PACHNER MOVES ON GEOMETRIC TRIANGULATIONS

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Date: December 9, 2019

Dr. Tejas Kalelkar
Thesis Supervisor

Declaration

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Contents

1

Introduction

A foundational theorem of Pachner, Theorem 5.5 of [35] showed that PL-triangulations of PL homeomorphic manifolds are related by local combinatorial transformations called bistellar moves or Pachner moves. Whitehead in Theorem 8 of [43] showed that smooth triangulations of diffeomorphic smooth manifolds are related by smooth Pachner moves. The same question can be asked for other classes of manifolds as well, for example the constant curvature Riemannian manifolds with geometric triangulations.

Triangulations of a manifold are well understood structures, they have been actively used in research for a long time. We can use them to convert topological or geometric problems of the manifold into combinatorial ones. The books by Rourke and Sanderson [37] and Ziegler [46] are good sources of introduction to the theory of piecewise linear topology.

Pachner defined bistellar moves or Pachner moves in [35], which can be performed on the triangulation of a manifold to get a new triangulation. For a dimension n manifold there are $n+1$ such local moves. Pachner moves involve removing a certain subcomplex and replacing it with another simplicial complex. Pachner moves were used by Tuarev and Viro [42] to define 3-manifold invariants. To define an invariant for PL 3-manifold it suffices to show that the quantity is unchanged by these Pachner moves. We will look at Pachner moves in detail in Chapter 2 .

Polytopes and polytopal complexes have been an important class of topological spaces in our research. We have extended some of the results for polytopes to star-convex polyhedra. We have extensively used the theory of *shellability* of polytopes and polyhedra.

Shelling of a triangulated polytope, introduced in the seminal 1971 paper of Bruggesser and Mani [5], is a way to inductively remove simplexes σ_i from the triangulation such that at each stage $\partial\sigma_i$ intersects what remains in a pure $(n-1)-$dimensional complex. It is easy to see that 2-dimensional polytopes are shellable. Higher dimensional PL polytopes are not in general shellable. The earliest example of nonshellable topological subdivisions of 3-polytopes were given by Newman [34] way back in 1926. Later Rudin [38] showed that even linear subdivisions of a 3-simplex may not be shellable. For spheres, Lickorish [25] has given several examples of unshellable triangulations. These examples illustrate that even in the simplest of cases, the property of shellability may not hold. Recently though Adiprasito and Benedetti [1] have shown that linear subdivisions of convex polytopes are shellable up to subdivision. We have used shellable triangulations of polyhedra for counting the number of Pachner moves required in changing the triangulation of these polyhedra.

It is known that triangulations of convex polytopes are related by stellar exchanges [32] [44], Theorem 2.2.3 in the thesis. If P is a polytope, then idea is to subdivide a triangulated simplicial cobordism $P \times [0,1]$ to a regular triangulation by stellar subdivision and realise stellar moves as projection of upper boundary onto $P \times \{0\}$ under a sequence which inductively removes simplexes from top to bottom, where upper boundary is the boundary visible from above. We call a triangulation *regular* if there is a piecewise linear function from the triangulation to $\mathbb{R}$ which is strictly convex across codimension one simplexes. We have given detailed definitions and examples in Section 2.2 of regular triangulations. We prove a result similar to Theorem 2.2.3 for *star-convex flat polyhedra*, extending this result to geometric manifolds is the main aim of Chapter 2. We show that geometric triangulations of isometric constant curvature Riemanannian manifolds are related by geometric Pachner moves up to derived subdivisions and for low dimension these triangulations are related directly by geometric Pachner moves. A *geometric triangulation* of a Riemannian manifold is a finite triangulation where the interior of each simplex is a totally geodesic disk. A *totally geodesic disk* is a disk where every geodesic under the induced Riemannian metric is also a geodesic of the manifold.

The problem of determining if two given manifolds are homeomorphic has been extensively studied. Using ideas from Perelman's proof of the geometrization of closed irreducible three dimensional manifolds, Scott and Short [40] have built on work by Manning, Jaco, Oertel and others to give an algorithm for the homeomorphism problem of such manifolds. More recently, Kuperberg [23] has given a self-contained proof using only the statement of geometrization to show that the homeomorphism problem for 3-manifolds has computational complexity that is bounded by a bounded tower of exponentials in the number of tetrahedra. Mijatovic in a series of papers gives such a bound for a large class of 3-manifolds [28] [29] [30] [31]. The bounds he obtain are also in terms of bounded towers of exponentials on the number of tetrahedra. In 1958, Markov [26] had shown that the homeomorphism problem is unsolvable for manifolds of dimension greater than 3. This curtailed the search for a general algorithm applicable to manifolds of all dimension. For closed hyperbolic manifolds, the fundamental group is a complete invariant but it is not easy to algorithmically check if two Kleinian groups are isomorphic.

In Chapter 3 we ask the next natural question, can we get a bound on the number of Pachner moves required to relate geometric trianguations? We answer this by showing that any two geometric triangulations of a closed hyperbolic, spherical or Euclidean manifold are related by a sequence of combinatorial Pachner moves and barycentric subdivisions of bounded length. This bound is in terms of the dimension of the manifold, the number of top dimensional simplexes and bound on the lengths of edges of the triangulation. If we also have an upper diameter bound and lower volume bound for the manifold, then this bound is a cubic polynomial in the number of simplexes and doubly exponential in the upper length bound of edges. This leads to an algorithm to check if two geometrically triangulated closed hyperbolic or low dimensional spherical manifolds are isometric or not. We give an algorithmic solution for the homeomorphism problem on the restricted class of geometrically triangulated constant curvature manifolds, by obtaining a bound on the number of barycentric subdivisions and Pachner moves needed to relate them.

A *Once-punctured torus bundle* is a fiber bundles over S^1 with fiber as once-punctured torus. Most of the Dehn fillings performed on once-punctured torus bundles give manifolds which are not Haken or reducible. Culler, Jaco, Rubinstein in [6] studied once-punctured torus bundles and listed all possible essential surfaces, Definition 4.1.3. Floyd and Hatcher [11] studied essential surfaces in hyperbolic once-punctured torus bundles. Essential surfaces.

In 1961, Wolfgang Haken introduced Haken manifolds. In 1962, Haken showed that Haken manifolds have a set of incompressible surfaces such that cutting along them gives

3-balls, this set of incompressible surfaces is known as a *Haken hierarchy* of the manifold. William Jaco and Ulrich Oertel in [16] gave an algorithm to determine if a given 3-manifold is Haken. Haken manifolds is a category of well studied manifolds, Thurston proved hyperbolization conjecture for atoroidal Haken manifolds, which led him to ask the same question for each component of prime, closed 3-manifolds when they are cut along essential tori. An important class of manifolds we get from cutting along these tori are Siefert fibered spaces. Mijatovic studied Siefert Fibered Spaces and gave a bound on the number of Pachner moves required to relate two of their triangulations. Our attempt is to prove a similar result for once punctured torus bundles using ideas inspired by Mijatovic's result [29].

Mijatovic's well known result [31], Theorem 4.1.1 for knot complements gives a bound on number of Pachner moves required to relate two of their triangulations. In Chapter 4, we attempt at generalizing Mijatovic's result for bounds on number of Pachner moves required to relate two triangulations of once-punctured torus bundles. Our idea is to use a Haken hierarchy for once-punctured torus bundles given by Jaco, Culler and Rubinstein [6] and cut along them to simplify these manifolds.

In the thesis, as the chapters have different backgrounds, we have added definitions and theorems chapter-wise to ensure the completeness of each chapter.

2

Geometric Pachner Moves

2.1 Introduction

We know that every constant curvature manifold has a geometric triangulation. In the converse direction, Cartan has shown that if for every point p in a Riemannian manifold M and every subspace V of T_pM there exists a totally geodesic submanifold S through p with $T_pS = V$, then M must have constant curvature; which seems to suggest that the only manifolds which have many geometric triangulations are the constant curvature ones. A *common subcomplex* of simplicial triangulations K_1 and K_2 of M is a simplicial complex structure L on a subspace of M such that $K_1|_{|L|} = K_2|_{|L|} = L$. The content of this chapter corresponds to our work from the paper [19]. Following is the main result of the chapter:

Theorem 2.1.1. *Let K_1 and K_2 be geometric simplicial triangulations of a compact constant curvature manifold M with a (possibly empty) common subcomplex L with $|L| \supset \partial M$. When M is spherical we assume that the diameter of the star of each simplex is less than π. Then for some $s \in \mathbb{N}$, the s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric Pachner moves which keep $\beta^s L$ fixed.*

We also have similar results for cusped hyperbolic manifolds. We call K the geometric triangulation of a cusped hyperbolic manifold M if for some subset V' of the set of vertices of K, $M = |K| \setminus |V'|$ and the interior of each simplex of K is a totally geodesic disk in M. Cusped finite volume hyperbolic manifolds have canonical ideal polyhedral decompositions [10]. Further decomposing this into ideal triangulations without introducing new vertices may result in degenerate flat tetrahedra. If however we allow genuine vertices, simply taking a barycentric subdivision of this polyhedral decomposition gives a geometric triangulation for any cusped manifold. For cusped manifolds we have the following weaker result:

Theorem 2.1.2. *Let K_1 and K_2 be geometric simplicial triangulations of a cusped hyperbolic manifold which have a common geometric subdivision. Then for some $s \in \mathbb{N}$, the s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are related by geometric Pachner moves.*

In low dimension we get a stronger result because of the fact that derived subdivisions can be realised by geometric Pachner moves, so we get the following immediate corollary:

Corollary 2.1.3. *Let K_1 and K_2 be geometric simplicial triangulations of a closed constant curvature 3-manifold M. When M is spherical we assume that the diameter of the star of each simplex is less than π. Then K_1 is related to K_2 by geometric Pachner moves.*

An *abstract simplicial complex* consists of a finite set K^0 (the vertices) and a family K of subsets of K^0 (the simplexes) such that if $B \subset A \in K$ then $B \in K$. A *simplicial isomorphism* between simplicial complexes is a bijection between their vertices which induces a bijection between their simplexes. A *realisation* of a simplicial complex K is a subspace $|K|$ of some $\mathbb{R}^N$, where K^0 is represented by a finite subset of $\mathbb{R}^N$ and vertices of each simplex are in general position and represented by the linear simplex which is their convex hull. Every simplicial complex has a realisation in $\mathbb{R}^N$ where N is the size of K_0, by representing K_0 as a basis of $\mathbb{R}^N$. Any two realisations of a simplicial complex are simplicially isomorphic. For A a simplex of K, we denote by ∂A the *boundary complex* of A. When the context is clear, we shall use the same symbol A to denote the simplex and the simplicial complex $A \cup \partial A$. We call K a *simplicial triangulation* of a manifold M if there exists a homeomorphism from a realisation $|K|$ of K to M. The simplexes of this triangulation are the images of simplexes of $|K|$ under this homeomorphism.

Definition 2.1.4. For A and B simplexes of a simplicial complex K, we denote their

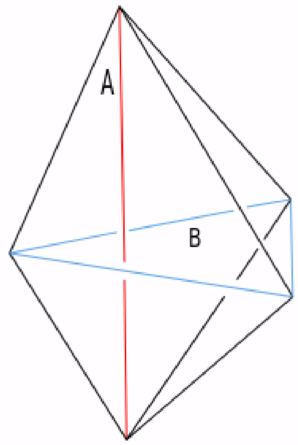

Figure 2.1: Join $A \star \partial B$.

join $A \star B$ as the simplex $A \cup B$. Let $K^0 = \{a_1, \dots, a_n, b_1, \dots, b_s\}$ be the set of vertices of a simplicial complex K, $A = \{a_1, \dots, a_l\}$ and $B = \{b_1, \dots, b_t\}$, then the join $A \star B$ is defined as the simplex formed by set $A \cup B = \{a_1, \dots, a_l, b_1, \dots, b_t\}$. As the join of totally geodesic disks in a constant curvature manifold gives a totally geodesic disk, operations involving joins are well-defined in the class of geometric triangulations of a constant curvature manifold.

The link of a simplex A in a simplicial complex K is the simplicial complex defined by $lk(A, K) = \{B \in K : A \star B \in K\}$. The (closed) star of A in K is the simplicial complex defined by $st(A, K) = A \star lk(A, K)$.

Definition 2.1.5. Let A be a non-empty simplex in a combinatorial n-manifold M such that $lk(A, K) = \partial B \star L$ for some non-empty simplex B such that $B \notin K$ but $B \subset M$ and some complex $L \in K$. Then K is related to K' by a *stellar exchange* $\kappa(A, B)$, if K' is obtained by replacing $A \star \partial B \star L$ with $\partial A \star B \star L$.

Definition 2.1.6. Let A be an r-simplex in a simplicial complex K of dimension n then a stellar subdivision on A gives the geometric triangulation $(A, a)K$ by replacing $st(A, K)$ with $a \star \partial A \star lk(A, K)$ for $a \in int(A)$. The inverse of this operation $(A, a)^{-1}K$ is called a stellar weld and they both are together called stellar moves.

A *Pachner move* is defined as the stellar exchange $K(A, B)$ when $lk(A, K) = \partial B$

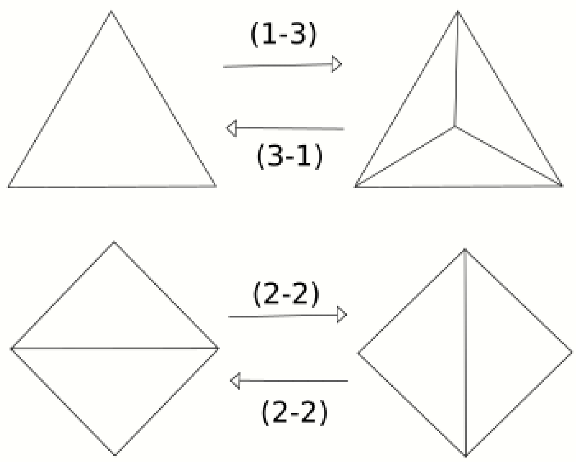

Figure 2.2: 2–Dimensional Pachner Moves.

for some $n-r$ dimensional geometric simplex $B \notin K$, i.e. when $L = \emptyset$, it consists of changing K by replacing $A \star \partial B$ with $\partial A \star B$. Note that the Pachner move $\kappa(A, B)$ is the composition of a stellar subdivision and a stellar weld, namely $(B, a)^{-1}(A, a)$. We list the Pachner moves in dimension 2 and 3 in Figure 2.2 and Figure 2.3, where a $(k-l)$ move represents replacing a complex with k triangles resp. tetrahedra with a complex with l triangles resp. tetrahedra.

