ddot{Q}_{ij}^{TT}(t_r). \tag{2.2.17}$$

This is called *Einstein's quadrupolar formula.* Using equation (2.1.10) it can be shown that, total radiated power (Luminosity) in quadrupolar approximation is given by,

$$P_{quad} = \frac{G}{5c^5} < \dddot{Q}_{ij}\dddot{Q}_{ij} > . \tag{2.2.18}$$

2.3 Generation of GWs from Post Newtonian Sources

In linearized theory, the spacetime in which sources move was taken to be the Minkowski spacetime. Physically, we were assuming that sources responsible for GW emission, will not alter the background space-time significantly. We then discussed production of GWs writing a multipole expansion (2.2.10), in the parameter v/c. We discovered that leading order contribution comes from mass quadrupole term. Note that this algorithm assumes internal velocity v of the source and background curvature as independent. It takes into account v/c corrections, while still assuming background spacetime as flat one. When the system is governed by non-gravitational forces (like system of accelerated charged particles), the above mentioned algorithm is sufficient.

But most of the astrophysical systems, which are potential candidates for GW detection are gravitationally bound systems. When internal velocities of these systems become moderately higher, then the systems contribute significantly to background curvature. So our assumption of treating background curvature and velocity of the system independently in linearized theory needs modification.

In this section, we go beyond LT and discuss post-Newtonian (PN) formalism, to describe dynamics of the weakly self gravitating system having moderate internal velocities. In post-Newtonian formalism, we calculate corrections to the equation of motions given by Newtonian gravity, in higher powers of small parameter v/c ($\equiv R_s/d$).

Through out this section we assume that system is of finite spatial extent (d) and matter composition is smooth inside. In previous section we found that reduced wavelength $\bar{\lambda}$ (2.2.4) of GW is much bigger than the extent of source(d). Let us classify the region of space into two zones, near zone ($d < r << \bar{\lambda}$) and far zone ($r >> \bar{\lambda}$). In the near zone where retardation effects are negligible, we use PN formalism to describe the GWs. In the far zone, retardation effects are crucial and we will adapt to a method called post-Minkowskian expansion.

2.3.1 The relaxed Einstein equations

Let us define the field $\mathbf{h}^{\alpha\beta}$ which is a measure of the deviation from flat space metric by,

$$\mathbf{h}^{\alpha\beta} \equiv (-g)^{1/2} g^{\alpha\beta} - \eta^{\alpha\beta}. \tag{2.3.1}$$

Note that this $\mathbf{h}$ is different from the one we used in linearized theory and also relation between $\mathbf{h}^{\mu\nu}$ and $g_{\mu\nu}$ is non linear. We are not making any assumptions about $\mathbf{h}$ here. But it reduces to negative of $\bar{h}^{\mu\nu}$ when $\mathbf{h}^{\mu\nu}$ is small. We choose harmonic gauge (2.1.3) or de Donder gauge, which in terms of $\mathbf{h}^{\mu\nu}$ becomes,

$$\partial_\beta \mathbf{h}^{\alpha\beta} = 0. \tag{2.3.2}$$

In this gauge, the Einstein equations take the form,

$$\Box \mathbf{h}^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{2.3.3}$$

where,

$$\tau^{\alpha\beta} \equiv (-g)T^{\alpha\beta} + \frac{c^4}{16\pi G}\mathcal{P}^{\alpha\beta}, \tag{2.3.4}$$

with $T^{\alpha\beta}$ is the matter energy-momentum (E-M) tensor. The tensor $\mathcal{P}$ is independent of source and is given by,

$$\mathcal{P}^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL} + (\partial_\nu \mathbf{h}^{\alpha\mu}\partial_\mu \mathbf{h}^{\beta\nu} - \mathbf{h}^{\mu\nu}\partial_\mu\partial_\nu \mathbf{h}^{\alpha\beta}), \tag{2.3.5}$$

where $t^{\alpha\beta}_{LL}$ is called Landau-Lifshitz pseudo energy momentum tensor [10], which is highly non linear functional of $\mathbf{h}^{\mu\nu}$. Einstein equations are completely equivalent to equation (2.3.3), together with harmonic gauge condition (2.3.2). Note that we can first solve Eq.(2.3.3) without requiring that, the harmonic condition is satisfied. Then we can apply gauge condition (2.3.2) to get full exact solution. Therefore these 10 independent equations (2.3.3) are called as *the relaxed Einstein's equations*.

Finding the exact solution to Eq. (2.3.3) is very difficult as right hand side is highly non-linear function of the $\mathbf{h}^{\mu\nu}$, for which we are solving for. So we must adopt some approximate methods depending on which region we are solving the equation. In near field, since retardation effects are small, we make use of PN formalism. In far regions, where retardation effects are crucial we make use of post-Minkowskian expansion technique. Then in intermediate region where both the methods hold, we match the solutions from both the methods.

2.3.2 Post-Newtonian Expansion in near zone

Near the sources, the background metric will be very complicated and it will have enormous deviation $\mathbf{h}^{\mu\nu}$ from flat space metric. Solving the Einstein equations (2.0.1) analytically will be very difficult. So we will use a approximation technique called post-Newtonian expansion to solve the Einstein equations (2.0.1). In PN expansion technique [10, 2] one finds the background metric where the source lie, as a expansion in a small expansion parameter,

$$\epsilon \sim (R_s/d)^{1/2} \sim v/c. \tag{2.3.6}$$

The procedure involved in the technique is iterative. We start from Newtonian equations for motion and we add the general relativistic corrections to those equations of motion. As we get the modified EOMs, we compute the the new energy momentum tensor for the source and we solve for new metric corrections. With corrected metric, again we calculate the modified EOMs, then new E-M tensor and then new metric corrections, and this procedure goes on. We can calculate the metric and EOMs with desired PN order.

2.3.3 Post-Minkowskian expansion outside the source

Now we consider the region outside the source *i.e* the region given by $d < r < \infty$. Outside the source, matter E-M tensor vanishes and we are left with relaxed vacuum Einstein equation,

$$\Box \mathbf{h}^{\alpha\beta} = \mathcal{P}^{\alpha\beta}. \tag{2.3.7}$$

At a distance r, deviation from Minkowski metric can be expressed as an expansion in the parameter R_s/r. Since $R_s = 2Gm/c^2$ we can as well write expansion in terms of parameter G. Therefore we write,

$$\mathbf{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}. \tag{2.3.8}$$

We plug this into Eq.(2.3.3) and we equate the terms of same order in G. The tensor $\mathcal{P}^{\alpha\beta}$ depends on $g^{\alpha\beta}$ which is highly non linear functional of $\mathbf{h}^{\alpha\beta}$, it starts from terms which are quadratic in h. Thus we can write,

$$\mathcal{P}^{\alpha\beta} = N^{\alpha\beta}[h,h] + M^{\alpha\beta}[h,h,h] + L^{\alpha\beta}[h,h,h,h] + O(h^5), \tag{2.3.9}$$

where N,M and L etc can be found by the calculation of $\mathcal{P}^{\alpha\beta}$. Since $\mathcal{P}$ does not contain terms of order G, so at lowest order we have,

$$\Box h_1^{\alpha\beta} = 0, \tag{2.3.10}$$

and for next order in G we get,

$$\Box h_2^{\alpha\beta} = N^{\alpha\beta}[h_1, h_1], \tag{2.3.11}$$

$$\Box h_3^{\alpha\beta} = M^{\alpha\beta}[h_1, h_1, h_1] + N^{\alpha\beta}[h_1, h_2] + N^{\alpha\beta}[h_2, h_1], \tag{2.3.12}$$

and so on.

We make use of a iterative algorithm called multipolar post-Minkowskian (MPM) expansion algorithm to generate most general solution $\mathbf{h}^{\alpha\beta}$ of the field equation (2.3.3). In this algorithm as a first step we solve for h_1 and then compute h_2 using Eq.(2.3.11). Using both h_1 and h_2 we solve for h_3 and so on. Finally we get the full solution to vacuum field

Einstein equations parametrized by six sets of symmetric and trace free (STF) moments I_L, J_L known as source moments and W_L, X_L, Y_L, Z_L known as gauge moments. Together these moments are called as *the algorithmic moments* and are arbitrary functions of retarded time t_r. The full solution looks like,

$$\mathbf{h}^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}[I_L, J_L, W_L, X_L, Y_L, Z_L]. \tag{2.3.13}$$

.

2.3.4 Matching of the solutions

We know that post-Newtonian expansion is valid both inside the source and in the external near zone. The post-Minkowskian expansion uses the parameter G which is related to R_s/r at a distance r outside the source. So in order to compare the PN solution with the post- Minkowskian solution it is necessary that both the expansions should have the same expansion parameter. So in the near external region($d < r < R$) of the source, we expand the post-Newtonian solution using R_s/r or d/r as an expansion parameter, which gives rise to "multipolar post-Newtonian expansion".

Now we are armed with the solution to $\mathbf{h}^{\alpha\beta}$ in both near zone and external zone of the source. In external zone solution is characterized by the algorithmic multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$. But as of now these moments are just some arbitrary functions of retarded time and are yet to be determined. However in the near zone the solution is in terms of post-Newtonian expansion, and E-M tensor of source directly enters the expansion. So we match both solutions in near external zone where both solutions are valid and fix the STF multipole moments $(I_L, J_L, W_L, X_L, Y_L, Z_L)$ in terms of matter and gravitational fields of source. The explicit expressions for these moments are given in [24, 25]

It is possible to parametrize the six sets of source and gauge moments in terms of only two sets of STF moments called *canonical mass-type* and *canonical current-type* multipole moments, M_L and S_L respectively. It can be shown that [16],

$$M_L = I_L + \mathcal{M}[I_L, J_L, W_L, X_L, Y_L, Z_L], \tag{2.3.14}$$

$$S_L = J_L + \mathcal{S}[I_L, J_L, W_L, X_L, Y_L, Z_L], \tag{2.3.15}$$

where, $\mathcal{M}$ and $\mathcal{S}$ are some non-linear functionals of source and gauge moments. They are at least quadratic and start only at 2.5 PN order.

