Automorphic Forms and L-functions for Symplectic Groups of Genus 3

A thesis

submitted in partial fulfillment of the requirements

of the degree of

Doctor of Philosophy

by

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INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

2020

Dedicated to

My Parents, My Grandparents and My Uncle

Certificate

Certified that the work incorporated in the thesis entitled "*Automorphic Forms and L-functions for Symplectic Groups of Genus* 3", submitted by *Manidipa Pal* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

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Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Acknowledgements

First and foremost, I would like to express my sincere gratitude towards my thesis supervisor Dr. Baskar Balasubramanyam for his continuous support. He was always ready to discuss and was patient enough to explain Mathematics to me. The questions studied in this thesis are formulated by him. Besides my supervisor, I would like to thank the rest of my research advisory committee members: Prof. A. Raghuram and Dr. Kaneenika Sinha for their insightful comments. I had the opportunity to talk mathematics with several people. I would like to thank them for their support. In particular, I would like to thank Dr. Ameya Pitale, Dr. Vivek Mohan Mallick and Dr. Anindya Goswami for their helpful mathematical discussions. I am thankful to CSIR for the financial support in the form of the research fellowship. I would like to acknowledge the support of the institute and its administrative staff members for their cooperations.I could not express my gratitude in words to my teachers starting from my school days till date for having faith in me and guiding me in the right direction. I thank all my batchmates and students at IISER Pune for being enthusiastic about discussing mathematics with me. I especially thank Rohit and Sushil for their valuable help and mathematical discussions. It was an enriching experience to discuss mathematics with Rohit. Last but not the least, I must express my deepest gratitude to my parents and sister for providing me with unconditional support and constant encouragement throughout my years of study and through the process of research. They will always remain my unshakeable source of strength. Without their immense help and encouragement, this thesis would not ever have been possible.

Manidipa Pal

Contents

Abstract

For the similitude symplectic group GSp_6 over a totally real number field F, we establish the meromorphic continuation of the standard L-function and the spin L-function which are Langlands L-functions associated to the automorphic representation of $\mathrm{PGSp}_6(\mathbb{A}_{\mathbb{F}})$. In the second part of this thesis we compute the dimesion of the spaces of automorphic forms for rank 3 unitary groups where the entries of the group are from a definite quaternion algebra B over $\mathbb{Q}$. This group is an inner form of GSp_6 over $\mathbb{Q}$.

Notation

F : a field (char $\neq 2$)

$\bar{F}$: algebraic closure of F

$\mathbb{Z}$: integers

$\mathbb{Q}$: the field of rational numbers

$\mathbb{R}$: the real field

$\mathbb{C}$: the complex field

$\mathbb{Q}_p$: p-adic fields

$\otimes$: tensor product

$\oplus$: direct sum

$\cong$: isomorphism

$\widehat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z}$, finite adèles of $\mathbb{Z}$

$\widehat{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$, finite adèles of $\mathbb{Q}$

$\mathbb{A} := \mathbb{R} \times \widehat{\mathbb{Q}}$, the ring of adeles of $\mathbb{Q}$

g^t : transpose of a matrix g

$\square$: end of a proof

Chapter 1

Introduction

This thesis deals with two questions on which I was working during my Ph.D. Both of these two questions are related to Siegel modular forms of genus 3.

The idea of the first question came after reading the paper by Asgari–Schmidt [AS01]. In their paper, they start with a Siegel modular Hecke eigen form f of degree n for the full modular group $\mathrm{Sp}_{2n}(\mathbb{Z})$ with trivial nebentypus character. Then using the strong approximation property for Sp_{2n}, they associate to f a function Φ_f on $\mathrm{PGSp}_{2n}(\mathbb{A})$ which may be thought of as the adélic version of f. Moreover, they construct an automorphic representation $\pi(f)$ of $\mathrm{PGSp}_{2n}(\mathbb{A})$ via Φ_f. Using classical Hecke operators acting on f they considered the associated standard L-function and spin L-function of degree $(2n+1)$ and 2^n respectively. Then they deal with $n = 3$ situation. The Langlands dual of PGSp_6 is Spin_7. Let $\rho_1 : \mathrm{Spin}_7 \xrightarrow{\mathrm{Std}} \mathrm{SO}_7(\mathbb{C})$ be the standard representation and $\rho_2 : \mathrm{Spin}_7 \xrightarrow{\mathrm{spin}} \mathrm{SO}_8(\mathbb{C})$ be the spin representation. Let $L(s,\pi(f),\rho_1)$ and $L(s,\pi(f),\rho_2)$ ([AS01, Section 4.6]) be two Langlands L-functions associated with them. These Langlands L-functions are related to classical standard L-function and spin L-function with a shifting in $s \in \mathbb{C}$. The main goal of their paper [AS01] is to prove the meromorphic continuation of L-functions $L(s,\pi(f),\rho_1)$ and $L(s,\pi(f),\rho_2)$ to all of $\mathbb{C}$ via Langlands theory of Euler products [AS01, Theorem 4].

The goal of the first question is to generalise their result in the case of Siegel-Hilbert modular forms, i.e., replace the base field $\mathbb{Q}$ by a totally real number field. Let F be a totally real number field of degree d over $\mathbb{Q}$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6)$ be the Weil restriction of scalars from F to $\mathbb{Q}$ of the algebraic group GSp_6. Then $G(\mathbb{A}) = \mathrm{GSp}_6(\mathbb{A}_F)$. We recall the necessary theory of scalar valued Siegel-Hilbert modular forms of genus 3 and weight

$k = (k_1, k_2, \ldots, k_d)$ where k_i's are nonnegative integers. These forms are functions on $\mathscr{H}_3^d$ satisfying the usual transformation property with respect to congruence subgroups. Here $\mathscr{H}_3^d$ is the d-copies of Siegel upper half-space. Let us start with a tuple $\underline{f} = (f_1, f_2, \ldots, f_h)$ of Siegel-Hilbert modular forms with trivial characters, then utilizing the strong approximation theorem for Sp_6, an adélic Siegel-Hilbert automorphic form $\Phi_{\underline{f}} : G(\mathbb{A}) \to \mathbb{C}$ may be realised as this tuple. Here h denotes the narrow class number of F. By Borel and Jacquet [BJ79], we associate an automorphic representation $\pi(\underline{f})$ of $G(\mathbb{A})$ with $\Phi_{\underline{f}}$. Since $\pi(\underline{f})$ has a trivial central character so we consider $\pi(\underline{f})$ as an automorphic representation of $\overline{G}(\mathbb{A}) = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{PGSp}_6)(\mathbb{A})$. Here the L-group of $\overline{G}$ is

$$^L\overline{G} = (\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q}),$$

where F' is a finite Galois extension of $\mathbb{Q}$ such that F' contains F. Let S denote the set of places of $\mathbb{Q}$ which include Archimedean place ∞, the ramified primes p and those finite places p where $\pi(\underline{f})_p$ is not spherical.

