Symmetries in Gravity and Supergravity in the light-cone gauge

A thesis
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by

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To my mother, for all her love and support.

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all the principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Certificate

Certified that the work incorporated in the thesis entitled "*Symmetries in gravity and supergravity in the light-cone gauge*", submitted by "*Sucheta Majumdar*" was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

Date : JUNE 04, 2018

Sudarshan Ananth

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Abstract

This thesis examines the symmetries in gravity and supergravity in four dimensions in the light-cone gauge. The initial focus of the thesis is revisiting pure gravity in four dimensions in the light-cone gauge. We describe how the light-cone Hamiltonian for pure gravity can be expressed as a positive definite quadratic form. We also discuss second-order corrections to residual reparametrizations, which leave the light-cone Hamiltonian invariant. We comment on possible links this quadratic form structure might have to hidden symmetries in gravity. This is in light of some recent studies which suggest improved ultraviolet behaviour in pure gravity.

The second part of the thesis examines the symmetries in maximal supergravity theories, which is our key focus. The maximal supergravity theory in four dimensions, $\mathcal{N} = 8$ supergravity, has excellent ultraviolet properties, not all of which can be traced back to the known symmetries in the theory. We first study the symmetries of $\mathcal{N} = 8$ supergravity in the light-cone superspace. We then argue that the theory possesses a larger symmetry than previously believed. The proof involves dimensional reduction of the theory to three dimensions, a field redefinition in $d = 3$ and oxidation back to $d = 4$. Finally, we extend our analysis to $d = 11$ to argue that there is a hidden exceptional symmetry in eleven-dimensional supergravity. We explain how the exceptional symmetries in these theories are as fundamental as supersymmetry itself.

Publications

The thesis is primarily based on the following publications by the author

- S. Ananth, L. Brink and S. Majumdar, *E_8 in $\mathcal{N} = 8$ supergravity in four dimensions,* **JHEP 1801 (2018) 024**

- S. Ananth, A. Kar, S. Majumdar and N. Shah, *Deriving spin-1 quartic interaction vertices from closure of the Poincaré algebra* , **Nucl. Phys. B926 (2018) 11**

- S. Ananth, L. Brink, S. Majumdar, M. Mali and N. Shah, *Gravitation and Quadratic forms,* **JHEP 1703 (2017) 169**

- S. Ananth, L. Brink and S. Majumdar, *Exceptional versus superPoincarè algebra as the defining symmetry of maximal supergravity* , **JHEP 1603 (2016) 051**

and the following peer-reviewed essay.

- S. Ananth, L. Brink and S. Majumdar, *Maximal supergravity and the quest for finiteness*, to appear in a special edition of **IJMPD** - Gravity Research Foundation (2018)

Contents

Chapter 1

Introduction

1.1 Quantum field theory and the Standard Model

The Standard Model of Particle Physics which describes the electromagnetic, weak and strong interactions between elementary particles, is the most successful quantum field theory known to date. Experiments performed in various particle accelerators and the discovery of the Higgs boson at the LHC validate the predictions made by this model to a high degree of precision. Nevertheless, the Standard Model in its present form cannot be the complete theory of all the fundamental interactions in Nature, since it does not include gravity. Instead, the Standard Model should be viewed as a low energy effective theory of an underlying bigger theory that also accounts for the quantum behaviour of gravity at the Planck scale.

At the classical level, Einstein's theory of general relativity provides an excellent description of gravity as a force originating from the geometry of the spacetime. The recent experimental detection of gravitational waves by LIGO, almost a century after their prediction by general relativity, further consolidates our faith in this classical picture of gravity. However, any attempts to reconcile general relativity with the laws of quantum mechanics within the framework of quantum field theory are met with intractable ultraviolet divergences. These divergences stem from positive length dimension of the coupling constant in the theory of gravity. The coupling constant, which is related to the Newton's gravitational constant G_N, has a length dimension of one. Therefore in a perturbative expansion, there is an infinite series of interaction terms where the dimensions of the fields can be compensated by an appropriate power of the coupling constant, rendering the theory non-renormalizable. In [1], explicit loop calculations were performed to show that the theory diverges at two-loop order and that there exists a non-vanishing counterterm in the gravity Lagrangian at this loop order.

Over the past few decades, there have been many developments in search of a quantum theory of gravity. The most promising approaches include string theory, higher spin theories, higher derivative gravity and supergravity. In this context symmetries play a central role, since divergence cancellations in quantum field theories can most often be traced back to some symmetry in the theory. For example, loop diagrams in QED, which appear to be quadratically divergent from naive power counting, turn out to diverge only logarithmically due to the gauge symmetry of the photons. Also, from Noether's theorem, we know that symmetries lead to conservation laws, which in turn put constraints on the physical observables of the theory. In quantum field theories, symmetries can thus constrain and/or rule out possible counterterms in the Lagrangian. Therefore in order to better understand the quantum nature of gravity and fix the divergences, a precise knowledge of all the symmetries of the theory is indispensable.

In this regard supersymmetry, which links a boson to a fermion, offers exciting prospects for a theory of gravity with better ultraviolet properties, as it brings about systematic divergence cancellations in quantum field theories.

1.2 Supersymmetry and ultraviolet divergences

The idea of supersymmetry has its origin in elementary particle Physics as one of the possible resolutions to the "hierarchy problem" in the Standard Model, pertaining to loop corrections to the mass of the Higgs boson. The mass of the scalar Higgs boson, unlike that of the gauge bosons and fermions in the Standard Model, is not protected from quantum corrections as it does not possess any gauge or chiral symmetry. Hence the Higgs boson will receive enormous loop contributions to its mass from each particle that couples to it. This is in contradiction with the observed mass of Higgs boson, $m_H \sim 125$ GeV. The two types of corrections that contribute to the mass are [2]

- Contribution from a fermion of mass m_f coupled to the Higgs boson with a coupling λ_f :

$$\Delta m_H^2 \;\sim\; -\,\frac{\lambda_f^2}{8\pi^2}\,\Lambda_{UV}^2 \;, \tag{1.1}$$

- Contribution from a boson (say, a scalar) of mass m_b coupled to the Higgs boson with a coupling λ_b:

$$\Delta m_H^2 \ \sim \ + \frac{\lambda_b}{16\pi^2} \, \Lambda_{UV}^2 \ . \tag{1.2}$$

Supersymmetry proposes a resolution to this problem in a special way so that these corrections cancel due to the relative minus sign between a fermionic and a bosonic loop. For each particle in the Standard Model, supersymmetry predicts a superpartner, which differs from it by spin one-half. As a result of this, the contribution to m_H from any particle cancels with that coming from its superpartner.

Supersymmetry transformations converts a fermion into a boson and vice-versa

$$\mathcal{Q}\,|Fermion\rangle = |Boson\rangle \ \ ; \qquad \mathcal{Q}\,|Boson\rangle = |Fermion\rangle \ .$$

Since it changes the spin of the particle by one-half, supersymmetry is fermionic in nature, as opposed to the bosonic symmetries (Poincaré symmetry, gauge symmetry etc.) usually encountered in Physics.

The first example of a supersymmetric interacting field theory was the Wess-Zumino model in four dimensions involving a complex scalar and a fermion [3], where the $\mathcal{N}$ = 1 supersymmetry in the theory renders it renormalizable. Similarly, supersymmetric gauge theories are found to be remarkably well-behaved in the ultraviolet regime[1]. In fact, the maximally supersymmetric gauge theory in four dimensions, $\mathcal{N} = 4$ superYang-Mills theory is an ultraviolet finite theory to all orders in perturbation theory [4, 5]. This motivates us to study supersymmetric theories of gravity, that have softer divergences in the ultraviolet than pure gravity.

The theory that is of most interest to us is the maximally supersymmetric theory of gravity in four dimensions, $\mathcal{N} = 8$ supergravity. As the name suggests, this theory contains the maximum amount of supersymmetry allowed in four dimensions. Owing to its high degree of symmetry, this theory has the best ultraviolet behaviour of any field theoretic extension of Einstein's gravity in $d = 4$. Using some advanced techniques developed in the recent years for computing higher loop amplitudes, the four-graviton scattering amplitude for this theory was shown to be finite up to four

[1]It is important to mention that there has been no experimental evidence for supersymmetry so far. Nevertheless, supersymmetry as a theoretical construct offers useful insights into the study of ultraviolet divergences in quantum field theories

loops [6]. Some recent studies based on on-shell superspace power-counting even predict that the onset of divergences in this theory can be delayed to seven-loop order [7]. Much of this improved behaviour can be traced back to the symmetries that this theory possesses, namely Poincaré symmetry, maximal supersymmetry, $SU(8)$ R-symmetry and an exceptional symmetry, $E_{7(7)}$. Although at this point we cannot definitively answer the question whether the $\mathcal{N} = 8$ theory is finite or not, counterterm arguments based on $E_{7(7)}$ and maximal supersymmetry suggest that the perturbation series must be more finite than expected [8]. However, these higher loop calculations show that there are unexpected cancellations between the divergent pieces [6], which are not fully explained by all the known symmetries of the theory. It was also observed that up to four-loop order the degree of divergence in the $\mathcal{N} = 8$ theory is no worse than that of $\mathcal{N} = 4$ superYang-Mills theory, which is a conformally invariant and an ultraviolet finite theory. These observations suggest that the remarkable ultraviolet properties of this theory might be a manifestation of some hidden symmetry. It is crucial to understand what this symmetry could be and how this symmetry appears in the theory.

At this point, we pose two important questions which constitute the primary theme of this thesis.

- Some recent developments indicate that pure gravity is better behaved[2] than expected in the ultraviolet regime [6]. Also supersymmetry alone cannot explain all the nice properties of the $\mathcal{N} = 8$ theory. This hints at an intriguing possibility that these properties may be attributed to some hidden symmetry or simple structures in pure gravity itself [10]. Given these facts, we ask :

  Are there hidden symmetries in the theory of gravity in four dimensions, which carries over to $\mathcal{N} = 8$ supergravity and explains its ultraviolet properties?

- Since the known symmetries in the $\mathcal{N} = 8$ theory can only partially account for all the improved ultraviolet properties, we raise the following question :

  Is there a larger hidden symmetry in $\mathcal{N} = 8$ supergravity, which could be responsible for the improved ultraviolet behaviour?

To address the first question, we look for simple structures in the Hamiltonian for

[2] For instance, gravity amplitudes fall off as $\frac{1}{z^2}$, as opposed to a $\frac{1}{z}$ fall-off for gauge theory amplitudes in on-shell recursion relations as complexified momentum $p(z)$ goes to infinity [9].

pure gravity, similar to the ones found in Yang-Mills theory. This idea is motivated by the perturbative links between gravity and Yang-Mills amplitudes, namely the KLT relations [11] and the color-kinematic duality [12] which roughly state that

$$\text{(Gravity)} \;=\; \text{(Yang-Mills)} \;\times\; \text{(Yang-Mills)}\;. \tag{1.3}$$

In order to answer the second question, we turn our attention to the exceptional symmetry already present in $\mathcal{N}=8$ supergravity. We start with this $E_{7(7)}$ symmetry in four dimensions and examine if this symmetry can be enhanced to a larger symmetry. To motivate this idea, we consider the more general case of maximal supergravity theories in different spacetime dimensions, where such exceptional symmetries occur in abundance.

1.3 Maximal supergravity and exceptional symmetries

The largest spacetime dimensions allowed for supergravity involving fields of spin two or lower, is eleven [13]. Maximal supergravity theories in all dimensions, $d < 11$ are descendants of the eleven-dimensional $\mathcal{N}=1$ supergravity theory. Since there is only one theory in eleven dimensions, maximal supergravity in any particular dimension is unique. This eleven-dimensional theory first formulated in [14] is believed to be the effective description of the M-theory in the low energy limit [15]. The large Poincaré group in eleven dimensions on reduction to lower dimensions gives rise to interesting internal symmetries, called R-symmetry in the dimensionally reduced supergravity theories. In addition to this, the Lagrangian for the scalar fields in the theories exhibits a non-linear sigma-model symmetry. Taking into account the sigma-model symmetry of the scalars, the internal R-symmetry groups can be enhanced to Lie algebras of the exceptional type [16].

$\mathcal{N}=8$ supergravity in four dimensions is one such theory in a series of maximal supergravity theories obtained by dimensional reduction from the eleven-dimensional parent. In any given spacetime dimension d, the maximal supergravity theory possesses an $E_{11-d(11-d)}$ exceptional symmetry[3]. Some recent studies suggest that on reduction to two or lower dimensions, these symmetries can be further extended to

[3]Here the numeral in the brackets refers to the number of non-compact generators minus the number of compact generators of the exceptional Lie group, which also happens to be $11-d$.

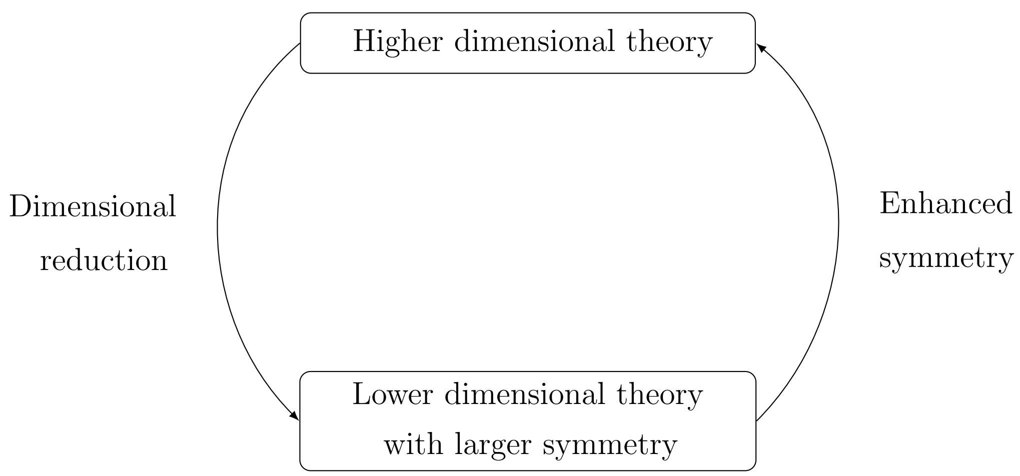
infinite-dimensional exceptional groups, $E_9\,, E_{10}$ and E_{11}, which may shed light on the very origins of space [17, 18].

$$
\begin{array}{lcl}
(\mathcal{N}=1, \mathrm{d}=11)\ \ \text{supergravity} & & \\
\qquad\qquad\quad\big\downarrow & & \\
(\mathcal{N}=8, \mathrm{d}=5)\ \text{supergravity} & \rightarrow & E_{6(6)} \\
(\mathcal{N}=8, \mathrm{d}=4)\ \ \text{supergravity} & \rightarrow & E_{7(7)} \\
(\mathcal{N}=16, \mathrm{d}=3)\ \ \text{supergravity} & \rightarrow & E_{8(8)} \\
\qquad\qquad\quad\downarrow & & \\
(\mathcal{N}=32, \mathrm{d}=1)\ \ \text{supergravity} & \rightarrow & E_{10}/E_{11}\ (?)
\end{array}
$$

A striking feature of these dimensionally reduced supersymmetric theories is that they retain a lot of information about the higher dimensional parent theories. We notice that the exceptional symmetries become larger as we go down in dimensions and ask the following question : Are these enhanced symmetries indicative of a larger symmetry in the parent theory itself?

This is the central idea of this thesis : to realize a larger symmetry, that originally manifests itself in a lower-dimensional theory, in the higher-dimensional parent theory as well. We start with a theory in d-dimensions and reduce it to a lower dimensional theory, where there is a larger symmetry present (as shown in the flowchart above). The next step is to carefully lift this lower dimensional theory back to d-dimensions such that the symmetry is not affected.

Our plan of action is depicted schematically in the diagram below.



We first implement this idea to enhance the $E_{7(7)}$ symmetry in four dimensions to an $E_{8(8)}$ symmetry, which originally appears in the three-dimensional theory. We then extend our method to look for similar symmetry enhancement in dimensions other than four.

1.4 Outline of the thesis

Here, we present an overview of the results discussed in the thesis. A suitable choice of the light-cone gauge brings out many nice features of a theory, which may be difficult to appreciate in the covariant formalism. While working in the light-cone gauge, we can describe the theory using the physical fields only. Thus, there is no need to introduce ghost or auxiliary fields. Moreover, many scattering amplitude structures appear in the light-cone field theories at the level of the Lagrangian, for example the KLT relations and MHV amplitude structures [19, 20]. In the light-cone field theories, the symmetries are non-linearly realized on the physical fields, which makes this formulation ideal to look for new structures or symmetries, as we will demonstrate with several examples.

In chapter 2, we review the basics of light-cone formulation of Yang-Mills theory and gravity. We discuss the light-cone representation of the Poincaré algebra in four dimensions. We also briefly mention how the non-linear realization of the Poincaré symmetry offers a unique framework for deriving interacting theories in the light-cone gauge.

In chapter 3, we discuss some interesting aspects of pure gravity in $d = 4$ when studied in the light-cone gauge. We describe how the light-cone Hamiltonian for pure gravity, in four dimensions, can be expressed as a positive definite quadratic form to second order in the coupling constant in analogy to pure Yang-Mills theory in four dimensions. We also present the corrections to residual reparametrization transformations to second order in the coupling constant.

In chapter 4, we first introduce the light-cone superspace formalism, which make the supersymmetry of the theory manifest . This superspace formalism adapted to $\mathcal{N} = 4$ superYang-Mills theory was used to prove the all-order finiteness of this theory [4, 5]. Next, we formulate the $\mathcal{N} = 8$ supergravity theory in $d = 4$ in the light-cone superspace and study its symmetries.

In chapter 5, we argue that $\mathcal{N} = 8$ supergravity shows signs of an $E_{8(8)}$ symmetry enhanced from the original $E_{7(7)}$ symmetry. The proof involves a series of three steps, including dimensional reduction to d=3, a field redefinition in d=3 and oxidation back to d=4 preserving the $E_{8(8)}$ symmetry. We comment on some possible implications this enhanced symmetry might have for the ultraviolet behaviour of the $\mathcal{N} = 8$ theory.

In the next chapter, we generalize this idea for symmetry enhancement to uncover an $E_{7(7)}$ symmetry in eleven-dimensional supergravity. We show that we must choose between maximal supersymmetry and $E_{7(7)}$ symmetry to "oxidize" the $(\mathcal{N} = 8, d = 4)$ supergravity to $d = 11$. Thus, there exist two equivalent formulations of the $d = 11$ theory with one of the two symmetries manifest and these must be related by a field redefinition. This analysis can offer insights into the origin of the exceptional symmetries in maximal supergravity.

We summarize our results in the last chapter. We conclude with some remarks about our findings and frame some questions that need to be addressed in future.

Chapter 2

Field theories in the light-cone gauge

We present a short review of Yang-Mills theory and gravity in four dimensions in the light-cone gauge. There are two different approaches to constructing light-cone field theories : gauge-fixing the covariant Lagrangian to the light-cone gauge and deriving interacting field theories from symmetry principles. Both the approaches greatly rely on the fact that the theory can be described using only the physical degrees of freedom. In this chapter, we mainly focus on the first approach for both gravity and Yang-Mills theory. We close the chapter with a brief note on the second approach to deriving Lagrangians, the details of which are presented in appendix A.

2.1 Poincaré algebra in the light-cone gauge

Dirac proposed in his famous paper [21] that for a relativistically invariant theory, any direction within the light-cone can be chosen as the "time" or evolution parameter. In particular, one can choose one of the light-cone directions to be the time, which came to be known as the "light-cone frame" or "infinite momentum frame".

With the metric signature $(-,+,+,+)$, the light-cone coordinates are defined as

$$x^{\pm} = \frac{1}{\sqrt{2}}\,(\,x^0 \pm x^3\,)\ ; \quad x = \frac{1}{\sqrt{2}}\,(\,x_1\,+\,i\,x_2\,)\ ; \quad \bar{x} \;\;=\;\; \frac{1}{\sqrt{2}}\,(\,x_1\,-\,i\,x_2\,)\ , \quad (2.1)$$

and introduce the following light-cone derivatives

$$\partial^{\pm} = \frac{1}{\sqrt{2}}\,(\,-\partial_0 \pm \partial_3\,)\ ; \quad \bar{\partial} = \frac{1}{\sqrt{2}}\,(\,\partial_1\, -\, i\,\partial_2\,)\ ; \quad \partial = \frac{1}{\sqrt{2}}\,(\,\partial_1\, +\, i\,\partial_2\,)\ , \tag{2.2}$$

with

$$\partial^- x^+ = \partial^+ x^- = -1\ ; \quad \partial\bar{x} = \bar{\partial} x = 1\ . \tag{2.3}$$

The coordinate x^+ is taken to be time and the corresponding derivative $\partial_+ = -\partial^-$ becomes the time derivative. The light-cone metric thus takes a completely off-diagonal form

$$\eta_{\mu\nu} \;=\; \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\ . \tag{2.4}$$

Any scalar product $A_\mu B^\mu$ in the light-cone coordinates reads $A\bar{B} + \bar{A}B - A^+B^- - A^-B^+$.

We now consider the Poincaré algebra in covariant notation

$$\begin{aligned} \frac{1}{i}\,[J^{\mu\nu}\,,\,P^\rho] \;&=\; \eta^{\mu\rho}P^\nu \;-\; \eta^{\nu\rho}P^\mu\ , \\ \frac{1}{i}\,[J^{\mu\nu}\,,\,J^{\rho\sigma}] \;&=\; \eta^{\mu\rho}J^{\nu\sigma} \;-\; \eta^{\nu\rho}J^{\mu\sigma} \;-\; \eta^{\nu\sigma}J^{\mu\sigma} \;+\; \eta^{\nu\sigma}J^{\mu\rho}\ . \end{aligned}$$

The momenta P^μ given as

$$P^\mu \;=\; -i\partial^\mu \tag{2.5}$$

satisfy the massless condition $P^2 = 0$. In the light-cone language, we can use this condition to solve for the momentum conjugate to time, P^- in the following way

$$\begin{aligned} P^2 = P_\mu P^\mu \;&=\; (P^+P^- - P\bar{P}) = 0 \\ \Rightarrow\ P^- \;&=\; \frac{P\bar{P}}{P^+}\ . \end{aligned} \tag{2.6}$$

This expression does not involve any square roots, unlike in the covariant formalism where we find

$$P^0 = \pm\sqrt{(P^1)^2 + (P^2)^2 + (P^3)^2}\ . \tag{2.7}$$

Note that for the purpose of this thesis we will consider massless fields only.

The Lorentz generators $J^{\mu\nu}$ are decomposed into an orbital part $L^{\mu\nu}$

$$L^{\mu\nu} \;=\; -i(x^\mu\partial^\nu \;-\; x^\nu\partial^\mu) \tag{2.8}$$

and a spin part $S^{\mu\nu}$, which specifies the correct representation of the Poincaré algebra for a particle of a given spin λ. In case of massless fields, this part of the generator is given in terms of the helicity of the field.

The Poincaré generators in the light-cone coordinates are the momenta [22]

$$\begin{aligned} P^- \;=\; -\,i\,\frac{\partial\bar\partial}{\partial^+} \;=\; -\,P_+ \quad ; \quad P^+ \;=\; -\,i\,\partial^+ \;=\; -\,P_- \\ P \;=\; -\,i\,\partial \quad ; \quad \bar P = -i\,\bar\partial\ . \end{aligned} \tag{2.9}$$

For the Lorentz generators, we first define

$$J^+ \;=\; \frac{J^{+1} \,+\, i\ J^{+2}}{\sqrt{2}}\ ; \quad J^- \;=\; \frac{J^{-1} \,+\, i\ J^{-2}}{\sqrt{2}}\ ; \quad J \;=\; J^{12}\ .$$

The explicit form of the Lorentz generators are given by

$$\begin{aligned} J \;&=\; i\,(x\bar\partial - \bar x\partial - \lambda)\ , \\ J^+ \;&=\; i\,(x\,\partial^+ - x^+\partial)\ , \\ J^{+-} \;&=\; i\,(x^-\partial^+ - x^+\frac{\partial\bar\partial}{\partial^+})\ , \\ J^- \;&=\; i\,(x\frac{\partial\bar\partial}{\partial^+} - x^-\partial - \lambda\frac{\partial}{\partial^+})\ , \\ \bar J^+ \;&=\; (J^+)^*\ , \quad \bar J^- \;=\; (J^-)^*\ . \end{aligned} \tag{2.10}$$

Here λ denotes the helicity of the field the generators act on.

We choose to work on the surface of constant time, $x^+ = 0$ which simplifies our calculations. The Poincaré generators that do not involve time derivatives, ∂^- are called the kinematical generators

$$P^- \ , \ \ P \ , \ \ \bar{P} \ , \ \ J \ , \ \ J^+ \ , \ \ \bar{J}^+ \ \text{and} \ \ J^{+-} \tag{2.11}$$

and those that depend on time derivatives are the dynamical generators

$$P^- \equiv H \ , \ \ J^- \ , \ \ \bar{J}^- \ . \tag{2.12}$$

These generators pick up corrections order by order, in the coupling constant when we include interactions in the theory. The non-linear dependence of these dynamical generators on the fields may seem to be an unnecessary complication in light-cone field theories. On the contrary, this non-linear representation of the Poincaré algebra turns out to be a boon. It leads the way to a very powerful framework for deriving interacting theories in the light-cone gauge just from the symmetries of the theory, as discussed in section 2.4.

Following is the list of the non-vanishing commutators of the Poincaré algebra in the light-cone notation [22]

$$\begin{aligned}
[H, J^{+-}] &= -iH \ , \qquad [H, J^+] = -iP \ , \quad [H, \bar{J}^+] = -i\bar{P} \ , \\
[P^+, J^{+-}] &= iP^+ \ , \qquad [P^+, J^-] = -iP \ , \quad [P^+, \bar{J}^-] = -i\bar{P} \ , \\
[P, \bar{J}^-] &= -iH \ , \qquad [P, \bar{J}^+] = -iP^+ \ , \quad [P, J] = P \ , \\
[\bar{P}, J^-] &= -iH \ , \qquad [\bar{P}, J^+] = -iP^+ \ , \quad [\bar{P}, J] = -\bar{P} \ , \\
[J^-, J^{+-}] &= -iJ^- \ , \qquad [J^-, \bar{J}^+] = iJ^{+-} + J \ , \quad [J^-, J] = J^- \ , \\
[\bar{J}^-, J^{+-}] &= -i\bar{J}^- \ , \qquad [\bar{J}^-, J^+] = iJ^{+-} - J \ , \quad [\bar{J}^-, J] = -\bar{J}^- \ , \\
[J^{+-}, J^+] &= -iJ^+ \ , \qquad [J^{+-}, \bar{J}^+] = -i\bar{J}^+ \ , \\
[J^+, J] &= J^+ \ , \qquad [\bar{J}^+, J] = -\bar{J}^+ \ .
\end{aligned} \tag{2.13}$$

$\frac{1}{\partial^+}$ **operator :**

The $\frac{1}{\partial^+} = -\frac{1}{\partial_-}$ is an artifact of the light-cone frame. We can see from (2.6) that such an operator arises naturally in this choice of coordinates. This operator is formally defined in terms of the Heaviside step function $\epsilon(x^- - x'^-)$. Consider two functions $g(x)$ and $f(x)$ such that

$$\partial_- g(x) = f(x) \ .$$

The $\frac{1}{\partial_-}$ allows us to solve for $g(x)$ in terms of $f(x)$ up to an arbitrary function $h(x)$, which is independent of x^-

$$g(x) = \frac{1}{\partial_-} f(x) \ + \ h(x) = \int_{-\infty}^{\infty} \epsilon(x^- - x'^-) f(x'^-) dx'^- + \ h(x) \ .$$

The function $h(x)$ can be removed using suitable boundary conditions. Thus the $\frac{1}{\partial_-}$ is interpreted as an integral operator, not a differential operator. We can use partial integrations with ∂_- without having to worry about boundary terms. While working in the momentum space, we make use of the well-defined pole prescription for $\frac{1}{p_-}$ following [5].

