Stiefel-Whitney classes of representations of dihedral and symmetric groups

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Certificate

This is to certify that this dissertation entitled Stiefel-Whitney classes of representations of dihedral and symmetric groups towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sujeet Bhalerao at Indian Institute of Science Education and Research under the supervision of Dr. Steven Spallone, Associate Professor, Department of Mathematics, during the academic year 2019-2020.

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Declaration

I hereby declare that the matter embodied in the report entitled Stiefel-Whitney classes of representations of dihedral and symmetric groups are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Steven Spallone and the same has not been submitted elsewhere for any other degree.

Sujeet Bhalerao

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Abstract

We compute Stiefel-Whitney classes of irreducible representations of dihedral groups and symmetric groups S_4 and S_5. We give character formulas for all Stiefel-Whitney classes of representations of the cyclic group of order 2, the Klein four-group, and odd dihedral groups. For representations of even dihedral groups, we give a character formula for the first and second Stiefel-Whitney class. We also give a new proof of Theorem 6.4 in [GS20], which gives a character formula for the second Stiefel-Whitney class of a representation of S_n for $n \geq 4$.

Contents

Introduction

A finite dimensional real representation (π, V) of a finite group G is said to be orthogonal if there exists an inner product $\langle\ ,\ \rangle$ on V such that $\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle$ for all $g \in G$ and $v, w \in V$. Whenever we write "a representation" we mean "a real orthogonal representation" unless stated otherwise. The orthogonal group $\mathrm{O}(V)$ has a double cover $\mathrm{Pin}(V)$ known as the Pin group with covering map $\rho : \mathrm{Pin}(V) \to \mathrm{O}(V)$. A real orthogonal representation is said to be *spinorial* if it lifts to the Pin group, that is, if there exists a homomorphism $\hat{\pi} : G \to \mathrm{Pin}(V)$ such that $\rho \circ \hat{\pi} = \pi$. The problem of spinoriality of orthogonal representations has been studied previously. In [PR95], the authors mention the lifting of representations of finite groups and in particular of symmetric groups. In [JS19], a criterion for the lifting of an orthogonal representation of a connected reductive group over a field of characteristic zero is given in terms of the highest weights of the irreducible constituents of the representation. The notion of a Stiefel-Whitney class of a real orthogonal representation arises in connection with the problem of lifting of representations to the Pin group. In [GS20], the authors address the lifting problem for symmetric and alternating groups. They give a criterion for the spinoriality of representations in terms of the first and second Stiefel-Whitney classes of representations of finite groups. They also prove a character formula for the second Stiefel-Whitney class of a real orthogonal representation of the symmetric group S_n for $n \geq 4$.

In this thesis, we give a character formula for Stiefel-Whitney classes of real orthogonal representations of the dihedral group D_n of order $2n$ which has the presentation $D_n = \langle r, s \mid r^n = s^2 = e, srs = r^{-1}\rangle$, where e denotes the identity in the group. We also compute Stiefel-Whitney classes of irreducible representations of symmetric groups S_4 and S_5. The key results of the thesis are given below. In each of these theorems, for a representation π of a finite group G, if $s \in G$, we write g_s for the multiplicity of the -1-eigenspace of $\pi(s)$.

Theorem 1. Suppose π is a real representation of $C_2 \times C_2 = \langle a, b \mid a^2 = b^2 = e, ab = ba\rangle$. Then

$$w_2(\pi) = \left[\frac{g_a}{2}\right]\alpha^2 + \left[\frac{g_b}{2}\right]\beta^2 + \left(\left[\frac{g_{ab}}{2}\right] + \left[\frac{g_a}{2}\right] + \left[\frac{g_b}{2}\right]\right)\alpha\beta$$

where $\alpha = w_1(\phi_a)$, $\beta = w_1(\phi_b)$, with ϕ_a being the representation of $C_2 \times C_2$ which sends a to -1 and b to 1 and ϕ_b being the representation of $C_2 \times C_2$ which sends b to -1 and a

to 1.

Theorem 2. For a real representation π of an odd dihedral group, that is, $D_n = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$ with odd n, we have

$$w_m(\pi) = \binom{g_s}{m} w_1(\rho_s)^m$$

where ρ_s is the representation of D_n which sends r to 1 and s to -1.

Theorem 3. Let π be a real representation of an even dihedral group, by which we mean D_n where n is a power of 2. Then

$$\begin{aligned} w_1(\pi) &= g_s x + (g_{rs} + g_s) y \\ w_2(\pi) &= \left(\left[\frac{g_{rs}}{2} \right] + \left[\frac{g_s}{2} \right] \right) x^2 + \left[\frac{g_s}{2} \right] y^2 + \left[\frac{g_{r^{2^{k-1}}}}{2} \right] w \end{aligned}$$

where $x = w_1(\rho_r)$ $y = w_1(\rho_s)$, $w = w_2(\sigma_1)$ with the representations ρ_r, ρ_s and σ_1 given by

$$\begin{aligned} \rho_r(r) &= -1 & \rho_r(s) &= 1 \\ \rho_s(r) &= 1 & \rho_s(s) &= -1 \\ \sigma_1(r) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} & \sigma_1(s) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

with $\theta = 2\pi/n$.

We also give a different proof of the following theorem from [GS20].

Theorem 4. For π a real orthogonal representation of S_n for $n \geq 4$ we have

$$w_2(\pi) = \left[\frac{g_s}{2} \right] w_1(\mathrm{sgn})^2 + \frac{g_{rs}}{2} w_2(\pi_n)$$

where $s = (12)$ and $rs = (12)(34)$, and π_n denotes the standard n-dimensional representation of S_n.

Organization of the thesis. In Chapter 1, we review properties of the Pin group. We also recall the definition of Stiefel-Whitney classes of vector bundles. We state the properties satisfied by Stiefel-Whitney classes of representations along with the criterion for spinoriality of a real representation of a finite group G in terms of the first and second Stiefel-Whitney class of the representation.
In Chapter 2, we review definitions from group cohomology and give a short informal introduction to spectral sequences. We then give two examples of the Lyndon-Hochschild-Serre spectral sequence in action: we use it to compute the integral cohomology of odd

dihedral groups with $\mathbb{Z}$ coefficients and the mod 2-cohomology of odd dihedral groups. Finally we introduce the notion of a detection theorem, and prove that the cohomology of a finite group G is detected by its 2-Sylow subgroup.

In Chapter 3, we address the spinoriality of irreducible representations of dihedral groups. We then determine the Stiefel-Whitney classes of irreducible representations of both odd and even dihedral groups as elements of their respective cohomology rings. We then give a character formula for the first and second Stiefel-Whitney classes of real representations of the cyclic group of order 2, the Klein four-group, and dihedral groups. The chapter ends with a character formula for higher Stiefel-Whitney classes for C_2, $C_2 \times C_2$ and odd dihedral groups.

In Chapter 4, we describe the structure of 2-Sylow subgroups of symmetric groups and their representation theory. We compute Stiefel-Whitney classes of irreducible representations of S_4 and S_5. The chapter ends with a different proof of Theorem 6.4 in [GS20], which gives a character formula for the 2nd Stiefel-Whitney class of a real representation of S_n for $n \geq 4$.

In the final Chapter, we mention some problems related to those addressed in this thesis and state partial results in this direction.

Chapter 1

Vector bundles, Stiefel-Whitney classes and spinoriality of representations

1.1 Vector bundles and Stiefel-Whitney classes: a review

One can associate to a real vector bundle ξ with total space $E(\xi)$, base space $B(\xi)$ and projection map $E(\xi) \to B(\xi)$ certain cohomology classes $w_i(\xi)$ called *Stiefel-Whitney classes* which lie in the singular cohomology $H^i(B(\xi), \mathbb{Z}/2\mathbb{Z})$ of the base space. These are uniquely characterized by the following axioms (see [[MS74], Chapter 4]):

- **Axiom 1.** Corresponding to each vector bundle ξ there is a sequence of cohomology classes

  $$w_i(\xi) \in H^i(B(\xi), \mathbb{Z}/2\mathbb{Z}), \quad i = 0, 1, 2, ...$$

  called the Stiefel-Whitney classes of ξ. The zeroth Stiefel-Whitney class $w_0(\xi)$ is the element $1 \in H^0(B, \mathbb{Z}/2\mathbb{Z})$ and for a vector bundle ξ of rank n, we have $w_i(\xi) = 0$ for $i > n$.

- **Axiom 2. (The naturality axiom)** If we have a map $f : B(\xi) \to B(\eta)$ between base spaces of two real vector bundles ξ and η which is covered by a bundle map $g : E(\xi) \to E(\eta)$ then we have

  $$w_i(\xi) = f^*(w_i(\eta)).$$

- **Axiom 3.(The Whitney product theorem)** If ξ and η are vector bundles over the same base space, then

  $$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta).$$

- **Axiom 4.** For the line bundle γ_1^1 over $\mathbb{RP}^1$, the Stiefel-Whitney class $w_1(\gamma_1^1)$ is nonzero.

The total Stiefel-Whitney class of a bundle ξ is defined to be the element $w_0(\xi) + w_0(\xi) + \cdots$ in $\bigoplus_{i=1}^{\infty} H^i(B(\xi))$. In terms of the total Stiefel-Whitney class, the Whitney product theorem can be rephrased as $w(\xi \oplus \eta) = w(\xi) \cup w(\eta)$, where $\cup$ denotes the cup product in group cohomology (see [Chapter 5, [Bro12]] for a definition of cup product).

1.2 Stiefel-Whitney classes of a representation and spinoriality

1.2.1 The Pin group

We briefly review the definition and basic properties (without proof) of the Pin group following the exposition in [BD]. Given a finite dimensional real vector space V with a norm $|\cdot|$, we define the *Clifford algebra* $C(V)$ as the quotient of the tensor algebra $T(V)$ of V by the two sided ideal generated by the set $\{v \otimes v + |v|^2 \mid v \in V\}$. Then $C(V)$ is an $\mathbb{R}$-algebra with a unique anti-automorphism $t : C(V) \to C(V)$ which satisfies $t(x \cdot y) = t(y) \cdot t(x)$ and $t^2 = \mathrm{id}$. This anti-automorphism is uniquely determined by $t(x) = x$ for $x \in i(V)$ where i is the natural inclusion of V in $C(V)$. It turns out that the Clifford algebra also has a unique automorphism $\alpha : C(V) \to C(V)$ which satisfies $\alpha^2 = \mathrm{id}$ and $\alpha(x) = -x$ for $x \in i(V)$. One then observes that we have $t\alpha = \alpha t$, and that this composition is also an algebra anti-automorphism. We introduce new notation for the composition $t\alpha$ and define $\bar{x} = t\alpha(x)$ for $x \in C(V)$. This allows us to define a "norm" map

$$N : C(V) \to C(V) \qquad \text{by}$$

$$\mathrm{N}(x) = x \cdot \bar{x}.$$

Consider now the subgroup Γ_V of the group of units $C(V)^*$ of $C(V)$ given by

$$\Gamma_V = \{x \in C(V)^* \mid \alpha(x) \cdot v \cdot x^{-1} \in V \text{ for all } v \in V\}.$$

This group comes with a natural representation $\rho : \Gamma_V \to \mathrm{GL}(V)$ given by $\rho(x)(v) = \alpha(x) \cdot v \cdot x^{-1}$ for $x \in \Gamma_V$ and $v \in V$. We will restrict our attention to the case when $V = \mathbb{R}^n$. We now state a series of lemmas from [Chapter 1, Section 6, [BD]] that will lead us to the definition of the Pin group.

Lemma 1. The maps α and t induce an automorphism and anti-automorphism of Γ_V.

Lemma 2. The kernel of $\rho : \Gamma_V \to \mathrm{GL}(V)$ is $\mathbb{R}^*$ in $C(V)$.

Lemma 3. If $x \in \Gamma_V$ then $\mathrm{N}(x) \in \mathbb{R}^*$.

Lemma 4. The map $\mathrm{N}|_{\Gamma_V} : \Gamma_V \to \mathbb{R}^*$ is a homomorphism and $\mathrm{N}(\alpha(x)) = N(x)$.

Lemma 5. We have $\mathbb{R}^n - \{0\} \subset \Gamma_V$ and if $\mathbb{R}^n - \{0\}$ then $\rho(x)$ is the reflection in the hyperplane orthogonal to x. Also, $\rho(\Gamma_V) \subset \mathrm{O}(V)$.

We now have sufficient background to state the definition of the Pin group.

Definition 1. We define $\mathrm{Pin}(n)$ to be the kernel of $\mathrm{N} : \Gamma_V \to \mathbb{R}^*$ for $n \geq 1$.

An important property of the Pin group is the following

Proposition 1 (Chapter 1, Theorem 6.15, [BD])**.** The map $\rho|_{\mathrm{Pin}(n)}$ has image $\mathrm{O}(n)$ and the kernel equal to $\mathbb{Z}/2\mathbb{Z}$. Thus we have a short exact sequence of groups

$$\{e\} \to \mathbb{Z}/2\mathbb{Z} \to \mathrm{Pin}(n) \to \mathrm{O}(n) \to \{e\}.$$

1.2.2 Stiefel-Whitney classes of a real representation

In this section we will see how one can define Stiefel-Whitney classes of a real representation of a finite group. We also state properties that these Stiefel-Whitney classes satisfy. In the rest of the thesis we will rely solely on these properties without making reference to the definition.

Milnor showed (see [Mil56]) that given any topological group G there exists a contractible space EG with a free right G action. The quotient EG/G is called a classifying space of G, denoted by BG. We thus obtain a principal G-bundle $EG \to BG$. For a finite group (in fact, for any discrete group) G, we have a model for BG given by the Eilenberg-MacLane space $K(\mathrm{G}, 1)$ (see Example 1B.7 of [Hat] for an explicit construction). This space $K(G, 1)$ is characterized up to homotopy equivalence by having fundamental group G and trivial higher homotopy groups. It is also known that BG is unique up to homotopy equivalence. A neat fact (see [[Ben91],Theorem 2.2.3]) which relates the singular cohomology of BG to the group cohomology of G is that these are isomorphic as groups: we have $H^i_{\mathrm{Top}}(BG, R) \cong H^i_{\mathrm{Grp}}(G, R)$ for a coefficient ring R considered as a trivial G-module, where H^i_{Top} denotes singular cohomology and $H^i_{\mathrm{Grp}}(G, R)$ refers to group cohomology.

We can now define Stiefel-Whitney classes of a representation of a finite group. Given a finite group G and a finite-dimensional real representation (π, V) we see that the space $EG \times V$ has a natural right G-action and the orbit space of this action is denoted $EG \times_G V$. The fiber bundle $EG \times_G V \to BG$ is called the associated fiber bundle over BG with fiber V. We define for each $i = 0, 1, 2, ...$ the Stiefel-Whitney classes $w_i(\pi) \in H^i(BG, \mathbb{Z}/2\mathbb{Z})$ of the representation (π, V) to be the Stiefel-Whitney classes of the associated bundle $EG \times_G V \to BG$ with fiber V. Using the isomorphism $H^i_{\mathrm{Top}}(BG, R) \cong H^i_{\mathrm{Grp}}(G, R)$, we can consider characteristic classes of representations to lie in the group cohomology of G. The total Stiefel-Whitney class $w(\pi)$ is an element in the $\mathbb{Z}/2\mathbb{Z}$-cohomology ring and is defined to be

$$w(\pi) = w_0(\pi) + w_1(\pi) + \cdots \in H^*(\mathrm{G}, \mathbb{Z}/2\mathbb{Z}) = \bigoplus_{i=1}^{\infty} H^i(G, \mathbb{Z}/2\mathbb{Z})$$

Since we restrict our attention to only Stiefel-Whitney classes of real representations which lie in $H^*(G, \mathbb{Z}/2\mathbb{Z})$, we omit the coefficients $\mathbb{Z}/2\mathbb{Z}$. Thus $H^*(G)$ is to be understood as $H^*(G, \mathbb{Z}/2\mathbb{Z})$. Stiefel-Whitney classes of a real orthogonal representation π of a finite group G satisfy the following properties (see for example [GS20], [GKT89]):

1. $w_0(\pi) = 1$.
2. $w_1(\pi) = \det \pi$, which is an element of $H^1(\mathrm{G}, \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}(G, \pm 1)$.
3. If π' is another real representation of G, then $w(\pi \oplus \pi') = w(\pi) \cup w(\pi')$.
4. If $f : G' \to G$ is a group homomorphism, then $w(\pi \circ f) = f^*(w(\pi))$ where f^* is the induced map on cohomology.

Stiefel-Whitney classes of a representation arise when addressing the problem of spinoriality of representations. From [GS20], we also have the following spinoriality criterion.

Proposition 2 (Proposition 6.1, [GS20])**.** A real representation of a finite group G is spinorial if and only if $w_2(\pi) = w_1(\pi) \cup w_1(\pi)$.

Chapter 2

Group cohomology and spectral sequences

2.1 Introduction

We begin by recalling the definition of group cohomology. One can define the group cohomology of a finite group G to be the singular cohomology of the classifying space BG (as in [AM04].) One can also give a purely algebraic definition using Ext functors, which we will now describe. Suppose G is a finite group. An abelian group A with an action of G is called a *G-module*. Then the group cohomology of G with coefficients in A is defined as

$$H^n(G, A) = \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$$

where $\mathbb{Z}$ is considered as a trivial $\mathbb{Z}G$ module. It is also possible to give an explicit definition of group cohomology without reference to a projective resolution of $\mathbb{Z}$. For a finite group G and a G-module A, define $C^n(G, A) = A$ for $n = 0$ and for $n \geq 1$, define $C^n(G, A)$ to be the set of all maps from $G \times G \times ... \times G$ (n copies) to A. Note that $C^n(G, A)$ is an abelian group with the operation given by pointwise addition of functions. Define the nth coboundary homomorphism $d_n(f) : C^n(G, A) \to C^{n+1}(G, A)$ by

$$\begin{aligned} d_n(f)\,(g_1, \ldots, g_{n+1}) =& g_1 \cdot f\,(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f\,(g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} f\,(g_1, \ldots, g_n)\,. \end{aligned}$$

One checks that this map is indeed a homomorphism. We then define the group cohomology of G with coefficients in A to be

$$H^n(G, A) = \frac{\ker d_n}{\operatorname{im} d_{n-1}}.$$

2.2 The Lyndon-Hochschild-Serre spectral sequence

We have seen previously that characteristic classes of a representation of a group G lie in the group cohomology of G. If the group cohomology ring (with say $\mathbb{Z}/2\mathbb{Z}$ or $\mathbb{Z}$ coefficients) is known and has an explicit description in terms of some generators and relations, one can try and describe characteristic classes of representations in terms of these generators of the cohomology ring. For instance, we shall see in the case for dihedral groups that the cohomology ring is a polynomial ring generated by Stiefel-Whitney classes of certain special representations and the Stiefel-Whithey classes of all other irreducible representations can be described in terms of these generators. Indispensable to any description of the kind mentioned above is a computation of the group cohomology of the group, and the product structure of the cohomology ring. This section is devoted to our main tool for such computations: spectral sequences. We will focus on one spectral sequence; the Lyndon-Hochschild-Serre spectral sequence (henceforth abbreviated as the LHS spectral sequence) in group cohomology . Since the spectral sequence that we consider is first quadrant and cohomological, we will implicitly assume this is the case for all further discussion. For a formal definition of a spectral sequence and related notions, we refer the reader to [Wei94] and [McC00]. Our emphasis will be on merely using spectral sequences as a tool.

