Quadratic Order Constraints on Gravity from Entanglement

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A thesis submitted towards partial fulfilment of

BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Quadratic order constraints on gravity from Entanglement" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Abhijith Gandrakota at Indian Institute of Science, under the supervision of Prof.Aninda Sinha during the academic year 2014-2015.

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Declaration

I hereby declare that the matter embodied in the report entitled "Quadratic order constraints on gravity from Entanglement" are the results of the investigations carried out by me at the Center for High Energy Physics, Indian Institute of Science, under the supervision of Prof.Aninda Sinha and the same has not been submitted elsewhere for any other degree.

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Abstract

Entanglement Entropy in the recent year has been a powerful tool to provide with new insights in various areas of Physics. One class of such attempts was to provide us insights about gravity in the context of gravity. Inspired by the approach taken by Ted Jacobson in deriving the Einstein's equations starting from Thermodynamical argument, We are exploring such a possibility starting from the Quantum informatic arguments.

We take condition of positivity of relative entropy arising from the quantum mechanical arguments and imposing these arguments in the context of holography with the help of Ryu-Takayanagi formulation. Imposing these relative entropy conditions on the Quantum Field theories with Holographic dual, we impose constrains on the gravitational theories allowed in the holographic bulk.

Following the approach developed by A. Sinha et al and A. Kaviraj et al, We systematically introduce Higher derivative perturbations in the Boundary field theory to study the relative entropy and there by obtain constraints at non linear level for gravitational dual. We also compute the greens function required to compute the second order corrections to the Minimal surface area in the bulk homologous to the entangling region of boundary field theory. There by using the greens function we were able to compute the second order corrections to the relative entropy, Thus enabling impose constraints and check in the bulk gravitational theory is necessarly differmorphism Invariant.

Contents

Chapter 1

Introduction

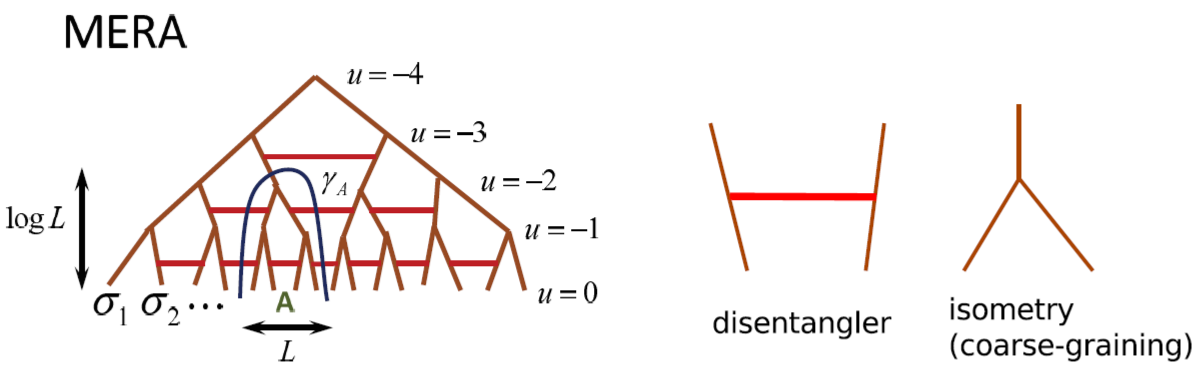
Entanglement Entropy is a measure of Entanglement in a quantum system. The particular formalism of entanglement entropy that is widely used is the Von-Neumann entropy. we follow this formalism through out the calculations. It is defined as following for a quantum subsystem A in the system AB with ρ_A as it's density matrix.

$$S(\rho) = -Tr\rho_A \log \rho_A \tag{1.1}$$

Calculating Entanglement entropy for Quantum Field theory has given very interesting results. It was both analytically and numerically observed that Entanglement entropy is proportional to the Area of the entangling region [6, 7]. This is similar to the blackhole entropy, hence Entanglement entropy was long proposed to be the origin of blackhole entropy.

In the recent past Entanglement Entropy has proved to be a powerful tool to give new insights into the various areas of physics. In the recent years, Entanglement entropy is frequently calculated in condensed matter systems for classification of quantum phases. It is also used as a diagnostic to characterize the quantum critical points and topological phases. Entanglement entropy has been instrumental in exploring various topics in Quantum field theory such as structure of renormalisation group flows and as a useful probe for gauge transitions in gauge theories. It played an important role in establishing c-theorems in three and higher dimensions. In the context of AdS/CFT, Holographic Entanglement Entropy was considered in the Holographic descriptions of quantum gravity and was used to classify holographic field theories. It has been suggested that at a fundamental level it could be used to understand the quantum structure of the space time.

Recently there were very interesting results for holographic Entanglement entropy from MERA (Multiscale Entanglement Renormalisation Ansatz) [5].It was employed to find a ground state of interacting spin system on a one dimensional lattice with 2^m sites. To deal with exponentially large Hilbert space an iterative procedure is used to look for description in fewer effective degrees of freedom, by the process of coarse-graining, where quantum correlation of spins needs to be taken care of. This coarse-graining is achieved through unitary transformation known as "disentangler", which is to remove quantum entanglement in a given scale. This is a naive application of real space renormalisation group on quantum systems moving from UV to IR. Hence, the ground state is described by a structure consisting of course graining and disentanglers acting at different scales. The Iterative steps would be related to the extra dimension of AdS.This is carried on to QFT by cMERA (continuous MERA). Assuming a QFT with a Hamiltonian given, a UV cut off $\Lambda = \frac{1}{\epsilon}$ is imposed, where ϵ is defined as the lattice constant. There is a striking connection between the procedure involved in calculating Holographic Entanglement Entropy and estimation of Entanglement entropy in MERA.



Source of Image : [5]

Where we have the entropy in this case given as,

$$S_a \propto Min_{\gamma_A}[\#Bonds] \tag{1.2}$$

In the past it was shown by Ted Jacobson [8], that it is possible to arrive at derivations of full Einstein's equations starting from pure Thermodynamic arguments. Extending this question, Can one derive Einstein's Equations starting from Quantum Mechanical arguments ? This question makes sense in the light of AdS/CFT correspondence and Quantum Information using the Holographic Entanglement Entropy . Recently there were attempts made to see what Entanglement would teach us about gravity[?, 3, 4]. As we know, Ryu-Tagayanaki Prescription gives us a way to calculate Entanglement Entropy of a Quantum Field theory with gravitational dual. We can turn the question around and understand what Entanglement Entropy of the Quantum Field Theory would tell us about the space-time dynamics of the Holographic Dual without making any assumptions about it. In particular, it would be intriguing to know if the Holographic dual necessarily follows Einstein gravity or allows a wider class of theories.

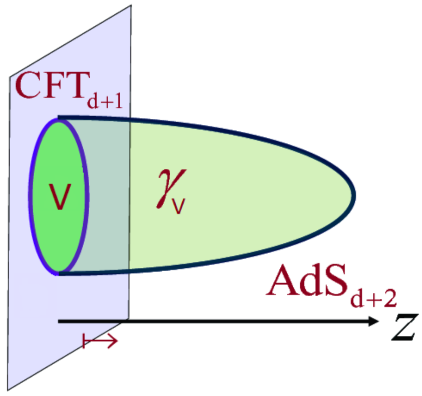
1.1 Theory

1.1.1 Ryu-Tkayanagi Entropy formulation

The Ryu-Takayanagi Prescription gives us an elegant and a simpler way to calculate the Holographic Entanglement entropy. The Entanglement Entropy between a spatial region V in the boundary field theory and it's compliment $\overline{V}$ would be the Von-Neumann entropy of it's density matrix with the degrees of freedom in the region $\overline{V}$ traced out. In the Holographic prescription, to yield the same entropy class of surfaces U are considered. Which extend into the holographic bulk, such that they are homologous to the region V in boundary, such that $\partial V = \partial U$. Then the area of U is extremised to obtain the Entanglement entropy.

$$S(V) = \frac{2\pi}{l_p^{d-1}} \int_{Ext[V]} dx^{d-1}\sqrt{h} \tag{1.3}$$

where we follow the convention that $l_p^{d-1} = 8\pi G_N$ and d is the spacetime dimension in which boundary field theory is present.



Source of Image : [11] An Example calculation for Entanglement Entropy in AdS_3/CFT_2 is show in the following section

1.1.2 Relative Entropy

To probe the questions previously posed, it would be convenient to use another measure, called "Relative Entropy", which is a derivative from Von-Neumann entropy. Relative entropy is the fundamental statistical measure of the 'distance' between two states sharing the same Hilbert space. Relative entropy between two states with density matrices ρ_0 and ρ_1 is given as the following

$$S\left(\rho_1|\rho_0\right) = \text{Tr}\left(\rho_1 \log \rho_1\right) \text{ - Tr}\left(\rho_1 \log \rho_0\right) \tag{1.4}$$

From Quantum Mechanics we know that relative entropy must always be positive and zero only if both the sates are the same.

$$S\left(\rho_1|\rho_0\right) \geq\ 0 \text{ (zero only when } \rho_1 = \rho_0) \tag{1.5}$$

Given a state which is thermal with respect to it's Hamiltonian H, then one can express it's density matrix as $\rho = \dfrac{e^{-H/T}}{Tr(e^{-H/T})}$. In this case we could express the relative entropy between ρ_0 and ρ_1 as the following.

$$S(\rho_1|\rho_0) \ = \ \frac{1}{T}(F(\rho_1) - F(\rho_0)) \quad \text{(Where F is the Free energy.)} \tag{1.6}$$

We also know that Free energy can be represented as $F(\rho) = Tr(\rho H) - T\ S(\rho)$. Hence we'll have

$$S\left(\rho_1|\rho_0\right) = \Delta\langle \text{H}\rangle - \text{T}\ \Delta\text{S} \tag{1.7}$$

The reduced density matrix of a Quantum Field theory on region V could also be written as

$$\rho = \frac{e^{-H}}{Tr(e^{-H})} \qquad \text{(where } H \text{ is a particular Hermitian operator)} \tag{1.8}$$

This is justified as the density matrix would be both Hermitian and Positive semidefinite. The Hermitian operator(H) that would give reduced density matrix is known as 'Modular Hamiltonian', which is not a local operator.The modular hamiltonian is only known for a few cases. The further calculations will require the knowledge of modular hamiltonian for spherical entangling region, which is given as

$$H = \int_{r<R} d^{d-1}x\ \frac{R^2 - r^2}{2R} T_{00} \tag{1.9}$$

where T_{00} is the time-time component of d-dimensional field theory stress tensor.