2.2 Yang-Mills theory in the light-cone gauge

In this section, we discuss non-abelian gauge theories in the light-cone gauge. Yang-Mills theory is at the core of the Standard model, in which case the relevant gauge group is $U(1) \times SU(2) \times SU(3)$. The $U(1)$ corresponds to the electromagnetic force, SU(2) to the weak interactions and the $SU(3)$ is the gauge group for strong interactions. We start with the Lagrangian for pure Yang-Mills theory in four dimensions.

$$\mathcal{S} \ = -\frac{1}{4} \int d^4x \ F^a_{\mu\nu} \ F^{\mu\nu a} \ . \tag{2.14}$$

The anti-symmetric field strength is defined as

$$F^a_{\mu\nu} \ = \ \partial_\mu A^a_\nu \ - \ \partial_\nu A^a_\mu \ + \ g\, f^{abc} \ A^b_\mu A^c_\nu \ , \tag{2.15}$$

the indices $a, b, c, \ldots$ run from 1 to $N^2 - 1$ for the corresponding gauge group $SU(N)$, since the gauge fields transform under the adjoint representation.

The Euler-Lagrange equations of motion for the Yang-Mills action is

$$\partial_\mu F^{\mu\nu a} \; + \; g \; f^{abc} A^b_\mu \; F^{\mu\nu c} \; = \; 0 \; . \tag{2.16}$$

The action is invariant under the gauge transformation

$$A^a \; \rightarrow A'^a_\mu \; = \; A^a_\mu \; + \partial_\mu \Lambda^a \; - i \, g \, f^{abc} \, A^b_\mu \, \Lambda^c \; , \tag{2.17}$$

which allows us to choose one gauge parameter Λ^a. We choose the light-cone gauge by setting

$$A^{+a} \; = \; -A_-{}^a \; = \; 0 \; . \tag{2.18}$$

As a result of this gauge choice, the equations of motion splits into two kinds: constraint relations which do not contain time derivatives and dynamical equations of motion which contain time derivatives. The $\nu = +$ equation in (2.16) becomes such a constraint relation, which can be used to eliminate the A^a_+ component from the theory in favour of the two transverse components A_i, $(i = 1, 2)$

$$A^a_+ \; = \; \frac{\partial_i}{\partial_-} \; A^a_i \; + \; g \, f^{abc} \, \frac{1}{{\partial_-}^2} (A^b_i \, \partial_- \, A^c_i \,) \; . \tag{2.19}$$

Thus we can express the Yang-Mills Lagrangian solely in terms of the two remaining degrees of freedom, A_1 and A_2. We further combine these two components in a helicity basis

$$\begin{aligned} A^a \; &= \; \frac{1}{\sqrt{2}} ({A_1}^a \; + \; i{A_2}^a) \; ; \\ \bar{A}^a \; &= \; \frac{1}{\sqrt{2}} ({A_1}^a \; - \; i{A_2}^a) \; . \end{aligned} \tag{2.20}$$

Under the little group $SO(2)$, these combinations A^a and $\bar{A}^a$ have helicity $+1$ and -1 respectively.

In terms of these helicity states, the Yang-Mills Lagrangian takes the form

$$
\begin{aligned}
\mathcal{L} \;=\; & \bar{A}^a \Box A^a \;-\; 2\,g\,f^{abc}\left(\frac{\partial}{\partial_-}\,\bar{A}^a\,\partial_- A^b\,\bar{A}^c \;+\; \frac{\bar{\partial}}{\partial_-}\,A^a\,\partial_-\,\bar{A}^b\,A^c\right) \\
& -2\,g\,f^{abc}\,f^{ade}\frac{1}{\partial_-}(\partial_-\,A^b\bar{A}^c)\frac{1}{\partial_-}(\partial_-\bar{A}^d\,A^e)\ . \qquad (2.21)
\end{aligned}
$$

The same Lagrangian can alternatively be derived starting from the closure of the Poincaré algebra in four dimensions, as we will discuss in section 2.4.

2.3 Gravitation in the light-cone gauge

In this section, we discuss the light-cone formulation of gravity in four dimensions. We present a detailed account of how we obtain the light-cone Lagrangian for pure gravity by gauge-fixing the Einstein-Hilbert action.

On a Minkowski background with vanishing cosmological constant Λ, the Einstein-Hilbert action reads

$$
S_{EH} = \int d^4x\;\mathcal{L} \;=\; \frac{1}{2\,\kappa^2}\int d^4x\;\sqrt{-g}\;\mathcal{R}\ , \qquad (2.22)
$$

where g is the determinant of the metric. $\mathcal{R}$ is the curvature scalar and the coupling constant is derived from the Newton's gravitational constant, $\kappa = \sqrt{8\pi G}$.

Einstein's field equations derived from the action principle reads

$$
\mathcal{R}_{\mu\nu} \;-\; \frac{1}{2}\,g_{\mu\nu}\mathcal{R} \;=\; 0\ . \qquad (2.23)
$$

The Einstein-Hilbert action enjoys a symmetry under general coordinate transformations

$$
x^\mu \rightarrow x'^\mu \;=\; x^\mu \;+\; \alpha^\mu(x)\ , \qquad (2.24)
$$

which allows us to make four gauge choices corresponding to the 4-vector, α^μ.

We use the light-cone gauge, where out of the four gauge choices we impose the following *three* [23, 24]

$$
g_{--} \;=\; g_{-i} \;=\; 0 \quad ,\; i = 1, 2\ . \qquad (2.25)
$$

These choices are made keeping in mind that $\eta_{--} = \eta_{-i} = 0$. We parametrize the metric as

$$\begin{aligned} g_{+-} &= -\,e^{\phi}\ , \\ g_{i\,j} &= e^{\psi}\,\gamma_{ij}\ . \end{aligned} \tag{2.26}$$

ϕ, ψ are real parameters and γ^{ij} is a 2×2 real, symmetric matrix with unit determinant. Just like in the case of Yang-Mills theory, Einstein's field equations (2.23) also split into constraint relations and dynamical equations of motion. The $\mu\!=\!\nu\!=\!-$ equation becomes a constraint equation which reads

$$2\,\partial_-\phi\,\partial_-\psi\,-\,2\,\partial_-^2\psi\,-\,(\partial_-\psi)^2\,+\,\frac{1}{2}\partial_-\gamma^{ij}\,\partial_-\gamma_{ij}\,=\,0\ . \tag{2.27}$$

At this point we make our *fourth* gauge choice

$$\phi\,=\,\frac{\psi}{2}\ , \tag{2.28}$$

which allows us to solve for ψ in (2.27)

$$\psi\,=\,\frac{1}{4}\,\frac{1}{\partial_-^2}\,(\partial_-\gamma^{ij}\,\partial_-\gamma_{ij})\ . \tag{2.29}$$

The other constraint relations help us eliminate some of the remaining metric components. For example, $\mu = i,\ \nu = -$ in (2.23) gives

$$\begin{aligned} g^{-i} \;&=\; \mathrm{e}^{-\,\phi}\,\frac{1}{\partial_-}\Bigg[\gamma^{ij}\,\mathrm{e}^{\phi\,-\,2\,\psi}\,\frac{1}{\partial_-}\,\Big\{\,\mathrm{e}^{\psi}\,\Big(\,\frac{1}{2}\,\partial_-\,\gamma^{kl}\,\partial_j\,\gamma_{kl}\,-\,\partial_-\,\partial_j\,\phi \\ &\qquad -\,\partial_-\,\partial_j\,\psi\,+\,\partial_j\phi\,\partial_-\,\psi\,\Big)\,+\,\partial_l\,\Big(\mathrm{e}^{\psi}\,\gamma^{kl}\,\partial_-\,\gamma_{jk}\Big)\,\Big\}\Bigg]\ . \end{aligned} \tag{2.30}$$

Thus we have expressed all the non-zero components of the metric in terms of γ_{ij}, which has two degrees of freedom.

All the metric components are now substituted into the Einstein-Hilbert action to obtain [24]

$$\begin{aligned} S \;=\; &\frac{1}{2\kappa^2}\int d^4x\;e^{\psi}\left(2\,\partial_+\partial_-\phi\,+\,\partial_+\partial_-\psi-\frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\ &-e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi+\frac{1}{2}\partial_i\phi\partial_j\phi-\partial_i\phi\partial_j\psi-\frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl}+\frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\ &-\frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\ , \end{aligned} \tag{2.31}$$

where

$$R_i \equiv e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})\ .$$

This is the closed form expression for the action for gravity in the light-cone gauge, in terms of the two physical degrees of freedom in the theory. In the next chapter, we present a perturbative expansion of the above closed form expression and discuss some interesting properties of the resulting Hamiltonian.

2.4 Deriving interacting theories from symmetries

It is evident from our earlier analysis of the Yang-Mills theory and gravity that Poincaré invariance is not manifest in the light-cone gauge. Thus in light-cone field theories one needs to explicitly check invariance under the Poincaré group. However, we can use this fact in our favour and devise a method to derive interaction vertices in light-cone field theories that respect Poincaré symmetry. The key point is that the Hamiltonian itself appears as an element of the Poincaré algebra. Constraints obtained from the closure of the Poincaré algebra can thus be used to determine the Hamiltonian. This offers a unique framework for deriving interacting theories from a first-principles approach.

This approach was first present in [22], where cubic interaction vertices for arbitrary spin fields were derived to first order in coupling constant. The action, to cubic order was found to be

$$S = \int d^4x\left(\frac{1}{2}\bar{\phi}\,\Box\,\phi + \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\bar{\phi}(\partial^+)^{\lambda}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi\frac{\bar{\partial}^n}{\partial^{+n}}\phi\right] + c.c.\right), \quad (2.32)$$

for even integer spin λ and

$$S = \int d^4x\left(\frac{1}{2}\bar{\phi}^a\Box\phi^a + \alpha f^{abc}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\bar{\phi}^a(\partial^+)^{\lambda}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi^b\frac{\bar{\partial}^n}{\partial^{+n}}\phi^c\right] + c.c.\right), \quad (2.33)$$

for odd integer spin λ.

Here ϕ and $\bar{\phi}$ represent the positive and negative helicity components of a field of arbitrary spin λ. At cubic order,we notice that the algebra enforces the introduction of an antisymmetric constant f^{abc} in case of odd spin fields. This approach was then applied to derive cubic interactions in supersymmetric field theories in the light-cone gauge in [25]. In appendix A, we demonstrate how this framework when extended to the quartic order for the specific case of spin-1 fields naturally leads to the emergence of a gauge group in the theory [26], which serves as a proof of concept for this powerful approach. We describe an entire method, where we start with an ansatz for the interaction vertices and fix the ansatz using just the closure of the Poincaré algebra. For the case of spin-2 fields, this framework yields the light-cone Lagrangian for pure gravity which matches exactly with the one obtained previously (3.9) by gauge-fixing the Einstein-Hilbert action.

This approach when adapted to curved spacetimes, could potentially lead to a Lagrangian formulation for arbitrary higher spin fields, where we expect to uncover the higher spin symmetry as a consequence of closure of the isometry algebra. A similar endeavour in six spacetime dimensions might give us a better handle on the elusive $\mathcal{N} = (2,0)$ superYang-Mills theory [27]. Although very exciting, these topics are outside the scope of this thesis.

Chapter 3

Perturbative gravity, in $d = 4$, in the light-cone gauge

The material presented here is primarily based on work done by the author in [28].

In this chapter, we present a perturbative expansion of gravity in the light-cone gauge. We then discuss some properties of the light-cone Hamiltonian for gravity to second order in perturbation theory. We conclude the chapter with some brief remarks about our results.

3.1 Perturbative expansion

We start with the light-cone action for gravity obtained in the previous chapter.

$$\begin{aligned} S \;=\; & \frac{1}{2\kappa^2}\int d^4x\; e^{\psi}\left(2\,\partial_+\partial_-\phi \,+\, \partial_+\partial_-\psi - \frac{1}{2}\,\partial_+\gamma^{ij}\partial_-\gamma_{ij}\right) \\ & -e^{\phi}\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right) \\ & -\frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-}R_j\ , \end{aligned} \tag{3.1}$$

where

$$R_i \;\equiv\; e^{\psi}\left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi\right) + \partial_k(e^{\psi}\,\gamma^{jk}\partial_-\gamma_{ij})\ .$$

We now consider a perturbative expansion of the above closed form result. The matrix γ_{ij} is parametrized as [24]

$$\gamma_{ij} \;=\; (e^{H})_{ij}\ , \tag{3.2}$$

where H is a traceless matrix since $det\,(\gamma_{ij}) = 1$. We make the choice

$$H \;=\; \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & -h_{11} \end{pmatrix} \;. \tag{3.3}$$

We define the following linear combinations of h_{11} and h_{12}

$$h \;=\; \frac{(h_{11} + i\,h_{12})}{\sqrt{2}} \;, \qquad \bar{h} \;=\; \frac{(h_{11} - i\,h_{12})}{\sqrt{2}} \;, \tag{3.4}$$

which correspond to the positive and negative helicity states of a graviton respectively. From (2.29), ψ now takes the form

$$\psi \;=\; -\frac{1}{\partial_-^2}(\partial_- h\,\partial_- \bar{h}) \;+\; \mathcal{O}(h^4) \;. \tag{3.5}$$

The fields are rescaled by

$$h \;\rightarrow\; \frac{h}{\kappa} \;. \tag{3.6}$$

The Lagrangian can be now perturbatively expanded around the flat spacetime as

$$\mathcal{L} \;\sim\; \mathcal{L}^{(0)} \;+\; \mathcal{L}^{(\kappa)} \;+\; \mathcal{L}^{(\kappa^2)} \;+\; \cdots \tag{3.7}$$

Unlike in case of Yang-Mills theory where the perturbation series terminates at second order in the coupling constant, the gravity Lagrangian is an infinite series of interaction terms where the increasing power of the dimensionful coupling constant κ appropriately compensates for the dimensions of the fields at each order.

The Lagrangian (density) at lowest order reads

$$\mathcal{L}^{(0)} \;=\; \frac{1}{2}\,\bar{h}\,\Box\,h \;, \tag{3.8}$$

where the d'Alembertian $\Box \;=\; 2\,(\,\partial\,\bar{\partial} \;-\; \partial_+\,\partial_-\,)$. At order κ, the Lagrangian reads

$$\mathcal{L}^{(\kappa)} \;=\; 2\,\kappa\,\bar{h}\,\partial_-^2 \left[-\,h\,\frac{\bar{\partial}^2}{\partial_-^2}h \;+\; \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right] \;+\; \text{complex conjugate} \;, \tag{3.9}$$

where the d'Alembertian $\Box \;=\; 2\,(\,\partial\,\bar{\partial} \;-\; \partial_+\,\partial_-\,)$. The Lagrangian at order κ^2 was first presented in [24] while the order κ^3 results were derived in [29].

3.2 The Hamiltonian - first order in the coupling constant

In this section, we discuss some properties of light-cone gravity to first order in the coupling constant. We shall discuss similar results for light-cone gravity at order κ^2 in the next section.

3.2.1 Quadratic form

With x^+ as the time coordinate, the conjugate momenta are given as

$$\pi \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+ h)}\;;\quad \bar{\phi} \;=\; \frac{\delta\mathcal{L}}{\delta(\partial_+ \bar{h})}\;. \tag{3.10}$$

The corresponding Hamiltonian (density) is defined as

$$\mathcal{H} \;=\; \pi\,\partial_+ h \;+\; \bar{\phi}\,\partial_+\bar{h} \;-\; \mathcal{L}\;. \tag{3.11}$$

When expanded to order κ, the Hamiltonian

$$\begin{aligned}\mathcal{H}^{(\kappa)} \;&=\partial\bar{h}\;\bar{\partial}h \;+2\;\kappa\;\bar{\partial}h\;\frac{1}{\partial_-^2}\;\left(\frac{\bar{\partial}}{\partial_-}h\;\partial_-^3\bar{h}\;-\;h\;\partial_-^2\bar{\partial}\bar{h}\right)\\ &-2\kappa\;\partial\bar{h}\left(\frac{\partial}{\partial_-}\bar{h}\;\partial_-^3 h\;-\;\bar{h}\;\partial_-^2\partial h\right)\;,\end{aligned} \tag{3.12}$$

after some simplifications can be be written in a compact form [30]

$$\mathcal{H} \;=\; \int d^3x\;\;\mathcal{D}\bar{h}\;\bar{\mathcal{D}}h \tag{3.13}$$

with

$$\mathcal{D}\bar{h} \;=\; \partial\bar{h} \;+\; 2\kappa\,\frac{1}{\partial_-^2}\;\big(\frac{\bar{\partial}}{\partial_-}h\,\partial_-^3\bar{h}\;-\;h\,\partial_-^2\bar{\partial}\bar{h}\big)\;, \tag{3.14}$$

where $\bar{\mathcal{D}}h$ is just the complex conjugate of the expression above.

3.2.2 Residual reparametrization invariance

After fixing the gauge to the light-cone gauge and eliminating all the unphysical degrees of freedom, there remains an infinitesimal reparametrization invariance in the theory. In [30], the framework discussed in Appendix A, was extended to derive all possible counterterms that can be added to the light-cone Hamiltonian. Such an analysis requires precise knowledge of all the symmetries of the theory. The residual reparametrization symmetry was found to be crucial in classifying all the counterterms and choosing the correct ones that satisfy this invariance. Such a reparametrization invariance when correctly taken into account, reproduces the well-known two-loop counterterm present in the theory of gravity [1].

We now study the invariance of the light-cone Hamiltonian under these reparametrizations[1],

$$x \to x \;+\; \xi(\bar{x}) \;, \qquad \bar{x} \to \bar{x} \;+\; \bar{\xi}(x) \;. \tag{3.15}$$

From the transformation of the metric under the shift in coordinates given above, we can find how h and $\bar{h}$ transform to lowest order in κ [30]

$$\delta h \;=\; \frac{1}{2\kappa}\partial\xi \;+\; \xi\bar{\partial}h \;+\; \bar{\xi}\partial h \;, \tag{3.16}$$

$$\delta \bar{h} \;=\; \frac{1}{2\kappa}\bar{\partial}\bar{\xi} \;+\; \xi\bar{\partial}\bar{h} \;+\; \bar{\xi}\partial\bar{h} \;, \tag{3.17}$$

where the parameter ξ satisfies the following constraints

$$\partial_-\xi \;=\; 0 \;, \qquad \bar{\partial}\,\xi \;=\; 0 \;. \tag{3.18}$$

The variation of the Hamiltonian to the zeroth order in the coupling constant is given by[2]

$$\delta\mathcal{H}^{(\kappa^0)} \;=\; \delta^{\kappa^0}\,(\partial\bar{h}\,\bar{\partial}h) + 2\kappa\,\delta^{\kappa^{-1}}\,\left\{\,\bar{h}\,\partial_-{}^2\left(h\frac{\bar{\partial}^2}{\partial_-{}^2}h \;-\; \frac{\bar{\partial}}{\partial_-}h\frac{\bar{\partial}}{\partial_-}h\right) \;+\; \text{c.c.}\,\right\} \;. \tag{3.19}$$

The variation of the first term in (3.19) gives

$$-\,\partial\xi\;\bar{h}\;\bar{\partial}^2 h \;-\; \bar{\partial}\bar{\xi}\;h\;\partial^2\bar{h} \;, \tag{3.20}$$

whereas the contribution from the second term in (3.19) and its complex conjugate is exactly equal and opposite to the terms above, proving that

$$\delta\mathcal{H}^{(\kappa^0)} \;=\; 0 \;. \tag{3.21}$$

[1]Here we consider only a special class of reparametrizations for simplicity. However, our analysis holds for any generic form of residual reparametrizations as well.

[2]We use the notation δ^{κ^0} to denote variation of the Hamiltonian at order κ^0.

Thus the residual reparametrizations in (3.15) leave the Hamiltonian invariant to order κ^0. We also note that the derivative, $\mathcal{D}h$ introduced in (3.14) transforms "covariantly"

$$\delta(\bar{\mathcal{D}}h) \,=\, (\xi\bar{\partial} + \,\bar{\xi}\partial\,)\,\bar{\mathcal{D}}\,h\;, \tag{3.22}$$

which is in agreement with a similar analysis for Yang-Mills theories done in [**?**].

3.3 The Hamiltonian - second order in the coupling constant

We now move on to the next order in perturbation theory and examine if the theory still exhibits some nice features similar to the ones described in the last section. At the next order in the coupling constant, we find that time derivatives appear in the Lagrangian, which can be removed by suitable field redefinitions as shown in [24, 31]. The Hamiltonian corresponding to the field-redefined Lagrangian to order κ^2 reads [24]

$$\begin{aligned}
\mathcal{H} \;=\;\; & \partial\bar{h}\,\bar{\partial}h \,-\, 2\,\kappa\,\bar{h}\,\partial_-^2\left\{-\,h\,\frac{\bar{\partial}^2}{\partial_-^2}\;h \,+\, \frac{\bar{\partial}}{\partial_-}h\,\frac{\bar{\partial}}{\partial_-}h\right\} \,-\, 2\,\kappa\,h\,\partial_-^2\left\{-\,\bar{h}\,\frac{\partial^2}{\partial_-^2}\;\bar{h} \,+\, \frac{\partial}{\partial_-}\bar{h}\,\frac{\partial}{\partial_-}\bar{h}\right\} \\
& -\,4\kappa^2\left\{-\,2\,\frac{1}{\partial_-^2}\left(\frac{\bar{\partial}}{\partial_-}\,h\,\partial_-^3\,\bar{h} \,-\, h\,\partial_-^2\,\bar{\partial}\bar{h}\right)\frac{1}{\partial_-^2}\left(\frac{\partial}{\partial_-}\,\bar{h}\,\partial_-^3\,h \,-\, \bar{h}\,\partial_-^2\,\partial h\right)\right. \\
& +\frac{1}{\partial_-^2}(\bar{\partial}h\,\partial_-^2\,\bar{h} \,-\, \partial_-h\,\partial_-\bar{\partial}\bar{h})\,\frac{1}{\partial_-^2}\,(\partial\bar{h}\,\partial_-^2\,h \,-\, \partial_-\bar{h}\,\partial_-\partial h) \,-\, 3\,\frac{1}{\partial_-}(\bar{\partial}h\,\partial_-\bar{h})\,\frac{1}{\partial_-}\,(\partial_-h\,\partial\bar{h}) \\
& +\,\frac{1}{\partial_-}(\bar{\partial}h\,\partial_-\bar{h} \,-\, \partial_-h\,\bar{\partial}\bar{h})\,\frac{1}{\partial_-}\,(\partial\bar{h}\,\partial_-h \,-\, \partial_-\bar{h}\,\partial h) \,+\, 3\,\frac{1}{\partial_-}\,(\partial_-h\,\partial_-\bar{h})\,\frac{1}{\partial_-}\,(\bar{\partial}h\,\partial\bar{h}) \\
& \left. +\,\left[\frac{1}{\partial_-^2}(\partial_-h\,\partial_-\bar{h}) \,-\, h\,\bar{h}\right](\bar{\partial}h\,\partial\bar{h} \,+\, \partial h\,\bar{\partial}\,\bar{h} \,-\, \partial_-h\,\frac{\partial\,\bar{\partial}}{\partial_-}\,\bar{h} \,-\, \partial_-\bar{h}\,\frac{\partial\,\bar{\partial}}{\partial_-}\,h)\right\}\;.
\end{aligned} \tag{3.23}$$

We shall now investigate the properties of this Hamiltonian at order κ^2. We discuss how the quadratic form structure and the residual reparametrization invariance work at this order.

3.3.1 Quadratic form

In this section, we demonstrate how the quadratic form structure of the Hamiltonian at order κ (3.13) also extends to the next order. The Hamiltonian in (3.23), to order κ^2 may indeed be expressed as a quadratic form

$$\mathcal{H} \,=\, \int d^3x \;\; \mathcal{D}\bar{h} \; \bar{\mathcal{D}}h \;\; , \tag{3.24}$$

where $\mathcal{D}\bar{h}$ now contains terms of order κ^2, which are explicitly computed below.

Computation of $\mathcal{D}\bar{h}$ at order κ^2

We need to compute contributions to $\mathcal{D}\bar{h}$ from each line of the Hamiltonian. $\mathcal{D}\bar{h}$ is already known to order κ. The product $\mathcal{D}\bar{h}\,(\kappa)\;\bar{\mathcal{D}}h\,(\kappa)$ accounts for one-half of the second line in (3.23). This computation then shows that (the remaining) half of the second line and all the other terms, of order κ^2, in (3.23) may be put together in the form

$$\mathcal{D}\bar{h}\,(\kappa^2)\,\bar{\partial}h \,+\, \partial\bar{h}\,\bar{\mathcal{D}}h\,(\kappa^2)\;. \tag{3.25}$$

After some partial integrations and simple $\frac{1}{\partial_-}$ manipulations, we can categorize all the order κ^2 terms in the Hamiltonian in the following way.

In expression (3.23) :

from line 2

Contribution to $\mathcal{D}\bar{h}$

$$+2\,\kappa^2\,\frac{1}{\partial_-}\{\partial_-^2\,\bar{h}\frac{1}{\partial_-^3}(\partial_-^3\,h\frac{\partial}{\partial_-}\bar{h} - \partial_-^2\,\partial h\bar{h}\,)\,\} \;+\; 2\,\kappa^2\,\frac{1}{\partial_-}\{\,\frac{\partial}{\partial_-^4}(\bar{h}\partial_-^2\,h)\partial_-^3\,\bar{h}\}$$

Extra terms (that cannot be written in the form $X\bar{\partial}h$ or $Y\partial\bar{h}$)

$$-2\,\kappa^2\,h\partial_-^2\,\bar{h}\,\frac{\bar{\partial}}{\partial_-^4}(\partial_-^2\,\partial h\bar{h}) + \text{c.c.}$$

from line 3

Contribution to $\mathcal{D}\bar{h}$

$$-2\,\kappa^2\,\partial_-^2\,\bar{h}\,\frac{1}{\partial_-^4}(\partial_-^2\,h\partial\bar{h} - 2\partial_-\partial h\partial_-\bar{h}\,)$$

Extra terms

$$-4\,\kappa^2\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{\partial}\bar{h})\,\frac{1}{\partial_-^2}(\partial_-\partial h\partial_-\bar{h})$$

from line 4

Contribution to $\mathcal{D}\bar{h}$

$$+2\,\kappa^2\,\partial_-\bar{h}\frac{1}{\partial_-^2}(\partial_- h\partial\bar{h} - 2\partial h\partial_-\bar{h})$$

Extra terms

$$-4\,\kappa^2\,\frac{1}{\partial_-}(\partial_- h\,\bar{\partial}\bar{h})\,\frac{1}{\partial_-}(\partial h\partial_-\bar{h})$$

from line 5

Contribution to $\mathcal{D}\bar{h}$

$$+6\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\,\partial\bar{h}\;\;-\;\;6\,\kappa^2\,\partial_-\bar{h}\,\frac{1}{\partial_-^2}(\partial_- h\partial\bar{h})$$

from line 6

Contribution to $\mathcal{D}\bar{h}$

$$-2\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\partial\bar{h}\;+\;4\kappa^2\,h\bar{h}\,\partial\bar{h}\;+\;4\kappa^2\,\frac{\partial}{\partial_-}\{\partial_-\bar{h}\,(\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h}) - h\bar{h})\,\}$$

Extra terms

$$-4\kappa^2\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\partial h\bar{\partial}\bar{h}$$

The "Extra terms" from each line when combined together yield the desired structures, $X\bar{\partial}h$ or $Y\partial\bar{h}$, which simply add factors of X or Y to $\mathcal{D}\bar{h}$ or $\bar{\mathcal{D}}h$.