One way to think of a spectral sequence is as a book containing a sequence of pages, and on each page one has a coordinate system, the first quadrant of which consists of an abelian group at each lattice point, that is, for each pair of non-negative integers (p, q) we have an abelian group. We consider the abelian groups present at the lattice points in all other quadrants to be the trivial group. Furthermore, there is an arrow (a map, also known as a *differential*) originating from each abelian group and also one ending at each abelian group. The arrows on the r^{th} page map the abelian group at the $(p, q)^{\text{th}}$ position to the abelian group at the $(p + r, q - r + 1)^{\text{th}}$ position. In words, on the r^{th} page the arrows go r places to the right and $r - 1$ places down. On each page, these differentials have the property that the composition of any two successive differentials is zero, thus these differentials on each page form a complex. We denote the group at the $(p, q)^{th}$ position on the r^{th} page by $E_r^{p,q}$ and the differential on the r^{th} page which originates from $E_r^{p,q}$ by $d_r^{p,q}$. Consecutive pages are not unrelated; the relation between the groups $E_r^{p,q}$ and $E_{r+1}^{p,q}$ is that one obtains $E_{r+1}^{p,q}$ by taking homology at the $(p, q)^{th}$ spot on the r^{th} page, that is, $E_{r+1}^{p,q} \approx \ker d_r^{p,q} / \operatorname{im} d_r^{p-r,q+r-1}$. Note that for a fixed (p, q), as r increases, the length of the differentials originating from and ending at $E_r^{p,q}$ also increases. Eventually, for large enough r, the differential originating from $E_r^{p,q}$ maps to a group outside the first quadrant (i.e., into a trivial group) and the differential ending at $E_r^{p,q}$ originates from outside the first quadrant (i.e., from a trivial group). Thus, we have that $E_k^{p,q} = E_{k+1}^{p,q}$ for all $k \geq r$. This stable value of $E_r^{p,q}$ is denoted $E_\infty^{p,q}$. To state the existence of the LHS spectral sequence, we require the notion of convergence of a spectral sequence. We say a spectral sequence *converges* to H^* if there exists a family of abelian groups H^n each having a finite filtration

$$0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n$$

so that

$$E_\infty^{p,q} \cong F^pH^{p+q}/F^{p+1}H^{p+q}.$$

The above information that makes up the definition of convergence of a spectral sequence is traditionally abbreviated as

$$E_r^{p,q} \Longrightarrow H^{p+q}.$$

We now give the statement of the Lyndon-Hochschild-Serre spectral sequence.

Proposition 3 (Chapter 6, Section 8, [Wei94])**.** For a G-module A, for every normal subgroup H of a group G there is a spectral sequence with

$$E_2^{p,q} = H^p\left(G/H, H^q(H,A)\right) \Longrightarrow H^{p+q}(G,A).$$

For a description of the action of G/H on $H^q(H,A)$, see [Example 6.7.7, [Wei94]]. Thus, given any finite group G and normal subgroup H, we obtain a description of the group cohomology of G in terms of that H and G/H. Sometimes, as is the case for cyclic groups using the short exact sequence of groups $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n \to 0$ one can work backwards and find the cohomology of G/H or that of H if the cohomology of the other two groups are known. The reader is warned that the convergence of the LHS spectral sequence to a filtered module $H^{p+q}(G)$ does not necessarily mean that we can compute $H^{p+q}(G)$, we only obtain successive quotients of the associated filtration, and if we are to determine $H^{p+q}(G)$ then we must solve a sequence of "extension" problems. The ease with which this can be done also depends on the coefficient module that we consider. For instance, if we have coefficients in a field k, then there are no non trivial extension problems.

2.3 Group cohomology of dihedral groups D_n

In this section we will see the LHS spectral sequence in action. We will use it to compute the integral cohomology and mod 2-cohomology of odd dihedral groups.

2.3.1 Integral cohomology of odd dihedral groups

We first recall the definition of the invariants and coinvariants functors from the category G-**mod** of G-modules to the category **Ab** of abelian groups.

Definition 2. The *invariant subgroup* A^G of a G-module A is defined as

$$A^G = \{a \in A \,|\, ga = a \text{ for all } g \in G \text{ and } a \in A\}.$$

Definition 3. The *coinvariants* A_G of a G-module A is defined as

$$A_G = A/\langle ga - a \,|\, g \in G, a \in A \rangle.$$

Let us recall first the cohomology of cyclic groups .

Proposition 4 (Theorem 6.2.2, [Wei94])**.** The cohomology groups of a finite cyclic group G of order n with coefficients in a G-module A are given by

$$H^m(G,A) = \begin{cases} \frac{A^G}{NA}, & \text{if } m \text{ is even and } m \geq 2 \\ \frac{{}_NA}{(\sigma-1)A}, & \text{if } m \text{ is odd and } m \geq 1, \\ A^G, & \text{if } m = 0, \end{cases} \tag{2.1}$$

where A^G is the set $\{a \in A \,|\, g \cdot a = a \text{ for all } g \in G\}$ of fixed points of A under the action of G and ${}_NA = \{a \in A \,|\, N \cdot a = 0\}$. In particular for integral coefficients we have

$$H^m(G,\mathbb{Z}) = \begin{cases} \frac{\mathbb{Z}}{n\mathbb{Z}}, & \text{if } \; m = 2,4,6,... \\ 0, & \text{if } m = 1,3,5,..., \\ \mathbb{Z}, & \text{if } m = 0. \end{cases} \tag{2.2}$$

Proposition 5 (Example 6.7.10, [Wei94])**.** Action of C_2 on cohomology group $H^{2q}(C_n)$ is given by multiplication by $(-1)^q$.

The figure below shows the E_2 page of the Lyndon-Hochschild-Serre spectral sequence stated in Theorem 3 applied with $G = D_n$ with n odd, and $H = C_n$.

q				
4	$H^0(C_2, H^4(C_n, \mathbb{Z}))$	$H^1(C_2, H^4(C_n, \mathbb{Z}))$	$H^2(C_2, H^4(C_n, \mathbb{Z}))$	$H^3(C_2, H^4(C_n, \mathbb{Z}))$
3	$H^0(C_2, H^3(C_n, \mathbb{Z}))$	$H^1(C_2, H^3(C_n, \mathbb{Z}))$	$H^2(C_2, H^3(C_n, \mathbb{Z}))$	$H^3(C_2, H^3(C_n, \mathbb{Z}))$
2	$H^0(C_2, H^2(C_n, \mathbb{Z}))$	$H^1(C_2, H^2(C_n, \mathbb{Z}))$	$H^2(C_2, H^2(C_n, \mathbb{Z}))$	$H^3(C_2, H^2(C_n, \mathbb{Z}))$
1	$H^0(C_2, H^1(C_n, \mathbb{Z}))$	$H^1(C_2, H^1(C_n, \mathbb{Z}))$	$H^2(C_2, H^1(C_n, \mathbb{Z}))$	$H^3(C_2, H^1(C_n, \mathbb{Z}))$
0	$H^0(C_2, H^0(C_n, \mathbb{Z}))$	$H^1(C_2, H^0(C_n, \mathbb{Z}))$	$H^2(C_2, H^0(C_n, \mathbb{Z}))$	$H^3(C_2, H^0(C_n, \mathbb{Z}))$
E_2 page	0	1	2	3 k

We make a few observations:

1. All terms on the page apart from those lying on the 0th row and 0th column are zero.

   Proof. First suppose that $q > 1$ and q is odd. Then from Theorem 4 we see that $H^q(C_n, \mathbb{Z}) = 0$, so $E_2^{pq} = H^p(C_2, H^q(C_n, \mathbb{Z})) = 0$. If q is even (say $q = 2i$ for an integer i) we consider two cases for $p \geq 1$: for p even and p odd. If p is odd, then we have $H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{{}_N H^q(C_n, \mathbb{Z})}{(\sigma - 1)H^q(C_n, \mathbb{Z})}$ where σ is the nonzero element of C_2. More explicitly, we have

   $$H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{\{g \in H^q(C_n, \mathbb{Z}) \mid (1 + \sigma) \cdot g = 0\}}{\{(\sigma - 1) \cdot g \mid g \in H^q(C_n, \mathbb{Z})\}}.$$

   Using the computation in Theorem 5, we can rewrite this as

   $$H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{\{g \in H^q(C_n, \mathbb{Z}) \mid g + (-1)^i g = 0\}}{\{(-1)^i g - g \mid g \in H^q(C_n, \mathbb{Z})\}}.$$

   Consider two further cases; one for even i and the other for odd i. If i is even, the above equation gives

   $$H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{\{g \in H^q(C_n, \mathbb{Z}) \mid 2g = 0\}}{\{g - g \mid g \in H^q(C_n, \mathbb{Z})\}} = \{g \in H^q(C_n, \mathbb{Z}) \mid 2g = 0\}.$$

   Now from Theorem 4 we know $H^q(C_n, \mathbb{Z}) = C_n$ with n odd, and thus 2 is a unit in $H^q(C_n, \mathbb{Z})$. Therefore, we get

   $$H^p(C_2, H^q(C_n, \mathbb{Z})) = 0.$$

For odd i, we have

$$H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{\{g \in H^q(C_n, \mathbb{Z}) \mid g = g\}}{\{-2g \mid g \in H^q(C_n, \mathbb{Z})\}}.$$

Again, using that 2 is a unit in $H^q(C_n, \mathbb{Z}) = C_n$, we see that the group in the denominator is $H^q(C_n, \mathbb{Z})$ and so is the numerator, which gives

$$H^p(C_2, H^q(C_n, \mathbb{Z})) = 0.$$

The case where p is even remains, which we now deal with. We have $H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{H^q(C_n,\mathbb{Z})^{C_2}}{(1+\sigma)H^q(C_n,\mathbb{Z})}$ where σ is the nonzero element of C_2. More explicitly, this gives

$$H^p(C_2, H^q(C_n, \mathbb{Z})) = \frac{\{g \in H^q(C_n, \mathbb{Z}) \mid (-1)^i g = g\}}{\{g + (-1)^i g \mid g \in H^q(C_n, \mathbb{Z})\}}.$$

As done previously, if i is odd the numerator is the trivial group and hence so is E_2^{pq}. If i is even, both the numerator and denominator are the full group (that 2 is a unit in C_n with n odd is used here), and thus the quotient E_2^{pq} is trivial.

■

2. Along the 0th row, terms lying on odd numbered columns must be zero, and terms lying along even numbered columns must be $\mathbb{Z}/2$.

   Proof. We have $q = 0$. We use that $H^0(G, A) = A^G$ for any group G and G-module A, to obtain $E_2^{p0} = H^p(C_2, H^0(C_n, \mathbb{Z})) = H^p(C_2, \mathbb{Z})$. From Theorem 4, we obtain the desired result. ■

3. The differentials on the second page are all 0.

   Proof. This is true since the only nonzero terms are those lying on the first row and first column. ■

4. Along the 0th column, terms lying on rows whose index is not divisible by 4 are 0, while terms lying on rows numbered $4k$ (for some $k > 1$) are $\mathbb{Z}/m$.

   Proof. We have $p = 0$. We have already seen that if q is odd, $E_2^{pq} = H^p(C_2, H^q(C_n, \mathbb{Z})) = 0$. If q is even, we use that $H^0(G, A) = A^G$ for any group G and G-module A, to obtain $E_2^{pq} = H^0(C_2, H^q(C_n, \mathbb{Z})) = H^q(C_n, \mathbb{Z})^{C_2}$. If we have $q \equiv 0 (\bmod 4)$ then $H^q(C_n, \mathbb{Z})^{C_2} = H^q(C_n, \mathbb{Z})$, since the action of C_2 is trivial in this case. If instead we have $q \equiv 2 (\bmod 4)$ then $H^q(C_n, \mathbb{Z})^{C_2} = 0$ since the action of C_2 in this case is given by $g \cdot a = -a$. ■

In view of these observations, the lower left quadrant of E_2 is given by the following figure.

q \ k	0	1	2	3	4	5	6	7
4	$\mathbb{Z}/m\mathbb{Z}$	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0
0	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0
E_2 page	0	1	2	3	4	5	6	7

Proposition 6. The integral cohomology of the dihedral group D_m for $m \geq 3$ odd is given by

$$H^n(D_m, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ \mathbb{Z}/2m\mathbb{Z}, & \text{if } n \equiv 0 \pmod 4 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } n \equiv 2 \pmod 4 \\ 0, & \text{otherwise} \end{cases}$$

Proof. Recall what it means for a first quadrant cohomology spectral sequence to converge to H^* : H^n has a finite filtration

$$0 = F^{n+1}H^n \subset F^nH^n \subset \cdots \subset F^1H^n \subset F^0H^n = H^n.$$

The last term $F^nH^n \cong E_\infty^{n0}$ of the filtration lies on the x-axis and the top term $E_\infty^{0n} \cong H^n/F^1H^n$ lies on the y-axis. Moreover, we have $E_\infty^{pq} \cong F^pH^{p+q}/F^{p+1}H^{p+q}$. Since each of the differentials on page 2 is 0, we have $E_\infty^{pq} \cong E_2^{pq}$. Also since for nonzero p and q we have $E_2^{pq} = 0$, we see that $E_\infty^{n-1,1} \cong F^{n-1}H^n/E_\infty^{n0}$ and hence $E_\infty^{n0} = F^1H^n$. Thus we have $E_\infty^{0n} \cong H^n/E_\infty^{n0}$, which gives an exact sequence

$$0 \to E_\infty^{n0} \to H^n \to E_\infty^{0n} \to 0.$$

Since if $n \equiv 0 \pmod 4$ (which is the only nontrivial case) we have $E_\infty^{0n} = \mathbb{Z}/m\mathbb{Z}$ and $E_\infty^{n0} = \mathbb{Z}/2$, we conclude that $H^n = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2m\mathbb{Z}$. The Schur-Zassenhaus theorem (Theorem 6.6.9 in [Wei94]) guarantees that the short exact sequence splits. ■

Remark. Example 6.8.5 of [Wei94] computes integral homology of D_n for n odd by the same argument. We could have used this example along with the universal coefficient theorem to find the integral cohomology of D_n.

2.3.2 mod 2-Cohomology ring of odd dihedral groups

We use the Lyndon-Hochschild-Serre spectral sequence for the normal subgroup $\langle r \rangle$ of order n in $D_n = \langle r, s \mid r^n = s^2 = 1, srs = r^{-1} \rangle$ to determine the $\mathbb{Z}/2\mathbb{Z}$-cohomology ring of D_n when n is odd.
Recall that the LHS spectral sequence states that we have

$$E_2^{p,q} = H^p(G/H, H^q(H)) \implies H^{p+q}(G).$$

Thus we find that the second page is as shown below.

q				
4	$H^0(C_2, H^4(C_n))$	$H^1(C_2, H^4(C_n))$	$H^2(C_2, H^4(C_n,))$	$H^3(C_2, H^4(C_n))$
3	$H^0(C_2, H^3(C_n))$	$H^1(C_2, H^3(C_n))$	$H^2(C_2, H^3(C_n))$	$H^3(C_2, H^3(C_n))$
2	$H^0(C_2, H^2(C_n))$	$H^1(C_2, H^2(C_n))$	$H^2(C_2, H^2(C_n))$	$H^3(C_2, H^2(C_n))$
1	$H^0(C_2, H^1(C_n))$	$H^1(C_2, H^1(C_n))$	$H^2(C_2, H^1(C_n))$	$H^3(C_2, H^1(C_n))$
0	$H^0(C_2, H^0(C_n))$	$H^1(C_2, H^0(C_n))$	$H^2(C_2, H^0(C_n))$	$H^3(C_2, H^0(C_n))$
E_2 page	0	1	2	3 k

Using that $H^p(C_n, C_2) = C_{gcd(n,2)}$ from [Example 6.2.3, [Wei94]], we see that all terms with $q \geq 1$ have trivial coefficients and are thus trivial groups. This simplification gives the following 2^{nd} page.

q				
4	0	0	0	0
3	0	0	0	0
2	0	0	0	0
1	0	0	0	0
0	$H^0(C_2, H^0(C_n))$	$H^1(C_2, H^0(C_n))$	$H^2(C_2, H^0(C_n))$	$H^3(C_2, H^0(C_n))$
E_2 page	0	1	2	3 k

From this page it is clear that $E_2 = E_\infty$ and that we have an isomorphism $H^*(D_n) = H^*(\mathbb{Z}_2)$. Thus $H^*(D_n, \mathbb{Z}/2\mathbb{Z})$ is a polynomial ring in one variable.

2.4 Detection theorems

For a finite group G we say that $H^*(G, \mathbb{Z}/p\mathbb{Z})$ *is detected by abelian subgroups* if there is a family of abelian subgroups $H_i \subset G$ so that

$$\bigsqcup_i \left(res^G_{H_i}\right)^* : H^*(G, \mathbb{Z}/p\mathbb{Z}) \to \bigsqcup_i H^*(H_i \mathbb{Z}/p\mathbb{Z})$$

is an injection. Detection theorems will be our primary method of obtaining a character formula for Stiefel-Whitney classes of real representations.

2.4.1 Detection of group cohomology with coefficients in $\mathbb{Z}/p\mathbb{Z}$ by a p-Sylow subgroup

This note is aimed at proving Theorem 8, which states that p-Sylow subgroups detect mod p-cohomology.

Definition 4. Let G and G' be two finite groups. Suppose A is a G-module and A' is a G'-module. The group homomorphisms $\phi : G' \to G$ and $\psi : A \to A'$ are said to be *compatible* if ψ is a G'-module homomorphism when A is made into a G'-module via ϕ, ,i.e., if $\psi(\phi(g')a) = g'\psi(a)$.

Compatible homomorphisms ϕ and ψ induce the homomorphism

$$\lambda_n : C^n(G, A) \to C^n(G', A')$$

$$f \mapsto \psi \circ f \circ \phi^n$$

at the level of chain groups. One can check that compatibility of ϕ and ψ ensures that λ_n commutes with the coboundary operator; λ_n maps cocycles to cocycles and coboundaries to coboundaries and hence induces a group homomorphism on cohomology

$$\lambda_n : H^n(G, A) \to H^n(G, A).$$

Definition 5. A G-module A is also an H-module for any subgroup H of G. It is easily seen that the inclusion map $i : H \to G$ and the identity map $id : A \to A$ are compatible homomorphisms. These maps induce the *restriction homomorphism* on the cohomology groups:

$$\text{res} : H^n(G, A) \to H^n(H, A), \quad n \geq 0.$$

Before we define the corestriction homomorphism, we introduce the notion of an induced module and prove an important property of group cohomology with coefficients in an induced module.

Definition 6. If H is a sugroup of G, and A is a an H-module, we define the *induced G-module* $M_H^G(A)$ to be $\text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)$.

If H has finite index in G, then we have $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$. We also have that for subgroups $K \leq H \leq G$, $M_H^G(M_K^H(A)) = M_K^G(A)$.
The following proposition illustrates an important property of cohomology with coefficients in an induced module. We will later make use of a map defined in the proof, so we reproduce the proof from [DF04].

Lemma 6. (Shapiro's lemma)[Proposition 23, 17.2, [DF04]]
For any subgroup H of G and any H-module A, we have $H^n(G, M_H^G(A)) = H^n(H, A)$.

Proof. Consider a resolution

$$\cdots \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

of $\mathbb{Z}$ by projective G-modules. Recall that if we apply the functor $\text{Hom}_{\mathbb{Z}G}(-, M_H^G(A))$ to this resolution and then consider the cohomology groups of the resulting cochain complex, we obtain $H^n(G, M_H^G(A))$. If instead we apply the functor $\text{Hom}_{\mathbb{Z}H}(-, A)$ to the same resolution and consider cohomology groups of the resulting cochain complex, we get $H^n(H, A)$. To show that the cohomology groups are isomorphic, it suffices to define isomorphisms between the cochain groups of the aforementioned cochain complexes and show that these isomorphisms commute with cochain maps in the complexes. The desired isomorphism

$$\phi : \text{Hom}_{\mathbb{Z}G}(P_n, \text{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A)) \to \text{Hom}_{\mathbb{Z}H}(P_n, A)$$

between the cochain groups is given by

$$\phi(f)(p) = f(p)(1)$$

for all $f \in \mathrm{Hom}_{\mathbb{Z}G}(P_n, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A))$ and $p \in P_n$. The inverse map

$$\Psi = \phi^{-1} : \mathrm{Hom}_{\mathbb{Z}H}(P_n, A) \to \mathrm{Hom}_{\mathbb{Z}G}(P_n, \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, A))$$

is given by $(\Psi(f')(p))(g) = f'(gp)$ for all $f' \in \mathrm{Hom}_{\mathbb{Z}H}(P_n, A)$ and $p \in P_n$. ■

We now have all the tools to define the corestriction homomorphism.