Considering the expression for the reduced density matrix, it is obvious that the relative entropy in this case would have as the same form as the thermal relative entropy with $T = 1$, which would be

$$S\left(\rho_1|\rho_0\right) = \Delta\langle H\rangle\ -\ \Delta S \tag{1.10}$$

where, $\Delta S = S(\rho_1) - S(\rho_0)$, which we know from the Ryu-Takayanagi prescription will be $\Delta S\ =\ \frac{2\pi}{l_p^{d-1}}\ \Delta\text{Area}[\gamma_a]$ (γ_a is the minimal surface) and $\Delta\langle H\rangle = Tr(\rho_1 H) - Tr(\rho_0 H)$. As previously stated, the positivity of relative entropy will lead to

$$S\left(\rho_1|\rho_0\right) \geq\ 0 \Rightarrow\ \Delta\langle H\rangle \geq \Delta S \tag{1.11}$$

Taking ρ_0 as a fixed state and consider moving ρ_1 through a family of states with an affine parameter λ such that $\rho_1(\lambda = 0) = \rho_0$. Then it is very evident that $S(\rho_1(\lambda)|\rho_0) > 0$, $\forall\ \lambda \neq 0$ and $S(\rho_1(\lambda)|\rho_0) = 0$, for $\lambda = 0$. If $S(\rho_1(\lambda)|\rho_0)$ describes a smooth curve with respect to λ, then the first derivative should vanish at $\lambda = 0$, which implies for nearby states that

$$\Rightarrow \Delta\langle \mathrm{H} \rangle = \Delta \mathrm{S}$$

This would be true for the first order expansion. This inequality as stated above have it's origins from Quantum information. Since it is a inequality, it might not be possible to recover full Einstein's equations starting from it. But this inequality holds the key for imposing constraints on the gravitational theories allowed in the holographic dual.

1.2 Literature Survey

1.2.1 Linear order

It was recently shown by Robert. C. Myers et al in [2], that using the equality $\Delta\langle H \rangle = \Delta S$, it is possible to recover linear Einstein equations for holographic dual at first order. What follows is the summary of their work.

Taking ρ_0 as the density matrix of vacuum state in the spherical region with radius R in the boundary CFT. The extremal surface homologous to this, on the holographic bulk is given by

$$z = \sqrt{R^2 - r^2}, \qquad \text{(where, } r^2 = x_i^2\text{)} \tag{1.12}$$

Taking ρ_1 whose small deviation of the vacuum state ρ_0 is characterised by expectation value of the stress tensor $T^0_{\mu\nu}$ in the boundary CFT. For general analysis we use Fefferman-Graham expansion for the metric defining the bulk, given as,

$$ds^2 = \frac{L^2}{z^2}(dz^2 + g_{\mu\nu}\ dx^\mu dx^\nu) \tag{1.13}$$

When $z \simeq 0$, the above would describe asymptotic geometry. As the asymptotic metric is chosen to be flat, we may write

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\delta g_{\mu\nu} \tag{1.14}$$

Here ϵ controls the strength of perturbation. As FG expansion is used, the deviation of the bulk metric from the pure AdS in eq. 1.14 takes the form

$$\delta g_{\mu\nu} = \frac{2}{d}\frac{l_P^{d-1}}{L^{d-1}} z^d \sum_{n=0} z^{2n} T_{\mu\nu}^{(n)} \tag{1.15}$$

On solving the Einstein's equations with the perturbed metric, we have,

$$T_{\mu\nu}^{(n)} = \frac{(-1)^n \Gamma[d/2+1]}{2^{2n} n! \Gamma[d/2+n+1]} \Box^n T_{\mu\nu} \tag{1.16}$$

With the above expansion of the metric, on calculating ΔS and $\Delta\langle H\rangle$ to the linear order, it is found that

$$\Delta S = \Delta\langle H\rangle \tag{1.17}$$

Thus it was observed that by using positivity of relative entropy it is possible to retrieve Einstein's equations at linear order.

1.2.2 Quadratic order

As we have seen in the previous attempt, from the positivity of the relative entropy, at the first order,only linear Einstein's equations have been recovered. A step further has been taken in the work done by A. Sinha et al , to move to the quadratic order in [3]. Following is the summary of their work.

In the fist order, we have $\Delta S = \Delta\langle H\rangle$. But we need $\Delta\langle \mathrm{H}\rangle \geq \Delta\mathrm{S}$. So, in the second order we'll have

$$\Delta^{(2)} S < 0 \tag{1.18}$$

And to calculate the quadratic correction to the entanglement entropy, the metric 1.14 needss to be further perturbed as given

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon\delta \mathrm{g}_{\mu\nu} + \epsilon^2 \delta^{(2)} g_{\mu\nu} \tag{1.19}$$

which expands as,

$$\delta^{(2)} g_{\mu\nu} = a^2 z^{2d} \left(n_1 T_{\mu\alpha} T^{\alpha}{}_{\nu} + n_2 \eta_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta}\right) \text{ (where } (a{=}\frac{2}{d}\frac{{l_p}^{d-1}}{L^{d-1}}) \text{ and } T_{\mu\nu} \text{ is constant.)} \tag{1.20}$$

when Einstein's equation is solved with the above metric, then n1 and n2 obtain the following values,

$$n_1 = \frac{1}{2}; \quad n_2 = -\frac{1}{8(d-1)} \quad \text{where } d{=}4 \tag{1.21}$$

As we are moving on the quadratic order in T , extremal surface gets perturbed, as bulk is altered. In the linear order case, minimal surface could be described by $z(x^i)$ in a simple way 1.12 as function of radial coordinate of boundary theory. Due to the present perturbative expansion, we can expand z as

$$z(x^i) = z_0(x^i) + \epsilon z_1(x^i) \qquad \text{where } z_0(x^i) = \sqrt{R^2 - r^2} \tag{1.22}$$

From the ryu-Takayanagi prescription 1.3, we know that entropy depends on the $\sqrt{h}$ (h is the induced metric). To compute the quadratic correction to the entropy, $\sqrt{h}$ is Taylor expanded, which leads to

$$\int d^{d-1}\delta^{(2)}\sqrt{h} = \int d^{d-1}\left(\frac{1}{8}\sqrt{h}\left(h^{\mathrm{ij}}\delta \mathrm{h}_{\mathrm{ij}}\right)^2 + \frac{1}{4}\sqrt{h}\delta \mathrm{h}^{\mathrm{ij}}\delta \mathrm{h}_{\mathrm{ij}} + \frac{1}{4}\sqrt{h}h^{\mathrm{ij}}\delta^{(2)}h_{\mathrm{ij}}\right) \tag{1.23}$$

The induced metric is given as

$$h_{ij} = \frac{L^2}{z^2}(g_{ij} + \partial_i z \partial_j z) \tag{1.24}$$

The above is evaluated at extremal surface $z = z_0 + \epsilon z_1 \quad (z_0 = \sqrt{R^2 - r^2})$. The contributions of $\Delta^2 S$ is categorised into 3 second order contributions based on powers of z_1 as the following,

$$\int d^{d-1}\delta^{(2)}\sqrt{h} = A_{(2,0)} + A_{(2,1)} + A_{(2,2)} \tag{1.25}$$

Thus the z_1 can be found by minimizing $A_{(2,1)} + A_{(2,2)}$. This gives us,

$$z_1 = \frac{-aR^2 {z_0}^{d-1}}{2(d+1)}(T + T_x) \;\; (\text{where } T = {T_i}^i \;, T_x = T_{\mathrm{ij}}\frac{x^i x^j}{R^2}) \tag{1.26}$$

Plugging the solution of z_1, calculating the $\Delta^{(2)}S$ and imposing the condition $\Delta^{(2)}S < 0$

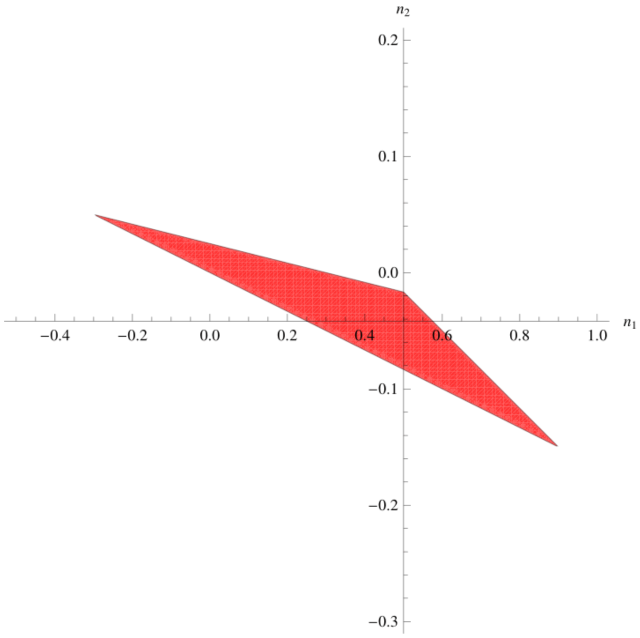
gives the folloing inequality,

$$n_1 + 2(d-1)n_2 \geq 0\,, \tag{1.27}$$

$$2d + 1 - 4(d+1)n_1 - 4(d^2-1)n_2 \geq 0\,, \tag{1.28}$$

$$d + 2 - 4(d+1)n_1 - 4d(d^2-1)n_2 \geq 0\,, \tag{1.29}$$

This inequality represents a triangle in the parameter space of n_1 and n_2



Source of Image : [4]

1.2.3 Non-constant stress Tensor

As we have seen that, in quadratic case we have not obtained the Einstein's equations rather obtained constraints on the class of gravitational theories allowed in the holographic bulk This is due to the inequality. The following work by Aprathim Kaviraj et al in [4], is to minimise these constraints by taking the non-constant stress tensor and obtaining non-linear constraints.The ρ_o is still taken as the same.