The derived subdivision βK of K is obtained from K by performing a stellar subdivision on all r-simplexes, and ranging r inductively from n down to 1, Figure 2.4 shows the derived subdivision of a 2-dimensional triangulation.

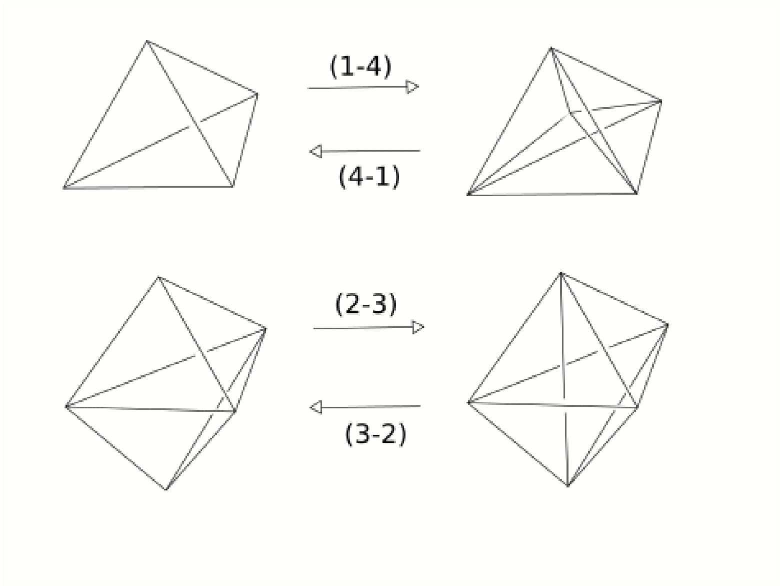

Figure 2.3: 3–Dimensional Pachner Moves

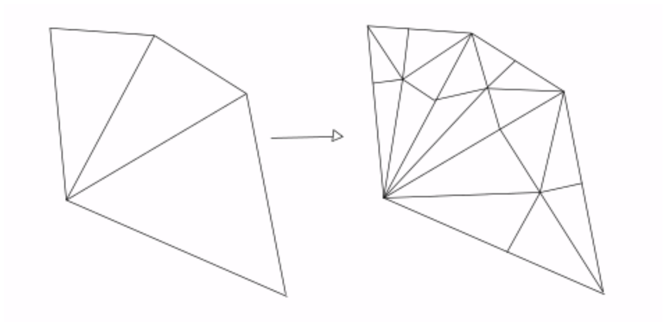

Figure 2.4: Derived subdivision of a simplicial complex.

All stellar and Pachner moves we consider are geometric in nature. Not every combinatorial Pachner move in a geometric manifold can be expressed by geometric Pachner moves (see Figure 2.5). For details of geometric Pachner moves see Santos [39], where he has also shown that in dimension at least 5, there exist geometric triangulations of a polyhedron that can not be related by geometric Pachner moves which do not introduce new vertices. Among cusped hyperbolic manifolds, it is remarked in [7] that the Figure Eight knot complement has geometric ideal triangulations which can not be related by geometric Pachner moves which do not introduce genuine vertices.

2.1.1 Outline of the proof

It is known that linear triangulations of a convex polytope $P \subset \mathbb{R}^N$ are related by stellar exchanges [32] [44]. The idea of the proof is to take a geometrically triangulated simplicial cobordism $P \times I$, subdivide it to a regular triangulation with stellar subdivisions and then realise the stellar moves as projections of the upper boundary onto P under a sequence which inductively removes simplexes from the top to the bottom. As the supports in $\mathbb{R}^N$ of two triangulations of a manifold may be different so when the manifold is not a polytope we can not take a linear cobordism between them. We observe that simplicially triangulated constant curvature manifolds are made up of star-convex polyhedra. We consider the cones of such polyhedra to get a cobordism connecting the cone over the boundary of the polyhedron and the polyhedron. We show that with enough derived subdivisions, the cone of a triangulations of the polyhedron can be made into regular. So we get a regular cobordism connecting the subdivided cone over the boundary of polyhedron and the subdivided polyhedron.

Given two geometric triangulations K_1 and K_2 of a Riemannian manifold M. We get a common subdivision K_C of K_1 and K_2 by intersecting simplexes of K_1 and K_2. Then we show that we can go from K_i to K_C using geometric Pachner moves. A subtle point here is that even if we obtain a common geometric refinement of two geometric triangulations, the refinement may not be a simplicial subdivision of the corresponding simplicial complexes. To see a topological subdivision which is not a simplicial subdivision, observe that there exists a simplicial triangulation K of a 3-simplex Δ which contains in its 1-skeleton a trefoil with just 3 edges [25]. If K were a simplicial subdivision of Δ there

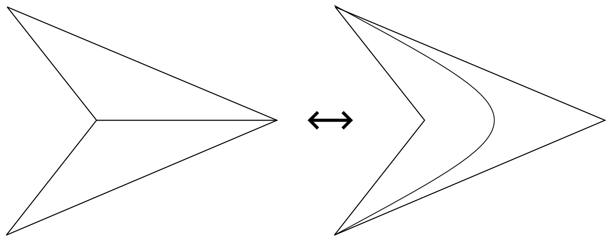

Figure 2.5: Pachner move which is not geometric

would exist a linear embedding of Δ in some $\mathbb{R}^N$ which takes simplexes of K to linear simplexes in $\mathbb{R}^N$. As the stick number of a trefoil is 6, there can exist no such embedding. While there may not exist such a global embedding of a geometric triangulation K as a simplicial complex in $\mathbb{R}^N$ which takes geometric subdivisions to linear subdivisions, for constant curvature manifolds there does exist such a local embedding on stars of simplexes of K. So we can take the intersection of K_1 and K_2 to get a common subdivision. Then we prove that we can relate some barycentric subdivision of K_i and K_C.

2.2 Star-Convex Flat Polyhedra

In this section we will prove the main Theorem 2.1.1 for star convex flat polyhedra. We can observe that star convex flat polyhedra are building blocks of any simplicially triangulated constant curvature manifold. The important property we use here is *regularity* of a triangulation. Regularity gives us a way to change the triangulation of a polyhedron to the cone over its boundary.

A *Polytope* is a convex hull of finitely many points in some $\mathbb{R}^n$, and *polyhedron* is a polytopal complex that is homeomorphic to a ball where a polytopal complex is a union of polytopes such that the intersection of two polytopes ia a face of each of them.

Definition 2.2.1. We call a polyhedron P in $\mathbb{R}^n$ strictly star-convex with respect to a point x in its interior if for any $y \in P$, the interior of the segment $[x, y]$ lies in the interior of P. We call the polyhedron $P \subset \mathbb{R}^n$ flat if it is n-dimensional.

We call a triangulation K of P regular if there is a function $h : |K| \to \mathbb{R}$ that is piecewise linear with respect to K and strictly convex across codimension one simplexes of K. Figure 2.6 is an example of non-regular triangulation of a triangle.

Example 2.2.2. Consider a regular tetrahedron in $\mathbb{R}^3$, then a simplicial disk formed by two faces of this tetrahedron is not a convex flat polyhedron.

In their proof of the weak Oda conjecture, Morelli and Wlodrczyk proved the following:

Theorem 2.2.3. *[32] [44] Any two triangulations of a convex polyhedron are related by a sequence of stellar moves.*

Our aim in this section is to show that their techniques also give a boundary relative version for triangulations of strictly star-convex flat polyhedra, with the stronger notion of bistellar equivalence in place of stellar equivalence. The main theorem of this section is the following:

Theorem 2.2.4. *Let $P \subset \mathbb{R}^n$ be a strictly star-convex flat polyhedron. Let K_1 and K_2 be triangulations of P that agree on the boundary. Then for some $s \in \mathbb{N}$, their s-th derived subdivisions $\beta^s K_1$ and $\beta^s K_2$ are bistellar equivalent.*

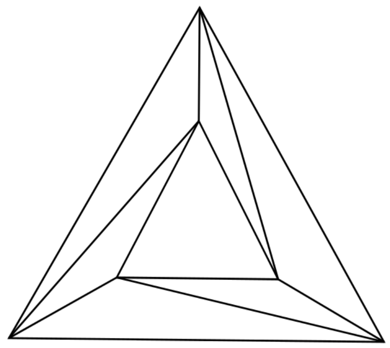

Figure 2.6: Non regular triangulation

We use the following simple observation in the proof:

Lemma 2.2.5. *[Lemma 4, Ch 1 of [45]] Let K and L denote two simplicial complexes with $|K| \subset |L|$. Then there exists $r \in \mathbb{N}$ and a subdivision K' of K such that K' is a subcomplex of $\beta^r L$.*

Proof. Proof is by induction on number of simplexes of K. When $K = \emptyset$, trivially we have the result. Let A be a top dimensional simplex in K, then subdivision of $K \setminus A$ is subcomplex of $\beta^{r-1}L$. Let $\beta^r L$ be derived subdivision of $\beta^{r-1}L$ formed by starring each simplex $B \in \beta^{r-1}L$ using some point in the interior of B. This way, $\beta^r L$ gives a subdivision of A formed by subdividing each $B \in \beta^{r-1}L$ which intersects with $int(A)$. Subivisions of A and $K \setminus A$ gives us a subdivision of K which is a subcomplex of $\beta^r L$. Because of the induction is on number of simplexes of K, we also get that r is bounded by number of simplexes in K □

Theorem 2.2.6. *[Theorem 1 of [2]] For every Pl sphere Δ, there exist a $k \geq 0$ such that $sd^k\Delta$ is polytopal, i.e., it is combinatorially equivalent to the boundary complex of some convex polytope.*

Claim 3 in the proof of Theorem 2.2.6 proves that if D is PL homeomorphic to boundary of some $(d+1)$-simplex and $\beta^l D$ is derived subdivision of some regular subdivision D' of D, then $\beta^l D$ is also regular.

Lemma 2.2.7. *Let K denote a triangulated flat polyhedron. Then for some $s \in \mathbb{N}$, its*

s-th derived subdivision $\beta^s K$ is regular.

Proof. Let Δ be an n-simplex with $|\Delta| \supset |K|$. By Lemma 2.2.5, there exists an $r \in \mathbb{N}$ and subdivision K' of K which is a subcomplex of $\beta^r\Delta$. As Δ is trivially a regular triangulation, so its stellar subdivision $\beta^r\Delta$ is also regular. Restricting its regular function to the subcomplex K' we get K' to be regular, as codimension one simplexes of K' are also codimension one simplexes of $\beta^r\Delta$. As $|K| = |K'|$ so applying Lemma 2.2.5 a second time, we get $s \in \mathbb{N}$ such that $\beta^s K$ is a subdivision of K'. Finally as $\beta^s K$ is the subdivision of a regular subdivision K' of K so by Claim 3 in the proof of Theorem 2.2.6, $\beta^s K$ is a regular triangulation. □

Proof of 2.2.4. The techniques in this proof are essentially those of Morelli and Wlodarczyk as detailed in Section 2 of [20].

Choose $a \in \mathbb{R}^{n+1}$ outside K_1 such that the orthogonal projection map $pr : \mathbb{R}^{n+1} \to \mathbb{R}^n$ takes the support of $C(K_1) = a \star K_1 \subset \mathbb{R}^{n+1}$ onto P and takes a to the interior of an n-simplex of K_1 i.e., we can choose a such that $a \in (a', t) \subset \mathbb{R}^{n+1}$ where a' is in the complement of $n-1-$ skeleton of K_1 in K_1. By Lemma 2.2.7, there exists $s \in \mathbb{N}$ so that $K = \beta^s C(K_1)$ is a regular triangulation with boundary $\beta^s K_1 \cup \beta^s C(\partial K_1)$. Choose new vertices of the derived subdivision K such that for any simplex $A \in K$ of dimension less than $n+1$, $pr(A)$ is a simplex of the same dimension as A. This can be done by choosing vertices in the interior of each simplex.

Let $h : |K| \to \mathbb{R}$ be a regular function for K. If a simplex σ' has some point above a simplex σ (in the direction of x^{n+1}) then $\frac{\partial h}{\partial x_{n+1}}$ on σ' is greater than $\frac{\partial h}{\partial x_{n+1}}$ on σ. So inductively removing simplexes in non-increasing order of the vertical derivative of h we ensure that the projection of the upper boundary onto P is always one-to-one. That is, we get a sequence of triangulations $\Sigma_0 = K$, Σ_1, ... , $\Sigma_N = K_1$ such that $\Sigma_{i+1} = \Sigma_i \setminus \sigma_i$ and the orthogonal projection $pr : \partial^+\Sigma_i \to P$ from the upper boundary of Σ_i onto P is one-to-one for every i. Removing an $n+1$-simplex σ_i from K corresponds to a bistellar move on $\partial^+\Sigma_i$. As the projection map is linear so it also corresponds to a bistellar move taking $pr(\partial^+\Sigma_i)$ to $pr(\partial^+\Sigma_{i+1})$. Therefore $pr(\partial^+\Sigma_0) = \beta^s C(\partial K_1)$ is bistellar equivalent to $pr(\partial^+\Sigma_N) = \beta^s K_1$. Consequently, $\beta^s K_1$ is bistellar equivalent to $\beta^s K_2$ via $\beta^s C(\partial K_1) = \beta^s C(\partial K_2)$. □

2.3 Geometric Manifolds

Definition 2.3.1. Let K be a geometric triangulation of a Riemannian manifold M and let L be a subcomplex of K. We call K locally geodesically-flat relative to L if for each simplex A of $K \setminus L$, $st(A, K) \setminus (\partial A \star lk(st(A, K)))$ is simplicially isomorphic to the interior of a star-convex flat polyhedron in $\mathbb{R}^n$ by a map which takes geodesics to straight lines.