Definition 7. Suppose H is a subgroup G of index m and that A is a G-module. Let $g_1, g_2, ..., g_m$ be representatives for the left cosets of H in G. Define a map

$$\pi : M_H^G(A) \to A$$

$$f \mapsto \sum_{i=1}^{m} g_i \cdot f(g_i^{-1}).$$

It is easy to see that π is a well defined G-module homomorphism. Thus, it induces a group homomorphism from $H^n(G, M_H^G(A))$ to $H^n(G, A)$. Since A is also an H-module, we have an isomorphism $\xi : H^n(H, A) \xrightarrow{\cong} H^n(G, M_H^G(A))$ induced by the map Ψ in the proof of Shapiro's lemma. The composition of these two maps is called the *corestriction homomorphism*: $\mathrm{Cor} = \pi \circ \xi : H^n(H, A) \to H^n(G, A)$.

We can give an explicit description of the corestriction homomorphism as follows. For a cocycle $f \in \mathrm{Hom}_{\mathbb{Z}H}(P_n, A)$ representing a cohomology class $c \in H^n(H, A)$, a representative $\mathrm{Cor}(f)$ for the class $\mathrm{Cor}(c) \in H^n(G, A)$ is given by

$$\mathrm{Cor}(f)(p) = \sum_{i=1}^{m} g_i \cdot \psi(f)(p)(g_i^{-1}) = \sum_{i=1}^{m} g_i f(g_i^{-1} p).$$

We also have the following result which states that if A has exponent p for a prime p then $H^n(G, A)$ has exponent dividing p.

Lemma 7 (Proposition 20, 17.2, [DF04])**.** Let G be a finite group and A be a G-module. Suppose $mA = 0$ for some integer $m \geq 1$(i.e., the G-module A has exponent dividing m as an abelian group). Then

$$mZ^n(G, A) = mB^n(G, A) = mH^n(G, A) = 0 \text{ for all } n \geq 0.$$

We will now establish an important relation between the restriction and corestriction maps.

Proposition 7 (Proposition 26,17.2, [DF04])**.** Suppose H is a subgroup of G of index m. Then $\mathrm{Cor} \circ \mathrm{res} = m$, i.e. , if c is a cohomology class in $H^n(G, A)$ for some G-module A, then

$$\mathrm{Cor}(\mathrm{res}(c)) = mc \in H^n(G, A) \text{ for all } n \geq 0.$$

We first recall some structure theory of finite abelian groups: if G is a finite abelian group, then for each prime p the elements of order p^n in G for some $n \in \mathbb{N}$ form a subgroup $G_p = \{g \in G \mid p^n g = 0 \text{ for some } n \in \mathbb{N}\}$. We call G_p the p-primary components of G. It is a well known fact that a finite abelian group G is a direct sum of its p-primary components, i.e., $G = \bigoplus_p G_p$. Let us now try to use Theorem 7 to prove the following theorem.

Proposition 8 (Exercise 19, 17.2, [DF04])**.** Let p be a prime and let P be a Sylow p-subgroup of the finite group G. Show that for any G-module A and all $n \geq 0$ the map $\mathrm{res} : H^n(G, A) \to H^n(P, A)$ is injective on the p-primary component of $H^n(G, A)$. Deduce that if $|A| = p^a$ then the restriction map is injective on $H^n(G, A)$.

Proof of Theorem 8. We know P is a Sylow p-subgroup of a finite group G. Suppose we have $|G| = p^r m$, where m is coprime to p. Then we have that $|P| = p^r$. Thus P is a subgroup of index m. From this we obtain the injectivity of the restriction map on the p-primary component of the $H^n(G, A)$. Indeed, if we have $\mathrm{res}(c_1) = \mathrm{res}(c_2)$, we can apply the corestriction map to obtain $\mathrm{Cor}(\mathrm{res}(c_1)) = \mathrm{Cor}(\mathrm{res}(c_2))$, which means $mc_1 = mc_2$. Since c_1 and c_2 belong to the p-primary component of $H^n(G, A)$, the equation $m \cdot (c_1 - c_2) = 0$ gives us that p^n divides m. However, we know m is coprime to p, which forces that $c_1 = c_2$.

For the second part of the exercise, using Lemma 7 we see that any element of $H^n(G, A)$ must have order dividing p^a. If we have $\mathrm{res}(c_1) = \mathrm{res}(c_2)$, applying the corestriction map to both sides gives $mc_1 = mc_2$, that is, $m \cdot (c_1 - c_2) = 0$ which implies that some power of p divides m. But this forces that $c_1 = c_2$, since m is coprime to p. ■

2.4.2 Examples of detection theorems

The following theorem can be proved in the same manner as Theorem 8. For $p = 2$, the theorem states that a subgroup of odd index detects mod 2-cohomology.

Proposition 9 (Corollary 5.2, II.6, [AM04])**.** Let $p \mid |G|$ but assume $[G : H]$ is prime to p, then

$$H^*(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\left(\mathrm{res}^G_{H_i}\right)^*} H^*(H, \mathbb{Z}/p\mathbb{Z})$$

is injective if F_p is the trivial $\mathbb{Z}(G)$ module.

We give three concrete examples of detection theorems.

Proposition 10 (Proposition 6.3, [GS20])**.** The map

$$\Phi : H^2(S_n) \to H^2(\langle(12)\rangle) \oplus H^2(\langle(12)(34)\rangle)$$

given by the two restriction is an isomorphism for $n \geq 4$.

Proposition 11 (Proposition 6.4.1, [Gan19])**.** The map

$$\Phi : H^2(S_n) \to H^2(\langle(12)\rangle) \oplus H^2(\langle A_n \rangle)$$

given by the two restrictions is an isomorphism for $n \geq 4$.

The next theorem is one that we will use extensively in later chapters.

Proposition 12 (Chapter 6, Proposition 3.3, [FP78])**.** The groups $E_1 = \{1, s, r^{2^{k-1}}, sr^{2^{k-1}}\}$ and $E_2 = \{1, rs, r^{2^{k-1}}, rsr^{2^{k-1}}\}$ detect the mod 2-cohomology of even dihedral groups D_{2^k}, that is, the restriction map $\text{res}^* : H^*(D_{2^k}) \to H^*(E_1) \oplus H^*(E_2)$ is an injection.

Remark. An example of subgroups that do not detect cohomology follows. For n a power of 2, the map $H^2(D_n, \mathbb{Z}/2) \to H^2(C_n, \mathbb{Z}/2) \oplus H^2(\langle s \rangle, \mathbb{Z}/2)$ given by the restriction map in each coordinate is not an injection. A proof will be evident once we state the mod 2-cohomology ring of D_n with n a power of 2. This will be done in the next chapter.

Chapter 3

Stiefel-Whitney classes of representations of dihedral groups

3.1 Preliminaries from representation theory of finite groups

We first recall some basic facts from the representation theory of finite groups.

Proposition 13 (Corollary 11, 18.2, [DF04])**.** The number of inequivalent one dimensional representations of a group G is equal to the index of the commutator subgroup $[G, G]$ in G.

The cyclic group C_2

The cyclic group $C_2 = \langle a \mid a^2 = e \rangle$ of order 2 has two irreducible representations, both of dimension 1. The trivial representation $\mathbb{1}_{C_2}$ sends a to 1 and the non-trivial representation sgn_a sends a to -1.

The Klein four-group $C_2 \times C_2$

The Klein four-group $C_2 \times C_2 = \langle a, b \mid a^2 = b^2 = e\ ,\ ab = ba \rangle$ has four irreducible representations, each of dimension 1. They are given by

$$\mathbb{1} : (a, b) \mapsto (1, 1) \qquad \phi_a : (a, b) \mapsto (-1, 1)$$
$$\phi_b : (a, b) \mapsto (1, -1) \qquad \phi_{ab} : (a, b) \mapsto (-1, -1).$$

Dihedral groups D_n

Lemma 8 (Theorem 9, Chapter 3, [Ser77].)**.** Let A be an abelian subgroup of G. Each irreducible representation of G has order at most $\frac{|G|}{|A|}$ Thus all irreducible representations of D_n have dimension 1 or 2.

The following two lemmas give us the number of one dimensional representations of D_n.

Lemma 9. The commutator subgroup of D_n is $[D_n, D_n] = \langle r^2 \rangle$. When n is odd, we have $\langle r^2 \rangle = \langle r \rangle$. When n is even, $\langle r^2 \rangle$ is a proper subgroup of $\langle r \rangle$.

A straightforward consequence of having determined the commutator subgroup is that we obtain the abelianisation of dihedral groups.

Lemma 10. The abelianisation $D_n/[D_n, D_n]$ of D_n is $\mathbb{Z}/2\mathbb{Z}$ for odd n, and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ for even n.

We now list all irreducible representations of D_n. Recall that the squares of the degrees of irreducible representations add up to the order of the group; this fact ensures that we have a complete list of irreducible representations. We treat the cases for n odd and n even separately.

Proposition 14. Assume n is odd. There are two 1-dimensional irreducible representations of D_n. There are $(n-1)/2$ two dimensional irreducible representations of D_n. The 1 dimensional representations are given by

$$\mathbb{1} : (r, s) \mapsto (1, 1),$$

$$\rho_s : (r, s) \mapsto (1, -1).$$

The 2-dimensional irreducible representations σ_k are given by

$$\sigma_k(r) = \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ \sin(2\pi k/n) & \cos(2\pi k/n) \end{pmatrix}$$

$$\sigma_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for each $k = 1, ..., (n-1)/2$.

Proposition 15. Assume n is even. There are four 1-dimensional irreducible representations of D_n. There are $(n-2)/2$ two dimensional irreducible representations of D_n. The 1 dimensional representations are given by

$$\mathbb{1} : (r, s) \mapsto (1, 1),$$

$$\rho_s : (r, s) \mapsto (1, -1),$$

$$\rho_r : (r, s) \mapsto (-1, 1),$$

$$\rho_{rs} : (r, s) \mapsto (-1, -1).$$

The 2-dimensional irreducible representations σ_k are given by

$$\sigma_k(r) = \begin{pmatrix} \cos(2\pi k/n) & -\sin(2\pi k/n) \\ \sin(2\pi k/n) & \cos(2\pi k/n) \end{pmatrix}$$

$$\sigma_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for each $k = 1, ..., (n-2)/2$.

Remark on notation. We will often have to work with different copies of C_2 inside various groups. If the non-trivial element in this copy of C_2 is denoted s, then we write sgn_s for the non-trivial representation of C_2. This notational modification will also apply to the Klein four-group and the dihedral groups.

3.2 Spinoriality of irreducible representations of dihedral groups

We will now determine which of the irreducible representations of dihedral groups are spinorial. Recall that a real representation $\pi : G \to \mathrm{O}(V)$ is said to be *spinorial* if there exists a homomorphism $\hat{\pi} : G \to \mathrm{Pin}(V)$ such that the following diagram commutes:

$$\begin{array}{ccc} & & \mathrm{Pin}(V) \\ & \overset{\hat{\pi}}{\nearrow} & \downarrow \rho \\ G & \xrightarrow{\pi} & \mathrm{O}(V). \end{array}$$

Note that we have $\sigma_k(s)^2 = \mathbb{1}$, which implies that the eigenvalues of the matrix $\sigma_k(s)$ are ± 1. The dimension of the -1 eigenspace of $\sigma_k(s)$ is 1. Let $\{u\}$ denote an orthonormal basis for the -1 eigenspace of $\sigma_k(s)$. Extend u to an orthonormal basis $\{u, v\}$ of the representation space V. With respect to this extended basis, $\sigma_k(s)$ is of the form

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Recall from Section 1.2.1 of Chapter 1 that the map $\rho : \mathrm{Pin}(V) \to \mathrm{O}(V)$ is given by $\rho(x)(v) = \alpha(x) \cdot v \cdot x^{-1}$, where α denotes the unique automorphism of the Clifford algebra of V which satisfies $\alpha^2 = 1$ and $\alpha(x) = -x$ for $x \in i(V)$. We claim that we have

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \rho(u).$$

This is equivalent to the two claims that $\rho(u)(u) = -u$ and $\rho(u)(v) = v$. Indeed, we have

$$\begin{aligned} \rho(u)(u) &= \alpha(u) \cdot u \cdot u^{-1} && \text{using the definition of } \rho \\ &= -u \cdot u \cdot -u && \text{since } \alpha(x) = -x \text{ for } x \in i(V), \text{ and } u^{-1} = -u. \\ &= -u. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \rho(u)(v) &= \alpha(u) \cdot v \cdot u^{-1} && \text{using the definition of } \rho \\ &= -u \cdot v \cdot -u && \text{since } \alpha(x) = -x \text{ for } x \in i(V), \text{ and } u^{-1} = -u. \\ &= v \cdot u \cdot -u && \text{since } u \cdot v = -v \cdot u. \\ &= v. \end{aligned}$$

Therefore, we have $\rho(u) = \sigma_k(s)$. Suppose for the sake of contradiction that the representation σ_k is spinorial. Denote the lift of σ_k to the Pin group by $\hat{\sigma_k}$. Then it is necessary that $\rho(\hat{\sigma_k}(s)) = \sigma_k(s) = \rho(u)$. Since the kernel of ρ is $\{\pm 1\}$, we have $\hat{\sigma_k}(s) = \pm u$. Since $\hat{\sigma_k}$ is a homomorphism and s satisfies the relation $s^2 = e \in D_n$, we must have $\hat{\sigma_k}(s^2) = \hat{\sigma_k}(s)^2 = (\pm u)^2 = 1$. But we know that in the Pin group, $(\pm u)^2 = -1$, which is a contradiction. Thus σ_k is not spinorial. This proves the following

Theorem 5. None of the 2-dimensional irreducible representations of dihedral groups are spinorial.

We draw the reader's attention to the fact that the above proof relies only on the dihedral groups containing an element of order 2. Indeed, the previous theorem is a consequence of the following more general phenomenon.

Lemma 11. Let G be a finite group containing an element s of order 2. Let $\pi : G \to \mathrm{O}(V)$ be a real representation of G. Let g_s denote the multiplicity of the eigenvalue -1 of $\pi(s)$. If the representation π is spinorial then $g_\pi \equiv 0$ or $3 \pmod 4$.

Note that for the 2-dimensional irreducible representations σ_k of dihedral groups, we have $g_{\sigma_k} = 1$.

3.3 Determining Stiefel-Whitney classes as elements of the cohomology ring

We first state the cohomology ring of C_2 and $C_2 \times C_2$.

Proposition 16 (Theorem 4.4, II.4 [AM04]). We have

$$H^*(C_2) = \mathbb{Z}/2\mathbb{Z}[\eta]$$
$$H^*(C_2 \times C_2) = \mathbb{Z}/2\mathbb{Z}[\alpha, \beta]$$

where $\eta = w_1(\text{sgn}_a)$, $\alpha = w_1(\phi_a)$, and $\beta = w_1(\phi_b)$.

3.3.1 Stiefel-Whitney classes of irreducible representations of odd dihedral groups

Recall from Chapter 2 that the restriction map $\text{res}^* : H^*(D_n) \to H^*(\langle s \rangle)$ is an injection and in fact an isomorphism. We claim that for any 2-dimensional irreducible representation σ_k of D_n, we have $w_2(\sigma_k) = 0$. There are (at least) two ways of proving this.

One way is to use the spinoriality criterion. We know that a real representation ϕ of a finite group G is spinorial iff $w_2(\phi) = w_1(\phi) \cup w_1(\phi)$. We know that $H^1(D_n) = H^2(D_n) = \mathbb{Z}/2\mathbb{Z}$. Since σ_k has non-trivial determinant, we must have that the non-zero element of $H^1(D_n)$ is $w_1(\sigma_k)$. Now since the cohomology ring of odd dihedral groups is a polyonomial ring in one variable, we know that $w_1(\sigma_k) \cup w_1(\sigma_k)$ is the non zero element in $H^2(D_n)$, and using aspinoriality of σ_k along with the spinoriality criterion, we also know that $w_2(\sigma_k) \neq w_1(\sigma_k) \cup w_1(\sigma_k)$. This leaves only one choice for $w_2(\sigma_k)$; it must be zero.

Another way of arriving at the result that $w_2(\sigma_k) = 0$ is to use the detection by the subgroup $\langle s \rangle$. This is done by first computing the second Stiefel-Whitney class of the restriction of σ_k to $\langle s \rangle$. This will turn out to be zero, and then it follows from the injectivity of the restriction map that $w_2(\sigma_k)$ must also be zero. The details of this approach are given below.
First, we start by computing the second Stiefel-Whitney class of the restriction of σ_k to $\langle s \rangle$ by decomposing the restriction into its constituent 1 dimensional irreducible representations. We have

$$\sigma_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus we have $\sigma_k|_{\langle s \rangle} = \mathbb{1} \oplus \text{sgn}_s$. Then the total Stiefel-Whitney class of $\sigma_k|_{\langle s \rangle}$ is

$$\begin{aligned} w(\sigma_k|_{\langle s \rangle}) &= w(\mathbb{1}) \cup w(\text{sgn}_s) \\ &= 1 \cup (1 + \eta) \\ &= 1 + \eta. \end{aligned}$$

Thus, we see that $w_2(\sigma_k|_{\langle s \rangle}) = 0$. The injectivity of the restriction map forces $w_2(\sigma_k)$ to be

0, in accordance with the other approach.

3.3.2 Stiefel-Whitney classes of irreducible representations of D_n with $n = 2^m$

The following result from [FP78] describes the cohomology ring of a dihedral group of order 2^{m+1}.
Let $D = D_n$ denote the dihedral group of order $2n = 2^{m+1}, m \geq 2$. Let $\sigma : D_n \to O_2(\mathbb{R})$ denote the standard representation of D given by

$$\sigma(r) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{with } \theta = 2\pi/2^m$$
$$\sigma(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now define $x, y \in H^1(D_n)$ by

$$\langle x, r\rangle = 1 = \langle y, s\rangle$$
$$\langle x, s\rangle = 0 = \langle y, r\rangle.$$

Note that since for a representation ϕ of a group G we have $w_1(\phi) = \det\phi \in H^1(G)$ we obtain that $x = w_1(\rho_r)$ and $y = w_1(\rho_s)$. Let $w \in H^2(D_n)$ denote the second Stiefel-Whitney class of σ (i.e $w = \sigma^*(w_2)$ where $w_2 \in BO_2(\mathbb{R})$) is a degree 2 universal Stiefel-Whitney class. Recall that we have $H^*(BO_2(\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}[w_1, w_2]$ where w_1, w_2 are the universal Stiefel-Whitney classes of degrees 1 and 2 respectively.

Proposition 17 (Chapter 6, Proposition 3.1, [FP78])**.** With notation as above, we have

$$H^*(D_m) = \frac{\mathbb{Z}/2\mathbb{Z}[x, y, w]}{(x^2 + xy)}.$$

We restate Theorem 12 which gives a detection theorem for even dihedral groups.

Theorem 6 (Chapter 6, Proposition 3.3, [FP78])**.** The groups $E_1 = \{1, s, r^{2^{m-1}}, sr^{2^{m-1}}\}$ and $E_2 = \{1, rs, r^{2^{m-1}}, rsr^{2^{m-1}}\}$ detect the mod 2-cohomology of even dihedral groups D_{2^m}, that is, the restriction map $\text{res}^* : H^*(D_{2^m}) \to H^*(E_1) \oplus H^*(E_2)$ is an injection.

We will use the cohomology rings of E_1 and E_2 in the following form:

$$H^*(E_1) = \mathbb{Z}/2\mathbb{Z}[\alpha_1, \beta_1] \tag{3.1}$$
$$H^*(E_2) = \mathbb{Z}/2\mathbb{Z}[\alpha_2, \beta_2] \tag{3.2}$$

where $\beta_1 = w_1(\phi_s)$, $\beta_2 = w_1(\phi_{rs})$, $\alpha_1 = w_1(\phi_{r^{2^{m-1}}})$ and $\alpha_2 = w_1(\phi_{r^{2^{m-1}}})$.

