An Investigation of Stability of Anti-de Sitter Spacetime

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "An Investigation of Stability of Anti-de Sitter Spacetime" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by " Pranav Kumar" at Indian Institute of Science Education and Research Pune, under the supervision of Dr. Suneeta Vardarajan during the academic year 2014-2015.

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Declaration

I hereby declare that the matter embodied in the report entitled An Investigation of Stability of AdS Spacetime are the results of the investigations carried out by me at the Department of Physics, Indian Institute of Science Education and Research Pune, under the supervision of Dr. Suneeta Vardarajan and the same has not been submitted elsewhere for any other degree.

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Acknowledgements

First and foremost I would like to thank my supervisor Dr. Suneeta Vardarajan. Her guidance has been invaluable over the entire period I have worked with her. Due to her encouragement and her advice, I have been able to learn abundantly in the past year. Her deep understanding of the subject and experience has shaped this thesis, and has given me an introduction to research in theoretical physics, for which I will always be grateful.

Additionally, I would like to thank Dr. Arjun Bagchi for taking time to mentor me during the initial stages of this project. Without his advice and comments, it would have been difficult to progress further in my fifth year project. I would like to thanks Ms Amruta Sadhu and Ms Dhanya Menon for their valuable time and discussions during the course of my entire fifth year.

I would like to thank my family for their unending support and encouragement . I would also like to express my gratitude towards my friends, although that rarely suffices, for their encouragement and enthusiasm throughout these five years. Their support has kept me going through the whole of my course at IISER.

I would also like to thank IISER Pune and KVPY for supporting this project.

Abstract

In this project, we analyze the structure of non spherically-symmetric Tensor Gravitational perturbations in the Transverse Traceless(TT) gauge beyond the linearized order on Anti-de Sitter(AdS) background in a generic dimension. The spherical symmetry of the background is used to simplify the perturbation equations. We start with introducing AdS and it's causal structure, and then review some work on scalar field perturbations which lead to instability in that spacetime in the first chapter. Additionally, we present the calculation for higher order equations for this system. In the second chapter, we derive the dynamical equations of gravitational perturbations in TT gauge upto second order for the given background. After that, a formalism developed in [1] is introduced and applied to simplify the dynamical equations at linear and nonlinear order. Finally, we conclude with the analysis of the structure of second order equations, and its results.

Contents

CHAPTER 1

Chapter 1

Introduction

The issue of stability of spacetimes has always been an important topic of study in the field of General Relativity. At first, a lot of effort was given in the study of linearized stability of Schwarzschild and other asymptotically flat spacetimes containing a black-hole[2],[3][4]. Later, the linearized perturbations on de sitter spacetime were studied[5]. According to the results of these extensive studies, these spacetimes are stable under linear perturbations.
In contrast to previous studies, the issue of stability of Minkowski spacetime has been studied upto full non-linear formalism in great detail[6]. The results of this study establish that Minkowski are stable under small perturbations. For Minkowski spacetime, the mechanism for stability is the dispersion of perturbation towards infinity. This leads to the inital perturbations slowly decaying to zero with time, a phenomenon referred to as Asymptotic stability.

In recent years, Anti-de Sitter spacetime has been subject to much analysis and research due to the AdS-CFT correspondence [7]. A lot of effort has gone into developing dynamical theories which contain various matter fields, considering AdS as a background. In order to to make such a consistent theory, it needs to be checked apriori that AdS is stable under such perturbations or not, i.e. do small perturbations grow indefinetly in time or not. In the past few years, the issue of stability of Anti de-Sitter spacetime has become very important[8],[9]. In general, full non-linear analytical treatment of this problem is very complicated as finding the complete set of solutions for the partial differential equations is very hard, in general. Hence, most of the progress is made by using numerical techniques. Only under certain simplifying approximations can anything conclusive be said about the issue. A number of numerical and analytic studies have found that under certain class of scalar field perturbations, for a given set of boundary conditions and initial con-

ditions, AdS is unstable, i.e. indefinitely small perturbations grow linearly in time[10],[11]. On the other hand, a few studies have also suggested that there are a class of perturbations for Asymptotically AdS spacetimes which do not result in unstable modes.[12]. Hence, it important to study when such instabilities are generated, and what's the reason.
We begin by describing the basic properties of Anti-de Sitter spacetime, and why studying the stability of AdS is more difficult.

1.1 Introduction to AdS

Anti-de Sitter spacetime is the unique maximally symmetric Lorentzian manifold with constant negative scalar curvature[13]. In $n+2$ dimensions, it can be represented as

$$X_1^2 + X_2^2 + ... + X_d^2 - U^2 - V^2 = -\ell^2 \tag{1.1}$$

embedded in a a flat $n+3$ dimensional space with metric:

$$ds^2 = dX_1^2 + ... + dX_d^2 - dU^2 - dV^2. \tag{1.2}$$

Here, ℓ represents the radius of Anti deSitter spacetime. After a series of coordinate transformations, the metric can be written in a form where the causal structure is apparent, which is

$$ds^2 = \frac{\ell^2}{\cos x^2}(-dt^2 + dx^2 + \sin x^2 d\Omega^2); \tag{1.3}$$

where $d\Omega^2$ is the metric on a n dimensional sphere and $(t, x) \in R \times [0, \frac{\pi}{2})$. This metric solves the Einstein Equations with a negative cosmological constant $\Lambda = -\frac{n(n+1)}{2\ell^2}$. Using this form of the metric, we can study the causal structure of this spacetime.

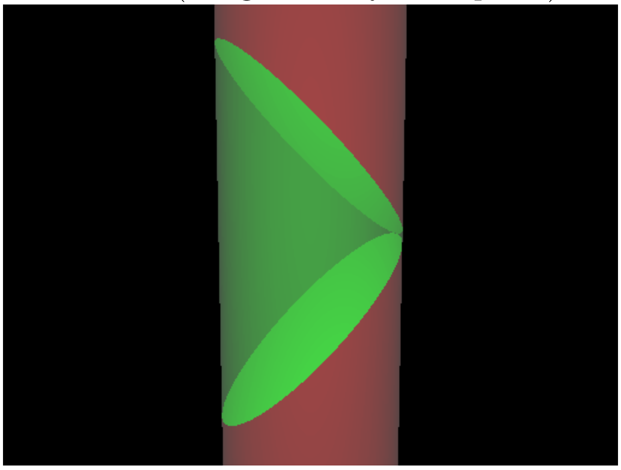
The conformal infinity $\mathbb{I}(x = \pi/2)$ is the timelike cylinder $\mathbb{R} \times S^n$. From the figure [1] on the next page, it is apparent that AdS can be conformally matched to half of the Einstein static universe .

1.2 Initial and Boundary Conditions

The time taken by null geodesics to reach the origin from the boundary is finite, and this can be seen by the following calculation.
Consider null geodesics in the metric:

$$ds^2 = \frac{\ell^2}{\cos x^2}(-dt^2 + dx^2 + \sin x^2 d\Omega^2);$$

Figure 1.1: The causal diagram of AdS, where the spherical dimensions have been represented as a circle. As we can see, the region shaded with green corresponds to Anti-de Sitter spacetime, and the full cylinder is the whole Einstein Static Universe. The surface of the cylinder is the boundary of AdS. The two green disks are the null geodesics, and timelike geodesics are contained within the disks. (Image Curtsey: Wikipedia)



The equation for a null geodesic with constant spherical coordinates is $dt = \pm dx$. To get the time interval for light rays to reach the origin, we need to solve $\Delta t = \int dt = \int_{\pi/2}^{0} dx$. Since the limits on the integral is finite, this expression has a finite value.

To be able to define dynamics for a system, we need to prescribe initial conditions at $t = 0$. But, in the case of AdS, since information from the boundary takes only finite time to reach inside, that information could effect the dynamics that have been defined by intitial conditions prescribed by us. Hence, in addition to Initial Conditions, we also need to prescribe Boundary conditions for the system such that the mathematical problem of defining dynamics is well posed,[14],[15]. This makes the study of AdS instability more interesting and complicated.

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Chapter 2

AdS Instability by a Scalar field

In this chapter, we will present an example of an analytic study,[11] where, for a certain class of scalar field perturbations; given a specific set of initial conditions and reflecting boundary conditions, an instability is found for AdS in more than three dimensions.

2.1 The Scalar field model

A simple model of a scalar field as the perturbation on the vacuum solution is considered, where the field is spherically symmetric. Although in this section, we are working in a four dimensional metric, the procedure is easily extendable to higher dimensions[16]. We are trying to solve the Einstein-Scalar system governed by the following equations:

$$G_{ab} + \Lambda g_{ab} = 8\pi G(\partial_a\phi\partial_b\phi - \frac{1}{2}g_{ab}(\partial\phi)^2) \tag{2.1}$$

where $\Lambda = -3/\ell^2$.
The scalar field obeys the equation of motion: $g^{ab}\nabla_a\nabla_b\phi = 0$.