Taking the stress tensor to be non-constant will add more terms to $\delta^{(2)}g_{\mu\nu}$ at the quadratic order. In this paper, terms with maximum of two derivatives are considered. Hence it would change the metric as follows,

$$\delta^{(2)} g_{\mu\nu} = a^2 z^{2d} \left(n_1 T_{\mu\alpha} T^{\alpha}{}_{\nu} + n_2 \eta_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} + z^2 \mathcal{T}_{\mu\nu}{}^{(1)} \right). \tag{1.30}$$

Where $\mathcal{T}_{\mu\nu}$ expands as

$$\begin{aligned}\mathcal{T}_{\mu\nu} =& n_3 \left(T_{\mu\alpha} \Box T_{\nu}{}^{\alpha} + T_{\nu\alpha} \Box T_{\mu}{}^{\alpha} \right) + n_4 \eta_{\mu\nu} T_{\alpha\beta} \Box T^{\alpha\beta} + n_5 \partial_\mu T_{\alpha\beta} \partial_\nu T^{\alpha\beta} + n_6 \partial_\alpha T_{\mu\beta} \partial^\beta T_\nu{}^\alpha \\ &+ n_7 \partial_\mu \partial_\nu T_{\alpha\beta} T^{\alpha\beta} + n_8 \partial_\alpha T_{\mu\beta} \partial^\alpha T_\nu{}^\beta + n_9 \left(\partial_\mu T_{\alpha\beta} \partial^\beta T_\nu{}^\alpha + \partial_\nu T_{\alpha\beta} \partial^\beta T_\mu{}^\alpha \right) + n_{10} \eta_{\mu\nu} \partial_\alpha T_{\beta\gamma} \partial^\alpha T^{\beta\gamma} \\ &+ n_{11} \partial_\alpha T_{\gamma\beta} \partial^\beta T^{\gamma\alpha} \eta_{\mu\nu} + n_{12} \left(T^{\beta\alpha} \partial_\alpha \partial_\mu T_{\nu\beta} + T^{\beta\alpha} \partial_\alpha \partial_\nu T_{\mu\beta} \right) + n_{13} T^{\alpha\beta} \partial_\alpha \partial_\beta T_{\mu\nu} \end{aligned} \tag{1.31}$$

on solving the Einsteins equation with the above metric, we'll that the n's obtain the following values,

$$\begin{aligned} & n_3 = -\frac{1}{24}, n_4 = \frac{1}{180}, n_5 = -\frac{1}{180}, n_6 = -\frac{1}{60}, n_7 = \frac{1}{360}, \\ & n_8 = 0, n_9 = \frac{1}{120}, n_{10} = \frac{1}{720}, n_{11} = 0, n_{12} = -\frac{1}{120}, n_{13} = \frac{1}{60} \end{aligned} \tag{1.32}$$

As in the previous case, minimising $A_{(2,1)} + A_{(2,2)}$ with respect to z_1 gives the following equation.

$$\begin{aligned} & \frac{1}{{z_0}^{d-1} R} \left(\partial^2 \left(z_0 z_1 \right) - \frac{x^i x^j}{R^2} \partial_i \partial_j \left(z_0 z_1 \right) \right) = \\ & \frac{z_0}{2R} \left(T \left(d-2 \right) + T_x \left(d+2 \right) - \frac{z_0^2}{12} \left(d \partial^2 T + (d+4) \frac{x^i x^j}{R^2} \partial^2 T_{ij} \right) + x^i \partial_i T + 2 x^i \partial_0 T_{0j} + \frac{1}{R^2} x^i x^j x^k \partial_k T_{ij} \right), \end{aligned} \tag{1.33}$$

The solution of z_1 that satisfies the above equation with other constraints is the following (the process of obtaining this solution will be explained in the coming chapters)

$$\begin{aligned} z_1 = -{z_0}^3 R^2 & \left(\frac{T + T_x}{10} + \frac{1}{12} \left(x^i \partial_i T + x^i x^j x^k \frac{\partial_k T_{ij}}{R^2} \right) \right. \\ & \left. + \frac{1}{28} \left(x^i x^j \partial_i \partial_j T + x^i x^j x^k x^l \frac{\partial_i \partial_j T_{kl}}{R^2} \right) - \frac{(R^2 - r^2)}{168} \left(\partial^2 T + x^i x^j \frac{\partial^2 T_{ij}}{R^2} \right) \right). \end{aligned} \tag{1.34}$$

On calculating $\Delta^{(2)}S$ we get that,

$$\Delta^{(2)}S_1 = \frac{8\pi^2 L^3 R^8}{4725\ell_P^3}\left(-160(n_1+6n_2)\left(T_{i0}\right)^2 + 8(-9+20n_1+60n_2)\left(T_{ij}\right)^2 + 8(1+60n_2)T^2\right) +$$
$$+\frac{8\pi^2 L^3 R^{10}}{31185\ell_P^3}\left[(10-12n_1+2160n_{11}+720n_6+1440n_9)\left(\partial_i T_{jk}\partial^k T^{ji}\right) + 48\left(7n_2+45n_4+15n_7\right)T\partial^2 T\right.$$
$$+\left(-120n_1-672n_2-1440n_3-4320n_4-1440n_7\right)T^{0i}\partial^2 T_{0i} + \left(-12+720n_{13}\right)T^{ij}\partial_i\partial_j T$$
$$+\left(-55+120n_1+2160n_{10}+336n_2+720n_5+720n_8\right)\left(\partial_i T_{jk}\right)^2 + \left(12n_1-2160n_{11}-1440n_9\right)\partial_i T_{0j}\partial^j T^{0i}$$
$$+\left(5+2160n_{10}+336n_2+720n_5\right)\left(\partial_i T\right)^2 + \left(120n_1+336n_2+1440n_3+2160n_4+720n_7\right)T^{ij}\partial^2 T_{ij}$$
$$\left.+\left(-120n_1-4320n_{10}-672n_2-1440n_5-720n_8\right)\left(\partial_i T_{0j}\right)^2\right]. \tag{1.35}$$

substituting the values from 1.32, it reduces to

$$\Delta^{(2)}S = -16\pi^2 R^{10}\frac{L^3}{\ell_P^3}\left[\frac{6T^2+20\left(T_{i0}\right)^2+6\left(T_{ij}\right)^2}{4725R^2}\right.$$
$$\left.+\frac{\left(5\left(\partial_i T\right)^2+15(\partial_i T_{0j})^2+3\partial_i T_{0j}\partial^j T_0{}^i+5(\partial_i T_{jk})^2-2\partial_i T_{kj}\partial^k T^{ij}\right)}{31185}\right]. \tag{1.36}$$

As there are no $T\partial\partial T$ terms present, it is possible to show that when Einstein values are substituted the $\Delta^2 S$ can be shown as a negative definite quantity by completing the squares. Due this would not be possible in the 1.35 case, as such terms are present and $\partial\partial T\partial\partial T$ are also required to complete the squares. Hence, by making an assumption that $T_{\mu\nu}(\vec{x}=0)=0$, reducing the parameter space to n_1 and n_2 and imposing the condition $\Delta^{(2)}S < 0$ gives us the new constrain region,

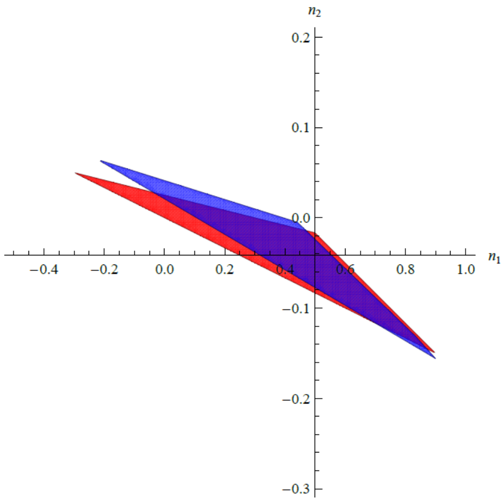


Figure 1.1: The blue (upper) triangle is from constant stress tensor case and the red one from non-constant case. The intersecting part is the net allowed region for n_1 and n_2. Source of Image : [4]

1.3 Motivation

As we have seen in the various scenarios above, the Einsteins equations starting from the positivity of relative entropy are recovered in linear order in $\mathcal{O}(T)$. Where as on moving to the quadratic order, we see that we obtain constraints for a class of gravitational theories allowed in the holographic dual. To recover Einstein's equation at the quadratic order, the constrained area in the parameter space of n_i 's has to be reduced to the Einstein values/point. And the $\Delta^{(2)}S$ needs to be shown as a negative definite quantity. This was not possible due to the terms of order $\mathcal{O}(T\partial\partial T)$ are present. As they have made the assumption $T_{\mu\nu}(\vec{x}=0)=0$, which restricts the stress tensor to a special class, resulted in a new triangular region of constrained area which is rotated around the old at the Einstein point as in 1.1

As initially stated, we are not assuming anything about the holographic bulk, hence we don't even take diffeomorphism of the Bulk into consideration. Thus, we need to add higher derivative ($4-\partial$'s acting on $2-T$'s) perturbation to the metric, so that we'll have $\partial\partial T\partial\partial T$ terms to complete the squares. By doing so, we still don't have to restrict ourselves with a particular class of stress tensors such as $T_{\mu\nu}(\vec{x}=0)=0$, but can consider a wider variety of them. As a result, we may end up new constrained areas as the following,

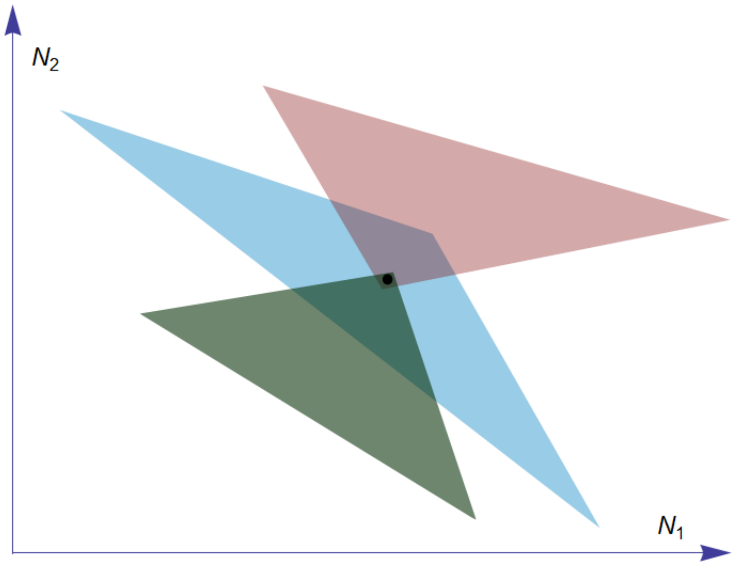


Figure 1.2: Here we have the Blackdot representing the Einstein point. Blue - representing the old case and red, green representing the new one

Doing so would help us constrain the allowed region in the parameter space to a great extent around the Einstein values and show that the . It might also happen that we can recover the full Einstein's equations and show that the bulk theory is diffeomorphism invariant up till the quadratic order . But we should remind ourselves that as we are dealing with inequality, that we also end up with parameter space restricting itself to the unbounded space.

Chapter 2

Higher derivative perturbations

In order to constrain the area around the parameter space, it is required to move to higher derivative perturbation. In this chapter we initially deal with introducing Higher Derivative perturbation and solving the Einstein's equations using the new perturbed metric. Further on we use the new induced metric to find the Equations of motion for z_1. Later, a method is illustrated on how to obtain the solution for new z_1 and the solution is found.