Thus, we find that at order κ^2, $\mathcal{D}\bar{h}$ reads

$$
\begin{aligned}
&+2\,\kappa^2\,\frac{1}{\partial_-}\{\partial_-^2\bar{h}\frac{1}{\partial_-^3}(\partial_-^3 h\frac{\partial}{\partial_-}\bar{h}-\partial_-^2\partial h\bar{h}\,)\,\}+2\,\kappa^2\,\frac{1}{\partial_-}\{\,\frac{\partial}{\partial_-^4}(\bar{h}\partial_-^2 h)\partial_-^3\bar{h}\} \\
&-2\,\kappa^2\,\partial_-^2\bar{h}\,\frac{1}{\partial_-^4}(\partial_-^2 h\partial\bar{h}-2\partial_-\partial h\partial_-\bar{h}\,)+2\,\kappa^2\,\partial_-\bar{h}\frac{1}{\partial_-^2}(\partial_- h\partial\bar{h}-2\partial h\partial_-\bar{h}) \\
&+6\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\,\partial\bar{h}-6\,\kappa^2\,\partial_-\bar{h}\,\frac{1}{\partial_-^2}(\partial_- h\partial\bar{h})-2\,\kappa^2\,\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})\partial\bar{h} \\
&+4\kappa^2\,h\bar{h}\,\partial\bar{h}+4\kappa^2\,\frac{\partial}{\partial_-}\{\partial_-\bar{h}\,(\frac{1}{\partial_-^2}(\partial_- h\partial_-\bar{h})-h\bar{h})\,\}+2\kappa^2\,\partial_-^2\bar{h}\frac{1}{\partial_-^4}(\partial_-^2\partial h\bar{h}) \\
&-2\kappa^2\,\partial_-\{\partial_-\bar{h}\frac{1}{\partial_-^2}(\bar{h}\partial h)\}-2\kappa^2\,\partial\{\bar{h}\frac{1}{\partial_-^2}(\partial_-\bar{h}\partial_- h)\}-2\kappa^2\,\partial_-^2\bar{h}\frac{1}{\partial_-^3}(\partial_-\partial h\bar{h}) \\
&+2\kappa^2\partial_-\partial\{\bar{h}\frac{1}{\partial_-^3}(h\partial_-^2\bar{h})\}+2\kappa^2\partial\{\partial_-\bar{h}\frac{1}{\partial_-^3}(\bar{h}\partial_-^2 h)\}+2\kappa^2\partial_-^2\{\bar{h}\frac{1}{\partial_-^3}(\partial_-\bar{h}\partial h)\}\;.
\end{aligned}
\tag{3.26}
$$

Thus we show that the light-cone Hamiltonian for pure gravity in $d=4$ can be expressed as a quadratic form up to order κ^2. This is an interesting results because in four dimensions very few special theories show such quadratic form structures. We shall revisit this point in the last chapter.

3.3.2 Residual reparametrization invariance

We now turn our attention to residual reparametrization invariance of the light-cone Hamiltonian. We find that the quartic interaction terms in (3.23) spoil the invariance of the Hamiltonian under the reparametrizations introduced in the previous section. To illustrate this, we first consider how the cubic and quartic interaction vertices transform at order κ.

$$\delta\mathcal{H}_{c,q}^{(\kappa)} \;=\; \delta^{\kappa^0}(\text{cubic terms}) \;+\; \delta^{\kappa^{-1}}(\text{quartic terms}) \tag{3.27}$$

Contributions from cubic terms

$$
\begin{aligned}
&\delta^{\kappa^0}(\text{cubic terms})\\
&= 2\kappa(\bar\xi\partial\bar h \;+\; \xi\bar\partial\bar h)\partial_-{}^2\left(h\frac{\bar\partial^2}{\partial_-{}^2}h \;-\; \frac{\bar\partial}{\partial_-}h\frac{\bar\partial}{\partial_-}h\right)\\
&\quad + 2\kappa\bar h\;\partial_-{}^2\Bigg((\xi\bar\partial h \;+\; \bar\xi\partial h)\frac{\bar\partial^2}{\partial_-{}^2}h \;+\; h\frac{\bar\partial^2}{\partial_-{}^2}(\xi\bar\partial h \;+\; \bar\xi\partial h)\\
&\quad - 2\;\frac{\bar\partial}{\partial_-}(\xi\bar\partial h \;+\; \bar\xi\partial h)\frac{\bar\partial}{\partial_-}h\Bigg)\\
&= 2\kappa\;\bar\xi\;\partial\bar h\;\partial_-{}^2\left(h\frac{\bar\partial^2}{\partial_-{}^2}h \;-\; \frac{\bar\partial}{\partial_-}h\frac{\bar\partial}{\partial_-}h\right)\\
&\quad + 2\kappa\bar h\;\partial_-{}^2\left(\bar\xi\partial h\frac{\bar\partial^2}{\partial_-{}^2}h \;+\; h\frac{\bar\partial^2}{\partial_-{}^2}(\bar\xi\partial h) \;-\; 2\;\frac{\bar\partial}{\partial_-}(\bar\xi\partial h)\frac{\bar\partial}{\partial_-}h\right) \;+W(\xi)\\
&= \mathcal{X} \;+\; \mathcal{Y} \;+\; \mathcal{W}(\xi)\;.
\end{aligned}
\tag{3.28}
$$

$\mathcal{W}$ contains all the ξ-dependent terms

$$
\begin{aligned}
\mathcal{W} \;&=\; 2\kappa\;\xi\bar\partial\bar h\;\partial_-^2\left(h\frac{\bar\partial^2}{\partial_-^2}h \;-\; \frac{\bar\partial}{\partial_-}h\frac{\bar\partial}{\partial_-}h\right)\\
&\qquad + 2\kappa\;\bar h\;\partial_-^2\left(\xi\bar\partial h\frac{\bar\partial^2}{\partial_-^2}h \;+\; h\frac{\bar\partial^2}{\partial_-^2}(\xi\bar\partial h) \;-2\frac{\bar\partial}{\partial_-}(\xi\bar\partial h)\frac{\bar\partial}{\partial_-}h\right)\\
&=\; 0\;.
\end{aligned}
\tag{3.29}
$$

On partially integrating the $\bar\partial$ in the first line, it easily follows that all the terms in (3.29) cancel among each other. Similarly, the variation of the other cubic term in the Hamiltonian will have no $\bar\xi$ terms. After some partial integrations, $\mathcal{X}$ and $\mathcal{Y}$ take the form

$$
\mathcal{X} \;=\; -2\kappa\;\bar\xi\;\bar h\;\partial_-{}^2\partial\left(h\frac{\bar\partial^2}{\partial_-{}^2}h\right) + 2\kappa\;\bar\xi\;\bar h\;\partial_-{}^2\partial\left(\frac{\bar\partial}{\partial_-}h\frac{\bar\partial}{\partial_-}h\right)
\tag{3.30}
$$

and

$$
\begin{aligned}
\mathcal{Y} \;&=\; 2\kappa\bar h\;\bar\xi\partial_-{}^2\;\partial\left(h\frac{\bar\partial^2}{\partial_-{}^2}h\right) \;-\; 2\kappa\bar h\bar\xi\partial_-{}^2\;\partial\left(\frac{\bar\partial}{\partial_-}h\frac{\bar\partial}{\partial_-}h\right)\\
&\qquad - 4\kappa\;\bar\partial\bar\xi\;\frac{\partial}{\partial_-}h\;\frac{\bar\partial}{\partial_-}h\;\partial_-^2\bar h \;+\; 2\kappa\;\bar\partial^2\bar\xi\;\frac{\partial}{\partial_-^2}h\;h\;\partial_-^2\bar h \;+\; 4\kappa\;\bar\partial\bar\xi\;\frac{\partial\bar\partial}{\partial_-^2}h\;h\;\partial_-^2\bar h\;.
\end{aligned}
\tag{3.31}
$$

Finally, we find that the net contribution from the cubic vertex at order κ is as follows

$$\begin{aligned}\delta^{\kappa^0}(\text{cubic term}) &= \mathcal{X} + \mathcal{Y} \\ &= 4\kappa\,\bar\partial\bar\xi\,\frac{\partial}{\partial_-}h\,\frac{\bar\partial}{\partial_-}h\,\partial_-^2\bar h - 2\kappa\,\bar\partial^2\bar\xi\,\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar h - 4\kappa\,\bar\partial\bar\xi\,\frac{\partial\bar\partial}{\partial_-^2}h\,h\,\partial_-^2\bar h\,.\end{aligned} \tag{3.32}$$

Contribution from quartic terms

We now consider the κ^{-1} variation of the quartic terms in (3.23). We restrict ourselves to the $\bar\xi$ terms only since the $\xi-$dependent terms simply follow from complex conjugation. It can be seen from (3.18) that only those terms, where $\bar h$ appears without any derivative or with only a $\bar\partial$ will contribute.

$$\begin{aligned}\delta^{\kappa^{-1}}(\text{quartic}) &= -8\kappa^2\,\frac{1}{\partial_-{}^2}\left(\frac{\bar\partial}{\partial_-}h\,\partial_-{}^3\,\bar h - h\,\partial_-{}^2\bar\partial\bar h\right)\frac{1}{\partial_-{}^2}\left(\delta\bar h\,\partial_-{}^2\partial h\right) \\ &\quad + 4\kappa^2\,\frac{1}{\partial_-}\left(\partial_- h\,\bar\partial\delta\bar h\right)\frac{1}{\partial_-}\left(\partial\bar h\,\partial_- h - \partial_-\bar h\,\partial h\right) \\ &\quad + 4\kappa^2 h\,\delta\bar h\left(\bar\partial h\,\partial\bar h + \partial h\,\bar\partial\bar h - \partial_-\bar h\,\frac{\partial\bar\partial}{\partial_-}h - \partial_- h\,\frac{\partial\bar\partial}{\partial_-}\,\bar h\right) \\ &\quad - 4\kappa^2\left(\frac{1}{\partial_-{}^2}\left(\partial_- h\,\partial_-\bar h\right) - h\,\bar h\right)\partial h\,\bar\partial\,(\delta\bar h) \\ &= \mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}\end{aligned} \tag{3.33}$$

Let us focus on $\mathcal{A}$ and $\mathcal{D}$ first.

$$\begin{aligned}\mathcal{A} &= -\,4\kappa\,\bar\partial\bar\xi\,\partial h\,\frac{1}{\partial_-{}^2}\left(\frac{\bar\partial}{\partial_-}h\,\partial_-{}^3\,\bar h - h\,\partial_-{}^2\bar\partial\bar h\right) \\ &= 4\kappa\,\bar\partial\bar\xi\,\frac{\partial}{\partial_-}h\,\frac{\bar\partial}{\partial_-}h\,\partial_-^2\bar h - 4\kappa\,\bar\partial^2\bar\xi\,\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar h - 4\kappa\,\bar\partial\bar\xi\,\frac{\partial\bar\partial}{\partial_-^2}h\,h\,\partial_-^2\bar h\end{aligned} \tag{3.34}$$

$$\begin{aligned}\mathcal{D} &= -\,2\kappa^2\,\partial h\,\bar\partial^2\bar\xi\left(\frac{1}{\partial_-{}^2}\left(\partial_- h\,\partial_-\bar h\right) - h\,\bar h\right) \\ &= -\,2\kappa\,\bar\partial^2\bar\xi\,\frac{\partial}{\partial_-}h\,\partial_- h\,\bar h + 2\kappa\,\bar\partial^2\bar\xi\,\frac{\partial}{\partial_-^2}h\,h\,\partial_-^2\bar h\end{aligned} \tag{3.35}$$

We immediately note that the terms in (3.34) along with the second term in (3.35) exactly cancel against the terms in (3.32). So, we are left with the following terms

$$\mathcal{B} \;+\; \mathcal{C} \;-\; 2\kappa\, \bar\partial^2 \bar\xi\, \frac{\partial}{\partial_-} h\, \partial_- h\, \bar h\; . \tag{3.36}$$

These terms can be further simplified as

$$\begin{aligned}\mathcal{B} -\; 2\kappa\, \bar\partial^2 \bar\xi\, \frac{\partial}{\partial_-} h\, \partial_- h\, \bar h \;&=\; -2\kappa\, \bar\partial^2 \bar\xi\, \frac{1}{\partial_-} h\, (\partial \bar h\, \partial_- h \;-\; \partial_- \bar h\, \partial h) \;-\; 2\kappa\, \bar\partial^2 \bar\xi\, \frac{\partial}{\partial_-} h\, \partial_- h\, \bar h \\ &=\; -\; 2\kappa \bar\partial^2 \bar\xi\; h\, \bar h\, \partial h\; , \end{aligned} \tag{3.37}$$

$$\mathcal{C} \;=\; 2\kappa h\, \bar\partial \bar\xi\, \left(\, \bar\partial h\, \partial \bar h \;+\; \partial h\, \bar\partial \bar h \;-\; \partial_- \bar h\, \frac{\partial \bar\partial}{\partial_-} h \;-\; \partial_- h\, \frac{\partial \bar\partial}{\partial_-}\, \bar h \right)\; . \tag{3.38}$$

Consider the last term in (5.22)

$$-2\kappa\, h\, \bar\partial \bar\xi\, \partial_- h\, \frac{\partial \bar\partial}{\partial_-}\, \bar h \;=\; -\kappa\, \bar\partial \bar\xi\, \partial_- (hh) \frac{\partial \bar\partial}{\partial_-}\, \bar h \;=\; -\kappa\, \bar\partial \bar\xi\, \partial (hh)\, \bar\partial \bar h \;=\; -2\kappa\, \bar\partial \bar\xi\, h\, \partial h\, \bar\partial \bar h\, . \tag{3.39}$$

This cancels with the second term in (5.22), which simplifies $\mathcal{C}$ to

$$\mathcal{C} \;=\; 2\kappa h\, \bar\partial \bar\xi\, \bar\partial h\, \partial \bar h \;-\; 2\kappa h\, \bar\partial \bar\xi\, \partial_- \bar h\, \frac{\partial \bar\partial}{\partial_-} h\; . \tag{3.40}$$

Finally, from (3.37) and (5.22), we are left with

$$-\; 2\kappa \bar\partial^2 \bar\xi\; h\, \bar h\, \partial h \;+\; 2\kappa h\, \bar\partial \bar\xi\, \bar\partial h\, \partial \bar h \;-\; 2\kappa h\, \bar\partial \bar\xi\, \partial_- \bar h\, \frac{\partial \bar\partial}{\partial_-} h\; . \tag{3.41}$$

Thus, the variation of the Hamiltonian to order κ so obtained is

$$\delta H^{(\kappa)} \;=\; -\; 2\kappa \bar\partial^2 \bar\xi\; h\, \bar h\, \partial h \;+\; 2\kappa h\, \bar\partial \bar\xi\, \bar\partial h\, \partial \bar h \;-\; 2\kappa h\, \bar\partial \bar\xi\, \partial_- \bar h\, \frac{\partial \bar\partial}{\partial_-} h \;+\; c.c. \tag{3.42}$$

Evidently, the Hamiltonian is not invariant under the transformations in (3.16) at this order. In order to restore the invariance, we introduce new terms of order κ, to the r.h.s of (3.16). We find the correction terms to be

$$\delta h \;=\; \frac{1}{2\kappa} \partial \xi \;+\; \xi \bar\partial h \;+\; \bar\xi \partial h \;\;-\; \kappa\, \bar\partial \bar\xi\; hh \;+\; 2\kappa\, \partial \xi\, \frac{1}{\partial_-} (\bar h\, \partial_- h) \tag{3.43}$$

and

$$\delta\bar{h} \;=\; \frac{1}{2\kappa}\bar{\partial}\bar{\xi} \;+\; \xi\bar{\partial}\bar{h} \;+\; \bar{\xi}\partial\bar{h} \;\; - \;\kappa\,\partial\xi\;\bar{h}\bar{h} \;+\; 2\kappa\;\bar{\partial}\bar{\xi}\;\frac{1}{\partial_-}(h\;\partial_-\bar{h})\;. \tag{3.44}$$

These new corrections terms, when substituted in the kinetic term of (3.23), contribute at the same order as those in (3.42). On varying the kinetic term $\delta^\kappa(\partial\bar{h}\,\bar{\partial}h)$ we find that it cancels exactly against the terms in (3.42), confirming that

$$\delta\mathcal{H}^{(\kappa)} \;=\; 0\;. \tag{3.45}$$

This proves that the light-cone Hamiltonian to order κ^2 is invariant under the "corrected" residual reparametrizations (3.43) and (3.44).

We can easily check that two such transformations at order κ close on another transformation at the same order

$$[\;\delta_1(\xi_1)\;,\;\delta_2(\xi_2)\;]h \;=\; \delta_{12}(\xi_{12})h \tag{3.46}$$

with the parameter

$$\xi_{12} \;=\; \overline{\xi_2}\partial\xi_1 \;-\; \overline{\xi_1}\partial\xi_2\;, \tag{3.47}$$

which shows that this is indeed a residual reparametrization symmetry, albeit an infinitesimal one. From the discussion above, it is evident that we must keep on adding correction terms to the transformations (3.43) and (3.44) at every order in perturbation to maintain the invariance of the Hamiltonian, thus forming an infinite series of terms which represents the full symmetry. The fact that this symmetry constrains the possible counterterms in the Hamiltonian suggests that it might be possible to integrate this infinitesimal symmetry to a finite one. Such a finite symmetry can have important implications for the macroscopic properties of the theory. We wish to return to this question in future and examine possible links to the work of [32].

3.4 Transformation properties of the quadratic form

Given that the Hamiltonian at order κ^2 is invariant under residual reparametrization transformations, we now examine the transformation properties of the $\mathcal{D}\bar{h}$ operator under these transformations. Unlike at order κ, we find that $\mathcal{D}\bar{h}$ does not transform "covariantly" at this order, that is, the derivative does not transform like the field.

Explicitly varying the derivative (3.26) we obtain

$$\delta\,(\,\Delta\bar{h}\,)^{\kappa} = \quad +\kappa\,\partial\xi\,\partial\{\,\partial_{-}\bar{h}\frac{1}{\partial_{-}}\bar{h}\,\}$$
$$+\,2\kappa\,\bar{\partial}\bar{\xi}\,h\,\partial\,\bar{h}\,+\,\kappa\,\bar{\partial}\,\bar{\xi}\,\partial\,\partial_{-}\,\bar{h}\,\frac{1}{\partial_{-}}\,h\,-\,\kappa\,\bar{\partial}\,\bar{\xi}\,\frac{1}{\partial_{-}}\,\{\,\partial_{-}\,\partial\,\bar{h}\,h\,\} \quad (3.48)$$

and it can be easily verified that

$$\delta\mathcal{H}^{\kappa} \;=\; \int d^3x\;[\,\delta(\Delta\bar{h})\,\bar{\mathcal{D}}h \;+\; \Delta\bar{h}\,\delta(\bar{\mathcal{D}}h)\,]^{\kappa} \;=\; 0\;. \quad (3.49)$$

In this section, we argue on general grounds, why this transformation property in (3.49) is justified.

We start with the most general ansatz for $\delta(\bar{\mathcal{D}}h)$ and $\delta(\mathcal{D}\bar{h})$ from (3.43) and (3.44) (we ignore all the ξ-dependent terms)

$$\delta(\bar{\mathcal{D}}h) \;=\; 0\;+\;(\,\xi\bar{\partial}\;+\;\bar{\xi}\partial\,)\,\bar{\mathcal{D}}h\;-\;\kappa\,\bar{\partial}\bar{\xi}\;\sum_i\,\alpha_i\,\hat{A}_i\;(\hat{B}_i h\;\hat{C}_i h)$$
$$+\,2\kappa\,\partial\xi\,\sum_j\,\beta_j\,\hat{P}_j(\hat{Q}_j\bar{h}\,\hat{R}_j h) \quad (3.50)$$

and

$$\delta(\mathcal{D}\bar{h}) \;=\; 0\,+\,(\,\xi\bar{\partial}\;+\;\bar{\xi}\partial\,)\,\mathcal{D}\bar{h}\;-\;\kappa\,\partial\xi\,\sum_i\,\alpha_i\,\bar{\hat{A}}_i\,(\bar{\hat{B}}_i\bar{h}\,\bar{\hat{C}}_i\bar{h}\,)$$
$$+\;2\kappa\;\bar{\partial}\bar{\xi}\;\sum_j\,\beta_j\,\bar{\hat{P}}_j(\bar{\hat{Q}}_j h\bar{\hat{R}}_j\bar{h})\;, \quad (3.51)$$

where the α and β are constants and the $\hat{A}_i,\ldots$ are operators to be determined later. It is easy to note that this ansatz transforms "covariantly" (like the field) for one particular choice of operators

$$\alpha = 1\,,\;\hat{A}\;=\;\bar{\partial}\;,\;\hat{B}\;=\;\hat{C}\;=\;1\;,$$
$$\beta = 1\,,\;\hat{P}\;=\;\frac{1}{\partial_{-}}\;,\;\hat{Q}\;=\;1\;,\;\hat{R}\;=\;\partial_{-}\bar{\partial}\;. \quad (3.52)$$

From the invariance of the Hamiltonian under (3.43) and (3.44), we have

$$\delta\mathcal{H} = 0 \;\;\Longrightarrow\;\; \int d^3x\;[\,\delta(\Delta\bar{h})\,\bar{\mathcal{D}}h\;+\;\Delta\bar{h}\,\delta(\bar{\mathcal{D}}h)\,]\;. \quad (3.53)$$

At order κ^0, (3.53) yields

$$\delta\mathcal{H} = \int d^3x \; [\, (\delta(\Delta\bar{h}))^{\kappa^0}\bar{\partial}h \; + \partial\bar{h}(\delta(\bar{\mathcal{D}}h))^{\kappa^0} \,]$$
$$= \int d^3x \; [\bar{\xi}\partial^2\bar{h}\bar{\partial}h \; + \; \partial\bar{h}\xi\partial\bar{\partial}h\,] \; . \tag{3.54}$$

Integrating a ∂ from the $\bar{h}$ in the first term gives us $(\delta\mathcal{H})^{\kappa^0} \; = \; 0$.

At order $\mathcal{O}(\kappa)$, we start with

$$(\delta\mathcal{H})^{\kappa} \;\; = \;\; \int d^3x \; [\, (\delta(\Delta\bar{h}))^{\kappa}\bar{\partial}h \; + \; (\delta(\Delta\bar{h}))^{\kappa^0}(\bar{\mathcal{D}}h)^{\kappa} \; + \; (\Delta\bar{h})^{\kappa}(\delta(\bar{\mathcal{D}}h))^{\kappa^0} \; + \partial\bar{h}(\delta(\bar{\mathcal{D}}h))^{\kappa}\,]$$
$$= \;\; \kappa\int d^3x \; \{\,[\,\bar{\xi}\partial(\Delta\bar{h})^{\kappa} + \; 2\bar{\partial}\bar{\xi} \; \sum_j \beta_j\,\hat{\bar{P}}_j(\hat{\bar{Q}}_j h\hat{\bar{R}}_j\bar{h})\,]\bar{\partial}h \; + \; \bar{\xi}\partial^2\bar{h}(\bar{\mathcal{D}}h)^{\kappa}$$
$$+ \; [\,\xi\partial(\bar{\mathcal{D}}h)^{\kappa} - \; \bar{\partial}\bar{\xi} \; \sum_i \alpha_i\,\hat{A}_i\;(\hat{B}_i h\;\hat{C}_i h)]\;\partial\bar{h} \; + \; (\Delta\bar{h})^{\kappa}\xi\partial\bar{\partial}h\,\} \; . \tag{3.55}$$

On integrating a ∂ from $\bar{h}$ in the last term of (3.55) it cancels against the first term in (3.55). Similarly, the last term in (3.55) cancels against the first term in (3.55) by integrating a ∂. So, we are left with

$$(\delta\mathcal{H})^{\kappa} \;\; = \;\; \kappa\int d^3x \; [\,2\bar{\partial}\bar{\xi} \; \sum_j \beta_j\,\hat{\bar{P}}_j(\hat{\bar{Q}}_j h\hat{\bar{R}}_j\bar{h})\bar{\partial}h \; - \; \bar{\partial}\bar{\xi} \; \sum_i \alpha_i\,\hat{A}_i\;(\hat{B}_i h\;\hat{C}_i h)\,\partial\bar{h}] \; . \tag{3.56}$$

Substituting (3.52) into (3.56) , we see that

$$(\delta\mathcal{H})^{\kappa} = +2\,\kappa\int d^3x \;\; \bar{\partial}\bar{\xi}\,\frac{1}{\partial_-}(\,\partial_- h\,\partial\bar{h}\,)\,\bar{\partial}h \; + \; \text{c.c.} \; \neq \; 0 \; . \tag{3.57}$$

Thus the Hamiltonian for gravity is indeed a quadratic form but not the square of a "covariant derivative". In fact, if we substitute (3.48) into (3.56), the invariance is restored

$$(\delta\mathcal{H})^{\kappa} \; = \; 0 \; . \tag{3.58}$$

Hence we learn from this analysis that the residual reparametrization invariance of the Hamiltonian forces $\mathcal{D}\bar{h}$ to have transformation properties different from that of a "covariant derivative".

This point is clearly in stark contrast with Yang-Mills theory. Both the pure and maximally supersymmetric Yang-Mills theory admit quadratic form Hamiltonian which transform "covariantly' under the residual gauge transformations [33]. This mismatch points to the fact that the non-trivial dissimilarities between Yang-Mills theory and gravity show up first at the quartic order. The tree-level amplitudes in Yang-Mills theory can be written solely in terms of the "square" or "angular" brackets. The cubic amplitude in gravity does have the same property, but the quartic and higher vertices involve a mixture of both kinds of brackets. Thus the fact that the derivative introduced in gravity does not transform like that in Yang-Mills theory is in keeping with the MHV amplitude structures [20].

With this chapter, we conclude our discussion on pure gravity in the light-cone gauge. In the last chapter, we briefly comment on some of the results presented here. In the rest of the thesis, our focus lies on theories of gravity with supersymmetry.

Chapter 4

Supergravity, in $d = 4$, in the light-cone gauge

In this chapter, we review some relevant results for $\mathcal{N} = 8$ supergravity in four dimensions from existing literature, which will be used extensively for the discussions in the subsequent chapters. We begin with a brief review of the supersymmetry algebra and then introduce the light-cone superspace in context of the $\mathcal{N} = 8$ theory.

4.1 The supersymmetry algebra

In a relativistic field theory, the symmetries of the S-matrix include the Poincaré symmetry (P_μ and $M_{\mu\nu}$) and some internal symmetries, T^a. The Coleman-Mandula theorem states that the spacetime and internal symmetries can be combined only in a trivial manner through a direct product of the Poincaré group with the internal symmetry group [34]. All the corresponding conserved quantities are Lorentz scalars and the internal symmetry generators trivially commute with the Poincaré generators.

However, there is a possible extension of the Poincaré group without violating this no-go theorem by including generators, $\mathcal{Q}_\alpha$ and $\overline{\mathcal{Q}}_{\dot\alpha}$ that transform as spinors under the Lorentz group. The Lie algebra of the Poincaré group is thus enlarged to the superPoincaré algebra, which now includes anticommutators between the spinorial generators.

The $\mathcal{Q}_\alpha$ transforms in the $(\frac{1}{2}, 0)$ representation of the Lorentz group and the $\overline{\mathcal{Q}}_{\dot\alpha}$ in the $(0, \frac{1}{2})$ representation. Therefore, the anticommutator of these generators must transform as a four-vector in the $(\frac{1}{2}, \frac{1}{2})$ representation, i e. the translation generator in the Poincaré group. This defines the supersymmetry algebra.

$$\{\mathcal{Q}_\alpha, \overline{\mathcal{Q}}_{\dot\alpha}\} \;=\; 2(\sigma^\mu)_{\alpha\dot\alpha}\, P_\mu \;. \tag{4.1}$$

The commutation relations of the supersymmetry generators with momenta P_μ are

$$\begin{aligned} [\, P_\mu, \mathcal{Q}_\alpha\,] &= 0 \;, \\ [\, P_\mu, \overline{\mathcal{Q}}_{\dot\beta}\,] &= 0 \end{aligned}$$

and with the Lorentz generators $M_{\mu\nu}$ are

$$\begin{aligned} [\; M_{\mu\nu} \;,\; \mathcal{Q}_\alpha] &= i\,(\sigma_{\mu\nu})_\alpha{}^\beta\; \mathcal{Q}_\beta \;, \\ [\; M_{\mu\nu} \;,\; \overline{\mathcal{Q}}^{\dot\alpha}] &= i\,(\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\; \overline{\mathcal{Q}}^{\dot\beta} \;, \end{aligned}$$

where the symbols have their usual meaning. The spinor indices (dotted and undotted) are raised and lowered using the Levi-Civita tensors, $\epsilon^{\alpha\beta}$ and $\epsilon^{\dot\alpha\dot\beta}$. The commutation relations with $M_{\mu\nu}$ clearly show that $\mathcal{Q}$ and $\overline{\mathcal{Q}}$ transform in the spinor representation of the Lorentz group.

In the case of extended supersymmetry where the number of supersymmetries $\mathcal{N}$ is greater than one, the supersymmetry generators are written as $\mathcal{Q}_\alpha{}^m$ (and $\overline{\mathcal{Q}}_{m\dot\alpha}$), which transform as the representation $\mathcal{N}$ (and $\overline{\mathcal{N}}$) under the internal R-symmetry group $SU(\mathcal{N})$.