The perturbation is assumed to be of a special kind, where the metric components g_{00} and g_{11} are perturbed, and the perturbation is assumed to have a form:

$$ds^2 = \frac{\ell^2}{\cos^2 x}(-A\exp^{-2\delta} dt^2 + A^{-1}dx^2 + \sin^2 x d\Omega^2) \tag{2.2}$$

The perturbation coefficients A, δ, and Φ are functions of (t,x). For pure AdS case A=1; $\delta = 0$, $\phi = 0$. This ansatz corresponds to a spherically symmetric perturbation by a spherically symmetric field.

The equations governing the dynamics of the system are :

$$\dot{\Pi} = \frac{1}{\tan^2 x}(\tan^2 x A e^{-\delta}\Phi)' \tag{2.3}$$

This is the scalar wave equation in AdS background, where $\Pi := A^{-1}e^{\delta}\dot{\phi}$ and $\Phi := \phi'$. Einstein's equations yield two constraints, which are:

$$A' = \frac{1+2\sin^2 x}{\sin x \cos x}(1-A) - \sin x \cos x A(\Phi^2 + \Pi^2) \tag{2.4}$$

$$\delta' = -\sin x \cos x(\Phi^2 + \Pi^2) \tag{2.5}$$

Note that the AdS length scale ℓ drops out of the equations.
To find a solution of the above equations, a perturbative expansions of the unknown variables is assumed. The expansion is :

$$\phi(t,x) = \sum_{j=0}^{\infty} \phi_{2j+1}(t,x)\epsilon^{2j+1}; A(t,x) = 1 - \sum_{j=1}^{\infty} A_{2j}(t,x)\epsilon^{2j}; \delta(t,x) = \sum_{j=1}^{\infty} \delta_{2j}(t,x)\epsilon^{2j} \tag{2.6}$$

The justification of this expansion is that at first order, there is no back-reaction to the metric. So, at linear order, the system corresponds to a massless scalar field in the AdS background with no back-reaction to the metric. The back-reaction comes at the second order. And hence, the metric perturbation coefficients start from second order. Also, the reason why the field is only at odd orders is that having a coefficient of the scalar field to the even order parameter will give us nothing new in the system as that equation will be identical to the equation corresponding to one order lower than that.

2.1.1 Equations at Linear Order

Inserting this expansion into the set of dynamical equations, and collecting terms of the same order, we get, to the lowest order:

$$\ddot{\phi}_1 + L\phi_1 = 0; L = -\frac{1}{\tan^2 x}\partial_x(\tan^2 x \partial_x) \tag{2.7}$$

This Sturm-Liouville operator is self adjoint on $L^2([0,\pi/2], \tan^2 x dx)$. Eigenvalues of this operator are: $\omega_j^2 = (3+2j)^2; j \in (0,1,2..)$. Since the eigen values are real, this implies that the spacetime is linearly stable. The eigenfunctions of this self-adjoint operator are $e_j(x) = d_j \cos^3 x \, {}_2F_1(-j, 3+j, 3/2; \sin^2 x)$, where d_j is the normalization constant. These eigenfunctions are hypergeometric functions, which in our case simplify, and are Jacobi Polynomials.

Since these functions are eigen-functions of a self adjoint operator, they form a complete set of functions, i.e. any function can be written as a linear combination of the set of jacobi polynomials. The fact that the linear order operator is Self Adjoint, will be of great use in the next section and the subsequent chapters.
The full solution can now be written as: $\phi_1(t,x) = \sum_{j=0}^{\inf} a_j \cos{(\omega_j t + b_j)} e_j(x)$, where a's and b's are determined by initial conditions prescribed by us.

2.1.2 Back reaction to the metric

The back reaction to the metric appears at the second order, and the expression can be determined from the two constrained equations by taking the lowest order non-trivial terms. Integrating the equations, we get

$$A_2(t,x) = \frac{\cos^3 x}{\sin x} \int_0^x (\dot{\phi_1}(t,y)^2 + \phi_1'(t,y)^2) \tan^2 y dy, \tag{2.8}$$

$$\delta_2(t,x) = -\int_0^x (\dot{\phi_1}(t,y)^2 + \phi_1'(t,y)^2) \sin y \cos y dy \tag{2.9}$$

These equations have now given us a relation between the metric perturbation coefficients at second order and the scalar field perturbation at first order. Now we take the third order terms from the scalar wave equation. This is an inhomogeneous differential equation with the same differential operator on left hand side. The right hand side can be thought of as the source at third order. It consists of both second and first order terms. The exact equation is:

$$\ddot{\phi_3} + L\phi_3 = S(A_2, \delta_2, \phi_1), \tag{2.10}$$

where $S := -2(\delta_2 + A_2)\ddot{\phi_1} - (\dot{A_2} + \dot{\delta_2})\dot{\phi_1} - (A_2' + \delta_2')\phi_1'$
Substituting the expressions for A_2 and δ_2, we see that the whole source term can be written sum over as products of terms of the form $a_j \cos\omega_j t + b_j e_j(x)$. After projecting the third order equation on the eigen-basis, we get an infinite set of decoupled forced harmonic oscillator equations for the Fourier coefficients (ϕ_3, e_j),

$$\ddot{c_j} + \omega_j^2 c_j = S_j := (S, e_j). \tag{2.11}$$

This equation has the structure of a forced harmonic oscillator. On the right hand side, by combination of ω_j's in the source terms, if the resonant frequency can be produced, then due to secular terms the scalar field perturbation will grow linearly in time indefinitely.
Writing down the full expression of the (s, e_j), we get that for a given ω_l, the secular terms can arise for $\omega_l = \omega_i + \omega_j - \omega_k$, $2\omega_i - \omega_k$. For one mode initial

data, the resonant term is removable through some techniques, but for two or higher mode initial data, the resonant term is non-removable.
Also, from this calculation, we can see how the energy shifts to higher scales. Let us take an example to see this. Let's consider that the resonance is due to a two mode initial data $\cos\omega_5 + \cos\omega_3$. Then the secular terms arise for $2\omega_i - \omega_k$, which are $2(3+2\times 5) - (3+2\times 3) = 3+2\times 7$, hence due to initial data of $i = 5$ and $j = 3$, the resonance happens at $l = 7$, which is a higher frequency.

This last calculation suggests that energy is transferred from the low frequency modes to high frequency modes. This can also be interpreted as concentration of energy into smaller scales. This process of concentration of an excessive amount of energy can lead to formation of a black-hole. Further numerical calculations from similar studies show this phenomenon in detail.

2.2 Higher order structure

After doing perturbative analysis of the system, it was found that the system has a set of natural frequencies at the third order. But this is a restricted statement, i.e. doesn't apply to the full problem. To say something about the full problem, even perturbatively, we have to see if the same structure arises at every order. Hence, we need to go beyond the previous study to higher orders, where we can see the frequency spectrum of the full problem. The natural frequencies of the full system will be given as an expansion in the parameter ϵ with the coefficients being the spectrum of frequencies at each order.
In order to further explore that, we performed the fourth and fifth order calculations. Demanding the perturbative expansion(1.8) for the variables, we find that the fourth order equations are :

$$A_4 = \frac{\cos^3 x}{\sin x} \int_0^x (\dot{\phi}_1(t,y)^2(A_2(t,y) + 2\delta_2(t,y)) - A_2(t,y)\phi_1'(t,y)^2) \tan^2 y dy \tag{2.12}$$

$$\delta_4 = -\int_0^x (2(\dot{\phi}_1)^2(A_2 + \delta_2) + \phi_1'\phi_3') \sin y \cos y dy \tag{2.13}$$

These equations have been derived from expanding the Einstein equations and writing them as an integral. The fifth order equation comes from expanding the scalar wave equation, which is

$$\ddot{\phi}_5 + L\phi_5 = T(A_4, \delta_4, A_2, \delta_2, \phi_3, \phi_1) \tag{2.14}$$

where $T := -2\ddot{\phi}_3(A_2+\delta_2) - \dot{\phi}_3(\dot{A}_2+\dot{\delta}_2) - 2\ddot{\phi}_1(A_4+\delta_4) - \ddot{\phi}_1 A_2^2 - \dot{\phi}_1(\dot{\delta}_4 + \dot{A}_4 + 2A_2\dot{A}_2 + \delta_2\dot{A}_2 + A_2\dot{\delta}_2) - \phi_3'(A_2'+\delta_2') - \phi_1'(A_4'+\delta_4' - A_2\delta_2' - \delta_2 A_2')$.

This equation is of the same form as the one at third order. The difference comes because of the source which is more complicated at higher orders, as combining the various order components of the metric perturbations has more possibilities. The common forced harmonic oscillator structure comes because of the scalar wave equation which ensures the exact combination occurs on the left hand side.
So we see that the same harmonic oscillator structure arises here as well. And looking at the structure of initial Einstein Equations (2.4) and (2.5) and the wave equation (2.3), we can say that this structure will continue to appear at every odd order, with the source terms getting more and more complicated. And it is those source terms that determine what will be the spectrum of resonant frequencies at that order.