2.1 New terms to metric

Adding Higher derivative perturbation would mean to add new terms to metric in quadratic order in T with $4-\partial$'s acting on them. Hence the metric 1.19 would get perturbed as,

$$\delta^{(2)} g_{\mu\nu} = a^2 z^{2d} \left(n_1 T_{\mu\alpha} T^{\alpha}{}_{\nu} + n_2 \eta_{\mu\nu} T_{\alpha\beta} T^{\alpha\beta} + z^2 \mathcal{T}_{\mu\nu}{}^{(1)} + z^4 \mathcal{T}_{\mu\nu}{}^{(2)} \right) . \tag{2.1}$$

$\mathcal{T}^{(2)}_{\mu\nu}$ is constructed by taking into account all the possibilities of $4-\partial$s acting on $2-T$s, the total non-zero contributions to the metric would be the following

$$
\begin{aligned}
\mathcal{T}^{(2)}_{\mu\nu} =\; & n_{14}\;\eta_{\mu\nu}\Box T_{ab}\Box T^{ab} + n_{15}\;\eta_{\mu\nu}\partial_a\partial_b T^{cq}\partial_c\partial_q T^{ab} + n_{16}\;\eta_{\mu\nu}\partial_a\partial_b T^{cq}\partial^a\partial^b T_{cq} + n_{17}\;\eta_{\mu\nu}\partial_a\partial_b T^{cq}\partial^a\partial_c T^b_q \\
& + n_{18}\;\partial_a\partial_b T_{\mu\nu}\Box T^{ab} + n_{19}\;\partial_a\partial_c T^b_\mu\partial^c\partial_b T^a_\nu + n_{20}\;\Box T_{\mu a}\Box T^a_\nu + n_{21}\;\partial_b\partial_c T_{\mu a}\partial^c\partial^b T^a_\nu + n_{22}\;\partial_\mu\partial_\nu T_{ab}\Box T^{ab} \\
& + n_{23}\;\partial_c\partial_\mu T^{ab}\partial_a\partial_\nu T^c_b + n_{24}\;\partial_a\partial_\mu T^{bc}\partial^a\partial_\nu T_{bc} + n_{25}\;(\partial_c\partial_\mu T_{\nu a}\Box T^{ac}) + n_{26}\;(\partial_a\partial_\mu T^b_c\partial^a\partial_b T^c_\nu) \\
& + n_{27}\;(\partial_a\partial_\mu T^{bc}\partial_c\partial_b T^a_\nu) + n_{28}\;\eta_{\mu\nu}\Box\partial_c T_{ab}\partial^c T^{ab} + n_{29}\;\eta_{\mu\nu}\Box\partial_a T_{bc}\partial^c T^{ab} + n_{30}\;\partial_a\partial_b\partial_c T_{\mu\nu}\partial^c T^{ab} \\
& + n_{31}\;(\Box\partial_b T_{\mu a}\partial^b T^a_\nu) + n_{32}\;(\Box\partial_a T^b_\mu\partial^b T^a_\nu) + n_{33}\;\partial_c\partial_\mu\partial_\nu T^{ab}\partial^c T_{ab} + n_{34}\;\partial_c\partial_\mu\partial_\nu T^{ab}\partial^a T^c_b \\
& + n_{35}\;(\Box\partial_\mu T^{ab}\partial^\nu T_{ab}) + n_{36}\;(\partial_a\partial_b\partial_\mu T_{\nu c}\partial^c Tab) + n_{37}\;(\partial_a\partial_b\partial_\mu T_{\nu c}\partial^b Tac) + n_{38}\;(\Box\partial_\mu T^{ab}\partial_a T\nu b) \\
& + n_{39}\;\eta_{\mu\nu}\Box^2 T_{ab}T^{ab} + n_{40}\;\Box\partial_a\partial_b T_{\mu\nu}T^{ab} + n_{41}\;(\Box^2 T_{\mu a}T^a_\nu) + n_{42}\;\Box\partial_\mu\partial_\nu T_{ab}T^{ab} \\
& + n_{43}\;(\Box\partial_a\partial_\mu T_{\nu b}T^{ab}) + n_{44}\;(\Box\partial_b T_{\mu c}\partial_\nu T^{bc}) \qquad (2.2)
\end{aligned}
$$

The terms in brackets are symmetrised by addition.

2.1.1 Solving for Einstein values

The coefficients infront of each term, in the higher derivative perturbation needs to be found. One primary reason being that, to check if $\Delta^{(2)}S$ is satisfying the negativity condition on Einstein values and secondly, to see effect of new terms on n_1 and n_2 , so a comparative study could be made with the previous cases. As we could see, without a efficient way, it would be nearly impossible to fix these coefficients by taking arbitrary stress tensor.

Hence, we take special cases of Stress tensor, i.e $T_{\mu\nu}$ and use them as the trial functions to solve the Einstein equation for empty AdS. The trial functions as taken as,

$$
T_{\mu\nu} = \alpha_{\mu\nu} e^{(\beta^{(1)}_{\mu\nu}x^n \;+\; \beta^{(2)}_{\mu\nu}y^n \;+\; \beta^{(3)}_{\mu\nu}w^n)\kappa} \qquad \text{(where x,y,w are space cordinates of CFT)} \qquad (2.3)
$$

Where we took, $\alpha_{\mu\nu}0$ or 1 to either turn on or off the component and $\beta^{(i)}_{\mu\nu} = (0, 1)$ and $n = 1, 2, 3, 4$. These values are chosen such that, we have the stress tensor to be Traceless $(T^i_i = 0)$ and divergence less $(\partial_i T^{ij} = 0)$.

As they are taken as infinitesimal perturbation, we can set $R_{\mu\nu} - \frac{1}{2}R\ g_{\mu\nu} + \Lambda\ g_{\mu\nu}$ to zero. By this way on solving Einstein's equations we can segregate coefficients of exponential functions to a particular order in κ depending on n and set them to zero. This would give us multiple linear equations in terms of n_i's which would be easier to solve. Hence doing so, we'll get the following values for the coefficients of higher order perturbations

$$\begin{aligned}
&n_{14} = -\frac{1}{4608} - \alpha\ ,\ n_{15} = \frac{1}{34560} + \alpha\ ,\ n_{16} = \alpha\ ,\ n_{17} = -\frac{1}{17280} - 2\alpha\ ,\ n_{18} = -\frac{17}{17280} + 2\alpha\ ,\\
&n_{19} = \frac{7}{17280} + 2\ ,\ n_{20} = \frac{61}{17280} + 2\alpha\ ,\ n_{21} = -\frac{1}{17280} - 2\alpha\ ,\ n_{22} = -\frac{1}{4320} + 2\alpha\ ,\ n_{23} = \frac{1}{17280} + 2\alpha\ ,\\
&n_{24} = \frac{1}{17280} - 2\alpha\ ,\ n_{25} = \frac{19}{138240} - \frac{\alpha}{2}\ ,\ n_{26} = -\frac{1}{8640} + 2\alpha\ ,\ n_{27} = -\frac{1}{17280} - 2\alpha\ ,\ n_{28} = -\frac{1}{6912}\ ,\ n_{29} = 0\ ,\\
&n_{30} = -\frac{1}{2880}\ ,\ n_{31} = 0\ ,\ n_{32} = \frac{1}{960}\ ,\ n_{33} = -\frac{1}{17280}\ ,\ n_{34} = 0\ ,\ n_{35} = \frac{11}{34560}\ ,\ n_{36} = 0\ ,\ n_{37} = \frac{1}{5760}\ ,\\
&n_{38} = -\frac{1}{2304}\ ,\ n_{39} = -\frac{1}{6912}\ ,\ n_{40} = -\frac{1}{960}\ ,\ n_{41} = \frac{1}{768}\ ,\ n_{42} = -\frac{1}{8640}\ ,\ n_{43} = \frac{1}{2304}\ ,\ n_{44} = -\frac{7}{11520}
\end{aligned} \tag{2.4}$$

Where α is found to be a free parameter.Due to diffeomorpism we expects all the values of n's to be fixed, but this α seems to be a gauge artefact, which on selection of proper gauge transformation can be gauged away.

To check the above values,a stress tensor with arbitrary functions can be taken such that it satisfies trace, divergence less and symmetry conditions. Such as

$$T_{\mu\nu} = f_{\mu\nu}(x, y, w) \tag{2.5}$$

We were able to solve Einstein's equations, on assuming above conditions and the values obtained to see that they perfectly agree and go to zero.Hence establishing a check for the above values.

2.2 EOM for z_1

Since higher order perturbations are introduced to the metric, this will in turn change the induced metric for extremal surface in the holographic dual. This would mean new Equations of motion for z_1 has to be derived for the current scenario by minimising $A_{(2,1)} + A_{(2,2)}$.

2.2.1 Induced metric

The induced metric for the extremal surface can be defined as the following.

$$h_{ij} = \frac{L^2}{z^2}(g_{ij} + \partial_i z \partial_j z) \tag{2.6}$$

Here we have g_{ij} expanded as $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon \delta \mathrm{g}_{\mu\nu} + \epsilon^2 \delta^{(2)} g_{\mu\nu}$ and z gets expanded as $z = z_0 + \epsilon z_1$. On collecting terms expanding the above equation and segregating with different powers of ϵ we'll have h^0_{ij} , δh_{ij} and $\delta^{(2)} h_{ij}$, which is give as the following

$$\begin{aligned}
h^0_{ij} &= \frac{L^2}{z_0^2}(\eta_{ij} + \frac{x_i x_j}{z_0^2})\,, && (2.7)\\
\delta h_{ij} &= \frac{L^2}{z_0^2}(\delta g_{ij} - \frac{x_i \partial_j z_1}{z_0} - \frac{x_j \partial_i z_1}{z_0} - \frac{2z_1}{z_0}(\eta_{ij} + \frac{x_i x_j}{z_0^2}))\,, && (2.8)\\
\delta^{(2)} h_{ij} &= \frac{L^2}{z_0^2}(\delta^{(2)} g_{ij} + \partial_i z_1 \partial_j z_1 \frac{2z_1}{z_0}(\delta g_{ij} - \frac{x_i \partial_j z_1}{z_0} - \frac{x_j \partial_i z_1}{z_0}) + \frac{3z_1^2}{z_0^2}(\eta_{ij} + \frac{x_i x_j}{z_0^2}))\\
&\quad + z_1 \sum_{n=0}^{2} (d+2n) z_0^{d+2n-1} (T^{(n)}_{ij}) && (2.9)
\end{aligned}$$

As we know that induced metric satisfies the condition $h_{ij} h^{jk} = h^0_{ij} (h^0)^{jk} = \delta^k_i$. So from this we'll have ,

$$\begin{aligned}
h^{0\ ij} &= \frac{z_0^2}{L^2}(\eta_{ij} - \frac{x_i x_j}{R^2})\,, && (2.10)\\
\delta h^{ij} &= \delta h_{lm}\ h^{0\ il}\ h^{0\ mj} && (2.11)
\end{aligned}$$