3.3.3 Computing $j^*(w)$

First we compute $w_2(\sigma|_{E_i})$ for $i = 1, 2$. We start with the restriction to E_1 for which we set up some notation: let $e_1 = r^{2^{m-1}}$ and $\overline{e_1} = s$, so that E_1 is the Klein four-group with generators e_1 and $\overline{e_1}$. Then $H^1(E_1) \simeq \mathrm{Hom}(E_1, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the Klein four-group generated by ϕ_{e_1} and $\phi_{\overline{e_1}}$. Note that we have $w_1(\phi_{e_1}) = \alpha_1$ and $w_1(\phi_{\overline{e_1}}) = \beta_1$. Using that $\theta = 2\pi/2^m$ gives

$$
\begin{aligned}
\sigma|_{E_1}(r^{2^{m-1}}) &= \begin{pmatrix} \cos 2^{m-1}\theta & -\sin 2^{m-1}\theta \\ \sin 2^{m-1}\theta & \cos 2^{m-1}\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
$$

and

$$
\sigma|_{E_1}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
$$

Thus $\sigma|_{E_1}$ decomposes into characters of E_1 as

$$
\sigma|_{E_1} = \phi_{e_1} \oplus \phi_{e_1\overline{e_1}}.
$$

By the Whitney sum formula we have

$$
\begin{aligned}
w(\sigma|_{E_1}) &= w(\phi_{e_1}) \cup w(\phi_{e_1\overline{e_1}}) \\
&= (1 + \alpha_1) \cup (1 + \alpha_1 + \beta_1) \\
&= 1 + \alpha_1 + \beta_1 + \alpha_1 + \alpha_2 + \alpha_1\beta_1 \\
&= 1 + \beta_1 + (\alpha_1^2 + \alpha_1\beta_1).
\end{aligned}
$$

Hence we have

$$
w_2(\sigma|_{E_1}) = \alpha_1^2 + \alpha_1\beta_1.
$$

We now compute $w_2(\sigma|_{E_2})$, for which notation is similar to that in the previous case: let $e_2 = r^{2^{m-1}}$ and $\overline{e_2} = rs$ so that E_2 is the Klein four-group with generators e_2 and $\overline{e_2}$. Then $\mathrm{Hom}(E_2, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is the Klein four-group generated by ϕ_{e_2} and $\phi_{\overline{e_2}}$ where ϕ_{e_2} denotes the 1-dimensional representation which sends e_2 to -1 and $\overline{e_2}$ to 1.

Note that we have $w_1(\phi_{e_2}) = \alpha_2$ and $w_1(\phi_{\overline{e_2}}) = \beta_2$. We have

$$\begin{aligned}
\sigma|_{E_2}(r^{2^{m-1}}) &= \begin{pmatrix} \cos 2^{m-1}\theta & -\sin 2^{m-1}\theta \\ \sin 2^{m-1}\theta & \cos 2^{m-1}\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos \pi & -\sin \pi \\ \sin \pi & \cos \pi \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
\sigma|_{E_2}(rs) &= \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\sin \theta & \cos \theta \\ \cos \theta & \sin \theta \end{pmatrix}.
\end{aligned}$$

Therefore, $\sigma|_{E_2}$ decomposes into characters of E_2 as

$$\sigma|_{E_2} = \phi_{e_2} \oplus \phi_{e_2\overline{e_2}}.$$

Then the total Stiefel-Whitney class of $\sigma|_{E_2}$ is

$$\begin{aligned}
w(\sigma|_{E_2}) &= w(\phi_{e_2}) \cup w(\phi_{e_2\overline{e_2}}) \\
&= (1 + \alpha_2) \cup (1 + \alpha_2 + \beta_2) \\
&= 1 + \alpha_2 + \beta_2 + \alpha_2 + \alpha_2 + \alpha_2\beta_2 \\
&= 1 + \beta_2 + (\alpha_2^2 + \alpha_2\beta_2).
\end{aligned}$$

Hence we have

$$w_2(\sigma|_{E_2}) = \alpha_2^2 + \alpha_2\beta_2.$$

Using the above calculations gives

$$\begin{aligned}
j^*(w) &= (j_1^*(w), j_2^*(w)) \\
&= (j_1^*(w_2(\sigma)), j_2^*(w_2(\sigma))) \\
&= (w_2(\sigma|_{E_1}), w_2(\sigma|_{E_2})) \\
&= (\alpha_1^2 + \alpha_1\beta_1, \alpha_2^2 + \alpha_2\beta_2).
\end{aligned}$$

We have $H^1(D_m) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In fact, $H^1(D_m)$ has generators x and y. We can then determine the first Stiefel-Whitney class of σ_k to be $w_1(\sigma_k) = \det \sigma_k = y$. To determine the second Stiefel-Whitney class w_2 of σ_k, we first note that $w_2(\sigma_k)$ belongs to $H^2(D_m) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Note that from the description of the cohomology ring in Theorem 17, we see that $H^2(D_m)$ is 3 dimensional as a $\mathbb{Z}/2\mathbb{Z}$ vector space and is generated by x^2, y^2, and w. Since we have $w_2(\sigma_k) \in H^2(D_m)$, we can write $w_2(\sigma_k) = Ax^2 + By^2 + Cw$ for some $A, B, C \in \mathbb{Z}/2\mathbb{Z}$. We wish to determine the coefficients A, B and C. In order to do so, we will compute the image $j^*(w_2(\sigma_k)) \in H^2(E_1) \oplus H^2(E_2)$ in two different ways and equate what we obtain in each case. First, we use the naturality of Stiefel-Whitney classes to write

$$j^*(w_2(\sigma_k)) = (w_2(\sigma_k|_{E_1}), w_2(\sigma_k|_{E_2})).$$

To determine $w_2(\sigma_k|_{E_i})$ for $i = 1, 2$, we consider two cases: for the first case we consider odd k and the second case deals with even k.

If k is odd, then the restriction of σ_k to E_1 and E_2 is the same as the restriction of σ to E_1 and E_2, and hence $w_2(\sigma_k|_{E_i}) = w_2(\sigma|_{E_i})$ for $i = 1, 2$.

Now suppose k is even. Then for the restriction to E_1 we compute that

$$\sigma_k(r^{2^{m-1}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_k(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Evidently we have $\sigma_k|_{E_1} = \mathbb{1} \oplus \phi_{\overline{e_1}}$. Then the total Stiefel-Whitney class of σ_k is

$$\begin{aligned} w(\sigma_k|_{E_1}) &= w(\mathbb{1}) \cup w(\phi_{\overline{e_1}}) \\ &= 1 \cup (1 + \beta_1) \\ &= 1 + \beta_1. \end{aligned}$$

Hence, for even k,

$$w_2(\sigma_k|_{E_1}) = 0.$$

Similarly for the restriction to E_2 when k is even we see that

$$\sigma_k(r^{2^{m-1}}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_k(rs) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On comparing characters we get $\sigma_k|_{E_2} = \mathbb{1} \oplus \phi_{\overline{e_2}}$.Then the total Stiefel-Whitney class of σ_k

is

$$\begin{aligned} w(\sigma_k|_{E_2}) &= w(\mathbb{1}) \cup w(\phi_{\overline{e_2}}) \\ &= 1 \cup (1+\beta_2) \\ &= 1+\beta_2. \end{aligned}$$

Thus, for even k,

$$w_2(\sigma_k|_{E_2}) = 0.$$

On the other hand, we can also write

$$\begin{aligned} j^*(w_2(\sigma_k)) &= j^*(Ax^2 + By^2 + Cw) \\ &= Aj^*(x^2) + Bj^*(y^2) + Cj^*(w) \\ &= A(0, \beta_2^2) + B(\beta_1^2, \beta_2^2) + C(\alpha_1^2 + \alpha_1\beta_1, \alpha_2^2 + \alpha_2\beta_2) \\ &= (B\beta_1^2 + C(\alpha_1^2 + \alpha_1\beta_1), (A+B)\beta_2^2 + C(\alpha_2^2 + \alpha_2\beta_2)) \end{aligned}$$

Using the computations of $j^*(w_2(\sigma_k))$ we get

$$(B\beta_1^2 + C(\alpha_1^2 + \alpha_1\beta_1), (A+B)\beta_2^2 + C(\alpha_2^2 + \alpha_2\beta_2)) = (w_2(\sigma_k|_{E_1}), w_2(\sigma_k|_{E_2})). \tag{3.3}$$

If k is even, then the right hand side of the equation above is 0, which forces the left side to be zero and we obtain that the coefficients A, B and C are all 0. Thus

$$w_2(\sigma_k) = 0.$$

If k is odd, then $w_2(\sigma_k|_{E_i}) = \alpha_i^2 + \alpha_i\beta_i$ which gives the coefficents $C = 1, B = A = 0$. Hence we have

$$w_2(\sigma_k) = w.$$

We summarize our computation in this section in the following theorem.

Theorem 7. For odd dihedral groups. For any 2-dimensional irreducible representation σ_k of an odd dihedral group with $1 \leq k \leq \frac{n-1}{2}$, we have

$$w_2(\sigma_k) = 0.$$

For even dihedral groups. The second Stiefel-Whitney class of the 2-dimensional irreducible representation σ_k of D_{2^m} with $1 \leq k \leq \frac{n-2}{2}$ is given by

$$w_2(\sigma_k) = \begin{cases} 0, & \text{for even } k \\ w, & \text{for odd } k. \end{cases}$$

Now that we have determined the Stiefel-Whitney classes of irreducible representations of D_n with $n = 2^m$, one might ask what happens in the case for D_n with even n, but when n is not a power of 2. We explain how this case can be reduced to the case when n is a power of 2 by showing that if n is even, a dihedral group of the form D_{2^l} is a 2-Sylow subgroup of D_n.

Suppose $n = 2q$ for some integer q, where $q = 2^l t$ for integers l,t with $t > 1$ odd. Then it is easy to see that the group generated by the two elements

$$\langle r^t, s \rangle$$

is a dihedral group of order 2^{l+1}, and it is thus a 2-Sylow subgroup of D_n. Since we have proved in Chapter 2 that the cohomology of a group is detected by its 2-Sylow subgroup, our calculation of Stiefel-Whitney classes when n is a power of 2 determines Stiefel-Whitney classes when n is even but not a power of 2.

3.4 A character formula for first and second Stiefel-Whitney classes of representations of D_n

Roughly speaking, by a character formula for the kth Stiefel-Whitney class of a representation π we mean that we can write $w_k(\pi)$ in terms of a basis of the kth cohomology group such that the coefficients of each basis element can be described in terms of character vales of π.

3.4.1 Character formula for cyclic group of order 2

We will first give a character formula for Stiefel-Whitney classes of representations of the cyclic group $C_2 = \langle a \mid a^2 = e \rangle$ of order 2.
Suppose π is a real representation of C_2. We write $\pi = n_1 \cdot \mathbb{1} \oplus n_2 \cdot \mathrm{sgn}_a$ for some non negative integers n_1 and n_2. In what follows, we will use the fact that the mod 2-cohomology ring of C_2 is given by $H^*(C_2) = \mathbb{Z}/2\mathbb{Z}[\eta]$ where $\eta = w_1(\mathrm{sgn}_a)$. Using the Whitney sum formula, we have $w(\pi) = w(n_1 \cdot \mathbb{1}) \cup w(n_2 \cdot \mathrm{sgn}_a)$, which gives

$$\begin{aligned} w(\pi) &= \underbrace{w(\mathrm{sgn}_a) \cup w(\mathrm{sgn}_a) \cup \cdots \cup w(\mathrm{sgn}_a)}_{n_2 \text{ times}} \\ &= (1 + w_1(\mathrm{sgn}_a))^{n_2} \\ &= 1 + n_2 w_1(\mathrm{sgn}_a) + \binom{n_2}{2} w_1(\mathrm{sgn}_a) \cup w_1(\mathrm{sgn}_a) + \cdots . \end{aligned}$$

Thus,

$$w_1(\pi) = n_2 w_1(\mathrm{sgn}_a)$$

and

$$w_2(\pi) = \binom{n_2}{2} w_1(\mathrm{sgn}_a) \cup w_1(\mathrm{sgn}_a).$$

Note that the integer n_2, which is the multiplicity of the representation sgn_a as a constituent of π, is also the multiplicity of the eigenvalue -1 for $\pi(a)$ which we denote by g_a. Thus we can write $n_2 = g_a = \frac{\chi_\pi(1)-\chi_\pi(a)}{2}$. Substituting this value of n_2 in the formulas for $w_1(\pi)$ and $w_2(\pi)$ gives us the following theorem.

Theorem 8. The first and second Stiefel-Whitney classes of a real representation π of C_2 are given by:

$$\begin{aligned} w_1(\pi) &= g_a \cdot w_1(\mathrm{sgn}_a) \\ &= \frac{\chi_\pi(1) - \chi_\pi(a)}{2} \cdot w_1(\mathrm{sgn}_a). \end{aligned}$$

$$\begin{aligned} w_2(\pi) &= \frac{(g_a)(g_a - 1)}{2} \cdot w_1(\mathrm{sgn}_a) \cup w_1(\mathrm{sgn}_a) \\ &= \frac{(\chi_\pi(1) - \chi_\pi(a))(\chi_\pi(1) - \chi_\pi(a) - 2)}{8} \cdot w_1(\mathrm{sgn}_a) \cup w_1(\mathrm{sgn}_a). \end{aligned}$$

Remark. Recall that from the spinoriality criterion we know that π is spinorial if and only if $w_2(\pi) = w_1(\pi) \cup w_1(\pi)$, which in this case gives π is spinorial if and only $n_2^2 \equiv \binom{n_2}{2} \pmod 2$. Thus we have the following spinoriality criterion for a real representation π of C_2, which will be used extensively when determining a character formula for first and second Stiefel-Whitney classes of representations of the Klein four-group $C_2 \times C_2$ and dihedral groups.

Theorem 9. A real representation π of $C_2 = \langle a \mid a^2 = e \rangle$ is spinorial if and only if $g_a \equiv 0$ or $3 \pmod 4$, where $g_a = \frac{\chi_\pi(1)-\chi_\pi(a)}{2}$ is the multiplicity of the eigenvalue -1 of $\pi(a)$.

3.4.2 Character formula for odd dihedral groups

We will now obtain a character formula for the first and second Stiefel-Whitney classes of irreducible representations of odd dihedral groups $D_n = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$. We begin by recalling that the mod 2-cohomology ring of odd dihedral groups is given by $\mathbb{Z}/2\mathbb{Z}[x]$ where $x = w_1(\rho_s)$. Similarly, the mod 2-cohomology ring of the order 2 subgroup $C_2 = \langle s \mid s^2 = e \rangle$ of D_n is $\mathbb{Z}/2\mathbb{Z}[\eta]$ where $\eta = w_1(\mathrm{sgn}_s)$. We also have the detection theorem which says that when n is odd, the restriction map $\mathrm{res}^* : H^*(D_n) \to H^*(\langle s \rangle)$ which takes x to η is an injection, and in fact an isomorphism of rings. In particular, we have injections $\mathrm{res}^* : H^1(D_n) \to H^1(\langle s \rangle)$ and $\mathrm{res}^* : H^2(D_n) \to H^2(\langle s \rangle)$.

We start by giving a character formula for the first Stiefel-Whitney class. Suppose π is a real representation of D_n. Then $w_1(\pi)$, being an element of the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^1(D_n)$, can be written as

$$w_1(\pi) = c \cdot x$$

for some $c \in \mathbb{Z}/2\mathbb{Z}$. We apply the restriction map $\text{res}^* : H^1(D_n) \to H^1(\langle s \rangle)$ to both sides of this equation and use the naturality axiom on the left side and that $\text{res}^*(x) = \eta$ on the right side to obtain

$$w_1(\pi|_{\langle s \rangle}) = c \cdot \eta.$$

From the previous subsection, we know that the left hand side is equal to $g_s \cdot \eta$. Thus comparing coefficients on both sides gives $c \equiv g_s \pmod 2$, where g_s is the multiplicity of eigenvalue -1 of $\pi(s)$. We have proved the following theorem.

Theorem 10. The first Stiefel-Whitney class of a real representation π of an odd dihedral group is given by

$$\begin{aligned} w_1(\pi) &= g_s \cdot x \\ &= \frac{\chi_\pi(1) - \chi_\pi(s)}{2} \cdot x, \end{aligned}$$

where $x = w_1(\rho_s)$.

Next, we present two methods of obtaining the second Stiefel-Whitney class of a real representation of D_n in terms of character values, after which we will demonstrate that both methods yield equivalent answers.

First method. Suppose π is a real representation of D_n which is achiral. Since $w_2(\pi)$ is an element of the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^2(D_n)$ which has as a basis the single element x^2, we can write

$$w_2(\pi) = c \cdot x^2$$

for some $c \in \mathbb{Z}/2\mathbb{Z}$. We now apply the restriction map $\text{res}^* : H^2(D_n) \to H^2(\langle s \rangle)$ to both sides of this equation. Use the naturality axiom of Stiefel-Whitney classes on the left side and use that $\text{res}^*(x^2) = \eta^2$ on the right side to obtain

$$w_2(\pi|_{\langle s \rangle}) = c \cdot \eta^2.$$

Notice now that we have $c \equiv 0 \pmod 2$ if and only $w_2(\pi|_{\langle s \rangle}) = 0$. Since we assumed that π is achiral, we see $\pi|_{\langle s \rangle}$ is also achiral. We summarize the relevant conclusions below.

1. We have $c \equiv 0 \pmod 2$ if and only $w_2(\pi|_{\langle s \rangle}) = 0$.

2. The spinoriality criterion then implies that $\pi|_{\langle s \rangle}$ is spinorial if and only if $w_2(\pi|_{\langle s \rangle}) = 0$.

3. A real representation π of $C_2 = \langle s \mid s^2 = e \rangle$ is spinorial if and only if $g_s \equiv 0$ or 3 (mod 4), where $g_s = \frac{\chi_\pi(1) - \chi_\pi(s)}{2}$ is the multiplicity of the eigenvalue -1 of $\pi(s)$.

4. For an achiral representation π of a group G with an element s of order 2, the multiplicity of the eigenvalue -1 of $\pi(s)$ must be even.

Thus, we have the following chain of equivalences:

$$c \equiv 0 \pmod 2 \iff w_2(\pi|_{\langle s \rangle}) = 0 \iff \pi|_{\langle s \rangle} \text{ is spinorial } \iff g_s \equiv 0 \pmod 4.$$

Observe that this implies that c and $\frac{g_s}{2}$ have the same parity. Thus we have $c = \frac{g_s}{2}$. This gives the following character formula for $w_2(\pi)$ when π is achiral.

Lemma 12. For an achiral real representation π of D_n, we have

$$\begin{aligned} w_2(\pi) &= \frac{g_s}{2} \cdot x \\ &= \frac{\chi_\pi(1) - \chi_\pi(s)}{4} \cdot x^2. \end{aligned}$$

We now turn to the case when we have a chiral real representation π of D_n. Our reasoning will be along the following lines: we will obtain from π an achiral representation by taking the direct sum of π with an appropriate representation whose second Stiefel-Whitney class is known. We then use the Whitney sum formula to obtain the second Stiefel-Whitney class of π. The following lemmas will dictate our choice of the direct summand of the achiral representation we wish to obtain.

Lemma 13. Suppose π is a chiral real representation of D_n. Then $\det \pi(r) = 1$. Since π is chiral, this implies $\det \pi(s) = -1$.

Proof. Since π is an orthogonal representation, we must have $\det \pi(r) \in \{\pm 1\}$. Since we have $r^n = e$, we get $\pi(r)^n = \mathbb{1}$. Thus $\det(\pi(r)^n) = 1$. Since n is odd, it cannot be that case that $\det \pi(r) = -1$. ■

Lemma 14. For a chiral real representation π, the representation $\pi' = \pi \oplus \rho_s$ is achiral.