Now corresponding to the representations ρ_1 and ρ_2, let us define another two representations,

$$\phi_1 : (\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q}) \to \mathrm{GL}_{7d}(\mathbb{C})$$

and

$$\phi_2 : (\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q}) \to \mathrm{GL}_{8d}(\mathbb{C}).$$

The representations ϕ_1 and ϕ_2 are constructed out of ρ_1 and ρ_2. So, in our setting, they are the analogues of standard representation and spin representation. Now the local components of $\pi(\underline{f})$ which are spherical representations of local groups $\overline{G}(\mathbb{Q}_p)$ can be attached to a unique semisimple conjugacy class denoted by (t_p^0, Fr_p) in the local L-group of $\overline{G}$. Then corresponding to these two representations ϕ_1, ϕ_2 we have two Langlands L-functions associated to the automorphic representation $\pi(\underline{f}) = \otimes'_p \pi_p(\underline{f})$ of $\overline{G}(\mathbb{A}_\mathbb{Q})$.

One is the standard L-function

$$L^S(s, \pi(\underline{f}), \phi_1) := \prod_{p \notin S} L_p(s, \pi(\underline{f})_p, \phi_{1p})$$

for $s \in \mathbb{C}$, where the local Euler factors attached to $\pi(\underline{f})_p$ and ϕ_{1p} are defined as

$$L_p(s, \pi(\underline{f})_p, \phi_{1p}) := \det\left(I - \phi_{1p}(t_p^0, \mathrm{Fr}_p)p^{-s}\right)^{-1}.$$

and another one is the spin L-function,

$$L^S(s, \pi(\underline{f}), \phi_2) := \prod_{p \notin S} L_p(s, \pi(\underline{f})_p, \phi_{2p})$$

for $s \in \mathbb{C}$, where the local Euler factors attached to $\pi(\underline{f})_p$ and ϕ_{2p} are defined as

$$L_p(s, \pi(\underline{f})_p, \phi_{2p}) := \det\left(I - \phi_{2p}(t_p^0, \mathrm{Fr}_p)p^{-s}\right)^{-1}.$$

Our main aim is to prove the meromorphic continuation of $L^S(s, \pi(\underline{f}), \phi_1)$ and $L^S(s, \pi(\underline{f}), \phi_2)$ to all of $\mathbb{C}$ using Langlands theory. Our Theorem 3.2.2 is a straightforward generalisation of [AS01, Theorem 4] by Asgari-Schmidt.

In this context, we mention that one of the results in Kret–Shin [KS16] is the meromorphic continuation of the spin L-function for GSp_{2n} over totally real number field F under a local hypothesis that at the Archimedean place there is a Steinberg component twisted by a character.

The second question studied in this thesis is algorithmic and more computational in nature. Here we compute the dimension of the spaces of automorphic forms for rank 3 unitary groups where the entries of the elements of the group are from a quaternion algebra. The idea of this second problem came after reading various papers based on the dimension calculation of the spaces of modular forms for different groups (for example see [CD09, Dem05, Dem14, Loe08]). For a reductive algebraic group $\mathtt{G}$ over $\mathbb{Q}$ the space of automorphic forms for $\mathtt{G}$ of a given level and weight is known to be finite dimensional. However, for most of the groups how to calculate this dimension explicitly is less known. For the case of classical modular forms, for GL_2 there are well-known algorithms based on modular symbols (See [Ste07]), but in general for other groups very little is known. For computational convenience Gross in [Gro99] has developed a theory of modular forms totally algebraically. His theory deals with a connected reductive group over $\mathbb{Q}$ where the group satisfies the condition that all its arithmetic subgroups are finite. He defined the space of algebraic modular forms for these groups. Theoretically, this space is computable. Carrying out Gross's theory, Loeffler [Loe08] has given an algorithm for computing the full space of automorphic forms of full level for definite unitary groups over $\mathbb{Q}$. He has applied this algorithm of a rank 3 definite unitary group and calculated dimension for various small weights. Cunningham and Dembélé have their subsequent papers for the algorithmic calculations in the case of GSp_4 under the assumption of conjectural Jacquet-Langlands correspondence. We refer the readers to [CD09] where the authors have presented an algorithm for the computation of the space of genus 2 Siegel-Hilbert cusp forms over a real quadratic field of narrow class number 1 and then for compact inner forms of GSp_4 over totally real number fields (cf. [Dem14]).

In the same spirit, we want to calculate the dimensions of the space of genus 3 Siegel automorphic forms for various small weights for the group GSp_6 over $\mathbb{Q}$. We can not compute this space directly. To be able to apply Gross's theory we take a definite quaternion algebra B over $\mathbb{Q}$ which is ramified exactly at a prime p and ∞ and unramified at all other places. Let G^B over $\mathbb{Q}$ be the algebraic group whose $\mathbb{Q}$-rational points are given by the unitary similitude group $\mathrm{GU}_3(B)$. The group G^B is an inner form of GSp_6 over $\mathbb{Q}$ such that $G^B(\mathbb{R})$ is compact modulo center. We check that every arithmetic subgroup of G^B is finite. Now fixing an irreducible algebraic representation (ρ, V) of $G^B(\mathbb{Q})$ and $\underline{K} := \underline{G}^B(\widehat{\mathbb{Z}}) = \prod_{p<\infty} \underline{G}^B(\mathbb{Z}_p)$ maximal compact open subgroup of $G^B(\widehat{\mathbb{Q}})$ the space of algebraic automorphic forms of weight V, genus 3 and level $\underline{K}$ is then defined by Gross as,

$$M_{G^B}(V) = \{f : G^B(\mathbb{A})/(G^B(\mathbb{R})_+ \times \underline{G}^B(\widehat{\mathbb{Z}})) \to V \mid f(\gamma g) = \gamma f(g) \text{ for } \gamma \in G^B(\mathbb{Q})\}.$$

By the conjectural Jacquet-Langlands correspondence for similitude symplectic groups, computing the dimension of the space of Siegel automorphic forms amounts to computing the dimension of the space of algebraic automorphic forms on B. Then under the assumption of the existence of a Jacquet-Langlands correspondence between G^B and $\mathrm{GSp}_6/\mathbb{Q}$, our goal is to compute the dimension of the space of algebraic automorphic forms $M_{G^B}(V)$. In

Chapter 5, we give a Table 5.1 of dimensions of the spaces of cuspidal algebraic automorphic forms of full level and for various small weights V. The weights V are parametrized by non-negative integers a,b,c,d with no condition on b and with the condition that $a+c$ to be even. We fix d to be 0. The main idea of Chpater 5 is to give an algorithm to compute the dimensions of the space $M_{G^B}(V)$ which takes values for a,b,c as inputs and gives dimensions as outputs.

Chapter 2

Theory of Siegel-Hilbert automorphic forms

The purpose of this chapter is to include the preliminaries related to Siegel–Hilbert modular forms and then describe the procedure of associating a Siegel–Hilbert automorphic form with an automorphic representation of $\mathrm{GSp}_6(F)$, where F is a totally real number field. To describe the matters in details, let us fix the following notations.