Chapter 3

Inspiralling compact binaries

In this chapter we discuss the dynamics of inspiralling compact binaries (ICBs). We consider a binary system, consisting of two neutron stars (NS) or two black-holes (BH) or a neutron star and a black-hole. We assume that individual compact objects are non-spinning. Let their masses to be m_1 and m_2 and positions to be $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. To find out the equations of motion, we go to the center of mass frame (CM) where, in Newtonian approximation the problem becomes equivalent to that of a particle with mass $\mu = m_1 m_2/m$ called reduced mass moving in gravitational potential of mass $m = m_1 + m_2$ at a distance $r = |\mathbf{r}_1 - \mathbf{r}_2|$. The Newtonian equation of motion is given by,

$$\ddot{\mathbf{r}} = -\frac{Gm}{r^3}\mathbf{r}. \tag{3.0.1}$$

Let us assume that binaries are in circular orbit. Then orbital frequency ω_s is related to radius of the orbit R as,

$$\omega_s^2 = \frac{Gm}{R^3}. \tag{3.0.2}$$

We calculate the quadrupole moment of the system using (2.2.14) and compute the gravitational waveform using linearized theory. Therefore quadrupole waveform in Newtonian approximation is given by,

$$h_+ = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\frac{1+\cos\theta^2}{2}\cos\left(2\pi f_{gw} t_r + 2\phi\right), \tag{3.0.3}$$

$$h_\times = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{5/3}\left(\frac{\pi f_{gw}}{c}\right)^{2/3}\cos\theta\sin\left(2\pi f_{gw} t_r + 2\phi\right), \tag{3.0.4}$$

here,

$$M_c = \mu^{3/5} m^{2/5} = \nu^{3/5} m, \tag{3.0.5}$$

called the *chirp mass* where,

$$\nu = \frac{m_1 m_2}{m^2}, \tag{3.0.6}$$

is called the symmetric mass ratio. θ , ϕ and r indicate the location of the source in the sky. Also $\omega_{gw} = 2\omega_s$ *i.e* gravitational wave frequency is double that of source frequency. Using luminosity equation (2.2.18), it can be shown that, total power radiated in the form of GWs by the system is given by,

$$P_{binary} = \frac{32c^5}{5G}\left(\frac{GM_c\omega_{gw}}{2c^3}\right)^{10/3} \tag{3.0.7}$$

3.1 Adiabatic inspiral quasi circular orbits

Even though the binary system starts in an elliptical orbit, over the time with emission of gravitational radiation, system loses energy. Energy of the system is given by,

$$E_{sys} = E_{kin} + E_{pot}, \tag{3.1.1}$$
$$= -\frac{Gm_1m_2}{2R}. \tag{3.1.2}$$

It is easy to see that as the system loses energy, radius must decrease so that energy becomes more and more negative, and Eq.(3.0.2) implies that as radius decreases the orbital frequency (ω_s) increases. On the other hand, by Eq.(3.0.7) it is clear that as orbital frequency increases, system loses more and more energy and radius decreases further. So it is a runaway process, which after a sufficiently long time leads to coalescence of the binary system.
Using Eq.(3.0.2) it is easy to write,

$$\dot{R} = -\frac{2}{3}R\frac{\dot{\omega}_s}{\omega_s}, \tag{3.1.3}$$
$$= -\frac{2}{3}\omega_s R\frac{\dot{\omega}_s}{\omega_s^2}. \tag{3.1.4}$$

So it clear that if the condition $\dot{\omega}_s << \omega_s^2$ is fulfilled, then $\dot{R}$ is much lesser than tangential velocity $\omega_s R$ and we say that the system is in adiabatic quasi circular orbit. Let us assume that orbit of the system is in the X-Y plane. Let $\mathbf{n} = \mathbf{r}/r$ be the unit vector along the relative separation vector $\mathbf{r}$. So

$$\mathbf{n} = \cos\phi \;\; \mathbf{e}_X + \sin\phi \;\; \mathbf{e}_Y, \tag{3.1.5}$$

where, ϕ is the orbital phase of the system. Now we can describe the motion of the binary using the rotating orthogonal basis given by $(\mathbf{n}, \boldsymbol{\lambda}, \mathbf{e}_Z)$ where $\boldsymbol{\lambda} = \ \mathbf{e}_Z \times \mathbf{n}$. Using simple vector algebra we can derive the expressions for the relative position $\mathbf{x}$, velocity $\mathbf{v}$, and acceleration $\mathbf{a}$, as follows,

$$\mathbf{x} = r\ \mathbf{n}, \tag{3.1.6}$$

$$\mathbf{v} = \dot{r}\ \mathbf{n} + r\omega\boldsymbol{\lambda}, \tag{3.1.7}$$

$$\mathbf{a} = (\ddot{r} - r\omega^2)\ \mathbf{n} + (r\dot{\omega} + 2\dot{r}\omega)\boldsymbol{\lambda}, \tag{3.1.8}$$

where the orbital frequency ω is given by, $\omega = \dot{\phi}$. In adiabatic approximation, till 2PN order, we can describe the orbit of the binary system as a circular orbit, so $\ddot{r} = \dot{r} = \dot{\omega} = 0$ and $r\omega^2 = -\ \mathbf{n} \cdot\ \mathbf{a}$.

But at 2.5PN order, the back reaction of GWs comes into picture, so we must consider inspiral motion of the binary system. In Newtonian gravity the orbital energy is given by,

$$E = -\frac{1}{2}\nu\frac{Gm^2}{r} + O(\epsilon). \tag{3.1.9}$$

Using Eq.(3.0.2) in Eq.(3.0.7) it can be shown that in quadrupolar approximation the gravitational luminosity is given by,

$$\mathcal{L} = \frac{32}{5}\nu^2\frac{G^4m^5}{r^5c^5} + O(\epsilon). \tag{3.1.10}$$

Now we assume that energy carried away by GWs is balanced by the reduction in the orbital energy (i.e. $dE/dt = -\mathcal{L}$). Therefore

$$\dot{r} = (\frac{dE}{dt}/\frac{dE}{dr}) = -\frac{64}{5}\nu\frac{G^3m^3}{r^3c^5} + O(\epsilon^{7/2}) \tag{3.1.11}$$

and using the above equation we deduce the rate of change of the orbital frequency as,

$$\dot{\omega} = (\frac{d\omega}{dr}/\frac{dr}{dt}) = \frac{96}{5}\nu\left(\frac{G^7m^7}{c^{10}r^{11}}\right)^{1/2} + O(\epsilon^{7/2}). \tag{3.1.12}$$

Substituting above results into Eqs.(3.1.7) and (3.1.8) we derive the expressions for the relative velocity and acceleration till 3PN order as follows,

$$\mathbf{v} = r\omega\boldsymbol{\lambda} - \frac{64}{5}\nu\frac{G^3m^3}{r^3c^5}\ \mathbf{n} + O(\epsilon^{7/2}), \tag{3.1.13}$$

$$\mathbf{a} = -\omega^2\mathbf{x} - \frac{32}{5}\nu\frac{G^3m^3}{c^5r^4}\ \mathbf{v} + O(\epsilon^{7/2}). \tag{3.1.14}$$

The orbital angular frequency till 3 PN order [21, 20] is given as,

$$\begin{aligned}\omega^2 &= \frac{Gm}{r^3}\Bigg\{1+\gamma\left(-3+\nu\right)+\gamma^2\left(6+\frac{41}{4}\nu+\nu^2\right)\\ &+\gamma^3\left(-10+\left[-\frac{75707}{840}+\frac{41}{64}\pi^2+22\ln\left(\frac{r}{r_0'}\right)\right]\nu\right.\\ &\left.+\frac{19}{2}\nu^2+\nu^3\right)\Bigg\}+O(\epsilon^4),\end{aligned} \tag{3.1.15}$$

where we used the post Newtonian parameter

$$\gamma \equiv \frac{Gm}{rc^2}, \tag{3.1.16}$$

and the constant r_0' is explained in [21, 20]. In terms of another post-Newtonian parameter

$$x \equiv \left(\frac{Gm\omega}{c^3}\right)^{2/3} \tag{3.1.17}$$

γ is given by,

$$\begin{aligned}\gamma &= x\Bigg\{1+x\left(1-\frac{1}{3}\nu\right)+x^2\left(1-\frac{65}{12}\nu\right)\\ &+x^3\left(1+\left[-\frac{2203}{2520}-\frac{41}{192}\pi^2-\frac{22}{3}\ln\left(\frac{r}{r_0'}\right)\right]\nu\right.\\ &\left.+\frac{229}{36}\nu^2+\frac{1}{81}\nu^3\right)\Bigg\}+O(\epsilon^4).\end{aligned} \tag{3.1.18}$$

3.2 Source mass multipole and source current multipole moments for ICBs

For ICBs in circular orbits, general expressions till 1 PN order for source multipole moments (I_L, J_L) are available in literature. Source mass multipole moments (I_L) at 1 PN are given in appendix A of Kidder[15] as,

$$\begin{aligned}I_L = \nu\tilde{m}\Bigg\{&\left[f_{\ell-1}(\nu)-\gamma f_\ell(\nu)+\gamma\frac{5\ell^2+6\ell+9}{2(\ell+1)(2\ell+3)}f_{\ell+1}(\nu)\right]x_{\langle L\rangle}\\ &+\frac{\ell(\ell-1)(\ell+9)}{2(\ell+1)(2\ell+3)}f_{\ell+1}(\nu)\frac{r^2}{c^2}x_{\langle L-2}v_{i_{\ell-1}i_\ell\rangle}\Bigg\}+O(\epsilon^2),\end{aligned} \tag{3.2.1}$$

where,

$$\{\tilde{m}, f_k(\nu)\} = \begin{cases} \{m, s_k(\nu)\} & \text{for } \ell \text{ even,} \\ \{-\delta m, d_k(\nu)\} & \text{for } \ell \text{ odd,} \end{cases} \tag{3.2.2}$$

where $s_\ell = (m_1^\ell + m_2^\ell)/m^\ell$ and $d_\ell = (m_1^\ell - m_2^\ell)/m^\ell$. These can be written as functions of ν as

$$\begin{aligned} s_\ell(\nu) &= 1 + \sum_{k=1}^{\ell/2} \left[\binom{\ell-k-1}{k-1} + \binom{\ell-k}{k} \right] (-\nu)^k, \\ d_\ell(\nu) &= \sum_{k=0}^{\ell/2} \binom{\ell-k-1}{k} (-\nu)^k, \end{aligned} \tag{3.2.3}$$

x_i and v_i denote the components of relative position $\mathbf{x}$ (3.1.6) and relative velocity $\mathbf{v}$ (3.1.7) of the binary system. Similarly, source current multipole moments (J_L) till 1 PN is given in appendix A of the paper by Damour, Iyer and Nagar [23].