Example 2.3.2. *Let K be some triangulation of an unit sphere S^2 in $\mathbb{R}^3$ such that each set of three vertices represents a unique 2-simplex, let $S = (0, 0, -1)$ and $L = K \setminus st(S, K)$ and $\theta : K \setminus L \to \mathbb{R}^2$ be radial projection from the center of the ball bounded by the sphere. It is easy to see that θ is a simplicial isomorphism between $K \setminus L$ to $\theta(K \setminus L)$ and it takes $st(A \setminus K) \setminus (\partial A \star lk(st(A, K)))$ to the interior of a start convex flat polyhedron in $\mathbb{R}^2$ with geodesics going to straight lines.*

Definition 2.3.3. Let L be a subcomplex of K containing ∂K and let αK be a subdivision of K which agrees with K on L and A be a simplex in K. Let $\beta_r^\alpha K$ be the subdivision of K such that, if A is a simplex in L or $dim(A) \leq r$, then $\beta_r^\alpha A = \alpha A$. If A is not in L and $dim(A) > r$ then $\beta_r^\alpha A = a \star \beta_r^\alpha \partial \alpha A$, i.e. it is subdivided as the cone on the already defined subdivision of its boundary. Observe that $\beta_n^\alpha K$ is αK while $\beta_0^\alpha K = \beta_L K$ is a barycentric subdivision of K relative to L.

Lemma 2.3.4. *Let K be a locally geodesically-flat simplicial complex relative to a subcomplex L which contains ∂K. Let αK be a geometric subdivision of K which agrees with K on L. Then there exists $s \in \mathbb{N}$ for which $\beta^s \alpha K$ is related to $\beta^s K$ by bistellar moves which keep $\beta^s L$ fixed.*

Proof. For A a positive dimensional r-simplex in $K \setminus L$, $st(A, \beta_r^\alpha K)$ is a strictly star-convex subset of $st(A, K)$. As K is locally geodesically-flat relative to L, there exists a geodesic embedding taking $st(A, \beta_r^\alpha K)$ to a strictly star-convex flat polyhedron of $\mathbb{R}^n$. By Theorem 2.2.4, $\beta^s st(A, \beta_r^\alpha K)$ is bistellar equivalent to $\beta^s C(\partial st(A, \beta_r^\alpha K))$. As A is not in L so no interior simplex of $st(A, \beta_r^\alpha K)$ is in L and consequently these bistellar moves keep $\beta^s L$ fixed. Taking all simplexes A in $K \setminus L$ of dimension $r = n$, we get a sequence of bistellar moves taking $\beta^s \beta_r^\alpha K$ to $\beta^s \beta_{r-1}^\alpha K$. Ranging r from n down to 1, we inductively obtain a sequence of bistellar moves taking $\beta^s \alpha K = \beta^s \beta_n^\alpha K$ to $\beta^s \beta_L K = \beta^s \beta_0^\alpha K$, which

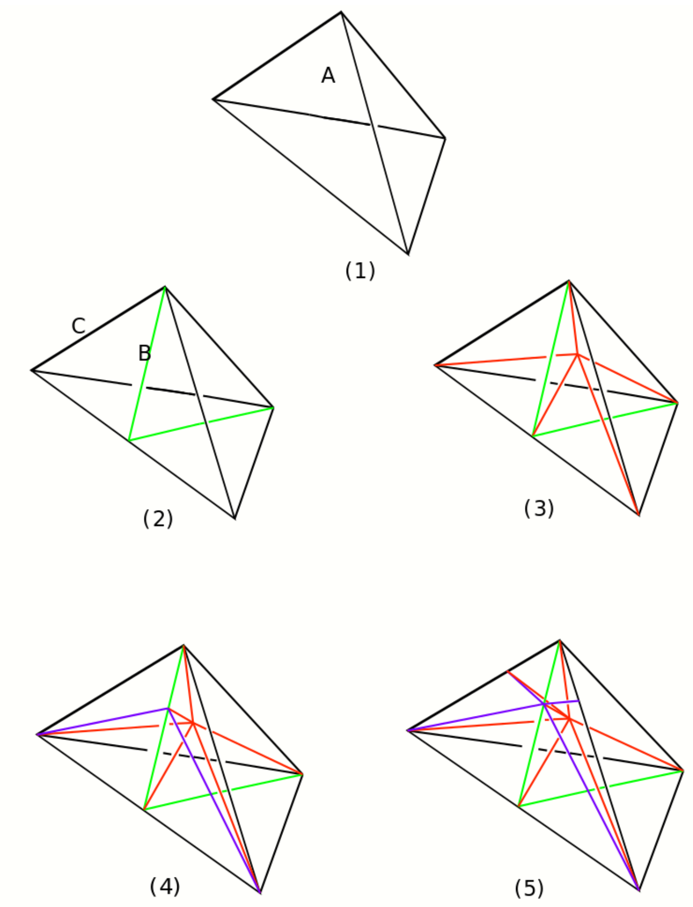

Figure 2.7: Subdivision to Barycentric Subdivision

keeps $\beta^s L$ fixed. And finally, arguing as above with the trivial subdivision $\alpha K = K$, we get $\beta^s \beta_L K$ from $\beta^s K$ by bistellar moves which keep $\beta^s L$ fixed.

This process is illustrated in Figure 2.7. We can go from the subdivision of a triangulation to its barycentric subdivision, where (1) represents a 3-simplex $A = K$, (2) represents a subdivision $\alpha K = \beta_3^\alpha K$ of K, (3) represents $\beta_2^\alpha K = a \star \beta_2^\alpha \partial \alpha A = a \star \partial \alpha A$, (4) represents an intermediate step towards $\beta_1^\alpha K$ where $st(B, \beta_2^\alpha K)$ is changed to the cone over its boundary, (5) represents an intermediate step towards $\beta_0^\alpha K$ where $st(C, \beta_1^\alpha K)$ is changed to the cone over its boundary. $\square$

The following simple observation allows us to treat the star of a simplex in a geometric triangulation as the linear triangulation of a star-convex polytope in $\mathbb{R}^n$ and bistellar moves in the manifold as bistellar moves of the polytope.

Lemma 2.3.5. *Let K be a geometric simplicial triangulation of a spherical, hyperbolic or Euclidean n-manifold M and let L be a subcomplex of K containing ∂K. When M is spherical we require the star of each positive dimensional simplex of $K \setminus L$ to have diameter less than π. When M is cusped we include the ideal vertices in L. Then K is locally geodesically-flat relative to L.*

Proof. Let K be a geometric triangulation of M and let B be the interior of the star of a simplex in $K \setminus L$. As K is simplicial, B is an open n-ball.

When M is hyperbolic, let $\phi : B \to \mathbb{H}^n$ be the lift of B to the hyperbolic space in the Klein model. As geodesics in the Klein model are Euclidean straight lines (as sets) so ϕ is the required embedding.

When M is spherical, let D be the southern hemisphere of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, let T be the hyperplane $x_{n+1} = -1$ and let $p : D \to T$ be the radial projection map (gnomonic projection) which takes spherical geodesics to Euclidean straight lines. As B is small enough, lift B to D and compose with the projection p to obtain the required embedding ϕ from B to $T \simeq \mathbb{E}^n$.

When B is Euclidean let ϕ be the lift of B to $\mathbb{R}^n$, which is an isometry. □

Theorem 2.3.6. *[Theorem 4 of [2]]*

1. *If $d \leq 3$, then there is no k that would depend only on d such that all PL d-spheres become polytopal after k derived subdivisions.*

2. *For $d = 3$, the number $k = k(\Delta)$ of derived subdivisions needed to make a PL sphere Δ polytopal can be bounded from above by*

$$k(\Delta) \leq a * 2^{b*f_3(\Delta)*2^{c*f_3^2(\Delta)*2^{d*f_3^2(\Delta)}}} + c * f_3^2(\Delta) * 2^{d*f_3^2(\Delta)}$$

   , where a, b, c, $d \geq 0$ are constants independent of Δ.

3. *If $d \geq 5$, then the number of derived subdivisions that makes a PL d-sphere Δ polytopal is not (Turing machine) computable from Δ.*

where $f_i()$ denote the number of i-dimensional faces of simplicial complex.

Theorem 2.3.6 shows that for simplicial complexes of dimension at least 5 the number of derived subdivisions required to make the link of a vertex combinatorially isomorphic to a convex polyhedron is not (Turing machine) computable. So in particular, the stars of simplexes of a geometric triangulation may not even be combinatorially isomorphic to convex polyhedra, which is why we need to work with star-convex polyhedra instead.

Given a Riemannian manifold M, a *geometric polytopal complex* C of M is a finite collection of geometric convex polytopes in M whose union is all of M and such that for every $P \in C$, C contains all faces of P and the intersection of two polytopes is a face of each of them.

Proof of 2.1.1 and 2.1.2. By Lemma 2.3.5, K_1 and K_2 are locally geodesically flat simplicial complexes. Let C be the geometric polytopal complex obtained by intersecting the simplexes of K_1 and K_2. Then $K = \beta_L C$, the barycentric subdivision of C relative to L is a common geometric subdivision of K_1 and K_2. When M is a cusped manifold we assume that we are given such a common geometric subdivision K as C might have infinitely many polytopes. By Lemma 2.3.4 then, there exists $s \in \mathbb{N}$ so that $\beta^s K_1$ and $\beta^s K_2$ are bistellar equivalent via $\beta^s K$ by bistellar moves which leave $\beta^s L$ fixed. In the cusped situation we can take L as the set of ideal vertices. $\square$

3

Bound On Pachner Moves

3.1 Introduction

We give an algorithmic solution for the homeomorphism problem on the restricted class of geometrically triangulated constant curvature manifolds, by obtaining a bound on the number of barycentric subdivisions and Pachner moves needed to relate them. The content of this chapter corresponds to our work from the paper [18]. Following is the main result in the chapter:

Theorem 3.1.1. *Let M be closed spherical, Euclidean or hyperbolic n-manifold with geometric triangulations K_1 and K_2. Let K_1 and K_2 have p and q many n-simplexes respectively with lengths of edges bounded above by Λ and let $inj(M)$ be the injectivity radius of M. When M is spherical, we require $\Lambda \leq \pi/2$. Then the 2^{n+1}-th barycentric subdivisions of K_1 and K_2 are related by less than $2^n(n+1)!^{4+3m'}pq(p+q)$ Pachner moves which do not remove common vertices. When $n \leq 4$, then K_1 and K_2 are directly related by $2^n(n+1)!^{4+3m}pq(p+q)$ Pachner moves which do not remove common vertices. Here $m' = max(2^{n+1}, m)$ and m is an integer greater than $\mu \ln(\Lambda/inj(M))$ where μ is as follows:*

1. *When M is Euclidean, $\mu = n + 1$*
2. *When M is Spherical, $\mu = 2n + 1$*

3. *When M is Hyperbolic,* $\mu = ncosh^{n-1}(\Lambda) + 1$

From Lemma 3.3.12 and Theorem 3.3.13 it follows that $inj(M) > \pi vol(M)/\delta vol(\mathbb{S}^n)$ which gives the following corollary in terms of the volume and diameter of the manifold.

Corollary 3.1.2. *With notations as in Theorem 3.1.1, we can take m to be an integer greater than $\mu \ln(\Lambda\delta vol(\mathbb{S}^n)/(\pi vol(M)))$, with δ as follows:*

1. *When M is Euclidean,* $\delta = diam(M)$
2. *When M is Spherical,* $\delta = sin^{n-1}(diam(M))$
3. *When M is Hyperbolic,* $\delta = sinh^{n-1}(diam(M))$

Remark 3.1.3. To express m entirely in terms of the triangulation, we therefore need an upper diameter bound and a lower volume bound as a function of n, p and bounds on lengths of edges. In the lower bound for m we can replace $diam(M)$ by $p\Lambda$, because a shortest path between two points of M has length less than a piecewise geodesic path that intersects each simplex at most once and by Lemma 3.4.4, the diameter of a simplex is bounded by the maximum length of its edges. We have a lower bound for $vol(M)$ by the volume of Δ_λ, a regular tetrahedron of length λ, where λ is a lower bound on the length of edges of the triangulation.

Remark 3.1.4. For M a closed orientable hyperbolic 3-manifold, volume is bounded below by $w = 0.9427$ [12], so we can take $m > (3cosh^2(\Lambda) + 1)ln(2\pi p\Lambda^2/w)$. For even dimensional closed hyperbolic manifolds, Hopf's generalised Gauss Bonnet formula gives us $vol(M) = (-1)^{n/2} vol(\mathbb{S}^n)\chi(M)/2$ where $\chi(M)$ is the Euler characteristic of M, so we can take $m > max(2^{n+1}, (ncosh^{n-1}(\Lambda) + 1)\ln(2p\Lambda^2/\pi))$. In general for closed hyperbolic n-manifolds, volume is universally bounded below by $vol(\mathbb{S}^{n-1})/(n(n+3)^n\pi^{n(n-1)})$ [21]. So for $n > 2$ as $2vol(\mathbb{S}^n)/(\pi vol(\mathbb{S}^{n-1})) \leq 1$, we can take $m > max(2^{n+1}, (ncosh^{n-1}(\Lambda) + 1)ln(2p\Lambda^2 n(n+3)^n\pi^{n(n-1)}))$.

Theorem 3.1.5. *Let M be closed spherical, Euclidean or hyperbolic n-manifold with geometric triangulations K_1 and K_2 having p and q many n-simplexes respectively and lengths of edges bounded below by λ and above by Λ. When M is spherical, we require $\Lambda \leq \pi/2$. Then 2^{n+1}-th barycentric subdivisions $\beta^{2^{n+1}}K_1$ and $\beta^{2^{n+1}}K_2$ are related by at most $f(p, q, n, \lambda, \Lambda)$ Pachner moves (which do not remove common vertices). When*

$n \leq 4$, then K_1 and K_2 are directly related by $f(p,q,n,\lambda,\Lambda)$ Pachner moves (which do not remove common vertices), where $f(p,q,n,\lambda,\Lambda) = 2^n(n+1)!^{4+3m}pq(p+q)$ with m as below and Δ_λ a regular n-simplex of edge length λ:

1. *When M is Hyperbolic, $m > max(2^{n+1}, (ncosh^{n-1}(\Lambda)+1)ln(2p\Lambda^2 n(n+3)^n \pi^{n(n-1)}))$*
2. *When M is Spherical, $m > max(2^{n+1}, (2n+1)ln(11\Lambda/vol(\Delta_\lambda)))$*
3. *When M is Euclidean, $m > max(2^{n+1}, (n+1)ln(11p\Lambda^2/vol(\Delta_\lambda)))$*

Corollary 3.1.6. *For closed hyperbolic 3-manifolds, we can take*

$$f(p,q,n,\Lambda) = 3 \cdot 10^6 \cdot 24^{3(e(2\Lambda)+4)ln(7p\Lambda^2)}pq(p+q)$$

We must point out that as Pachner moves are combinatorial in nature, the intermediate triangulations we obtain may not be geometric. But as they are just local combinatorial operations, such a bound gives a naive algorithm to check if given hyperbolic or low dimensional spherical manifolds are isometric.