2.1 Notations

The similitude symplectic group of degree n is given by,

$$\mathrm{GSp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid \exists\, \mu(g) \in \mathrm{GL}_1 \;\; gJg^t = \mu(g)J\},$$

where

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \;\; I_n \text{ is the } n \times n \text{ identity matrix.}$$

Let F denote a totally real number field of degree d over $\mathbb{Q}$ and $\mathcal{O}_F$ be its ring of integers. The set of real embeddings of F is denoted by $S_\infty = \{\sigma_1, \sigma_2, \ldots, \sigma_d\}$. This is the set of all archimedean places of F. Let F^+ denote the set of all totally positive elements in F. By totally positive we mean all those elements a in F such that, $\sigma_i(a) > 0$ for all $i = 1, 2, \ldots, d$. Let $F_\infty = \prod_{v \in S_\infty} F_v = \prod_{j=1}^{d} F_{\sigma_j} \cong \mathbb{R}^d$. Now $F_\infty^+ \subset F_\infty$ is such that, $F_\infty^+ = \{(x_1, \ldots, x_d) \in F_\infty \mid x_j > 0\, \forall j\}$. Let $\mathbb{A}_F$ denote the adèle ring of F, $\mathbb{A}_{f,F}$ denotes the finite adèles of F. Let us call, $G' := \mathrm{GSp}_6$. Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6)$ [$\overline{G} := \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{PGSp}_6)$] which is the Weil restriction of scalars from F to $\mathbb{Q}$ of the F-algebraic group GSp_6 [of the F-algebraic group PGSp_6]. Then, $G(\mathbb{Q}) = \mathrm{GSp}_6(F)$ and more generally for any $\mathbb{Q}$-algebra A, $G(A) = \mathrm{GSp}_6(A \otimes_{\mathbb{Q}} F)$. Hence $G(\mathbb{A}) = \mathrm{GSp}_6(\mathbb{A}_F)$. Let $G(\mathbb{A}_f)$ denote the finite part of $G(\mathbb{A})$, where $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$. Let $\mathfrak{p}$ denote a prime ideal of $\mathcal{O}_F$ and $\mathcal{O}_{F_\mathfrak{p}}$ denotes the completion of $\mathcal{O}_F$ at $\mathfrak{p}$. Then $\mathcal{O}_{F_\mathfrak{p}}$ is the ring of integers of $F_\mathfrak{p}$. For any prime p in $\mathbb{Q}$, $G(\mathbb{Q}_p) = \mathrm{GSp}_6(\mathbb{Q}_p \otimes_{\mathbb{Q}} F) = \prod_{\mathfrak{p}|p} \mathrm{GSp}_6(F_\mathfrak{p})$ where $\mathfrak{p}|p$ denotes prime ideals $\mathfrak{p}$ lying over p. Let $G_\infty = G(\mathbb{R})$ and let G_∞^+ denote the matrices in $G(\mathbb{R})$ which have positive similitudes at each place $\sigma \in S_\infty$. Let $G(\mathbb{Q})_+ = G(\mathbb{Q}) \cap G_\infty^+$. Let K_f be an open compact subgroup of $G(\mathbb{A}_f)$. We choose K_f to be $\mathrm{GSp}_6(\widehat{\mathcal{O}}_F)$ and we fix the choice. Let $K_\infty = \prod \mathrm{GU}_3(\mathbb{R})$ denote the maximal compact subgroup of $G(\mathbb{R})$; $K_\infty^+ = \prod \mathrm{U}_3(\mathbb{R})$ will denote the connected component of the identity element. Let $\mathbf{Z}$ and Z_∞ denote the center of G and G_∞, respectively.

2.2 Siegel-Hilbert automorphic forms

Siegel modular forms are certain holomorphic functions on the Siegel upper half space $\mathscr{H}_n$ of genus n. The Siegel upper half space is by definition

$$\mathscr{H}_n := \{Z = X + iY \in \mathrm{M}_n(\mathbb{C}) \mid Z = Z^t,\ Y \text{ is positive definite}\}.$$

To know basic facts about classical Siegel modular forms, we refer the readers to see Klingen [Kli90], Andrianov [And09], [And74]. We will first recall the definition of Siegel-Hilbert modular forms which are generalisations of Siegel modular forms in some sense. We are interested in genus-3 case only. We regard F as a subring of $\mathbb{R}^d$ by means of embeddings $\alpha \mapsto (\sigma_1(\alpha), \ldots, \sigma_d(\alpha))$ for α in F. Via these σ_i's we have a map, $\mathrm{GSp}_6(F) \hookrightarrow \mathrm{GSp}_6(\mathbb{R})^d$ such that,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \left(\begin{pmatrix} \sigma_1(A) & \sigma_1(B) \\ \sigma_1(C) & \sigma_1(D) \end{pmatrix}, \begin{pmatrix} \sigma_2(A) & \sigma_2(B) \\ \sigma_2(C) & \sigma_2(D) \end{pmatrix}, \ldots, \begin{pmatrix} \sigma_d(A) & \sigma_d(B) \\ \sigma_d(C) & \sigma_d(D) \end{pmatrix} \right).$$

The group $\mathrm{GSp}_6^+(\mathbb{R})$ acts on $\mathscr{H}_3$ via linear fractional transformations defined as following,

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : Z \mapsto g\langle Z\rangle := (AZ+B)(CZ+D)^{-1}.$$

Remark 2.2.1.

(1) This is a bonafide group action, i.e., $g_1g_2\langle Z\rangle = g_1\langle g_2\langle Z\rangle\rangle$ for any $g_1, g_2 \in \mathrm{GSp}_6^+(\mathbb{R})$ and $Z \in \mathscr{H}_3$.

(2) For any such symplectic map, $Z \mapsto g\langle Z\rangle$ let us define the function $\mathbf{j}$ by $\mathbf{j}(g,Z) := \det(CZ+D)$, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_6^+(\mathbb{R})$. The function $\mathbf{j}$ satisfies the cocycle relation: $\mathbf{j}(g_1g_2,Z) = \mathbf{j}(g_1, g_2\langle Z\rangle)\,\mathbf{j}(g_2,Z)$ for all $g_1, g_2 \in \mathrm{GSp}_6^+(\mathbb{R})$, and $Z \in \mathscr{H}_3$.

(3) $\mathrm{GSp}_6^+(\mathbb{R})\langle iI_3\rangle = \mathscr{H}_3$, i.e., if we vary $g_\infty \in \mathrm{GSp}_6^+(\mathbb{R})$ and apply it on iI_3 we will get entire Siegel upper half space.

(4) $\mathrm{Stab}_{\mathrm{Sp}_6(\mathbb{R})}(iI_3) = K_{\infty,\mathrm{Sp}_6} \cong \mathrm{U}(3)$, where $\mathrm{U}(3)$ is the maximal compact subgroup of $\mathrm{Sp}_6(\mathbb{R})$.