$$\begin{aligned} J_L = \; & \nu M \epsilon_{ab\langle i_\ell} x_{\underline{a}} v_{\underline{b}} \Bigg\{ x_{L-1\rangle} \Bigg[c_{\ell+1}(\nu) + \gamma \left(-\frac{\nu}{2\ell} c_{\ell+1}(\nu) + \frac{2\ell+3}{2\ell} b_{\ell+1}(\nu) \right. \\ & \left. + 2\nu \frac{\ell+1}{\ell} b_{\ell-1}(\nu) + \frac{1}{2} \left(\frac{\ell+1}{\ell} - \frac{(\ell-1)(\ell+4)}{(\ell+2)(2\ell+3)} \right) c_{\ell+3}(\nu) \right] \\ & + \frac{r^2}{c^2} x_{L-3} v_{i_{\ell-2}} v_{i_{\ell-1}\rangle} \frac{(\ell-1)(\ell-2)(\ell+4)}{2(\ell+2)(2\ell+3)} c_{\ell+3}(\nu) + O(\epsilon^2) \Bigg\}, \end{aligned} \tag{3.2.4}$$

where,

$$b_\ell(\nu) \equiv X_2^\ell + (-)^\ell X_1^\ell, \tag{3.2.5}$$

$$c_\ell(\nu) \equiv X_2^{\ell-1} + (-)^\ell X_1^{\ell-1}, \tag{3.2.6}$$

with

$$X_1 \equiv m_1/m, \tag{3.2.7}$$

$$X_2 \equiv m_2/m. \tag{3.2.8}$$

In literature, general expressions for I_L and J_L are available only till 1 PN. However some of the individual multipoles are known to high accuracy. For example I_{ij} is known up to 3.5 PN accuracy [26] and V_{ij} upto 2.5 PN accuracy [16], *etc.*

Chapter 4

Gravitational Waveform and Computing spherical harmonic modes for 3.5 PN accurate gravitational waveform for non-spinning ICBs

The multipolar post-Minkowskan solutions $\mathbf{h}^{\alpha\beta}$ is valid in the exterior region and so at future null infinity, i.e at $r \to \infty$ with $t - r/c$ fixed, where our detectors are located. But these solutions exhibit a logarithmic character $\sim \ln r^p / r^k$ at these distances in harmonic coordinates $x^\mu = (t, r)$. Appearance of these $\ln r$ terms in the solution is due to the coordinate effect. It is possible to choose coordinate system, at futute null infinity called *radiative* coordinates, in which one can make the log terms absent. We denote such coordinates by $X^\mu = (T, R)$ and let $R = |X|$ and retarded time $T_R \equiv T - R/c$.

We now introduce two sets of STF multipole moments called as *radiative multipole moments* denoted by U_L (mass type) and V_L (current type) and these are functions of retarded time T_R. In terms of these moments the waveform is given by,

$$h^{\rm TT}_{ij} = \frac{4G}{c^2 R}\Lambda^{\rm TT}_{ijkl}(\mathbf{N}) \sum_{\ell=2}^{+\infty} \frac{1}{c^\ell \ell!}\left\{ N_{L-2}\, U_{klL-2}(T_R) - \frac{2\ell}{c(\ell+1)}\, N_{aL-2}\, \varepsilon_{ab(k}\, V_{l)bL-2}(T_R)\right\} \;\; + \mathcal{O}\left(\tfrac{1}{R^2}\right) . \tag{4.0.1}$$

Here $\mathbf{N} = \mathbf{X}/R = (N_i)$ is the unit vector pointing from the source to the far away detector along which GWs travel from the source.

In terms of the two gravitational wave polarizations, h_+ and $h_\times$, the *spin weighted spherical harmonic modes* (simply called as *spherical harmonic modes* or sometime just *modes*) h^{lm}s are given by,

$$h_+ - ih_\times = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h^{lm} {}_{-2}Y^{lm}(\Theta, \Phi), \tag{4.0.2}$$

where, ${}_{-2}Y^{lm}(\Theta, \Phi)$ is spin weighted spherical harmonic of spin weight -2.[3] The h^{lm}s are related to radiative multipole moments U_L and V_L as follows,

$$h^{lm} = \frac{G}{\sqrt{2}Rc^{l+2}} \left(U^{lm}(T_R) - \frac{i}{c} V^{lm}(T_R) \right), \tag{4.0.3}$$

with,

$$U^{lm} = \frac{16\pi}{(2l+1)!!} \sqrt{\frac{(l+1)(l+2)}{2l(l-1)}} U_L Y_L^{lm*}, \tag{4.0.4}$$

$$V^{lm} = \frac{-32\pi l}{(2l+1)!!} \sqrt{\frac{(l+2)}{2l(l+1)(l-1)}} V_L Y_L^{lm*}, \tag{4.0.5}$$

where Y_L^{lm} are the STF spherical harmonics which are related to the scalar spherical harmonics by,

$$Y^{lm}(\Theta, \Phi) = Y_L^{lm} N_L. \tag{4.0.6}$$

Alternatively,

$$Y_L^{lm} \quad = \quad \frac{4\pi l!}{(2l+1)!!} \int d\Omega N_{<L>} Y^{lm}. \tag{4.0.7}$$

[3]We define the spin-weighted spherical harmonics of spin weight s using the Wigner d-functions as follows [15],

$${}_{-s}Y^{\ell m}(\Theta, \Phi) = (-1)^s \sqrt{\frac{2\ell+1}{4\pi}} d^{\ell}_{ms}(\Theta) e^{im\Phi},$$

where

$$\begin{aligned} d^{\ell}_{ms}(\Theta) &= \sqrt{(\ell+m)!(\ell-m)!(\ell+s)!(\ell-s)!} \\ \times \sum_{k=k_i}^{k_f} & \frac{(-1)^k (\sin\frac{\Theta}{2})^{2k+s-m} (\cos\frac{\Theta}{2})^{2\ell+m-s-2k}}{k!(\ell+m-k)!(\ell-s-k)!(s-m+k)!}, \end{aligned}$$

where $k_i = \max(0, m-s)$ and $k_f = \min(\ell+m, \ell-s)$, (R, Θ, Φ) are usual spherical coordinates.

4.1 Radiative multipole moments

The radiative moments(U_L,V_L) are non-linear functionals of canonical moments(M_L,S_L). The contributions to radiative multipole moments can be classified into instantaneous, tail, tail-tail and memory terms as follows,

$$U_L = U_L^{inst} + U_L^{tail} + U_L^{tail-tail} + U_L^{memory}, \tag{4.1.1}$$

$$V_L = V_L^{inst} + V_L^{tail} + V_L^{tail-tail} + V_L^{memory}. \tag{4.1.2}$$

Various terms contribute to radiative moments at various PN order. Instantaneous terms involve canonical moments, those are to be evaluated at retarded time. Tail terms arise at 1.5 PN, due to scattering of GWs from the static mass monopole of the source. Tail-tail terms comes from cubic interactions between two monopoles M and one static multipole. They contribute at 3 PN order. Memory terms arise due to quadratic interaction between two radiative multipoles at 1.5 PN order. Apart from instantaneous terms all other terms are collectively called as *hereditary terms*, for the reason, they all depend on past history of source.

In this chapter we discuss the calculation of the spin-weighted harmonics h^{lm}s, that contribute to the 3.5 PN accurate gravitational waveform of binaries in circular orbits. Previously Kidder [15] has calculated the spherical harmonic modes for 2.5 PN accurate gravitational waveform for ICBs in circular orbit. Then Blanchet et al [16] went on to compute the spherical modes that contribute to 3 PN accurate waveform. In here, I go half a PN order higher and calculate spherical harmonic mode for 3.5 accurate waveform with available source moments in literature.

4.2 Accuracy of radiative multipole moments required for 3.5 PN waveform.

The Equation (4.0.1) can be schematically written as,

$$h_{ij}^{TT} \sim K\left[U_{ij} + \frac{1}{c}\left(V_{ij} + N_k U_{ijk}\right) + \frac{1}{c^2}\left(N_k V_{ijk} + N_{kl} U_{ijkl}\right) + \cdots\right] \tag{4.2.1}$$

From this, it is eveident that, for a particular PN order waveform, only a finite number of multipoles will contribute. Higher the multipolarity of multipoles, at higher the order they will contribute to the waveform. The table below lists out the accuracy of different multipoles required to obtain the 3.5 PN accurate waveform.

Radiative Multipoles	Required PN accuracy
U_{ij}	3.5
U_{ijk},V_{ij}	3
U_{ijkl},V_{ijk}	2.5
U_5,V_4	2
U_6,V_5	1.5
U_7,V_6	1
U_8,V_7	0.5
U_9,V_8	0

. (4.2.2)

So the modes from $l = 2$ to $l = 9$ will contribute to the 3.5 PN accurate gravitational waveform. FBI [22] have listed down all instantaneous terms and hereditary terms (tail terms, tail-tail terms, memory terms) which will contribute to radiative moments of the above mentioned accuracy. Here I am giving some of them, which I require for my work.