Corollary 3.1.7. *Let (M, K_M) and (N, K_N) be geometrically triangulated closed hyperbolic manifolds of dimension at least 3 or closed spherical manifolds of dimension at most 6 and edge length at most $\pi/2$. Then M is isometric to N if and only if the 2^{n+1}-th barycentric subdivisions of K_M and K_N are related by $2^n(n+1)!^{4+3m'}pq(p+q)$ Pachner moves followed by a simplicial isomorphism, with m', p and q as defined in Theorem 3.1.1*

We have shown that geometric triangulations can be related by geometric Pachner moves (in [19] and Chapter 2) using simplicial cobordisms. We follow a different approach in this chapter, using shellings and relating via combinatorial Pachner moves instead as it leads to a tighter bound.

3.1.1 Outline of Proof

Given geometric triangulations K_1 and K_2 of M, we first take repeated barycentric subdivisions till each simplex lies in a strongly convex ball. This is where we crucially use the upper length bound on the edges to handle tall thin 'needle-shaped' tetrahedra.

The factor by which these subdivisions scale simplexes is worked out in Section 3.4. Next we consider the geometric polyhedral complex $K_1 \cap K_2$ obtained by intersecting the simplexes of K_1 and K_2, which we further subdivide to a common geometric subdivision K'. As simplexes of K_1 and K_2 are strongly convex they intersect at most once, which gives a bound on the number of simplexes in K'.

While every simplex of K' lies in some simplex of K_i, to see that it is in fact a simplicial subdivision (and hence K_i are PL-equivalent) we would need an embedding of K_i in some $\mathbb{R}^n$ which is linear on both simplexes of K_i and of K'. For constant curvature manifolds however, there do exist local embeddings in $\mathbb{R}^n$ which are linear on both K_i and K'. This allows us to treat geometric subdivisions of geometric simplexes in the manifold as simplicial subdivisions of linear simplexes in $\mathbb{R}^n$.

Theorem 3.1.8. *[Theorem A of [1]] If C is any subdivision of a convex polytope, the second derived subdivision of C is shellable. If $dimC = 3$, already the first derived subdivision of C is shellable.*

We first take repeated barycentric subdivisions to make the link of every simplex of K shellable. Given a geometric subdivision αK of K, we next define partial barycentric subdivisions $\beta_r^\alpha K$ by putting the given subdivision αA on simplexes A of dimension at most r and the barycentric subdivision βA on the rest. By Theorem 3.1.8, αA is shellable up to subdivisions and as link of A in $\beta_r^\alpha K$ is also shellable so we can extend shellability to 'star neighbourhoods' of αA in $\beta_r^\alpha K$. When a polytope is shellable it is easy to see that it is starrable, i.e., there exists a sequence of Pachner moves which takes the subdivision of a star neighbourhood to the cone over its boundary. Using this, we get a sequence of Pachner moves which takes a star neighbourhood of αA to a cone on its boundary and varying A over all r simplexes of K, a sequence of moves from $\beta_r^\alpha K$ to $\beta_{r-1}^\alpha K$. This gives a sequence of moves from $\beta_n^\alpha K = \alpha K$ to $\beta_0^\alpha K = \beta K$. Taking αK as the common geometric subdivision K' of K_1 and K_2, we get a sequence of moves from βK_1 to βK_2 of controlled length as required.

3.2 Links Of Simplexes

In this section we prove results about links of partial barycentric subdivisions of simplicial complexes. The main result of this section is that after taking sufficiently many barycentric subdivisions, the link of every simplex of a geometric triangulation is shellable. Books by Rourke and Sanderson [37] and Ziegler [46] are good sources of introduction to the theory of piecewise linear topology.

Definition 3.2.1. [24] Suppose that A and B are simplexes of a simplicial triangulation of an n-manifold M with boundary ∂M, that $A \star B$ is an n-simplex of M, that $A \cap \partial M = \partial A$ and that $B \star \partial A \subset \partial M$. Then the manifold M' obtained from M by elementary shelling along B is the closure of $M \setminus (A \star B)$. Closure here means adding the simplexes of $A \star \partial B$. The relation between M and M' will be denoted by $M \xrightarrow{(shB)} M'$. An n-ball is said to be shellable if it can be reduced to an n-simplex by a sequence of elementary shellings. An n-sphere is shellable if removing some n-simplex from it gives a shellable n-ball.

We reproduce the proof of the statement that shellable balls are starrable from [24] for completeness and to record the number of Pachner moves required in the starring process.

In our proof we need the links of positive dimensional simplexes to be shellable spheres. We show that after sufficiently many barycentric subdivisions the links of simplexes do become shellable. As triangulated spheres of dimension at most 2 are always shellable, so for manifold dimension $n \leq 4$ the links of positive dimensional simplexes are shellable and we do not need to take these initial subdivisions.

Lemma 3.2.2. *[Lemma 5.7 of [24]] Let K be a shellable triangulation of an n-ball with r many n-simplexes, then $v \star \partial K$ is related to K by a sequence of r Pachner moves.*

Proof. We prove this by induction on the number r of n-simplexes of K. If $r = 1$, then K is a n-simplex and a single Pachner move changes K to $v \star \partial K$.

Suppose that the first elementary shelling of K is $K \xrightarrow{(shB)} K_1$, where $A \star B$ is a n-simplex of X, $A \cap \partial K = \partial A$ and $B \star \partial A \in \partial K$ (see Figure 3.1). By the induction on r, K_1 is simplicially isomorphic to $v \star \partial K_1$ after at most $r - 1$ Pachner moves. Observe that $v \star \partial K_1 \cup A \star B$ is changed to $v \star \partial K$ by the single Pachner move $\kappa(A, v \star B)$. $\square$

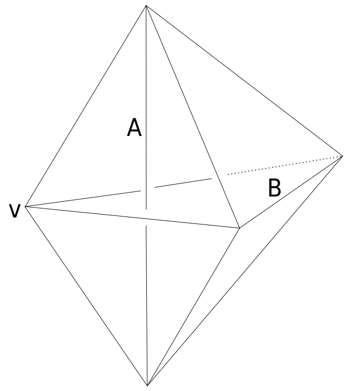

Figure 3.1: $A \cap \partial K = \partial A$ and $B \star \partial A \in \partial K$

Definition 3.2.3. Let αK be a geometric subdivision of K. Let $\beta_r^\alpha K$ be the geometric subdivision of K such that, if A is a simplex in K and $dim(A) \leq r$, then $\beta_r^\alpha A = \alpha A$ and if $dim(A) > r$ then $\beta_r^\alpha A = a \star \beta_r^\alpha \partial \alpha A$, i.e. it is subdivided as the geometric cone on the already defined subdivision of its boundary. In other words, fix a point $a \in A$ and for each simplex $B \in \beta_r \partial \alpha A$, introduce the geometric simplex $a \star B$ by taking the union of geodesics in A which start at a and end at some point in B. As M is of constant curvature the geometric join of a point with a totally geodesic disk is again a totally geodesic disk, so $\beta_r^\alpha K$ is a geometric triangulation of M. Observe that $\beta_n^\alpha K$ is αK while $\beta_0^\alpha K = \beta K$ is the geometric barycentric subdivision of K. When $\alpha K = K$, we denote $\beta_r^\alpha K$ by $\beta_r K$ and call it a partial barycentric subdivision.

The following lemma relates the links of simplexes in a partial barycentric subdivision with the barycentric subdivision of the links in the original simplicial complex, as can be seen in Figure 3.2.

Lemma 3.2.4. *Let A be a r-simplex in a simplicial complex K. Then $lk(A, \beta_r K)$ is simplicially isomorphic to $\beta lk(A, K)$.*

Proof. Observe that as A is r-dimensional, $\beta_r A = A$ and we can take A to be a simplex of both $\beta_r K$ and K.

Let B be a simplex in $lk(A, K)$. The barycentric subdivision βB of B is given by $b \star \beta \partial B$. So the vertices of $\beta lk(A, K)$ are exactly such points b, one for each simplex B in

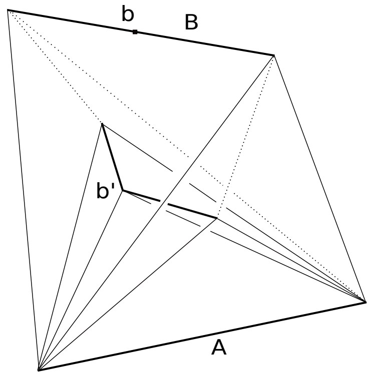

Figure 3.2: When $K = A * B$, βB is isomorphic to $lk(A, \beta_1 K)$

$lk(A, K)$. As $A \star B$ has dimension greater than r, so $\beta_r(A \star B) = b' \star \beta_r(\partial(A \star B))$. And as A is unchanged by β_r, so $A \in \beta_r(\partial(A \star B))$ and consequently $b' \star A \in \beta_r(A \star B) \subset \beta_r K$. So given $B \in lk(A, K)$, we obtain a vertex b' of $lk(A, \beta_r K)$. Conversely, given a vertex b' of $lk(A, \beta_r K)$, $b' \star A$ is a simplex in $\beta_r K$ of dimension more than r. So there exists some $B \in lk(A, K)$ such that $\beta_r(A \star B) = b' \star \beta_r(\partial(A \star B))$.

Define ϕ as this bijection from the vertex set of $\beta lk(A, K)$ to the vertex set of $lk(A, \beta_r K)$ which sends the vertex b corresponding to $B \in lk(A, K)$ to the vertex b' of $\beta_r(A \star B)$. We claim that ϕ extends to a simplicial isomorphism from $\beta lk(A, K)$ to $lk(A, \beta_r K)$. See Figure 3.2 for the case when $K = A * B$ and $r = 1$.

As ϕ is a bijection on the vertices it is a simplicial isomorphism on the 0-skeleton of $\beta lk(A, K)$. Let $B \in lk(A, K)$ be m dimensional and assume that ϕ is a simplicial isomorphism on the $m - 1$ skeleton of $\beta lk(A, K)$. As $\beta_r(A \star B) = b' \star \beta_r \partial(A \star B) = (b' \star \beta_r(\partial A \star B)) \cup (b' \star \beta_r(A \star \partial B))$ so each simplex of $\beta_r(A \star B)$ lies entirely in $(b' \star \beta_r(\partial A \star B))$ or $(b' \star \beta_r(A \star \partial B))$ (or both). So if $A \star C \in \beta_r(A \star B)$ then as A belongs to $b' \star \beta_r(A \star \partial B)$ so C belongs to it as well, and we get $lk(A, \beta_r(A \star B)) = lk(A, b' \star \beta_r(A \star \partial B))$. As $A \in \beta_r(A \star \partial B)$ so $lk(A, b' \star \beta_r(A \star \partial B)) = b' \star lk(A, \beta_r(A \star \partial B))$. By assumption, ϕ restricted to $\beta(\partial B)$ is simplicially isomorphic to $lk(A, \beta_r(A \star \partial B))$. So $\beta B = b \star \beta(\partial B)$ is simplicially isomorphic via ϕ to $b' \star lk(A, \beta_r(A \star \partial B)) = lk(A, \beta_r(A \star B))$. Varying B over all m-simplexes, shows that ϕ a simplicial isomorphism on the m-skeleton of $\beta lk(A, K)$. So by induction taking $m = n$, we get a simplicial isomorphism from $\beta lk(A, K)$ to $lk(A, \beta_r K)$. □

The following result proved in [45] for simplicial complexes also works for spherical triangulations:

Lemma 3.2.5. *Let K and L be geometric triangulations of a sphere. Then for $s \in \mathbb{N}$ denoting the total number of simplexes of K, the s-th derived subdivision of L is a subdivision of K.*

Proof. $|A|$ represents underlying topology of A. We have $|K| = |L|$. By Lemma 2.2.5 for $|K| \subset |L|$, we get that some subdivision K' of K is subcomplex of $\beta^s L$ and $|K'| = |\beta^s L|$ which implies that $K' = \beta^s L$. As K' is a subdivision of K, which proves our claim that $\beta^s L$ is a subdivision of K.

□

Lemma 3.2.6. *Let K be the geometric triangulation of a Riemannian n-manifold. For all vertices v of K, $\beta^{m-2} lk(v, K)$ is simplicially isomorphic to a subdivision of the boundary of an n-simplex, where $m = 2^{n+1}$.*

Proof. As K is a geometric triangulation, $|int(st(v, K))|$ is a neighbourhood of v. Let $L \subset int(st(v, K))$ be a geometrically triangulated sphere centered at v which is isomorphic to $lk(v, K)$ under radial projection. Let δ be a spherical triangulation of $|L|$ as the boundary of a spherical n-simplex, so in particular δ has $m - 2$ simplexes. By Lemma 3.2.5, $\beta^{m-2} L$ is a subdivision of δ which is isomorphic to the boundary of a simplex. □

Proposition 3.2.7. *[Proposition 1 of [5]] Every decomposition of an n-cell and every decomposition of an n-sphere contains a shellable subdivision.*

Theorem 3.2.8. *Links of all simplexes in $\beta^m K$ are shellable for $m = 2^{n+1}$.*

Proof. We first prove that all vertex links in $\beta^m K$ are shellable. For v a vertex of K, by Lemma 3.2.6 $st(v, \beta^{m-2} K) = v \star lk(v, \beta^{m-2} K) = v \star \beta^{m-2} lk(v, K)$ is isomorphic to the subdivision of a simplex. By Theorem 3.1.8 , $\beta^2 st(v, \beta^{m-2} K)$ is shellable. As links of vertices of shellable complexes are shellable, so $lk(v, \beta^2 st(v, \beta^{m-2} K)) = \beta^2 lk(v, \beta^{m-2} K) = lk(v, \beta^m K)$ is shellable for any vertex $v \in K$.