(5) $\mathrm{Stab}_{\mathrm{GSp}_6^+(\mathbb{R})}(iI_3) = K_{\infty,\mathrm{Sp}_6} \cdot Z^\infty$, where Z^∞ is the center of $\mathrm{GSp}_6^+(\mathbb{R})$.

For $Z = (Z_1, Z_2, \ldots, Z_d) \in \mathscr{H}_3^d$, where $\mathscr{H}_3^d$ is the d-fold product of Siegel upper half space, there is a group action of $\mathrm{GSp}_6^+(F)$ on $\mathscr{H}_3^d$ defined by

$$g\langle Z\rangle := (\sigma_1(g)\langle Z_1\rangle, \sigma_2(g)\langle Z_2\rangle, \ldots, \sigma_d(g)\langle Z_d\rangle),$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_6^+(F)$. Explicitly, we have,

$$\begin{aligned}\begin{pmatrix} A & B \\ C & D \end{pmatrix}\langle Z_1, Z_2, \ldots, Z_d\rangle := \Big(&(\sigma_1(A)Z_1 + \sigma_1(B))(\sigma_1(C)Z_1 + \sigma_1(D))^{-1}, \\ &(\sigma_2(A)Z_2 + \sigma_2(B))(\sigma_2(C)Z_2 + \sigma_2(D))^{-1}, \\ &\ldots, (\sigma_d(A)Z_d + \sigma_d(B))(\sigma_d(C)Z_d + \sigma_d(D))^{-1}\Big).\end{aligned}$$

This induces an action of $\mathrm{GSp}_6^+(F)$ on the space of functions $\{f : \mathscr{H}_3^d \to \mathbb{C}\}$.

$$\mathrm{GSp}_6^+(F) \times \{\mathscr{H}_3^d \to \mathbb{C}\} \longrightarrow \{\mathscr{H}_3^d \to \mathbb{C}\}$$

$$(g, f) \mapsto f|_k g,$$

where $k = (k_1, \ldots, k_d)$ and $k_1, k_2, \ldots, k_d$ are non-negative integers. The function $f|_k g$ is defined by

$$\begin{aligned} f|_k g(Z) &= f|_k g(Z_1, Z_2, \ldots, Z_d) \\ &= \prod_{l=1}^{d} \mu(\sigma_l(g))^{3k_l/2} \mathbf{j}(\sigma_l(g), Z_l)^{-k_l} f(g\langle Z\rangle). \end{aligned}$$

Definition 2.2.2. *A **Siegel-Hilbert modular form** of weight $k = (k_1, \ldots, k_d)$, genus* 3*, level* 1 *is an analytic function $f : \mathscr{H}_3^d \to \mathbb{C}$ such that $f|_k \gamma = f$ for all $\gamma \in \mathrm{GSp}_6^+(\mathcal{O}_F)$. i.e., a Siegel Hilbert modular form f of weight $(k_1, \ldots, k_d)$ is a complex valued function such that*

(1) f is an analytic function on $\mathscr{H}_3^d$.

(2) $f(\gamma\langle Z\rangle) = \prod_{l=1}^{d} \mu(\sigma_l(\gamma))^{-3k_l/2} \mathbf{j}(\sigma_l(\gamma), Z_l)^{k_l} f(Z)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathrm{GSp}_6^+(\mathcal{O}_F)$.

Remark 2.2.3. We remark that the exponent of μ as showing up in the definition of Siegel–Hilbert modular form is chosen this way so that the center of $G(\mathbb{A}) = \mathrm{GSp}_6(\mathbb{A}_F)$ acts trivially. Note that the center of GSp_6 consists of scalar matrices. The integers $k_1, k_2 \ldots, k_d$ have the same parity so that the space of Siegel–Hilbert modular forms is nonzero.

Using the strong approximation theorem of Sp_6 (See Kneser[Kne66]) one may find $t_i \in G(\mathbb{A})$, where t_i is of the form $t_i = \begin{pmatrix} a_i I_3 & 0 \\ 0 & I_3 \end{pmatrix}$ with $\mu(t_i) = a_i$ and a_i's are from ideles chosen as representatives of the narrow class group of F such that

$$G(\mathbb{A}) = \bigsqcup_{l=1}^{h} G(\mathbb{Q}) t_l \mathrm{GSp}_6(\widehat{\mathcal{O}}_F) G_\infty^+. \tag{2.2.1}$$

Note that, since $\mu(\mathrm{GSp}_6(\widehat{\mathcal{O}}_F)) = \widehat{\mathcal{O}}_F^\times$, where $\widehat{\mathcal{O}}_F = \prod_{\mathfrak{p}} \mathcal{O}_{F_{\mathfrak{p}}}$ denotes the product of all completions of $\mathcal{O}_F$, h is just the narrow class number of F, where narrow class number is the cardinality of the narrow class group $F^\times \backslash \mathbb{A}_F^\times / \widehat{\mathcal{O}}_F^\times F_\infty^{+\times}$ of F.

Now set, $\Gamma_l = G(\mathbb{Q})_+ \cap t_l K_f G_\infty^+ t_l^{-1}$. This Γ_l is an arithmetic subgroup of $G(\mathbb{Q})$. Let us denote by $Z_0 = (iI_3, iI_3, \ldots, iI_3)$, the base point in $\mathscr{H}_3^d$. Note that, $\mathscr{H}_3^d \cong G_\infty^+ / K_\infty^+ Z_\infty$. Then the map,

$$\gamma t_l u_f g_\infty \mapsto g_\infty \langle Z_0 \rangle$$

for $\gamma \in G(\mathbb{Q}), u_f \in K_f$ and $g_\infty \in G_\infty^+$, induces a decomposition,

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_f K_\infty^+ Z_\infty \cong \bigsqcup_{l=1}^{h} \Gamma_l \backslash \mathscr{H}_3^d. \tag{2.2.2}$$

We put $\mathbf{M}_k(\Gamma_l)$ to be the space of Siegel Hilbert modular forms of weight $k = (k_1, \ldots, k_d)$ with respect to Γ_l by which we mean a space of functions f that are holomorphic on $\mathscr{H}_3^d$ and satisfy $f|_k\gamma = f$ for all $\gamma \in \Gamma_l$ (Definition 2.2.2). Every $f \in \mathbf{M}_k(\Gamma_l)$ admits a Fourier expansion, which by the Koecher principle takes the form,

$$f(Z) = \sum_{\{Q\} \cup \{0\}} a_Q e^{2\pi i \mathrm{Tr}(QZ)},$$

where Q runs over all half-integral symmetric totally positive matrices and Tr denotes the trace of a matrix.

Definition 2.2.4. *A Siegel-Hilbert modular form is called a* ***cusp form*** *if for all $\gamma \in \Gamma_l$, the constant term in the Fourier expansion of $f|_k\gamma$ vanishes.*

We denote the space of Siegel Hilbert cusp forms by $\mathbf{S}_k(\Gamma_l)$.