4.2.1 Instantaneous terms

The mass-type moments are given by.

$$\begin{aligned}
U^{\text{inst}}_{ijklmn} &= M^{(6)}_{ijklmn} \\
&+ \frac{G}{c^3}\Big[-\frac{45}{28} M_{\langle ij} M^{(7)}_{klmn\rangle} - \frac{111}{28} M^{(1)}_{\langle ij} M^{(6)}_{klmn\rangle} - \frac{561}{28} M^{(2)}_{\langle ij} M^{(5)}_{klmn\rangle} \\
&\quad - \frac{1595}{28} M^{(3)}_{\langle ij} M^{(4)}_{klmn\rangle} - \frac{2505}{28} M^{(4)}_{\langle ij} M^{(3)}_{klmn\rangle} - \frac{2115}{28} M^{(5)}_{\langle ij} M^{(2)}_{klmn\rangle} \\
&\quad - \frac{909}{28} M^{(6)}_{\langle ij} M^{(1)}_{klmn\rangle} - \frac{159}{28} M^{(7)}_{\langle ij} M_{klmn\rangle} - \frac{15}{7} M_{\langle ijk} M^{(7)}_{lmn\rangle} - \frac{75}{7} M^{(1)}_{\langle ijk} M^{(6)}_{lmn\rangle} \\
&\quad - \frac{135}{7} M^{(2)}_{\langle ijk} M^{(5)}_{lmn\rangle} - \frac{505}{21} M^{(3)}_{\langle ijk} M^{(4)}_{lmn\rangle} \Big], \qquad (4.2.3)
\end{aligned}$$

$$U_L = M^{(l)}_L \quad \text{for} \quad l \geq 7. \qquad (4.2.4)$$

Current type moments are given by,

$$\begin{aligned}
V^{\text{inst}}_{ijklm} &= S^{(5)}_{ijklm} \\
&+ \frac{G}{c^3}\Big[-\frac{3}{2} M_{\langle ij} S^{(6)}_{klm\rangle} - \frac{33}{10} M^{(1)}_{\langle ij} S^{(5)}_{klm\rangle} - 12 M^{(2)}_{\langle ij} S^{(4)}_{klm\rangle} - 27 M^{(3)}_{\langle ij} S^{(3)}_{klm\rangle} \\
&\quad - \frac{69}{2} M^{(4)}_{\langle ij} S^{(2)}_{klm\rangle} - \frac{39}{2} M^{(5)}_{\langle ij} S^{(1)}_{klm\rangle} - \frac{21}{5} M^{(6)}_{\langle ij} S_{klm\rangle} - \frac{4}{3} S_{\langle ij} M^{(6)}_{klm\rangle} \\
&\quad - \frac{76}{15} S^{(1)}_{\langle ij} M^{(5)}_{klm\rangle} - \frac{16}{3} S^{(2)}_{\langle ij} M^{(4)}_{klm\rangle} - 8 S^{(3)}_{\langle ij} M^{(3)}_{klm\rangle} - \frac{28}{3} S^{(4)}_{\langle ij} M^{(2)}_{klm\rangle}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{20}{3}S^{(5)}_{\langle ij}M^{(1)}_{klm\rangle}-\frac{8}{5}S^{(6)}_{\langle ij}M_{klm\rangle}-\frac{3}{5}S_{\langle i}M^{(6)}_{jklm\rangle}\\
&+\varepsilon_{ab\langle i}\bigg(\frac{1}{14}M_{j\underline{a}}M^{(7)}_{klm\rangle b}+\frac{1}{2}M^{(1)}_{j\underline{a}}M^{(6)}_{klm\rangle b}-\frac{3}{5}M^{(2)}_{j\underline{a}}M^{(5)}_{klm\rangle b}-\frac{4}{3}M^{(3)}_{j\underline{a}}M^{(4)}_{klm\rangle b}\\
&\quad-\frac{3}{2}M^{(4)}_{j\underline{a}}M^{(3)}_{klm\rangle b}-\frac{1}{2}M^{(5)}_{j\underline{a}}M^{(2)}_{klm\rangle b}+\frac{1}{35}M^{(7)}_{j\underline{a}}M_{klm\rangle b}+\frac{1}{7}M_{jk\underline{a}}M^{(7)}_{lm\rangle b}\\
&\quad+\frac{2}{3}M^{(1)}_{jk\underline{a}}M^{(6)}_{lm\rangle b}+\frac{4}{3}M^{(2)}_{jk\underline{a}}M^{(5)}_{lm\rangle b}+\frac{1}{3}M^{(3)}_{jk\underline{a}}M^{(4)}_{lm\rangle b}\bigg)\bigg],
\end{aligned}
\tag{4.2.5}
$$

$$
S_L = I_L^{(l)} \quad \text{for} \quad l \geq 5. \tag{4.2.6}
$$

4.2.2 Tail terms

$$
U_L^{\text{tail}}(T_R) = \frac{2GM}{c^3}\int_{-\infty}^{T_R} M_L^{(\ell+2)}(\tau)\left[\ln\left(\frac{T_R-\tau}{2b}\right)+\kappa_\ell\right]\tau\,, \tag{4.2.7}
$$

$$
V_L^{\text{tail}}(T_R) = \frac{2GM}{c^3}\int_{-\infty}^{T_R} S_L^{(\ell+2)}(\tau)\left[\ln\left(\frac{T_R-\tau}{2b}\right)+\pi_\ell\right]\tau\,. \tag{4.2.8}
$$

Note that constant b comes in a relation between T_R and t_r. κ_ℓ and π_ℓ are given for general ℓ, in *harmonic coordinates*, as [22]

$$
\kappa_\ell = \frac{2\ell^2+5\ell+4}{\ell(\ell+1)(\ell+2)} + H_{\ell-2}\,, \qquad \pi_\ell = \frac{\ell-1}{\ell(\ell+1)} + H_{\ell-1} \tag{4.2.9}
$$

where $H_k \equiv \sum_{j=1}^{k}\frac{1}{j}$.

4.2.3 Memory terms

$$
U^{\text{mem}}_{ijklmn}(T_R) = \frac{G}{c^3}\int_{-\infty}^{T_R}\left[\frac{5}{7}M^{(4)}_{\langle ijk}(\tau)M^{(4)}_{lmn\rangle}(\tau)-\frac{15}{14}M^{(4)}_{\langle ij}(\tau)M^{(4)}_{klmn\rangle}(\tau)\right]\tau. \tag{4.2.10}
$$

4.3 Computing 3.5 PN spherical harmonic modes for non-spinning ICBs

In this section I will discuss the availability source moments for ICBs, mode separation and the procedure that I have followed to calculate spherical harmonic modes of 3.5 PN accuracy.

4.3.1 Availability of source moments

Careful observations of expressions for instantaneous and hereditary terms (from (4.2.3) to (4.2.10)) tells that, radiative moments depend only on canonical moments (M_L, S_L), which in tern depend on source moments(I_L, J_L). So the procedure is clear, we first have to gather the source moments of desired accuracy then derive canonical moments. Then from the help of these canonical moments, we will construct radiative moments (U_L, V_L) which are essential to compute the spherical harmonic modes. In the table below I am listing the accuracy of source multipole moments of non-spinning ICBs, required to compute 3.5 PN accurate gravitational waveform and their availability.

I_L	Required PN accuracy	Availability	J_L	Required PN accuracy	Availability
I_{ij}	3.5	✓	J_{ij}	3	×
I_{ijk}	3	✓	J_{ijk}	2.5	×
I_{ijkl}	2.5	×	J_{ijkl}	2	×
I_5	2	×	J_5	1.5	✓
I_6	1.5	✓	J_6	5	✓
I_7	1	✓	J_7	0.5	✓
I_8	0.5	✓	J_8	0	✓
I_9	0	✓			

(4.3.1)

From above table (4.3.1) it is clear that not all the required source moments are available in literature in order to calculate 3.5 PN harmonic modes. Note that we have general formula for source moments till 1 PN (3.2.1),(3.2.4) and next order correction to these source moments comes at 2 PN, so effectively we have general expressions for source moments till 1.5 PN.

4.3.2 Mode separation for non-spinning binaries

Using parity invariance of Einsteins equations it can be proved [26] that, for non-spinning binaries, for particular l the mode h^{lm} is given only by mass radiative moments (U_L) when $l+m$ is even, and only by current radiative moments (V_L) when $l+m$ is odd. *i.e*

$$h^{lm} = \frac{G}{\sqrt{2}Rc^{l+2}} U^{lm}(T_R) \quad \text{when} \quad l+m \quad \text{is even} \tag{4.3.2}$$

$$h^{lm} = -\frac{iG}{\sqrt{2}Rc^{l+3}} V^{lm}(T_R) \quad \text{when} \quad l+m \quad \text{is odd} \tag{4.3.3}$$

Above equations imply that, to have all the modes for particular l, with need to have both the radiative moments U_L and V_L with desired accuracy. If one of them is not available then we can only have modes, either for $l+m$ even or for $l+m$ odd, for particular l.

4.3.3 Available radiative moments for ICBs.

By noting down the availability and non-availability of various source moments (which are same as canonical moments till 2 PN order) from table 2 (4.3.1), and by observing the instantaneous and the hereditary terms eqs (from (4.2.3) to(4.2.10)), we can decide which radiative moments can be computed with desired accuracy listed in (4.2.2). and which cannot. In the table below I am listing the availability of radiative moments for non-spinning ICBs in the current literature.