As $\beta^m(v \star lk(v, K))$ is the stellar subdivision of $v \star \beta^m lk(v, K)$ and as stellar subdivisions preserve shellability (see proof of Proposition 3.2.7) so $\beta^m(st(v, K))$ is shellable. For any vertex $w \in int(\beta^m st(v, K))$, $lk(w, \beta^m K) = lk(w, \beta^m st(v, K))$ is shellable as it is the link of a vertex in a shellable complex. As $|K| = |\beta^m K|$ is covered by $|int(st(v,K))| = |int(\beta^m st(v,K))|$ so the link of any vertex w of $\beta^m K$ is shellable.

Assume that links of all simplexes of dimension less than $r > 0$ are shellable. Let $A = B * b$ be an r-simplex in $\beta^m K$ for b a vertex. Then $lk(A, \beta^m K) = lk(b, lk(B, \beta^m K))$. As B is a simplex of dimension less than r, so by induction $lk(B, \beta^m K)$ is shellable and as links of vertices of shellable complexes are shellable, so $lk(A, \beta^m K)$ is also shellable.

□

Theorem 3.2.9. *[Theorem 5.1 of [3]] Let K be a shellable simplicial complex. Then the barycentric subdivision sd K is shellable.*

We end this section with a statement about shellability links of simplexes of partial barycentric subdivisions.

Lemma 3.2.10. *Let K be a simplicial complex such that the link of each vertex is shellable. Let A be an r-simplex in K, then $lk(A, \beta_r K)$ is shellable.*

Proof. For A any simplex of K, $lk(A, \beta_r K) = \beta lk(A, K)$ by Lemma 3.2.4. By arguments as in Theorem 3.2.8, as vertex links of K are shellable, so $lk(A, K)$ is shellable. And by Theorem 3.2.9, the barycentric subdivision of a shellable complex is shellable so $lk(A, \beta_r K)$ is shellable. □

3.3 Common Geometric Subdivision

Given two abstract simplicial complexes, there is no canonical notion of a common subdivision. In this section we use the geometry of the manifold to get a common geometric subdivision of two geometric triangulations. This allows us to relate them via a bounded length sequence of Pachner moves through the common subdivision. We must caution here that even though the terminal triangulations of this sequence are geometric in nature, the intermediate triangulations we obtain are merely topological triangulations.

Definition 3.3.1. A subset C of a Riemannian manifold M convex, if any two points in C are connected by a unique geodesic in C. We call it strongly convex, if any two points in C are connected by a unique minimizing geodesic in M which also happens to lie entirely in C.

A hyperbolic, spherical or Euclidean k-simplex in $\mathbb{H}^n$, $\mathbb{S}^n$ or $\mathbb{E}^n$ is the convex hull of a generic set of $k+1$ points. In the spherical case, we further assume that the diameter of the simplex is at most $\pi/2$.

Definition 3.3.2. A geometric simplicial triangulation K of a hyperbolic, spherical or Euclidean manifold M is a simplicial triangulation of M where each simplex is isometric to a hyperbolic, spherical or Euclidean simplex respectively. We say a geometric simplicial triangulation K' of M is a geometric subdivision of K if each simplex of K' is isometrically embedded in some simplex of K.

When M is a closed spherical, Euclidean or hyperbolic manifold then M has a geometric triangulation. See Theorem 3.3.3 where a strongly essential geometric triangulation is obtained. We henceforth fix the notation (M, K) to refer to the geometric simplicial triangulation K of a closed hyperbolic, spherical or Euclidean manifold M of dimension n.

Theorem 3.3.3. *[Theorem 7.3 of [14]] Suppose that M is a closed Riemannian manifold of dimension 3 with metric with constant curvature -1, 0 or +1. In the case of curvature +1, also assume that the diameter of the manifold is less than π. Then the cell decomposition dual to the cut locus of any point x_0 can be subdivided to give a strongly essential one-vertex triangulation.*

The simple observation spelled out in Lemma 2.3.5 allows us to treat the geometric triangulation of a convex polytope in M as the linear triangulation of a convex polytope in $\mathbb{E}^n$.

We have already seen the proof of Lemma 2.3.5, an outline of which is that every geometric simplex lifts to either $\mathbb{E}^n$, the Klein model of $\mathbb{H}^n$ or a hemisphere in $\mathbb{S}^n$ followed by the radial/gnomonic projection. In either case we get a map from the geometric simplex in M to a linear simplex in $\mathbb{E}^n$ which takes geodesics to straight lines. So in particular, it takes a geometric subdivision of the simplex to a simplicial subdivision of the corresponding linear simplex.

Lemma 3.3.4. *When K has p_i many i-simplexes, βK has $(i+1)!p_i$ many i-simplexes in the i-skeleton of K.*

Proof. To obtain the barycentric subdivision βK of K we replace each simplex of K with the cone on its boundary, starting with vertices and inductively going up to simplexes of dimension n.

For an i-simplex A, let a_i be the number of i simplexes in βA. As there are $i+1$ many codimension one faces of A so $a_i = (i+1)a_{i-1}$ and $a_0 = 1$. This gives $a_i = (i+1)!$. So if there are p_i many i-simplexes in K, there are $(i+1)!p_i$ many i-simplexes of βK in the i-skeleton of K. □

Lemma 3.3.5. *[Lemma 4.4 of [24]] Let K be an n dimensional simplicial complex, let αK be a simplicial subdivision of K with property that, for each simplex A in K, αA is a stellar ball. Then αK is stellar equivalent to K.*

The following is an effective version of Lemma 3.3.5 using the stronger notion of shellability instead of starrability, to get bistellar equivalence in place of stellar equivalence.

Lemma 3.3.6. *Let K be a geometric triangulation where the link of every positive dimensional simplex is shellable. Let αK be a geometric subdivision of K such that for each simplex $A \in K$, αA is shellable. Let p_i be the number of i-simplexes of K, with $p_{-1} = 1$. Let s_i be the number of i-simplexes of αK in the i-skeleton of K. Then αK is related to βK by $\sum_{i=1}^{n}(n-i)!p_{n-i-1}s_i$ Pachner moves. Furthermore, none of these Pachner moves remove any vertex of K.*

Proof. Our aim is to bound the number of Pachner moves needed to go from $\beta_r^\alpha K$ to $\beta_{r-1}^\alpha K$ for $1 \leq r \leq n$. This would give us a bound on the number of moves relating $\beta_n^\alpha K = \alpha K$ and $\beta_0^\alpha K = \beta K$.

As links of simplexes in K are given to be shellable, so for any r simplex $A \in K$, by Lemma 3.2.10, $lk(A, \beta_r K) = \beta lk(A, K)$ is shellable. As αA is given to be shellable so $S(A) = \alpha A \star lk(A, \beta_r K)$, the join of shellable complexes, is shellable as well. $S(A)$ should morally be thought of as the star neighbourhood of αA in $\beta_r K$.

Let m_A be the number of r-simplexes of αA in A. The number of $(n-r-1)$ simplexes in $lk(A, K)$ is at most p_{n-r-1}, so by Lemma 3.3.4 the number of $(n-r-1)$ simplexes in $\beta lk(A, K)$ is at most $(n-r)!p_{n-r-1}$. By Lemma 3.2.4, $\beta lk(A, K) = lk(A, \beta_r K)$, so $S(A)$ has at most $(n-r)!p_{n-r-1}m_A$ many n-simplexes.

By Lemma 3.2.2, there is a sequence of as many Pachner moves which changes $S(A)$ to $a \star \partial S(A) = a \star \partial\alpha A \star lk(A, \beta_r K)$, for a a point in the interior of A. Making this change for each r-simplex A of K replaces each αA with $a \star \partial\alpha A = a \star \beta_{r-1}^\alpha \partial\alpha A$ while higher dimensional simplexes of K remain subdivided as cones on their boundary. This gives us $\beta_{r-1}^\alpha K$ from $\beta_r^\alpha K$ by at most $(n-r)!s_r p_{n-r-1}$ Pachner moves, where s_r is the total number of r-simplexes of αK in the r-skeleton of K. So $\beta_n^\alpha K = \alpha K$ is related to $\beta_0^\alpha K = \beta K$ by $\sum_{r=1}^n (n-r)!s_r p_{n-r-1}$ Pachner moves. When $r = n$, $lk(A, \beta_r K)$ is empty so we take $p_{-1} = 1$. Note that as none of these Pachner moves remove any vertices of A so they never remove any vertex of K. $\square$

Lemma 3.3.7. *Let K be a simplicial complex where link of every positive dimensional simplex is shellable. The m-th barycentric subdivision $\beta^m K$ is related to K by $(n+1)!^{2m+2}p_n^2$ Pachner moves.*

Proof. Taking $\alpha K = K$ in Lemma 3.3.6, for A a simplex of K, $\alpha A = A$ is trivially shellable. Also $s_i = p_i$, so that K is related to βK by $\sum_{i=1}^n (n-i)!p_{n-i-1}p_i$ many Pachner moves. Bounding p_i by $\binom{n+1}{i+1}p_n$ and $(2n+2)!$ by $4(n+1)!^3$ we get:

$$\begin{array}{rclcl} \sum_{i=1}^n (n-i)!p_i p_{n-i-1} & < & \sum_{i=1}^n (n-i)!\binom{n+1}{i+1}\binom{n+1}{n-i}p_n^2 & < & (n-1)!p_n^2 \sum_{i=1}^n \binom{n+1}{i+1}^2 \\ & < & (n-1)!p_n^2\binom{2n+2}{n+1} & < & \frac{4}{n(n+1)}(n+1)!^2 p_n^2 \\ & < & (n+1)!^2 p_n^2 & & \end{array}$$

By Lemma 3.3.4, the number of n-simplexes p_n changes to $(n+1)!p_n$ on taking a barycentric subdivision. So on taking m subdivisions the bound on the number of moves relating K and $\beta^m K$ becomes:

$$p_n^2[(n+1)!^2 + (n+1)!^4 + ... + (n+1)!^{2m}] < (n+1)!^{2(m+1)}p_n^2$$

□

We now use Theorem 3.1.8 to bound the number of Pachner moves needed to relate a locally shellable geometric triangulation with its subdivision.

Theorem 3.3.8. *Let K be a geometric triangulation where the link of every positive dimensional simplex is shellable. Let K' be a geometric subdivision of K. Let p_i be the number of i-simplexes of K for $i > 0$, with $p_0 = 2$ and $p_{-1} = 1$. Let s_i be the number of i-simplexes of K' that lie in the i-skeleton of K. Then $\beta^2 K'$ is related to βK by $\sum_{i=1}^{n}(n-i)!(i+1)!(i+1)!p_{n-i-1}s_i$ many Pachner moves none of which remove any vertex of K.*

Proof. By Lemma 3.3.4, $\beta K'$ has less than $(i+1)!s_i$ many i-simplexes in the i-skeleton of K' and applied a second time, $\beta^2 K'$ has less than $(i+1)!(i+1)!s_i$ many i-simplexes in the i-skeleton of K'.

Let $\alpha K = K'$. For each simplex A of K, by Lemma 2.3.5 there is a simplicial isomorphism from αA to a linear subdivision of a convex polytope in $\mathbb{E}^n$. By Theorem 3.1.8, its second barycentric subdivision $\beta^2\alpha A$ is shellable and so replacing s_i in Lemma 3.3.6 with $(i+1)!(i+1)!s_i$ we get the required bounds.

□

In the rest of this section, we obtain a common subdivision with a controlled number of simplexes from a given pair of geometric triangulations.

Definition 3.3.9. Given a Riemannian manifold M, a geometric polytopal complex C of M is a finite collection of geometric convex polytopes in M whose union is all of M and such that for every $P \in C$, C contains all faces of P and the intersection of two polytopes is a face of each of them.

When each simplex of the geometric triangulations is strongly convex, any two simplexes intersect at most once. We can therefore bound the number of simplexes in the common geometric subdivision $K' = \beta(K_1 \cap K_2)$.

Lemma 3.3.10. *When K_1 and K_2 are strongly convex geometric triangulations with p_i and q_i many i-simplexes respectively, then they have a common geometric subdivision K' with s_i many i-dimensional simplexes that lie in the i-skeleton of K_1 where*

$$s_i < (2^n - 1)(n+1)!^2 p_i q_n$$

Proof. Let A be a linear k-simplex and B a linear l-simplex in $\mathbb{R}^N$. Suppose that B intersects A in a k-dimensional polytope P. So $l \geq k$ and the interiors of A and B intersect transversely inside a subspace $V(A+B)$ of $\mathbb{R}^N$ spanned by vectors in A and B (assume $0 \in A \cap B$). As their intersection P is k dimensional, so $V(P) = V(A) \cap V(B)$ is a k-dimensional space and by the Rank-Nullity theorem $V(A+B)$ is l-dimensional. Therefore any $(k-1)$ face of P is obtained by intersecting an $(l-1)$ simplex of B with the k-simplex of A or by intersecting a $(k-1)$-simplex of A with the l-simplex of B. There are therefore at most $(k+1)+(l+1)$ codimension one faces of P.

The barycentric subdivision βP of P is a simplicial complex. Observe that P has at most $k+l+2$ codimension one faces, each of which has $(k-1)+l+2$ codimension one faces by above reasoning, and so on down to $k=1$ which has exactly 2 codimension one faces (the end points of the edge). So the number of k dimensional simplexes of βP is bounded by $(k+l+2)((k-1)+l+2)...(2+l+2)(2) = 2(k+l+2)!/(l+3)!$ by reasoning similar to that of Lemma 3.3.4.

Note that strongly convex geometric triangulations are simplicial triangulations. Let $K_1 \cap K_2$ be a geometric polytopal complex of M obtained by intersecting the geometric simplexes of K_1 and K_2. Observe that as the polytopes of $K_1 \cap K_2$ are obtained by the intersection of convex simplexes so they are convex in M and their barycentric subdivision $K' = \beta(K_1 \cap K_2)$ is a geometric simplicial complex which is a common geometric subdivision of both K_1 and K_2.

Let s_i be the number of i-dimensional simplexes of K' that lie in K_1. As each i-polytope P of $K_1 \cap K_2$ that lies in the i-skeleton of K_1 is the intersection of a i-simplex of K_1 with some j simplex of K_2 for $j \geq i$, so by above arguments its barycentric subdivision βP has $2(i+j+2)!/(j+3)!$ many i-dimensional simplexes. As each simplex of K_1 and

K_2 is strongly convex, their intersection is convex and hence connected. So there are at most $\sum_{j=i}^{n} p_i q_j$ many i-polytopes of $K_1 \cap K_2$ that lie in the i-skeleton of K_1. We therefore get $s_i \leq \sum_{j=i}^{n} \frac{2(i+j+2)!}{(j+3)!} p_i q_j$.