Proof. Using the previous lemma we have

$$\det \pi'(r) = \det \pi(r) \cdot \det \rho_s(r) = 1$$

and

$$\det \pi'(s) = \det \pi(s) \cdot \det \rho_s(s) = -1 \cdot -1 = 1.$$

■

We make two observations:

1. The multiplicity of the eigenvalue -1 of $\pi'(s)$ is g_s+1 where g_s is the multiplicity of the eigenvalue -1 of $\pi(s)$.

2. We have $w_1(\rho_s) = w_1(\pi) = x \in H^1(D_n)$.

The first of these observations together with the character formula for w_2 of an achiral representation from Lemma 12 gives

$$w_2(\pi') = \frac{g_s+1}{2} \cdot x^2.$$

Using the Whitney sum formula for the representation $\pi' = \pi \oplus \rho_s$ gives

$$\begin{aligned} w_2(\pi') &= w_2(\pi) + (w_1(\pi) \cup w_1(\rho_s)) \\ \frac{g_s+1}{2} \cdot x^2 &= w_2(\pi) + x^2. \end{aligned}$$

Thus for an achiral real representation π' of D_n we have

$$w_2(\pi) = \frac{g_s-1}{2} \cdot x^2.$$

The next lemma allows us to combine the chiral and achiral cases into one uniform result.

Lemma 15. Let $[\cdot]$ denote the greatest integer function. If π is an achiral real representation of D_n, we have $\left[\frac{g_s}{2}\right] \equiv \frac{g_s}{2}$. If π is a chiral real representation of D_n, we have $\left[\frac{g_s}{2}\right] \equiv \frac{g_s-1}{2}$.

Proof. If π is achiral, then we know that g_s is even. Thus $\frac{g_s}{2}$ is an integer, which gives $\left[\frac{g_s}{2}\right] \equiv \frac{g_s}{2}$. If π is chiral, then we know that g_s is odd, say $g_s = 2k+1$ for some integer k. Then we have $\left[\frac{g_s}{2}\right] = \left[\frac{2k+1}{2}\right] = \left[k+\frac{1}{2}\right] = k = \frac{g_s-1}{2}$. ∎

This completes the proof of the following theorem.

Theorem 11. Let π be a real representation of an odd dihedral group. Then we have

$$\begin{aligned} w_2(\pi) &= \left[\frac{g_s}{2}\right] w_1(\rho_s) \cup w_1(\rho_s) \\ &= \left[\frac{\chi_\pi(1) - \chi_\pi(s)}{4}\right] w_1(\rho_s) \cup w_1(\rho_s). \end{aligned}$$

We mentioned previously that there are two ways of obtaining a character formula for the second Stiefel-Whitney class of odd dihedral groups. We have so far discussed the first method. Let us now pursue the second method, and show that the character formula obtained from each method is compatible.

Second method. This method will differ from the first one in that we do not deal with the chiral and achiral cases separately. Suppose π is a real representation of D_n. Since $w_2(\pi)$

is an element of the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^2(D_n)$ which has as a basis the single element x^2, we can write

$$w_2(\pi) = c \cdot x^2$$

for some $c \in \mathbb{Z}/2\mathbb{Z}$. We now apply the restriction map $\text{res}^* : H^2(D_n) \to H^2(\langle s \rangle)$ to both sides of this equation. Use the naturality axiom of Stiefel-Whitney classes on the left side and use that $\text{res}^*(x^2) = \eta^2$ on the right side to obtain

$$w_2(\pi|_{\langle s \rangle}) = c \cdot \eta^2.$$

It is at this point that our reasoning diverges from that in the first method. We know that $\pi|_{\langle s \rangle}$ is a real representation of a cyclic group of order 2. We have already obtained a character formula for the second Stiefel-Whitney class of a real representation of a cyclic group of order 2 in Theorem 8. We use this to write

$$\frac{(g_s)(g_s - 1)}{2} \cdot \eta^2 = c \cdot \eta^2$$

which on comparing coefficients gives $c = \frac{(g_s)(g_s-1)}{2}$. We thus obtain the following theorem.

Theorem 12. Let π be a real representation of an odd dihedral group. Then we have

$$\begin{aligned} w_2(\pi) &= \frac{(g_s)(g_s - 1)}{2} \cdot w_1(\rho_s) \cup w_1(\rho_s) \\ &= \frac{(\chi_\pi(1) - \chi_\pi(s))(\chi_\pi(1) - \chi_\pi(s) - 2)}{8} \cdot w_1(\rho_s) \cup w_1(\rho_s). \end{aligned}$$

The following lemma shows the compatibility of the character formula obtained from either method.

Lemma 16. Let π be a real representation of an odd dihedral group. Then we have

$$\frac{(g_s)(g_s - 1)}{2} \equiv \left[\frac{g_s}{2}\right] \pmod 2.$$

Proof. Suppose g_s is odd, with $g_s = 2k + 1$ for some integer k. Then the right hand side is $\left[\frac{g_s}{2}\right] = \left[\frac{2k+1}{2}\right] = k$. The left hand side is $\frac{(g_s)(g_s-1)}{2} = (2k+1)(k)$ has the same parity as k, which is equal to the right hand side.

Suppose now g_s is even, with $g_s = 2k$ for some integer k. The left side simplifies to $k(2k-1)$ which has the same parity as $k = \left[\frac{g_s}{2}\right]$. ∎

Remark. We provide some clarification regarding the essential difference between the two methods. We shall see that it is possible to use the spinoriality criterion for representations of C_2 to obtain a more succinct but equivalent character formula for the second Stiefel-Whitney class of a real representation π of $C_2 = \langle a \mid a^2 = e \rangle$. The argument is identical to the "first method" above and is briefly reproduced here. Suppose π is achiral. As before, we write $w_2(\pi) = c \cdot w_1(\mathrm{sgn}_a)^2$ for some $c \in \mathbb{Z}/2\mathbb{Z}$. We have the following chain of equivalences

$$c = 0 \pmod 2 \iff w_2(\pi) = 0 \iff \pi \text{ is spinorial } \iff g_a \equiv 0 \pmod 4$$

which show that we have $c = \frac{g_a}{2}$. This gives the formula $w_2(\pi) = \frac{g_a}{2} \cdot w_1(\mathrm{sgn}_a)^2$ when π is achiral. If π is chiral, applying the Whitney sum formula for the second Stiefel-Whitney class of the achiral representation $\pi' := \pi \oplus \mathrm{sgn}_a$ in combination with the formula in the achiral case yields $\frac{g_a+1}{2} \cdot w_1(\mathrm{sgn}_a)^2 = w_2(\pi) + w_1(\mathrm{sgn}_a)^2$. On rearranging we get $w_2(\pi) = \frac{g_a-1}{2} \cdot w_1(\mathrm{sgn}_a)^2$. As in the case of odd dihedral groups, these can be combined into the following theorem.

Theorem 13. Let π be a real representation of C_2. Then

$$\begin{aligned} w_2(\pi) &= \left[\frac{g_a}{2}\right] w_1(\mathrm{sgn}_a) \cup w_1(\mathrm{sgn}_a) \\ &= \left[\frac{\chi_\pi(1) - \chi_\pi(a)}{4}\right] w_1(\mathrm{sgn}_a) \cup w_1(\mathrm{sgn}_a). \end{aligned}$$

Thus the two "methods" correspond precisely to the two choices of a character formula for the second Stiefel-Whitney class of the restriction of π to C_2.

Example 1. To illustrate the use of the character formula we have found, we determine the first and second Stiefel-Whitney classes of the regular representation of an odd dihedral group D_n and check whether the regular representation is spinorial. Recall that the dimension of the regular representation π_{reg} of a finite group G is equal to the order $|G|$ of the group. Also, we know $\chi_{\pi_{\mathrm{reg}}}(s) = 0$ for $s \neq e$. Thus the character formula for the first and second Stiefel-Whitney classes give:

$$\begin{aligned} w_1(\pi_{\mathrm{reg}}) &= \frac{\chi_\pi(1) - \chi_\pi(s)}{2} \cdot w_1(\rho_s) \\ &= \frac{2n-0}{2} \cdot w_1(\rho_s) \\ &= n \cdot w_1(\rho_s). \\ w_2(\pi_{\mathrm{reg}}) &= \left[\frac{\chi_\pi(1) - \chi_\pi(s)}{4}\right] \cdot w_1(\rho_s) \cup w_1(\rho_s) \\ &= \left[\frac{2n-0}{4}\right] \cdot w_1(\rho_s) \cup w_1(\rho_s) \\ &= \left[\frac{n}{2}\right] \cdot w_1(\rho_s) \cup w_1(\rho_s). \end{aligned}$$

By the spinoriality criterion, π_{reg} is spinorial if and only if $n^2 \equiv \left[\frac{n}{2}\right] \pmod 2$. Since n is odd, we have $n \equiv 1$ or $3 \pmod 4$. The next lemma provides further simplification.

Lemma 17. Since n is odd, we have $n^2 \equiv 1 \pmod 2$. We have $\left[\frac{n}{2}\right] \equiv 1 \pmod 2$ if and only if $n \equiv 3 \pmod 4$.

Proof. If $n \equiv 1 \pmod 4$, then $\left[\frac{n}{2}\right] = \left[\frac{4k+1}{2}\right] = \left[2k + \frac{1}{2}\right] = 2k$ is even. Therefore $\left[\frac{n}{2}\right] \equiv 1 \pmod 2$ implies $n \equiv 3 \pmod 4$. For the converse, if $n \equiv 3 \pmod 4$, then $\left[\frac{n}{2}\right] = \left[\frac{4k+3}{2}\right] = 2k+1$ which is odd . ∎

The preceding discussion gives the following characterisation of the spinoriality of the regular representation of odd dihedral groups.

Theorem 14. The regular representation π_{reg} of an odd dihedral group is spinorial if and only if $n \equiv 3 \pmod 4$.

3.4.3 Character formula for the Klein four-group $C_2 \times C_2$

Note that $C_2 \times C_2$ has three subgroups of order 2; generated by a, b and ab respectively. The mod 2-cohomology ring of $C_2 \times C_2$ is given by

$$H^*(C_2 \times C_2) = \mathbb{Z}/2\mathbb{Z}[\alpha, \beta]$$

where $\alpha = w_1(\phi_a)$ and $\beta = w_1(\phi_b)$. The mod 2-cohomology ring of each of the three subgroups of order 2 is given by

$$\begin{aligned} H^*(\langle a \rangle) &= \mathbb{Z}/2\mathbb{Z}[\eta_1] \\ H^*(\langle b \rangle) &= \mathbb{Z}/2\mathbb{Z}[\eta_2] \\ H^*(\langle ab \rangle) &= \mathbb{Z}/2\mathbb{Z}[\eta_3] \end{aligned}$$

where $\eta_1 = w_1(\text{sgn}_a)$, $\eta_2 = w_1(\text{sgn}_b)$ and $\eta_3 = w_1(\text{sgn}_{ab})$. We will begin by giving a character formula for the first Stiefel-Whitney class of a real representation of $C_2 \times C_2$. We first prove the following detection theorem.

Theorem 15. The restriction map $\text{res}^* : H^1(C_2 \times C_2) \to H^1(\langle a \rangle) \oplus H^1(\langle b \rangle)$ is an isomorphism.

Proof. Observe that $H^1(C_2 \times C_2)$ is a two dimensional $\mathbb{Z}/2\mathbb{Z}-$vector space with basis elements α and β. The codomain $H^1(\langle a \rangle) \oplus H^1(\langle b \rangle)$ is also a two dimensional $\mathbb{Z}/2\mathbb{Z}-$vector space with basis elements η_1 and η_2. To show that res^*, which is a linear map between two dimensional $\mathbb{Z}/2\mathbb{Z}-$vector spaces, is an isomorphism, it is enough to show that it is surjective. We will show that each basis element η_1 and η_2 of the codomain has a preimage. We know that $\alpha = w_1(\phi_a)$ and $\beta = w_1(\phi_b)$. The image of α under the restriction map is $\text{res}^*(\alpha) = \text{res}^*(w_1(\phi_a)) = w_1(\phi_a|_{\langle a \rangle}) + w_1(\phi_a|_{\langle b \rangle})$. Clearly, $\phi_a|_{\langle a \rangle} = \text{sgn}_a$ and $\phi_a|_{\langle b \rangle} = \mathbb{1}$. Thus $\text{res}^*(\alpha) = \eta_1$. Similarly, $\text{res}^*(\beta) = \text{res}^*(w_1(\phi_b)) = w_1(\phi_b|_{\langle a \rangle}) + w_1(\phi_b|_{\langle b \rangle})$. Using that $\phi_b|_{\langle a \rangle} = \mathbb{1}$ and $\phi_b|_{\langle b \rangle} = \text{sgn}_b$ gives $\text{res}^*(\beta) = \eta_2$. ∎

We can now state a character formula for the first Stiefel-Whitney class.

Theorem 16. Suppose π is a real representation of $C_2 \times C_2$. Then we have

$$\begin{aligned} w_1(\pi) &= g_a\, w_1(\phi_a) + g_b\, w_1(\phi_b) \\ &= \frac{\chi_\pi(1) - \chi_\pi(a)}{2}\, w_1(\phi_a) + \frac{\chi_\pi(1) - \chi_\pi(b)}{2}\, w_1(\phi_b). \end{aligned}$$

Proof. Since $w_1(\pi)$ lies in $H^1(C_2 \times C_2)$, we can write

$$w_1(\pi) = c_1\alpha + c_2\beta \qquad \text{with } c_1,\ c_2 \in \mathbb{Z}/2\mathbb{Z}.$$

Apply the restriction map $\text{res}^* : H^1(C_2 \times C_2) \to H^1(\langle a \rangle) \oplus H^1(\langle b \rangle)$ on both sides, and use the naturality axiom to get

$$\begin{aligned} (w_1(\pi|_{\langle a \rangle}), w_1(\pi|_{\langle b \rangle})) &= c_1(\eta_1, 0) + c_2(0, \eta_2) \\ &= (c_1\eta_1, c_2\eta_2). \end{aligned}$$

Substituting the formula for first Stiefel-Whitney class of a representation of C_2 from Theorem 8 gives $(g_a\eta_1, g_b\eta_2) = (c_1\eta_1, c_2\eta_2)$ which gives $c_1 = g_a$ and $c_2 = g_b$. ■

Next we prove a detection theorem for the second cohomology group of $C_2 \times C_2$.

Theorem 17. The restriction map $\text{res}^* : H^2(C_2 \times C_2) \to H^2(\langle a \rangle) \oplus H^2(\langle b \rangle) \oplus H^2(\langle ab \rangle)$ is an isomorphism.

Proof. Observe that $H^2(C_2 \times C_2)$ is a three dimensional $\mathbb{Z}/2\mathbb{Z}-$vector space with basis elements α^2, β^2 and $\alpha\beta$. The codomain $H^2(\langle a \rangle) \oplus H^2(\langle b \rangle) \oplus H^2(\langle ab \rangle)$ is also a three dimensional $\mathbb{Z}/2\mathbb{Z}-$vector space with basis elements η_1^2, η_2^2 and η_3^2. The generators of $H^2(C_2 \times C_2)$ can be identified as Stiefel-Whitney classes of some representation as follows. Consider the representations $\phi_a \oplus \phi_a, \phi_b \oplus \phi_b$ and $\phi_a \oplus \phi_b$ of $C_2 \times C_2$. The second Stiefel-Whitney class of these representations is

$$\begin{aligned} w_2(\phi_a \oplus \phi_a) &= w_1(\phi_a) \cup w_1(\phi_a) \\ &= \alpha^2, \\ w_2(\phi_b \oplus \phi_b) &= w_1(\phi_b) \cup w_1(\phi_b) \\ &= \beta^2 \qquad \text{and} \\ w_2(\phi_a \oplus \phi_b) &= w_1(\phi_a) \cup w_1(\phi_b) \\ &= \alpha\beta. \end{aligned}$$

We now compute the image of each of α^2, β^2 and $\alpha\beta$ under the restriction map using the naturality axiom. We have

$$\begin{aligned}
\mathrm{res}^*(\alpha^2) &= \mathrm{res}^*(w_2(\phi_a \oplus \phi_a)) \\
&= (w_2(\phi_a \oplus \phi_a|_{\langle a\rangle}), w_2(\phi_a \oplus \phi_a|_{\langle b\rangle}), w_2(\phi_a \oplus \phi_a|_{\langle ab\rangle})) \\
&= (w_2(\mathrm{sgn}_a \oplus \mathrm{sgn}_a), w_2(\mathbb{1} \oplus \mathbb{1}), w_2(\mathrm{sgn}_{ab} \oplus \mathrm{sgn}_{ab})) \\
&= (\eta_1^2, 0, \eta_3^2) \\
\mathrm{res}^*(\beta^2) &= \mathrm{res}^*(w_2(\phi_b \oplus \phi_b)) \\
&= (w_2(\phi_b \oplus \phi_b|_{\langle a\rangle}), w_2(\phi_b \oplus \phi_b|_{\langle b\rangle}), w_2(\phi_b \oplus \phi_b|_{\langle ab\rangle})) \\
&= (w_2(\mathbb{1} \oplus \mathbb{1}), w_2(\mathrm{sgn}_b \oplus \mathrm{sgn}_b), w_2(\mathrm{sgn}_{ab} \oplus \mathrm{sgn}_{ab})) \\
&= (0, \eta_2^2, \eta_3^2) \qquad \text{and} \\
\mathrm{res}^*(\alpha\beta) &= \mathrm{res}^*(w_2(\phi_a \oplus \phi_b)) \\
&= (w_2(\phi_a \oplus \phi_b|_{\langle a\rangle}), w_2(\phi_a \oplus \phi_b|_{\langle b\rangle}), w_2(\phi_a \oplus \phi_b|_{\langle ab\rangle})) \\
&= (w_2(\mathrm{sgn}_a \oplus \mathbb{1}), w_2(\mathbb{1} \oplus \mathrm{sgn}_b)), w_2(\mathrm{sgn}_{ab} \oplus \mathrm{sgn}_{ab})) \\
&= (0, 0, \eta_3^2).
\end{aligned}$$

Observe that we have $\mathrm{res}^*(\alpha^2 + \alpha\beta) = (\eta_1^2, 0, 0)$, $\mathrm{res}^*(\beta^2 + \alpha\beta) = (0, \eta_2^2, 0)$ and $\mathrm{res}^*(\alpha\beta) = (0, 0, \eta_3^2)$ which shows that the restriction map has full rank and hence must be an isomorphism. ■

A character formula for the second Stiefel-Whitney class of a representation of $C_2 \times C_2$ is stated below.

Theorem 18. Suppose π is a real representation of $C_2 \times C_2$. Then its second Stiefel-Whitney class is given by

$$w_2(\pi) = \left[\frac{g_a}{2}\right]\alpha^2 + \left[\frac{g_b}{2}\right]\beta^2 + \left(\left[\frac{g_{ab}}{2}\right] + \left[\frac{g_a}{2}\right] + \left[\frac{g_b}{2}\right]\right)\alpha\beta$$

where $\alpha = w_1(\phi_a)$, $\beta = w_1(\phi_b)$.