With these expressions of induced metric, it would easy to expand and calculate $\delta^{(2)}\sqrt{h}$, by estimating $A_{(2,0)}$,$A_{(2,1)}$and$A_{(2,1)}$

Calculating EOM of z_1

As we know that the second order perturbation of $\sqrt{h}$ expands as

$$\int d^{d-1} \delta^{(2)} \sqrt{h} = \int d^{d-1} \left(\frac{1}{8}\sqrt{h}\left(h^{\mathrm{ij}} \delta \mathrm{h}_{\mathrm{ij}}\right)^2 + \frac{1}{4}\sqrt{h}\delta \mathrm{h}^{\mathrm{ij}} \delta \mathrm{h}_{\mathrm{ij}} + \frac{1}{4}\sqrt{h} h^{\mathrm{ij}} \delta^{(2)} h_{\mathrm{ij}} \right) \tag{2.12}$$

On segregating the resultant expression in the powers of z_1, we would get,

$$\int d^{d-1} \delta^{(2)} \sqrt{h} = A_{(2,0)} + A_{(2,1)} + A_{(2,2)} \tag{2.13}$$

For calculating the Equations of Motion for z_1, we need only $A_{(2,1)}$ and $A_{(2,2)}$ as the other terms would not contain z_1 or $\partial_i z_1$. On plugging in Induced metric expressions 2.7 in the Taylor expansion of $\sqrt{h}$ 2.12, we get that

$$\begin{aligned} A_{2,1} = L^{d-1} a \int d^{d-1}x \frac{R}{2z_0} \Bigg(T \left(z_1 - \frac{z_0^2}{R^2} x^i \partial_i z_1 \right) - \frac{z_0^2}{12} \partial^2 T \left(3z_1 - \frac{z_0^2}{R^2} x^i \partial_i z_1 \right) \\ + \frac{z_0^4}{384} \partial^4 T \left(5z_1 - \frac{z_0^2}{R^2} x^i \partial_i z_1 \right) + T_{ij} \left(2z_0^2 x^i \partial^j z_1 / R^2 - \frac{z_1 x^i x^j}{R^2} - \frac{z_0^2 x^i x^j x^k \partial_k z_1}{R^4} \right) \\ - \frac{z_0^2}{12} \partial^2 T_{ij} \left(2z_0^2 x^i \partial^j z_1 / R^2 - 3\frac{z_1 x^i x^j}{R^2} - \frac{z_0^2 x^i x^j x^k \partial_k z_1}{R^4} \right) \\ + \frac{z_0^4}{384} \partial^4 T_{ij} \left(2z_0^2 x^i \partial^j z_1 / R^2 - 5\frac{z_1 x^i x^j}{R^2} - \frac{z_0^2 x^i x^j x^k \partial_k z_1}{R^4} \right) \Bigg) . \end{aligned} \tag{2.14}$$

and

$$A_{2,2} = L^3 \int d^3x \frac{R}{z_0^4} \left(\frac{d(d-1){z_1}^2}{{z_0}^2} + \frac{{z_0}^2 \left(\partial_i z_1\right)}{2R^2} - \frac{{z_0}^2}{2R^4} \left(x^i \partial_i z_1\right)^2 + \frac{(d-1)}{R^2} z_1 x^i \partial_i z_1 \right) . \tag{2.15}$$

As we are taking boundary spacetime dimensions to be 4, we set d=4. and now we need to minimise the sum of above two with respect to z_1 by substituting in the Euler-Lagrange, i.e

$$\frac{\partial \mathcal{L}}{\partial z_1} - \partial_i \left(\frac{\partial \mathcal{L}}{\partial \partial_i z_1} \right) = 0 \qquad \text{with} \qquad \mathcal{L} = A_{2,1} + A_{2,2} \,. \tag{2.16}$$

On solving the above equation, we get equations of motion for z_1 as the following,

$$\begin{aligned} \frac{1}{{z_0}^{d-1} R} \left(\partial^2 \left(z_0 z_1\right) - \frac{x^i x^j}{R^2} \partial_i \partial_j \left(z_0 z_1\right) \right) = \\ \frac{z_0}{2R} \Bigg(T\left(d-2\right) + T_x \left(d+2\right) - \frac{z_0^2}{12} \left(d\partial^2 T + (d+4) \frac{x^i x^j}{R^2} \partial^2 T_{ij} \right) \\ + \frac{z_0^4}{384} \left((d+2) \Box^2 T + (d+6) \frac{x^i x^j}{R^2} \Box^2 T_{ij} \right) + x^i \partial_i T + 2x^i \partial_0 T_{0j} + \frac{1}{R^2} x^i x^j x^k \partial_k T_{ij} \\ - \frac{z_0^2}{12} \left(x^i \partial_i \Box T + 2x^i \partial_0 \Box T_{0j} + \frac{1}{R^2} x^i x^j x^k \partial_k \Box T_{ij} \right) \Bigg) , \end{aligned} \tag{2.17}$$

As in the previous notation, $T = T^i_i$ and $T_x = x^i x^j \frac{T_{ij}}{R^2}$. Hence, a solution for z_1 should satisfy the above equations of motion.

2.3 Methodology to find z_1 solution

Here we illustrate and follow the method of finding z_1 developed by Kaviraj et al in [4].

The minimality constrain that we obtained above can be written as the following,

$$z_0 \left(\partial^2 (z_0 z_1) - x^i x^j \partial_i \partial_j (z_0 z_1)\right) = J \,. \tag{2.18}$$

The J we have, contains the details of how bulk metric is deformed due to perturbations of stress-tensor from the boundary. By using a particular coordinate transformation, which will be illustrated in the latter part of the section, the above equation can be written as

$$(-\Delta_{H^3} + 3) z_1 = J \,, \tag{2.19}$$

where Δ_{H^3} is scalar Laplacian on AdS_3. Since we are dealing with the time-independent perturbations and then parametrise minimal surface we'll have the AdS_3 Laplacian instead of a 5 dimensional one. By the form of above equation , z_1 can be thought of as a scalar field propagating on the z_0 surface. So, we can see that it's a equation of field propagating AdS_3 with $m = 3$, m being the mass of it. By this the general solution can be written as,

$$z_1 = \int G_{bulk-bulk} J + f_{hom} \,, \tag{2.20}$$

$G_{bulk-bulk}$ is the bulk to bulk propagator for a massive scalar field. As the $z_1$vanishes on the entangling surface, we can set f_{hom} to zero.

2.3.1 Greens function for z_1

From the above, the general solution for z_1 can be written as,

$$z_1(x) = \int d\mu_{\hat{x}} G(x, \hat{x}) J(\hat{x}) \,, \tag{2.21}$$

As we could see that $d\mu_{\hat{x}}$ is the riemannian volume element on AdS_3 and x collectively describes the intrinsic coordinates, An AdS_3 of unit radius is described by a hyperboloid Minkowski space $R^{3,1}$ given by,

$$X_1^2 + X_2^2 + X_3^2 - X_4^2 = -1 \,, \tag{2.22}$$

where the X_i's are coordinates of $R^{3,1}$. Thus we can take the intinsic coordinates of AdS_3 as

$$X_1 = \sin\theta\ \cos\phi\ \sinh\eta,\ X_2 = \sin\theta\ \sin\phi\ \sinh\eta,\ X_3 = \cos\theta\ \sinh\eta,\ X_4 = \cosh\eta\,. \tag{2.23}$$

Where we have $\eta = \tanh^{-1}(r)$ and θ,ϕ are angular coordinates.The metric in terms of these coordinates is the following

$$ds^2 = d\eta^2 + \sinh^2\eta(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{2.24}$$

The geodesic distance between two points x and $\hat{x}$ on AdS_3 is given by the relation

$$\cosh\ d(x,\hat{x}) = -X.\hat{X}\,, \tag{2.25}$$

we have X and $\hat{X}$ are respective intrinsic coordinates. As the greens function is the bulk-bulk propagator for a scalar field with $m^2 = 3$(as for present case this is the value of m^2 2.20) is given by,

$$G(x,\hat{x}) = \frac{1}{4\pi}\frac{e^{-2d(x,\hat{x})}}{\sinh\ d(x,\hat{x})}\,. \tag{2.26}$$

We can take point x to have $\eta = 0$, which translates to $X_1 = X_2 = X_3 = 0, X_4 = 1$. The calculations at this point would get simpler as it is the lowest point in the hyperboloid and we'll have $\cosh d = \cosh\hat{\eta}$. As we know, $\hat{\eta}$ is the intrinsic coordinates corresponding to $\hat{x}$. So, we'll have

$$z_1(0) = \int d\hat{\Omega}_2 \int_0^\infty d\hat{\eta}\sinh^2\hat{\eta}\frac{1}{4\pi}\frac{e^{-2\hat{\eta}}}{\sinh\ \hat{\eta}}J(\hat{\eta}) \qquad \text{where} \quad J(\hat{\eta}) = \text{sech}^5\hat{\eta}(1+\tanh^2\hat{\eta})\,. \tag{2.27}$$

The process evaluating the above integral around origin is justifiable as we can always make a coordinate transformation such that the point of evaluation is origin.

2.4 Solution for z_1

As We have in the above equations, 2.18 and 2.17. We know that from the minimality constrain the source function that we would obtain will be the following,

$$J = \frac{z_0}{2R}\left(T\left(d-2\right) + T_x\left(d+2\right) - \frac{z_0^2}{12}\left(d\partial^2 T + (d+4)\frac{x^i x^j}{R^2}\partial^2 T_{ij}\right)\right.$$
$$+\frac{z_0^4}{384}\left((d+2)\Box^2 T + (d+6)\frac{x^i x^j}{R^2}\Box^2 T_{ij}\right) + x^i\partial_i T + 2x^i\partial_0 T_{0j} + \frac{1}{R^2}x^i x^j x^k \partial_k T_{ij}$$
$$\left. -\frac{z_0^2}{12}\left(x^i\partial_i\Box T + 2x^i\partial_0\Box T_{0j} + \frac{1}{R^2}x^i x^j x^k\partial_k\Box T_{ij}\right)\right)\,, \quad (2.28)$$

Now we make use of the formula 2.27 with the above source to find z_1. To make calculation simpler we move to the Fourier space we write the above J in the form of derivatives of exponentials to find the solution.