Simplifying this by bounding q_j with $\binom{n+1}{j+1} q_n$ and $(2n+2)!$ with $4(n+1)!^3$ gives us:

$$\begin{array}{rcccl} s_i & < & p_i \sum_{j=1}^{n} \frac{2(n+j+2)!}{(j+3)!(n-1)!}(n-1)!\binom{n+1}{j+1} q_n & < & 2(n-1)!\binom{2n+2}{n-1}(2^{n+1}-2) p_i q_n \\ & < & \frac{4(2^n-1)(2n+2)!}{(n+3)!} p_i q_n & < & \frac{16}{(n+2)(n+3)}(2^n-1)(n+1)!^2 p_i q_n \end{array}$$

As $n \geq 2$, we get the required bound.

□

We now present some relations between the convexity radius and other invariants of the manifold.

Definition 3.3.11. [9] For a Riemannian manifold M, the injectivity radius at $p \in M$ is given by $inj(p) = \max\{R > 0 \mid exp_p|_{B(0,s)}$ is injective for all $0 < s < R\}$, the convexity radius at p is given by $r(p) = \max\{R > 0 \mid B(p,s)$ is strongly convex for all $0 < s < R\}$ where $B(0,s) \subset T_pM$ denotes the Euclidean ball of radius s around the origin and $B(p,s) \subset M$ denotes the ball of radius s around p. The focal radius at p is defined as $r_f(p) = \min\{T > 0 \mid \exists$ a non-trivial normal Jacobi field J along a unit speed geodesic γ with $\gamma(0) = p$, $J(0) = 0$, and $||J||'(T) = 0\}$. If such a Jacobi field does not exist, then the focal radius is defined to be infinite. Globally, let $inj(M) = \inf_{p\in M} inj(p)$, $r(M) = \inf_{p\in M} r(p)$ and let $r_f(M) = \inf_{p\in M} r_f(p)$ respectively be the injectivity radius, convexity radius and focal radius of the manifold M.

Applying the results of Dibble [9] and Klingenberg [22] to constant curvature manifolds, we get the following relation between convexity radius, injectivity radius and l_c, the length of smallest closed geodesic.

Lemma 3.3.12. *For M a spherical, Euclidean or hyperbolic closed manifold*

$$r(M) = \frac{1}{2} inj(M) = \frac{1}{4} l_c(M)$$

Proof. Theorem 2.6 of [9] shows that when M is compact, the convexity radius $r(M)$ equals $\min\{r_f(M), \frac{1}{4} l_c(M)\}$. When M is hyperbolic or Euclidean, $r_f(M) = \infty$. When M is spherical $r_f(M) = \pi/2$ and $l_c(M)/4 \leq 2diam(M)/4 \leq \pi/2$. So in either case,

$r(M) = \frac{1}{4}l_c(M)$. Klingenberg [22] has shown that $inj(M) = min\{r_c(M), \frac{1}{2}l_c(M)\}$. For hyperbolic and Euclidean manifolds $r_c(M) = \infty$ and for spherical manifolds $r_c(M) = \pi$ and $\frac{1}{2}l_c(M) \leq \pi$, so in either case $inj(M) = \frac{1}{2}l_c(M)$. □

Cheeger's inequality roughly says that when we have an upper diameter bound, lower section curvature bound and lower volume bound we get a lower injectivity radius bound. The following is a sharper bound by Heintze and Karcher (Corollary 2.3.2 of [13]) which we state here only for constant curvature manifolds:

Theorem 3.3.13. *[13] Let M be a complete spherical, Euclidean or hyperbolic n-manifold and let γ be a closed geodesic in M. Then $l(\gamma) \geq 2\pi vol(M)/(\delta vol(\mathbb{S}^n))$ where $\mathbb{S}^n$ is the the round n-sphere and*

$$\delta = \begin{cases} diam(M) & \text{for } M \text{ Euclidean} \\ sin^{n-1}(diam(M)) & \text{for } M \text{ spherical} \\ sinh^{n-1}(diam(M)) & \text{for } M \text{ hyperbolic} \end{cases}$$

We use Lemma 3.3.12 and Theorem 3.3.13 to get a lower injectivity radius bound which is used to derive Corollary 3.1.2 from Theorem 3.1.1. In order to prove Theorem 3.1.1, we first subdivide the given geometric triangulations sufficiently many times so that each simplex lies in a strongly convex ball. To bound the rate at which barycentric subdivisions scale the diameter of the simplex, we need the following theorem which we prove in Section 3.4.

Theorem 3.3.14. *Let $\beta^m\Delta$ be the m-th geometric barycentric subdivision of an n simplex Δ with new vertices added at the centroid of simplexes. Let Λ be an upper bound on the length of edges of Δ. Then the diameter of simplexes of $\beta^m\Delta$ is at most $\kappa^m\Lambda$ where*

$$\kappa = \begin{cases} \frac{n}{n+1} & \text{for } M \text{ Euclidean} \\ \frac{2n}{2n+1} & \text{for } M \text{ spherical} \\ \frac{ncosh^{n-1}(\Lambda)}{ncosh^{n-1}(\Lambda)+1} & \text{for } M \text{ hyperbolic} \end{cases}$$

We finally prove the main Theorem of this chapter below.

Proof of Theorem 3.1.1. First assume that K_1 and K_2 are strongly convex geometric triangulations where links of all simplexes are shellable. By Lemma 3.3.10, there exists a common geometric subdivision K' of K_1 and K_2 with s_i many i-simplexes in the i-skeleton of K_1. Using Lemma 3.3.8 next we get a bound on the number of Pachner moves relating βK_i and $\beta^2 K'$. Vertices that are common to both K_1 and K_2 are not removed by these Pachner moves.

Plugging in the bounds for s_i from Lemma 3.3.10 in the formula obtained in Lemma 3.3.8, and then bounding $(2n+2)!$ by $4(n+1)!^3$, p_i by $\binom{n+1}{i+1}p_n$ and q_i similarly, we get the following bound for the number of moves relating βK_1 and $\beta^2 K'$.

$$\begin{aligned}
\textstyle\sum_{i=1}^{n}(n-i)!p_{n-i-1}s_i &< (2^n-1)(n+1)!^2 q_n \textstyle\sum_{i=1}^{n}(n-i)!p_{n-i-1}p_i \\
&< (2^n-1)(n+1)!^2(n-1)!\binom{2n+2}{n+1}p_n^2 q_n \\
&< (2^n-1)(n+1)!^4 p_n^2 q_n \cdot 4/(n(n+1)) \\
&< (2^n-1)(n+1)!^4 p_n^2 q_n
\end{aligned}$$

Exchanging the roles of p_i and q_i we get a bound on the number of moves relating βK_2 and $\beta^2 K'$. Summing them up we get the total number of moves needed to go from βK_1 to βK_2 as $(2^n-1)(n+1)!^4 p_n q_n(p_n+q_n)$. To simplify notation we henceforth denote p_n by p and q_n by q.

Given geometric triangulations K_1 and K_2 which may not be strongly convex, we need an integer m such that $\beta^m K_i$ is strongly convex. That is, we need m such that each simplex in $\beta^m K_i$ lies in a strongly convex ball or by Theorem 3.3.14, $\kappa^m \Lambda < 2r(M)$ where Λ is an upper bound on the length of edges of K_1 and K_2. So we take m to be any integer greater than $(\ln(2r(M)) - \ln(\Lambda))/\ln(\kappa)$, as $\ln(\kappa) < 0$. For $a > 0$, $ln(a+1) - ln(a) = \int_a^{a+1} 1/x > 1/(a+1)$. So we get $-1/\ln(\kappa) \leq \mu$ and we can take m to be an integer greater than $\mu(\ln(\Lambda) - \ln(2r(M)))$ or by Lemma 3.3.12 we can take m to be an integer greater than $\mu \ln(\Lambda/inj(M))$. To ensure that links of simplexes are shellable, by Theorem 3.2.8, we assume that m is also greater than 2^{n+1}. As the simplexes of the subdivision are strongly convex, it is a simplicial geometric triangulation.

By Lemma 3.3.4, the locally shellable complexes $\beta^m K_1$ and $\beta^m K_2$ have $(n+1)!^m p$ and $(n+1)!^m q$ many n-simplexes which are all strongly convex. So to go between $\beta\beta^m K_i$ we need $(2^n-1)(n+1)!^{4+3m}pq(p+q)$ moves. By Lemma 3.3.7, when K_1 is locally shellable, K_1 and $\beta^{m+1}K_1$ are related by $(n+1)!^{2(m+2)}p^2$ moves (and similarly for K_2). When $n \leq 4$, the links of positive dimensional simplexes are spheres of dimension at most 2 and are therefore shellable. This gives a bound on moves to go from K_1 to K_2 as $2^n(n+1)!^{4+3m}pq(p+q)$ when $n \leq 4$.

For a general n, by Lemma 3.3.7, $\beta^{2^{n+1}}K_1$ and $\beta^{m+1}K_1$ are related by $(n+1)!^{2(m+1-2^{n+1}+1)} < (n+1)!^{2(m+2)}p^2$ moves (and similarly for K_2). So by the above arguments $\beta^{2^{n+1}}K_i$ are also related by less than $2^n(n+1)!^{4+3m}pq(p+q)$ moves.

□

Proof of Corollary 3.1.7. If $F : M \to N$ is an isometry then $F^{-1}(K_N)$ is a geometric triangulation of M. So by Theorem 3.1.1, $K_1 = K_M$ and $K_2 = F^{-1}(K_N)$ are related by the given bounded number of Pachner moves. As F is a simplicial isomorphism from K_2 to K_N, we get the required result.

When M and N are complete finite volume hyperbolic manifolds of dimension at least 3, then by Mostow-Prasad [33] [36] rigidity, every homeomorphism is isotopic to an isometry. So if K_M and K_N are related by Pachner moves and simplicial isomorphisms, then M and N are homeomorphic and hence isometric.

For dimensions up to 6, the PL and DIFF categories are isomorphic and by a theorem of De Rham [8] diffeomorphic spherical manifolds are isometric, so the converse also holds for spherical manifolds of dimension at most 6.

The converse is not true in the Euclidean case in any dimension as there are simplicially isomorphic flat tori which are not isometric. □

3.4 Subdivisions In Constant Curvature Geometries

The aim of this section is to prove Theorem 3.3.14 which gives the scaling factor for diameter of simplexes in the model geometries upon taking barycentric subdivisions.

Definition 3.4.1. Let $\Delta = [v_0, ..., v_n]$ be a geometric n-simplex. We define medians and centroids of faces of Δ inductively. Each vertex v_i is defined to be its own centroid. We define the centroid of an edge of Δ as the midpoint of the edge. Having defined centroids of k dimensional faces of Δ, we define the medians of a $k+1$ dimensional face σ as the geodesics in σ joining a vertex of σ to the centroid of its opposing k dimensional face in σ. We define the centroid $c(\sigma)$ of σ as the common intersection of all medians of σ. We shall show that such a common intersection exists for hyperbolic, spherical and Euclidean tetrahedra. Given simplexes A and B such that $\sigma = A * B$, we define the medial segment joining A and B as the geodesic in σ that connects the centroids $c(A)$ of A and $c(B)$ of B. When A or B is a vertex the medial segment is a median.

Lemma 3.4.2. *Let Δ be a Euclidean, hyperbolic or spherical n dimensional simplex. All medial segments of Δ intersect at a common point $c(\Delta)$. Furthermore if Λ is an upper bound for the length of the edges of Δ (with $\Lambda \leq \pi/2$ for Δ spherical) and $\Delta = a * B$ for a vertex a and B an $n-1$ dimensional face, then $d(a, c(\Delta))/d(a, c(B)) \leq \kappa$ where κ is as in Theorem 3.3.14.*

Proof. Case I: Δ is Euclidean. Realise Δ as a linear combination of basis vectors (v_i) in $\mathbb{R}^{n+1}$. For each face $\sigma = [v_{i_0}, ..., v_{i_k}]$ of Δ, let $c(\sigma) = (v_{i_0} + ... + v_{i_k})/(k+1)$. By inducting on the dimension of σ, we shall show that $c(\sigma)$ is the centroid of σ.

When σ is a vertex or an edge, $c(\sigma)$ is by definition the centroid of σ. Assume that the centroid is well defined for all faces of Δ of dimension less than k. After relabeling the vertices, assume that $\sigma = [v_0, ..., v_k]$ and $\sigma = A * B$ with $A = [v_0, ..., v_p]$ and $B = [v_{p+1}, ..., v_k]$. The dimensions of A and B are p and $q = k - (p+1)$. We can express $c(\sigma)$ as a convex linear combination of $c(A)$ and $c(B)$ as below:

$$\begin{aligned} c(\sigma) &= \frac{\sum_{i=0}^{k} v_i}{k+1} \\ &= \frac{p+1}{k+1}\frac{\sum_{i=0}^{p} v_i}{p+1} + \frac{(k+1)-(p+1)}{k+1}\frac{\sum_{i=p+1}^{k} v_i}{k-p} \\ &= \frac{p+1}{k+1}c(A) + \frac{q+1}{k+1}c(B) \end{aligned}$$

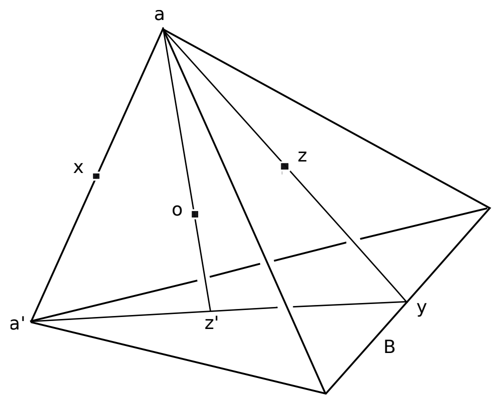

Figure 3.3: An n-simplex $\Delta = a * a' * B$ with $x = c(a * a')$, $y = c(B)$, $z = c(a * B)$, $z' = c(a' * B)$ and $o = c(\Delta)$, points on $\delta = [aa'y]$.

The point $c(\sigma)$ therefore lies on the medial segment connecting the centroids of A and B. Furthermore it divides the medial segment $[c(A), c(B)]$ in the ratio $(q+1)/(p+1)$. Taking $\sigma = \Delta$ and A as a vertex a we get $d(a, c(\Delta))/d(c(\Delta), c(B)) = n$, so that taking reciprocals and adding one on both sides gives $d(a, c(\Delta))/d(a, c(B)) = n/n{+}1$ as required.