2.3 Results and Limitations of this approach

We can see that the structure of equations for all the variables remains essentially the same at each order. It seems obvious now, as they are just expansions of the same equation which is true to all orders. Also, in this model, the ansatz for the metric was chosen so as to get a specific form of equations.
But, as we will see in the later chapters, this is a generic feature of dynamical equations for a wide class of perturbations, and follows fundamentally from the structure of Einstein's Equations .

Till now we have an example of scalar field perturbations that lead to an instability in the metric. This is a very special class of perturbations for AdS, where the scalar field only perturbs specific components of the metric. We need to study a larger class of perturbations that include gravitational perturbations. In order to do this, we need to use the General Relativity formalism for deriving perturbations equations. We start with the most general class, and then simplify it further using properties of the background spacetime, in the next chapter.

Chapter 3

Gravitational Perturbation Equations

As we saw in the previous chapters, that there are a certain class of perturbations, under which, for specific initial and boundary conditions, AdS is unstable. The reason for this instability is the presence of a forced harmonic oscillator structure where the source terms can have secular terms at the resonant frequencies. We now study the gravitational perturbations on AdS in the TT gauge.

We would now like to consider a wider class of perturbations, and see if such a structure arises. In order to do that, we have to first derive the basic dynamical equations for the most general perturbations on Anti-de Sitter spacetime. As we will see, these equations are almost impossible to solve without using simplifying measures. Before doing that, we would first like to classify different kinds of perturbations. To simplify these equations, we use a method described in Wald and Ishibashi,[1] where they take advantage of the spherical symmetry of the metric.

3.1 Formalism in GR

Anti-de Sitter spacetime is the maximally symmetric solution to the vacuum Einstein equations with a negative cosmological constant. Assuming that as the background spacetime, we wish to analyze the structure of dynamical equations for perturbations at first and second order. We are working with a $n+2$ dimensional spacetime, whose metric can be written as

$$ds_{n+2}^2 = \frac{\ell^2}{\sin^2 x}(-dt^2 + dx^2 + \cos^2 x \gamma_{ij} dz^i dz^j)$$

The range of x is $(0, \frac{\pi}{2}]$ and the conformal boundary is located at $x = 0$. This coordinate system is slightly in contrast with the one mentioned in the previous chapter as here, the coordinate x has been transformed to $\frac{\pi}{2} - x$. Therefore, in this coordinate system, the boundary is at $x = 0$, and the origin of the coordinates is at $x = \frac{\pi}{2}$. This choice of coordinates makes the spherical symmetry explicit in the problem. Due to this structure, we can bring this metric in a useful form by a simple coordinate transformation, which we will see in the next chapter. So, for the rest of the document, we will refer to the above expression as the reference metric for pure AdS.

We are trying to solve the equation

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{3.1}$$

This equation further translates to $R_{\mu\nu} = \frac{2}{n}\Lambda g_{\mu\nu}$. After inserting the value of $\Lambda = -\frac{n(n+1)}{2\ell^2}$, the equation simplifies to

$$R_{\mu\nu} + \frac{n+1}{\ell^2} g_{\mu\nu} = 0 \tag{3.2}$$

where ℓ is the radius of AdS.

3.1.1 Background Derivative Procedure

The dynamical equation (3.2) obtained above is for a metric different from the pure AdS solution, i.e. the derivatives and the connections in (3.2) correspond to the full metric, and not the maximally symmetric one. We want to write the equations in orders of pertubation around the background spacetime, so as to make use of it's symmetry properties. To do so, we would like to express the derivative corresponding to the full metric,${}^{\lambda}\nabla_{\mu}$ in terms of the derivative corresponding to the background metric,$\bar{\nabla}_{\mu}$. Such a procedure is given in [17], Chapter 2, where the difference between the two derivatives can be expressed by a three index object which depends on both the metrics. So, the expression can be written as :

$${}^{\lambda}\nabla_{\mu}\omega_{\nu} = \bar{\nabla}_{\mu}\omega_{\nu} - C^{\rho}_{\mu\nu}\omega_{\rho} \tag{3.3}$$

and when acting on rank two tensors, the expression is:

$${}^{\lambda}\nabla_{\mu}t_{\alpha\beta} = \bar{\nabla}_{\mu}t_{\alpha\beta} - C^{\nu}_{\mu\alpha}t_{\beta\nu} - C^{\nu}_{\mu\beta}t_{\alpha\nu} \tag{3.4}$$

where C can be expressed as :

$$C^{\rho}_{\mu\nu}(\lambda) = \frac{1}{2}g^{\rho\beta}(\lambda)(\bar{\nabla}_{\mu}g_{\nu\beta}(\lambda) + \bar{\nabla}_{\nu}g_{\mu\beta}(\lambda) - \bar{\nabla}_{\beta}g_{\mu\nu}(\lambda)) \tag{3.5}$$

Note that this object is symmetric in the two bottom indices. Also, if the background is flat, this reduces to the expression of the Christoffel Symbol.

3.1.2 Manipulation of Riemann Tensor

We have an expression for the Riemann tensor in terms of the full derivative. We would like to write it in terms of the background derivative. The Riemann tensor for a given metric is defined as:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\omega_\alpha = R^\rho_{\mu\nu\alpha}\omega_\rho \tag{3.6}$$

Substituting equation (3.3) and (3.4) in (3.6), we get

$$\begin{aligned}(\bar{\nabla}_\mu \bar{\nabla}\nu - \bar{\nabla}_\nu \bar{\nabla}\mu)\omega_\alpha - \bar{\nabla}_\mu (C^\rho_{\nu\alpha}\omega_\rho) + \bar{\nabla}_\nu (C^\rho_{\mu\alpha}\omega_\rho) \\ -C^\rho_{\mu\alpha}\bar{\nabla}_\nu \omega_\rho + C^\rho_{\nu\alpha}\bar{\nabla}_\mu \omega_\rho + C^\beta_{\mu\alpha}C^\rho_{\nu\beta}\omega_\rho - C^\beta_{\nu\alpha}C^\rho_{\mu\beta}\omega_\rho\end{aligned} \tag{3.7}$$

The first term is just the Riemann tensor of the background metric. Now, using the Leibnitz rule for Covariant derivatives, we can simplify this expression such that the full Riemann tensor can be written as:

$$R^\rho_{\mu\beta\nu}(\lambda) = \bar{R}^\rho_{\mu\beta\nu} - 2\bar{\nabla}_{[\mu}C^\rho_{\beta]\nu} + 2C^\alpha_{\nu[\mu}C^\rho_{\beta]\alpha} \tag{3.8}$$

Thus, the Ricci tensor is given by:

$$R_{\mu\nu}(\lambda) = \bar{R}_{\mu\nu} - 2\bar{\nabla}_{[\mu}C^\rho_{\rho]\nu} + 2C^\alpha_{\nu[\mu}C^\rho_{\rho]\alpha} \tag{3.9}$$

Finally, the dynamical equation, (3.2) that we have to solve, takes the form:

$$\bar{R}_{\mu\nu} - \bar{\nabla}_\mu C^\rho_{\rho\nu} + \bar{\nabla}_\rho C^\rho_{\mu\nu} + C^\alpha_{\nu\mu}C\rho_{\rho\alpha} - C^\alpha_{\nu\rho}C^\rho_{\mu\alpha} + \frac{n+1}{\ell^2}g_{\mu\nu} = 0 \tag{3.10}$$

. This is a most general expression where the Ricci tensor of a given metric is written in terms of derivatives corresponding to another metric. Apriori in the above expression, there need not be a connection between the two metrics. Now we'll analyze this equation, assuming that the two metrics correspond to the full and the background metric.