The stress tensor is taken to be of the form,

$$T_{ij} = \epsilon_{ij}(\vec{k})e^{i\vec{k}.\vec{x}}, \quad \epsilon_{ij}(\vec{k})k_j = 0\,. \quad (2.29)$$

On Fourier transforming the source, z_1 is given by

$$z_1 = \left(2\epsilon - \frac{6\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j}{R^2}\right) - \frac{1}{12}\left(4\left(-k^2\right)R^2\epsilon - k^2 4\epsilon\overline{\partial}^2 + k^2 8\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j + \frac{8\epsilon_{\rm ij}k^2\overline{\partial}^i\partial^j\overline{\partial}^2}{R^2}\right)$$
$$+\frac{1}{384}\left(6k^4R^4\epsilon + 2k^4R^2 6\epsilon\overline{\partial}^2 + k^4 6\epsilon\overline{\partial}^4 - k^4R^2 10\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j - 20\epsilon_{\rm ij}k^4\overline{\partial}^i\overline{\partial}^j\overline{\partial}^2 - \frac{k^4 10\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j\overline{\partial}^4}{R^2}\right)$$
$$+\left(k_i\epsilon\overline{\partial}^i - \frac{k_l\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j\overline{\partial}^l}{R^2} + 2k_0\epsilon_{\rm oj}\overline{\partial}^i\right) - \frac{1}{12}\left(-k^2k_iR^2\epsilon\overline{\partial}^i - k^2k_i\epsilon\overline{\partial}^2\overline{\partial}^i + k^2k_l\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j\overline{\partial}^l + \frac{k^2k_l\epsilon_{\rm ij}\overline{\partial}^i\overline{\partial}^j\overline{\partial}^l\overline{\partial}^2}{R^2}\right.$$
$$\left. -2\overline{\partial}^i k^2R^2k_0\epsilon_{\rm oj} - 2k_0\epsilon_{\rm oj}k^2\overline{\partial}^i\overline{\partial}^2\right)\int d^3\hat{x}G(x,\hat{x})({z_0}^5\ e^{i\vec{k}.\vec{\hat{x}}}) \quad (2.30)$$

where we have $\overline{\partial}^i$ denoting $\partial/\partial k_i$

To simplify things, we can assume $\vec{k}$ to be in the direction of x_3. Then the integral part of the equation simplifies to

$$\int d^3\hat{x}G(x,\hat{x}){z_0}^5 \; e^{i\vec{k}.\vec{\hat{x}}} = \int d^3\hat{x}G(x,\hat{x})(R^2-\hat{r}^2)^{5/2} \; e^{ik\hat{x}_3} \,. \tag{2.31}$$

As previously discussed, these transformations has to be made to the new intrinsic coordinates (η',θ',ϕ') , where we have $\vec{x}=(r,\theta,\phi)=(R\tanh\eta,\theta,\phi)$ becoming the origin. In these coordinates

$$\left(1-\frac{r^2}{R^2}\right) = \frac{1}{(\cosh\eta\cosh\eta'+\cos\theta'\sinh\eta\sinh\eta')^2} \tag{2.32}$$

$$\frac{x_3}{R} = \frac{\cos\theta\cosh\eta'\sinh\eta+\cos\theta\cos\theta'\cosh\eta\sinh\eta'-\cos\phi'\sin\theta\sin\theta'\mathrm{sinh}\eta'}{\cosh\eta\cosh\eta'+\cos\theta'\sinh\eta\sinh\eta'} \tag{2.33}$$

The integrand is expanded in powers of k up till 8 powers of k. The expression has been evaluated upto 4 powers of k in [4], and the results are given bellow,

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta'-\sinh\eta')^2}{4\pi}\left(R^2-r(\eta',\theta',\phi')^2\right)^{5/2} = \frac{R^5\mathrm{sech}^3\eta}{12} \quad , \tag{2.34}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta'-\sinh\eta')^2}{4\pi} ik\; x_3(\eta',\theta',\phi')\left(R^2-r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{ikR^6}{20}\;\cos\theta\mathrm{sech}^3\eta\tanh\eta \quad , \tag{2.35}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta'-\sinh\eta')^2}{4\pi}\left(ik\; x_3(\eta',\theta',\phi')\right)^2\left(R^2-r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{(ik)^2R^7}{360}\;\mathrm{sech}^3\eta\left(1+6\cos^2\theta\tanh^2\eta\right) \quad . \tag{2.36}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta'-\sinh\eta')^2}{4\pi}\left(ik\; x_3(\eta',\theta',\phi')\right)^3\left(R^2-r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= -\frac{(ik)^3R^8\cos\theta\mathrm{sech}^3\eta\tanh\eta\left(3+10\cos^2\theta\tanh^2\eta\right)}{2520} \quad , \tag{2.37}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta'-\sinh\eta')^2}{4\pi}\left(ik\; x_3(\eta',\theta',\phi')\right)^4\left(R^2-r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{(ik)^4R^9\mathrm{sech}^3\alpha\left(1+6\cos^2\beta\tanh^2\alpha+15\cos^4\beta\tanh^4\alpha\right)}{20160} \quad . \tag{2.38}$$

We need to evaluate it for further 4 more powers up to k^8

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta' - \sinh\eta')^2}{4\pi} \left(ik\ x_3(\eta',\theta',\phi')\right)^5 \left(R^2 - r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{ik^5 R^{10}\cos\theta \mathrm{sech}^3\alpha \tanh\alpha \left(3 + 10\cos^2\theta\tanh^2\alpha + 21\cos^4\theta\tanh^4\alpha\right)}{181440} \quad . \tag{2.39}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta' - \sinh\eta')^2}{4\pi} \left(ik\ x_3(\eta',\theta',\phi')\right)^6 \left(R^2 - r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{(ik)^6 R^{11}\mathrm{sech}^3\alpha \left(1 + 6\cos^2\theta\tanh^2\alpha + 15\cos^4\theta\tanh^4\alpha + 28\cos^6\theta\tanh^6\alpha\right)}{1814400} \quad . \tag{2.40}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta' - \sinh\eta')^2}{4\pi} \left(ik\ x_3(\eta',\theta',\phi')\right)^7 \left(R^2 - r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{ik^7 R^{12}\cos\theta \mathrm{sech}^3\alpha \tanh\alpha \left(3 + 10\cos^2\theta\tanh^2\alpha + 21\cos^4\theta\tanh\alpha + 36\cos^6\theta\tanh^6\alpha\right)}{19958400} \quad , \tag{2.41}$$

$$\int d\eta' d\theta' d\phi' \frac{\sinh\eta'(\cosh\eta' - \sinh\eta')^2}{4\pi} \left(ik\ x_3(\eta',\theta',\phi')\right)^8 \left(R^2 - r(\eta',\theta',\phi')^2\right)^{5/2}$$
$$= \frac{(ik)^8 R^{13}\mathrm{sech}^3\alpha \left(1 + 6\cos^2\theta\tanh^2\alpha + 15\cos^4\theta\tanh^4\alpha + 28\cos^6\theta\tanh^6\alpha + 45\cos^8\theta\tanh^8\alpha\right)}{239500800} \quad . \tag{2.42}$$

we can see that $\mathrm{sech}^3\eta$ is nothing but z_0^3 and $k\cos\theta\tanh\eta$ is nothing but $(k \cdot r)$. On transforming back and acting the source on integral we'll have.

$$z_1 = -{z_0}^3 R^2 \Big(-\frac{k^4R^4}{6912} + \frac{k^4R^2x^2}{3456} - \frac{k^4x^4}{6912} + \frac{1}{432}k^2(k\cdot r)^2R^2 - \frac{1}{432}k^2(k\cdot r)^2x^2$$
$$- \frac{1}{192}ik^2(k\cdot r)R^2 + \frac{1}{192}ik^2(k\cdot r)x^2 - \frac{k^2R^2}{168} + \frac{k^2x^2}{168} - \frac{(k\cdot r)^4}{432} + \frac{i(k\cdot r)^3}{96} + \frac{(k\cdot r)^2}{28} - \frac{i(k\cdot r)}{12}$$
$$- \frac{1}{10}\Big)(T + T_x) \quad . \tag{2.43}$$

On performing inverse Fourier transform, we'll have

$$
\begin{aligned}
z_1 = z_0^3 R^2 \Bigg(& -\frac{1}{10}\left(\frac{x^i x^j T_{\rm ij}}{R^2} + T\right) + \frac{1}{168}\left(R^2 - x^2\right)\left(\partial^2 T + x^i x^j \partial^2 \frac{T_{\rm ij}}{R^2}\right) \\
& -\frac{R^4 - 2R^2x^2 + x^4}{6912}\left(\partial^4 T + x^i x^j \partial^4 \frac{T_{\rm ij}}{R^2}\right) - \frac{1}{12}\left(x^i x^j x^l \frac{\partial}{\partial x^l}\frac{T_{\rm ij}}{R^2} + x^l \frac{\partial T}{\partial x^l}\right) \\
& +\frac{1}{192}\left(R^2 - x^2\right)\left(x^l \partial_l \partial^2 T + x^l x^i x^j \partial_l \partial^2 \frac{T_{\rm ij}}{R^2}\right) - \frac{1}{28}\left(x^i x^j x^l x^m \partial_l \partial_m \frac{T_{\rm ij}}{R^2} + x^l x^m \partial_l \frac{\partial T}{\partial x^m}\right) \\
& +\frac{1}{432}\left(R^2 - x^2\right)\left(x^l x^m \partial_l \partial_m \partial^2 T + x^l x^m x^i x^j \partial_l \partial_m \partial^2 \frac{T_{\rm ij}}{R^2}\right) \\
& -\frac{1}{96}\left(x^i x^j x^l x^m x^p \partial_l \partial_m \partial_p \frac{T_{\rm ij}}{R^2} + x^l x^m x^p \partial_l \partial_m \partial_p T\right) \\
& -\frac{1}{432}\left(x^i x^j x^l x^m x^p x^q \partial_l \partial_m \partial_p \partial_q \frac{T_{\rm ij}}{R^2} + x^l x^m x^p x^q \partial_l \partial_m \partial_p \partial_q T\right)\Bigg)
\end{aligned}
\tag{2.44}
$$

As we now have z_1 in Functional form , we can proceed to calculating $\Delta^2 S$.

Chapter 3

$\Delta^{(2)}S$ and Results

3.1 Calculating $\Delta^{(2)}S$

As we now have the z_1 We can proceed to calculate the Area functional in terms of Stress tensor and it's derivatives.