So far we have discussed the concept of supersymmetry as a global symmetry. Supersymmetric gauge theories involving spin-1 fields are based on global supersymmetry. Now we consider the case when the supersymmetry is local, such that the supersymmetry transformation parameters ϵ^m and $\bar\epsilon_m$ depend on spacetime coordinates. The underlying supersymmetry algebra (4.1) implies that two such local transformations will close on a local translation. Thus, local supersymmetry naturally leads to a theory of gravity, namely supergravity. The spin-2 graviton field in supergravity theory are accompanied by a spin-$\frac{3}{2}$ superpartner, called the gravitino. The gravitino is the gauge field associated with local supersymmetry. Hence the number of gravitinos in a supergravity theory is equal to the number of supersymmetries, $\mathcal{N}$.

4.2 $\mathcal{N} = 8$ supergravity in the light-cone gauge

The $\mathcal{N} = 8$ theory is the maximally supersymmetric extension of Einstein's gravity in four dimension [16]. This theory possesses eight supersymmetries, which is the most any theory in four dimensions involving no fields of spin greater than two, can have. The R-symmetry group in this case is $SU(8)$. The supermultiplet contains 128 bosonic and 128 fermionic degrees of freedom, together forming 256 physical states.

In table (4.1) we categorize all the physical states according to their helicity.

Table 4.1: The $\mathcal{N} = 8$ supermultiplet

Spin	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2
Field	h	$\overline{\psi}_m$	$\overline{A}_{mn}$	$\overline{\chi}_{mnp}$	C_{mnpq}	χ^{mnp}	A^{mn}	ψ^m	$\overline{h}$
	1	8	28	56	70	56	28	8	1

Here, h and $\bar{h}$ denote the two degrees of freedom of the graviton, $\bar{\psi}_m$ correspond to the 8 spin-$\frac{3}{2}$ gravitinos, $\bar{A}_{mn}$ are the 28 abelian gauge fields with $\bar{\chi}_{mnp}$ being the corresponding 56 gauginos and $\bar{C}_{mnpq}$ represents the 70 scalar fields in the theory. The SU(8) indices $m, n, p, q...$ take values from 1 through 8.

Now we introduce some basic tools that we use to study the properties of this theory in the light-cone gauge.

4.2.1 Light-cone superspace

In appendix B, we show how a spinor in the light-cone frame can be split into "+" and "-" components using suitable projection operators. Similarly, the supersymmetry generators $\mathcal{Q}_\alpha$ and $\overline{\mathcal{Q}}_{\dot\beta}$ can be decomposed into two two-component complex spinors

$$\mathcal{Q}_\alpha \;=\; \mathcal{Q}_{+\alpha} \;+\; \mathcal{Q}_{-\alpha}\;; \qquad \overline{\mathcal{Q}}_{\dot\beta} \;=\; \overline{\mathcal{Q}}_{+\dot\beta} \;+\; \overline{\mathcal{Q}}_{-\dot\beta}\;.$$

The light-cone supersymmetry algebra then follows directly from (4.1)

$$\{\, \mathcal{Q}_+^m \,,\, \overline{\mathcal{Q}}_{+n} \,\} \;=\; -\sqrt{2}\,\delta^m_n\, P^+ \,, \tag{4.2}$$

$$\{\, \mathcal{Q}_-^m \,,\, \overline{\mathcal{Q}}_{-n} \,\} \;=\; -\sqrt{2}\,\delta^m_n\, P^- \,, \tag{4.3}$$

$$\{\, \mathcal{Q}_+^m \,,\, \overline{\mathcal{Q}}_{-n} \,\} \;=\; -\sqrt{2}\,\delta^m_n\, P \,. \tag{4.4}$$

The indices m, n run from 1 to $\mathcal{N}$. Except for the complex conjugate of the last commutator listed above, all other anticommutators are zero. Note that we have suppressed the spinor indices $(\alpha,\, \dot{\beta})$ on the generators for simplicity. From the algebra, we can see that the $\mathcal{Q}_+$ generators are kinematical supersymmetries, which generate the spectrum and the $\mathcal{Q}_-$ generators are the dynamical ones, which close on the Hamiltonian P^-. Thus, the dynamical supersymmetries take the fields forward in light-cone time.

Supersymmetric field theories can be described elegantly using the superspace formalism. In this approach, the supersymmetry operators are treated as the generators of translation in anticommuting coordinates θ^m, just as momentum P^μ generates translation in ordinary spacetime coordinates x^μ.

In the rest of the section we discuss the light-cone superspace adapted to $\mathcal{N} = 8$ supergravity in four dimensions first presented in [35]. The light-cone superspace is spanned by eight Grassmann variables θ^m and their complex conjugates $\bar{\theta}_m$ ($m = 1, \ldots, 8$), which transform as a $\mathbf{8}$ and a $\bar{\mathbf{8}}$ of $SU(8)$ respectively. These variables follow the relations

$$\{\, \theta^m \,,\, \theta^n \,\} = \{\, \bar{\theta}_m \,,\, \bar{\theta}_n \,\} \;=\; \{\, \theta^m \,,\, \bar{\theta}_n \,\} \;=\; 0 \,. \tag{4.5}$$

An alternative choice would be to define the superspace with just θ's and no $\bar{\theta}$'s, but that makes the notion of complex conjugation more complicated. The advantage of working in a superspace with both θ and $\bar{\theta}$ is that it makes the $SU(8)$ R-symmetry of the theory manifest. We can define chiral derivatives anticommuting with the $\mathcal{Q}$'s, such that the theory can be described by just *one* constrained superfield as we will show below.

The kinematical supersymmetries can be represented on this superspace as

$$q_+^m \;=\; -\frac{\partial}{\partial\,\bar{\theta}_m} + \frac{i}{\sqrt{2}}\,\theta^m\,\partial^+; \qquad \bar{q}_{+n} = \;\frac{\partial}{\partial\,\theta^n} - \frac{i}{\sqrt{2}}\,\bar{\theta}_n\,\partial^+ \tag{4.6}$$

and the dynamical ones as

$$q_{-}^{m} \;=\; \frac{\bar{\partial}}{\partial^{+}}\, q_{+}^{m}\,, \quad \bar{q}_{-\,n} \;=\; \frac{\partial}{\partial^{+}}\, \bar{q}_{+\,n}\,. \tag{4.7}$$

(Note that we use the notation with the lower case letters when working with the explicit form of operators.)

We also define the following chiral derivatives in the superspace

$$d^{m} \;=\; -\,\frac{\partial}{\partial\,\bar{\theta}_{m}} \;-\; \frac{i}{\sqrt{2}}\,\theta^{m}\,\partial^{+} \;\; ; \qquad \bar{d}_{n} \;=\; \frac{\partial}{\partial\,\theta^{n}} \;+\; \frac{i}{\sqrt{2}}\,\bar{\theta}_{n}\,\partial^{+} \;. \tag{4.8}$$

The chiral derivatives satisfy the anticommutation relation

$$\{\, d^{m} \,,\, \bar{d}_{n}\,\} \quad = \quad -\,i\sqrt{2}\delta_{n}^{m}\;\partial^{+}\;. \tag{4.9}$$

These derivatives anticommute with the supercharges q^{m} and $\bar{q}_{n}$.

The superfield

All the 256 physical degrees of freedom in the $\mathcal{N}\,=\,8$ theory listed before can be encaptured in a single superfield [25, 35] in terms of θ^{m}

$$\begin{aligned}
\phi\,(\,y\,) \;=& \frac{1}{\partial^{+2}}\,h\,(y) \;+\; i\,\theta^{m}\,\frac{1}{\partial^{+2}}\,\bar{\psi}_{m}\,(y) \;+\; \frac{i}{2}\,\theta^{m}\,\theta^{n}\,\frac{1}{\partial^{+}}\,\bar{A}_{mn}\,(y)\;, \\
&-\;\frac{1}{3!}\,\theta^{m}\,\theta^{n}\,\theta^{p}\,\frac{1}{\partial^{+}}\,\bar{\chi}_{mnp}\,(y) \;-\; \frac{1}{4!}\,\theta^{m}\,\theta^{n}\,\theta^{p}\,\theta^{q}\,\overline{C}_{mnpq}\,(y)\;, \\
&+\;\frac{i}{5!}\,\theta^{m}\,\theta^{n}\,\theta^{p}\,\theta^{q}\,\theta^{r}\,\epsilon_{mnpqrstu}\,\chi^{stu}\,(y)\;, \\
&+\;\frac{i}{6!}\,\theta^{m}\,\theta^{n}\,\theta^{p}\,\theta^{q}\,\theta^{r}\,\theta^{s}\,\epsilon_{mnpqrstu}\,\partial^{+}\,A^{tu}\,(y)\;, \\
&+\;\frac{1}{7!}\,\theta^{m}\,\theta^{n}\,\theta^{p}\,\theta^{q}\,\theta^{r}\,\theta^{s}\,\theta^{t}\,\epsilon_{mnpqrstu}\,\partial^{+}\,\psi^{u}\,(y)\;, \\
&+\;\frac{4}{8!}\,\theta^{m}\,\theta^{n}\,\theta^{p}\,\theta^{q}\,\theta^{r}\,\theta^{s}\,\theta^{t}\,\theta^{u}\,\epsilon_{mnpqrstu}\,\partial^{+2}\,\bar{h}\,(y)\;.
\end{aligned} \tag{4.10}$$

The superfield by construction makes the $SU(8)$ decomposition of the fields manifest. For instance, the graviton which appears with no θ is a singlet under $SU(8)$, the spin-$\frac{3}{2}$ fields being a $\mathbf{8}$ of $SU(8)$ appear with a θ^{m} and so on. The superfield can thus be viewed as a representation of $SU(8)$ in the following way.

$$\mathbf{256} \;\;=\;\; \mathbf{1} \;+\; \mathbf{8} \;+\; \mathbf{28} \;+\; \mathbf{56} \;+\; \mathbf{70} \;+\; \mathbf{56} \;+\; \mathbf{28} \;+\; \mathbf{8} \;+\; \mathbf{1} \tag{4.11}$$

All the fields are all local in the coordinates

$$y \;=\; (\,x,\,\bar{x},\,x^+,\,y^- \equiv x^- - \frac{i}{\sqrt{2}}\,\theta^m\,\bar{\theta}_m\,)\;. \tag{4.12}$$

This follows from the "chirality" condition satisfied by the superfield ϕ and its conjugate $\bar{\phi}$

$$d^m\,\phi\,(\,y\,) \;=\; 0\;\;; \qquad \bar{d}_n\,\bar{\phi}\,(\,y\,) \;=\; 0\;. \tag{4.13}$$

ϕ and $\bar{\phi}$ are further related through the "inside-out" constraint,

$$\phi \;=\; \frac{1}{4}\,\frac{(\,d\,)^8}{\partial^{+4}}\,\bar{\phi}\;, \tag{4.14}$$

where $(\,d\,)^8 \;=\; d^1\,d^2\,\ldots\,d^8$. This constraint is unique to maximally supersymmetric theories. which is a consequence of the self-duality of the scalar fields in the theory

$$C^{mnpq} \;\;=\;\; \frac{1}{4!}\,\epsilon^{mnpqrstu}\,\overline{C}_{rstu}\;. \tag{4.15}$$

4.2.2 SuperPoincaré algebra in the light-cone gauge

We now construct the lowest order representation of the superPoincaré algebra in four dimensions by augmenting the Lorentz generators with θ-terms. We choose to work on the constant time surface $x^+ = 0$.

We first consider the kinematical generators.

- The three kinematical momenta remain the same

$$p^+ \;=\; -\,i\,\partial^+\;, \qquad p \;=\; -\,i\,\partial\;, \qquad \bar{p} \;=\; -\,i\,\bar{\partial}\;. \tag{4.16}$$

- The transverse space rotation is given by

$$j \;=\; x\,\bar{\partial} - \bar{x}\,\partial + S^{12}\;, \tag{4.17}$$

where the spin part gets θ-corrections

$$S^{12} \;=\; \frac{1}{2}\,(\,\theta^\alpha\,\bar{\partial}_\alpha \,-\, \bar{\theta}_\alpha\,\partial^\alpha\,) + \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^\alpha\,\bar{d}_\alpha - \bar{d}_\alpha\,d^\alpha\,) \tag{4.18}$$

such that

$$[\, j\,,\, d^\alpha\,] \;=\; [\, j\,,\, \bar{d}_\beta\,] \;=\; 0\;. \tag{4.19}$$

• The kinematical rotations read

$$j^+ \;=\; i\,x\,\partial^+\;, \qquad \bar{j}^+ \;=\; i\,\bar{x}\,\partial^+\;, \tag{4.20}$$

$$j^{+-} \;=\; i\,x^-\,\partial^+ - \frac{i}{2}\,(\,\theta^\alpha\bar{\partial}_\alpha + \bar{\theta}_\alpha\,\partial^\alpha\,)\;, \tag{4.21}$$

which obey

$$\begin{aligned}
[\, j^{+-}\,,\, y^-\,] \;&=\; -\,i\,y^-\;,\\
[\, j^{+-}\,,\, d^\alpha\,] \;&=\; \frac{i}{2}\,d^\alpha\;, \qquad [\, j^{+-}\,,\, \bar{d}_\beta\,] \;=\; \frac{i}{2}\,\bar{d}_\beta\;.
\end{aligned} \tag{4.22}$$

The dynamical generators at the free order are listed below.

• The light-cone Hamiltonian is

$$p^- \;=\; -\,i\frac{\partial\bar{\partial}}{\partial^+}\;. \tag{4.23}$$

• The dynamical boosts

$$\begin{aligned}
j^- \;&=\; i\,x\,\frac{\partial\bar{\partial}}{\partial^+} \;-\; i\,x^-\,\partial \;+\; i\left(\theta^\alpha\bar{\partial}_\alpha \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^\alpha\,\bar{d}_\alpha - \bar{d}_\alpha\,d^\alpha\,)\right)\frac{\partial}{\partial^+}\;,\\
\bar{j}^- \;&=\; i\,\bar{x}\,\frac{\partial\bar{\partial}}{\partial^+} \;-\; i\,x^-\,\bar{\partial} \;+\; i\left(\bar{\theta}_\beta\partial^\beta + \frac{i}{4\sqrt{2}\,\partial^+}\,(\,d^\beta\,\bar{d}_\beta - \bar{d}_\beta\,d^\beta\,)\right)\frac{\bar{\partial}}{\partial^+}\;,
\end{aligned} \tag{4.24}$$

satisfy the following commutation relations

$$[\, j^-\,,\, \bar{j}^+\,] \;=\; -\,i\,j^{+-} - j\;, \qquad [\, j^-\,,\, j^{+-}\,] \;=\; i\,j^-\;. \tag{4.25}$$

• The dynamical supersymmetries constructed from the kinematical ones read

$$\begin{aligned}
q_-^{\,m} \equiv i\,[\,\bar{j}^-\,,\, q_+^{\,m}\,] \;&=\; \frac{\bar{\partial}}{\partial^+}\,q_+^{\,m}\,,\\
\bar{q}_{-\,n} \equiv i\,[\,j^-\,,\, \bar{q}_{+\,n}\,] \;&=\; \frac{\partial}{\partial^+}\,\bar{q}_{+\,n}\,.
\end{aligned} \tag{4.26}$$

This completes the light-cone representation of the superPoincaré algebra in $d = 4$ to the lowest order. At higher orders, all the dynamical generators will pick up corrections.

4.3 $\mathcal{N} = 8$ supergravity action

The light-cone action for $\mathcal{N} = 8$ supergravity to order κ in terms of the superfield ϕ and its conjugate $\bar{\phi}$ reads [25]

$$-\frac{1}{64}\int d^4x \int d^8\theta\, d^8\bar{\theta}\,\mathcal{L}\ , \tag{4.27}$$

where

$$\mathcal{L} \;=\; -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;-\; 2\,\kappa\,(\,\frac{1}{\partial^{+2}}\,\bar{\phi}\;\;\bar{\partial}\,\phi\;\bar{\partial}\,\phi + \frac{1}{\partial^{+2}}\;\phi\,\partial\,\bar{\phi}\,\partial\,\bar{\phi})\ . \tag{4.28}$$

The Grassmann integration is normalized such that

$$\int d^8\theta\,(\theta)^8 = 1\ . \tag{4.29}$$

The dynamical generators get non-linear corrections at order κ accordingly. In particular, the dynamical supersymmetry generator at this order reads

$$\bar{q}_{-m}{}^{(\kappa)}\phi = \frac{1}{\partial^{+}}(\bar{\partial}\bar{q}_m\phi\partial^{+2}\phi - \partial^{+}\bar{q}_m\phi\partial^{+}\bar{\partial}\phi). \tag{4.30}$$

Note that we have suppressed the + index on the kinematical supersymmetries to simplify our notation. The complex conjugate of this formula yields $q_{-}^{m\,(\kappa)}\bar{\phi}$. We can then derive $q^{m\,(\kappa)}\phi$ and $\bar{q}_m{}^{(\kappa)}\bar{\phi}$ by using the "inside-out" constraint (4.14). From the anticommutation relation (4.2), we can obtain $P^{-}\phi$ at order κ. The corrections to the other dynamical generators follow similarly.

4.4 The Hamiltonian as a quadratic form

In the light-cone superspace, the Hamiltonian of the theory can similarly be constructed as a perturbative expansion in κ. This has been done up to the four-point coupling [31]. In this section, we show how the light-cone Hamiltonian for $\mathcal{N} = 8$ supergravity can be written as a quadratic form [31] to order κ^2, similar to the case of pure gravity discussed in the previous chapter. We show the proof of the quadratic form only to the lowest order and avoid some of the technical details in our discussion.

At lowest order, the light-cone Hamiltonian reads

$$\mathcal{H}^0 \;=\; \int d^4x\, d^8\theta\, d^8\bar{\theta}\,\bar{\phi}\,\frac{2\,\partial\bar{\partial}}{\partial_{-}{}^{4}}\,\phi\ . \tag{4.31}$$

We claim that this Hamiltonian can be written in a compact form

$$\mathcal{H} \;=\; \frac{1}{4\,\sqrt{2}}\,(\,\mathcal{W}_m\,,\,\mathcal{W}_m\,)\;, \tag{4.32}$$

with

$$\mathcal{W}_m \;=\; \bar{Q}_{-\,m}\,\phi\;, \tag{4.33}$$

where the inner product defined as

$$(\,\phi\,,\,\xi\,)\;\equiv -\,2i\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\,\bar{\phi}\,\frac{1}{{\partial_-}^3}\xi\;. \tag{4.34}$$

To prove this, we start with (4.32) at the lowest order

$$\mathcal{H}^0 \;=\; \frac{1}{4\,\sqrt{2}}\,(\,\mathcal{W}^0_m\,,\,\mathcal{W}^0_m\,)\;=\;-\,\frac{2i}{4\,\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\;\;Q^{\,m}_-\,\bar{\phi}\,\frac{1}{{\partial_-}^3}\,\bar{Q}_{-\,m}\,\phi\;, \tag{4.35}$$

and rewrite it as

$$\mathcal{H}^0 \;= -\,\frac{i}{4\,\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\;\Big(Q^{\,m}_-\,\bar{\phi}\,\frac{1}{{\partial_-}^3}\,\bar{Q}_{-\,m}\,\phi\,+\,Q^{\,m}_-\,\bar{\phi}\,\frac{1}{{\partial_-}^3}\,\bar{Q}_{-\,m}\,\phi\Big)\;. \tag{4.36}$$

We now make use of the "inside-out" constraint (4.14), which is unique to maximally supersymmetric theories. The second term in (4.36) yields

$$\mathcal{H}^0 \;= -\,\frac{i}{4\,\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\;\Big(Q^{\,m}_-\,\bar{\phi}\,\frac{1}{{\partial_-}^3}\,\bar{Q}_{-\,m}\,\phi\,+\,\frac{1}{{\partial_-}^4}\,Q^{\,m}_-\,\phi\,\partial_-\,\bar{Q}_{-\,m}\,\bar{\phi}\Big)\;. \tag{4.37}$$

Using (4.26), we substitute for Q^m_- in terms of kinematical supersymmetries

$$\mathcal{H}^0 \;= -\,\frac{i}{4\,\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\;\Big(\frac{\bar{\partial}}{\partial_-}\,q^{\,m}_+\,\bar{\phi}\,\frac{\partial}{{\partial_-}^4}\,\bar{q}_{+\,m}\,\phi\,+\,\frac{\bar{\partial}}{{\partial_-}^5}\,q^{\,m}_+\,\phi\,\partial\,\bar{q}_{+\,m}\,\bar{\phi}\Big)\;.$$

After some integration by parts and $\frac{1}{\partial_-}$ manipulations, we find

$$\begin{aligned}\mathcal{H}^0 \;&=\; -\,\frac{i}{4\,\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\,\frac{\partial\,\bar{\partial}}{{\partial_-}^5}\,\bar{\phi}\,\{\,q^{\,m}_+\,,\,\bar{q}_{+\,m}\,\}\,\phi\\ &=\;\int d^4x\,d^8\theta\,d^8\bar{\theta}\,\bar{\phi}\,\frac{2\,\partial\bar{\partial}}{{\partial_-}^4}\,\phi\;.\end{aligned}$$

In the last step, we use the anticommutation relation $\{\, q_+^m \,,\, \bar{q}_{+\,m} \,\}\, \phi \;=\; -\, i\, 8\, \sqrt{2}\, \partial_-\, \phi$.

Hence, we proved that the free Hamiltonian can be expressed as a quadratic form. This property extend to the next order as well. We do not show the details here but simply present the results. At order κ the operator $\mathcal{W}_m$, which is nothing but the dynamical supersymmetry, reads

$$\mathcal{W}_m \;=\; -\, \frac{\partial}{\partial_-} \bar{q}_{+\,m}\, \phi \;-\; \kappa\, \frac{1}{\partial_-} \left(\bar{\partial}\, \bar{d}_m\, \phi\, \partial_-{}^2\, \phi \;-\; \partial_-\, \bar{d}_m\, \phi\, \partial_-\, \bar{\partial}\, \phi \right) \;+\; \mathcal{O}(\kappa^2) \;, \tag{4.38}$$

$$\overline{\mathcal{W}}^m \;=\; -\, \frac{\bar{\partial}}{\partial_-}\, q_+^m\, \bar{\phi} \;-\; \kappa\, \frac{1}{\partial_-} \left(\partial\, d^m\, \bar{\phi}\, \partial_-{}^2\, \bar{\phi} \;-\; \partial_-\, d^m\, \bar{\phi}\, \partial_-\, \partial\, \bar{\phi} \right) \;+\; \mathcal{O}(\kappa^2) \;. \tag{4.39}$$

We can now compute the quadratic form with these operators and find the Hamiltonian to order κ

$$\frac{1}{4\sqrt{2}}\, (\, \mathcal{W} \,,\, \mathcal{W}\,) \;=\; -\, \frac{2\, i}{4\sqrt{2}} \int d^4x\, d^8\theta\, d^8\, \bar{\theta}\; \overline{\mathcal{W}}\, \frac{1}{\partial_-{}^3}\, \mathcal{W} \;. \tag{4.40}$$

The existence of a quadratic form structure is different from the statement that the Hamiltonian is the anticommutator of the supersymmetries

$$\{\, Q_-^m \,,\, \overline{Q}_{-n} \,\} \;=\; -\sqrt{2}\, \delta^m_n\; P^- \;,$$

since the above relation is a sum of two products, unlike the expression in (4.40). The key ingredient in the proof of the quadratic form structure is the "inside-out" relation, which is special to maximally supersymmetric theories. The only other supersymmetric theory in four dimensions, which admits such a quadratic form Hamiltonian is the $\mathcal{N} = 4$ superYang-Mills theory in the light-cone gauge as was shown in [33]. The superfield in this case also satisfies a similar "inside-out" constraint which is crucial for the quadratic form structure to work.

4.5 $E_{7(7)}$ symmetry

The non-linear $E_{7(7)}$ was discovered by Cremmer and Julia [16], which turned out to play a pivotal role in the construction of the $\mathcal{N} = 8$ supergravity theory in the covariant formalism. The $E_{7(7)}$ symmetry in this formulation acts only on the scalar and vector fields of the theory. It is a symmetry at the level of equations of motion and not the action. $E_{7(7)}$ is a non-compact 133-dimensional exceptional Lie group. The $E_{7(7)}$ can be understood in the following way.

$$E_{7(7)} \;=\; SU(8)\;\times\;\frac{E_{7(7)}}{SU(8)}\;;\quad \mathbf{133}\;=\;\mathbf{63}\;+\;\mathbf{70} \tag{4.41}$$

The $SU(8)$ is the R-symmetry group, which rotate the supersymmetries into each other. It is important to note that $SU(8)$ is the maximal compact subgroup of E_7. The '(7)' in the $E_{7(7)}$, stands for $70 - 63 = 7$, which is the number of non-compact generators minus the number of compact generators in the group. The 70 coset transformations are related to a duality symmetry of the vector fields and a sigma-model symmetry of the 70 scalar fields in the theory. Two of these coset transformations close on a $SU(8)$ transformation

$$[\,\delta_{E_{7(7)}/SU8}\,,\,\delta_{E_{7(7)}/SU8}\,]\;\;=\;\;\delta_{SU(8)}\;. \tag{4.42}$$

$E_{7(7)}$ symmetry in the light-cone superspace

The $E_{7(7)}$ symmetry was formulated in the light-cone superspace in [36], where some important differences from the covariant description were observed. We start with the Lagrangian and the $E_{7(7)}/SU(8)$ coset transformations in the covariant formulation. After light-cone gauge-fixing the Lagrangian and eliminating time derivatives from the resulting Lagrangian using field redefinitions, we find that the coset transformation for the scalars involves terms quadratic in the vector fields. This sort of mixing does not occur in the covariant formalism. This indicates that the $E_{7(7)}/SU(8)$ variations are incomplete. We must include the variation of the other fields in the supermultiplet for the $E_{7(7)}$ algebra to close properly. We find the variation of the other fields from the requirement that the kinematical supersymmetries must commute with the coset variations

$$[\delta^{kin}_{\bar q}\;,\;\delta_{E_{7(7)}/SU8}]\;\phi\;\;=\;\;0\;. \tag{4.43}$$

The kinematical supersymmetries[1] are given by the linear action of $\bar q_m$, q^m on ϕ

$$\delta^{kin}_{\bar s}\phi(y)\;=\;\bar\epsilon_m q^m\phi(y)\;,\quad \delta^{kin}_{s}\phi(y)\;=\;\epsilon^m\bar q_m\phi(y)\;. \tag{4.44}$$

[1] We use the notation 's' for $\epsilon^m\bar q_m$ and '$\bar s$' for $\bar\epsilon_m q^m$.

The non-linear $E_{7(7)}/SU(8)$ transformations for all the fields to order κ can be written in the superfield language as [36]

$$\delta\phi = -\frac{2}{\kappa}\,\theta^{klmn}\,\overline{\Xi}_{klmn} + \frac{\kappa}{4!}\,\Xi^{mnpq}\frac{1}{\partial^{+2}}\left(\overline{d}_{mnpq}\frac{1}{\partial^{+}}\phi\,\partial^{+3}\phi \,-\, 4\,\overline{d}_{mnp}\phi\,\overline{d}_q\partial^{+2}\phi \,+\, 3\,\overline{d}_{mn}\partial^{+}\phi\,\overline{d}_{pq}\partial^{+}\phi\right), \tag{4.45}$$

where

$$\theta^{klmn} = \theta^k\theta^l\theta^m\theta^n\ , \quad \overline{d}_{m_1\ldots m_n} = \bar{d}_{m_1}\ldots.\bar{d}_{m_n}$$

and

$$\overline{\Xi}_{klmn} = \frac{1}{2}\epsilon_{klmnpqrs}\,\Xi^{pqrs}$$

are 70 real parameters. Note that only the scalars $\overline{C}^{mnpq}$ have an order κ^{-1}, which is in keeping with the fact that these fields exhibit a sigma-model like symmetry.

The $SU(8)$ generators are given in terms of the kinematical supersymmtries

$$T^m{}_n \;=\; \frac{i}{2\sqrt{2}\,\delta^+}\left(q^m\bar{q}_n \,-\, \frac{1}{8}\,\delta^m{}_n\,q^p\bar{q}_p\right)\ , \tag{4.46}$$

which satisfy the algebra

$$[\,T^m{}_n\,,\,T^p{}_q\,] \;=\; \delta^p{}_n\,T^m{}_q - \delta^m{}_q\,T^p{}_n\ . \tag{4.47}$$

The chiral superfield transform linearly under the $SU(8)$

$$\delta_{SU(8)}\;\phi(y) \;=\; \omega^j{}_i\,T^i_j\;\phi(y)$$

where $\omega^j{}_i$ are the 63 $SU(8)$ parameters. These transformations along with the 70 non-linear coset transformations constitute the entire $E_{7(7)}$ algebra.