Proof. We write $w_2(\pi)$ as a linear combination of a basis of $H^2(C_2 \times C_2)$ as

$$w_2(\pi) = c_1\alpha^2 + c_2\beta^2 + c_3\alpha\beta \qquad \text{with } c_1,\ c_2,\ c_3 \in \mathbb{Z}/2\mathbb{Z}.$$

Applying the restriction map from the previous theorem to both sides, and using the naturality axiom gives

$$\begin{aligned}
(w_2(\pi|_{\langle a\rangle}), w_2(\pi|_{\langle b\rangle}), w_2(\pi|_{\langle ab\rangle})) &= c_1(\eta_1^2, 0, \eta_3^2) + c_2(0, \eta_2^2, \eta_3^2) + c_3(0, 0, \eta_3^2) \\
&= (c_1\eta_1^2, c_2\eta_2^2, (c_1 + c_2 + c_3)\eta_3^2).
\end{aligned}$$

We rewrite the left hand side using the character formula for a representation of a cyclic group of order 2 from Theorem 8. This yields

$$\left(\left[\frac{g_a}{2}\right]\eta_1^2, \left[\frac{g_b}{2}\right]\eta_2^2, \left[\frac{g_{ab}}{2}\right]\eta_3^2\right) = (c_1\eta_1^2, c_2\eta_2^2, (c_1+c_2+c_3)\eta_3^2).$$

This gives

$$\begin{aligned} c_1 &= \left[\frac{g_a}{2}\right] \\ c_2 &= \left[\frac{g_b}{2}\right] \\ c_1+c_2+c_3 &= \left[\frac{g_{ab}}{2}\right]. \end{aligned}$$

Substituting the first two equations into the last equation gives

$$c_3 = \left[\frac{g_{ab}}{2}\right] + \left[\frac{g_a}{2}\right] + \left[\frac{g_b}{2}\right].$$

■

3.4.4 Character formula for even dihedral groups

The goal of this section is to state and prove a character formula for the first and second Stiefel-Whitney classes of irreducible representations of even dihedral groups D_n when $n = 2^m$ for some integer m. We first reproduce some facts about the group cohomology of even dihedral groups from [FP78]. Let $\sigma : D_n \to O_2(\mathbb{R})$ denote the standard representation of D given by

$$\sigma(s) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \text{with } \theta = 2\pi/2^m$$

$$\sigma(b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The mod 2-cohomology ring of even dihedral groups is given by

$$H^*(D_{2^m}) = \frac{\mathbb{Z}/2\mathbb{Z}[x,y,w]}{(x^2+xy)}$$

where $x = w_1(\rho_r), y = w_1(\rho_s)$ and w is the second Stiefel-Whitney class of the standard representation. We have the following detection theorem from [FP78].

Proposition 18 (Chapter 6, Proposition 3.3, [FP78])**.** The groups $E_1 = \{1, s, r^{2^{m-1}}, sr^{2^{m-1}}\}$ and $E_2 = \{1, rs, r^{2^{m-1}}, rsr^{2^{m-1}}\}$ detect the mod 2-cohomology of even dihedral groups, that is, the restriction map $\text{res}^* : H^*(D_{2^m}) \to H^*(E_1) \oplus H^*(E_2)$ is an injection. We also have

$$\begin{aligned} \text{res}^*(x) &= (0, \beta_2) \\ \text{res}^*(x) &= (\beta_1, \beta_2) \\ \text{res}^*(w) &= (\alpha_1^2 + \alpha_1\beta_1, \alpha_2^2 + \alpha_2\beta_2). \end{aligned}$$

We write the mod 2-cohomology rings of the groups E_1 and E_2 as

$$\begin{aligned} H^*(E_1) &= \mathbb{Z}/2\mathbb{Z}[\alpha_1, \beta_1] \\ H^*(E_2) &= \mathbb{Z}/2\mathbb{Z}[\alpha_2, \beta_2] \end{aligned}$$

where $\beta_1 = w_1(\phi_s)$, $\beta_2 = w_1(\phi_{rs})$, $\alpha_1 = w_1(\phi_{r^{2^{m-1}}})$ and $\alpha_2 = w_1(\phi_{r^{2^{m-1}}})$.
where $w_1(\pi_1) = \beta_1$, $w_1(\pi_2) = \alpha_1$, $w_1(\pi_1') = \beta_2$ and $w_1(\pi_2') = \alpha_2$. The next theorem gives character formulas for both the first and second Stiefel-Whitney classes.

Theorem 19. Let π be a real representation of an even dihedral group. Then its first and second Stiefel-Whitney classes are given by

$$\begin{aligned} w_1(\pi) &= g_s x + (g_{rs} + g_s) y \\ w_2(\pi) &= \left(\left[\frac{g_{rs}}{2}\right] + \left[\frac{g_s}{2}\right] \right) x^2 + \left[\frac{g_s}{2}\right] y^2 + \left[\frac{g_{r^{2^{m-1}}}}{2}\right] w. \end{aligned}$$

Proof. Note that $H^1(D_{2^k}) = \langle x, y \rangle \simeq \mathbb{Z}/2\mathbb{Z}^2$, and $H^2(D_{2^k}) = \langle x^2,\ y^2,\ w \rangle \simeq \mathbb{Z}/2\mathbb{Z}^3$. We write the first Stiefel-Whitney class of π as

$$w_1(\pi) = c_1 x + c_2 y \qquad \text{with } c_1,\ c_2 \in \mathbb{Z}/2\mathbb{Z}.$$

Apply the restriction map from the detection theorem in Theorem 12 to both sides. Then use the naturality axiom along with the character formula for the first Stiefel-Whitney class of a representation of a Klein four group to get

$$\begin{aligned} (w_1(\pi|_{E_1}), w_1(\pi|_{E_2})) &= c_1(0, \beta_2) + c_2(\beta_1, \beta_2) \\ (g_{r^{2^{m-1}}}\alpha_1 + g_s\beta_1, g_{r^{2^{m-1}}}\alpha_2 + g_{rs}\beta_2) &= (c_1\beta_1, (c_1 + c_2)\beta_2). \end{aligned}$$

This gives $c_1 = g_s$ and $c_1 + c_2 = g_{rs}$. We conclude that $c_2 = g_{rs} + g_s$. Also note $g_{r^{2^{m-1}}}$ is even.
The second Stiefel-Whitney class of π can be written as

$$w_2(\pi) = m_1 x^2 + m_2 y^2 + m_3 w \qquad \text{with } m_1,\ m_2,\ m_3, \in \mathbb{Z}/2\mathbb{Z}.$$

As usual, we apply the restriction map and use naturality:

$$(w_2(\pi|_{E_1}), w_2(\pi|_{E_2})) = m_1(0, \beta_2^2) + m_2(\beta_1^2, \beta_2^2) + m_3(\alpha_1^2 + \alpha_1\beta_1, \alpha_2^2 + \alpha_2\beta_2).$$

We rewrite equality of each coordinate below:

$$\left[\frac{g_{r^{2^{m-1}}}}{2}\right]\alpha_1^2 + \left[\frac{g_s}{2}\right]\beta_1^2 + \left(\left[\frac{g_{r^{2^{m-1}}s}}{2}\right] + \left[\frac{g_{r^{2^{m-1}}}}{2}\right] + \left[\frac{g_s}{2}\right]\right)\alpha_1\beta_1 = m_3(\alpha_1^2 + \alpha_1\beta_1) + m_2\beta_1^2 \quad \text{and}$$
$$\left[\frac{g_{r^{2^{m-1}}}}{2}\right]\alpha_2^2 + \left[\frac{g_{rs}}{2}\right]\beta_2^2 + \left(\left[\frac{g_{rsr^{2^{m-1}}}}{2}\right] + \left[\frac{g_{r^{2^{m-1}}}}{2}\right] + \left[\frac{g_{rs}}{2}\right]\right)\alpha_2\beta_2 = (m_1 + m_2)\beta_2^2 + m_3(\alpha_2^2 + \alpha_2\beta_2).$$

Thus we have

$$\begin{aligned} m_1 &= \left[\frac{g_{rs}}{2}\right] + \left[\frac{g_s}{2}\right] \\ m_2 &= \left[\frac{g_s}{2}\right] \text{ and} \\ m_3 &= \left[\frac{g_{r^{2^{m-1}}}}{2}\right]. \end{aligned}$$

■

Example 2. Once again, we illustrate the theory through the example of the regular representation π_{reg} of D_{2^m} for $m > 1$. Note that since $\chi_{\pi_{\text{reg}}}(s) = 0$ for $s \neq e$, we have $g_{rs} = g_s = g_{r^{2^{m-1}}} = \frac{\dim \pi_{\text{reg}}}{2} = 2^m$, which is even. The first and second Stiefel-Whitney classes are

$$\begin{aligned} w_1(\pi_{\text{reg}}) &= g_s x + (g_{rs} + g_s)y \\ &= 0. \\ w_2(\pi_{\text{reg}}) &= \left(\left[\frac{g_{rs}}{2}\right] + \left[\frac{g_s}{2}\right]\right)x^2 + \left[\frac{g_s}{2}\right]y^2 + \left[\frac{g_{r^{2^{m-1}}}}{2}\right]w. \\ &= 0. \end{aligned}$$

Thus we have $w_2(\pi_{\text{reg}}) = w_1(\pi_{\text{reg}})^2$, implying that the regular representation of D_{2^m} is achiral and spinorial for all $m > 1$.

3.5 Higher Stiefel-Whitney classes

The techniques used in the previous sections can be used to give a character formula for all Stiefel-Whitney classes of C_2, $C_2 \times C_2$ and odd dihedral groups. The difference is that we no longer have an analogue of the spinoriality criterion to independently determine Stiefel-Whitney classes. We will have to rely on a more primitive approach for the groups C_2, and $C_2 \times C_2$. We will then use detection theorems to obtain results for dihedral groups.

We begin with the cyclic group $C_2 = \langle a \mid a^2 = e \rangle$ of order 2. Any representation π of C_2 decomposes as $\pi = n_1 \mathbb{1} \oplus n_2 \operatorname{sgn}_a$, where $n_2 = g_a$. The total Stiefel-Whitney class of π is

$$\begin{aligned} w(\pi) &= \underbrace{w(\operatorname{sgn}_a) \cup w(\operatorname{sgn}_a) \cup \cdots \cup w(\operatorname{sgn}_a)}_{g_a \text{ times}} \\ &= (1 + w_1(\operatorname{sgn}_a))^{g_a}. \end{aligned}$$

By the binomial theorem the m^{th} Stiefel-Whitney class is

$$w_m(\pi) = \binom{g_a}{m} \eta^m$$

with $\eta = w_1(\operatorname{sgn}_a)$.
From this we can obtain higher Stiefel-Whitney classes for $D_n = \langle r, s \mid r^n = s^2 = e, srs = r^{-1} \rangle$ when n is odd in the following manner. We know that the map $\operatorname{res}^* : H^*(D_n) \to H^*(\langle s \rangle)$ is an injection. If π is a real representation of D_n, we write its m^{th} Stiefel-Whitney class as

$$w_m(\pi) = cx^m \text{ for } c \in \mathbb{Z}/2\mathbb{Z}$$

where $x = w_1(\rho_s)$. Apply the restriction map (which we know sends $x \in H^1(D_n)$ to $w_1(\operatorname{sgn}_a) \in H^1(\langle s \rangle)$) to get

$$\begin{aligned} \operatorname{res}^*(w_m(\pi)) &= c(w_1(\operatorname{sgn}_a))^m \\ w_m(\pi|_{\langle s \rangle}) &= c(w_1(\operatorname{sgn}_a))^m \\ \binom{g_s}{m}(w_1(\operatorname{sgn}_a))^m &= c(w_1(\operatorname{sgn}_a))^m, \end{aligned}$$

which gives $c = \binom{g_s}{m}$.
We now turn to the Klein four-group $C_2 \times C_2 = \langle a, b \mid a^2 = b^2 = e,\ ab = ba \rangle$. We know this group has four 1-dimensional representations given by

$$\begin{aligned} &\mathbb{1} : (a, b) \mapsto (1, 1) && \phi_a : (a, b) \mapsto (-1, 1) \\ &\phi_b : (a, b) \mapsto (1, -1) && \phi_{ab} : (a, b) \mapsto (-1, -1). \end{aligned}$$

So any real representation π of $C_2 \times C_2$ decomposes as

$$\pi = n_1 \mathbb{1} \oplus\ n_2 \phi_a \oplus\ n_3 \phi_b \oplus\ n_4 \phi_{ab}.$$

Recall the multiplicity n_2 of the representation ϕ_a is given by the following inner product of characters

$$\begin{aligned} n_2 &= \langle \chi_{\phi_a}, \chi_\pi \rangle_{C_2\times C_2} \\ &= \frac{1}{4}\left(\sum_{g\in C_2\times C_2} \chi_{\phi_a}(g)\chi_\pi(g) \right) \\ &= \frac{1}{4}(\chi_\pi(g) - \chi_\pi(a) + \chi_\pi(b) - \chi_\pi(ab)). \end{aligned}$$

Straightforward manipulation of this equation along with similar equations for n_3 and n_4 yields the following relations between n_j's and g_a, g_b and g_{ab}:

$$\begin{aligned} g_a &= n_2 + n_4 \\ g_b &= n_3 + n_4 \\ g_{ab} &= n_2 + n_3. \end{aligned}$$

Solving for n_1, n_2 and n_3 we get

$$\begin{aligned} n_2 &= \frac{g_a + g_{ab} - g_b}{2} \\ n_3 &= \frac{g_b + g_{ab} - g_a}{2} \\ n_4 &= \frac{g_a + g_b - g_{ab}}{2}. \end{aligned}$$

We will use the notation from Theorem 16 for the cohomology ring of $C_2 \times C_2$. The total Stiefel-Whitney class of π is

$$\begin{aligned} w(\pi) &= w(\phi_a)^{n_2} w(\phi_b)^{n_3} w(\phi_{ab})^{n_4} \\ &= (1+\alpha)^{n_2}(1+\beta)^{n_3}(1+\alpha+\beta)^{n_4}. \end{aligned}$$

The m^{th} cohomology group is generated by elements $\alpha^i\beta^{m-i}$ for $i = 0, 1, ..., m$. Thus determining the m^{th} Stiefel-Whitney class amounts to finding the coefficient of a term $\alpha^i\beta^{m-i}$ for $i = 0, ..., m$ in the product

$$\begin{aligned} w(\pi) &= (1+\alpha)^{n_2}(1+\beta)^{n_3}(1+\alpha+\beta)^{n_4} \\ &= \left(\sum_{r_2=0}^{n_2} \binom{n_2}{r_2} \alpha^{r_2} \right) \left(\sum_{r_3=0}^{n_3} \binom{n_3}{r_3} \beta^{r_3} \right) \left(\sum_{r_4+r_5+r_6=n_4} \binom{n_4}{r_4, r_5, r_6} \alpha^{r_5}\beta^{r_6} \right) \end{aligned}$$

where we have used the binomial theorem for the first two terms and the multinomial theorem for the last term. An easy counting argument gives

$$w_m(\pi) = \sum_{i=0}^{m} \left[\sum_{r=0}^{i} \binom{n_2}{r} \binom{n_4}{i-r} \binom{n_3 + n_4 - (i-r)}{m-i} \right] \alpha^i \beta^{m-i}.$$

For example, let $m = 1$. Then

$$
\begin{aligned}
w_1(\pi) &= \left[\binom{n_2}{0}\binom{n_4}{0}\binom{n_3+n_4-0}{1}\right]\alpha^0\beta^1 + \left[\binom{n_4}{1}\binom{n_3+n_4-1}{0} + \binom{n_2}{1}\binom{n_3+n_4}{0}\right]\alpha^1\beta^0 \\
&= (n_3+n_4)\beta + (n_4+n_2)\alpha \\
&= g_a\alpha + g_b\beta,
\end{aligned}
$$

which is precisely what we obtained in Theorem 18. For $m = 2$ we obtain

$$
\begin{aligned}
\text{coefficient of } \alpha^2 &= \sum_{r=0}^{2}\binom{n_2}{r}\binom{n_4}{2-r}\binom{n_3+n_4-(2-r)}{0} \\
&= \binom{n_2+n_4}{2} = \binom{g_a}{2}, \\
\text{coefficient of } \alpha\beta &= \sum_{r=0}^{1}\binom{n_2}{r}\binom{n_4}{1-r}\binom{n_3+n_4-(1-r)}{1} \\
&= \binom{n_2}{0}\binom{n_4}{1}\binom{n_3+n_4-1}{1} + \binom{n_2}{1}\binom{n_4}{0}\binom{n_3+n_4}{1} \\
&= n_4(n_3+n_4-1) + n_2(n_3+n_4) \\
&= (n_2+n_4)(n_3+n_4) - n_4 \\
&= g_a g_b - \frac{g_a+g_b-g_{ab}}{2}, \\
\text{coefficient of } \beta^2 &= \binom{n_2}{0}\binom{n_4}{0}\binom{n_3+n_4}{2} = \binom{g_b}{2}.
\end{aligned}
$$

We have seen in Lemma 16 that $\binom{g_a}{2} \equiv \left[\frac{g_a}{2}\right] \pmod 2$ and $\binom{g_b}{2} \equiv \left[\frac{g_b}{2}\right] \pmod 2$. To recover the results in Theorem 18 it remains to show that

$$
g_a g_b - \frac{g_a+g_b-g_{ab}}{2} \equiv \left[\frac{g_{ab}}{2}\right] + \left[\frac{g_a}{2}\right] + \left[\frac{g_b}{2}\right] \pmod 2.
$$

The proof, which we omit, is a straightforward exhaustion of 8 cases which arise from each of g_a, g_b and g_{ab} being even or odd.

For even dihedral groups, it is in principle possible to write down a formula as follows. First, we must determine the generators for the m^{th} cohomology group using the description of the cohomology ring. Then using that the cohomology of these dihedral groups is detected by two Klein four groups, and using that we have a character formula for Klein four-groups we obtain a character formula for these dihedral groups. The formula for the Klein four-group is complicated which makes this calculation tedious. We omit it.
We collect the results of this section in the following theorems.

Theorem 20. Let π be real representation of $C_2 = \langle a \mid a^2 = e\rangle$. Then its m^{th} Stiefel-

Whitney class is given by

$$w_m(\pi) = \binom{g_a}{m} w_1(\mathrm{sgn}_a)^m.$$

Theorem 21. For a real representation π of an odd dihedral group the m^{th} Stiefel-Whitney class is given by

$$w_m(\pi) = \binom{g_s}{m} w_1(\rho_s)^m.$$

Theorem 22. For a real representation π of $C_2 \times C_2 = \langle a, b \mid a^2 = b^2 = e, ab = ba \rangle$ the m^{th} Stiefel-Whitney class is given by

$$w_m(\pi) = \sum_{i=0}^{m} \left[\sum_{r=0}^{i} \binom{n_2}{r} \binom{n_4}{i-r} \binom{n_3 + n_4 - (i-r)}{m-i} \right] \alpha^i \beta^{m-i}$$

where $\alpha = w_1(\phi_a)$, $\beta = w_1(\phi_b)$.

Remark. Computing Stiefel-Whitney classes for dihedral groups yields answers to the following questions about the extent to which Stiefel-Whitney classes of a representation characterise the representation.

Question 1. Suppose π and π' are irreducible real representations of a finite group G. Is it true that $w(\pi) = w(\pi')$ implies $\pi \simeq \pi'$?

Answer. No. For the odd dihedral groups, all 2-dimensional irreducible representations have the same total Stiefel-Whitney class, that is, we have for each $k \neq k'$ in $1, ..., \frac{n-1}{2}$, $w(\sigma_k) = w(\sigma_{k'})$. For the even dihedral groups, when k and k' are either both odd or both even, we have $w(\sigma_k) = w(\sigma_{k'})$.

Question 2. Suppose π is an irreducible real representation of a finite group G. Is it true that $w(\pi) = 1$ implies π is the trivial representation?

Answer. This is clearly true for the odd dihedral groups and C_2. In fact, it is true for these groups without the hypothesis of irreduciblity.

Proof. We will prove that non-trivial real representations of C_2 and odd dihedral groups have non-trivial total Stiefel-Whitney class.

Suppose π is a non-trivial real representation of C_2. Then it decomposes as $\pi = n_1 \mathbb{1} \oplus n_2 \,\mathrm{sgn}_a$. Since π is non-trivial, n_2 must be non zero. The total Stiefel-Whitney class is

$$w(\pi) = (1 + w_1(\mathrm{sgn}_a))^{n_2}.$$

Evidently the n_2^{th} Stiefel-Whitney class will be non-zero. Similarly for a non-trivial real representation π' of D_n when n is odd, it decomposes into irreducible representations as

$$\pi' = n_1 \mathbb{1} \oplus n_2 \rho_s \oplus \bigoplus_{k=1}^{\frac{n-1}{2}} m_k \sigma_k,$$

and non-triviality implies that the sum $S := n_2 + \sum_{k=1}^{\frac{n-1}{2}} m_k$ is non zero. The total Stiefel-Whitney class of π' is

$$\begin{aligned} w(\pi') &= w(\rho_s)^{n_2} \prod_{k=1}^{\frac{n-1}{2}} w(\sigma_k)^{m_k} \\ &= w(\rho_s)^S, \end{aligned}$$

since $w(\sigma_k) = w(\rho_s)$. Evidently the S^{th} Stiefel-Whitney class of π' is non-zero. ■

We end this chapter with two examples. It is a theorem that (see Problem 8-B in [MS74]) that for the first non-zero Stiefel-Whitney class w_n, n must be a power of 2. We demonstrate via the first example that for each integer $k > 1$ there exists a representation of D_n with odd n such that $w_2, w_4, w_8, ..., w_{2^{k-1}}$ are all zero and w_{2^k} is nonzero.