As we have Constructed the metric till two derivatives of T_{ij}, we will evaluate the area functional up till that order. The Area functional as mentioned previously in 2.12 and 2.13 will have contributions from $A_{(2,0)}$ $A_{(2,1)}$ $A_{(2,2)}$. Hence the better way would be to evaluate these individually and sum the contributions. If we recollect, The solution for z_1 depends on the derivatives of the stress tensor at the origin. Thus, this should be followed for area functional as well. To implement this we need to Taylor Expand the $T_{ij}(\vec{x})$'s appearing in the area functional and then integrate it. On Taylor expanding $T_{ij}(\vec{x})$ around origin up to four derivatives we'll have,

$$T_{ij}(\vec{x}) = T_{ij} + x^k\partial_k T_{ij} + \frac{1}{2!}x^k x^l \partial_k\partial_l T_{ij} + \frac{1}{3!}x^k x^l x^m \partial_k\partial_l\partial_m T_{ij} + \frac{1}{4!}x^k x^l x^m x^n \partial_k\partial_l\partial_m\partial_n T_{ij} \tag{3.1}$$

3.1.1 Contribution from $A_{(2,1)}$

We have previously seen that $A_{(2,1)}$ is of the form

$$A_{(2,1)} = L^{d-1}a\int d^{d-1}x\frac{R}{2z_0}\left(T\left(\vec{x}\right)\left(z_1 - \frac{z_0^2}{R^2}x^i\partial_i z_1\right) - \frac{z_0^2}{12}\partial^2 T\left(\vec{x}\right)\left(3z_1 - \frac{z_0^2}{R^2}x^i\partial_i z_1\right)\right.$$
$$+\frac{z_0^4}{384}\partial^4 T\left(\vec{x}\right)\left(5z_1 - \frac{z_0^2}{R^2}x^i\partial_i z_1\right) + T_{ij}\left(\vec{x}\right)\left(2z_0^2x^i\partial^j z_1/R^2 - \frac{z_1x^ix^j}{R^2} - \frac{z_0^2x^ix^jx^k\partial_k z_1}{R^4}\right)$$
$$-\frac{z_0^2}{12}\partial^2 T_{ij}\left(\vec{x}\right)\left(2z_0^2x^i\partial^j z_1/R^2 - 3\frac{z_1x^ix^j}{R^2} - \frac{z_0^2x^ix^jx^k\partial_k z_1}{R^4}\right)$$
$$\left.+\frac{z_0^4}{384}\partial^4 T_{ij}\left(\vec{x}\right)\left(2z_0^2x^i\partial^j z_1/R^2 - 5\frac{z_1x^ix^j}{R^2} - \frac{z_0^2x^ix^jx^k\partial_k z_1}{R^4}\right)\right) \; . \tag{3.2}$$

Now, replace $T_{ij}(\vec{x})$ with it's Fourier expansion, described in 3.1 and expand it further by replacing z_1 with it's solution given in 2.44 . On expanding this would give us with explosion of terms. We hand pick terms with four derivatives resulting from this expansion, as terms with two derivative corrections has already been dealt with in [4]. As expected there is no appearance of the three(odd) derivative terms. We are not writing the expanded form of $A_{(2,1)}$, As the resultant expression is huge.
We us the the following tick to simplify the integrand further

$$\int d^{(d-1)}x\ f(r)x^ix^jx^kx^l\cdots n \text{ pairs} = N_n(\partial_{ij}\partial_{kl}\cdots + \text{permutations})\int d^{d-1}x\ f(r)r^{2n}\, , \tag{3.3}$$

Where N_n is a normalisation constant.

$$N_1 = \frac{1}{d-1} \qquad \text{for } n=1 \tag{3.4}$$
$$N_2 = \frac{1}{((d-1)^2+2(d-1))} \qquad \text{for } n=2 \tag{3.5}$$
$$N_3 = \frac{1}{((d-1)^3+6(d-1)^2+8(d-1))} \qquad \text{for } n=3 \tag{3.6}$$
$$N_4 = \frac{1}{-15-8d+14d^2+8d^3+d^4} \qquad \text{for } n=4\,. \tag{3.7}$$

By this way we turn our integrands with r^n's as the only integrating variables. Hence making it easy to integrate.

On following the above and integrating the expression of $A_{(2,1)}$ with r going from $(0, R)$ We'll obtain the contribution from $A_{(2,1)}$.

3.1.2 Contribution from $A_{(2,2)}$

We've noted that the functional form of $A_{(2,2)}$ is given by the following

$$A_{(2,2)} = L^3 \int d^3x \frac{R}{z_0^4} \left(\frac{d(d-1){z_1}^2}{{z_0}^2} + \frac{{z_0}^2 \left(\partial_i z_1\right)}{2R^2} - \frac{{z_0}^2}{2R^4} \left(x^i \partial_i z_1\right)^2 + \frac{(d-1)}{R^2} z_1 x^i \partial_i z_1 \right) . \tag{3.8}$$

And we follow the same procedure as we did in the case of $A_{(2,1)}$ and evaluate the Area functional for the above expression.

3.1.3 Contribution from $A_{(2,0)}$

The expression for $A_{(2,0)}$ in compressed form can be written as the following.

$$A_{(2,0)} = L^3 \int d^3x \;\; \frac{R}{z_0^4} (\frac{1}{8}((\delta g)^2 - 2\delta g \delta g_x + (\delta g_x)^2) + \frac{1}{4}(\delta g_{ij} \delta g^{ij} + \delta^{(2)} g - \delta^{(2)} g_x)) \tag{3.9}$$

where we have $\delta g = \delta g^i_i$ and $\delta g_x = \frac{x^i x^j}{R^2} \delta g_{ij}$, Similarly for $\delta^{(2)} g$ and $\delta^{(2)} g_x$. we need to Taylor expand $T_{ij}(\vec{x})$ and use the above trick to obtain the contribution from $A_{(2,0)}$.

3.1.4 A_2

On summing up the contributions from $A_{(2,0)}$,$A_{(2,1)}$and $A_{(2,2)}$, We will have A_2 which is the correction tot he minimal surface due to new perturbations. For the current considerations it is relavent for to consider only terms of the order $\mathcal{O}(\partial\partial T \partial\partial T)$, as only these would be useful in completing the squares, hence keeping terms uptill this order we will have A_2 as the following.

$$A_2 = \frac{R^{12}}{85135050} \left(97(\partial^2 T)^2 + 2\left(3\left(97\left(\partial_i\partial_j T\right)^2 + 1364\left(\partial_i\partial_j T_{lm}\right)^2 - \partial_i\partial_j T_{lm}\left(4\partial^i\partial^l T^{\mathrm{jm}} + \partial^l\partial^m T^{\mathrm{ij}}\right)\right.\right.\right.$$
$$+ 210\left(\left(\partial^2 T_{0i}\right)^2\left(3n_1 + 8n_2 + 72n_3 + 102n_4 - 6n_5 + 36n_7 + 1728n_{14} + 576n_{20} + 576n_{22}\right)\right.$$
$$+2\left(6\left(17n_{10} + 144n_{16}\right)\left(\partial_i\partial_j T_{00}\right)^2 + \left(\partial^2 T_{00}\right)^2\left(4n_2 + 51n_4 - 3n_5 + 18n_7 + 864n_{14} + 288n_{22}\right)\right)$$
$$+2\left(11340\partial^2 T_{\mathrm{ij}}\partial^i\partial^j T\left(n_{13} + 16n_{18}\right) + \left(\partial^2 T_{\mathrm{ij}}\right)^2\left(945n_1 + 2520n_2 + 22680n_3 + 32130n_4 - 1890n_5\right.\right.$$
$$\left.+11340n_7 + 544320n_{14} + 181440n_{20} + 181440n_{22} + 341\right) + 315\left(2\left(8n_2 + 33n_5 - 3n_7 + 288n_{24}\right)\left(\partial_i\partial_j T_{00}\right)^2\right.$$
$$+ 2\left(3n_1 + 8n_2 + 33n_5 - 3n_7 + 39n_8 + 102n_{10} + 864n_{16} + 288n_{21} + 288n_{24}\right)\left(\partial_i\partial_j T_{0l}\right)^2$$
$$+ \partial_i\partial_j T_{0l}\partial^i\partial^l T^{0j}\left(6\left(13n_6 + 22n_9 + 34n_{11} - 2n_{12} + 288n_{17} + 96n_{23} + 192n_{26}\right) - n_1\right)$$
$$+ \partial_i\partial_j T_{lm}\left(-6\partial^l\partial^m T^{\mathrm{ij}}\left(n_6 + n_{13} - 96\left(3n_{15} + 2n_{27}\right)\right) + 2\partial^i\partial^j T^{\mathrm{lm}}\left(3n_1\right.\right.$$
$$\left. + 8n_2 + 33n_5 - 3n_7 + 36n_8 + 102n_{10} + 864n_{16} + 288n_{21} + 288n_{24}\right)$$
$$\left.\left.+\partial^i\partial^l T^{\mathrm{jm}}\left(6\left(12n_6 - n_8 + 22n_9 + 34n_{11} - 2n_{12} + 288n_{17} + 96n_{19} + 96n_{23} + 192n_{26}\right) - n_1\right)\right)\right) \tag{3.10}$$

From the above A_2 we can have the $\mathcal{O}(\partial\partial T\partial\partial T)$ contribution to $\Delta^{(2)}S$ as the following

$$\Delta^{(2)}S_2 = \frac{8\pi^2 L^3}{l_P^3} A_2 \tag{3.11}$$

The total $\Delta^{(2)}S$ up to four derivative corrections is given by

$$\Delta^{(2)}S = \Delta^{(2)}S_1 + \Delta^{(2)}S_2 \tag{3.12}$$

where $\Delta^{(2)}S_1$ is previously given in 1.35

3.2 Results

We have seen that we have obtained a equation describing the new minimal area, resulted due to perturbation introduced by higher order derivatives. The equation describing the new minimal surface is as follows,

$$z = z_0 + \epsilon z_1 \qquad \text{where } z_0 \text{ is given by } z_0 = R^2 - r^2 \tag{3.13}$$

and we have z_1 which we previously described as,

$$z_1 = z_0^3 R^2 \left(-\frac{1}{10} \left(\frac{x^i x^j T_{\text{ij}}}{R^2} + T \right) + \frac{1}{168} \left(R^2 - x^2 \right) \left(\partial^2 T + x^i x^j \partial^2 \frac{T_{\text{ij}}}{R^2} \right) \right.$$
$$-\frac{R^4 - 2R^2 x^2 + x^4}{6912} \left(\partial^4 T + x^i x^j \partial^4 \frac{T_{\text{ij}}}{R^2} \right) - \frac{1}{12} \left(x^i x^j x^l \frac{\partial}{\partial x^l} \frac{T_{\text{ij}}}{R^2} + x^l \frac{\partial T}{\partial x^l} \right)$$
$$+\frac{1}{192} \left(R^2 - x^2 \right) \left(x^l \partial_l \partial^2 T + x^l x^i x^j \partial_l \partial^2 \frac{T_{\text{ij}}}{R^2} \right) - \frac{1}{28} \left(x^i x^j x^l x^m \partial_l \partial_m \frac{T_{\text{ij}}}{R^2} + x^l x^m \partial_l \frac{\partial T}{\partial x^m} \right)$$
$$+\frac{1}{432} \left(R^2 - x^2 \right) \left(x^l x^m \partial_l \partial_m \partial^2 T + x^l x^m x^i x^j \partial_l \partial_m \partial^2 \frac{T_{\text{ij}}}{R^2} \right)$$
$$-\frac{1}{96} \left(x^i x^j x^l x^m x^p \partial_l \partial_m \partial_p \frac{T_{\text{ij}}}{R^2} + x^l x^m x^p \partial_l \partial_m \partial_p T \right)$$
$$\left. -\frac{1}{432} \left(x^i x^j x^l x^m x^p x^q \partial_l \partial_m \partial_p \partial_q \frac{T_{\text{ij}}}{R^2} + x^l x^m x^p x^q \partial_l \partial_m \partial_p \partial_q T \right) \right) \tag{3.14}$$