Now choose a function, $f_l \in \mathbf{S}_k(\Gamma_l)$ for each $l \in \{1, \ldots, h\}$ and put $\Phi_f := (f_1, f_2, \ldots, f_h)$. Then using the decompositions (2.2.1) and (2.2.2), let us define, $\Phi_f : G(\mathbb{A}) \to \mathbb{C}$ by

$$\Phi_f(\gamma t_l u_f g_\infty) = f_l|_k g_\infty(Z_0)$$

for $\gamma \in G(\mathbb{Q}), u_f \in K_f$ and $g_\infty \in G_\infty^+$.

Definition 2.2.5. *A **Siegel–Hilbert automorphic cusp form** of weight $k = (k_1, \ldots, k_d)$ and level* 1 *is a function $\Phi : G(\mathbb{A}) \to \mathbb{C}$ satisfying the following properties:*

(1) $\Phi(\gamma g) = \Phi(g)$ *for all* $\gamma \in G(\mathbb{Q})$.

(2) $\Phi(zg) = \Phi(g)$ *for all* $z \in \mathbf{Z}(\mathbb{A})$.

(3) $\Phi(gu_f) = \Phi(g)$ *for all* $u_f \in K_f$.

(4) $\Phi(gu_\infty) = \prod_{l=1}^{d} j(u_\infty^l, iI_3)^{-k_l}\Phi(g)$, *where* $u_\infty \in K_\infty^+$ *and* $u_\infty^l := \sigma_l(u_\infty), \sigma_l \in S_\infty$.

(5) Φ *has vanishing constant terms, i.e., for each* $g \in G(\mathbb{A})$, $\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Phi(ng)\mathrm{d}n = 0$, *where N is the unipotent radical of B_∞, the standard Borel subgroup of G.*

We denote the space of Siegel-Hilbert cuspidal automorphic forms by $\mathcal{S}_k(K_f)$.

2.3 Hecke algebras

Let Φ be a cusp form of weight $k = (k_1, k_2, \ldots, k_d)$ and of level 1. The space of cuspidal automorphic forms, denoted by $\mathcal{S}_k(K_f)$ comes equipped with a Hecke-algebra action. First, we will recall the definition of Hecke algebra and then the action of it on $\mathcal{S}_k(K_f)$. Now, let $\Delta_f = G(\mathbb{A}_f) \cap \mathrm{M}_6(\widehat{\mathcal{O}}_F)$, let $K_f\backslash\Delta_f/K_f$ denote the space of double cosets of K_f in Δ_f. Define the Hecke algebra,

$$\mathscr{H}(\Delta_f, K_f) := \mathbb{Z}[K_f\backslash\Delta_f/K_f]$$

to be the free abelian group with basis the set of double cosets of K_f in Δ_f. For a double coset, $K_f g K_f \in K_f\backslash\Delta_f/K_f$, let $[K_f g K_f]$ denote the corresponding basis element. The algebra structure on $\mathscr{H}(\Delta_f, K_f)$ is given by customary convolution formula

$$[K_f g_\alpha K_f] * [K_f g_\beta K_f] = \sum c_{\alpha\beta\gamma}[K_f g_\gamma K_f],$$

where the coefficients $c_{\alpha\beta\gamma}$ are computed as follows:

The group $K_{f_\alpha} = K_f \cap g_\alpha K_f g_\alpha^{-1}$ is compact and open, hence of finite index in K_f. Hence there exists finite number of elements $x_1, x_2, \ldots, x_m$ of K_f such that $K_f = \sqcup_{j=1}^{m} x_j K_{f_\alpha}$.

Therefore $K_f g K_f = \sqcup x_j g_\alpha K_f$. Define similarly K_{f_β} and get $y_1, \ldots, y_n$. Then $c_{\alpha\beta\gamma}$ is the number of pairs (j,l) such that $g_\gamma^{-1} x_j g_\alpha y_l g_\beta \in K_f$ (see Cartier [Car79, p. 116] and Shimura [Shi71]). Now for each integral ideal $\mathfrak{m}$, let $T_\mathfrak{m} = \sum_g [K_f g K_f]$, where the sum is taken over all distinct double cosets with $g \in \Delta_f$ such that $(\mu(g))\mathcal{O}_F = \mathfrak{m}$. Noting that summand $K_f g K_f$ can be expressed as a disjoint union of left cosets, i.e., $K_f g K_f = \sqcup_l g_l K_f$, mentioned earlier, we can define Hecke action on Φ as

$$(\Phi|_{[K_f g K_f]})(x) = \sum_l \Phi(x g_l).$$

We can define the Hecke algebra, locally as following.

For each prime $\mathfrak{p}$ of $\mathcal{O}_F$, let $\varpi_\mathfrak{p}$ be the uniformizer of $\mathcal{O}_{F_\mathfrak{p}}$. Let $G_\mathfrak{p} := \mathrm{GSp}_6(F_\mathfrak{p})$ and $K_\mathfrak{p} := \mathrm{GSp}_6(\mathcal{O}_{F_\mathfrak{p}})$. Let $\mathscr{H}_\mathfrak{p}(G_\mathfrak{p}, K_\mathfrak{p})$ be the unramified Hecke algebra consisting of compactly supported functions which are bi-$K_\mathfrak{p}$ invariant, i.e., $T(kgk') = T(g)$ for all $g \in G_\mathfrak{p}$ and $k, k' \in K_\mathfrak{p}$. The definition assures that T vanishes off a finite union of double cosets $K_\mathfrak{p} g K_\mathfrak{p}$. The multiplication in $\mathscr{H}_\mathfrak{p}(G_\mathfrak{p}, K_\mathfrak{p})$ is defined by the customary convolution formula,

$$(T_1 * T_2)(x) = \int_{G_\mathfrak{p}} T_1(xy) T_2(y^{-1}) dy,$$

for $T_1, T_2 \in \mathscr{H}_\mathfrak{p}(G_\mathfrak{p}, K_\mathfrak{p})$. The integral makes sense since as a function of y the integrand is locally constant and compactly supported. In our case, we have four Hecke operators corresponding to the double $K_\mathfrak{p}$ cosets of the 6×6 symplectic similitude matrices which are,

$$\begin{aligned}
T_{0,\mathfrak{p}} &= \mathrm{diag}(1,1,1,\varpi_\mathfrak{p},\varpi_\mathfrak{p},\varpi_\mathfrak{p}) \\
T_{1,\mathfrak{p}} &= \mathrm{diag}(\varpi_\mathfrak{p},1,1,\varpi_\mathfrak{p}^{-1},1,1) \\
T_{2,\mathfrak{p}} &= \mathrm{diag}(1,\varpi_\mathfrak{p},1,1,\varpi_\mathfrak{p}^{-1},1) \\
T_{3,\mathfrak{p}} &= \mathrm{diag}(1,1,\varpi_\mathfrak{p},1,1,\varpi_\mathfrak{p}^{-1}).
\end{aligned}$$