The coset transformation (4.45) can be expressed in a compact coherent-state like notation[2]

$$\delta\phi \;=\; -\frac{2}{\kappa}\,\theta^{mnpq}\,\overline{\Xi}_{mnpq} + \frac{\kappa}{4!}\,\Xi^{mnpq}\left(\frac{\partial}{\partial\eta}\right)_{mnpq}\frac{1}{\partial^{+2}}\left(e^{\eta\hat{\bar{d}}}\partial^{+3}\phi\,e^{-\eta\hat{\bar{d}}}\partial^{+3}\phi\right)\Bigg|_{\eta=0} + \mathcal{O}(\kappa^2), \tag{4.48}$$

[2]In this notation, $\big|_{\eta=0}$ means we differentiate with respect to η four times and then set η to zero.

where

$$\eta\hat{\bar{d}} = \eta^m \frac{\bar{d}_m}{\partial^+}, \quad \text{and} \quad \left(\frac{\partial}{\partial\eta}\right)_{mnpq} \equiv \frac{\partial}{\partial\eta^m}\frac{\partial}{\partial\eta^n}\frac{\partial}{\partial\eta^p}\frac{\partial}{\partial\eta^q} \ . \tag{4.49}$$

In this formalism, the $E_{7(7)}$ symmetry transforms all the physical fields in the supermultiplet including the graviton. This is an important point of difference from the covariant formulation of $\mathcal{N} = 8$ supergravity, where this symmetry only affects the vector and scalar fields in the theory. It can be checked that the $E_{7(7)}$ symmetry leaves the light-cone action invariant at least up to order κ^2. Therefore, in the light-cone frame it is a symmetry at the level of the action and not just the equations of motion. In that sense, the $E_{7((7)}$ symmetry is as genuine a symmetry as is the supersymmetry. Another important point to note is that in the covariant formalism, the process of upgrading the $SU(8)$ to the $E_{7(7)}$ involves invoking the duality symmetry of the vector fields. There are no such duality transformations in the light-cone formalism as we have to deal with the real degrees of freedom of the theory only. The effect of the duality transformations is thus achieved by a field redefinition in the light-cone superspace.

Dynamical supersymmetry

We can now exploit the $E_{7(7)}$ symmetry to derive interaction terms that appear in the light-cone Hamiltonian. This is achieved by extending the commutativity of $E_{7(7)}/SU(8)$ coset with the supersymmetries to the dynamical generators. By requiring that the dynamical supersymmetries commute with the coset variation of the superfield

$$[\,\delta^{dyn}_{\bar{q}}\ ,\ \delta_{E_{7(7)}/SU(8)}\,]\,\phi \ \ = \ \ 0\ , \tag{4.50}$$

we can fix the form of $\delta^{dyn}_q\,\phi$. Here, the non-linearity of the $E_{7(7)}$ transformations turns out to be an advantage because these transformations link interaction terms of different order in κ. Hence, we can start from the lowest order and build up the interactions with higher powers of κ by closing the commutator (4.50) at the appropriate order in perturbation.

The action of the dynamical supersymmetries on the superfield reads

$$\begin{aligned} \delta^{dyn}_s\,\phi \ &= \ \delta^{dyn(0)}_s\,\phi \ + \ \delta^{dyn(1)}_s\,\phi \ + \ \delta^{dyn(2)}_s\,\phi \ + \mathcal{O}(\kappa^3) \qquad (4.51) \\ &= \ \epsilon^m \left\{ \frac{\partial}{\partial^+}\,\bar{q}_m\phi \ + \ \kappa\,\frac{1}{\partial^+}\,\left(\bar{\partial}\bar{d}_m\phi\,\partial^{+2}\phi \ - \ \partial^+\bar{d}_m\phi\,\partial^+\bar{\partial}\phi\right) \ + \ \mathcal{O}(\kappa^2) \right\} \ . \end{aligned}$$

The order κ terms were derived from the light-cone action [25], but the commutator (4.50) also leads to the same expression. Note that in (4.30), the dynamical supersymmetry generator is written in terms of $\bar{q}_m$, instead of $\bar{d}_m$. But it can be checked that both the forms are equivalent.

Next we derive the interaction terms of $\mathcal{O}(\kappa^2)$ from the $E_{7(7)}$ symmetry by closing the commutator (4.50) to order κ

$$[\,\delta^{(-1)}_{E_{7(7)}/SU(8)}\,,\,\delta^{dyn(2)}_s\,]\,\phi\;+\;[\,\delta^{(1)}_{E_{7(7)}/SU(8)}\,,\,\delta^{dyn(0)}_s\,]\,\phi\;\;=\;\;0\;. \tag{4.52}$$

The above relation completely fixes the form of $\delta^{dyn(2)}_s\,\phi$. Thus the dynamical supersymmetry to order κ^2 written in the coherent-state like notation reads [36]

$$\begin{aligned}\delta^{dyn}_s\,\phi\;\;&=\;\;\frac{\partial}{\partial a}\frac{\partial}{\partial b}\Bigg\{e^{a\hat{\partial}}e^{b\epsilon\hat{\bar{q}}\partial^+\phi}\;+\;\frac{\kappa}{2}\frac{1}{\partial^+}\Big(e^{a\hat{\partial}+b\epsilon\hat{\bar{q}}}\partial^{+2}\phi\,e^{-a\hat{\partial}-b\epsilon\hat{\bar{q}}}\partial^{+2}\phi\Big)\\&\quad+\;\frac{\kappa^2}{2.4!}\left(\frac{\partial}{\partial\eta}\right)_{ijkl}\frac{1}{\partial^{+4}}\Big(e^{a\hat{\partial}+b\epsilon\hat{\bar{q}}+\eta\hat{\bar{d}}}\partial^{+5}\phi\,e^{-a\hat{\partial}-b\epsilon\hat{\bar{q}}-\eta\hat{\bar{d}}}Z^{ijkl}\Big)\;+\;\mathcal{O}(\kappa^3)\Bigg\}\Bigg|_{a=b=\eta=0}\;.\end{aligned} \tag{4.53}$$

We can now invoke the quadratic form property of the light-cone Hamiltonian, which was explicitly shown to hold to order κ^2 [31].

$$\mathcal{H}\;\;=\;\;\frac{1}{4\sqrt{2}}\,(\,\mathcal{W}\,,\,\mathcal{W}\,)\;=-\,\frac{2\,i}{4\sqrt{2}}\int d^4x\,d^8\theta\,d^8\,\bar{\theta}\;\overline{\mathcal{W}}^m\,\frac{1}{\partial_-{}^3}\,\mathcal{W}_m\;, \tag{4.54}$$

where

$$\delta^{dyn}_s\phi\;\;=\;\;\epsilon^m\,\mathcal{W}_m\;. \tag{4.55}$$

Using this quadratic form structure the Hamiltonian to order κ^2 was found to be

$$\begin{aligned}\mathcal{H}^{\kappa^2}\;\;&=\;\;i\,\frac{\kappa^2}{4\sqrt{2}}\int\;d^8\theta\;d^8\bar{\theta}\;d^4x\;\frac{\partial}{\partial a}\,\frac{\partial}{\partial b}\\&\quad\Bigg\{\frac{1}{2}\frac{\partial}{\partial r}\frac{\partial}{\partial s}\frac{1}{\partial^{+5}}\Big(e^{a\hat{\partial}+b\hat{q}}\partial^{+2}\bar{\phi}\,e^{-a\hat{\partial}-b\hat{q}}\partial^{+2}\bar{\phi}\Big)\;\Big(e^{r\hat{\bar{\partial}}+s\hat{\bar{q}}}\partial^{+2}\phi\,e^{-r\hat{\bar{\partial}}-s\hat{\bar{q}}}\partial^{+2}\phi\Big)\\&\quad-\Bigg[\frac{1}{4!}\,\frac{\bar{\partial}}{\partial^{+4}}q^m d^{ijkl}\bar{\phi}\,\Big(e^{a\hat{\partial}+b\hat{\bar{q}}_m}\partial^{+}\bar{\phi}\,e^{-a\hat{\partial}-b\hat{\bar{q}}_m}\frac{1}{\partial^{+4}}Z_{ijkl}\Big)\;+\;c.c.\Bigg]\Bigg\}\Bigg|_{a=b=r=s=0}\;.\end{aligned} \tag{4.56}$$

This completes our discussion of ($\mathcal{N} = 8, d = 4$) supergravity in the light-cone formalism up to second order in the coupling constant. This formalism is particularly suitable for the study of the symmetries, since both the supersymmetry and the $E_{7(7)}$ symmetry are non-linearly realized on the light-cone superfield. In the next chapter, we show how this exceptional symmetry can be enhanced to a larger symmetry group.

Chapter 5

Enhancing the symmetry in $(\mathcal{N} = 8, d = 4)$ supergravity

The material presented here primarily based on work done by the author in [37].

In this chapter, we explore the key idea of realizing a larger symmetry in the four-dimensional theory, which is originally present in $d = 3$. We describe how the $E_{7(7)}$ symmetry in $\mathcal{N} = 8$ supergravity in four dimensions can be enhanced to an $E_{8(8)}$ symmetry. We devise a three-step method, which involves dimensional reduction to $d = 3$, a field redefinition in three dimensions and subsequent dimensional 'oxidation' to $d = 4$ with manifest $E_{8(8)}$. Our analysis leads us to an action for the maximal supergravity in four dimensions, which shows an $E_{8(8)}$ invariance at least up to second order in the coupling constant.

5.1 Maximal supergravity in $d = 3$: Version 1

We dimensionally now reduce this $d = 4$ theory to three dimensions by removing the dependence on one of the transverse derivatives, ∂_2. As a result, we are left with $\partial = \bar{\partial} = \partial_1$ by definition. Thus we obtain the three-dimensional action for the maximal supergravity theory.

$$\mathcal{S} \;=\; \int d^3x\; d^8\theta\; d^8\bar{\theta}\; \mathcal{L}\;, \tag{5.1}$$

where

$$\mathcal{L} \;=\; -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;+\; \frac{4}{3}\,\kappa\left(\frac{1}{\partial^{+4}}\bar{\phi}\;\partial^2\phi\;{\partial^{+}}^2\phi \;-\; \frac{1}{\partial^{+4}}\bar{\phi}\;\partial^+\partial\phi\;\partial^+\partial\phi \;+\; c.c.\right) \;+\cdots\,, \tag{5.2}$$

where the d'Alembertian in three dimensions is given by

$$\Box \;=\; 2(\partial^2 \;-\; \partial^+\partial^-)\ . \tag{5.3}$$

Although this action looks similar to the four-dimensional action, the important difference here is that there is only one transverse derivative, ∂ in the theory now. Accordingly, this formulation inherits a cubic interaction vertex from its four-dimensional parent theory and shows no manifest $E_{8(8)}$ symmetry.

5.2 Maximal supergravity in $d = 3$: Version 2

We shall now discuss the maximal supergravity theory in three dimensions, $\mathcal{N} = 16$ supergravity originally constructed by Marcus and Schwarz [38], which exhibits a full $E_{8(8)}$ symmetry. The supermultiplet for this theory contains 128 scalars and 128 spin one-half fermions, which form two inequivalent spinor representations under $SO(16)$, the R-symmetry group. As a consequence, the Lagrangian for this theory does not have vertices of odd order in κ (κ, κ^3 etc.), since we cannot form invariants from an odd number of such 128-dimensional spinors.

In [39], this theory was formulated in the light-cone superspace. The virtue of the light-cone superspace is that the same chiral superfield ϕ introduced in four dimensions for $\mathcal{N} = 8$ supergravity can now be used to describe all the degrees of freedom in the three-dimensional theory

$$\mathbf{256} \;=\; \mathbf{128_b} \;+\; \mathbf{128_f}\ . \tag{5.4}$$

As mentioned above, the action for this theory does not admit a three-point coupling. Thus in the light-cone formalism the Lagrangian takes the form

$$\mathcal{L} \;=\; -\,\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;+\; \mathcal{O}(\kappa^2)\ . \tag{5.5}$$

5.2.1 $SO(16)$ symmetry

We start with the Grassmann variables, θ^m and $\bar{\theta}_m$ in the $\mathcal{N} = 8$ superspace, which now form a **16** of $SO(16)$

$$SO(16)\;\supset\;SU(8)\,\times\,U(1)\ , \qquad \mathbf{16}\;=\;\mathbf{8}+\overline{\mathbf{8}}\ .$$

The quadratic action of the q^m, $\bar{q}_m$ generators on ϕ yield the $SO(16)$ transformations.

The $SO(16)$ is split in terms of $SU(8) \times U(1)$ as follows

$$\mathbf{120} \;=\; \mathbf{63_0} \;+\; \mathbf{28_{-1}} \;+\; \overline{\mathbf{28}}_\mathbf{1} \;+\; \mathbf{1_0} \;. \tag{5.6}$$

The subscripts stand for the $U(1)$ charges. The 63 $SU(8)$ generators discussed in context of $\mathcal{N}=8$ supergravity, are given in (4.46).

The $U(1)$ generator is [39]

$$T \;=\; \frac{i}{4\sqrt{2}\,\partial^+}\,[\,q^m\,,\,\bar{q}_m\,]\;, \qquad [\,T\,,\,T^m{}_n\,] \;=\; 0\;. \tag{5.7}$$

The quadratic operators

$$T^{mn} \;=\; \frac{1}{2}\frac{1}{\partial^+}q^m q^n\;, \qquad T_{mn} \;=\; \frac{1}{2}\frac{1}{\partial^+}\bar{q}_m\bar{q}_n\;, \tag{5.8}$$

generate the remaining part of $SO(16)$, i.e. the coset $SO(16)/(SU(8) \times U(1))$.

The generators T^{mn} and T_{mn} form the $\mathbf{28}$ and the $\overline{\mathbf{28}}$ representation of $SU(8)$ and close on $(SU(8) \times U(1))$

$$[\,T^{mn}\,,\,T_{pq}\,] \;=\; \delta^n{}_p T^m{}_q \;-\; \delta^m{}_p T^n{}_q \;-\; \delta^m{}_q T^n{}_p \;+\; \delta^m{}_q T^n{}_p \;+\; 2\,(\,\delta^n{}_p\delta^m{}_q \;-\; \delta^n{}_q\delta^m{}_p\,)\,T\;.$$

Thus the 120 linear $SO(16)$ transformations can be listed as

$$\delta_{SU_8}\,\varphi \;=\; \omega^n{}_m\,T^m{}_n\,\varphi \quad,\quad \delta_{U(1)}\,\varphi \;=\; T\,\varphi\;,$$

$$\delta_{\mathbf{28}}\,\varphi \;=\; \alpha_{mn}\,\frac{q^m q^n}{\partial^+}\,\varphi \quad,\quad \delta_{\overline{\mathbf{28}}}\,\varphi \;=\; \alpha^{mn}\,\frac{\bar{q}_m\bar{q}_n}{\partial^+}\,\varphi\;, \tag{5.9}$$

where $\omega^n{}_m$, α_{mn}, and α^{mn} are the corresponding transformation parameters.

5.2.2 $E_{8(8)}$ symmetry

This theory, which is obtained by dimensional reduction from the eleven-dimensional supergravity theory, shows an $E_{8(8)}$ symmetry, similar to the $E_{7(7)}$ in the four-dimensional theory. $E_{8(8)}$ is a non-compact 248-dimensional exceptional Lie group. The 120-dimensional R-symmetry group, $SO(16)$ is the maximal compact subgroup of E_8. Hence, the '(8)' in the $E_{8(8)}$, stands for $128 - 120 = 8$, which is the number of non-compact generators minus the number of compact generators in the group.

The $E_{8(8)}$ symmetry can be constructed in a way similar to the $E_{7(7)}$

$$E_{8(8)} \;=\; SO(16)\;\times\;\frac{E_{8(8)}}{SO(16)}\;;\qquad \mathbf{248}\;=\;\mathbf{120}\;+\;\mathbf{128}\;. \tag{5.10}$$

The 128 $E_{8(8)}/SO(16)$ coset transformations are related to a sigma-model symmetry of the 128 scalar fields in the theory.

Two of these non-linear coset transformations close on a linear $SO(16)$ transformation

$$[\,\delta_{E_{8(8)}/SO(16)}\,,\,\delta_{E_{8(8)}/SO(16)}\,]\;=\;\delta_{SO(16)}\;. \tag{5.11}$$

The coset $E_{8(8)}/SO(16)$ can also be decomposed in terms of $SU(8)\times U(1)$ representations

$$\mathbf{128} = \mathbf{1}'_{\mathbf{2}} + \mathbf{28}'_{\mathbf{1}} + \mathbf{70}_{\mathbf{0}} + \overline{\mathbf{28}}'_{\mathbf{-1}} + \overline{\mathbf{1}}'_{\mathbf{-2}}\quad . \tag{5.12}$$

The $\mathbf{70_0}$ is identified with the representation in $E_{7(7)}/SU(8)$; the rest of the coset contains two $U(1)$ singlets $\mathbf{1}'_{\mathbf{2}}$ and $\overline{\mathbf{1}}'_{\mathbf{-2}}$, a twenty-eight dimensional representation $\mathbf{28}'_{\mathbf{1}}$ and its complex conjugate $\overline{\mathbf{28}'}_{\mathbf{-1}}$.

Note that these representations are not the same as the $\mathbf{28}$ and $\overline{\mathbf{28}}$ of the linear$SO(16)$.

In the light-cone superspace, the 128 $E_{8(8)}/SO(16)$ coset transformations written in the coherent-state notation reads [39]

$$\begin{aligned}\delta_{E_{8(8)}/SO(16)}\,\phi \;&=\; \frac{1}{\kappa}\,F\;+\;\kappa\,\epsilon^{m_1 m_2\ldots m_8}\sum_{c=-2}^{2}\left(\hat{\bar{d}}_{m_1 m_2\cdots m_{2(c+2)}}\partial^{+c}\,F\right)\\ &\times\left\{\left(\frac{\delta}{\delta\,\eta}\right)_{m_{2c+5}\cdots m_8}\partial^{+(c-2)}\left(e^{\eta\cdot\hat{\bar{d}}}\,\partial^{+(3-c)}\phi\,e^{-\eta\cdot\hat{\bar{d}}}\partial^{+(3-c)}\phi\right)\Big|_{\eta=0}\;+\;\mathcal{O}(\kappa^2)\right\},\end{aligned} \tag{5.13}$$

where the sum runs over the $U(1)$ charges $c = 2, 1, 0 - 1, -2$ of the bosonic fields and

$$\hat{\bar{d}}_{m_1 m_2\cdots m_{2(c+2)}}\;\equiv\;\hat{\bar{d}}_{m_1}\hat{\bar{d}}_{m_2}\cdots\hat{\bar{d}}_{2(c+2)}\;. \tag{5.14}$$

Since the bosons in this theory are scalars with a sigma model-like symmetry, all the bosonic components in ϕ contain a constant term in the coset variation at the lowest order. This is reflected in the explicit form of the parameter F

$$
\begin{aligned}
F \;=\;& \frac{1}{\partial^{+2}}\,\beta\,(y^-) \;+\; i\,\theta^{mn}\,\frac{1}{\delta^+}\,\overline{\beta}_{mn}\,(y^-) - \;\theta^{mnpq}\,\overline{\beta}_{mnpq}\,(y^-) + \\
& + i\widetilde{\theta}_{\;mn}\,\delta^+\,\beta^{mn}\,(y^-) + \;4\,\widetilde{\theta}\,\delta^{+2}\,\bar{\beta}\,(y^-)\;,
\end{aligned}
\tag{5.15}
$$

F is a collection of all the 128 transformation parameters, which can be further decomposed into $SU(8)$ representations as shown in the expression above. For instance, the parameters β and $\bar{\beta}$ correspond to the $\mathbf{1'_2}$ and $\mathbf{\bar{1}'_{-2}}$ in (5.12).

Thus we learn that in the light-cone superspace the same $\mathcal{N}=8$ supermultiplet can be used to represent both the $E_{8(8)}$ as well as the $E_{7(7)}$ symmetry.

Dynamical supersymmetry in $d=3$

The non-linear $E_{8(8)}$ symmetry can now be used to construct the dynamical supersymmetry, $\delta_s^{dyn}\phi$ in three dimensions. The corrections to $\delta_s^{dyn}\phi$ up to order κ^2 were derived in [39]

$$
\begin{aligned}
\delta_s^{dyn}\,\phi \;&= \epsilon^m\,\frac{\partial}{\partial^+}\;\bar{q}_m\,\phi \\
&+\;\frac{\kappa^2}{2}\sum_{c=-2}^{2}\;\frac{1}{\partial^{+(c+4)}}\left\{\frac{\delta}{\delta a}\;\frac{\delta}{\delta b}\left(\frac{\delta}{\delta\eta}\right)_{m_1m_2\ldots m_{2(c+2)}}\left(E\partial^{+(c+5)}\phi\;E^{-1}\right)\Bigg|_{a=b=\eta=0}\right. \\
&\left.\times\frac{\epsilon^{m_1m_2\ldots m_8}}{(4-2c)!}\left(\frac{\delta}{\delta\eta}\right)_{m_{2c+5}\ldots m_8}\;\partial^{+2c}\;\left(E\partial^{+(4-c)}\phi E^{-1}\;\partial^{+(4-c)}\phi\;\right)\Bigg|_{\eta=0}\right\}\;,
\end{aligned}
\tag{5.16}
$$

where

$$
E\equiv e^{a\hat{\partial}\,+\,b\epsilon\hat{\bar{q}}\,+\,\eta\hat{\bar{d}}}\;\;\text{and}\;\;E^{-1}\equiv e^{-a\hat{\partial}\,-\,b\epsilon\hat{\bar{q}}\,-\,\eta\hat{\bar{d}}}\;,
$$

with

$$
a\,\hat{\partial}\;=\;a\frac{\partial}{\partial^+}\;,\qquad b\,\epsilon\hat{\bar{q}}\;=\;b\,\epsilon^m\frac{\bar{q}_m}{\partial^+},\qquad \eta\hat{\bar{d}}\;=\;\eta^m\frac{\bar{d}_m}{\partial^+}\;.
$$

It is apparent from the absence of an order κ term in the dynamical supersymmetry transformation that there cannot be a three-point coupling in the Lagrangian for this theory.

5.3 Relating the two versions : The field redefinition

It may seem puzzling that there exist two different forms of maximal supergravity in three dimensions, one obtained from dimensionally reducing ($\mathcal{N} = 8, d = 4$) supergravity and the $E_{8(8)}$ invariant one without a three-point coupling. We will now discuss how the two versions can be related by a field redefinition. We will show that the dimensionally reduced version also possesses an $SO(16)$ symmetry, albeit in a non-linearly realized manner.

The Lagrangian for the $E_{8(8)}$ invariant theory reads

$$\mathcal{L} \;=\; -\,\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi \;+\; \mathcal{O}(\kappa^2)\ . \tag{5.17}$$

We wish to find a suitable field redefinition that reproduces in (5.17) the order κ terms present in (5.2). As a starting point, we make the following ansatz on dimensional grounds

$$\phi \;=\; \phi' + \alpha\,\kappa\,\partial^{+A}\,(\partial^{+B}\phi'\,\partial^{+C}\phi') \;+\; \beta\,\kappa\,\partial^{+D}\,(\partial^{+E}\phi'\partial^{+F}\bar{\phi}')\ , \tag{5.18}$$

where α, β are some arbitrary constants and the integers A, B, C, D, E, F satisfy the following constraints

$$A + B + C = 2\ , \quad D + E + F = 2\ . \tag{5.19}$$

We find that the field redefinition [37]

$$\phi \rightarrow \phi \;=\; \phi' \;+\; \frac{1}{3}\,\kappa\,(\partial^{+}\phi'\,\partial^{+}\phi') \;+\; \frac{2}{3}\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}\,\phi'\,\partial^{+}\bar{\phi}'\right)\ , \tag{5.20}$$

correctly reproduces the cubic terms present in (5.2). The $(\phi'\bar{\phi}')$ term in the field redefinition achieves the action of replacing the time derivative ∂^{-} by $\frac{\partial^2}{\partial^+}$ in the interaction terms using the free equation of motion.

Verification of the field redefinition

Under the above field redefinition (5.20), the kinetic term in (5.2) takes the form

$$
\begin{aligned}
-\,\bar\phi\,\frac{\Box}{{\partial^+}^4}\phi \;&=\; -\,2\,\bar\phi\,\frac{(\partial^2\;-\;\partial^+\partial^-)}{{\partial^+}^4}\,\phi \\
&=\; -\,2\,\left\{\bar\phi'\;+\;\frac{1}{3}\,\kappa\,(\partial^+\bar\phi'\,\partial^+\bar\phi')\;+\;\frac{2}{3}\,\kappa\,{\partial^+}^4\left(\frac{1}{{\partial^+}^3}\bar\phi'\,\partial^+\phi'\right)\right\}\;\times \\
&\quad \frac{(\partial^2\;-\;\partial^+\partial^-)}{{\partial^+}^4}\;\left\{\phi'\;+\;\frac{1}{3}\,\kappa\,(\partial^+\phi'\,\partial^+\phi')\;+\;\frac{2}{3}\kappa\,{\partial^+}^4\left(\frac{1}{{\partial^+}^3}\,\phi'\,\partial^+\bar\phi'\right)\right\}\;.
\end{aligned}
$$

The lowest order term just reproduces the kinetic term. At order κ we only focus on the terms of the form $\bar\phi'\phi'\phi'$.

$$
-\,\frac{2}{3}\,\kappa\,\bar\phi'\,\frac{(\partial^2\;-\;\partial^+\partial^-)}{{\partial^+}^4}\,(\partial^+\phi'\,\partial^+\phi')-\frac{4}{3}\,\kappa\,{\partial^+}^4\left(\frac{1}{{\partial^+}^3}\bar\phi'\,\partial^+\phi'\right)\frac{(\partial^2\;-\;\partial^+\partial^-)}{{\partial^+}^4}\,\phi'\;=\;\mathcal{A}+\mathcal{B}
$$

The remaining terms of the kind $\bar\phi'\bar\phi'\phi'$ are simply the complex conjugate of these terms, which reproduce the other cubic vertex, $\kappa\;\bar\phi'\bar\phi'\phi'$ vertex in (5.2).

We further simplify $\mathcal{A}$ and $\mathcal{B}$ as follows.

$$
\begin{aligned}
\mathcal{A}\;&=\;-\,\frac{2}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\bar\phi'\,(\partial^2\;-\;\partial^+\partial^-)\,(\partial^+\phi'\,\partial^+\phi') \\
&=\;-\,\frac{4}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\phi'\,(\partial^+\partial^2\phi'\,\partial^+\phi'\;+\;\partial^+\partial\phi'\,\partial^+\partial\phi')\;+\;\frac{4}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\bar\phi'\,\partial^+(\partial^+\partial^-\phi'\,\partial^+\phi')
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{B}\;&=\;-\,\frac{4}{3}\,\kappa\,\left(\frac{1}{{\partial^+}^3}\bar\phi'\,\partial^+\phi'\right)(\partial^2\;-\;\partial^+\partial^-)\,\phi' \\
&=\;+\,\frac{4}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\bar\phi'\,\partial^+\,(\partial^2\phi'\,\partial^+\phi')\;-\;\frac{4}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\bar\phi'\,\partial^+\,(\partial^+\partial^-\phi'\,\partial^+\phi') \\
&=\;+\,\frac{4}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\bar\phi'\,(\partial^+\partial^2\phi'\,\partial^+\phi'\;+\;\partial^2\phi'\,{\partial^+}^2\phi')\;-\frac{4}{3}\,\kappa\,\frac{1}{{\partial^+}^4}\,\bar\phi'\,\partial^+\,(\partial^+\partial^-\phi'\,\partial^+\phi')
\end{aligned}
$$

Hence, we find that the field redefinition yields the following $\mathcal{O}(\kappa)$ terms

$$
\mathcal{A}\;+\;\mathcal{B}\;=\;\frac{4}{3}\,\kappa\left(\frac{1}{{\partial^+}^4}\bar\phi'\,\partial^2\phi'\,{\partial^+}^2\phi'\;-\;\frac{1}{{\partial^+}^4}\bar\phi'\,\partial^+\partial\phi'\,\partial^+\partial\phi'\;\right)\;.
$$

The new Lagrangian in terms of the redefined field ϕ' reads

$$\mathcal{L}' \;=\; -\bar{\phi}'\,\frac{\Box}{\partial^{+4}}\,\phi' \;+\; \frac{4}{3}\,\kappa\left(\frac{1}{\partial^{+4}}\bar{\phi}'\,\partial^2\phi'\,\partial^{+2}\phi' \;-\; \frac{1}{\partial^{+4}}\bar{\phi}'\,\partial^+\partial\phi'\,\partial^+\partial\phi' \;+\; c.c.\right)\;, \tag{5.21}$$

which is identical to (5.2), since at lowest order ϕ' is trivially equal to ϕ.