U_L	Desired PN accuracy	Available ?	V_L	Desired PN accuracy	Available ?
U_{ij}	3.5	✓	V_{ij}	3	×
U_{ijk}	3	✓	V_{ijk}	2.5	×
U_{ijkl}	2.5	×	V_{ijkl}	2	×
U_5	2	×	V_5	1.5	✓
U_6	1.5	✓	V_6	5	✓
U_7	1	✓	V_7	0.5	✓
U_8	0.5	✓	V_8	0	✓
U_9	0	✓			

(4.3.4)

Therefore in the above table in a particular row, if we have both the moments U_L and V_L are marked with ✓then only we can have all spherical harmonic modes for the corresponding l with full 3.5 PN accuracy (the reason is explained in subsection 4.3.2). For a particular l, if only U_L is marked with ✓then, we will only have modes with $l+m$ even with full 3.5 PN accuracy, on the hand if we have only V_L available, then we will only have modes with $l+m$ odd with full 3.5 PN accuracy. Therefore, for $l=2$ only (2,2) and (2,0), for $l=3$ only (3,3) and (3,1) are computable with 3.5 PN accuracy and they are already computed [22]. We cannot compute any modes for $l=4$ with full 3.5 PN accuracy as neither of the radiative moments U_{ijkl} and V_{ijkl} is available with required accuracy. For $l=5$ only (5,4), (5,2),(5,0) and for $l=6,7,8,9$ all the modes can be computed with 3.5 PN accuracy. In this work I am able to compute them I will display the results.

4.3.4 Procedure

We know that till 2 PN canonical moments (M_L, S_L) are same as source moments (I_L, J_L) (2.3.14). So, since we know 1.5 PN expressions for source moments (3.2.1),(3.2.4) for non-spinning ICBs, implies that we have expressions for 1.5 PN accurate canonical moments. From expressions (4.2.3) to (4.2.10) we calculate all the instantaneous terms and hereditary terms. Note that while taking the time derivatives of the source moments we make use of equations of motion (3.1.13), (3.1.13) and expression for ω (3.1.15). By taking the product between the radiative moments and STF spherical harmonics Y_L^{lm} (4.0.7), we construct U^{lm}, V^{lm}. Using these multipoles we calculate spherical harmonic modes with the help of equation (4.0.3)

I coded the general expressions for source moments(I_L, J_L) (3.2.1),(3.2.4) in *mathematica*. Using these expressions as inputs I am able to come with a *mathemaitca* code for spherical harmonics modes (h^{lm}s), calculable from available source moments (discussed in previous subsection 4.3.3) for 3.5 PN accurate gravitational waveform for ICBs. While performing computation, I have used integration techniques given in section IV and Appendix C of Kidder[15] to evaluate hereditary terms.

In the new phase variable ψ given by,

$$\psi \equiv \phi - 3x^{3/2}\big[1 - \frac{\nu x}{2}\big] \ln \frac{x}{x_0} \tag{4.3.5}$$

where, $x_0 = \left(\frac{Gm\omega_0}{c^3}\right)^{2/3}$ is an arbitrary constant linked to coordinate transformation from near zone to far zone (radiative) coordinates, the h^{lm} can be factorized as,

$$h^{lm} = \frac{2Gm\nu x}{Rc^2}\sqrt{\frac{16\pi}{5}}\hat{H}^{lm}e^{-im\psi}. \tag{4.3.6}$$

To have FWF with 3.5 PN accuracy, we need to have each of the $\hat{H}^{lm}$s from $l = 2$ to $l = 9$ with 3.5 PN accuracy.

4.3.5 Results and Discussion

Note that till 1 PN radiative moments U_L and V_L are just the l^{th} derivatives of the source moments I_L and J_L respectively. Since we have general expression for source moments for any l, we can compute all the radiative moments till 1PN irrespective of multipolarity (l) of the moment, without bothering about the hereditary terms as they will not come into picture at 1 PN. Therefore even though we need moments U_8, U_9 and V_7, V_8 with accuracy less than 1 PN order, we can use the full 1 PN expressions for these and come with the modes more accurate than 3.5 PN. Therefore spherical modes that I have computed for $l = 7, 8, 9$ are of higher accuracy than 3.5 PN.
Here I am displaying results for $\hat{H}^{lm}$ which are new results of this work and they all contribute to 3.5 PN full waveform. The modes for negative m are given by the relation,

$$h^{l,-m} = (-1)^l h^{lm*} \tag{4.3.7}$$

with * indicating the complex conjugate. Till 3 PN order, the results are in agreement with the modes given in BFIS [16].

$$\hat{H}^{54} = -\frac{32}{9\sqrt{165}}\Big[x^2\left(1-5\nu+5\nu^2\right)+x^3\left(-\frac{4451}{910}+\frac{3619\nu}{130}-\frac{521\nu^2}{13}+\frac{339\nu^3}{26}\right)$$
$$+x^{7/2}\left(4\pi-20\pi\nu+20\pi\nu^2 i\left(-\frac{52}{5}+8\log(2)+\nu\left(\frac{2882711}{53760}-40\log(2)\right)\right)\right)\Big]$$
$$+\mathcal{O}\left(x^4\right)\,, \tag{4.3.8}$$

$$\hat{H}^{52} = \frac{2}{27\sqrt{55}}\Big[x^2\left(1-5\nu+5\nu^2\right)+x^3\left(-\frac{3911}{910}+\frac{3079\nu}{130}-\frac{413\nu^2}{13}+\frac{231\nu^3}{26}\right)$$
$$+x^{7/2}\left(2\pi-10\pi\nu+10\pi\nu^2+i\left(-\frac{26}{5}+\frac{16237\nu}{336}-\frac{1861\nu^2}{20}\right)\right)\Big]+\mathcal{O}\left(x^4\right)\,, \tag{4.3.9}$$

$$\hat{H}^{66} = \frac{54}{5\sqrt{143}}\Big[x^2\left(1-5\nu+5\nu^2\right)+x^3\left(-\frac{113}{14}+\frac{91\nu}{2}-64\nu^2+\frac{39\nu^3}{2}\right)$$
$$+x^{7/2}\left(6\pi-30\pi\nu+30\pi\nu^2+i\left(-\frac{249}{14}+\nu\left(\frac{21678149}{217728}-60\log(3)\right)\right.\right.$$
$$\left.\left.+12\log 3+\nu^2\left(60\log(3)-\frac{2230871}{18144}\right)\right)\right]+\mathcal{O}\left(x^4\right)$$
$$\tag{4.3.10}$$

$$\hat{H}^{65} = \frac{3125 i\,\Delta}{504\sqrt{429}}\Big[x^{5/2}\left(1-4\nu+3\nu^2\right)$$
$$+x^{7/2}\left(-\frac{149}{24}+\frac{349\nu}{12}-\frac{409\nu^2}{12}+\frac{29\nu^3}{3}\right)\Big]+\mathcal{O}\left(x^4\right)\,, \tag{4.3.11}$$

$$\hat{H}^{64} = -\frac{128}{495}\sqrt{\frac{2}{39}}\Big[x^2\left(1-5\nu+5\nu^2\right)+x^3\left(-\frac{93}{14}+\frac{71\nu}{2}-44\nu^2+\frac{19\nu^3}{2}\right)$$
$$+x^{7/2}\left(4\pi-20\pi\nu+20\pi\nu^2+i\left(-\frac{83}{7}+\nu\left(\frac{23696105}{344064}-40\log(2)\right)\right.\right.$$
$$\left.\left.+8\log 2+\nu^2\left(40\log(2)-\frac{2566755}{28672}\right)\right)\right]+\mathcal{O}\left(x^4\right)$$
$$\tag{4.3.12}$$

$$\hat{H}^{63} = -\frac{81 i\,\Delta}{616\sqrt{65}}\Big[x^{5/2}\left(1-4\nu+3\nu^2\right)$$
$$+x^{7/2}\left(-\frac{133}{24}+\frac{301\nu}{12}-\frac{329\nu^2}{12}+7\nu^3\right)\Big]+\mathcal{O}\left(x^4\right)$$
$$\tag{4.3.13}$$

$$\hat{H}^{62} = \frac{2}{297\sqrt{65}}\Big[x^2\left(1-5\nu+5\nu^2\right)+x^3\left(-\frac{81}{14}+\frac{59\nu}{2}-32\nu^2+\frac{7\nu^3}{2}\right)$$

$$+ x^{7/2}\left(2\pi - 10\pi\nu + 10\pi\nu^2 + i\left(-\frac{83}{14} + \frac{1125139\nu}{13440} - \frac{242801\nu^2}{1120}\right)\right)\Bigg]$$
$$+ \mathcal{O}\left(x^4\right)$$

(4.3.14)

$$\hat{H}^{61} = \frac{i\,\Delta}{8316\sqrt{26}}\left[x^{5/2}\left(1 - 4\nu + 3\nu^2\right) + x^{7/2}\left(-\frac{125}{24} + \frac{277\nu}{12} - \frac{289\nu^2}{12} + \frac{17\nu^3}{3}\right)\right]$$
$$+ \mathcal{O}\left(x^4\right)$$

(4.3.15)

$$\hat{H}^{60} = \mathcal{O}\left(x^4\right)$$

(4.3.16)

$$\hat{H}^{77} = -\frac{16807i\,\Delta}{1440}\sqrt{\frac{7}{858}}\Bigg[x^{5/2}\left(1 - 4\nu + 3\nu^2\right)$$
$$+ x^{7/2}\left(-\frac{319}{34} + \frac{2225\nu}{51} - \frac{2558\nu^2}{51} + \frac{230\nu^3}{17}\right)\Bigg] + \mathcal{O}\left(x^4\right)$$

(4.3.17)

$$\hat{H}^{76} = \frac{81}{35}\sqrt{\frac{3}{143}}\Bigg[x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)$$
$$+ x^4\left(-\frac{1787}{238} + \frac{41177\nu}{714} - \frac{14189\nu^2}{102} + \frac{5873\nu^3}{51} - \frac{2735\nu^4}{102}\right)\Bigg] + \mathcal{O}\left(x^{9/2}\right)$$

(4.3.18)

$$\hat{H}^{75} = \frac{15625i\,\Delta}{26208\sqrt{66}}\Bigg[x^{5/2}\left(1 - 4\nu + 3\nu^2\right)$$
$$+ x^{7/2}\left(-\frac{271}{34} + \frac{1793\nu}{51} - \frac{1838\nu^2}{51} + \frac{134\nu^3}{17}\right)\Bigg] + \mathcal{O}\left(x^4\right)$$