Therefore,

$$\mathscr{H}_\mathfrak{p}(G_\mathfrak{p}, K_\mathfrak{p}) = \mathbb{C}[T_{0,\mathfrak{p}}, T_{1,\mathfrak{p}}, T_{2,\mathfrak{p}}, T_{3,\mathfrak{p}}]$$

Details are given in Asgari-Schmidt[AS01, p. 177]. The Hecke algebra $\mathscr{H}_\mathfrak{p}(G_\mathfrak{p}, K_\mathfrak{p})$ is generated by the operators $T_{0,\mathfrak{p}}, T_{1,\mathfrak{p}}, T_{2,\mathfrak{p}}, T_{3,\mathfrak{p}}$ and

$\mathscr{H}(\Delta_f, K_f) \otimes \mathbb{C} = \otimes_{\mathfrak{p}} \mathscr{H}_{\mathfrak{p}}(G_{\mathfrak{p}}, K_{\mathfrak{p}})$ (cf.[BJ79, p.194]) where $\mathfrak{p}$ runs over all primes in $\mathcal{O}_F$. There is a left action of $\mathscr{H}_{\mathfrak{p}}(G_{\mathfrak{p}}, K_{\mathfrak{p}})$ on $\Phi \in S_k(K_f)$, which is given by

$$(T\Phi)(x) = \int_{G_{\mathfrak{p}}} T(h)\Phi(xh)dh,$$

where $T \in \mathscr{H}_{\mathfrak{p}}(G_{\mathfrak{p}}, K_{\mathfrak{p}})$ and $x \in G(\mathbb{A})(= \mathrm{GSp}_6(\mathbb{A}_F))$. If T is a characteristic function of $K_{\mathfrak{p}} g K_{\mathfrak{p}}$ then writing $K_{\mathfrak{p}} g K_{\mathfrak{p}}$ as a disjoint union of left cosets, $K_{\mathfrak{p}} g K_{\mathfrak{p}} = \sqcup_l g_l K_{\mathfrak{p}}$ and noting that Φ is right $K_{\mathfrak{p}}$-invariant, we get $(T\Phi)(x) = \sum \Phi(x g_l)$. By Iwasawa decomposition of GSp_6, we may assume that, $g_l = \begin{pmatrix} A_l & B_l \\ 0 & \varpi_{\mathfrak{p}}^{d_{l0}}\, {}^t A_l^{-1} \end{pmatrix}$ with $A_l = \begin{pmatrix} \varpi_{\mathfrak{p}}^{d_{l1}} & 0 & 0 \\ * & \varpi_{\mathfrak{p}}^{d_{l2}} & 0 \\ * & * & \varpi_{\mathfrak{p}}^{d_{l3}} \end{pmatrix}$, where d_{lj} are integers, d_{l0} does not depend on l since it equals the valuation of $\mu(g)$. Here μ is the similitude factor (cf.[AS01, p. 178]).

2.4 Restriction of scalars and L-group

In this section, we will recall the definition of Langlands L-group and some necessary facts from Springer [Spr79] and Borel [Bor79, p. 34]. The notations and definitions are borrowed from the above mentioned references. We have already fixed our group to be $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6)$ and $G' = \mathrm{GSp}_6$, where F is a totally real number field. We denote the Galois group of $\overline{\mathbb{Q}}$ over F by $\Gamma_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ and $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Then Γ_F is an open subgroup of finite index of $\Gamma_{\mathbb{Q}}$.

Let $\sum_{F,\mathbb{Q}} = \Gamma_F \backslash \Gamma_{\mathbb{Q}}$ be the set of $\mathbb{Q}$-monomorphisms $F \to \overline{\mathbb{Q}}$. Then

$$G(\overline{\mathbb{Q}}) = \mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(G'(\overline{\mathbb{Q}})) = \prod_{\sigma \in \Gamma_F \backslash \Gamma_{\mathbb{Q}}} {}^{\sigma}G'(\overline{\mathbb{Q}}) = \prod_{\alpha : F \hookrightarrow \overline{\mathbb{Q}}} \alpha G'(\overline{\mathbb{Q}}),$$

where $\mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(G'(\overline{\mathbb{Q}})) := \{f : \Gamma_{\mathbb{Q}} \to G'(\overline{\mathbb{Q}}) \mid f(g'g) = g' \cdot f(g), g' \in \Gamma_F, g \in \Gamma_{\mathbb{Q}}\}$. For general definition of induced groups please see Borel[Bor79, p. 33]. Since, G' and G both are connected, reductive groups, it is possible to associate the root datum,

$$\psi(G') = (X^*(T'), \phi', X_*(T'), \phi'^{\vee}),$$

with G'. Here T' is a maximal torus of G' defined over $\overline{\mathbb{Q}}$, $X^*(T')$ denotes the group of characters of T' whereas $X_*(T')$ is the 1-parameter subgroup of T', $\phi, \phi'^{\vee}$ denote the set of roots and co-roots with respect to T' respectively. The choice of Borel subgroup $B' \supset T'$ (defined over $\overline{\mathbb{Q}}$) gives a basis Δ' of root datum, and so we can fix a based root data $\psi_0(G') = (X'^*, \Delta', X'_*, \Delta'^{\vee})$ associated to G'. Then, the root data corresponding to the group G is given by $\psi_0(G) = (X^*, \Delta, X_*, \Delta^{\vee})$, where

$$X = \mathrm{Ind}_{\Gamma_F}^{\Gamma_{\mathbb{Q}}}(X') \text{ and } \Delta = \cup_{a \in \Gamma_F \backslash \Gamma_{\mathbb{Q}}} \Delta' \cdot a. \tag{2.4.1}$$

For details, see Borel[Bor79, p. 35].

Remark 2.4.1. In our case, G' is an F-group, hence it is quasi-split over F. G is quasi-split over $\mathbb{Q}$. Note that G is not split over $\mathbb{Q}$. But G' is split over $\mathbb{Q}$.

Correspondingly, $B = \mathrm{Res}_{F/\mathbb{Q}} B'$ is a Borel subgroup of G and $T = \mathrm{Res}_{F/\mathbb{Q}}(T')$ will stand for torus in G. For any $\mathbb{Q}$-algebra A, we can talk about $B(A), T(A)$ as we did for $G(A)$. The inverse system to the based root datum $\psi_0(G')$ is $\psi_0(G')^{\vee} = (X'_*, \Delta'^{\vee}, X'^*, \Delta')$. To the $\overline{\mathbb{Q}}$-group G', we first associate the group ${}^L G'^0$ over $\mathbb{C}$ such that $\psi_0({}^L G'^0) = \psi_0(G')^{\vee}$. Let ${}^L T'^0, {}^L B'^0$ be the maximal torus and Borel subgroup defined by $\psi_0(G')^{\vee}$. We have a canonical bijection,

$$\mathrm{Aut}(\psi_0(G')^{\vee}) \cong \mathrm{Aut}({}^L G'^0, {}^L B'^0, {}^L T'^0, \{x_\alpha\}_{\alpha \in \Delta'^{\vee}})$$

and a homomorphism

$$\mu_{G'} : \Gamma_F \to \mathrm{Aut}(\psi_0(G')^{\vee}).$$

For details see [Bor79, Section 2.3]. Thus, we can define the Langlands dual group associated to G' as ${}^L G' = {}^L G'^0 \rtimes \Gamma_F = {}^L G'^0 \times \Gamma_F$ (Since G' splits over F, we get direct product) and associated to G as ${}^L G = {}^L G^0 \rtimes \Gamma_{\mathbb{Q}}$.