3.2 Expansion around the AdS background

After obtaining the general dynamical equation application to any system where there are two derivatives involved, we specialize to the case where one of the derivatives is the zeroth order approximation to the full derivative, corresponding to pure AdS spacetime, relevant to the problem. We remove the bar from the derivatives, as now, there is only one derivative in the problem. Inserting the expression for $C^\rho_{\mu\nu}(\lambda)$, we expand the previous equation to get:

$$
\begin{aligned}
\bar{R}_{\mu\nu} + \frac{1}{2}[& - \nabla_\mu g^{\rho\beta}\nabla_\rho g_{\nu\beta} - g^{\rho\beta}\nabla_\mu\nabla_\rho g_{\nu\beta} - \nabla_\mu g^{\rho\beta}\nabla_\nu g_{\rho\beta} - g^{\rho\beta}\nabla_\mu\nabla_\nu g_{\rho\beta} \\
& + \nabla_\mu g^{\rho\beta}\nabla_\beta g_{\nu\rho} + g^{\rho\beta}\nabla_\mu\nabla_\beta g_{\nu\rho} + \nabla_\rho g^{\rho\beta}\nabla_\mu g_{\nu\beta} + g^{\rho\beta}\nabla_\rho\nabla_\mu g_{\nu\beta} \\
& + \nabla_\rho g^{\rho\beta}\nabla_\nu g_{\mu\beta} + g^{\rho\beta}\nabla_\rho\nabla_\nu g_{\mu\beta} - \nabla_\rho g^{\rho\beta}\nabla_\beta g_{\mu\nu} - g^{\rho\beta}\nabla_\rho\nabla_\beta g_{\mu\nu}] \\
& + \frac{1}{4}g^{\alpha\gamma}g^{\rho\delta}[(\nabla_\nu g_{\mu\gamma} + \nabla_\mu g_{\nu\gamma} - \nabla_\gamma g_{\mu\nu})(\nabla_\rho g_{\alpha\delta} + \nabla_\alpha g_{\rho\delta} - \nabla_\delta g_{\rho\alpha}) \\
& - (\nabla_\nu g_{\rho\gamma} + \nabla_\rho g_{\nu\gamma} - \nabla_\gamma g_{\rho\nu})(\nabla_\mu g_{\alpha\delta} + \nabla_\alpha g_{\mu\delta} - \nabla_\delta g_{\mu\alpha})] \\
& + \frac{n+1}{\ell^2} g_{\mu\nu} = 0 \qquad (3.11)
\end{aligned}
$$

Please note that all the $g_{\mu\nu}$'s mentioned in the above equation are $g_{\mu\nu}(\lambda)$. Now, we can insert the perturbative expansions for $g_{\mu\nu}$ and $g^{\mu\nu}$ in the equation, and retain terms upto second order. The expansions are:

$$
\begin{aligned}
g_{\mu\nu}(\lambda) &= \bar{g}_{\mu\nu} + \lambda h_{\mu\nu} + \frac{\lambda^2}{2} f_{\mu\nu} + ... \qquad (3.12) \\
g^{\mu\nu}(\lambda) &= \bar{g}^{\mu\nu} - \lambda h^{\mu\nu} - \frac{\lambda^2}{2}(f_{\mu\nu} - h^\mu_\beta h^{\nu\beta}) \qquad (3.13)
\end{aligned}
$$

In the above expansion, the $\bar{g}_{\mu\nu}$ corresponds to the background piece. The derivative we are using now is defined w.r.t. that piece. The $h_{\mu\nu}$ is the first variation of the metric and $h_{\mu\nu} = \frac{dg_{\mu\nu}(\lambda)}{d\lambda}|_{\lambda=0}$. At second order in the perturbation parameter λ, we get the second variation in the metric$f_{\mu\nu}$, which can be obtained by $\frac{d^2 g_{\mu\nu}(\lambda)}{d\lambda^2}|_{\lambda=0}$.

The first and second variation in $g^{\mu\nu}(\lambda)$ is obtained by demanding that $g_{\mu\alpha}(\lambda)g^{\alpha\nu}(\lambda) = \delta^\nu_\mu + O(h^3)$. As we will see, the non-linear terms in $g^{\mu\nu}(\lambda)$ are not important in the analysis if we are restricting ourselves to second order. From now on, we drop the zeroth order term $\bar{R}_{\mu\nu}$ in the equation; as at that order, the equations are trivially satisfied. Now, substituting these expansions, (3.11) and (3.12) in (3.7), we get:

$$
\begin{aligned}
&\nabla_\mu h^{\rho\beta}\nabla_\rho h_{\nu\beta} + h^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} - \bar{g}^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} - \bar{g}^{\rho\beta}\nabla_\mu\nabla_\rho f_{\nu\beta} \\
&+\nabla_\mu h^{\rho\beta}\nabla_\nu h_{\rho\beta} + h^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} - \bar{g}^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} - \bar{g}^{\rho\beta}\nabla_\mu\nabla_\nu f_{\rho\beta} \\
&-\nabla_\mu h^{\rho\beta}\nabla_\beta h_{\nu\rho} - h^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} + \bar{g}^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} + \bar{g}^{\rho\beta}\nabla_\mu\nabla_\beta f_{\nu\rho} \\
&-\nabla_\rho h^{\rho\beta}\nabla_\mu h_{\nu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\mu f_{\nu\beta} \\
&-\nabla_\rho h^{\rho\beta}\nabla_\nu h_{\mu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\nu f_{\mu\beta} \\
&+\nabla_\rho h^{\rho\beta}\nabla_\beta h_{\mu\nu} + h^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} - \bar{g}^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} - g^{\rho\beta}\nabla_\rho\nabla_\beta f_{\mu\nu} \\
&+\frac{1}{2}\bar{g}^{\alpha\gamma}\bar{g}^{\rho\delta}[(\nabla_\nu h_{\mu\gamma} + \nabla_\mu h_{\nu\gamma} - \nabla_\gamma h_{\mu\nu})(\nabla_\rho h_{\alpha\delta} + \nabla_\alpha h_{\rho\delta} - \nabla_\delta h_{\rho\alpha}) \\
&\quad -(\nabla_\nu h_{\rho\gamma} + \nabla_\rho h_{\nu\gamma} - \nabla_\gamma h_{\rho\nu})(\nabla_\mu h_{\alpha\delta} + \nabla_\alpha h_{\mu\delta} - \nabla_\delta h_{\mu\alpha})] \\
&\quad +\frac{2(n+1)}{\ell^2}(h_{\mu\nu} + f_{\mu\nu}) = 0
\end{aligned}
\tag{3.14}
$$

We have made use of the fact that $\nabla_\beta \bar{g}_{\mu\nu} = 0$, as the derivative ∇ is defined w.r.t. the background metric. The reason we don't need the non-linear terms of $g^{\mu\nu}(\lambda)$ is that if we use $f^{\mu\nu}$ or $h^\mu_\beta h^{\nu\beta}$, then inside the derivative, there can only be $\bar{g}_{\mu\nu}$, if we are restricting ourselves to second order; and that term will be equal to zero as $\nabla_\beta \bar{g}_{\mu\nu} = 0$.

This equation can also be derived by taking the first and second variation of the Ricci tensor around the pure AdS background. The derivation is a simple one, and one can use it to perform a check on the above calculation. We consider the full Ricci tensor as a functional in the perturbation parameter λ, so the full Ricci tensor is $R_{\mu\nu}[g(\lambda)]$. We can write this as a taylor expansion in λ, and consider it's variations.

$$R_{\mu\nu}[g(\lambda)] = R_{\mu\nu}[\bar{g}] + \delta R_{\mu\nu} + \frac{\delta^2}{2} R_{\mu\nu} + .. \tag{3.15}$$

where the two variations can be derived as

$$\delta R_{\mu\nu} = \frac{\partial R_{\mu\nu}}{\partial g}\frac{dg}{d\lambda}|_{\lambda=0} d\lambda \tag{3.16}$$

$$\delta^2 R = \{(\frac{\partial R}{\partial g})^2 (\frac{dg}{d\lambda})^2 + \frac{\partial R}{\partial g}\frac{d^2 g}{d\lambda^2}\}|_{\lambda=0} d\lambda^2 \tag{3.17}$$

. from the above expressions, we can point out a correlation between the terms here and the terms we get from the full derivation done earlier. Using the definitions of the first and second variations of the metric, we can see that terms like $-\nabla_\rho h^{\rho\beta}\nabla_\nu h_{\mu\beta}$ correspond to $(\frac{\partial R}{\partial g})^2(\frac{dg}{d\lambda})^2$ in the taylor expansion. Terms involving $f_{\mu\nu}$ correspond to $\frac{\partial R}{\partial g}\frac{d^2 g}{d\lambda^2}$ here. This correspondence gives an

identification to the derived terms in the previous equation for the full Ricci tensor.
Having derived the equations, we can now simplify them.

What we derived is the full equation which we get from standard gravitational perturbation theory, without any assumption made about the perturbation or the background metric. Looking at the structure of the equations, we can now answer the point raised in the previous chapter. The reason that the Self-Adjoint operator was the same for the homogenous and non homogenous equations, can be attributed to the basic structure of Einstein equations themselves, as is apparent from the equation where the same derivative operators are acting on the linear $h_{\mu\nu}$ terms and the quadratic $f_{\mu\nu}$. Till now, we haven't said anything about the operator, as we haven't analyzed it in detail. So, we cannot say it is self-adjoint yet; but, we can say that it will be the same operator acting on the variation at every order.
Now, the first thing is to separate the first and the second order terms, as the equations have to be satisfied at every order. Then, we will impose the TT gauge conditions, which are: $h, f = 0$; $\nabla_\mu h^{\mu\nu} = 0$. This simplifies the equations a lot. In the calculations, we use the background metric to raise or lower indices on the perturbations and on the derivatives.
Also, here we can simplify the operator that acts on $h_{\mu\nu}$ at linear order, and on $f_{\mu\nu}$ at second order.

3.3 Linearized Equations for TT perturbations

We know take the linear order terms from the equation, and try to simplify them using the Transverse Traceless(TT) gauge conditions and symmetry of the background metric.