Thus the the dimensionally reduced action with a cubic vertex can be related to the $SO(16)$-invariant action sans a cubic vertex by means of a field redefinition.

5.3.1 $SO(16)$ symmetry revisited

We studied the linear realized $SO(16)$ transformations on the superfield ϕ (5.9). The $E_{8(8)}$-symmetric Lagrangian (5.17) is invariant under these transformations. At the free order, this implies

$$\delta\mathcal{L} \;=\; -\,(\delta\bar{\phi})\,\frac{\Box}{\partial^{+4}}\,\phi \;-\; \bar{\phi}\,\frac{\Box}{\partial^{+4}}\,(\delta\phi) \;=\; 0\;. \tag{5.22}$$

(Note that we use the notation $\delta_{SO(16)}\phi \,\equiv\, \delta\phi$ for simplicity.)

To examine the action of $SO(16)$ on the new superfield ϕ', we invert the field redefinition (5.20) and express ϕ in terms of ϕ'

$$\phi' \;=\; \phi \;-\; \frac{1}{3}\,\kappa\,(\partial^+\phi\,\partial^+\phi) \;-\; \frac{2}{3}\,\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}\phi\,\partial^+\bar{\phi}\right)\;, \tag{5.23}$$

$$\delta\phi' \;=\; \delta\phi - \frac{2}{3}\,\kappa\,(\partial^+(\delta\phi)\,\partial^+\phi) - \frac{2}{3}\,\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}(\delta\phi)\,\partial^+\bar{\phi}\right) - \frac{2}{3}\,\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}\phi\,\partial^+(\delta\bar{\phi})\right)\;.$$

We start with the variation of $\mathcal{L}'$ under the $SO(16)$ transformations

$$\delta\mathcal{L}' \;=\; \delta\mathcal{L}'_{kinetic} \;+\; \delta\mathcal{L}'_{cubic}\;, \tag{5.24}$$

where

$$\delta\mathcal{L}'_{kinetic} \;=\; -\,(\delta\bar{\phi}')\,\frac{\Box}{\partial^{+4}}\,\phi' \;-\; \bar{\phi}'\,\frac{\Box}{\partial^{+4}}\,(\delta\phi') \tag{5.25}$$

and

$$
\begin{aligned}
\delta\mathcal{L}'_{cubic} \;=\; &+\frac{4}{3}\,\kappa\Bigg(\frac{1}{\partial^{+4}}\overline{(\delta\phi')}\;\partial^2\phi'\;\partial^{+2}\phi' \;+\; \frac{1}{\partial^{+4}}\bar\phi'\;\partial^2(\delta\phi')\;\partial^{+2}\phi' \\
&+\frac{1}{\partial^{+4}}\bar\phi'\;\partial^2\phi'\partial^{+2}(\delta\phi') \;-\; \frac{1}{\partial^{+4}}(\delta\bar\phi')\;\partial^+\partial\phi'\;\partial^+\partial\phi' \\
&-2\,\frac{1}{\partial^{+4}}\bar\phi'\;\partial^+\partial(\delta\phi')\;\partial^+\partial\phi'\Bigg) \;+\; c.c.\;.
\end{aligned} \tag{5.26}
$$

Using (5.23) and retaining the terms only up to order κ, we find

$$
\begin{aligned}
\delta\mathcal{L}'_{kinetic} \;=\; &\left\{-\,(\delta\bar\phi)\,\frac{\Box}{\partial^{+4}}\,\phi \;-\; \bar\phi\,\frac{\Box}{\partial^{+4}}\,(\delta\phi)\right\} \\
&+\Bigg\{\frac{1}{3}\,\kappa\,(\delta\bar\phi)\,\frac{\Box}{\partial^{+4}}\,(\partial^+\phi\,\partial^+\phi) \;+\; \frac{2}{3}\,\kappa\,\bar\phi\,\frac{\Box}{\partial^{+4}}\,(\partial^+(\delta\phi)\,\partial^+\phi) \\
&+\frac{2}{3}\,\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}(\delta\bar\phi)\,\partial^+\phi\right)\frac{\Box}{\partial^{+4}}\phi \;+\frac{2}{3}\,\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}\bar\phi\,\partial^+(\delta\phi)\right)\frac{\Box}{\partial^{+4}}\phi \\
&+\frac{2}{3}\,\kappa\,\partial^{+4}\left(\frac{1}{\partial^{+3}}\bar\phi\,\partial^+\phi\right)\frac{\Box}{\partial^{+4}}(\delta\phi)\Bigg\} \;+\; c.c.\;.
\end{aligned} \tag{5.27}
$$

The free order terms cancel against each other, just like in eq. (5.22). Note that we have again considered the $(\bar\phi\phi\phi)$ terms in our discussion, as the other type $(\phi\bar\phi\bar\phi)$ are contained in the complex conjugate. Some simple manipulations lead us to

$$
\begin{aligned}
\delta\mathcal{L}'_{kinetic} \;=\; &-\frac{4}{3}\,\kappa\Bigg(\frac{1}{\partial^{+4}}\overline{(\delta\phi')}\;\partial^2\phi'\;\partial^{+2}\phi' + \frac{1}{\partial^{+4}}\bar\phi'\;\partial^2(\delta\phi')\;\partial^{+2}\phi' \\
&+\frac{1}{\partial^{+4}}\bar\phi'\;\partial^2\phi'\partial^{+2}(\delta\phi') \;-\; \frac{1}{\partial^{+4}}(\delta\bar\phi')\;\partial^+\partial\phi'\;\partial^+\partial\phi' \\
&-2\,\frac{1}{\partial^{+4}}\bar\phi'\;\partial^+\partial(\delta\phi')\;\partial^+\partial\phi'\Bigg) \;+\; c.c.\;,
\end{aligned} \tag{5.28}
$$

which exactly cancels against (5.26). This proves that the Lagrangian with a three-point coupling also possesses an $SO(16)$ symmetry, which is now non-linearly realized.

Before moving on to the $E_{8(8)}$, we would like to comment on this non-linear representation of the $SO(16)$. Supersymmetric theories in general have R-symmetry groups, which are manifest and are linearly realized on the supermultiplet. However, we find that the Lagrangian we obtained from the crude dimensional reduction of the $d = 4$ Lagrangian, does not have a manifest $SO(16)$ symmetry to start with. In order to 'see' this symmetry, we must resort to the field redefinition (5.20) which itself is non-linear in nature. Thus in the light-cone superspace approach we can study the same theory using different versions and examine how the symmetries of the theory are manifested with the help of (non-linear) field redefinitions.

5.3.2 $E_{8(8)}$ symmetry revisited

We now turn our attention to the $E_{8(8)}$ symmetry of the new Lagrangian $\mathcal{L}'$. We start with the 128 $E_{8(8)}/SO(16)$ coset transformations given in (5.13). Two of these coset transformations should close on $SO(16)$

$$[\delta'_1, \delta'_2]\phi \;=\; \delta_{SO(16)}\ \phi\ . \tag{5.29}$$

(We denote the coset transformations $\delta_{E_{8(8)}/SO(16)}$ here by $\delta'\phi$ for simplicity.)

We can use the inverse field redefinition (5.23) to express $\delta'\phi'$ in terms of $\delta'\phi$. Next we consider two coset transformations, δ'_1 and δ'_2 on ϕ'

$$\begin{aligned}[][\delta'_1, \delta'_2]\ \phi' \;&=\; [\delta'_1, \delta'_2]\ \phi \,-\, \frac{1}{3}\,\kappa\ [\delta'_1, \delta'_2](\partial^+\phi\ \partial^+\phi)\ \frac{2}{3}\,\kappa\ [\delta'_1, \delta'_2]\left\{\partial^{+4}\left(\frac{1}{\partial^{+3}}\phi\ \partial^+\bar{\phi}\right)\right\}\\ &=\; \delta_{SO(16)}\phi \,+\, \mathcal{X} \,+\, \mathcal{Y}\ .\end{aligned} \tag{5.30}$$

$\mathcal{X}$ and $\mathcal{Y}$ can be further simplified as follows.

$$\begin{aligned}\mathcal{X} \;&=\; -\,\frac{1}{3}\,\kappa\ [\delta'_1, \delta'_2](\partial^+\phi\ \partial^+\phi)\\ &=\; -\frac{2}{3}\,\kappa\ [\partial^+(\delta'_1\delta'_2\phi)\partial^+\phi \,+\, \partial^+(\delta'_2\phi)\partial^+(\delta'_1\phi) \,-\, \partial^+(\delta'_2\delta'_1\phi)\partial^+\phi \,-\, \partial^+(\delta'_1\phi)\partial^+(\delta'_2\phi)]\\ &=\; -\frac{2}{3}\,\kappa\ (\partial^+[\delta'_1, \delta'_2]\phi\ \partial^+\phi)\\ &=\; -\frac{1}{3}\,\kappa\ \delta_{SO(16)}(\partial^+\phi\ \partial^+\phi)\end{aligned}$$

and

$$\mathcal{Y} \;=\; -\frac{2}{3}\,\kappa\ [\delta'_1, \delta'_2]\left\{\partial^{+4}\left(\frac{1}{\partial^{+3}}\phi\ \partial^+\bar{\phi}\right)\right\} \;=\; -\frac{2}{3}\,\kappa\ \delta_{SO(16)}\left\{\partial^{+4}\left(\frac{1}{\partial^{+3}}\phi\ \partial^+\bar{\phi}\right)\right\}.$$

Using (5.23) we find

$$
\begin{aligned}
[\delta_1', \delta_2']\ \phi' &= \delta_{SO(16)}\phi\ - \frac{1}{3}\ \kappa\ \delta_{SO(16)}(\partial^+\phi\ \partial^+\phi) - \frac{2}{3}\ \kappa\ \delta_{SO(16)}\left\{\partial^{+4}\left(\frac{1}{\partial^{+3}}\phi\ \partial^+\bar{\phi}\right)\right\} \\
&= \delta_{SO(16)}\phi'\ ,
\end{aligned}
\tag{5.31}
$$

where the r.h.s. is simply the non-linear $SO(16)$ on the new superfield ϕ'. This renders $\mathcal{L}'$ invariant under the non-linear $E_{8(8)}$ symmetry.

Thus we have established a link between the dimensionally reduced Lagrangian with a cubic vertex and the Lagrangian without a cubic coupling, showing that the two versions are in fact equivalent at least up to order κ^2. Further, using the field redefinition we have proved that both the versions have $SO(16)$ and $E_{8(8)}$ symmetries, the only subtle difference being that the $SO(16)$ in one of the cases acts non-linearly on the fields.

5.4 Enhanced symmetry in four dimensions

Having discussed the two forms of the maximal supergravity Lagrangian in $d = 3$, we now move on to the third step in our analysis - to lift the theory back to $d = 4$ preserving the $E_{8(8)}$ symmetry. This is accomplished by carefully reintroducing the transverse derivative, ∂_2 in the theory.

We go back to the dynamical supersymmetry in three dimensions discussed before

$$
\begin{aligned}
\delta_s^{dyn}\,\phi\ &= \epsilon^m\frac{\partial}{\partial^+}\ \bar{q}_m\,\phi \\
&+\ \frac{\kappa^2}{2}\sum_{c=-2}^{2}\ \frac{1}{\partial^{+(c+4)}}\left\{\frac{\partial}{\partial a}\ \frac{\partial}{\partial b}\left(\frac{\partial}{\partial \eta}\right)_{m_1m_2\ldots m_{2(c+2)}}\left(E\partial^{+(c+5)}\phi\ E^{-1}\right)\Big|_{a=b=\eta=0}\right. \\
&\left.\times\frac{\epsilon^{m_1m_2\ldots m_8}}{(4-2c)!}\left(\frac{\partial}{\partial\eta}\right)_{m_{2c+5}\ldots m_8}\ \partial^{+2c}\ \left(E\partial^{+(4-c)}\phi E^{-1}\ \partial^{+(4-c)}\phi\ \right)\Big|_{\eta=0}\right\}\ .
\end{aligned}
\tag{5.32}
$$

We 'oxidize' the above expression to four dimensions by introducing a generalized derivative, in the spirit of [40, 41]

$$
\nabla\ \equiv\ \partial_1\ +\ i\ \partial_2\ , \tag{5.33}
$$

which replaces the transverse derivative $\partial\,(=\partial_1)$ in all the places, such that

$$\left[\delta_s^{dyn}\,\phi\,(\partial,\partial^+,\bar{q}_m,\bar{d}_m,\phi)\right]_{d=3}\;\longrightarrow\;\left[\delta_s^{dyn}\,\phi(\nabla,\partial^+,\bar{q}_m,\bar{d}_m,\phi)\right]_{d=4}\,.$$

Once we have obtained the expression for the dynamical supersymmetry in four dimensions, it is straightforward to write down the light-cone Hamiltonian using the quadratic form structure (4.40) discussed in the previous chapter.

$$\mathcal{H}\;=\;\frac{1}{4\sqrt{2}}\,(\mathcal{W}_m,\mathcal{W}_m)\;\equiv\;\frac{2i}{4\sqrt{2}}\int d^8\theta\;d^8\bar{\theta}\;d^3x\;\overline{\mathcal{W}}^m\,\frac{1}{\partial^{+3}}\,\mathcal{W}_m\;, \tag{5.34}$$

where

$$\delta_s^{dyn}\,\phi\;\equiv\;\epsilon^m\,\mathcal{W}_m\,. \tag{5.35}$$

In order to construct the Hamiltonian, we need to consider the complex conjugate of $\mathcal{W}_m$, which further introduces a "new" derivative in theory

$$\partial_1\;-\;i\,\partial_2\;\equiv\overline{\nabla}\;. \tag{5.36}$$

The exact form of these derivatives, ∇ and $\overline{\nabla}$ are uniquely fixed from the requirement that the superPoincaré algebra closes in four dimensions.

This method of oxidation evidently preserves both the $SO(16)$ and the full E_8 symmetry in four dimensions. The generalized derivatives do not involve any q^m, $\bar{q}_m$ (or d^m, $\bar{d}_m$) operators, which can affect invariance of the theory under these symmetry transformations. We thus arrive at a *Lorentz invariant* Hamiltonian for maximal supergravity in $d=4$ with the same field as in (4.56), but with manifest $E_{8(8)}$ symmetry at least up to the second order in the coupling constant. Since maximal supergravity theories in any given dimension is unique, these two formulations of $(\mathcal{N}=8,d=4)$ supergravity must be related via a field redefinition.

The reason why we chose to go down to three dimensions to achieve this symmetry enhancement is that this procedure allows us to render the larger E_8 symmetry manifest in the three-dimensional theory, which is not present in four dimensions. After studying the symmetry in the lower-dimensional theory, we can oxidize the theory in a careful manner preserving this enhanced symmetry in $d=4$. In order to see the E_8 symmetry manifestly in the original $(\mathcal{N}=8,d=4)$ supergravity Lagrangian, we must figure out a way to interpret the physical states of the theory as 128-dimensional $SO(16)$ spinors which mix fields with different spins, instead of the usual $SU(8)$ representations where this mixing does not occur. Such an interpretation is crucial, if we wish to look for signatures of this E_8 symmetry in the scattering amplitudes of the four-dimensional theory. We shall return to this point in the last chapter and make some remarks regarding the possible implications of this enhanced symmetry.

Chapter 6

Exceptional symmetries in eleven-dimensional supergravity

The material presented here is primarily based on work done by the author in [42]

In this chapter, we extend the idea for symmetry enhancement to a more general case, in order to realize an exceptional symmetry in eleven-dimensional supergravity. In [14], the classical action for ($\mathcal{N} = 1, d = 11$) supergravity was constructed, which is the only known theory in eleven dimensions with local supersymmetry containing no fields of spin higher than two. We first discuss how the eleven-dimensional theory on reduction to four dimensions leads to $\mathcal{N} = 8$ supergravity. With the $\mathcal{N} = 8$ theory as a starting point, we then present two approaches to 'oxidize' the theory to its eleven-dimensional parent, making either the maximal supersymmetry or an $E_{7(7)}$ symmetry manifest in the process.

6.1 Eleven-dimensional supergravity

The field content of $\mathcal{N} = 1$ supergravity in eleven dimensions consists of a graviton G^{MN}, an antisymmetric rank three tensor A^{MNP} and a spin three-half Rarita-Schwinger field Ψ^M, where the $M, N, P.. = 1, 2...9$ are $SO(9)$ indices. The fields are in the irreducible representations of the transverse little group $SO(9)$. The bosonic part of the supermultiplet corresponds to a **128** of $SO(9)$

$$\mathbf{128} \;=\; \mathbf{44} \;+\; \mathbf{84} \tag{6.1}$$

with the graviton transforming as a **44** and the 3-form as a **84**. Similarly, the fermionic field Ψ^M transforms as a spinorial **128** under the $SO(9)$. Together these form the 256 physical degrees of freedom of the theory.

The covariant action for $\mathcal{N}=1$ supergravity was first presented in [16]. In [43], the theory was constructed in the light-cone gauge in component form by gauge-fixing this covariant action. We do not present the explicit form of this action, as the details are not relevant for the purpose of this thesis. Instead, we focus on an alternative description of the theory formulated in the $\mathcal{N}=8$ light-cone superspace discussed before. Intuitively a superspace formulation for the $d=11$ theory with $\mathcal{N}=1$ supersymmetry may seem unnecessary at this point. However, we will illustrate how this superspace formulation helps us unveil some rich structures in the theory.

6.1.1 Dimensional reduction to $d = 4$

The dimensional reduction of $\mathcal{N}=1$ supergravity to four dimensions leads to the $\mathcal{N}=8$ supergravity theory. This approach of dimensional reduction was instrumental in the discovery of the $E_{7(7)}$ symmetry in four dimensions, which subsequently led to the elegant $\mathcal{N}=8$ supergravity Lagrangian in the covariant formalism [16]. On reduction to four dimensions, the $SO(9)$ little group in $d=11$ decomposes as

$$SO(9) \;\supset\; SO(2)\times SO(7)\ . \tag{6.2}$$

The $SO(2)$ is the little group in four dimensions and the $SO(7)$ part now forms an internal symmetry. Under the $SO(7)$, the eleven-dimensional states split as

- G^{MN} : $\mathbf{44}$ of $SO(9) \quad\rightarrow\quad$ 1 graviton, 7 vectors and 28 scalars

$$\mathbf{44} \;=\; [\mathbf{1}\times 2] \;+\; [\mathbf{7}\times 2] \;+\; [\mathbf{28}\times 1]$$

- A^{MNP} : $\mathbf{84}$ of $SO(9) \quad\rightarrow\quad$ 21 vectors and $7+35$ scalars

$$\mathbf{84} \;=\; [\mathbf{21}\times 2] \;+\; [\mathbf{7}\times 1] \;+\; [\mathbf{35}\times 1]$$

- Ψ^{M} : $\mathbf{128}$ of $SO(9) \quad\rightarrow\quad$ 8 spin three-half and 56 spin one-half fermions

$$\mathbf{128} \;=\; [\mathbf{8}\times 2] \;+\; [\mathbf{56}\times 2]\ .$$

The $SO(7)$ can be upgraded to an $SU(8)$, under which the vectors and scalars in the $SO(7)$ representations recombine and form a $\mathbf{28}$ and a $\mathbf{70}$ of $SU(8)$. The $SU(8)$

along with the sigma-model like symmetry of these 70 scalar fields further leads to the $E_{7(7)}$ symmetry as we discussed in chapter 4. Thus the **44** + **84** + **128** in eleven dimensions now describes the 256-dimensional supermultiplet in ($\mathcal{N} = 8, d = 4$) supergravity theory. In the subsequent sections, we will show how we can turn the argument around and describe the eleven-dimensional theory in terms of these four-dimensional states. The advantage of working in the light-cone gauge is that we do not have to include any auxiliary fields in our analysis and describe the dynamics solely in terms of the 256 physical degrees of freedom in any given spacetime dimension.

With the ($\mathcal{N} = 8, d = 4$) supergravity theory as a starting point, we now present two different approaches to construct the eleven-dimensional theory in the light-cone superspace.

6.2 Oxidation from $d = 4$ to $d = 11$: Method 1

In this section, we discuss a method to restore the $\mathcal{N} = 8$ theory to its eleven-dimensional parent without altering the superfield. This is achieved by enlarging the superPoincaré algebra by introducing an $SO(7)$ corresponding to seven 'new' coordinates x^m and their derivatives ∂^m. This $SO(7)$ part of the superPoincaré algebra on reduction from $d = 11$ to $d = 4$ was upgraded to an $SU(8)$. However, when 'oxidizing' the four-dimensional theory back to $d = 11$, it is the $SO(7)$ that is relevant. This $SO(7)$ along with the $SO(2)$ little group in four dimensions form the $SO(9)$ transverse little group in eleven dimensions.

$$SO(9) \;\supset\; SO(2) \times SO(7) \; . \tag{6.3}$$

So, we give up the notion of manifest $SU(8)$ and switch to $SO(7)$ in our discussion. To implement this, we now interpret the superspace coordinates θ as spinors, θ^α under the $SO(2) \times SO(7)$. Henceforth, we use α, β indices for the spinors and $m, n, p, q = 4, \ldots, 7$ for the vector indices of $SO(7)$ (not to be confused with the $SU(8)$ indices used previously). The explicit form of the superfield remains unchanged. However, all the fields now depend on the extra coordinates as well.

$$\phi\,(\, x, \bar{x}, x^m, y^-) \;=\; \frac{1}{\partial^{+2}}\; h(x, \bar{x}, x^m, y^-) \;+\; \cdots \tag{6.4}$$

6.2.1 SuperPoincaré algebra in $d = 11$

We now discuss the light-cone representation of the superPoincaré algebra in $d = 11$. The $SO(9)$ can be constructed in terms of the $(SO(2) \times SO(7))$ as follows [40].

$$SO(9) \;=\; (\,SO(2)\,\times\,SO(7)\,)\;\times\;\frac{SO(9)}{(\,SO(2)\,\times\,SO(7)\,)}\;.$$

The $SO(2)$ generator, J does not get modified. The $SO(7)$ generators are given by

$$J^{mn} \;=\; -\,i\,(\,x^m\,\partial^n\;-\;x^n\,\partial^n\,)\;-\;\frac{1}{2\sqrt{2}}\,q^\alpha\,(\gamma^{mn})^{\alpha\beta}\,\bar{q}_\beta. \qquad (6.5)$$

We need to introduce the generators for the coset $SO(9)/(SO(2) \times SO(7))$

$$\begin{aligned} J^m \;&=\; -\,i\,(\,x\,\partial^m\;-\;x^m\,\partial\,)\;+\;\frac{i}{4\sqrt{2}\,\partial^+}\,q_\alpha\,(\,\gamma^m)^{\alpha\,\beta}\,q_\beta\;, \\ \bar{J}^{\,n} \;&=\; -\,i\,(\,\bar{x}\,\partial^n\;-\;x^n\,\bar{\partial}\,)\;+\;\frac{i}{4\sqrt{2}\,\partial^+}\,\bar{q}_\alpha\,(\,\gamma^n)^{\alpha\,\beta}\,\bar{q}_\beta\;, \end{aligned} \qquad (6.6)$$

which satisfy

$$\begin{aligned} \Big[\,J\,,\,J^m\,\Big] \;&=\; J^m\;, \qquad \Big[\,J\,,\,\bar{J}^n\,\Big] \;=\; -\,\bar{J}^n\;, \\ \Big[\,J^{pq}\,,\,J^m\,\Big] \;&=\; \delta^{pm}\,J^q\;-\;\delta^{qm}\,J^p\;, \\ \Big[\,J^m\,,\,\bar{J}^n\,\Big] \;&=\; i\,J^{mn}\;+\;\delta^{mn}\,J\;. \end{aligned} \qquad (6.7)$$

Thus the entire $SO(9)$ transverse algebra is spanned by J, J^{mn}, J^m and $\bar{J}^n$. The action of these rotations preserve the chirality of the superfield ϕ and its conjugate $\bar{\phi}$. The rest of the kinematical generators remain the same

$$J^+\;=\;j^+\;, \qquad J^{+-}\;=\;j^{+-}\;. \qquad (6.8)$$

Also, there are new kinematical generators in the theory

$$J^{+\,m}\;=\;i\,x^m\,\partial^+\;; \qquad \bar{J}^{+\,n}\;=\;i\,\bar{x}^n\,\partial^+\;. \qquad (6.9)$$

The free part of the dynamical boosts get modified in the following way

$$
\begin{aligned}
J^- \;&=\; i\,x\,\frac{\partial\bar\partial\,+\,\frac{1}{2}\,\partial^m\,\partial^m}{\partial^+}\;-\;i\,x^-\,\partial+i\,\frac{\partial}{\partial^+}\left\{\theta^\alpha\,\bar\partial_\alpha\;+\;\frac{i}{4\sqrt{2}\,\partial^+}\,(d^\alpha\,\bar d_\alpha-\bar d_\alpha\,d^\alpha)\right\}\\
&\quad-\frac{1}{4}\,\frac{\partial_m}{\partial^+}\left\{\partial^+\,\theta^\alpha\,(\,\gamma^m)_{\alpha\,\beta}\,\theta^\beta\;-\;\frac{2}{\partial^+}\;\partial^\alpha\,(\,\gamma^m)_{\alpha\,\beta}\,\partial^\beta\;+\;\frac{1}{\partial^+}\;d^\alpha\,(\,\gamma^m)_{\alpha\,\beta}\,d^\beta\right\}\;.
\end{aligned}
\tag{6.10}
$$

The $SO(9)/(SO(2)\times SO(7))$ rotations yield the remaining boost operators

$$
J^{-\,m}\;=\;[\,J^-\,,\,J^m\,]\;;\qquad \bar J^{-\,n}\;=\;[\,\bar J^-\,,\,\bar J^n\,]\;. \tag{6.11}
$$

Therefore, the dynamical supersymmetries now read

$$
\begin{aligned}
[\,J^-\,,\,\bar q_{+\,\eta}\,]\;\equiv\;\overline{\mathcal{Q}}_\eta\;&=\;-\,i\,\frac{\partial}{\partial^+}\,\bar q_{+\,\eta}\;-\;\frac{i}{\sqrt{2}}\,(\,\gamma^n\,)_{\,\eta\,\rho}\,q_+{}^{\rho}\,\frac{\partial^n}{\partial^+}\;,\\
[\,\bar J^-\,,\,q_+^\alpha\,]\;\equiv\;\mathcal{Q}^\alpha\;&=\;i\,\frac{\bar\partial}{\partial^+}\,q_+{}^\alpha\;+\;\frac{i}{\sqrt{2}}\,(\,\gamma^m\,)^{\,\alpha\,\beta}\,\bar q_{+\,\beta}\,\frac{\partial^m}{\partial^+}\;,
\end{aligned}
\tag{6.12}
$$

satisfying

$$
\{\,\mathcal{Q}^\alpha\,,\,q_+^\eta\,\}\;=\;-\,(\,\gamma^m\,)^{\alpha\,\eta}\,\partial^m\;. \tag{6.13}
$$

Finally, the supersymmetry algebra takes the form

$$
\{\,\mathcal{Q}^\alpha\,,\,\overline{\mathcal{Q}}_\eta\,\}\;=\;i\,\sqrt{2}\;\delta^\alpha{}_\eta\,\frac{1}{\partial^+}\left(\partial\,\bar\partial\;+\;\frac{1}{2}\,\partial^m\,\partial^m\right)\;=\;-\,\sqrt{2}\,P^-\;. \tag{6.14}
$$

Note that we have suppressed the '-' indices on the dynamical supersymmetries to keep our notations simple.