From this we have proceeded on to calculating the Area Functional which would give us $\Delta^{(2)} S$. On substituting the Einstein values for all the n's expect for n_1 and n_2, We'll have the $\Delta^{(2)} S$ reduced to the following.

$$\Delta^{(2)} S = -\frac{8\pi^2 L^3}{l_P^3} \frac{R^{12}}{170270100} (194(\partial^2 T)^2 - 2667(\partial^2 T_{0i})^2 + 61(\partial^2 T_{\text{ij}})^2 + 84\partial^2 T_{\text{ij}} \partial^i \partial^j T + 6(194(\partial_i \partial_j T)^2$$
$$-28(\partial_i \partial_j T_{00})^2 - 7\partial_i \partial_j T_{0l}(6\partial^i \partial^j T^{0l} + 26\partial^i \partial^l T^{0j}) + \partial_i \partial_j T_{lm}(2686\partial^i \partial^j T^{\text{lm}} + 6\partial^i \partial^l T^{\text{jm}} - 9\partial^l \partial^m T^{\text{ij}}))$$
$$+2520(3(\partial^2 T_{0i})^2 + 3(\partial^2 T_{\text{ij}})^2 + 6(\partial_i \partial_j T_{0l})^2 + 6(\partial_i \partial_j T_{lm})^2 - \partial_i \partial_j T_{0l} \partial^i \partial^l T^{0j} - \partial_i \partial_j T_{lm} \partial^i \partial^l T^{\text{jm}}) n_1$$
$$+21((\partial^2 T_0)^2 - 69120\alpha + 960 n_2 - 11 + 960(((\partial^2 T_{0i})^2 + (\partial^2 T_{\text{ij}})^2 + 2((\partial_i \partial_j T_{0l})^2 + (\partial_i \partial_j T_{lm})^2 + (\partial_i \partial_j T_{00})^2)) n_2$$
$$+72((\partial^2 T_{0i})^2 + \partial^2 T_{\text{ij}}(\partial^2 T^{\text{ij}} + 2\partial^i \partial^j T) + (\partial_i \partial_j T_{00})^2 - (\partial_i \partial_j T_{0l})^2 - \partial_i \partial_j T_{lm}(\partial^i \partial^j T^{\text{lm}} - 2\partial^i \partial^l T^{\text{jm}} + \partial^l \partial^m T^{\text{ij}}))\alpha))) \tag{3.15}$$

The for restricting the parameter space to n_1 and n_2 is to, obtain a contained region and to make a comparative study with the previous cases.

3.3 Discussion

We could clearly see that by substituting the values of n_1 and n_2 in 3.15, not all the terms turn out to be positive definite. As is it not expected, but there is dependence on the free parameter α, which was not fixed on solving the Einstein's equations. We can use a different technique to prove that $\Delta^{(2)}S_2 < 0$. We can take $\Delta^{(2)}S = V^T M V$. where V is a column Vector containing elements of the form , $\partial_i\partial_j T_{k\mu}$ which are linearly independent of each other, made sure by the following constrains.

$$\partial^i T_{ij} = 0\,, \qquad \partial^i T_{i0} = 0 \qquad \text{and} \qquad \partial_i T_{jk} = \partial_i T_{kj}\,. \tag{3.16}$$

$\Delta^{(2)}S_1$ will have no $\mathcal{O}(\partial\partial T)$ contributions at Einstein point as we could see in 1.36. Hence the vector V is given by the following

$$\begin{aligned} V = \{&\partial_1^2 T_{0,1}, \partial_1\partial_2 T_{0,1}, \partial_2^2 T_{0,1}, \partial_1\partial_3 T_{0,1}, \partial_2\partial_3 T_{0,1}, \partial_3^2 T_{0,1}, \partial_1^2 T_{0,2}, \partial_1\partial_2 T_{0,2}, \partial_2^2 T_{0,2}, \partial_1\partial_3 T_{0,2}, \partial_2\partial_3 T_{0,2},\\ &\partial_3^2 T_{0,2}, \partial_1^2 T_{0,3}, \partial_1\partial_2 T_{0,3}, \partial_2^2 T_{0,3}, \partial_1^2 T_{1,1}, \partial_1\partial_2 T_{1,1}, \partial_2^2 T_{1,1}, \partial_1\partial_3 T_{1,1}, \partial_2\partial_3 T_{1,1}, \partial_3^2 T_{1,1}, \partial_1^2 T_{1,2},\\ &\partial_1\partial_2 T_{1,2}, \partial_2^2 T_{1,2}, \partial_1\partial_3 T_{1,2}, \partial_2\partial_3 T_{1,2}, \partial_3^2 T_{1,2}, \partial_1^2 T_{1,3}, \partial_1\partial_2 T_{1,3}, \partial_2^2 T_{1,3}, \partial_1^2 T_{2,2}, \partial_1\partial_2 T_{2,2}, \partial_2^2 T_{2,2}, \partial_1\partial_3 T_{2,2}\\ &, \partial_2\partial_3 T_{2,2}, \partial_3^2 T_{2,2}, \partial_1^2 T_{2,3}, \partial_1\partial_2 T_{2,3}, 2\partial_2^2 T_{2,3}, \partial_1^2 T_{3,3}, \partial_1\partial_2 T_{3,3}, \partial_2^2 T_{3,3}\} \end{aligned} \tag{3.17}$$

Now we diagonalize the 42×42 matrix M with a matrix U. Then we can write,

$$\Delta^{(2)}S = V^T M V = V^T U^T M_d U V = (UV)^T M_d (UV) = \sum_{i=1}^{23} \lambda_i (UV)_i{}^2\,. \tag{3.18}$$

where λ_i's are the eigenvalues of M. On calculating Eigenvalues for M at Einstein point, we see that few values are dependent on α. Of these values, a few of them are linearly dependent on α , rest in a Quadratic form. On plotting these functions of alpha, we see that it is not always a negative quantity. Hence, we observe that α , which was thought to be the free parameter in Einstein values in 2.4, is observed to take only restricted class of values.

Hence with out properly fixing the value of α, we can not obtain constraints on the parameter space. The work regarding the same is in progress. As we have all the nessary components of the order $\mathcal{O}(\partial\partial T \partial\partial T)$, We are expecting to complete the squares and show it as a negative definite quantity. Thus this would enable us to consider a wider class of stress tensors. This as stated initially can enable us to consider various class of stress tenors, which can lead to a situation in the parameter space as initially expected in [**?**] .Hence there is a possibility that atleast in a particular parameter slice (e.g: n1 and n2), we can recover the Einstein point.

Even though we are increasing the parameter space by including the new derivatives,by recovering Einstein point in one the parameter slice and Using the Feynman's arguments we can show that the bulk theory is diffeomorphism invariant up to the quadratic order.

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Appendix A

Appendix

A.1 Entanglement Entropy in AdS_3/CFT_2 (Sec. 1.1.1)

The following is the example calculation given in [1] For a (1+1)D CFT the entanglement entropy is given by the following expression

$$S_A = \frac{c}{3} \cdot \log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right), \tag{A.1}$$

According to AdS/CFT, gravitational theories on AdS_3 space of radius R are dual to (1+1)D CFTs with the central charge

$$c = \frac{3R}{2G_N^{(3)}}. \tag{A.2}$$

The Metric of AdS_3 in global coordinates can be expressed as the following

$$ds^2 = R^2\left(-\cosh\rho^2 dt^2 + d\rho^2 + \sinh\rho^2 d\theta^2\right). \tag{A.3}$$

As we go to the boundary, $\rho = \infty$ of the above metric is divergent. Hence a cut off is imposed on it, such that $\rho \leq \rho_0$ to regulate the space of the bounded region. This corresponds to imposing the UV cutoff on the CFT. If L is the total length of the system with both ends identified, and a is the lattice spacing (or UV cutoff) in the CFTs, we have the relation (up to a numerical factor)

$$e^{\rho_0} \sim L/a. \tag{A.4}$$

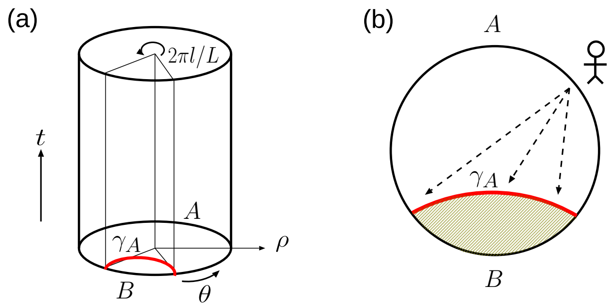

Figure A.1: (a) AdS_3 space and CFT_2 living on its boundary and (b) a geodesics γ_A as a holographic screen.

We take cylindrical coordinates for the CFT_2 at the boundary $\rho = \rho_0$ identifying with (t, θ). The subsystem A (is the region $0 \leq \theta \leq 2\pi l/L$. The minimal surface area γ_A in Eq. (1.3) is identified with the static geodesic that connects the boundary points $\theta = 0$ and $2\pi l/L$ in a particular time slice, travelling inside AdS_3 (Fig. A.1 (a)). With the cutoff ρ_0 introduced above, the geodesic distance L_{γ_A} is given by

$$\cosh\left(\frac{L_{\gamma_A}}{R}\right) = 1 + 2\sinh^2\rho_0 \;\sin^2\frac{\pi l}{L}. \tag{A.5}$$

Assuming that the UV cutoff(ρ_0) is large enough, such that $e^{\rho_0} \gg 1$. The Entanglement Entropy given by the formula [1], with the above described central charge is given by the following

$$S_A \simeq \frac{R}{4G_N^{(3)}}\log\left(e^{2\rho_0}\sin^2\frac{\pi l}{L}\right) = \frac{c}{3}\log\left(e^{\rho_0}\sin\frac{\pi l}{L}\right). \tag{A.6}$$

The resultant expression exactly coincides with the known CFT result A.1.