Remark 2.4.2. There are various variants of this notion, depending on the convenience of contexts. For instance, if we take a finite Galois extension F' of $\mathbb{Q}$ such that $F' \supset F$, then our group G splits over F' (G splits over F, so does over F' too). Now $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ acts trivially on $\psi_0(G,T)$ (since torus T is now defined over F'). Hence, we can divide $\Gamma_{\mathbb{Q}}$, by this closed normal subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/F')$ and take the action of $\mathrm{Gal}(F'/\mathbb{Q})$ on ${}^LG^0$.

By following the above remark, we can replace $\Gamma_{\mathbb{Q}}$ in the definition of L-group of G and can take the definition of L-group as

$${}^LG = {}^LG^0 \rtimes \mathrm{Gal}(F'/\mathbb{Q}).$$

Now,

$${}^LG = {}^L(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6)) = {}^L(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6))^0 \rtimes \mathrm{Gal}(F'/\mathbb{Q}).$$

Here

$$\begin{aligned} {}^LG^0 = {}^L(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6))^0 &\cong \prod_{\sigma \in \mathrm{Gal}(F'/F)\backslash \mathrm{Gal}(F'/\mathbb{Q})} {}^{\sigma}({}^L\mathrm{GSp}_6^0) \\ &= \prod_{\substack{\sigma : F \hookrightarrow F' \\ \mathbb{Q} \text{ embeddings}}} {}^{\sigma}\mathrm{GSpin}_7(\mathbb{C}) \\ &= \underbrace{\mathrm{GSpin}_7 \times \cdots \times \mathrm{GSpin}_7}_{d \text{ many copies}} \end{aligned}$$

(since $|\mathrm{Gal}(F'/F)\backslash \mathrm{Gal}(F'/\mathbb{Q})| = [F:\mathbb{Q}] = d$). Hence, ${}^LG = (\mathrm{GSpin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q})$. Here we have dropped $\mathbb{C}$ and simply written complex dual group $\mathrm{GSpin}_7(\mathbb{C})$ of GSp_6 as GSpin_7.

2.5 Parabolic subgroups of ${}^LG$ and Levi-decompositions

The notations and definitions in this section are borrowed from Borel [Bor79, p. 32]. We know that there is a canonical bijection between the set of conjugacy classes of parabolic $\overline{\mathbb{Q}}$-subgroups of G with respect to $G(\overline{\mathbb{Q}})$ and the subsets of Δ, Δ denoting the basis of root data corresponding to G. Let $J(\tilde{P})$ be the subset of Δ assigned to the class of $\tilde{P}$, where $\tilde{P}$ is any parabolic $\overline{\mathbb{Q}}$- subgroup of G.

Parabolic subgroups in L-dual: A parabolic subgroup P of ${}^LG$ is the normaliser of a parabolic subgroup P^0 in ${}^LG^0$ provided the normaliser meets every class of P modulo ${}^LG^0$. We call P to be standard parabolic if P contains Borel subgroup ${}^LB$. The standard parabolic subgroups are the subgroups ${}^LP^0 \rtimes \mathrm{Gal}(F'/\mathbb{Q})$, where ${}^LP^0$ runs through the standard parabolic subgroups of ${}^LG^0$ such that $J({}^LP^0) \subset \Delta^\vee$ is stable under $\mathrm{Gal}(F'/\mathbb{Q})$. Note that every parabolic subgroup of ${}^LG$ is a conjugate (under ${}^LG$ or ${}^LG^0$) to one and only one standard parabolic subgroup. So it is enough to talk about only standard parabolic subgroups (see [Bor79, p. 32, 33]).

Levi subgroups: Let P be a parabolic subgroup of ${}^LG$. The unipotent radical N of P^0 is normal in P. We call N to be the *unipotent radical* of P too in ${}^LG$. Then $P^0/N \cong M^0$ is Levi in ${}^LG^0$. In fact, $P \cong N \rtimes N_P(M^0)$, these normalizers $N_P(M^0)$ are Levi-subgroups of P. Let ${}^LP$ be the standard parabolic subgroup associated with parabolic subgroup P of G. Then ${}^LM = {}^LM^0 \rtimes \mathrm{Gal}(F'/\mathbb{Q})$ is identified with a Levi subgroup of ${}^LP$. Sometimes we replace the term "Levi subgroups of parabolic subgroup P in G" with "Levi-subgroup in G" for the sake of brevity.

2.6 Automorphic representations

We now associate a representation of $\overline{G}(\mathbb{A}) = \mathrm{PGSp}_6(\mathbb{A}_\mathbb{F})$ with Siegel–Hilbert automorphic form Φ defined in Section 2.2. Following Borel and Jacquet [BJ79], we say an irreducible representation of $G(\mathbb{A})$ is automorphic if it is isomorphic to an irreducible subquotient of the representation of $G(\mathbb{A})$ on its space of automorphic forms. Let Φ be an automorphic form on $G(\mathbb{A})$ which lies in $L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))$. Let V_Φ denote the subspace of this Hilbert space L^2 spanned by all right translates of Φ. Let π be an irreducible constituent of this representation. Let V_π be its representation space. Then π is an automorphic representation of $G(\mathbb{A}) = \mathrm{GSp}_6(\mathbb{A}_F)$, which is trivial on $\mathbf{Z}(\mathbb{A})$. Hence we can consider π as an automorphic representation of $\mathrm{PGSp}_6(\mathbb{A}_F)$. Now using the decomposition theorem by Flath (cf.[Fla79]), let us decompose $\pi = \pi_\infty \otimes \pi_f$, where $\pi_\infty = \prod_{\omega|\infty} \pi_\omega$ is an irreducible representation of $G(\mathbb{R}) = (\mathrm{GSp}_6(\mathbb{R}))^d$. In fact, we can write, $\pi_\infty = \otimes_{\sigma \in S_\infty} \pi_\sigma = \pi_{\sigma_1} \otimes \cdots \otimes \pi_{\sigma_d}$, where $S_\infty = \{\sigma_1, \ldots, \sigma_d\}$. Here each π_{σ_i} is an irreducible representation of $\mathrm{GSp}_6(\mathbb{R})$. The representation $\pi_f = \otimes'_p(\prod_{\mathfrak{p}|p} \pi_\mathfrak{p})$ is a restricted tensor product and an irreducible representation of $G(\mathbb{A}_f)$. Call $\pi_p := \prod_{\mathfrak{p}|p} \pi_{\mathfrak{p}}$, where $\pi_{\mathfrak{p}}$ is an irreducible representation of $\mathrm{GSp}_6(F_{\mathfrak{p}})$. The representations π_p are irreducible representations of $G(\mathbb{Q}_p)$ and by Flath's theorem, almost all of π_p's are unramified (spherical) [Fla79]. That means, for almost all prime p the representation space of π_p has a vector fixed by certain maximal compact subgroup $G(\mathbb{Z}_p)$. Let S denote the set of places of $\mathbb{Q}$ which include Archimedean place ∞, the ramified primes p and those finite places p where π_p is not spherical.