The linear order equation is:

$$\frac{1}{2}\bar{g}^{\rho\beta}[(\nabla_\rho\nabla\mu - \nabla_\mu\nabla_\rho)h_{\nu\beta} + \nabla_\rho\nabla_\nu h_{\mu\beta} - \nabla_\mu\nabla_\nu h_{\rho\beta} \quad - \quad \nabla_\rho\nabla_\beta h_{\mu\nu} + \nabla_\mu\nabla_\beta h_{\nu\rho}] \\ + \quad \frac{2(n+1)}{\ell^2}h_{\mu\nu} = 0 \quad (3.18)$$

Using the identity $(\nabla_\rho\nabla_\nu - \nabla_\nu\nabla_\rho)h_{\mu\beta} = R^\alpha_{\rho\nu\mu}h_{\alpha\beta} + R^\alpha_{\rho\nu\beta}h_{\mu\alpha}$, we can write the above equation as:

$$-\Delta h_{\mu\nu} \quad + \quad \frac{1}{2}\bar{g}^{\rho\beta}[R^\alpha_{\rho\mu\nu}h_{\alpha\beta} + R^\alpha_{\rho\nu\mu}h_{\alpha\beta} + R^\alpha_{\rho\mu\beta}h_{\alpha\nu} + R^\alpha_{\rho\nu\beta}h_{\alpha\mu}] \\ - \quad \nabla_\mu\nabla_\nu\mathbf{h} + \nabla_\nu\nabla_\beta h^\beta_\mu + \nabla_\mu\nabla_\beta h^\beta_\nu + \frac{2(n+1)}{\ell^2}h_{\mu\nu} = 0 \quad (3.19)$$

This is the standard linearized Einstein Equation without imposing any gauge conditions. This equation is identical to the ones derived in [18]. The difference from the cited reference being that we are dealing with a spacetime with a cosmological constant. Now, using the Lorentz gauge conditions, $\nabla_\nu h^{\mu\nu} = 0$ and the traceless-ness condition $\mathbf{h} = h^\mu_\mu = 0$, we can eliminate three terms from the above equation to get

$$\begin{aligned} -\Delta h_{\mu\nu} &+ \frac{1}{2}\bar{g}^{\rho\beta}[R^\alpha_{\rho\mu\nu}h_{\alpha\beta} + R^\alpha_{\rho\nu\mu}h_{\alpha\beta} + R^\alpha_{\rho\mu\beta}h_{\alpha\nu} + R^\alpha_{\rho\nu\beta}h_{\alpha\mu}] \\ &+ \frac{2(n+1)}{\ell^2}h_{\mu\nu} = 0 \end{aligned} \tag{3.20}$$

To further simplify this equation, we need to use the maximally symmetric property of the AdS Riemann tensor, i.e.

$$R_{\rho\mu\nu\alpha} = \frac{-1}{\ell^2}(g_{\rho\nu}g_{\mu\alpha} - g_{\rho\alpha}g_{\mu\nu}) \tag{3.21}$$

which translates to

$$R^\alpha_{\rho\mu\nu} = \frac{-1}{\ell^2}(g_{\rho\nu}g^\alpha_\mu - g^\alpha_\rho g_{\mu\nu})$$

Substituting these expressions in (3.15), we get

$$\frac{1}{2}\Delta h_{\mu\nu} + \frac{1}{\ell^2}h_{\mu\nu} = 0 \tag{3.22}$$

This is as far as we can come in simplifying the linear equations using general techniques. In the next chapter, we will decompose the derivatives into the AdS type derivatives and the derivatives on the sphere, and then bring this equation into a more solvable wave equation form.

3.4 Second order Equations in TT gauge

Keeping only the quadratic order terms in (3.14), we get

$$\begin{aligned} \nabla_\mu h^{\rho\beta}\nabla_\rho h_{\nu\beta} + h^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} - \bar{g}^{\rho\beta}\nabla_\mu\nabla_\rho f_{\nu\beta} + \nabla_\mu h^{\rho\beta}\nabla_\nu h_{\rho\beta} \\ +h^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} - \bar{g}^{\rho\beta}\nabla_\mu\nabla_\nu f_{\rho\beta} - \nabla_\mu h^{\rho\beta}\nabla_\beta h_{\nu\rho} - h^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} \\ +\bar{g}^{\rho\beta}\nabla_\mu\nabla_\beta f_{\nu\rho} - \nabla_\rho h^{\rho\beta}\nabla_\mu h_{\nu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\mu f_{\nu\beta} \\ -\nabla_\rho h^{\rho\beta}\nabla_\nu h_{\mu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\nu f_{\mu\beta} + \nabla_\rho h^{\rho\beta}\nabla_\beta h_{\mu\nu} \\ +h^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} - g^{\rho\beta}\nabla_\rho\nabla_\beta f_{\mu\nu} \\ +\frac{1}{2}\bar{g}^{\alpha\gamma}\bar{g}^{\rho\delta}[(\nabla_\nu h_{\mu\gamma} + \nabla_\mu h_{\nu\gamma} - \nabla_\gamma h_{\mu\nu})(\nabla_\rho h_{\alpha\delta} + \nabla_\alpha h_{\rho\delta} - \nabla_\delta h_{\rho\alpha}) \\ -(\nabla_\nu h_{\rho\gamma} + \nabla_\rho h_{\nu\gamma} - \nabla_\gamma h_{\rho\nu})(\nabla_\mu h_{\alpha\delta} + \nabla_\alpha h_{\mu\delta} - \nabla_\delta h_{\mu\alpha})] \\ +\frac{2(n+1)}{\ell^2}f_{\mu\nu} = 0 \end{aligned} \tag{3.23}$$

We can now impose the gauge conditions on the equations, to simplify them and get them in a form where the structure is apparent. After gauging away the terms, we get:

$$\begin{aligned}
&\nabla_\mu h^{\rho\beta}\nabla_\rho h_{\nu\beta} + h^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} + \nabla_\mu h^{\rho\beta}\nabla_\nu h_{\rho\beta} + h^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} \\
&-\nabla_\mu h^{\rho\beta}\nabla_\beta h_{\nu\rho} - h^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} - h^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\mu f_{\nu\beta} \\
&-h^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\nu f_{\mu\beta} + h^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} - g^{\rho\beta}\nabla_\rho\nabla_\beta f_{\mu\nu} \\
&-\frac{1}{2}\bar{g}^{\alpha\gamma}\bar{g}^{\rho\delta}(\nabla_\nu h_{\rho\gamma} + \nabla_\rho h_{\nu\gamma} - \nabla_\gamma h_{\rho\nu})(\nabla_\mu h_{\alpha\delta} + \nabla_\alpha h_{\mu\delta} - \nabla_\delta h_{\mu\alpha}) \\
&+\frac{2(n+1)}{\ell^2} f_{\mu\nu} = 0 \qquad (3.24)
\end{aligned}$$

As the first order calculation, we can now separate the terms and write them as the Lichnerowicz operator acting on the second perturbations, with the source terms on the right hand side.

$$\begin{aligned}
-\Delta f_{\mu\nu} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\mu f_{\nu\beta} + \bar{g}^{\rho\beta}\nabla_\rho\nabla_\nu f_{\mu\beta} + \frac{2(n+1)}{\ell^2} f_{\mu\nu} &= \nabla_\mu h^{\rho\beta}\nabla_\beta h_{\nu\rho} \\
-\nabla_\mu h^{\rho\beta}\nabla_\rho h_{\nu\beta} - h^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} - \nabla_\mu h^{\rho\beta}\nabla_\nu h_{\rho\beta} - h^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} \\
+h^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} + h^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} + h^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} \\
+\frac{1}{2}\bar{g}^{\alpha\gamma}\bar{g}^{\rho\delta}(\nabla_\nu h_{\rho\gamma} + \nabla_\rho h_{\nu\gamma} - \nabla_\gamma h_{\rho\nu})(\nabla_\mu h_{\alpha\delta} + \nabla_\alpha h_{\mu\delta} - \nabla_\delta h_{\mu\alpha}) \quad (3.25)
\end{aligned}$$

This equation has the form of an inhomogenous second order differential equation, with the source terms comprising of products of first order perturbations. After simplifying the Lichnerowicz operator, we get:

$$\begin{aligned}
- \quad & \Delta f_{\mu\nu} - \frac{1}{\ell^2} f_{\mu\nu} = \nabla_\mu h^{\rho\beta}\nabla_\beta h_{\nu\rho} - \nabla_\mu h^{\rho\beta}\nabla_\rho h_{\nu\beta} - h^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} \\
- \quad & \nabla_\mu h^{\rho\beta}\nabla_\nu h_{\rho\beta} - h^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} + h^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} + h^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} \\
+ \quad & h^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} \\
+ \quad & \frac{1}{2}\bar{g}^{\alpha\gamma}\bar{g}^{\rho\delta}(\nabla_\nu h_{\rho\gamma} + \nabla_\rho h_{\nu\gamma} - \nabla_\gamma h_{\rho\nu})(\nabla_\mu h_{\alpha\delta} + \nabla_\alpha h_{\mu\delta} - \nabla_\delta h_{\mu\alpha}) \quad (3.26)
\end{aligned}$$

At the end of this chapter, we have got the general perurbation equation(3.11) as well as the order by order dynamical equations for the TT perturbations. The full problem of study of stability of this spacetime under this whole set of gravitational perturbations. But since, in the current form, these equations are not analytically solvable, we will have to use simplifying measures. In order to do that, we will proceed towards introducing the general formalism of classification of second rank tensors on a spacetime with spherical symmetry. This analysis will help us determine the class of perturbations on such spacetimes, and what are the mathematical tools that can be used w.r.t. each of them. They will also tell us what is a consistent set of perturbations on this class of spacetimes, i.e. a subset of these perturbations that can be studied independently of others, if such a class exists. We will see that such decoupling happens, but only upto first order.