(4.3.19)

$$\hat{H}^{74} = -\frac{128}{1365}\sqrt{\frac{2}{33}}\Bigg[x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)$$
$$+ x^4\left(-\frac{14543}{2142} + \frac{36557\nu}{714} - \frac{12209\nu^2}{102} + \frac{1591\nu^3}{17} - \frac{2075\nu^4}{102}\right)\Bigg] + \mathcal{O}\left(x^{9/2}\right)$$

(4.3.20)

$$\hat{H}^{73} = -\frac{243i\,\Delta}{160160}\sqrt{\frac{3}{2}}\Bigg[x^{5/2}\left(1 - 4\nu + 3\nu^2\right)$$
$$+ x^{7/2}\left(-\frac{239}{34} + \frac{1505\nu}{51} - \frac{1358\nu^2}{51} + \frac{70\nu^3}{17}\right)\Bigg] + \mathcal{O}\left(x^4\right)$$

(4.3.21)

$$\hat{H}^{72} = \frac{1}{3003\sqrt{3}}\Bigg[x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right)$$
$$+ x^4\left(-\frac{13619}{2142} + \frac{33785\nu}{714} - \frac{11021\nu^2}{102} + \frac{1371\nu^3}{17} - \frac{1679\nu^4}{102}\right)\Bigg] + \mathcal{O}\left(x^{9/2}\right)$$

$$(4.3.22)$$

$$\hat{H}^{71} = \frac{i\,\Delta}{864864\sqrt{2}}\Bigg[x^{5/2}\left(1-4\nu+3\nu^2\right) + x^{7/2}\left(-\frac{223}{34}+\frac{1361\nu}{51}-\frac{1118\nu^2}{51}+\frac{38\nu^3}{17}\right)\Bigg]+\mathcal{O}\left(x^4\right) \tag{4.3.23}$$

$$\hat{H}^{70} = \mathcal{O}\left(x^4\right) \tag{4.3.24}$$

$$\hat{H}^{88} = -\frac{16384}{63}\sqrt{\frac{2}{85085}}\Bigg[x^3\left(1-7\nu+14\nu^2-7\nu^3\right) + x^4\left(-\frac{3653}{342}+\frac{9325\nu}{114}-\frac{22351\nu^2}{114}+\frac{9107\nu^3}{57}-\frac{4081\nu^4}{114}\right)\Bigg]+\mathcal{O}\left(x^{9/2}\right) \tag{4.3.25}$$

$$\hat{H}^{87} = -\frac{117649}{5184}i\,\Delta\sqrt{\frac{7}{24310}}\Bigg[x^{7/2}\left(1-6\nu+10\nu^2-4\nu^3\right) + x^{9/2}\left(-\frac{3343}{380}+\frac{44653\nu}{760}-\frac{46263\nu^2}{380}+\frac{6589\nu^3}{76}-\frac{337\nu^4}{19}\right)\Bigg]+\mathcal{O}\left(x^5\right) \tag{4.3.26}$$

$$\hat{H}^{86} = \frac{243}{35}\sqrt{\frac{3}{17017}}\Bigg[x^3\left(1-7\nu+14\nu^2-7\nu^3\right) + x^4\left(-\frac{353}{38}+\frac{7897\nu}{114}-\frac{18067\nu^2}{114}+\frac{6727\nu^3}{57}-\frac{2653\nu^4}{114}\right)\Bigg]+\mathcal{O}\left(x^{9/2}\right) \tag{4.3.27}$$

$$\hat{H}^{85} = \frac{78125}{36288\sqrt{4862}}i\,\Delta\Bigg[x^{7/2}\left(1-6\nu+10\nu^2-4\nu^3\right) + x^{9/2}\left(-\frac{611}{76}+\frac{8009\nu}{152}-\frac{8043\nu^2}{76}+\frac{5437\nu^3}{76}-\frac{265\nu^4}{19}\right)\Bigg]+\mathcal{O}\left(x^5\right) \tag{4.3.28}$$

$$\hat{H}^{84} = -\frac{128}{4095}\sqrt{\frac{2}{187}}\Bigg[x^3\left(1-7\nu+14\nu^2-7\nu^3\right) + x^4\left(-\frac{2837}{342}+\frac{6877\nu}{114}-\frac{15007\nu^2}{114}+\frac{5027\nu^3}{57}-\frac{1633\nu^4}{114}\right)\Bigg]+\mathcal{O}\left(x^{9/2}\right) \tag{4.3.29}$$

$$\hat{H}^{83} = -\frac{81}{5824}i\,\Delta\sqrt{\frac{3}{1870}}\Bigg[x^{7/2}\left(1-6\nu+10\nu^2-4\nu^3\right) + x^{9/2}\left(-\frac{2863}{380}+\frac{36973\nu}{760}-\frac{36183\nu^2}{380}+\frac{4669\nu^3}{76}-\frac{217\nu^4}{19}\right)\Bigg]+\mathcal{O}\left(x^5\right) \tag{4.3.30}$$

$$\hat{H}^{82} = \frac{i}{9009\sqrt{85}}\Bigg[x^3\left(1 - 7\nu + 14\nu^2 - 7\nu^3\right) + x^4\left(-\frac{2633}{342} + \frac{6265\nu}{114} - \frac{13171\nu^2}{114} + \frac{4007\nu^3}{57} - \frac{1021\nu^4}{114}\right)\Bigg] + \mathcal{O}\left(x^{9/2}\right) \tag{4.3.31}$$

$$\hat{H}^{81} = \frac{i\,\Delta}{741312\sqrt{238}}\Bigg[x^{7/2}\left(1 - 6\nu + 10\nu^2 - 4\nu^3\right) + x^{9/2}\left(-\frac{2767}{380} + \frac{35437\nu}{760} - \frac{34167\nu^2}{380} + \frac{4285\nu^3}{76} - \frac{193\nu^4}{19}\right)\Bigg] + \mathcal{O}\left(x^5\right) \tag{4.3.32}$$

$$\hat{H}^{80} = \mathcal{O}\left(x^5\right) \tag{4.3.33}$$

$$\hat{H}^{99} = \frac{1594323}{7168\sqrt{20995}}i\,\Delta\Bigg[x^{7/2}\left(1 - 6\nu + 10\nu^2 - 4\nu^3\right) + x^{9/2}\left(-\frac{419}{35} + \frac{2792\nu}{35} - \frac{822\nu^2}{5} + \frac{808\nu^3}{7} - \frac{160\nu^4}{7}\right)\Bigg] + \mathcal{O}\left(x^5\right) \tag{4.3.34}$$

$$\hat{H}^{98} = -\frac{131072}{2835}\sqrt{\frac{2}{20995}}\Bigg[x^4\left(1 - 9\nu + 27\nu^2 - 30\nu^3 + 9\nu^4\right) + x^5\left(-\frac{13969}{1386} + \frac{12289\nu}{126} - \frac{6956\nu^2}{21} + \frac{941\nu^3}{2} - \frac{3685\nu^4}{14} + \frac{1903\nu^5}{42}\right)\Bigg] + \mathcal{O}\left(x^{11/2}\right) \tag{4.3.35}$$

$$\hat{H}^{97} = -\frac{5764801}{1410048\sqrt{1235}}i\,\Delta\Bigg[x^{7/2}\left(1 - 6\nu + 10\nu^2 - 4\nu^3\right) + x^{9/2}\left(-\frac{53}{5} + \frac{344\nu}{5} - \frac{678\nu^2}{5} + 88\nu^3 - 16\nu^4\right)\Bigg] + \mathcal{O}\left(x^5\right) \tag{4.3.36}$$

$$\hat{H}^{96} = \frac{486}{595}\sqrt{\frac{3}{1235}}\Bigg[x^4\left(1 - 9\nu + 27\nu^2 - 30\nu^3 + 9\nu^4\right) + x^5\left(-\frac{12877}{1386} + \frac{11197\nu}{126} - \frac{2076\nu^2}{7} + \frac{2459\nu^3}{6} - \frac{9235\nu^4}{42} + \frac{513\nu^5}{14}\right)\Bigg] + \mathcal{O}\left(x^{11/2}\right) \tag{4.3.37}$$

$$\hat{H}^{95} = \frac{390625}{4935168\sqrt{247}}i\,\Delta\Bigg[x^{7/2}\left(1 - 6\nu + 10\nu^2 - 4\nu^3\right) + x^{9/2}\left(-\frac{67}{7} + \frac{424\nu}{7} - 114\nu^2 + \frac{472\nu^3}{7} - \frac{76\nu^4}{7}\right)\Bigg] + \mathcal{O}\left(x^5\right) \tag{4.3.38}$$

$$\hat{H}^{94} = -\frac{512}{6885}\sqrt{\frac{2}{8645}}\Bigg[x^4\left(1 - 9\nu + 27\nu^2 - 30\nu^3 + 9\nu^4\right)$$

$$+ x^5 \left(-\frac{12097}{1386} + \frac{10417\nu}{126} - \frac{5708\nu^2}{21} + \frac{733\nu^3}{2} - \frac{2645\nu^4}{14} + \frac{1279\nu^5}{42}\right)\Bigg] + \mathcal{O}\left(x^{11/2}\right) \tag{4.3.39}$$

$$\hat{H}^{93} = -\frac{81}{113152} i\,\Delta\sqrt{\frac{3}{665}}\Bigg[x^{7/2}\left(1 - 6\nu + 10\nu^2 - 4\nu^3\right) + x^{9/2}\left(-\frac{311}{35} + \frac{1928\nu}{35} - \frac{498\nu^2}{5} + \frac{376\nu^3}{7} - \frac{52\nu^4}{7}\right)\Bigg] + \mathcal{O}\left(x^5\right) \tag{4.3.40}$$

$$\hat{H}^{92} = \frac{2}{89505\sqrt{95}}\Bigg[x^4\left(1 - 9\nu + 27\nu^2 - 30\nu^3 + 9\nu^4\right) + x^5\left(-\frac{11629}{1386} + \frac{9949\nu}{126} - \frac{5396\nu^2}{21} + \frac{681\nu^3}{2} - \frac{2385\nu^4}{14} + \frac{1123\nu^5}{42}\right)\Bigg] + \mathcal{O}\left(x^{11/2}\right) \tag{4.3.41}$$

$$\hat{H}^{91} = \frac{i\,\Delta}{9165312\sqrt{2090}}\Bigg[x^{7/2}\left(1 - 6\nu + 10\nu^2 - 4\nu^3\right) + x^{9/2}\left(-\frac{299}{35} + \frac{1832\nu}{35} - \frac{462\nu^2}{5} + \frac{328\nu^3}{7} - \frac{40\nu^4}{7}\right)\Bigg] + \mathcal{O}\left(x^5\right) \tag{4.3.42}$$

$$\hat{H}^{90} = \mathcal{O}\left(x^5\right) \tag{4.3.43}$$

where $\Delta = \frac{m_1 - m_2}{m}$ mass difference ratio.