6.2.2 Dynamical supersymmetry in $d = 11$

We now focus on constructing the dynamical supersymmetry in $d = 11$ to first order in κ preserving the superPoincaré symmetry. Once we derive the explicit from of the dynamical supersymmetry transformations on the superfield, the Hamiltonian for the theory simply follows from the supersymmetry algebra (6.14). We start with a most general ansatz for $\overline{\mathcal{Q}}_\alpha\,\phi$ and fix the ansatz by closing the commutators of superPoincaré algebra in $d = 11$. Since we are interested only in two-derivative interaction terms,the dynamical supersymmetry generator must be linear in transverse derivatives at any given order in κ. Hence we consider the following three kinds of terms [42] :

- Terms with $\bar{\partial}$: These must remain the same as in four dimensions.
- Terms with ∂^n : These terms take into account the dependence on the new dimensions $4 \ldots 10$.
- Terms with ∂ : These terms do not appear in $d = 4$. But since in $d = 11$, there is a $SO(7)$ R-symmetry now instead of $SU(8)$, such a term can exist.

The last kind of terms can occur in $d = 11$ because the tranverse derivatives ∂, $\bar{\partial}$ and ∂^m now form an $SO(9)$ vector, instead of the $SO(2)$ vector in $d = 4$. The existence of these terms suggests that we cannot simply reduce the eleven-dimensional to the $d = 4$ theory with an $SU(8)$ R-symmetry. We need non-trivial duality transformations (or field redefinitions in the light-cone language) to realize the $SU(8)$ in four dimensions.

Schematically, the ansatz for the dynamical supersymmetry contains the following three pieces.

$$\overline{Q}_\alpha{}^{(\kappa)}\phi \;=\overline{Q}_\alpha{}^{\bar{\partial}} \;+\; \overline{Q}_\alpha{}^{\partial^n} \;+\; \overline{Q}_\alpha{}^{\partial} \;. \tag{6.15}$$

We begin by observing that these three kinds of terms do not mix when we check chirality and supersymmetry, since the spacetime derivatives trivially commute with the supersymmetries q^m, $\bar{q}_m$ and the chiral derivatives d^m, $\bar{d}_m$. The mixing occurs only when we consider the rotations with J^m and $\bar{J}^n$. The first type of terms are the known ones from (4.30) (at order κ).

$$\begin{aligned} \overline{Q}_\alpha{}^{\bar{\partial}}\phi &= \frac{1}{\partial^+}(\bar{\partial}\bar{q}_\alpha\phi\partial^{+2}\phi - \partial^+\bar{q}_\alpha\phi\partial^+\bar{\partial}\phi) && (6.16)\\ &= \frac{1}{\partial^+}(E\partial^+\bar{\partial}\phi E^{-1}\partial^{+2}\phi)|_{\rho^\alpha}\;, && (6.17) \end{aligned}$$

where

$$E = exp(\frac{\bar{q}\cdot\rho}{\partial^+})\;. \tag{6.18}$$

This expression stands for

$$\overline{Q}_\alpha{}^{\bar{\partial}}\phi = \frac{\partial}{\partial\rho^\alpha}\frac{1}{\partial^+}(E\partial^+\bar{\partial}\phi E^{-1}\partial^{+2}\phi)|_{\rho^\alpha}\;. \tag{6.19}$$

In this notation, $\big|_{\rho^\alpha}$ means we differentiate with respect to ρ once and then set ρ to zero. Using this technique to write the expressions makes the commutation relation with q^α straightforward. Since we use $\bar{q}$'s only, chirality is also automatically maintained. Further, the commutation relations with other kinematical generators are trivially satisfied. The only crucial commutators that remain to be explicitly checked are the rotations with J^m and $\bar{J}^n$. We start with the commutator

$$[\bar{J}^m, \overline{Q}_\alpha] = -\sqrt{2}(\gamma^m)_{\alpha\beta}Q^\beta, \tag{6.20}$$

where

$$\bar{J}^{\,n} \;=\; -\,i\,(\,\bar{x}\,\partial^n \;-\; x^n\,\bar{\partial}\,) \;+\; \frac{i}{4\sqrt{2}\,\partial^+}\,\bar{q}_\alpha\,(\,\gamma^n)^{\alpha\,\beta}\,\bar{q}_\beta\;. \tag{6.21}$$

For the $\bar{\partial}$- piece, the only non-trivial contribution comes from the $\bar{q}$ terms in $\bar{J}^n$

$$\begin{aligned}
& [\frac{i}{4\sqrt{2}\,\partial^+}\,\bar{q}_\beta\,(\,\gamma^n)^{\beta\,\gamma}\,\bar{q}_\gamma,\;\frac{1}{\partial^+}(E\partial^+\bar{\partial}\phi E^{-1}\partial^{+2}\phi)|_{\rho^\alpha}] \\
= \; & \frac{i}{4\sqrt{2}\,\partial^{+2}}\,(\,\gamma^n)^{\beta\gamma}\,(E\partial^{+2}\bar{\partial}\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta\,\rho^\gamma\,\rho^\alpha}\quad .
\end{aligned} \tag{6.22}$$

We now move on to the ∂^n-piece. In Appendix B, we discuss the gamma matrices γ^m in the $SO(7)$ space and prove some useful identities, which are liberally used in the calculation shown below. Using some gamma matrix manipulations (C.3), we also prove that any term with three $SO(7)$ spinors can be expressed either as a $|8\rangle$ or a $|48\rangle$ of $SO(7)$. We thus consider the most general expression by mixing the $|8\rangle$ and the $|48\rangle$ while keeping two free parameters, c_1 and c_2

$$\begin{aligned}
\overline{Q}_\alpha{}^{\partial^n}\phi \;&=\; c_1(\gamma^n)^{\beta\gamma}\frac{1}{\partial^{+2}}[E\partial^{+A}\partial^n\phi E^{-1}\partial^{+B}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\alpha} \\
&+\; c_2(\gamma^n\gamma^m)^{\alpha\delta}(\gamma^m)^{\beta\gamma}\frac{1}{\partial^{+2}}[E\partial^{+A}\partial^n\phi E^{-1}\partial^{+B}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\delta},
\end{aligned} \tag{6.23}$$

where $A+B=5$ on dimensional grounds and c_1 and c_2 are arbitrary constants to be determined by the algebra. We choose $A=2$ and $B=3$.

We now consider

$$
\begin{aligned}
&[-i\,(\,\bar{x}\,\partial^n\,-\,x^n\,\bar{\partial}\,)\,,c_1(\gamma^m)^{\beta\,\gamma}\frac{1}{\partial^{+2}}[E\partial^{+2}\partial^m\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\alpha}\\
+&\quad c_2(\gamma^m\gamma^p)^{\alpha\delta}(\gamma^p)^{\beta\gamma}\frac{1}{\partial^{+2}}[E\partial^{+2}\partial^m\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\delta}]\\
=&\quad \frac{ic_1}{\partial^{+2}}(\gamma^n)^{\beta\,\gamma}[E\partial^{+2}\bar{\partial}\phi E^{-1}\partial^{+3}\phi)|_{\rho\beta,\rho^\gamma,\rho^\alpha}\\
+&\quad \frac{ic_2}{\partial^{+2}}(\gamma^n\gamma^p)^{\alpha\delta}(\gamma^p)^{\beta\gamma}[E\partial^{+2}\bar{\partial}\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\delta}\,.
\end{aligned}
\tag{6.24}
$$

We then add (6.22) and (6.24). We notice that the term with a $|8\rangle$ is of the correct form in (6.20) and hence the terms with $|48\rangle$ must cancel against each other. This requirement gives us the relation

$$
\frac{1}{4\sqrt{2}}+c_1=0\ . \tag{6.25}
$$

Next we focus on the commutator with the $\bar{q}$ part of $\bar{J}^n$

$$
\begin{aligned}
&[\frac{i}{4\sqrt{2}\,\partial^+}\;\bar{q}_\beta\,(\,\gamma^n)^{\beta\,\gamma}\,\bar{q}_\gamma,\;c_1(\gamma^m)^{\delta\,\epsilon}\frac{1}{\partial^{+2}}(E\partial^{+2}\partial^m\phi E^{-1}\partial^{+3}\phi)|_{\rho^\delta,\rho^\epsilon,\rho^\alpha}]\\
&\quad=\;\frac{ic_1}{4\sqrt{2}\,\partial^{+3}}\,(\,\gamma^n)^{\beta\gamma}\,(\gamma^m)^{\delta\,\epsilon}(E\partial^{+3}\partial^m\phi E^{-1}\partial^{+4}\phi)|_{\rho^\beta\,\rho^\gamma\,\rho^\delta\,\rho^\epsilon\,\rho^\alpha}
\end{aligned}
\tag{6.26}
$$

and

$$
\begin{aligned}
&[\frac{i}{4\sqrt{2}\,\partial^+}\;\bar{q}_\beta\,(\,\gamma^n)^{\beta\,\gamma}\,\bar{q}_\gamma,\;c_2(\gamma^m\gamma^p)^{\alpha\delta}(\gamma^p)^{\epsilon\eta}\frac{1}{\partial^{+2}}[E\partial^{+A}\partial^m\phi E^{-1}\partial^{+B}\phi)|_{\rho^\epsilon,\rho^\eta,\rho^\delta}]\\
&\quad=\;\frac{ic_2}{4\sqrt{2}\,\partial^{+3}}\,(\,\gamma^n)^{\beta\gamma}\,(\gamma^m\gamma^p)^{\alpha\delta}(\gamma^p)^{\epsilon\eta}(E\partial^{+3}\partial^m\phi E^{-1}\partial^{+4}\phi)|_{\rho^\beta\,\rho^\gamma\,\rho^\delta\,\rho^\epsilon\,\rho^\eta}\quad .
\end{aligned}
\tag{6.27}
$$

We now turn to the third possible term : the ∂-dependent piece. We make the ansatz

$$
\overline{Q}_\alpha{}^{\partial}\phi=c_3(\gamma^n)^{\beta\,\gamma}(\gamma^n)^{\delta\,\epsilon}\,\frac{1}{\partial^{+3}}[E\partial^{+3}\partial\phi E^{-1}\partial^{+4}\phi]|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\alpha}\quad , \tag{6.28}
$$

where the free parameter c_3 is to be determined. Its commutator with first part of $\bar{J}^n$ will contribute

$$
\begin{aligned}
&[-i\,(\,\bar{x}\,\partial^n\,-\,x^n\,\bar{\partial}\,)\,,c_3(\gamma^m)^{\beta\,\gamma}(\gamma^m)^{\delta\,\epsilon}\,\frac{1}{\partial^{+3}}[E\partial^{+3}\partial\phi E^{-1}\partial^{+4}\phi]|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\alpha} \\
&\quad=\;-i\,c_3\,(\gamma^m)^{\beta\,\gamma}(\gamma^m)^{\delta\,\epsilon}\,\frac{1}{\partial^{+3}}[E\partial^{+3}\partial^n\phi E^{-1}\partial^{+4}\phi]|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\alpha}\quad.
\end{aligned}
\tag{6.29}
$$

Finally we add the three expressions (6.26), (6.27) and (6.29) and demand that the resulting expression reduces to the form in (C.9). The details of the calculation are presented in Appendix B. The value of coefficients are found to be

$$
\begin{aligned}
c_1 &= -\frac{1}{4\sqrt{2}}\,, \\
c_2 &= \frac{1}{36\sqrt{2}}\,, \\
c_3 &= -\frac{1}{288}\,.
\end{aligned}
\tag{6.30}
$$

The $\bar{q}$ part of $\bar{J}^n$ contributes one more term which contains seven γ-matrices

$$
\begin{aligned}
&[\frac{i}{4\sqrt{2}\,\partial^+}\,\bar{q}_\beta\,(\,\gamma^n)^{\beta\,\gamma}\,\bar{q}_\gamma,\;c_3(\gamma^m)^{\delta\,\epsilon}(\gamma^m)^{\eta\,\kappa}\,\frac{1}{\partial^{+3}}[E\partial^{+3}\partial\phi E^{-1}\partial^{+4}\phi]|_{\rho^\delta,\rho^\epsilon,\rho^\eta,\rho^\kappa,\rho^\alpha} \\
&\quad=\;\frac{i\,c_3}{4\sqrt{2}\,\partial^{+4}}\,(\gamma^n)^{\beta\,\gamma}\gamma^m)^{\delta\,\epsilon}(\gamma^m)^{\eta\,\kappa}\,[E\partial^{+4}\partial\phi E^{-1}\partial^{+5}\phi]|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\eta,\rho^\kappa,\rho^\alpha}\quad.
\end{aligned}
\tag{6.31}
$$

Using (C.14) from the Appendix, we find that the term with seven q's has the correct form. We thus put together all the three pieces and obtain the correct form for $\overline{Q}_\alpha$ in $d=11$ [42]

$$
\begin{aligned}
\overline{Q}_\alpha\phi \;=\; &\frac{1}{\partial^+}(E\partial^+\bar{\partial}\phi E^{-1}\partial^{+2}\phi)|_{\rho^\alpha} \\
&-\frac{1}{4\sqrt{2}}(\gamma^n)^{\beta\gamma}\frac{1}{\partial^{+2}}[E\partial^{+2}\partial^n\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\alpha} \\
&+\frac{1}{36\sqrt{2}}(\gamma^n\gamma^m)^{\alpha\delta}(\gamma^m)^{\beta\gamma}\frac{1}{\partial^{+2}}[E\partial^{+2}\partial^n\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\delta} \\
&-\frac{i}{288}(\gamma^n)^{\beta\,\gamma}(\gamma^n)^{\delta\,\epsilon}\,\frac{1}{\partial^{+3}}[E\partial^{+3}\partial\phi E^{-1}\partial^{+4}\phi]|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\alpha}\quad.
\end{aligned}
\tag{6.32}
$$

From (6.20), we can readily read off the explicit form of Q^α

$$
\begin{aligned}
Q^\alpha\phi &= \frac{i}{72\,\partial^{+2}}(\gamma^p)^{\alpha\beta}(\gamma^p)^{\gamma\delta}[E\partial^{+2}\bar{\partial}\phi E^{-1}\partial^{+3}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\delta} \\
&+\frac{i}{288\,\sqrt{2}\partial^{+3}}(\gamma^r)^{\alpha\beta}(\gamma^r)^{\gamma\delta}(\gamma^m)^{\epsilon\eta}\frac{1}{\partial^{+2}}[E\partial^{+3}\partial^m\phi E^{-1}\partial^{+4}\phi)|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\eta} \\
&+\frac{i}{16128\,\partial^{+4}}(\gamma^r)^{\alpha\beta}(\gamma^r)^{\gamma\delta}(\gamma^m)^{\epsilon\eta}(\gamma^m)^{\kappa\rho}[E\partial^{+4}\partial\phi E^{-1}\partial^{+5}\phi]|_{\rho^\beta,\rho^\gamma,\rho^\delta,\rho^\epsilon,\rho^\eta,\rho^\kappa,\rho^\rho}\quad .
\end{aligned}
\tag{6.33}
$$

Once we have constructed the dynamical supersymmetries to order κ, we can now construct the $d = 11$ Hamiltonian to that order. By construction, this formulation makes the maximal supersymmetry of the theory manifest. We can also check if the dynamical supersymmetry still commutes with the $E_{7(7)}$ transformations in (4.45). We immediately see that it will not since there is a bare field ϕ in the last term in (6.32), which transforms under the constant term in (4.45). There is no other term to cancel against it. Hence we arrive at the conclusion that this version of the eleven-dimensional theory is not invariant under the E_7 variations (4.45). In principle we should also derive the corrections to the other dynamical generators J^-, $\bar{J}^-$, J^{n-} and $\bar{J}^{n-}$. However, it is not necessary for our discussion as our main focus here is on maximal supersymmetry.

6.3 Oxidation from $d = 4$ to $d = 11$: Method 2

In this section, we discuss an alternative approach to "oxidize" the four-dimensional theory to $d = 11$ keeping the derivative structure of the $d = 4$ theory intact [40], with the hope of uncovering an $E_{7(7)}$ symmetry in the theory. The key ingredient in this approach is the 'generalized derivative' introduced in the same spirit as in the case of "oxidation" from $d = 3$ to $d = 4$ described in the last chapter. We start with the following ansatz for the derivative, which incorporates the extra seven derivatives ∂^m

$$
\overline{\nabla} \;=\; \bar{\partial} + \frac{\sigma}{16}\,\bar{d}_\alpha\,(\,\gamma^m\,)^{\alpha\,\beta}\,\bar{d}_\beta\,\frac{\partial^m}{\partial^+}\,, \tag{6.34}
$$

where σ is an arbitrary constant to be determined by the algebra. The commutator with J^m introduces some new derivatives

$$[\,\overline{\nabla}\,,\,J^m\,] \equiv \nabla^m \;=\; -\,i\,\partial^m \,+\, \frac{i\,\sigma}{16}\,\bar{d}_\alpha\,(\,\gamma^m\,)^{\alpha\,\beta}\,\bar{d}_\beta\,\frac{\partial}{\partial^+}\,. \qquad (6.35)$$

Note that the derivative ∂ is not introduced. These new derivatives $\overline{\nabla}$ and ∇^m form a vector under the $SO(9)$ little group in $d = 11$. The key idea here is to keep the form of the cubic vertex same as that in $d = 4$ and to obtain the eleven-dimensional vertex by simply replacing the transverse derivatives in the four-dimensional expression by the generalized derivatives. The cubic vertex in $d = 11$ thus reads

$$\mathcal{V} \;=-\;\frac{3}{2}\,\kappa\int d^{11}x\int d^8\theta\,d^8\bar{\theta}\;\frac{1}{\partial^{+2}}\;\overline{\phi}\;\;\overline{\nabla}\,\phi\;\overline{\nabla}\,\phi + \text{c.c.}\;. \qquad (6.36)$$

The $SO(2)$ invariance remains unaffected, while the $SO(7)$ part is covariantly realized in eleven dimensions. So the only part left to be checked explicitly is the invariance under the coset $SO(9)/(SO(7)\times SO(2))$.

Under the J^m rotation, we find

$$\delta_{J^m}\,\overline{\phi} \;=\; \frac{i}{2\,\sqrt{2}}\,\omega_m\,\frac{1}{\partial^+}\,q^\alpha\,(\,\gamma^m)_{\alpha\,\beta}\,q^\beta\,\overline{\phi} \;\equiv\; K(q)\;. \qquad (6.37)$$

The “inside-out” constraint (4.14) yields

$$\delta_{J^m}\,\phi \;\;=\;\; \frac{1}{4}\,\frac{(d)^8}{\partial^{+4}}\,(\delta_{J^m}\,\overline{\phi}) \;\equiv\; \frac{1}{4}\,\frac{(d)^8}{\partial^{+4}}\,K(q)\;. \qquad (6.38)$$

We now consider

$$\delta_{J^m}\,\overline{\nabla} \;\;=\;\; -\,\omega_m\,\nabla^m\;, \qquad (6.39)$$

where ω_m are the $SO(9)/(SO(7)\times SO(2))$ coset parameters. It is straightforward to verify that the relevant contribution comes from the terms that involve one $SO(2)$ derivative and one ∂^m. The net variation is given by

$$\delta_J\,\mathcal{V}\;\propto\;\int\left(\frac{1}{\sqrt{2}}i\sigma + i\right)\frac{1}{\partial^{+2}}\overline{\phi}\,\bar{\partial}\phi\,\partial^m\phi\;. \qquad (6.40)$$

The invariance under the coset group determines the value of σ by demanding that the above term vanish

$$\sigma = -\sqrt{2}\ . \tag{6.41}$$

This completes fixes the form of the generalized derivative. In this light-cone frame, the Lorentz invariance in eleven dimensions follows directly from the invariance under the little group.

Thus, the light-cone ($\mathcal{N}=1, d=11$) supergravity action to order κ reads

$$\beta \int d^{11}x \int d^8\theta\, d^8\bar{\theta}\, \mathcal{L}\ , \tag{6.42}$$

where $\beta\ =\ -\frac{1}{64}$ and

$$\mathcal{L}\ \ =\ \ -\bar{\phi}\,\frac{\Box}{\partial^{+4}}\,\phi\ -\ 2\,\kappa\,(\ \frac{1}{\partial^{+2}}\,\bar{\phi}\ \ \overline{\nabla}\,\phi\,\overline{\nabla}\,\phi + \ \frac{1}{\partial^{+2}}\,\phi\,\nabla\,\bar{\phi}\,\nabla\,\bar{\phi})\ , \tag{6.43}$$

with the eleven-dimensional d'Alembertian operator being

$$\Box\ =\ 2\,(\,\partial\,\bar{\partial}\ +\ \frac{1}{2}\,\partial^m\,\partial^m\ -\ \partial_+\,\partial_-\,)\ . \tag{6.44}$$

6.3.1 An $E_{7(7)}$ symmetry in eleven dimensions

We are now in a position to investigate a possible $E_{7(7)}$ symmetry of the Hamiltonian corresponding to the eleven-dimensional Lagrangian (6.43). The first check to perform is the invariance under the maximal subgroup $SU(8)$ of $E_{7(7)}$. Keeping in mind that the light-cone superspace is built on the underlying $SU(8)$ symmetry, the eleven-dimensional Hamiltonian can be considered to be function of θ which is a **8** of $SU(8)$. Thus the Hamiltonian in terms of the superfield ϕ, which is inherently a representation of $SU(8)$, respects the invariance under $SU(8)$ by construction. The $E_{7(7)}/SU(8)$ coset transformations (4.45) read

$$\begin{aligned}\delta\phi =\ &-\frac{2}{\kappa}\,\theta^{klmn}\,\Xi_{klmn}\\ &+\frac{\kappa}{4!}\,\Xi^{mnpq}\,\frac{1}{\partial^{+2}}\left(\overline{d}_{mnpq}\,\frac{1}{\partial^{+}}\phi\,\partial^{+3}\phi\ -\ 4\,\overline{d}_{mnp}\phi\,\overline{d}_q\partial^{+2}\phi\ +\ 3\,\overline{d}_{mn}\partial^{+}\phi\,\overline{d}_{pq}\partial^{+}\phi\right)\ .\end{aligned}$$

From the structure of these transformations it is evident that the eleven-dimensional Hamiltonian is invariant under the $E_{7(7)}$, since there are no ∂ mixed up with $\bar{\partial}$ in the cubic interaction term in (6.42). Also, ∇ contains only $\bar{d}_m$ derivatives which trivially anti-commute with the $\bar{d}$ derivatives in the coset transformations. Thus, we arrive at an action for $\mathcal{N} = 1$ supergravity in $d = 11$ which shows $E_{7(7)}$ invariance at least to order κ.

6.4 Exceptional versus superPoincaré symmetry

There exists only one theory of supergravity in eleven dimensions. So different versions of this theory must be related to each other by means of suitable field redefinitions. With this understanding, we now present a comparative analysis of the two methods of oxidation discussed in this chapter, which lead to two distinct supergravity actions in $d = 11$.

In the first method, the maximal supersymmetry of the theory is apparent as the explicit form of the dynamical supersymmetry generator is fixed from the requirement that the superPoincaré algebra closes in $d = 4$. The action derived using this method shows no signs of an exceptional symmetry in the theory.

The other method involves deriving the higher-dimensional action by replacing the transverse derivatives in $d = 4$ with "generalized derivatives". This method unveils a hidden exceptional symmetry, $E_{7(7)}$ in eleven dimensions, which is not obvious in the previous formulation. In order to relate the E_7-invariant action to the other action, we need to perform a field redefinition which will in effect conceal the exceptional symmetry and make the superPoincaré invariance manifest in the theory. Although one single formulation packed with all the symmetries is difficult to find, we can make a particular symmetry visible in one formulation and relate it to the others using field redefinitions.

The key point of our analysis is that both maximal supergravity and the exceptional symmetry are present in eleven dimensions. We have to choose one of the two symmetries to be made manifest while constructing the theory. Both the symmetries are non-linearly realized on the physical fields and can be exploited to find the Hamiltonian of the theory. Thus, we learn that in the light-cone formulation both maximal supersymmetry and the exceptional symmetry are equally important, which leaves us pondering over the question : which one is more fundamental?

Chapter 7

Outlook and future directions

We now discuss briefly a few remarks about the results presented in the thesis. We present some relevant open questions and research directions that we wish to pursue in future. We begin with enumerating some of the important lessons learnt from our study of gravity and maximal supergravity in the light-cone gauge. The key aspect of the light-cone formulation is that only physical degree of freedom appear in the theory. This leads to a lot of simplification and in the process brings out many nice features of the theories. For example,

- Symmetries are non-linearly realized on the physical fields which makes this formulation ideal for studying symmetries, both known and unknown.
- Interacting theories can be derived order by order in perturbation by demanding the closure of (super)Poincaré algebra.
- Exceptional symmetries in light-cone supergravity act on all the physical fields of the theory as opposed to the covariant formalism, where only the scalars and vector fields are affected. These are symmetries of the action and not just the equations of motion.
- In the light-cone superspace, we can use the same superfield ϕ to construct maximal supergravity in all the dimensions. In this sense, the superfield should be viewed as a collection of 256 physical states, which can be broken up into representations of the relevant R-symmetry group in a particular dimension.
- The light-cone superspace puts the exceptional symmetries on an equal footing as supersymmetry. Both the symmetries can be used to construct the interacting Hamiltonian by virtue of their non-linear action on the fields.

In these ways the light-cone formalism offers some useful insights into field theories. We shall now focus on some specific results discussed in the text.

Quadratic form Hamiltonian

In chapter 3, we showed the light-cone Hamiltonian for pure gravity in four dimensions can be expressed as a quadratic form up to order κ^2

$$\mathcal{H} = \int d^3x \;\; \mathcal{D}\bar{h} \; \bar{\mathcal{D}}h \; . \tag{7.1}$$

We also examined how the $\mathcal{D}\bar{h}$ operator transforms under residual reparameterizations to this order. The existence of a quadratic form Hamiltonian puts gravity in a very special class of theories which admit such Hamiltonians . In Table (7.1), we list down all such theories which show this feature.

Table 7.1: Field theories with a quadratic form Hamiltonian in four dimensions

Theory	**Quadratic form Hamiltonian**
Yang-Mills	Yes
Gravity	Yes (up to $\mathcal{O}(\kappa^2)$)
$\mathcal{N}=4$ superYang-Mills	Yes
$\mathcal{N}=8$ supergravity	Yes (up to $\mathcal{O}(\kappa)$)
Non-maximally supersymmetric theories	No

In four dimensions pure Yang-Mills theory, which is the basis of the Standard Model of particle Physics, and its maximally supersymmetric cousin, $\mathcal{N}=4$ superYang-Mills theory which happens to be an ultraviolet finite theory, have such quadratic form Hamiltonians. The fact that gravity and $\mathcal{N}=8$ supergravity Hamiltonians also share a similar feature, might be an indication of some hidden symmetry in the theory of gravity. In order to fully appreciate the quadratic form structure, we must understand what is the physical significance of the $\mathcal{D}\bar{h}$ operator. The quadratic form could also hint at some deeper links between gravity with gauge theories.

Interestingly, the quadratic form structure breaks down for theories with less than maximal supersymmetry. One possible explanation is that the supersymmetric truncation [44]

$$\int d^4x\, d^8\theta\, d^8\bar{\theta}\, \mathcal{L} \;=\; \frac{1}{16} \int d^4x\, d^7\theta\, d^7\bar{\theta}\, \bar{d}_4\, d^4\, \mathcal{L}\,\Bigg|_{\theta^8\,=\,\bar{\theta}_8\,=0} \,, \tag{7.2}$$

spoils the quadratic form structure. This point strongly suggests that maximal supergravity and superYang-Mills theories have rich structures, which could be responsible for their unique quantum properties.

Exceptional symmetry in eleven dimensions

In the previous chapter, we discussed how we can see the signs of an $E_{7(7)}$ symmetry in eleven-dimensional supergravity, starting form the $\mathcal{N}=8$ theory in four dimensions.

Apart from teaching us about the origin of exceptional symmetries in supergravity theories, the $d=11$ theory is interesting in its own right. The study of symmetries in $(\mathcal{N}=1, d=11)$ supergravity might offer new insights into M-theory whose actual structure eludes our understanding . Although we do not discuss any particular consequence of the E_7 symmetry for M-theory in our work, we believe that our analysis of exceptional symmetries in eleven dimensions could open up a new window to look into the underlying structure of M-theory.