Chapter 4

Decomposition of Derivatives and Dynamical Equations

Till now, we have gotten the Einstein equations for TT perturbations in the AdS background in the most general form. Now, we try to classify the perturbations into three types of perturbations. This simplifies the linear order equation a lot, and gives an ordinary differential equation with a self adjoint operator for the perturbation at linear and quadratic order.

4.1 Decomposition w.r.t S^n

To remind ourselves, we consider a system of spacetime dimension n+2, where n is dimension of the sphere. This formalism, as developed in [1] and [19] is applicable to maximally symmetric spaces whose line element can be written as

$$ds^2_{(n+2)} = g_{\mu\nu}dx^\mu dx^\nu = g_{ab}(y)dy^a dy^b + r^2(y)\gamma_{ij}(z)dz^i dz^j, \qquad (4.1)$$

where $\gamma_{ij}(z)dz^i dz^j$ is the metric on the unite sphere S^n and $r \equiv \ell\frac{\cos x}{\sin x}$. Using the spherical symmetry, metric perturbations can be decomposed into scalar, vector or tensor perturbations w.r.t action of a rotation group on the n-Sphere.

Notation We would like to clarify the notation used in this section and following part of the article. To denote the coordinates on the full manifold, "x" will be used. For the coordinates ranging t and r(x), "y" will be used and the spherical coordinates will be represented by "z". Greek indices on the derivatives refer to the full spacetime indices, i.e. $\nu, \mu, \alpha.. \in (0, 1, 2...., n+2)$. Indices a - h range over the t and r(x) variables and hence $a, b..h \in (0, 1)$

and indices i-p range over the coordinates of the sphere in the system, so $i, j, k, l \in (2, 3, 4, ..., n + 2)$.

4.1.1 Decomposition of Tensors on Compact Manifolds

We now quote two very important mathematical propositions that help us in characterizing the tensors on a compact manifold, which for our case, is an n-Sphere. The results have been cited here and their proofs are contained in [1].

1. Let(M, γ_{ij}) be a compact Riemannian Manifold, then any dual vector v_i can be written as

$$v_i = V_i + D_i S \tag{4.2}$$

, where $D^i V_i = 0$. We refer to V_i and S as the vector and scalar components of v_i

2. Let (M, γ_{ij}) be a compact Riemannian Einstein Space, i.e $R_{ij} = c\gamma_{ij}$ for any constant c, then we can write any second rank tensor field on M as

$$t_{ij} = T_{ij} + D_{(i}V_{j)} + (D_i D_j - \frac{\gamma_{ij} D^m D_m}{n})S + \frac{\gamma_{ij} t^m_m}{n}; \tag{4.3}$$

$$D^i T_{ij} = 0, T^i_i = 0, D^i V_i = 0$$

We refer to T_{ij}, V_j and (S, t^m_m) as the tensor, vector and scalar components of t_{ij}.

The uniqueness of this decomposition follows from the orthogonality of all the terms in (4.3), under the defined inner product of square integrable functions(L^2) on the compact manifold. It is important to note that the compactness of the sphere has been used extensively in this argument, especially in concluding that the operator is self-adjoint and that the spectrum of a self adjoint symmetric operator is discrete. This decomposition is very useful because, for any rotationally invariant operator on this compact manifold, no "mixing" is possible, i.e. scalar, vector and tensor cannot transform into each other by the action of any rotationally invariant operator. Since, at the first order the differential operator is invariant under rotations, the three kinds of perturbations decouple. Hence, the coupling between scalar, vector and tensor perturbations only starts at second order and at first order, we can study each kind of perturbation independently.

4.2 The System

We have seen that general tensorial TT perturbation can be decmomposed into three orthogonal components, i.e. terms of type h_{ij}. Additionally, we have seen that any rotationally invariant operator on the compact manifold can't couple these components at linear order. The full problem of stability of Anti-de Sitter spacetime reduces to the study of these three classes of perturbations. These perturbations can be studied separately at linear order. We now specialize to a one of these three classes and derive the dynamical equations.

In this project, we consider only the tensor component of metric perturbations. They provide a simplification of equations, and also provide a gauge independent model to work with, since the gauge freedom in General Relativity is $g_{\mu\nu} \to g_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)}$. This renders the equations of tensor components of perturbations unchanged in any gauge.

Tensor components can be written as a product of tensor spherical harmonics and scalar functions of the non spherical coordinates of the metric. The linearized Einstein equations then reduce to uncoupled wave equations for the scalar functions. So, this implies that we will be working with the follwing expressions of the perturbations at the linear level.

$$h_{ij} = S(t,r)\mathbf{T}_{K_T ij}$$
$$h_{ai} = 0; h_{ab} = 0$$

This choice implies that in our system, there are only tensor perturbations present. And hence, there will be no coupling even at higher orders due to absence of vector and scalar components.

4.2.1 Tensor Spherical Harmonics

The rotation group acts naturally on the Hilbert space of square-integrable rank-two symmetric tensor fields on S^n. Tensor Harmonics, $\mathbf{T}_{K_T ij}$ are defined to be an orthonormal basis of eigenvectors of $D^i D_i$ in L_T^2, i.e. they satisfy

$$(D^m D_m + k_T^2)\mathbf{T}_{K_T ij} = 0; D_j \mathbf{T}^j_{K_T i} = 0; \mathbf{T}^i_{K_T i} = 0 \tag{4.4}$$

$$k_T^2 = l(l+n-1) - 2; l = 2, 3, ...$$

The number of independent components of $\mathbf{T}_{K_T ij}$ is $\frac{(n-2)(n+1)}{2}$; so tensor spherical harmonics are only non trivial for a sphere of dimension 3 or higher.

4.2.2 Metric

For the specific section we are dealing with, we write the metric in form (4.1), where $r \equiv \ell \frac{\cos x}{\sin x}$, where ℓ is the radius of AdS. In the coordinates (t, r), the metric of non-sphere part takes the form:

$$g_{ab}(y)dy^a dy^b = -V^2\ell^2 dt^2 + \frac{dr^2}{V^2}; V^2 = 1 + \frac{r^2}{\ell^2} \tag{4.5}$$

4.3 Wave Equations at linear order for Tensor Perturbations

The perturbed line element can be written as

$$h_{\mu\nu}dx^\mu dx^\nu = h_{ab}dy^a dy^b + 2h_{ai}dy^a dz^i + h_{ij}dz^i dz^j \tag{4.6}$$

We use the prescription given earlier to decompose rank two tensors, and apply it to h_{ij}

$$h_{ij} = h_{ij}^{(2)} + 2D_{(i}h_{T\ j)}^{(1)} + h_L\gamma_{ij} + (D_iD_j - \frac{\gamma_{ij}D^m D_m}{n})h_T^0 \tag{4.7}$$

The purely tensorial component is $h_{ij}^{(2)}$, and it can be written as $h_{ij}^{(2)} = \sum H_{T_K}^{(2)}.\mathbf{T}_{Kij}$; where the summation is over all the possible K values. Substituting $H_T^{(2)} = r^{\frac{n+2}{2}}\phi_T$, the linearized equations are:

$$\nabla^a\nabla_a\phi_T - (\frac{n(n+2)}{4\ell^2} + [\frac{n(n-2)}{4} + l(l+n-1)]\frac{1}{r^2})\phi_T = 0 \tag{4.8}$$

Here, ∇^a represents the derivative w.r.t the non sphere part of the metric, i.e. $a \in (t, r)$
Expanding the derivative of the AdS part, we get a differential equation of the form:

$$\frac{\partial^2}{\partial t^2}\phi_T = (\frac{\partial^2}{\partial x^2} - \frac{\nu^2 - 1/4}{\sin^2 x} - \frac{\sigma^2 - 1/4}{\cos^2 x})\phi_T \tag{4.9}$$