Case II: Δ is hyperbolic. Let $E^{(n,1)}$ be the $(n,1)$ Minkowski space, i.e. $\mathbb{R}^{n+1}$ with the inner product $u.v = u_1v_1 + ... + u_nv_n - u_{n+1}v_{n+1}$. The n dimensional hyperbolic space $\mathbb{H}^n$ has a natural embedding in $E^{(n,1)}$ as the component of the hyperboloid $||x||^2 = -1$ which lies in the upper half space of $\mathbb{R}^{n+1}$. Let $T = \{v \in \mathbb{E}^{(n,1)} : ||v|| < 0\}$ and let (Euclidean) line segments in T with endpoints on $\mathbb{H}^n$ be called the chords of $\mathbb{H}^n$. Let $p : T \to \mathbb{H}^n$ be the radial projection $x \to \frac{\sqrt{-1}}{||x||}x$. It is easy to see that p takes chords to hyperbolic geodesic segments in $\mathbb{H}^n$. To see that p takes midpoints of chords to midpoints of the corresponding geodesic segment take x and y in $\mathbb{H}^n$ and let $r \in O^+(n,1)$ restrict to an isometry of $\mathbb{H}^n$ that exchanges x and y. Let $m = (x+y)/2$ be the midpoint of the chord joining x and y and let $z = p(m)$ be its image on the geodesic segment $[x, y]$. As there is a unique geodesic segment between pairs of points in $\mathbb{H}^n$, the isometry r reflects the geodesic segment $[x, y]$ fixing only the mid point of $[x, y]$. But as r is linear in $\mathbb{R}^{n+1}$, $r(z) = \frac{\sqrt{-1}}{||m||}r(m) = \frac{\sqrt{-1}}{||m||}m = z$, so z is the midpoint of $[x, y]$.

Given a hyperbolic simplex Δ in $\mathbb{H}^n$ with vertices v_i, let Δ_0 be the Euclidean convex linear combination of v_i in $\mathbb{R}^{n+1}$. As the homeomorphism $p|_{\Delta_0} : \Delta_0 \to \Delta$, fixes the vertices and takes midpoints of edges to midpoints of edges, by induction, it takes medial segments to medial segments and hence takes centroids to centroids. In particular, all the medial segments of Δ intersect at the common point $c(\Delta)$ as in the Euclidean case.

For points a, x, b in Δ, define the ratio $h(a, x, b) = sinh(d(a, x))/sinh(d(x, b))$. By inducting on the dimension of Δ we shall prove that if $\Delta = a * B$ with a a vertex and B an $n-1$ face, then $1 \leq h(a, c(\Delta), c(B)) \leq ncosh^{n-1}(\Lambda)$. When $\Delta = a * b$ is an edge, then $h(a, c(\Delta), b) = 1$. Let $\Delta = a * a' * B$ be an n dimensional simplex. Let δ be the geodesic triangle $[a, a', c(B)]$ in Δ. Let $x = c(a * a')$, $y = c(B)$, $z = c(a * B)$, $z' = c(a' * B)$ and $o = c(\Delta)$ be points of δ as in Figure 3.3. As the medial segments of Δ all intersect at the centroid o, the segments $[a, z']$, $[a', z]$ and $[x, y]$ of δ have a common intersection at o. By the hyperbolic version of van Obel's Theorem,

$$h(a, o, z') = cosh(d(a', z'))h(a, x, a') + cosh(d(z', y))h(a, z, y)$$

As x is the midpoint of $[a, a']$ so $h(a, x, a') = 1$ and by induction applied to the $n-1$ simplex $a * B$, $1 \leq h(a, z, y) = h(a, c(a * B), c(B)) \leq (n-1)cosh^{n-2}(\Lambda)$. As $1 \leq cosh$, $1 \leq h(a, o, z') \leq ncosh^{n-1}(\Lambda)$ as required.

Define $f(x) = sinh(x)/x$ for $x > 0$ and $f(0) = 1$. Then $f'(x) = (xcosh(x) - sinh(x))/x^2$ has positive numerator because it takes value 0 at 0 and it's derivative is positive. So f is an increasing function. For $0 < x \leq y$, $sinh(x)/x \leq sinh(y)/y$, i.e, $y/x \leq sinh(y)/sinh(x)$. As $h(a, o, z') \geq 1$, so $sinh(d(a, o)) \geq sinh(d(o, z'))$ and as $sinh$ is a strictly increasing function so $d(a, o) \geq d(o, z')$. By above arguments then $d(a, o)/d(o, z') \leq h(a, o, z') \leq ncosh^{n-1}(\Lambda)$. Taking reciprocals and adding one on both sides we get the required bound κ.

Case III: Δ is spherical. Taking the standard embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$ with $p : \mathbb{R}^{n+1} \setminus 0 \to \mathbb{S}^n$ as the radial projection $p(x) = \frac{x}{||x||}$ we can show that medial segments of a spherical simplex Δ have a common intersection at the centroid, as in the hyperbolic case.

Proceeding as in the hyperbolic case, using $s(a, x, b) = sin(d(a, x))/sin(d(x, b))$ instead of $h(a, x, b)$ and using the spherical van Obel theorem

$$s(a, o, z') = cos(d(a', z'))s(a, x, a') + cos(d(z', y))s(a, z, y)$$

we get the bound $s(a, o, z') \leq n$.

Suppose that for $0 < p, q \leq \pi/2$, we are given $sin(p)/sin(q) \leq n$. Then we shall show that $p/q \leq 2n$. As $sin(q) \leq q$ for $q > 0$, so $sin(p)/q \leq sin(p)/sin(q) \leq n$. Let $0 < t_0 < \pi/2$ be the point where $sin(t_0) = \pi/4$. When $t_0 \leq p \leq \pi/2$, $sin(t_0) \leq sin(p)$ so $sin(t_0)/q \leq sin(p)/q \leq n$ and we get $p/q \leq n\pi/(2sin(t_0)) = 2n$. When $0 < p \leq t_0$, $cos(t_0) \leq cos(p)$ and as $p \leq tan(p)$ (see the power series expansion of tan for this relation) so $pcos(p)/q \leq sin(p)/q \leq n$. We therefore get $p/q \leq n/cos(t_0) \leq 2n$ as $cos(t_0) \geq 1/2$. Taken together we conclude that $p/q \leq 2n$ as required. As $s(a, o, z') \leq n$, $d(a, o)/d(o, z') \leq 2n$ and adding one and taking reciprocals gives the required bound κ in the spherical case.

□

Lemma 3.4.3. *Let ABC be a hyperbolic, Euclidean or spherical triangle. When ABC is spherical we assume that the length of edges of ABC is at most $\pi/2$. Then for any point D on the segment $[B, C]$, $d(A, D) \leq \max(d(A, B), d(A, C))$.*

Proof. Suppose that ABC is a hyperbolic or Euclidean triangle for which the lemma is not true. Then the angle ADB is less than angle B and angle ADC is less than angle C which would imply that the sum of angles B and C is greater than π, a contradiction.

Let ABC be a spherical isosceles triangle in $S^2 \subset \mathbb{R}^3$ with A at the north pole and with base BC having z coordinate $z_0 \geq 0$. The plane containing the origin, B and C intersects S^2 in the spherical geodesic segment $[B, C]$ which lies in the half space $z \geq z_0$. So for any point $D \in [B, C]$, $d(A, D) \leq d(A, B)$. When ABC is an arbitrary spherical triangle with A at the north pole, side AB longer than side AC and z_0 as the z-coordinate of B, we extend the side AC to the point C' which has z coordinate z_0 so that ABC' is an isosceles triangle. For any point $D \in [B, C]$, extend the segment $[A, D]$ to $D' \in [BC']$, then by the above argument $d(A, D) \leq d(A, D') \leq d(A, B)$. □

Lemma 3.4.4. *Let Δ be a hyperbolic, spherical or Euclidean simplex. If Δ is spherical we assume the length of its edges is at most $\pi/2$. Then the diameter of Δ is the length of the longest edge of Δ.*

Proof. Let $[x, y]$ be a maximal segment in Δ and assume that it does not lie in any proper simplex of Δ. Let $x \in A$, $y \in B$ for simplexes A and B in $\partial\Delta$ then $\Delta = A * B$. If both x and y are vertices then trivially, $d(x, y) = l([x, y])$ is at most length of longest edge of Δ.

If x is not a vertex, then let $A = a * A'$ with a a vertex of A. Extend the segment $[a, x]$ to $x' \in A'$. Applying Lemma 3.4.3 to the triangle $[ax'y]$, $d(y, x) \leq max(d(y, a), d(y, x'))$. As dimensions of $a * B$ and $A' * B$ are both less than dimension of Δ, so by induction $d(y, x)$ is at most the length of the longest edge of Δ. □

Note that Lemma 3.4.4 is not true for spherical triangles with edges longer than $\pi/2$ as can be seen by taking an isosceles triangle with base length less than $\pi/2$ and the equal length edges of length more than $\pi/2$. The diameter of such a triangle is the length of the altitude on the base, which is greater than the length of all the edges.

We are finally in a position to prove the main Theorem of this section:

Proof of Theorem 3.3.14. We shall first show, by induction on the dimension of faces A of Δ, that $d(c(A), c(\Delta)) \leq \kappa\Lambda$. When A is a vertex, by Lemma 3.4.2 and Lemma 3.4.4, $d(a, c(\Delta)) \leq \kappa d(a, c(B)) \leq \kappa \text{diam}(\Delta) \leq \kappa\Lambda$. For $A = a * A'$, consider the triangle $T = [a, c(A'), c(\Delta)]$. As the medial segment $[a, c(A')]$ passes through $c(A)$, the segment $[c(\Delta), c(A)]$ lies in T and by Lemma 3.4.3, $d(c(\Delta), c(A))$ is at most $max(d(c(\Delta), a), d(c(\Delta), c(A')))$ which is in turn bounded by $\kappa\Lambda$ by induction.

Each edge of $\beta\Delta$ is a medial segment in some simplex $\delta \in \Delta$, of the kind $[c(\delta), c(A)]$ for $A \in \delta$. By above arguments, length of such edges is bounded by $\kappa\Lambda$. Repeating the argument for $\beta\Delta$ in place of Δ, taking $\kappa\Lambda$ as the upper bound for length of edges, we get the bound $\kappa^2\Lambda$ for edges of $\beta^2\Delta$. Repeating the argument m times and applying Lemma 3.4.4, we get the required upper bound for the diameter of simplexes of $\beta\Delta$. □

To see that the constant κ in the hyperbolic case can not be made independent of the length of the edges, consider a hyperbolic isosceles triangle $\Delta = ABC$ with base BC. Let a and b be the length of the sides opposite to vertices A and B, let m be the length of the median from A and let x be the distance from A to the centroid of ABC. Assume that $m = ya$ for some $y > 0$. By the hyperbolic version of Pythagoras theorem, $cosh(b) = cosh(a/2)cosh(m)$ which gives the following for all $a > 0$:

$$1 \leq b/m = \frac{cosh^{-1}(cosh(a/2)cosh(ya))}{ya} \leq \frac{cosh^{-1}(cosh(ya + a/2))}{ya} = 1 + \frac{1}{2y}$$

So for any fixed base length a and isosceles triangle as above with $m = ya$, $lim_{y\to\infty} m/b \to 1$. Also, as $sinh(x)/sinh(m) = 2cosh(a/2)/(2cosh(a/2) + 1) \to 1$ as

$a \to \infty$. So for large enough a and y, $x/b = (x/m)(m/b)$ is as close to 1 as required. In other words, the diameter of simplexes in $\beta\Delta$ can be made arbitrarily close to the diameter of Δ.

4

Algorithm For Once Punctured Torus Bundles

4.1 Introduction

Mijatovic [31] has given bounds on the number of Pachner moves required to relate two triangulations of knot complements. We attempt to give such a bound for once-punctured torus bundles, which are not covered by Mijatovic's result. Our main aim is to improve the bound significantly for such an important class of manifolds, as the bound in Mijatovic's result is a bounded tower of exponentials.

Theorem 4.1.1. *[Theorem 1.1 in [31]] Let P and Q be two triangulations of a knot complement M which contains p and q many tetrahedra respectively. Then there is a sequence of Pachner moves of length at most $e^{2^{ap}}(p) + e^{2^{aq}}(q)$ which transforms P into a triangulation isomorphic to Q. The constant a is bounded above by 200 and $e(x) = 2^x$. The homeomorphism that realises this simplicial isomorphism, is supported in the characteristic sub-manifold Σ of M and it does not permute the components of ∂M.*

In this chapter we will see the work done so far in lowering this bound.

4.1.1 Normal Surface Theory

Normal surface theory is a necessary tool in dealing with triangulations of 3-manifolds. Book by Sergei Matveev [27], papers by Mijaovic [28] and Bart and Schalemann [4] has covered the following section in details. A properly embedded disk D in a 3-simplex is called a *normal triangle* or *quadrilateral* if ∂D intersects exactly three or four edges transversely in a single point and remains disjoint from other 1-simplexes and vertices of 3-simplex. A *normal disk* is a normal triangle or a normal quadrilateral.

There are four types of normal triangles and three types of normal quadrilaterals up to isotopy through normal disks. Let M be a 3-manifold and let T be a triangulation of M. We say a properly embedded surface F in M is in *normal form* with respect to T, if it intersects each 3-simplex in a finite collection of disjoint normal disks.

Suppose F is normal surface in M with respect to T. Then F corresponds to a vector $X = (X_1, \ldots, X_{7t})$ in $\mathbb{R}^{7t}$, where t is the number of 3-simplexes in T and the indexing set $\{1, \ldots, 7t\}$ corresponds to all the possible normal disks types. The coordinate X_i corresponds to the number of copies of i-th disk type in F.