Finiteness analysis for $\mathcal{N}=8$ supergravity

We began our search for hidden symmetries in $\mathcal{N}=8$ supergravity with the hope that such a symmetry could explain the improved ultraviolet behavior of the theory. Using our method of dimensional reduction to $d=3$, field redefinition and dimensional oxidation, we were able to uncover an E_8 symmetry enhanced from the E_7 already present in four dimensions. It is important to note that the manifestly E_8 symmetric formulation is proven to be Lorentz invariant. Thus, we must further investigate the possible implications of this symmetry for $\mathcal{N}=8$ supergravity.

In the light-cone superspace, there exists a powerful finiteness analysis framework [4, 5, 45, 46], which was originally devised to prove that the $\mathcal{N}=4$ superYang-Mills theory finite to all orders. When we apply this analysis to $\mathcal{N}=8$ supergravity at the cubic order, the chirality constraint on the superfield (4.13) rules out the existence of a three-point counterterm in the theory [30], which is in keeping with the well-known results of [47]. Thus the first appearance of a counterterm could be at the four-point level. At this point, we must take into account the non-linear exceptional symmetries since these relate terms of different order in κ.

So, our immediate next step in this direction will be to incorporate these exceptional symmetries in this well established finiteness analysis for $\mathcal{N}=8$ supergravity, which is crucial for understanding the role of the E_7 and the new E_8 symmetry in the ultraviolet properties of this theory.

In this context, it is important to mention that a very recent paper on five-loop amplitudes in $\mathcal{N}=8$ supergravity [48] reaffirms the possibility of a seven-loop four-point counterterm in the theory, which is compatible with the E_7 symmetry [7]. An all-order finiteness analysis in the light-cone superspace (in light of the new E_8 symmetry) may prove to be useful in settling this issue - whether or not this seven-loop counterterm exists in $(\mathcal{N}=8, d=4)$ supergravity.

Appendix A

Deriving interacting theories from symmetry principles

The material presented here is primarily based on the work done by the author in [26].

Deriving spin-1 interaction vertices : An example

In this appendix, we briefly discuss the procedure for constructing the light-cone action for a theory of massless interacting spin-1 fields by demanding the closure of Poincaré algebra. Any massless field in four dimensions has two physical degrees of freedom ϕ and $\bar{\phi}$, which correspond to the $+$ and $-$ helicity states respectively.In the light-cone frame, the kinematical Poincaré generators that do not involve time derivatives are

$$P^+ \ , \ \ P \ , \ \ \bar{P} \ , \ \ J \ , \ \ J^+ \ , \ \ \bar{J}^+ \ \ \text{and} \ \ J^{+-} \ . \tag{A.1}$$

The dynamical ones which involve time derivatives are

$$P^- \equiv H \ , \ \ J^- \ , \ \ \bar{J}^- \ , \tag{A.2}$$

These generators pick up corrections when interactions are switched on. We also introduce the Hamiltonian variation

$$\delta_{\mathcal{H}}\phi \equiv \partial^-\phi = \{\phi, \mathcal{H}\} = \frac{\partial\bar{\partial}}{\partial^+}\,\phi \ , \tag{A.3}$$

where the second equality only holds for the free theory. In an interacting theory,we must add corrections to the the $\delta_{\mathcal{H}}$ operator order by order, in the coupling constant.

The main idea is to start with an ansatz for the operator $\delta_H\,\phi$, close all the commutators of the light-cone Poincaré algebra listed before (2.13) to fix the ansatz completely and thus determine the interacting part of the Hamiltonian variation..

Cubic interaction vertices

Cubic interaction vertices for fields of any arbitrary integer λ were derived in [22]. At order α, we start with a $\delta_{\mathcal{H}}\phi$ that is proportional to two fields, since a cubic vertex in the Hamiltonian involves three fields, $\bar{\phi}\phi\phi$.

$$\delta_H^{\alpha}\phi = \alpha\,K\;\partial^{+\mu}\left[\bar{\partial}^B\,\partial^C\,\partial^{+\rho}\,\phi\;\bar{\partial}^D\,\partial^E\,\partial^{+\sigma}\phi\right]\,, \tag{A.4}$$

where K is an arbitrary constant and μ, ρ, σ B, C, D, E are integers that will be fixed by the algebra[1]. On closing the commutator of this ansatz with $\delta_{J^{+-}}$, we get [22]

$$\mu+\rho+\sigma=-1\;. \tag{A.5}$$

The commutator with δ_J yields

$$B+D=1\quad;\qquad C=E=0\;. \tag{A.6}$$

The rest of the commutators determine the values of the remaining integers μ, ρ and σ. We thus obtain the cubic interaction vertices [22]

$$\delta_{\mathcal{H}}^{\alpha}\phi = \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}(\partial^+)^{(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi\frac{\bar{\partial}^n}{\partial^{+n}}\phi\right]\;, \tag{A.7}$$

for even λ.

Appearance of the "structure constant"

Surprisingly, it was found that there exists a non-trivial solution for odd-helicity fields *if and only if* we introduce an antisymmetric three-index object f^{abc} [22],

[1]The ansatz involving one ϕ and one $\bar{\phi}$ works in a similar way.

$$\delta^{\alpha}_{\mathcal{H}}\phi^a = \alpha f^{abc}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}(\partial^+)^{(\lambda-1)}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi^b\frac{\bar{\partial}^n}{\partial^{+n}}\phi^c\right] \ . \tag{A.8}$$

The same procedure can now be repeated to determine $\delta_H\phi$ corresponding to the $\alpha\,\bar{\phi}\phi$ structure, which leads to the other cubic vertex of the form $\bar{\phi}\bar{\phi}\phi$.

The action, to this order, then directly follows from (A.7) and (A.8)

$$S = \int d^4x\left(\frac{1}{2}\bar{\phi}\Box\phi + \alpha\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\bar{\phi}(\partial^+)^{\lambda}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi\frac{\bar{\partial}^n}{\partial^{+n}}\phi\right] + c.c.\right), \tag{A.9}$$

for even λ and

$$S = \int d^4x\left(\frac{1}{2}\bar{\phi}^a\Box\phi^a + \alpha f^{abc}\sum_{n=0}^{\lambda}(-1)^n\binom{\lambda}{n}\bar{\phi}^a(\partial^+)^{\lambda}\left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi^b\frac{\bar{\partial}^n}{\partial^{+n}}\phi^c\right] + c.c.\right), \tag{A.10}$$

for odd λ. These results, when applied to the cases of $\lambda = 2$ and $\lambda = 1$, match the gravity and Yang-Mills actions, obtained by light-cone gauge-fixing the covariant actions.

Quartic interaction vertices

We now extend this formalism to order α^2 for the specific case of $\lambda = 1$. We denote the two fields in the Yang-Mills theory by A and $\bar{A}$, which have helicity $+1$ and -1 respectively. The perturbation constant in this case, $2\,\alpha$ is identified with the dimensionless Yang-Mills coupling constant g . We start with the results discussed in the previous section,

$$\delta^g_H A^a = +g\,f^{abc}\left\{-A^c\,\frac{\bar{\partial}}{\partial^+}\,A^b + \frac{1}{\partial^{+2}}(\partial^{+2}A^b\,\frac{\partial}{\partial^+}\bar{A}^c) - \frac{1}{\partial^{+2}}(\partial\partial^+A^b\,\bar{A}^c)\ \right\}. \tag{A.11}$$

The key commutator in this case involves the Hamiltonian variation and J^-

$$[\delta_{J^-}\,,\,\delta_H]\,A^a = 0\ . \tag{A.12}$$

In this computation, we focus on the terms of the form $AA\bar{A}$ only[2]. We first consider the order g^2 contributions to this commutator from $[\delta^g_{J^-}\,,\,\delta_H{}^g]\,A^a$. This calculation involves the following two pieces : orbital part, L^- and the spin part, S^-. The orbital part is straightforward.

$$[\delta^g_{L^-}\,,\,\delta_H{}^g]\,A^a = [x\,\delta_H{}^g\,,\,\delta_H{}^g]\,A^a = -g\,f^{abc}\,A^c\frac{1}{\partial^+}(\delta_H{}^g\,A^b)\ , \tag{A.13}$$

For the spin part, we need to use the corrections to the spin generator at order g [22]

$$\delta^g_{\bar{S}^-}A^a = -g\,f^{abc}\,\frac{1}{\partial^{+2}}\left(\frac{1}{\partial^+}\,\bar{A}^c\,\partial^{+2}A^b + 3\,\bar{A}^c\partial^+A^b\right)\ , \tag{A.14}$$

$$\delta^g_{S^-}A^a = +g\,f^{abc}\,\frac{1}{\partial^+}\,A^b\,A^c\ . \tag{A.15}$$

Hence the spin part of the commutator (A.12) yields

$$\begin{aligned}
[\delta^g_{S^-}\,,\,\delta^g_H]\,A^a =&+g^2\,f^{abc}\left\{f^{bde}\frac{1}{\partial^{+2}}\left(\partial^{+2}(\frac{1}{\partial^+}A^d\,A^e)\frac{\partial}{\partial^+}\bar{A}^c\right)-f^{bde}\frac{1}{\partial^{+2}}\left(\partial\partial^+(\frac{1}{\partial^+}A^d\,A^e)\bar{A}^c\right)\right.\\
&-f^{cde}\,\frac{1}{\partial^{+2}}\left(\partial^{+2}\,A^b\,\frac{\partial}{\partial^{+3}}(\frac{1}{\partial^+}A^e\,\partial^{+2}\bar{A}^d + 3A^e\partial^+\bar{A}^d)\right)\\
&\left.+f^{cde}\,\frac{1}{\partial^{+2}}\left(\partial\partial^+\,A^b\,\frac{1}{\partial^{+2}}(\frac{1}{\partial^+}A^e\,\partial^{+2}\bar{A}^d + 3A^e\partial^+\bar{A}^d)\right)\right\}\\
&-g\,f^{abc}\,\delta^g_H A^c\,\frac{1}{\partial^+}A^b - g\,f^{abc}\,A^c\,\frac{1}{\partial^+}(\delta^g_H\,A^b)\ .
\end{aligned} \tag{A.16}$$

There are contributions to (A.12), which comes from commutators with one generator at order g^0 and one at order g^2. To evaluate these contributions, we need to construct an ansatz for $\delta^{g^2}_H$.

We begin with a general structure of the form [3]

$$\delta^{g^2}_H A^a = +g^2\,K\,f^{abc}\,f^{cde}\,\partial^{+\mu}\left[\bar{\partial}^B\,\partial^C\,\partial^{+\rho}A^b\ \partial^{+\sigma}\left(\bar{\partial}^D\,\partial^E\,\partial^{+\eta}A^d\ \bar{\partial}^F\,\partial^G\,\partial^{+\delta}\,\bar{A}^e\right)\right]\ , \tag{A.17}$$

[2] It can be checked that the terms of the form AAA vanish independently

[3] We may write down other combinations with the derivatives put in different places. But it can be shown that all such structures can be traced back to the form in (A.17).

with the constant K and integers μ, ρ, σ, η, δ, B, C, D, E, F, G to be fixed by the algebra.

We consider the commutator with δ_J, which gives the following constraints

$$B + D + F = C + E + G = \lambda - 1 \ . \tag{A.18}$$

Thus for $\lambda = 1$ no transverse derivatives are allowed. So, our ansatz reduces to

$$\delta_H^{g^2} A^a = +g^2\, K\, f^{abc}\, f^{cde}\, \partial^{+\mu} \left[\partial^{+\rho}\, A^b\ \partial^{+\sigma} \Big(\partial^{+\eta} A^d\ \partial^{+\delta}\, \bar{A}^e \Big) \right] \ . \tag{A.19}$$

The commutator with $\delta_{J^{+-}}$ put the following condition on the undetermined constants

$$\mu + \rho + \sigma + \eta + \delta = -1 \ . \tag{A.20}$$

Finally, we consider the last piece of the computation which involves

$$[\delta_{L^-}^{g^2}, \delta_H^0]\, A^a + [\delta_{J^-}^0, \delta_H^{g^2}]\, A^a \ , \tag{A.21}$$

Here, we have taken the correction to the spin generator at order g^2 to be zero in the first commutator (which is explained in details in the next section).

We find that the following solution which satisfies the commutator (A.12)

$$\begin{aligned} &(\mu = -1\, ; \rho = +1\, ; \sigma = -2\, ; \eta = 0\, ; \delta = +1) \\ &\quad + (\mu = 0\, ; \rho = 0\, ; \sigma = -2\, ; \eta = +1\, ; \delta = 0) \ , \end{aligned} \tag{A.22}$$

Thus our solution is a sum of two terms with the constants $\mu\,,\ \rho$... taking these two sets of values.We may try to consider a more general case, but such a computation is far more lengthy and unnecessary for our discussion.

We present the explicit computation of (A.21) for the aforementioned set of values.

$$
\begin{aligned}
& f^{abc} f^{cde} [-\frac{1}{\partial^{+2}}(\partial^+ \partial A^b \frac{1}{\partial^{+2}}(\bar{A}^e \partial^+ A^d)) + \frac{1}{\partial^{+2}}(\partial^+ \partial A^b \frac{1}{\partial^{+2}}(\partial^+ \bar{A}^e A^d)) \\
& + \frac{1}{\partial^{+2}}(\frac{\partial}{\partial^+} A^b \bar{A}^e \partial^+ A^d) + 2\frac{1}{\partial^{+2}}(\partial^+ A^b \frac{1}{\partial^{+2}}(\bar{A}^e \partial \partial^+ A^d)) - \frac{1}{\partial^{+2}}(\partial^+ A^b \frac{1}{\partial^+}(\partial^+ \bar{A}^e \frac{\partial}{\partial^+} A^d)) \\
& -2\frac{1}{\partial^{+2}}(\partial^+ A^b \frac{1}{\partial^+}(\frac{\partial}{\partial^+} \bar{A}^e \partial^+ A^d)) + \frac{1}{\partial^{+2}}(\partial^+ A^b \frac{1}{\partial^{+2}}(\partial^+ \partial \bar{A}^e A^d)) \\
& -2\frac{1}{\partial^{+2}}(\partial^{+2} A^b \frac{1}{\partial^{+3}}(\partial^+ \bar{A}^e \partial A^d)) + 4\frac{1}{\partial^{+2}}(\partial^{+2} A^b \frac{1}{\partial^{+3}}(\partial \bar{A}^e \partial^+ A^d)) \\
& +2\frac{1}{\partial^{+2}}(\partial^{+2} A^b \frac{1}{\partial^{+3}}(\bar{A}^e \partial \partial^+ A^d)) - \frac{1}{\partial^{+2}}(\partial^{+2} A^b \frac{1}{\partial^{+2}}(\partial^+ \bar{A}^e \frac{\partial}{\partial^+} A^d)) \\
& -\frac{1}{\partial^{+2}}(\partial^{+2} A^b \frac{1}{\partial^{+2}}(\frac{\partial}{\partial^+} \bar{A}^e \partial^+ A^d)) - \frac{1}{\partial^{+2}}(A^b \frac{\partial}{\partial^+} \bar{A}^e \partial^+ A^d) \\
& -\frac{1}{\partial^{+2}}(\partial^+ A^b \frac{1}{\partial^{+2}}(\partial^+ \bar{A}^e \partial A^d)) + 4\frac{1}{\partial^{+2}}(\partial^+ A^b \frac{1}{\partial^{+2}}(\partial \bar{A}^e \partial^+ A^d))] \ .
\end{aligned}
\tag{A.23}
$$

Emergence of a gauge group

The important point here is that the two expressions in (A.13) and (A.16) beautifully cancel against the huge mass of terms in (A.23), *if and only if* the f^{abc} introduced in (A.8) are assumed to satisfy the Jacobi identity,

$$
f^{abc}\, f^{bde} + f^{abd}\, f^{bec} + f^{abe}\, f^{bcd} \; = \; 0 \ . \tag{A.24}
$$

Also in order to prove that the terms of the form AAA vanish, the Jacobi identity is indispensable.

This requirement signals the emergence of a gauge group in theory. Thus the fields A^a and $\bar{A}^a$ describe a non-abelian gauge theory, namely the Yang-Mills theory.

Thus, we obtain the expression for the Hamiltonian variation at $\mathcal{O}(g^2)$

$$
\delta_H^{g^2} A^a = g^2\, f^{abc}\, f^{cde} \left[\frac{1}{\partial^+} \left(\partial^+ A^b \, \frac{1}{\partial^{+2}} (\partial^+ \bar{A}^e A^d) \right) - A^b \, \frac{1}{\partial^{+2}} (\bar{A}^e \, \partial^+ A^d) \right] \ , \tag{A.25}
$$

which eventually leads to the same quartic interaction terms in the action as those obtained by light-cone gauge-fixing the covariant Yang-Mills action.

The spin generator at order g^2

We now prove that $\delta_{S^-}^{g^2} = 0$. We first list below the helicities and dimensions of the objects involved in the analysis.

Quantity	Helicity	Dim $[L]$
x	$+1$	$+1$
$\bar{x}$	-1	$+1$
∂	$+1$	-1
$\bar{\partial}$	-1	-1
A	$+1$	-1
$\bar{A}$	-1	-1
∂^+	0	-1

At lowest order, the spin generators given by

$$\delta_{S^-}^{0} A^a = -\frac{\partial}{\partial^+} A^a \ . \tag{A.26}$$

It has helicity $+2$ and a length-dimension of -1. On these grounds, we make an ansatz at order g^2 which is of the form

$$\delta_{S^-}^{g^2} A^a \sim g^2 \, A \, A \, \partial \, \frac{1}{\partial^{+3}} \, \bar{A} \ . \tag{A.27}$$

The commutator with $\delta_{J^{+-}}$, however puts a rigid constraint on the structure of $\delta_{S^-}^{g^2} A^a$. The number of ∂^+'s in the denominator must be one greater than that in the numerator (see for example (A.20)) for this commutator to work. This immediately rules out the above ansatz.

Similarly, we can check the other commutators one by one. We observe that no combination of the objects in the table above (involving three fields) satisfies the Poincarè algebra. Thus a proof by exhaustion leads us to conclude that there are no corrections to the spin generator at the quartic order.

This completes the construction of the light-cone representation of the Poincaré algebra for spin-1 interacting fields. Therefore, using this framework we can derive the entire Yang-Mills theory in four dimensions, just by demanding the closure of the Poincarè algebra.

Appendix B

Fermions in the light-cone gauge

To deal with fermions, it is convenient to work with the gamma matrices expressed in the light-cone notation

$$\begin{aligned} \gamma^+ &= \frac{1}{\sqrt{2}}(\gamma^0 + \gamma^3) \; ; \quad \gamma^- = \frac{1}{\sqrt{2}}(\gamma^0 - \gamma^3) \\ \gamma &= \frac{1}{\sqrt{2}}(\gamma^1 - i\gamma^2) \; ; \quad \bar{\gamma} = \frac{1}{\sqrt{2}}(\gamma^0 - i\gamma^2) \; . \end{aligned} \tag{B.1}$$

The gamma matrices satisfy the Clifford algebra

$$\{ \gamma^\mu , \gamma^\nu \} = 2 \, \eta^{\mu\nu} \; ,$$

where $\eta^{\mu\nu}$ is the light-cone metric (2.4).

We can now define two projection operators

$$\pi_+ = -\frac{1}{2} \, \gamma_+\gamma_- \; ; \quad \pi_+ = -\frac{1}{2} \, \gamma_-\gamma_+ \tag{B.2}$$

with the properties

$${\pi_+}^2 = \pi_+ \; ; \quad \pi_+^2 = \pi_+ \; ; \quad \pi_+\pi_- = \pi_-\pi_+ = 0 \tag{B.3}$$

Using these operators, a spinor ψ in the light-cone frame can be split into "+" and "-" components

$$\psi_+ = P_+\psi \; ; \quad \psi_- = P_-\psi \; .$$

such that $\psi = \psi_+ + \psi_-$. This analysis will be useful when we discuss supersymmetry as the supersymmetry generators, which are fermionic in nature, have a similar decomposition in the light-cone formulation.

Free action for fermionic fields

We start with the Dirac Lagrangian for a spinor field in four dimensions and obtain the equations of motion. It turns out that the equation for ψ_- is an algebraic constraint relation, which can be used to eliminate ψ_- from the theory. ψ_+ contains two real components, which can be combined in a complex anticommuting field $\psi(x)$. The equation of motion for ψ looks the same as that of a bosonic field. However, the important difference is that the dimension of ψ is not the same as that of a bosonic field. Thus the free action for a fermion in four dimensions reads

$$\mathcal{S} \;=\; \int d^4x \; \partial^+\bar{\psi}(x)\, \frac{\Box}{\partial^+}\, \psi(x) \tag{B.4}$$

One can also derive interacting theories with fermionic fields in the light-cone formulation. These theories obviously cannot have three-point (or any odd order) self-interaction terms, but we can consider the coupling of fermionic fields to an integer spin field [49]. Such couplings occur naturally in theories with supersymmetry.

Appendix C

Gamma matrix manipulations in $d = 11$

In this appendix, we discuss some useful identities and gamma matrix manipulations for $SO(7)$ spinors. These identities are important for our calculations, because the kinematical supersymmetries, q^m and $\bar{q}_m$ transform in a 8-dimensional spinor representation under the $SO(7)$ in eleven dimensions.

We consider the 28 antisymmetric γ-matrices, γ^m and γ^{mn}, where the indices $m, n. \ldots$ run from 1 through 7.

The relevant Fierz identity for the product of two $SO(7)$ spinors is

$$A_\alpha \, A_\beta = -\frac{1}{8}\gamma^m{}_{\alpha\beta} \, A\gamma^m A + \frac{1}{16}\gamma^{mn}{}_{\alpha\beta} \, A\gamma^{mn} A. \tag{C.1}$$

where the $\alpha, \beta, ...$ are spinor indices.

We will now show how to manipulate expressions involving multiple spinors. In order to bring them to a convenient form, we must write all such expressions in terms of the $SO(7)$ irreducible representations.

In the case of two spinors, we can express the 28 different components as $\mathbf{7} + \mathbf{21}$ as shown in (C.1).

Expressions involving three spinors

Consider the following expression

$$\begin{aligned}
\gamma^n{}_{\beta\gamma}\,(E\partial^{+2}\bar{\partial}\phi E^{-1}\partial^{+3}\phi)|_{\rho^\alpha,\rho^\beta,\rho^\gamma} &= \gamma^n{}_{\beta\gamma}\frac{\partial}{\partial\rho^\alpha}\frac{\partial}{\partial\rho^\beta}\frac{\partial}{\partial\rho^\gamma}\,(E\partial^{+2}\bar{\partial}\phi E^{-1}\partial^{+3}\phi)|_{\rho=0} \\
&\equiv \gamma^n{}_{\beta\gamma}\,A_\alpha A_\beta A_\gamma\;B. \qquad \text{(C.2)}
\end{aligned}$$

In order to convert such expressions to the form (40), we use the Fierz identity to obtain a prefactor of the form $\gamma^n{}_{\alpha\beta}$.

With three spinors A we get 56 components which split into **8** and **48** as follows.

$$\begin{aligned}
|8\rangle_3 &= (\gamma^q A)_\alpha\, A\gamma^q A, \\
|48\rangle_3 &= A_\alpha A\gamma^p A - \frac{1}{7}(\gamma^p\gamma^q A)_\alpha\, A\gamma^q A.
\end{aligned}$$

Any expressions with $3A$'s can now be decomposed in terms of these states, for example

$$(\gamma^r A)_\alpha\, A\gamma^{rp} A = 5\left(A_\alpha A\gamma^p A - \frac{1}{7}(\gamma^p\gamma^q A)_\alpha\, A\gamma^q A\right) - \frac{2}{7}(\gamma^p\gamma^q A)_\alpha\, A\gamma^q A. \qquad \text{(C.3)}$$

Using the Fierz identity we also find

$$(\gamma^q A)_\alpha\, A\gamma^q A = \frac{1}{2}(\gamma^{qr} A)_\alpha\, A\gamma^{qr} A, \qquad \text{(C.4)}$$

Expressions involving four spinors

In an expression with 4 A's, there are 70 components. These can be split up as

$A\gamma^m A\, A\gamma^n A$ which is $\mathbf{1}+\mathbf{27}$.

$A\gamma^m A\, A\gamma^{mn} A$ which is **7**.

$A\gamma^{[m} A\, A\gamma^{np]} A$ which is **35**.

Expressions involving five spinors

We now focus on expressions with $5A$'s. The corresponding irreducible forms for five spinor combinations are

$$
\begin{aligned}
|8\rangle_5 &= A_\alpha\, A\gamma^r A\, A\gamma^r A \\
|48\rangle_5 &= (\gamma^r A)_\alpha A\gamma^p A\, A\gamma^r A - \frac{1}{7}(\gamma^p A)_\alpha\, A\gamma^r A\, A\gamma^r A.
\end{aligned}
$$

With the help of Fierzing an expression with $5A$'s decomposes as

$$
A_\alpha\, A\gamma^{pq} A A\gamma^q A = -\frac{2}{3}[(\gamma^r A)_\alpha A\gamma^p A\, A\gamma^r A - \frac{1}{7}(\gamma^p A)_\alpha\, A\gamma^r A\, A\gamma^r A] + \frac{4}{7}(\gamma^p A)_\alpha\, A\gamma^r A\, A\gamma^r A. \tag{C.5}
$$

This formula leads to an useful result

$$
(\gamma^p A)_\alpha\, A\gamma^{pr} A\, A\gamma^r A = 4A_\alpha\, A\gamma^r A\, A\gamma^r A \tag{C.6}
$$

We can derive two other useful formulae

$$
\begin{aligned}
A_\alpha\, A\gamma^p A\, A\gamma^q A &= \frac{1}{9}(\gamma^p\gamma^r A)_\alpha\, A\gamma^r A\, A\gamma^q A \\
&+\frac{1}{9}(\gamma^q\gamma^r A)_\alpha\, A\gamma^r A\, A\gamma^p A \\
&+\frac{1}{9}\delta^{pq}\, A_\alpha\, A\gamma^r A\, A\gamma^p A,
\end{aligned} \tag{C.7}
$$

and

$$
\begin{aligned}
A_\alpha\, A\gamma^{[pq} A\, A\gamma^{r]} A &= -\frac{1}{9}(\gamma^{pq}\gamma^s A)_\alpha\, A\gamma^s A\, A\gamma^r A \\
&-\frac{1}{9}(\gamma^{rp}\gamma^s A)_\alpha\, A\gamma^s A\, A\gamma^q A \\
&-\frac{1}{9}(\gamma^{qr}\gamma^s A)_\alpha\, A\gamma^s A\, A\gamma^p A.
\end{aligned} \tag{C.8}
$$

Using the above formulae, equations (6.26), (6.27) and (6.29) in chapter 6 yields the following expression (up to a common prefactor)

$$
\begin{aligned}
&c_1 A_\alpha\, A\gamma^n A\, A\gamma^m A\, B^m + c_2 (\gamma^m \gamma^p A)_\alpha\, A\gamma^n A\, A\gamma^p A\, B^m - c_3 A_\alpha\, A\gamma^m A\, A\gamma^m A\, B^n \\
&\quad = \frac{c_1}{9} (\gamma^n \gamma^r A)_\alpha\, A\gamma^r A\, A\gamma^m A\, B^m \\
&\quad + \left(\frac{c_1}{9} + c_2\right) (\gamma^m \gamma^r A)_\alpha\, A\gamma^n A\, A\gamma^r A\, B^m \\
&\quad + \left(\frac{c_1}{9} - 4\sqrt{2}\, c_3\right) A_\alpha\, A\gamma^r A\, A\gamma^r A\, B^n
\end{aligned} \tag{C.9}
$$

The first term on the r.h.s is of the desired form. Hence, the other two terms must vanish. This fixes the value of the free parameters in our ansatz.

$$ c_1 = -\frac{1}{4\sqrt{2}}, \tag{C.10} $$

$$ c_2 = \frac{1}{36\sqrt{2}}, \tag{C.11} $$

$$ c_3 = -\frac{1}{288}. \tag{C.12} $$

Expressions involving seven spinors

We now consider expressions with seven A's. Such an expression appears in (6.31). In this case, only an **8** is possible.

$$ |8\rangle_7 = (\gamma^m A)_\alpha\, A\gamma^m A\, A\gamma^n A\, A\gamma^n A. \tag{C.13} $$

Using the Fierz identity we immediately get

$$ A_\alpha\, A\gamma^n A\, A\gamma^m A\, A\gamma^m A \;=\; \frac{1}{7} (\gamma^n \gamma^r A)_\alpha\, A\gamma^r A\, A\gamma^m A\, A\gamma^m A. \tag{C.14} $$

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