Chapter 5

Fourier Transform of GW signal and Stationary phase approximation

In the previous chapter, we had only concentrated on improving amplitude part of the gravitational waves till 3.5 PN. In this chapter let us look at phase part of the GW. In matched filtering technique the template must remain in phase with the signal as long as possible for effective cross correlation. This forces us to model the phase of the GWs as accurately as possible for data analysis.

5.1 Time domain phasing formulas

By applying the PN approximation, it is possible to calculate the binding energy E and gravitational wave flux $\mathcal{F}$ as a series expansion in the parameter v for ICBs at early inspiral stages. Here v is characteristic velocity related to gravitational wave frequency F and total mass m as

$$v = (\pi m F)^{1/3}. \tag{5.1.1}$$

Currently E is available till 3 PN and $\mathcal{F}$ is available till 3.5 PN beyond Newtonian order for the binaries consisting of masses comparable to each other. Total relativistic energy of the system E_{tot} is related to binding energy E as $E_{tot} = m(1+E)$. In adiabatic approximation (section 3.1) Energy balance equation $dE_{tot}/dt = -\mathcal{F}$ gives us the following coupled differential equations for orbital phase ϕ [1, 3].

$$\frac{d\phi}{dt} - \frac{v^3}{m} = 0, \tag{5.1.2a}$$

$$\frac{dv}{dt} + \frac{\mathcal{F}(v)}{mE'(v)} = 0. \tag{5.1.2b}$$

The derivative in $E'(v)$ is taken with respect v.

Depending on how we treat the rational function $\mathcal{F}(v)/E'(v)$ while solving the phasing formulas 5.1.2 we will get different approximants such as TaylorT1, TaylorT2, TaylorF2 *etc* [1]. Particularly, TaylorT1 gives the expression for dv/dt and TaylorT2 gives the expression for phase ϕ as series expansion in v. TaylorF2 gives the phase of the GWs in frequency domain. Currently these approximants are known upto 3.5 PN for comparable mass binary systems[3], as we know the flux only till 3.5 PN and binding energy E only till 3 PN order for such systems.

5.2 Fourier transform of GW signal and usual SPA

Let us consider the GW signal given by,

$$h(t) = 2a(t)\cos\phi(t) = a(t)e^{-i\phi(t)} + a(t)e^{i\phi(t)} \tag{5.2.1}$$

$$\text{where,}\quad \frac{d\phi(t)}{dt} \equiv 2\pi F(t) > 0\,. \tag{5.2.2}$$

where $a(t)$, $\phi(t)$ and $F(t)$ indicate the amplitude, phase and instantaneous frequencies of signal respectively. Amplitude $(a(t))$ is related to the gravitational wave frequency $F(t)$ by,

$$a(t) = \mathcal{C}(\pi M_c F(t))^{2/3} \tag{5.2.3}$$

where $\mathcal{C}$ is constant which depends on orientation of the binary with respect to detector, distance and masses of the system. $F(t)$ is monotonically increasing function during the inspiral stages of the binary. We denote the Fourier transform of $h(t)$ as $\tilde{h}(f)$. Since $h(t)$ is real, we have $\tilde{h}(-f) = \tilde{h}(f)^*$. The Fourier transform of the signal (5.2.1) is given as,

$$\tilde{h}(f) = \tilde{h}_-(f) + \tilde{h}_+(f)\,, \tag{5.2.4}$$

$$\text{where,}\quad \tilde{h}_-(f) \equiv \int_{-\infty}^{\infty} dt\; a(t)e^{i(2\pi f t - \phi(t))}\,, \tag{5.2.5}$$

$$\tilde{h}_+(f) \equiv \int_{-\infty}^{\infty} dt\; a(t)e^{i(2\pi f t + \phi(t))}\,. \tag{5.2.6}$$

Integrands in both the integrals $\tilde{h}_-(f)$ and $\tilde{h}_+(f)$ are violently oscillating. So the dominant contribution to the integrals comes from the neighborhood of the points where their phase $(2\pi f t - \phi(t))$ has an extrema. We call this approximation as *stationary phase approximation (SPA)*. We usually call such points as *stationary points* or sometimes *saddle points*. We assume $f > 0$, so only $\tilde{h}_-(f)$ has such stationary point. Therefore, in the SPA approximation, Fourier transform of the signal (5.2.4) becomes,

$$\tilde{h}(f) \simeq \tilde{h}_-(f) \simeq \int_{-\infty}^{\infty} dt\; a(t)\, e^{i\psi_f(t)}\,, \tag{5.2.7}$$

$$\text{where}\,,\quad \psi_f(t) \equiv 2\pi f t - \phi(t)\,. \tag{5.2.8}$$

To find the stationary points we equate $d\psi_f(t)/dt$ to zero and evaluate for t. Let such point be denoted by t_f. Note that $d\psi_f(t)/dt = 0$ translates to $F(t_f) = f$, so basically, the stationary point t_f is the value of the time variable when the gravitational wave frequency becomes F equal to Fourier variable f. In the vicinity of stationary point t_f, we replace $a(t)$, $\psi_f(t)$ by their truncated Taylor expansions,

$$\psi_f(t) \simeq \psi_f(t_f) - \pi \dot{F}(t_f)(t - t_f)^2 \,, \tag{5.2.9}$$

$$a(t) \simeq a(t_f) \,. \tag{5.2.10}$$

and compute the equation (5.2.7), to get FT in stationary phase approximation. Note that we have kept only first term in the Taylor expansion of the amplitude (5.2.10). Because anyway the second term $\dot{a}(t_f)(t - t_f)$ does not contribute to the integral (5.2.7) as it makes the integrand odd and vanishes after integration.
These inputs turn the Eq.(5.2.7) into a Gaussian integral,

$$\tilde{h}(f) \simeq \int_{-\infty}^{\infty} dt \; a(t_f) \; e^{i\psi_f(t_f) - i\pi \dot{F}(t_f)(t-t_f)^2} \,. \tag{5.2.11}$$

Which upon solving gives the Fourier transform under stationary phase approximation as,

$$\tilde{h}^{\rm uspa}(f) = \frac{a(t_f)}{\sqrt{\dot{F}(t_f)}} e^{i\left[\psi_f(t_f) - \pi/4\right]} \,, \tag{5.2.12}$$

We call this expression as usual SPA, abbreviated as uSPA. In the adiabatic approximation using phasing formulas (5.1.2) it can be shown that,

$$t_f = t_{\rm c} + M \int_{v_f}^{v_{\rm c}} \frac{E'(v)}{\mathcal{F}(v)} \, dv, \tag{5.2.13}$$

$$\psi_f(t_f) = 2\pi f t_{\rm c} - \phi_{\rm c} + 2 \int_{v_f}^{v_{\rm ref}} (v_f^3 - v^3) \frac{E'(v)}{\mathcal{F}(v)} \, dv, \tag{5.2.14}$$

where

$$v_f = (\pi M f)^{1/3} \tag{5.2.3}$$

and ϕ_c, t_c are arbitrary constants. Currently the SPA phase (TaylorF2) $\Psi_{spa} = \psi_f(t_f) - \pi/4$ is known upto 3.5 PN for comparable mass binaries. For Newtonian signal the SPA phase is given by,

$$\Psi^N_{spa}(f) = 2\pi f t_{\rm c} - \phi_{\rm c} - \frac{\pi}{4} + \frac{3}{128 \nu v^5} \tag{5.2.4}$$

5.3 Extreme mass ratio inspirals (EMRIs)

Till now we had not assumed anything about the mass ratios of constituent masses of the ICBs. The stellar mass compact object inspiralling towards supermassive black hole (SMBH) is one of the potential sources of GWs for the evolved Laser Interferometric Space Antenna (eLISA). Such systems are known as extreme mass ratio inspirals (EMRIs). The GWs emitted from EMRIs can reveal the information about spin, mass and the environments near the central black-hole. The mass ratio of EMRIs is about 10^{-4} to 10^{-7}. For such systems GW emission can be calculated using *black-hole perturbation theory* instead of PN theory. Using black-hole perturbation theory for EMRIs one can calculate the binding energy E and gravitational wave flux $\mathcal{F}$ with more PN accuracy than PN theory for comparable mass ICBs. Gravitational waveforms for a test mass circling around Schwarzchild's black-hole, have been calculated till 22 PN using black-hole perturbation theory [4]. Recently Vijay et al [8] have come up with different template families (TaylorT1, TaylorT2,TaylorF2,etc) till 22 PN using the 22 PN expressions for energy and flux computed in ref[5]. 22 PN expressions for dv/dt, gravitational wave phase ϕ and uSPA phase Ψ_{spa} for EMRIs are available online [8].