Example 3. Consider the representation $\pi = 2^{n-1}\rho_s \oplus 2^{n-1}\sigma_1$ of D_n for odd n. This has total Stiefel-Whitney class

$$w(\pi) = (1 + w_1(\rho_s))^{2^n}.$$

Binomial coefficients have the property that $\binom{m}{k}$ is even for each $1 \leq k \leq m-1$ if and only if $m = 2^r$ for some integer r. Thus the only Stiefel-Whitney class which is non zero is w_{2^n}.

The final example is similar in spirit to an example in [MS74], which states that the orthogonal complement in the trivial bundle of rank $n+1$ of the line bundle γ_n^1 over $\mathbb{RP}^n$ has $w_k \neq 0$ for each $1 \leq k \leq n$. We give an example of a representation of D_n with odd n of dimension $2^n - 1$ which has $w_k \neq 0$ for each $1 \leq k \leq 2^n - 1$.

Example 4. Consider the representation

$$\pi = (2^n - n)\phi_s \oplus \bigoplus_{k=1}^{\frac{n-1}{2}} 2\sigma_k$$

of D_n for odd n. The total Stiefel-Whitney class is

$$\begin{aligned} w(\pi) &= w(\phi_s)^{2^n - n} \prod_{k=1}^{\frac{n-1}{2}} w(\sigma_k)^2 \\ &= w(\rho_s)^{2^n - n + n - 1} \\ &= w(\rho_s)^{2^n - 1} \\ &= (1 + w_1(\rho_s))^{2^n - 1}. \end{aligned}$$

We now use the property of binomial coefficients that $\binom{m}{k}$ is odd for each $0 \leq k \leq m$ if and only if $m = 2^r - 1$ for some integer r. Thus all Stiefel-Whitney classes for this representation are non zero.

Chapter 4

Stiefel-Whitney classes of representations of symmetric groups

4.1 Structure of 2-Sylow subgroups H_k of symmetric groups S_{2^k}

We start by defining the notion of a wreath product, which is crucial in describing the structure of 2-Sylow subgroups of S_n.

Definition 8. The wreath product $G \wr H$ of a finite group G with H where H is a subgroup of the symmetric group S_n is defined as the semidirect product $(\underbrace{G \times G \cdots \times G}_{n\text{times}}) \rtimes H$ where H acts on $G \times G \cdots \times G$ by permuting the coordinates.

We denote the 2-Sylow subgroup of S_n by P_n and the 2-Sylow subgroup of S_{2^k}, being of special interest, is denoted H_k. It is known (see [Kal48]) that if n has the binary expansion $n = 2^{k_1} + \cdots + 2^{k_s}$, then $P_n = \prod_{k_i} H_{k_i}$, where the product runs over all binary digits k_i of n. The groups H_k are known to be iterative wreath products of C_2's; we have $H_k = H_{k-1} \wr C_2$. We illustrate this recursive description of H_k for $k = 1, 2, 3$.
For $k = 1$, we have $S_2 = C_2 = \{e, (12)\}$, and so the 2-Sylow H_1 is S_2 itself. Denote the element (12) by g_1.
For $k = 2$, we know that S_4 consists of permutations of $\{1, 2, 3, 4\}$. We have the subgroup $S_2 \times S_2$ inside S_4 where the first factor S_2 is the set of permutations of $\{1, 2\}$ and the second factor S_2 is the set of permutations of $\{3, 4\}$ (which is generated by $g_1' = (34)$). From the inclusion $H_1 \hookrightarrow S_2$, we obtain a copy of $H_1 \times H_1'$ inside S_4, where H_1' denotes the 2-Sylow of the second factor in the product $S_2 \times S_2$. Consider now the element $g_2 = (13)(24)$ which 'switches' the two halves of {1,2,3,4}, and note that we have $g_2 g_1 g_2^{-1} = (13)(24)(12)(13)(24) = (34) = g_1'$. One checks that the set

$$H_1 \times H_1' \ \cup \ g_2(H_1 \times H_1')$$

is closed under multiplication, and is thus a subgroup of S_4. This is subgroup of size 8, and is thus a 2-Sylow subgroup of S_4. It can be checked that this is isomorphic to the dihedral group D_8 of order 8 generated by (12) and (1432). Note that H_2 is generated by g_1 and g_2.

For $k = 3$, we have the subgroup $S_4 \times S_4$ inside S_8, and thus the inclusion $H_2 \times H_2' \hookrightarrow S_8$. The element $g_3 := (15)(26)(37)(48)$ is such that $g_3 g_2 g_3^{-1} = g_2'$ and the set

$$H_2 \times H_2' \cup g_3(H_2 \times H_2')$$

is closed under multiplication, and is thus a subgroup of S_8. This is a subgroup of size 128 which is the correct size for it to be a 2-Sylow subgroup of S_8.

One can also describe 2-Sylow subgroups H_k of S_{2^k} as the automorphism group of the complete binary tree of height k as in [Nar17].

4.1.1 Conjugacy classes and representation theory of H_k

This subsection is devoted to summarizing results regarding the conjugacy classes and representations of H_k from [Nar17]. We have three types of conjugacy classes in H_k; representatives of each type of class, size and number of a conjugacy classes of each type are given in the following table from [Nar17]. The cardinality of the class $[\sigma]$ is denoted $c_k([\sigma])$ and the total number of conjugacy classes of the group H_k is denoted C_k in this table.

Type	Representative	# classes	Size of class(c_k)
I	$[(\sigma,\sigma)^1]$	C_{k-1}	$c_{k-1}([\sigma])^2$
II	$[(Id,\sigma)^{-1}]$	C_{k-1}	$\lvert H_{k-1}\rvert c_{k-1}([\sigma])$
III	$[(\sigma_1,\sigma_2)^1]$	$\binom{C_{k-1}}{2}$	$2c_{k-1}([\sigma_1])c_{k-1}([\sigma_2])$

Table 4.1: Conjugacy classes of H_k

It is well known that irreducible representations of a product $G_1 \times G_2$ of two groups are tensor products $\pi_1 \otimes \pi_2$ of representations, where π_1 is an irreducible representation of G_1 and π_2 is an irreducible representation of G_2. Consider a representation $\phi_1 \otimes \phi_2$ of $H_{k-1} \times H_{k-1}$. Since $H_{k-1} \times H_{k-1}$ is an index two subgroup of H_k, we have that for non-isomorphic irreducible representations ϕ_1 and ϕ_2 of H_{k-1}, then $\mathrm{Ind}_{H_{k-1}\times H_{k-1}}^{H_k} \phi_1 \otimes \phi_2$ is irreducible, and for an irreducible representation ϕ of H_{k-1} $\mathrm{Ind}_{H_{k-1}\times H_{k-1}}^{H_k} \phi \otimes \phi$ is reducible and decomposes as a direct sum of two irreducible representations of H_k, which are denoted Ext^+ and Ext^-. It turns out that all irreducible representations of H_k are obtained in this manner. Below we reproduce two tables from [Nar17], the first gives the character values of each type of irreducible representation of H_k, and the second is a recursive template for the character table of H_k in terms of H_{k-1}.

Table 4.2: Irreducible characters of H_k

Type	Notation	Description	Action on $(\sigma_1, \sigma_2)^1$	Action on $(\mathrm{Id}, \sigma)^{-1}$
I	$\mathrm{Ext}^+(\phi)$	Positive extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)$
II	$\mathrm{Ext}^-(\phi)$	Negative extension of $\phi \otimes \phi$	$\phi(\sigma_1)\phi(\sigma_2)$	$-\phi(\sigma)$
III	$\mathrm{Ind}(\phi_1, \phi_2)$	Induced from $\phi_1 \otimes \phi_2$	$\phi_1(\sigma_1)\phi_2(\sigma_2)$ $+\phi_1(\sigma_2)\phi_2(\sigma_1)$	0

Table 4.3: Template for the character table for H_k

	Type I	Type II	Type III
$\mathrm{Ext}^+(\phi)$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)\phi(\sigma)$	character table for H_{k-1}
$\mathrm{Ext}^-(\phi)$	$\phi(\sigma_1)\phi(\sigma_2)$	$\phi(\sigma)\phi(\sigma)$	-character table for H_{k-1}
$\mathrm{Ind}(\phi_1, \phi_2)$	$\phi_1(\sigma_1)\phi_2(\sigma_2) + \phi_1(\sigma_2)\phi_2(\sigma_1)$	$2\phi_1(\sigma)\phi_2(\sigma)$	0

4.1.2 Group cohomology of wreath products

Nakaoka showed in [Nak61] that the cohomology ring with coefficients in a field k of a wreath product $G \wr H$ with $H \subset S_n$ of groups is given by:

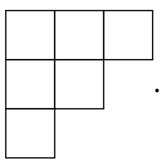
$$H^*(G \wr H, k) \simeq H^*(H, \bigotimes_n H^*(G, k)).$$

For details on how $\bigotimes_n H^*(G, k)$ is an H-module and a proof of this theorem, we refer the reader to [Eve91], Section 5.2.

4.2 A review of the representation theory of symmetric groups

The representation theory of symmetric groups is well-studied. We summarize relevant results here; the proofs can be found in [Pra15] or [Sag01].

There is a bijective correspondence between the set of representations of S_n and the set of partitions of n. We denote the representation corresponding to the partition λ of n by S^λ. A partition $\lambda = (\lambda_1, ..., \lambda_l)$ of n can be represented pictorially as a *Young diagram* (denoted $\mathcal{Y}(\lambda)$), which is a finite collection of boxes arranged in an array of left justified rows such that the ith row contains λ_i boxes. We say that the *shape* of $\mathcal{Y}(\lambda)$ is λ. For example, corresponding to the partition $(3, 2, 1)$ of 6 we have the following Young diagram $\mathcal{Y}(3, 2, 1)$:



Let λ be a partition of n. A *Young tableau* of shape λ is a Young diagram $\mathcal{Y}(\lambda)$ with its boxes filled by integers. A *standard Young tableau* (SYT) of shape λ is a Young diagram of shape λ filled with integers $\{1, 2, ..., n\}$ such that in each column the entries increase from top to bottom and in each row the entries increase from left to right. It is known that the dimension f_λ of the representation S^λ is equal to the number of standard Young tableau of shape λ. As an example, a standard Young tableau of shape $(3, 2, 1)$ is given below.

1	2	3
4	5	
6		

.

There is also a formula for the dimension of S^λ called the *hook length formula*, which we now describe.

The *hook length* of a box in a Young diagram is defined to be the number of boxes of the Young diagram to the right of the given box or directly below the given box plus one. In our example of $\mathcal{Y}(3, 2, 1)$, each box is filled in with its own hook length in the figure below:

5	3	1
3	1	
1		

.

The hook length formula states that the dimension f_λ of S_λ is given by

$$f_\lambda = \frac{n!}{\prod_{i,j} h_{ij}},$$

where $h_{i,j}$ denotes the hook length of the box in the ith row and jth column of the Young diagram $\mathcal{Y}(\lambda)$, and the product is over all boxes in $\mathcal{Y}(\lambda)$.

We will now describe the recursive Murnaghan-Nakayama rule which allows us to compute character values of irreducible representations of symmetric groups. For a box of a Young diagram in the (i,j)th position, we denote by rim_{ij} the set of boxes in positions (k, l) with $k \geq i$ and $l \geq j$ such that the Young diagram does not have a box in position $(k + 1, l + 1)$. As an example, for $\lambda = (5, 4, 3, 3)$, $\text{rim}_{2,2}$ consists of the boxes marked with '$\times$' in the figure below.

	$\cdot$	$\times$	$\times$	
		$\times$		
	$\times$	$\times$		

Observe that the number of boxes in rim_{ij} is equal to the hook length h_{ij}. Removing the boxes in $\text{rim}_{i,j}$ from the Young diagram of a partition λ of n yields a new Young diagram, which we denote by $\lambda - \text{rim}_{ij}$, of a partition of $n - h_{ij}$ The leg-length of a box in the (i, j)th position is defined to be the number of boxes directly below it plus one. Let $\lambda = (\lambda_1, ..., \lambda_l)$ and $\mu = (\mu_1, ..., \mu_m)$ be partitions of n. For any $i \in \{1, ..., m\}$, let $\hat{\mu}_i$ denote the partition obtained from μ by removing its ith part. Then the recursive Murnaghan-Nakayama rule states that the character value of S^λ on an element of cycle type μ to be

$$\chi_\lambda\left(w_\mu\right) = \sum_{h_{ij}=\mu_i} (-1)^{l_{ij}-1} \chi_{\lambda-\text{rim}_{ij}}\left(w_{\hat{\mu}_i}\right),$$

where w_μ is an element with cycle decomposition

$$(1 \cdots \mu_1)(\mu_1 + 1 \cdots \mu_1 + \mu_2) \cdots (\mu_1 + \cdots + \mu_{m-1} + 1 \cdots n).$$

We end this section with a result on the restriction of an irreducible representation of S_n to S_{n-1}. Let λ be a partition of n. Denote by λ^- the set of partitions whose Young diagrams can be obtained from the Young diagram of λ by removing one box. Denote by λ^+ the set of partitions whose Young diagrams can be obtained from the Young diagram of λ by adding one box. The restriction of the irreducible representation S^λ of S_n to S_{n-1} is a direct sum over all $\eta \in \lambda^-$ of irreducible representations S^η of S_{n-1}. Similarly, the induction of the irreducible representation S^λ of S_n to S_{n+1} is a direct sum over all $\eta \in \lambda^+$ of irreducible representations S^η of S_{n+1}.

4.3 Stiefel-Whitney classes of representations of symmetric groups S_n for small n

As an aid to computation, we will first prove the following lemma about the total Stiefel-Whitney class of a tensor product of a representation π of a finite group G with a degree 1 representation ϵ.

Lemma 18. Let π be a real representation of a finite group G of dimension m and ϵ be degree 1 real representation of G. Then

$$w(\pi \otimes \epsilon) = \sum_{k=0}^{m} (1 + w_1(\epsilon))^k \, w_{m-k}(\pi).$$

We will deduce this from [[MS74], Problem 7-C], which we reproduce below.

Problem 7-C. Let ξ^m and η^n be vector bundles over a paracompact base space. Show that the Stiefel-Whitney classes of the tensor product $\xi^m \otimes \eta^n$(or of the isomorphic bundle $\text{Hom}\,(\xi^m, \eta^n)$) can be computed as follows. If the fiber dimensions m and n are both 1, then

$$w_1\left(\xi^1 \otimes \eta^1\right) = w_1\left(\xi^1\right) + w_1\left(\eta^1\right).$$

More generally there is a universal formula of the form

$$w\left(\xi^m \otimes \eta^n\right) = p_{m,n}\left(w_1\left(\xi^m\right), \ldots, w_m\left(\xi^m\right), w_1\left(\eta^n\right), \ldots, w_n\left(\eta^n\right)\right)$$

where the polynomial $p_{m,n}$ in $m+n$ variables can be characterized as follows. If $\sigma_1, \ldots, \sigma_m$ are the elementary symmetric functions of indeterminates $t_1, \ldots, t_m$, and if $\sigma'_1, \ldots, \sigma'_n$ are the elementary symmetric functions of $t'_1, \ldots, t'_n$, then

$$p_{m,n}\left(\sigma_1, \ldots, \sigma_m, \sigma'_1, \ldots, \sigma'_n\right) = \prod_{i=1}^{m} \prod_{j=1}^{n} \left(1 + t_i + t'_j\right).$$

Proof of Lemma 18. Our proof relies on the fact that for variables $a, b_1, b_2, ..., b_m$ we have the following identity for polynomials in the variables $a, b_1, ..., b_m$ with mod 2- coefficients:

$$\prod_{i=1}^{m}(a + b_i) = \sum_{k=0}^{m} a^k \sigma_{r-k}(b_1, ..., b_m).$$

Let $n = 1$. Then the polynomial $p_{m,n}$ becomes

$$\begin{aligned} p_{m,n} &= \prod_{i=1}^{m}\left(1 + t_i + t'_1\right) \\ &= \prod_{i=1}^{m}\left(1 + t_i + t'_1\right) \\ &= \sum_{k=0}^{m}\left(1 + (t'_1)^k\right)\sigma_{m-k}(t_1, ..., t_m) \qquad \text{(using the aforementioned identity).} \end{aligned}$$

The observation that $w_{m-k}(\pi) = \sigma_{m-k}(t_1, ..., t_m)$ and $w_1(\epsilon) = \sigma'_1(t_1) = t'_1$ completes the proof. ■

4.3.1 S_4

We state the cohomology ring of each small symmetric group and the action of the restriction map from [KG]. We have

$$H^*(S_4) = \frac{\mathbb{Z}/2\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]}{(\sigma_1\sigma_3)}.$$

The 2-Sylow subgroup of S_4 is D_8, which has cohomology ring given by

$$H^*(D_8) = \frac{\mathbb{Z}/2\mathbb{Z}[x, y, w]}{(x^2 + xy)}$$

where $x = w_1(\rho_r), y = w_1(\rho_s)$ and w is the 2nd Stiefel-Whitney class of the standard representation of D_8.

The restriction map $\text{res}^* : H^*(S_4) \to H^*(D_8)$ is an injection and is given by

$$\begin{aligned}\text{res}^*(\sigma_1) &= x + y \\ \text{res}^*(\sigma_2) &= x^2 + w \text{ and} \\ \text{res}^*(\sigma_3) &= xw.\end{aligned}$$

The character table of D_8 along with Stiefel-Whitney classes of each irreducible representation is given stated as Table 4.4.

	()	(3,4)	(1,2)(3,4)	(1,3)(2,4)	(1,3,2,4)	w_1	w_2
χ_1	1	1	1	1	1	0	0
χ_2	1	-1	1	-1	1	y	0
χ_3	1	-1	1	1	-1	$x+y$	0
χ_4	1	1	1	-1	-1	x	0
χ_5	2	0	-2	0	0	y	w

Table 4.4: Stiefel-Whitney classes of irreducible representations of D_8.

Table 4.5 lists all irreducible representations of S_4 and all of their Stiefel-Whitney classes as elements of the cohomology ring.

Partition λ	$\dim(S^\lambda)$	$\text{Res}^{S_4}_{D_8}(S^\lambda)$	$w_1(S^\lambda)$	$w_2(S^\lambda)$	$w_3(S^\lambda)$
4	1	χ_1	0	0	0
$(3,1)$	3	$\chi_4 + \chi_5$	σ_1	σ_2	σ_3
$(2,2)$	2	$\chi_1 + \chi_3$	σ_1	0	0
$(2,1,1)$	3	$\chi_2 + \chi_5$	0	$\sigma_2 + \sigma_1^2$	$\sigma_2\sigma_1 + \sigma_3$
$(1,1,1,1)$	1	χ_3	σ_1	0	0

Table 4.5: Stiefel-Whitney classes of irreducible representations of S_4

4.3.2 S_5

Note that S_5 and S_4 have the same 2-Sylow subgroup. As a consequence, once we find restrictions of irreducible representations of S_4 to its 2-Sylow subgroup, we can use the final result of Section 4.2 to obtain restrictions of irreducible representations of S_5 to its 2-Sylow subgroup. The cohomology ring of S_5 is given by

$$H^*(S_5) = \frac{\mathbb{Z}/2\mathbb{Z}[\sigma_1, \sigma_2, \sigma_3]}{(\sigma_1\sigma_3)}.$$

The restriction map $\text{res}^* : H^*(S_4) \to H^*(D_8)$ is an injection and is given by

$$\text{res}^*(\sigma_1) = x + y$$
$$\text{res}^*(\sigma_2) = x^2 + w \text{ and}$$
$$\text{res}^*(\sigma_3) = xw.$$

We start by listing the decomposition of the restriction of irreducible representations of S_5 to D_8 in Table 4.6.