Each 2-simplex in T contains three types of normal arcs up to normal isotopy. If two 3-simplexes A and B share a 2-simplex, then number of normal disks of A which intersect the 2-simplex in the same type of normal arc should be the same as for B. If $\{x_i, y_i\}$ are number of normal triangles and number of normal quadrilaterals in two adjoining 3-simplexes. Then we should have, $x_1 + y_1 = x_2 + y_2$, we get a system of homogeneous linear equation for each arc type on each face. This system of homogeneous linear equations are called the *matching equations*. As quadrilateral of two distinct types in a 3-simplex would intersect so we add a quadrilateral constraint, which says that we can not have more than one quadrilateral type in a 3-simplex. Non-negative integer solution of the matching equations which satisfy the quadrilateral constraint gives rise to a normal surface and vector given by a normal surface is a non-negative solution of this system which satisfy the quadrilateral constraint. Hence, normal isotopy class of properly embedded normal surfaces are in one to one correspondence with non-negative integer solution of the matching equations with quadrilateral constraint.

Haken showed that all the non-negative integer solutions of this linear system are integer linear combination of a finite set of non-negative integer solutions $X_1', \ldots, X_n'$ called the *fundamental solutions*. Upon normalising the solution space of this linear system, it projects down to a compact linear cell which is called *the projective linear space*. Each fundamental solution gives rise to an embedded normal surface, which is called a *fundamental surface*. As each solution of the system can be written as the sum of fundamental solutions, we can write each embedded normal surface as a *Haken sum*

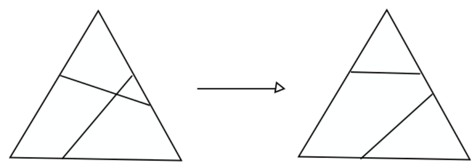

Figure 4.1: Regular switch

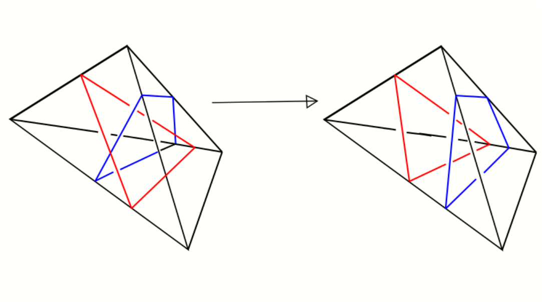

Figure 4.2: Sum of two normal surfaces

of fundamental surfaces. [15] has detailed description on the subject.

Given two surfaces F and F' in normal form with corresponding vectors X and X', define the sum $F + F'$ corresponding to the vector $X + X'$ in the following way:

A *regular switch* on a face of the 3-simplex is the operation depicted in Figure 4.1 In which we are cutting both the arcs at the intersection point and joining four arcs at the end points in a way to get two normal arcs. Regular switch on the faces extends to cut and paste operation on normal disks resulting in disjoint normal disks of the same normal isotopy type as shown in Figure 4.2, where two normal disks with red and blue boundary intersecting in two faces of a tetrahedron. Extending this procedure to every 3-simplex in the triangulation gives a normal surface which is the Haken sum $F + F'$ corresponding to the vector $X + X'$.

A *vertex surface* is a connected two sided normal surface that projects onto a vertex of the projective solution space of system of matching equations. We care about vertex surfaces and fundamental surfaces because we have a bound on number of normal disks of these surfaces.

Proposition 4.1.2. *[Lemma 6.1 of [15]] Let M be a compact triangulated 3-manifold* *containing t tetrahedra. Then each normal coordinate of a vertex surface in M is bounded above by* 2^{7t}. *If the normal surface is fundamental,* $7t2^{7t}$ *acts as an upper bound on all of its normal coordinates.*

Definition 4.1.3. A properly embedded surface S in M is called 2-sided if its normal I-bundle is trivial and it is called 1-sided otherwise, where $I = [0, 1]$. 2-sided surfaces which do not have sphere or disk components are called *incompressible* if for each embedded disk $D \subset M$ with $D \cap S = \partial D$ there is a disk $D' \subset S$ such that $\partial D' = \partial D$. A properly embedded surface S in M is called *∂-incompressible* if for each disk $D \subset M$ such that ∂D decomposes as union of two arcs α and β, with $D \cap S = \alpha$ and $D \cap \partial M = \beta$, there is a disk $D' \subset S$ with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial S$. We will call a surface in M *essential* if it is properly embedded, incompressible, boundary incompressible and not parallel to a surface in ∂M.

Definition 4.1.4. [$F-$ bundle over B] An F-bundle over B is a fibre bundle structure with topological spaces M, B, F and a projection map $\pi : M \to B$, such that for every $x \in M$, there is an open neighbourhood $U \subset B$ of $\pi(x)$ such that there is a homeomorphism $\phi : \pi^{-1}(U) \to U \times F$, $proj_1 : U \times F \to U$ and $proj_1 \circ \phi = \pi|_{\pi^{-1}(U)}$.

Let T be a punctured torus, two T-bundles over S^1 are equivalent if there is a fibre preserving homeomorphism between them which induces the identity on the base. We know that the equivalence classes of T-bundles over S^1 are in one-to-one correspondence with conjugacy classes of $GL_2(\mathbb{Z})$ and that every T-bundle over S^1 is of the form $T \times I/\eta$ with $\eta \in GL_2(\mathbb{Z})$. For more details refer to paper by Culler, Jaco and Rubinstein [6].

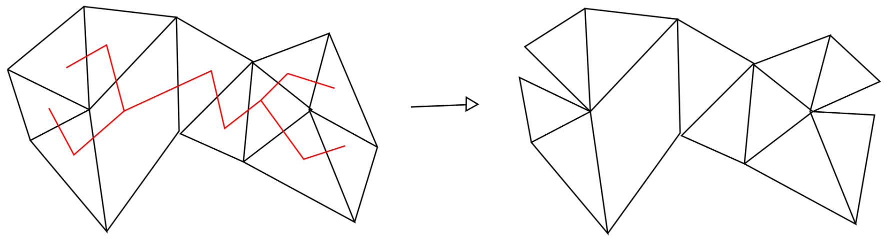

Figure 4.3: Maximal graph and Polygonal representation

4.2 Bound for Surfaces

This section deals with the bound on number of Pachner moves required to relate two triangulations of closed surfaces.

Let S be a closed surface and T be some triangulation of S. Let G_T be the dual graph of T, that is, take a vertex for each triangle in T and an edge between two vertices if their corresponding triangles share an edge. Let M_T be a maximal tree in the graph G_T. Consider the span $P(T)$ of M_T in T, that is an induced triangulation from T whose dual is the given graph M_T. Then $P(T)$ gives us a polygonal representation of S in the following way:

As $P(T)$ is an induced triangulation from T which contains all the triangles of T, we can go back and forth between $P(T)$ and T by identifying the necessary edges in $\partial P(T)$. See Figure 4.3. As dual graph M_T of $P(T)$ has no circuits so it is a disk.

Lemma 4.2.1. *[Lemma 3.2.2 for n = 2] Any combinatorial triangulation of a piecewise linear disk with n triangles can be changed to a cone over the boundary of the same disk by n Pachner moves.*

Proof. As the combinatorial triangulation of a disk is shellable, we can index all the simplexes in it by numbering them from 1 to n such that the ascending integers give a way to reduce the triangulation down to a single simplex. The 2-simplex which is left has index n. Make a (1–3) move on the index n simplex so that it is a cone over its boundary. Assume that we already have a cone over the boundary of last k simplexes and rest of the triangulation we started with is unchanged. If the triangle corresponding

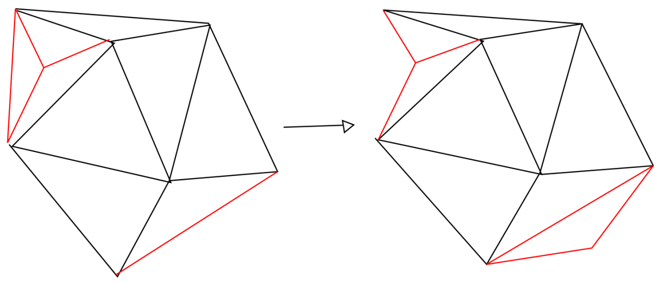

Figure 4.4: Changing Polygonal representation

to $k-1$ simplex has single edge in common with the coned sub complex then a (2–2) move gives a cone, if it shares two faces then (3–1) moves gives the cone. $\square$

Let T and T' be two combinatorial triangulations of S. Let $P(T)$ and $P(T')$ be two polygonal representations of S. Let us assume that, $e(\partial P(T)) = 2k > e(\partial P(T')) = 2k'$, e is the number of edges. Choose any $k-k'$ pairwise unidentified edges in the boundary of $P(T')$. Perform (1-3) Pachner moves on each triangle which intersects these edges and change the polygonal representation as shown in Figure 4.4.

After changing the polygonal representation of $P(T')$, we have a new polygonal triangulation, P' such that $\partial P(T)$ is isomorphic to $\partial P'$.

Theorem 4.2.2. *Let S be a closed surface, T and T' be combinatorial triangulations of S with t and t' many triangles respectively. Then, we get a sequence of at most $4t+2$ Pachner moves which takes T to T'.*

Proof. Let $P(T)$ and $P(T')$ be polygonal representations of P and P', with number of triangles t and t' respectively. Assume $t \geq t'$ then the number of edges in the boundary of $P(T)$ and $P(T')$ is bounded by $t+2$. We can change one of the polygonal representations with less than $(t+1)/2$ many (1–3) moves into a representation whose boundary is isomorphic to the boundary of the other polygonal representation.

As each (1–3) move adds two new triangles in the triangulation, the number of triangles in the new triangulation will be bounded by $2t+1 = t+1+t$ where, t is the bound on number of triangles of T and we need $(t+1)/2$ many (1-3) Pachner moves to change polygonal representation. Using Lemma 4.2.1, as assumed $t \geq t'$, we can relate polygonal representations via cone over their boundary in less than $4t+2$ Pachner moves. $\square$

4.3 Algorithm for once-punctured torus bundle

We know all the essential surfaces in once-punctured torus bundles thanks to Culler, Jaco and Rubinstein [6]. We know that essential surfaces are normal with respect to any triangulation. As the fibres of these bundles are essential, we can put these fibres in the two skeleton of the given triangulations using techniques of Mijatovic as in Lemma 4.3.1. In particular, we put that fibre in the 2-skeleton of the triangulation, which has a minimal weight boundary (least number of normal arcs) with respect to the boundary triangulation of once-punctured torus bundle.

Lemma 4.3.1. *[Lemma 4.1 in [28]] Let M be a 3-manifold with a triangulation T consisting of t tetrahedra. Assume that F is a properly embedded surface in M with respect to T which contains n normal pieces. Then we can obtain a subdivision T_1 of T, using less that 200nt Pachner moves, with the following properties: T_1 contains the surface F in its 2-skeleton and it consists of not more than 20(n+t) tetrahedra.*

If we start with two different triangulations T_1 and T_2 of the once-punctured torus bundle, assuming triangulations agree on the boundary, we can put the required fibre in the 2-skeleton using Lemma 4.3.1. Cutting along the fibre gives triangulations of $T \times I$ which agree on the boundary. We get a pair of compression disks for $T \times I$ which we put in the two skeleton using Lemma 4.3.1 and again cut along them to get triangulations of a 3-ball which agree on the boundary. Using Mijatovic's Theorem 4.3.2 for a 3-ball which gives a bound on number of Pachner moves required to go from any triangulation of a 3-ball to cone over its boundary, we get a bound on the number of Pachner moves required to relate triangulations of once-punctured torus bundles.

Theorem 4.3.2. *[Theorem 5.2 in [29]] Let T be triangulation of a 3-ball with t tetrahedra. Then it can be changed to cone on the bounding 2-sphere, without altering the induced triangulation of the boundary, by less than $at^2 2^{at^2}$ Pachner moves, where the constant a is bounded above by 6.10^6.*

In order to count the number of Pachner moves required, we need to count the number of simplexes at each step of our process. We need to know the number of simplexes when we put the required fibre in the 2-skeleton of the triangulations, we need to know the number of simplexes when we put compression disks in the two skeleton of $T \times I$ and finally we also need the triangulation on the boundary at each step to remain simplicially isomorphic.

4.3.1 Outline of the Algorithm

Let T be a compact once-punctured torus. Let $M = T \times I/\nu$, and let T_1 and T_2 be two triangulations of M. Assume that T_1 and T_2 agree on the boundary torus of M. Our algorithm works in the following way:

1. Fix a simple closed minimal weight curve c in ∂M such that c is not homotopic to a point and c bounds a fibre in M.
2. Show that the fibre F bounded by c is fundamental with respect to T_1 and T_2.
3. Subdivide T_1 and T_2 such that F lies in the 2-skeleton of both.
4. Change T_2 to T_2' by Pachner moves such that the triangulation of F induced by T_1 is isomorphic to the triangulation induced by T_2'.
5. Cut T_1 and T_2' along this fibre to get triangulations $T_1^{(1)}$ and $T_2^{(1)}$ of $T \times I = M'$.
6. Find a pair of disjoint compression disks in M' which are fundamental with respect to $T_1^{(1)}$ and $T_2^{(1)}$.
7. Subdivide $T_2^{(1)}$ such that the triangulation of these disks agree with the triangulation induced by $T_1^{(1)}$ on them.
8. Cut $T_1^{(1)}$ and subdivide $T_2^{(1)}$ along these disks to get triangulations $T_1^{(2)}$ and $T_2^{(2)}$ of a 3-ball such that $\partial T_1^{(2)} = \partial T_2^{(2)}$.
9. Use Theorem 4.3.2 to get a bound on the number of Pachner moves required to relate $T_1^{(2)}$ and $T_2^{(2)}$

4.4 Conclusion

As seen in the outline of the algorithm 4.3.1, our goal is to show that

- Boundary minimal weight fibres are fundamental with respect to given triangulations of M.
- Pairs of compression disks are fundamental with respect to triangulations of M'.
- Bounds depend on the triangulations $T_1^{(2)}$ and $T_2^{(2)}$ of the 3-ball.

In Mijatovic's result the number of Pachner moves relating cone over boundary of the 3-ball and the triangulation of 3-ball is a function of the number of 3-simplexes in the triangulation. So for counting the number of Pachner moves, we need the number of 3-simplexes in $T_1^{(2)}$ and $T_2^{(2)}$. This can be counted by keeping track of the subdivisions. Each time we put a surface in the 2-skeleton, we are adding some simplexes to the triangulation using Lemma 4.3.1. We keep a count of these new simplexes and then using Theorem 4.3.2 we attempt to get a bound.

This chapter is an ongoing work, to get a bound on number of Pachner moves relating two triangulations of once-punctured torus bundles. Solutions to above questions will give us our required result.

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