; which is defined on the interval$(0, \frac{\pi}{2})$, where $x = \frac{\pi}{2}$ corresponds to the origin of AdS and $x = 0$ is the boundary. In this particular case of tensor perturbations, non-trivial only in spacetime dimensions five or higher; $\sigma = l + \frac{n-1}{2}$ and $\nu = \frac{n+1}{2}$. This equation can also be derived by first decomposing the non gauged equation into the sphere and non-sphere type derivatives, and then inserting the special case of tensorial perturbations,[20],[19]. This equation is also similar to the one derived for maximally symmteric backgrounds in [21]. We get one such equation for each K in the summation. We now look for the solutions of these equations

4.4 Solutions to the Eigenvalue Problem

We now plan to analyze the solutions to above equation. We demand the solution of type $\phi_T = e^{i\omega t}\phi(r)$, thus obtaining the eigenvalue equation

$$\omega^2\phi(r) = (-\frac{\partial^2}{\partial x^2} + \frac{\nu^2 - 1/4}{\sin^2 x} + \frac{\sigma^2 - 1/4}{\cos^2 x})\phi(r) \tag{4.10}$$

We denote the differential operator on the right hand side, acting on $\phi(r)$, as L. We will be dealing with this operator in this and the next few sections. It can be shown that this is a self adjoint operator on the hilbert space of square integrable functions. Given that $\nu \geq 1$ in our case, if we demand regularity of solutions at the origin and the boundary, then we get the discrete spectrum of frequencies that can be allowed, which are:

$$\omega^2 = (2m + 1 + \nu + \sigma)^2; m = 0, 1, 2, ... \tag{4.11}$$

The eigenfunctions are :

$$\begin{aligned}
\phi(x) &= C.G_\nu(x)\frac{\Gamma(1+\sigma)\Gamma(-\nu)}{\Gamma(\zeta^{\omega}_{-\nu,\sigma})\Gamma(\zeta^{-\omega}_{-\nu,\sigma})}.(\sin x)^{2\nu}.F(\Gamma(\zeta^{\omega}_{\nu,\sigma}), \Gamma(\zeta^{-\omega}_{\nu,\sigma}), 1+\nu; \sin^2 x);\\
&= -C.G_\nu(x)\frac{\Gamma(1+\sigma)(-1)^{\nu})}{\Gamma(\zeta^{\omega}_{-\nu,\sigma})\Gamma(\zeta^{-\omega}_{-\nu,\sigma})}.(\sin x)^{2\nu}.\sum_{k=0}^{\infty}\frac{(\zeta^{\omega}_{-\nu,\sigma})_k(\zeta^{\omega}_{-\nu,\sigma})_k}{k!(\nu+k)!}. \qquad (4.12)\\
&\quad (\sin x)^{2k}(\log\sin^2 x - \psi(k+1) - \psi(k+\nu+1) + \psi(\zeta^{\omega}_{\nu,\sigma}+k) + \psi(\zeta^{-\omega}_{\nu,\sigma}+k))
\end{aligned}$$

The first equation corresponds to even dimensional spacetime, and second one corresponds to odd dimensional spacetime. Here

$$\begin{aligned}
\zeta^{\omega}_{\nu,\sigma} &\equiv \frac{\omega+\nu+\sigma+1}{2}\\
(\zeta)_k &\equiv \frac{\Gamma(\zeta+k)}{\Gamma(\zeta)}\\
\psi(z) &\equiv \frac{d}{dz}log(\Gamma(z))\\
G_\nu(x) &\equiv (\cos x)^{\sigma+1/2}.(\sin x)^{1/2-\nu}
\end{aligned}$$

So, we have the full solution of the linear order equation for tensorial component of TT perturbations. This solution is regular near the origin and the boundary. The positivity of the eigenvalue of the self adjoint operator guarantees that AdS is linearly stable. Also, sice the operator is self-adjoint, the eigenfunctions form a complete basis. This property will be put to much use, when we deal with the source terms in the second order equation.
To find any instability in the metric, we need to go beyond the first order and study that inhomogenous equation.

4.5 Decomposition of Second order Tensor Equations

Recall the second order dynamical equation

$$\begin{aligned}
- & \Delta f_{\mu\nu} - \frac{1}{\ell^2} f_{\mu\nu} = \nabla_\mu h^{\rho\beta}\nabla_\beta h_{\nu\rho} - \nabla_\mu h^{\rho\beta}\nabla_\rho h_{\nu\beta} - h^{\rho\beta}\nabla_\mu\nabla_\rho h_{\nu\beta} \\
- & \nabla_\mu h^{\rho\beta}\nabla_\nu h_{\rho\beta} - h^{\rho\beta}\nabla_\mu\nabla_\nu h_{\rho\beta} + h^{\rho\beta}\nabla_\mu\nabla_\beta h_{\nu\rho} + h^{\rho\beta}\nabla_\rho\nabla_\mu h_{\nu\beta} \\
+ & h^{\rho\beta}\nabla_\rho\nabla_\nu h_{\mu\beta} - h^{\rho\beta}\nabla_\rho\nabla_\beta h_{\mu\nu} \\
+ & \frac{1}{2}\bar{g}^{\alpha\gamma}\bar{g}^{\rho\delta}(\nabla_\nu h_{\rho\gamma} + \nabla_\rho h_{\nu\gamma} - \nabla_\gamma h_{\rho\nu})(\nabla_\mu h_{\alpha\delta} + \nabla_\alpha h_{\mu\delta} - \nabla_\delta h_{\mu\alpha})
\end{aligned}$$

.

Expanding these derivatives into the spherical and the non spherical coordinates, i.e. $\rho\beta \rightarrow ab, al, kb, kl,$, we get the full expansion which also contains the scalar and vector components. After performing this decomposition, since we are dealing with tensorial components, and we know that those components don't couple and can be studied separately, we now put the scalar and the vector components to zero in the calculations. So, only terms like $h_{i,j}$ in the expansion will remain. Alongside this exercise, we also have to put this equation in the form of the previous equation. Substituting $f_{ij}^{(2)} = \sum \mathbf{F^{(2)}_{T_K}}.\mathbf{T_{Kij}}$ and then $\mathbf{F^{(2)}_T} = \mathbf{r^{\frac{n+2}{2}}}\chi_\mathbf{T}$; where the summation is over all the K values, we get the second order equation, which is

$$\begin{aligned}
\sum_{\mathbf{K}_T}\{ & - \nabla^a\nabla_a\chi_T + (\frac{n(n+2)}{4\ell^2} + [\frac{n(n-2)}{4} + l(l+n-1)]\frac{1}{r^2})\chi_T\}\mathbf{T}_{Kij} = \nabla_i h^{kl}(\nabla_l h_{jk} \\
& - \nabla_k h_{jl} - \nabla_j h_{kl}) + h^{kl}((\nabla_k\nabla_i - \nabla_i\nabla_k)h_{jl} - \nabla_i\nabla_j h_{kl} + \nabla_i\nabla_l h_{jk} \qquad (4.13) \\
& + \nabla_k\nabla_j h_{il} + \nabla_k\nabla_l h_{ij}) + \frac{1}{2}\{\nabla_l h^k_j(\nabla_i h^l_k + \nabla_k h^l_i - \nabla^l h_{ik}) \\
& + \nabla^k h_{lj}(\nabla^l h_{ik} - \nabla_k h^l_i - \nabla_i h^l_k) + \nabla_j h^k_l(\nabla_i h^l_k + \nabla_k h^l_i - \nabla^l h_{ik}) - \nabla_b h^k_j \nabla^b h_{ik}\}.
\end{aligned}$$

Translating the above equation, and writing this as an ODE, we get:

$$\begin{aligned}
\sum_{\mathbf{K}_T}\{\frac{\partial^2}{\partial t^2}\chi_T + L\chi_T\}\mathbf{T}_{Kij} &= \nabla_i h^{kl}(\nabla_l h_{jk} - \nabla_k h_{jl} - \nabla_j h_{kl}) \qquad (4.14) \\
&+ h^{kl}((\nabla_k\nabla_i - \nabla_i\nabla_k)h_{jl} - \nabla_i\nabla_j h_{kl} + \nabla_i\nabla_l h_{jk} + \nabla_k\nabla_j h_{il} + \nabla_k\nabla_l h_{ij}) \\
&+ \frac{1}{2}\{\nabla_l h^k_j(\nabla_i h^l_k + \nabla_k h^l_i - \nabla^l h_{ik}) + \nabla^k h_{lj}(\nabla^l h_{ik} - \nabla_k h^l_i - \nabla_i h^l_k) \\
&+ \nabla_j h^k_l(\nabla_i h^l_k + \nabla_k h^l_i - \nabla^l h_{ik}) - \nabla_b h^k_j \nabla^b h_{ik}\};
\end{aligned}$$

where L is the differential operator identical to the first order equation. We can now expand χ in terms of the basis functions $e_m(x)$. So

$$\chi_T = \sum_m c_m(t) e_m(x) \tag{4.15}$$

so that the ODE can be written as

$$\begin{aligned}
&\sum_{\mathbf{K}_T} \sum_m \{ \frac{d^2}{t^2} c_m(t) e_m(x) + L c_m(t) e_m(x) \} \mathbf{T}_{Kij} = \nabla_i h^{kl} (\nabla_l h_{jk} - \nabla_k h_{jl} - \nabla_j h_{kl}) \\
+ \quad & h^{kl} ((\nabla_k \nabla_i - \nabla_i \nabla_k) h_{jl} - \nabla_i \nabla_j h_{kl} + \nabla_i \nabla_l h_{jk} + \nabla_k \nabla_j h_{il} + \nabla_k \nabla_l h_{ij}) \\
+ \quad & \frac{1}{2} \{ \nabla_l h^k_j (\nabla_i h^l_k + \nabla_k h^l_i - \nabla^l h_{ik}) + \nabla^k h_{lj} (\nabla^l h_{ik} - \nabla_k h^l_i - \nabla_i h^l_k) \\
+ \quad & \nabla_j h^k_l (\nabla_i h^l_k + \nabla_k h^l_i - \nabla^l h_{ik}) - \nabla_b h^k_j \nabla^b h_{ik} \};
\end{aligned}$$