Partition λ	$\text{Res}^{S_5}_{D_8}(S^\lambda)$
5	χ_1
$(4,1)$	$\chi_1 + \chi_4 + \chi_5$
$(3,2)$	$\chi_1 + \chi_3 + \chi_4 + \chi_5$
$(3,1,1)$	$\chi_2 + \chi_4 + \chi_5 + \chi_5$
$(2,2,1)$	$\chi_1 + \chi_2 + \chi_3 + \chi_5$
$(2,1,1,1)$	$\chi_2 + \chi_3 + \chi_4 + \chi_5$
$(1,1,1,1,1)$	χ_3

Table 4.6: Restrictions of irreducible representations of S_5 to D_8

Table 4.7 below lists all irreducible representations of S_5 and all of their Stiefel-Whitney classes as elements of the cohomology ring.

Partition λ	$\dim(S^\lambda)$	$w_1(S^\lambda)$	$w_2(S^\lambda)$	$w_3(S^\lambda)$	$w_4(S^\lambda)$	$w_5(S^\lambda)$	$w_6(S^\lambda)$
5	1	0	0	0	0	0	0
$(4,1)$	4	σ_1	σ_2	σ_3	0	0	0
$(3,2)$	5	0	$\sigma_1^2 + \sigma_2$	$\sigma_2\sigma_1 + \sigma_3$	0	0	0
$(3,1,1)$	6	σ_1	σ_2	σ_1^3	σ_2^2	$\sigma_1\sigma_2^2$	σ_3^2
$(2,2,1)$	5	σ_1	$\sigma_1^2 + \sigma_2$	$\sigma_1^3 + \sigma_3$	$\sigma_1^4 + \sigma_2\sigma_1^2$	σ_1^5	0
$(2,1,1,1)$	4	σ_1	σ_2	σ_3	$\sigma_2\sigma_1^2$	0	0
$(1,1,1,1,1)$	1	σ_1	0	0	0	0	0

Table 4.7: Stiefel-Whitney classes of irreducible representations of S_5

We verify this calculation for $w_6(S^\lambda)$ for the partition $\lambda = (3,1,1)$. The calculation for all other partitions for S_5 and also for S_4 is nearly identical and is omitted. Note that Lemma 18 can be used for computing Stiefel-Whitney classes of conjugate partitions. From the cohomology ring we have the following generators of the 6th cohomology group:

$$H^6(S_5) = \langle \sigma_1^6, \sigma_1^4\sigma_2, \sigma_3^2, \sigma_1^2\sigma_2^2 \rangle.$$

We have that $\mathrm{Res}^{S_5}_{D_8}(S^\lambda) = \chi_2 + \chi_4 + \chi_5 + \chi_5$. Write

$$w_6(S^\lambda) = c_1\sigma_1^6 + c_2\sigma_1^4\sigma_2 + c_3\sigma_3^2 + c_4\sigma_1^2\sigma_2^2$$

for $c_1, c_2, c_3, c_4 \in \mathbb{Z}/2\mathbb{Z}$. Apply the restriction map and evaluate $\mathrm{Res}^*(w_6(S^\lambda))$ in two ways, once by using linearity and then by using the naturality axiom. We then get

$$x^2w^2 = c_1(x^6 + y^6) + c_2((x^4 + y^4)(x^2 + w)) + c_3(x^2w^2) + c_4((x^2 + y^2)(x^4 + w^2)).$$

Comparing coefficients yields $c_1 = c_2 = c_4 = 0$ and $c_3 = 1$.

4.4 Character formula for the first and second Stiefel-Whitney class of representations of S_n

As an application of the character formula obtained in the previous section for even dihedral groups, we will obtain a character formula for the second Stiefel-Whitney class of real orthogonal representations of S_4 and then extend this result to S_n for $n \geq 4$. Recall that the 2-Sylow subgroup of S_4 is isomorphic to the dihedral group D_8 of order 8. We will use the copy $D_8 = \langle s = (12), r = (1432) \rangle$ inside S_4. We will use the cohomology ring and image of the restriction map as stated in the subsection 4.3.1.
Our usual technique yields the following character formula for the first and second Stiefel-Whitney classes.

Theorem 23. For π a real orthogonal representation of S_4 we have

$$w_1(\pi) = g_s\sigma_1$$
$$w_2(\pi) = \left[\frac{g_s}{2}\right]\sigma_1^2 + \frac{g_{rs}}{2}\sigma_2$$

where $s = (13)$ and $rs = (12)(34)$.

Proof. Since we have $H^1(S^4) = \langle \sigma_1 \rangle \simeq \mathbb{Z}/2\mathbb{Z}$ we write

$$w_1(\pi) = c\sigma_1 \quad \text{for } c \in \mathbb{Z}/2\mathbb{Z}.$$

Apply the restriction map to D_8 to get:

$$w_1(\pi|_{D_8}) = c(x + y).$$

Use the character formula for D_{2^k} to get

$$g_s x + (g_{rs} + g_s)y = c(x + y).$$

We know from Lemma 3.2 of [GS20] that g_{rs} is even. Thus we get that $c = g_s$.
For the second Stiefel-Whitney class write

$$w_2(\pi) = c_1\sigma_1^2 + c_2\sigma_2 \quad \text{for } c_1,\ c_2 \in \mathbb{Z}/2\mathbb{Z}.$$

Once again, we apply the restriction map to D_8 and use naturality along with the character formula for D_8:

$$w_2(\pi|_{D_8}) = c_1(x^2+y^2) + c_2(x^2+w)$$
$$\left(\left[\frac{g_s}{2}\right] + \left[\frac{g_{rs}}{2}\right]\right) x^2 + \left[\frac{g_s}{2}\right] y^2 + \left[\frac{g_{r^2}}{2}\right] w = (c_1+c_2)x^2 + c_1y^2 + c_2w.$$

where $s = (12)$, $r = (1432)$ and $r^2 = (13)(24)$.
Therefore we can choose

$$c_1 = \left[\frac{g_s}{2}\right]$$
$$c_2 = \left[\frac{g_{rs}}{2}\right] = \left[\frac{g_{r^2}}{2}\right].$$

Since g_{rs} is even, we have $\left[\frac{g_{rs}}{2}\right] = \frac{g_{rs}}{2}$.

■

We will now extend this result to S_n for $n \geq 4$. We start by recording generators of $H^2(S_4)$ in the next lemma. For a proof, one simply plugs in π_4 and $\mathrm{sgn} \oplus \mathrm{sgn}$ instead of π in Theorem 23.

Lemma 19. The generators of $H^2(S_4)$ are $w_2(\pi_4)$ and $w_2(\mathrm{sgn} \oplus \mathrm{sgn})$.

A theorem of Nakaoka (see Corollary 6.7 in [Nak60]) states that the cohomology of symmetric groups stabilises:

Proposition 19 (Nakaoka)**.** For $n > 2k$, the restriction map $H^k(S_n, \mathbb{F}) \to H^k(S_{n-1}, \mathbb{F})$ is an isomorphism, for any trivial coefficient module $\mathbb{F}$.

In particular, this implies that the restriction map $H^k(S_n, \mathbb{F}) \to H^k(S_{2k}, \mathbb{F})$ for $n > 2k$ is an isomorphism. For $k = 2$ this gives that the restriction map from $H^2(S_n, \mathbb{Z}/2\mathbb{Z})$ to $H^2(S_4, \mathbb{Z}/2\mathbb{Z})$ is an isomorphism. Observe that π_n restricts to S_4 as a direct sum of some copies of the trivial representation and π_4. Similarly the sign representation of S_n (denoted temporarily as sgn_n) restricts to the sign representation of S_4 (denoted temporarily as sgn_4). In particular, we have $\mathrm{res}^* w_2(\pi_n) = w_2(\pi_4)$ and $\mathrm{res}^* w_2(\mathrm{sgn}_n \oplus \mathrm{sgn}_n) = w_2(\mathrm{sgn}_4 \oplus \mathrm{sgn}_4)$. Thus we obtain generators for $H^2(S_n)$ for $n \geq 4$, which is recorded in the next lemma.

Lemma 20. A basis for the $\mathbb{Z}/2\mathbb{Z}$-vector space $H^2(S_n)$ is given by $w_2(\pi_n)$ and $w_2(\mathrm{sgn} \oplus \mathrm{sgn})$.

The next result is the previously promised extension of Theorem 23 to S_n for $n \geq 4$. Note that this recovers the formula in Theorem 6.4 of [GS20].

Theorem 24. For π a real representation of S_n for $n > 4$ we have

$$w_2(\pi) = \left[\frac{g_s}{2}\right]\sigma_1^2 + \frac{g_{rs}}{2}\sigma_2$$

where $s = (12)$ and $rs = (12)(34)$.

Proof. We know from Theorem 19 that the restriction map from $H^2(S_n)$ to $H^2(S_4)$ is an isomorphism. By Lemma 20 write $w_2(\pi) = c_1 w_2(\pi_n) \oplus c_2 w_2(\mathrm{sgn}_n \oplus \mathrm{sgn}_n)$. Apply the restriction map to get

$$w_2(\pi|_{S_4}) = c_1 w_2(\pi_4) \oplus c_2 w_2(\mathrm{sgn}_4 \oplus \mathrm{sgn}_4). \tag{4.1}$$

Use Theorem 23 to rewrite the left hand side, and compare coefficients. ∎

For the first Stiefel-Whitney class, note Theorem 19 for $k = 1$ gives that the restriction map $H^1(S_n) \to H^1(S_2)$ is an isomorphism for $n \geq 2$. For a real representation π of S_n, we get

$$w_1(\pi) = c_1 w_1(\mathrm{sgn}).$$

Applying the restriction map and using naturality gives $w_1(\pi) = g_{(12)} w_1(\mathrm{sgn})$.

Chapter 5

Conclusion

We conclude the thesis by stating some problems related to those addressed in the preceding chapters. We state partial results and suggest a possible approach for some of these problems.

Restriction of representations of S_{2^k} to H_k

The next lemma gives a sufficient condition for the vanishing of Stiefel-Whitney classes "from the top".

Lemma 21. Let π be a representation of a finite group G of dimension n. Suppose that the multiplicity of the trivial representation of $\mathrm{Syl}_2(G)$ in $\mathrm{Res}^G_{\mathrm{Syl}_2(G)}\pi$ is non-zero, that is, suppose we have

$$\langle \mathrm{Res}^G_{\mathrm{Syl}_2(G)}\pi, \mathbb{1}_{\mathrm{Syl}_2(G)}\rangle_{\mathrm{Syl}_2(G)} \neq 0.$$

Let $m = \langle \mathrm{Res}^G_{\mathrm{Syl}_2(G)}\pi, \mathbb{1}_{\mathrm{Syl}_2(G)}\rangle_{\mathrm{Syl}_2(G)}$, then we have

$$w_n(\pi) = w_{n-1}(\pi) = \cdots = w_{n-(m-1)}(\pi) = 0.$$

Proof. For convenience, we will assume that

$$\langle \mathrm{Res}^G_{\mathrm{Syl}_2(G)}\pi, \mathbb{1}_{\mathrm{Syl}_2(G)}\rangle_{\mathrm{Syl}_2(G)} = 1.$$

We have

$$\mathrm{Res}^G_{\mathrm{Syl}_2(G)}\pi = \mathbb{1}_{\mathrm{Syl}_2(G)} \oplus \pi'$$

where π' has dimension $n-1$. Our claim is that $w_n(\pi) = 0$. Using the restriction map

$\text{res}^* : H^n(G) \to H^n(\text{Syl}_2(G))$ and the naturality axiom, we have

$$\begin{aligned}\text{res}^*(w_n(\pi)) &= w_n(\text{Res}^G_{\text{Syl}_2(G)}\pi) \\ &= w_n(\mathbb{1}_{\text{Syl}_2(G)} \oplus \pi') \\ &= w_n(\pi') \\ &= 0. \qquad (\text{since } \dim(\pi') = n-1)\end{aligned}$$

■

The above lemma raises the question of finding a complete characterization of partitions λ of 2^k such that the trivial representation of H_k appears as a constituent in $\text{Res}^{S_{2^k}}_{H_k}(S^\lambda)$. This question is of interest for its own sake, and variants of it have been previously studied. For example, the representation theory of 2-Sylow subgroups of symmetric groups has been studied in [Nar17]. In particular, they give a recursive formula (Theorem 5.1) for the multiplicity of any irreducible representation of H_k in the restriction of odd-dimensional irreducible representations of S_{2^k}. One can show using this formula that the restriction of odd-dimensional representations of S_{2^k} to H_k does not contain the trivial representation as constituent. This is applicable to the representation corresponding to the partition $(2^k - 1, 1)$; the restriction of this representation to H_k never contains the trivial representation.

Another minor result in this direction is for S_n when n is a triangular number, that is, $n = \frac{m(m+1)}{2}$ for some integer m. In particular for the "staircase partitions", which are those of the form $(m, m-1, m-2, ..., 2, 1)$, it follows from the recursive Murnaghan-Nakayama rule that the restriction of these representations to the 2-Sylow subgroup has character value equal to 0 on all conjugacy classes except the trivial conjugacy class. Thus we get that the multiplicity of the trivial representation of the 2-Sylow subgroup of S_n in the restriction of representations corresponding to staircase partitions λ is $\frac{\dim(S^\lambda)}{|P_n|}$, after which we can apply Lemma 21.

In [GL19], the problem of finding the decomposition of restrictions of irreducible representations of symmetric groups to p-Sylow subgroups, where $p \geq 3$, has been addressed. The more general question of finding the full decomposition of the restriction of an irreducible representation of S_{2^k} to H_k is also of interest to us, since if we are to find the Stiefel-Whitney classes of irreducible representations of S_{2^k} using our technique of using the detection by the 2-Sylow subgroup, we have as an intermediary step the restriction problem in representation theory.

Stiefel-Whitney classes for representations of H_k

One approach to finding Stiefel-Whitney classes of S_{2^k} would be to first try and determine Stiefel-Whitney classes of irreducible representations of H_k. A formula for the total

Stiefel-Whitney class of an induced representation is given in [FM87] in terms of the norm map in group cohomology. A simplified version of the same formula for the case when the representation is induced from a subgroup of index 2 is given in [Gui10]. For the first Stiefel-Whitney class, the determinant of an induced representation in terms of the transfer map appears in [29.2, [BH06]].

Chern Classes

We have so far dealt with only real representations. As in the real case, one can associate to a complex vector bundle $p : E \to B$ cohomology classes known as *Chern classes* which lie in $H^*(B, \mathbb{Z})$. One can also define Chern classes of complex representations. Chern classes satisfy properties similar to those satisfied by Stiefel-Whitney classes. One can obtain from a complex representation π a real representation $\pi_{\mathbb{R}}$ of twice the dimension called the "realization". The relation between the Chern classes of a complex representation and the Stiefel-Whitney classes of its realization is given by the "coefficient homomorphism". The map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ which sends x to $x \pmod 2$ induces a homomorphism

$$\kappa : H^*(G, \mathbb{Z}) \to H^*(G, \mathbb{Z}/2\mathbb{Z}).$$

From [[MS74], Problem 14-B] we have that κ maps the total Chern class $c(\pi)$ to the total Stiefel-Whitney class $w(\pi_{\mathbb{R}})$. It is natural to ask if one can give, as we did for Stiefel-Whitney classes, a character formula for the Chern classes of a complex representation of a finite group.

Bibliography

[Kal48] Léo Kaloujnine. "La structure des p-groupes de Sylow des groupes symétriques finis". In: *Ann. Sci. Ecole Norm. Sup.* 65.3 (1948), 239–276.

[Mil56] John Milnor. "Construction of Universal Bundles, II." In: *Annals of Mathematics* 63.3 (1956), pp. 430–436.

[Nak60] Minoru Nakaoka. "Decomposition Theorem for Homology Groups of Symmetric Groups". In: *Annals of Mathematics* 71.1 (1960), pp. 16–42.

[Nak61] Minoru Nakaoka. "Homology of the infinite symmetric group". In: *The Annals of Mathematics* 73.2 (1961), 229–257.

[MS74] John Milnor and James Stasheff. *Characteristic classes*. Princeton University Press, 1974.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977.

[FP78] Z. Fiedorowicz and S. Priddy. *Homology of Classical Groups Over Finite Fields and Their Associated Infinite Loop Spaces*. Springer-Verlag Berlin Heidelberg, 1978.

[FM87] William Fulton and Robert MacPherson. "Characteristic classes of direct image bundles for covering maps". In: *Annals of mathematics* 125.1 (1987), pp. 1–92.

[GKT89] J Gunarwardena, B Kahn, and C Thomas. "Stiefel-Whitney classes of real representations of finite groups". In: *Journal of Algebra* 126.2 (1989), pp. 327 –347. ISSN: 0021-8693.

[Ben91] D. J. Benson. *Representations and Cohomology,Volume 2: Cohomology of Groups and Modules*. Cambridge University Press, 1991.

[Eve91] L. Evens. *The Cohomology of Groups*. Oxford mathematical monographs. Clarendon Press, 1991.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.

[PR95] Dipendra Prasad and Dinakar Ramakrishnan. "Lifting orthogonal representations to spin groups and local root numbers". In: *Proceedings of the Indian Academy of Sciences - Mathematical Sciences* 71.3 (1995), pp. 259–267.

[McC00] John McCleary. *A User's Guide to Spectral Sequences*. 2nd ed. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2000.

[Sag01] B. Sagan. *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer New York, 2001. ISBN: 9780387950679.

[AM04] Alejandro Adem and James Milgram. *Cohomology of Finite Groups*. Springer-Verlag Berlin Heidelberg, 2004.

[DF04] David S. Dummit and Richard M. Foote. *Abstract Algebra*. Wiley, 2004.

[BH06] Colin J. Bushnell and Guy Henniart. *The Local Langlands Conjecture for GL(2)*. Grundlehren der mathematischen Wissenschaften. Springer-Verlag Berlin Heidelberg, 2006.

[Gui10] Pierre Guillot. "The computation of Stiefel-Whitney classes". In: *Annales de l'institut Fourier* 60.2 (2010), pp. 565–606.

[Bro12] K.S. Brown. *Cohomology of Groups*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9781468493276.

[Pra15] Amritanshu Prasad. *Representation Theory: A Combinatorial Viewpoint*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2015.

[Nar17] Sridhar Narayanan. *The Representation Theory of 2-Sylow Subgroups of the Symmetric Group*. 2017. arXiv: 1712.02507v3 [math.CO, math.GR].

[Gan19] Jyotirmoy Ganguly. *Spinorial representations of symmetric and alternating groups*. PhD thesis, IISER Pune, 2019.

[GL19] Eugenio Giannelli and Stacey Law. *Sylow branching coefficients for symmetric groups*. 2019. arXiv: 1909.09446 [math.RT].

[JS19] Rohit Joshi and Steven Spallone. *Spinoriality of orthogonal representations of reductive groups*. 2019. arXiv: 1901.06232 [math.RT].

[GS20] Jyotirmoy Ganguly and Steven Spallone. "Spinorial representations of symmetric groups". In: *Journal of Algebra* 544 (2020), pp. 29 –46.

[BD] T. Bröcker and T.tom Dieck. *Representations of Compact Lie Groups*. Vol. 98. Graduate Texts in Mathematics. Springer-Verlag Berlin Heidelberg.

[Hat] Allen Hatcher. *Algebraic Topology*. URL: http://pi.math.cornell.edu/~hatcher/AT/AT.pdf.

[KG] Simon A. King and David J. Green. *Modular Cohomology Rings of some Finite Groups*. URL: https://users.fmi.uni-jena.de/~king/cohomology/nonprimepower/.