5.4 Adding corrections to the uSPA phase for EMRIs

Note that, while deriving the uSPA formula (5.2.12) we have used the truncated Taylor expressions for the phase $\phi(t)$ and amplitude $a(t)$. DIS[3] have retained the next two terms in both the expansions ((5.2.10),(5.2.9)) and calculated the integrals (5.2.7). They have proved that it is equivalent to multiplying the uSPA by a phase factor $e^{i\delta}$. So δ is basically the correction to uSPA phase Ψ_{spa} and is given by,

$$\delta = \frac{1}{2\pi \dot{F}(t_f)} \left[-\frac{1}{2}\frac{\ddot{a}}{a} + \frac{1}{2}\frac{\dot{a}}{a}\frac{\ddot{F}}{\dot{F}} + \frac{1}{8}\frac{\dddot{F}}{\dot{F}} - \frac{5}{24}\left(\frac{\ddot{F}}{\dot{F}}\right)^2 \right]_{t=t_f} . \tag{5.4.1}$$

where $\dot{a}$ indicates the time derivative of $a(t)$ and similarly for $F(t)$. I refer to δ as *the first order correction term.*

As mentioned earlier, Vijay et al [8] have calculated the uSPA phase Ψ_{spa} for EMRIs till 22 PN order, without considering this intrinsic correction δ. In the 2^{nd} part of project I am computing the corrections to the uSPA phase Ψ_{spa} for EMRIs using the first order correction term δ (5.4.1).

5.4.1 Procedure

For a moment let us concentrate on correction term δ. It has to be evaluated at t_f. Note t_f (5.2.13) is a function of v_f (5.2.3) but in SPA approximation it is equal to v (5.1.1) as Fourier variable f is same as the gravitational wave frequency F. So to calculate the correction term δ we first have to calculate t_f (5.2.13) and then calculate a, F which will be the functions of v. But explicit calculation of t_f is not necessary as explained below.

The δ requires amplitude (5.2.3) and gravitational wave frequency F(5.2.2). Since the amplitude (5.2.3) is the function of F, it means that we ideally need only instantaneous gravitational wave frequency $F(t)$ (5.2.2) for the calculation of δ. Since we know the gravitational wave phase $\phi(t)$ till 22 PN [8] as a expansion in parameter v, we can compute $F(t)$ till 22 PN using (5.2.2). So we will have both $a(t)$ and $F(t)$ till 22 PN as an expansion in parameter v. The time derivatives are calculated using the chain rule and expression for dv/dt which is again expansion in parameter v, for example

$$\dot{F} = \frac{dF}{dt} = \frac{dF}{dv}\frac{dv}{dt} \tag{5.4.2}$$

So we will have the all the terms in δ as functions of v directly. So we do not need to calculate t_f separately. As I have said earlier from gravitational phase ϕ, we can calculate F using (5.2.2) and from F we can calculate amplitude a using (5.2.3). For EMRIs gravitational phase ϕ is available till 22 PN in [8] as an expansion in parameter v. To calculate correction to the uSPA phase Ψ_{spa} for Newtonian signal we need to put Newtonian amplitude and Newtonian frequency F (which are calculated using Newtonian phase ϕ.) into first order correction term δ(5.4.1). Droz [6] et al have calculated the first order correction (δ) to uSPA phase Ψ_{spa} for Newtonian signal. Using *the method of steepest descents* [7] they showed the first order correction δ to be $(92/45)\eta v^5$, and they also showed 2^{nd} order correction term is of the order v^{10}. So compared to uSPA phase Ψ_{spa} for Newtonian signal (5.2.4)the correction is less by a factor of order v^{-10}. Since for Newtonian signal 2^{nd} order correction term appears at the order v^{10}, so I am assuming that for all post-Newtonian signals (PN) the 2^{nd} order correction term will appear at the order v^{10}. So the first order correction δ will be accurate till the order v^9 for all the PN signals till 22 PN.

I have written a *mathematica* code to calculate the correction to the uSPA phase Ψ_{spa} for each signals of different PN order. The code takes expression for 22 PN phase ϕ from [8] as input and calculates amplitude a and gravitational wave frequency F till required PN order. It calculates $\dot{a}, \ddot{a}, \dot{F}, \ddot{F}, \dddot{F}$ using expressions for dv/dt [8] as explained earlier. The code calculates n^{th} PN amplitude a and n^{th} frequency F using n^{th} PN phase ϕ, to calculate the first order correction term δ for n^{th} PN signal. The end result for first order correction term δ is expanded in parameter v and is truncated at order v^9 as at order v^{10} *second order correction terms* appear.

5.5 Results and Discussion

The mathematica code that I have written, takes n (PN order) as argument and gives first order correction term δ as a function of n. I am displaying the results for PN orders starting from Newtonian signal.

$$\delta[0] = \frac{92\nu v^5}{45} + \mathcal{O}(v)^{10}, \tag{5.5.1}$$

$$\delta[0.5] = \frac{92\nu v^5}{45} + \mathcal{O}(v)^{10}, \tag{5.5.2}$$

$$\delta[1] = \frac{92\nu v^5}{45} + \frac{17089\nu v^7}{3780} + \frac{12697127\nu v^9}{381024} + \mathcal{O}(v)^{10}, \tag{5.5.3}$$

$$\delta[1.5] = \frac{92\nu v^5}{45} + \frac{17089\nu v^7}{3780} - \frac{368}{45}\pi\nu v^8 + \frac{12697127\nu v^9}{381024} + \mathcal{O}(v)^{10}, \tag{5.5.4}$$

$$\delta[2] = \frac{92\nu v^5}{45} + \frac{17089\nu v^7}{3780} - \frac{368}{45}\pi\nu v^8 + \frac{70349479\nu v^9}{11430720} + \mathcal{O}(v)^{10}, \tag{5.5.5}$$

$$\delta[2.5] = \frac{92\nu v^5}{45} + \frac{17089\nu v^7}{3780} - \frac{368}{45}\pi\nu v^8 + \frac{70349479\nu v^9}{11430720} + \mathcal{O}(v)^{10}. \tag{5.5.6}$$

The first order correction to the uSPA phase Ψ_{spa} for Newtonian signal is in agreement with the result derived in [6].

Notations

- Greek letters stand for the spacetime indices and Latin letters for space indices.
- Subscript L denotes the multi index consisting of l spatial indices each ranging from 1 to 3, therefore $I_L = I_{i_1,\cdots,i_l}$, $I_{aL-1} = I_{ai_1,\cdots,i_{l-1}}$ and $x_L = x_{i_1}\cdots x_{i_l}$.
- The angular brackets $\langle\cdots\rangle$ around the indices indicate the symmetric and trace free (STF) projection of the expression with respect to those indices *i.e* $x_{\langle i}v_{j\rangle} = \frac{1}{2}(x_i v_j + x_j v_i) - \frac{1}{3}\delta_{ij}\mathbf{x}\cdot\mathbf{v}$.
- If any index is underlined in the angular brackets $\langle\cdots\rangle$ that index should be excluded from STF projection. *e.g.* $T_{\langle i\underline{a}j\rangle} = \frac{1}{2}(T_{iaj} + T_{jai}) - \frac{1}{3}\delta_{ij}T_{kak}$.

References

[1] A. Buonanno, B. R. Iyer, E. Ochsner, Y. Pan, and B. S.Sathyaprakash, Phys. Rev. D 80, 084043 (2009).

[2] "Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity" by Steven Weinberg.

[3] T. Damour, B. R. Iyer, and B. S. Sathyaprakash, Phys. Rev. D 63, 044023 (2001).

[4] M. Sasaki and H. Tagoshi, Living Rev. Rel. 6, 6 (2003), arXiv:gr-qc/0306120 [gr-qc].

[5] R. Fujita,Prog. Theor. Phys. 128, 971 (2012), arXiv:1211.5535 [gr-qc].

[6] S. Droz, D.J. Knapp, E. Poisson and B.J. Owen, Phys. Rev. D 59, 124016 (1999).

[7] "Advanced mathematical methods for scientists and engineers" by Carl M. Bender, Steven A. Orszag

[8] Vijay Varma, Ryuichi Fujita, Ashok Choudhary, Bala R. Iyer,arXiv:1304.5675 [gr-qc]

[9] "A First Course in General Relativity" by Bernard Schutz.

[10] "Gravitational Waves: Volume 1: Theory and Experiments" by Michele Maggiore

[11] J. H. Taylor J. M. Weisberg, ApJ, 253, 908 (1982)

[12] L. Blanchet, B. R. Iyer, C. M. Will, and A. G. Wiseman, Class. Quantum Grav. 13, 575 (1996), gr-qc/9602024.

[13] K. G. Arun, L. Blanchet, B. R. Iyer, and M. S. S. Qusailah, Class. Quantum Grav. 21, 3771 (2004), erratum-ibid. 22, 3115 (2005), gr-qc/0404185.

[14] L. E. Kidder, L. Blanchet, and B. R. Iyer, Class. Quant. Grav. 24, 5307 (2007), arXiv:0706.0726.

[15] L. E. Kidder, Phys. Rev. D D77, 044016 (2008), arXiv:0710.0614.

[16] L. Blanchet, G. Faye, B. R. Iyer, and S. Sinha, Class. Quant. Grav. 25, 165003 (2008), arXiv:0802.1249 [gr-qc].

[17] L. Blanchet, B. R. Iyer, and B. Joguet, Phys. Rev. D 65, 064005 (2002), Erratum-ibid 71, 129903(E) (2005), gr-qc/0105098.

[18] L. Blanchet, T. Damour, and B. R. Iyer, Phys. Rev. D 51, 5360 (1995), gr-qc/9501029.

[19] L. Blanchet and B. R. Iyer, Phys. Rev. D 71, 024004 (2004), gr-qc/0409094.

[20] L. Blanchet and G. Faye, Phys. Rev. D 63, 062005 (2001).

[21] L. Blanchet and G. Faye, Phys. Lett. A 271, 58 (2000)

[22] Guillaume Faye, Luc Blanchet, Bala R. Iyer, arXiv:1409.3546 [gr-qc]

[23] T. Damour, B. Iyer, and A. Nagar, Phys. Rev. D 79, 064004 (2009), arXiv:0811.2069 [gr-qc]

[24] L. Blanchet, Class. Quant. Grav. 15, 1971 (1998), gr-qc/9801101.

[25] O. Poujade and L. Blanchet, Phys. Rev. D 65, 124020 (2002), gr-qc/0112057

[26] Guillaume Faye, Sylvain Marsat, Luc Blanchet and Bala R Iyer, arXiv:1204.1043 [gr-qc]