We will use the relation $Le_m = \omega_m^2 e_m$, which was established in the previous section, to get the system into forced harmonic oscillator form. In the next chapter, we will analyze this equation further, projecting the source term onto the eigenbasis, and then we will get the equations in the forced harmonic oscillator form. We will then describe the spectrum of resonant frequencies in this model.

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Chapter 5

Results and Conclusions

Till now we have seen the structure of equations at first order. At second order, we have source terms which can cause an instability in the metric, for specific initial conditions. To study that, we will now derive the Forced Harmonic Oscillator structure at second order.

5.1 Forced Harmonic Oscillator Structure

Now we need to take the projection w.r.t. the inner product defined on this space. As has been pointed out, L is a self adjoint operator on the Hilbert space of square intergrable functions $L^2([0, \frac{\pi}{2}], dx)$. The measure factor comes out to be 1 in our case as a result of a simple calculation. So, the inner product between two first order perturbation terms is

$$(\psi, \phi) = \int_0^{\frac{\pi}{2}} \psi^* \phi dx \tag{5.1}$$

To get the projection, we need to perform this integral on the left hand side of the equation;

$$\{\sum_j (\ddot{c}_j(t) + \omega_j^2) \int_0^{\frac{\pi}{2}} e_j(x) e_m(x)\} \times \sum_{\mathbf{K}T} \{\int d^n\Omega \mathbf{T}_{Kij} \mathbf{T}^{ij}_{K\prime}\} = S(h, \nabla h) \tag{5.2}$$

Due to orthogonality of e_j's, i.e. $(e_j, e_m) = \delta_{jm}$; and of tensor spherical harmonics,[22], which says $(\mathbf{T}_{Kij}, \mathbf{T}^{ij}_{K\prime}) = c$; where c is a constant depending on the values of K and $K\prime$, we get the following equation.

$$\ddot{c_m}(t) + \omega_m^2 c_m(t) = S(h, \nabla h) \tag{5.3}$$

Here $S(h, \nabla h)$ is the projection of the source term onto the eigenbasis. Let us concentrate on the source term in the second order equation. It can be written as

$$\begin{aligned}
& \bar{g}^{ko}\bar{g}^{lp}\{\nabla_i h_{op}(\nabla_l h_{jk} - \nabla_k h_{jl} - \nabla_j h_{kl}) \\
+ \;& h_{op}\{(\nabla_k\nabla_i - \nabla_i\nabla_k)h_{jl} - \nabla_i\nabla_j h_{kl} + \nabla_i\nabla_l h_{jk} + \nabla_k\nabla_j h_{il} + \nabla_k\nabla_l h_{ij}\} \\
+ \;& \frac{1}{2}(\nabla_i h_{pk} + \nabla_k h_{ip} - \nabla_p h_{ik})(\nabla_j h_{jo} + \nabla_j h_{lo} - \nabla_o h_{lj})\} \\
- \;& \frac{1}{2}\bar{g}^{ko}\bar{g}^{ab}\nabla_b h_{oj}\nabla_a h_{ik}
\end{aligned} \tag{5.4}$$

We substitute the form of the solution we got for the first order perturbation.

$$h_{ij} = \sum_{K_T} r^{2-\frac{n}{2}}\phi(x)e^{i\omega t}T_{Kij}$$

where $r \equiv \ell\frac{\cos x}{\sin x}$ and $\phi(x) = \sum_m a_m(t)e_m(x)$; a_m being the normalization constant. The time dependence of the first order solution comes from the $e^{i\omega t}$ factor where the spectrum of frequencies allowed is discrete, and is given by (4.11). Since we know the full form of the eigenstates of the linear operator, and know that they form a complete basis set, we can expand the source terms into the eigenstates by using the inner product defined on the hilbert space.

Let us now take a generic term on the right hand side of the second order equation. We see that with the exception of the last term all the derivatives involved are w.r.t. coordinates on the sphere. Since the hypergeometric part and the time dependent part are independent of those coordinates, we can take them out of the derivatives. Finally, the structure of each term that we get is

$$e^{i\omega_{m1}t}e^{i\omega_{m2}t}\phi_{m1}(x)\phi_{m2}(x)f(x)Y(z_1, z_2, ..., z_n) \tag{5.5}$$

. As an example, if we consider the first source term, $\bar{g}^{ko}\bar{g}^{lp}\nabla_i h_{op}\nabla_l h_{jk}$, then each of the terms prescribed above correspond to

1. $e^{i\omega_{m1}t}$ is the time component of h_{op}, and $e^{i\omega_{m2}t}$ is for the second h factor.
2. $\phi_{m1}(x)$ is the space component of h_{op}, and similarly $\phi_{m2}(x)$ is for the second part.
3. f(x)$\equiv \bar{g}^{ko}\bar{g}^{lp} \times r^{4-n}$, one from each first order term.
4. $Y(z_1, z_2, ..., z_n) \equiv \nabla_i\mathbf{T}_{K_1op}\nabla_l\mathbf{T}_{K_2jk}\mathbf{T}^{ij}_{K\prime}$.

A generic term in it's expression can be written explicitly as

$$(\sum_{K_1}\sum_{K_2}\int d^n\Omega\nabla_i\mathbf{T}_{K_1op}\nabla_l\mathbf{T}_{K_2jk}\mathbf{T}^{ij}_{K\prime})\times(\int_0^{\frac{\pi}{2}} e_{m1}(x)e_{m2}(x)e_m(x)f(x)dx)\times e^{i\omega_{m1}t}e^{i\omega_{m2}t} \tag{5.6}$$

Now we have an infinite set of harmonic oscillators, each labeled by a value m from the discrete frequency spectrum. We will calculate the secular terms on the right hand, and determine the resonant frequency spectrum.

5.2 Resonant Frequency Spectrum

Given that we have the forced harmonic oscillator equation, we need to determine if the expression of the source contains secular terms, and we also need to find the resonant frequency spectrum. We start by recalling that the frequency spectrum at linear order looks like:

$$\omega_m = \pm(2m + 1 + \nu + \sigma) \tag{5.7}$$

where m is a non-negative integer. Substituting for ν and σ, we get

$$\omega_m = \pm(2m + 1 + l + n) \tag{5.8}$$

To recall, l refers to the harmonic in the expression of the tensor spherical harmonic.

The term $e^{i\omega_{m1}t}e^{i\omega_{m2}t}$ is present for all source terms on the right hand side of the second order equation, (4.14). We need to find out that for given values of m1 and m2, what value of m will the resonant frequency correspond to. A simple calculation shows that assuming the same value of l, i.e. considering coupling between same harmonics, we get, for this resonant term $+\omega_{m1} + \omega_{m2}$, that the resonant frequency occurs at $m = (m1 + m2 + \frac{l+n+1}{2})$, and for $+\omega_{m1} - \omega_{m2}$, the resonant frequency is $m = (m1 - m2 - \frac{l+n+1}{2})$. By $\pm\omega_m$, we mean the positive/negative frequency mode corresponding to that integer.
Of course the resonance will only be reached when the argument m is an integer. But, nevertheless, we can see that there are modes at which resonance will occur, and we can always produce such terms by taking different combinations of l_1 and l_2.

5.3 Conclusion

We have analytically studied the structure of tensor gravitaional perturbations in TT gauge upto second order. The AdS background and it's symmetry properties have allowed us to simplify the equations, and observe some patterns that emerge in the first and second order terms. The simplifying assumptions have allowed us to tackle this problem without numerical calculations. We see that these perturbations five or more dimensional AdS spacetime can cause an instability in the metric. The more interesting question is what will be the end point of this instability? A possibility of the end point is that a black hole is formed in the bulk. These topics need to studied as they will provide us with a greater understanding of dynamics in AdS, and the role it's conformal boundary plays in the evolution of matter fields in various theories.

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