In this decomposition of π, π_∞ is completely determined by the weights of Siegel Hilbert automorphic form Φ and π_p's are completely determined by the Satake parameters which we will be going to talk about in the next section.

2.7 *L*-functions

The isomorphism class of the spherical representations depends only on the unramified characters modulo the action of the Weyl group. It is further proved that each spherical representation is obtained in this way, for details see [AS01]. In our case, for $p \notin S$, each π_p is spherical, so π_p is obtained by unramified characters of $\mathbb{Q}_p^*$ (unramified characters are homomorphisms $\mathbb{Q}_p^* \to \mathbb{C}^*$, which are trivial on $\mathbb{Z}_p^*$). In fact, the Satake isomorphism attaches each π_p with a unique semisimple conjugacy class (known as Satake parameter) $t(\pi_p)$ in the local L-group ${}^LG_p$ (${}^LG_p$ is the L-group of G as a group defined over $\mathbb{Q}_p$), where $t_p := t(\pi_p) = (t_p^0, \mathrm{Fr}_p)$, $t_p^0 \in {}^LT^0, T = \mathrm{Res}_{F/\mathbb{Q}}T'$ and t_p^0 is determined up to conjugacy by ${}^LT^0$, Fr_p denotes the unique Frobenius conjugacy class in $\mathrm{Gal}(F'_{\mathfrak{p}}/\mathbb{Q}_p)$. We may further assume t_p^0 to be fixed by Fr_p (for details see Borel [Bor79, p. 35, Section 6] and Shahidi [Sha88, p. 553]).

Now let us take $\psi : {}^LG \to \mathrm{GL}_m(\mathbb{C})$ to be a finite dimensional complex representation of ${}^LG$. Let ψ_p denote the composite map ${}^LG_p \to {}^LG \to \mathrm{GL}_m(\mathbb{C})$ (since we have $G \hookrightarrow G_p$, hence we have a natural homomorphism, ${}^LG_p \to {}^LG$).

Then one can define partial Langlands L-function by

$$L^S(s, \pi, \psi) := \prod_{p \notin S} L_p^S(s, \pi_p, \psi_p) \tag{2.7.1}$$

for $s \in \mathbb{C}$, where the local Euler factors attached to π_p and ψ_p are defined as

$$L_p^S(s, \pi_p, \psi_p) := \det\left(I - \psi_p(t_p^0, \mathrm{Fr}_p) p^{-s}\right)^{-1}.$$

If p splits completely in $\mathcal{O}_F$, i.e., if $(p) = \mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_d$, where $d = [F : \mathbb{Q}]$, then $\pi_p = \pi_{\mathfrak{p}_1} \otimes \pi_{\mathfrak{p}_2} \otimes \cdots \otimes \pi_{\mathfrak{p}_d}$, where each $\pi_{\mathfrak{p}_i}$ is spherical, $t_p^0 = (t_{\mathfrak{p}_1}, t_{\mathfrak{p}_2}, \ldots, t_{\mathfrak{p}_d}) \in {}^L T^0$ with semisimple conjugacy classes $t_{\mathfrak{p}_i}$ associated to $\pi_{\mathfrak{p}_i}$, Fr_p =identity. By abuse of notation, we also denoted by $\pi = \otimes'_p \pi_p$ an automorphic representation of $\mathrm{PGSp}_6(\mathbb{A}_F)$ (Section 2.6) attached to a Siegel–Hilbert automorphic form Φ introduced in section 2.2.

Here the L-group of $\overline{G}$ is

$${}^L\overline{G} = (\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q});$$

Note that $\mathrm{Spin}_7 \subset \mathrm{GSpin}_7$ is also the derived group of GSpin_7 and ${}^L\overline{T}^0 := {}^L T^0 \cap (\mathrm{Spin}_7)^d$ is the maximal torus of ${}^L G^0$.

We are going to take two particular representations of our group ${}^L\overline{G}$. We will describe them now. Let

$$\rho_1 : \mathrm{Spin}_7(\mathbb{C}) \to \mathrm{SO}_7(\mathbb{C})$$

and

$$\rho_2 : \mathrm{Spin}_7(\mathbb{C}) \to \mathrm{SO}_8(\mathbb{C})$$

denote the first two fundamental representations of $\mathrm{Spin}_7(\mathbb{C})$, namely the "projective representation" ρ_1 and "spin representation" ρ_2, respectively.

Definition 2.7.1. *Define,* $\mathbb{P}_{\tau,n,d} :=$ *A block permutation matrix of order* $nd \times nd$*, where* τ *is some* $d \times d$ *permutation matrix which replaces each* 0 *and* 1 *by either null matrix* 0_n *or identity matrix* I_n*.*

Now corresponding to the representations ρ_1 and ρ_2, let us define another two representations as following,

$$\phi_1 : (\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q}) \to \mathrm{GL}_{7d}(\mathbb{C})$$

is defined by $\phi_1(g_1,g_2,\dots,g_d,1)=\mathrm{diag}(\rho_1(g_1),\rho_1(g_2),\dots,\rho_1(g_d))$ for $(g_1,\dots,g_d)\in(\mathrm{Spin}_7)^d$ and $\phi_1(1,\dots,1,\tau)=\mathbb{P}_{\tau,7,d}$ for $\tau\in\mathrm{Gal}(F'/\mathbb{Q})\subset S_d$, where S_d represents the symmetric group defined over $\{1,2,\dots,d\}$. Note that any element $(\tilde{g},\tau)$ of $(\mathrm{Spin}_7)^d\rtimes\mathrm{Gal}(F'/\mathbb{Q})$ can be written as $(\tilde{g},\tau)=(\tilde{g},1)\cdot(1,\tau)$ (by the definition of semi-direct product). So, it is enough to describe ϕ_1 on $(\tilde{g},1)$ and $(1,\tau)$ separately.

Similarly, we define

$$\phi_2:(\mathrm{Spin}_7)^d\rtimes\mathrm{Gal}(F'/\mathbb{Q})\to\mathrm{GL}_{8d}(\mathbb{C})$$

by $\phi_2(g_1,\dots,g_d,1)=\mathrm{diag}(\rho_2(g_1),\dots,\rho_2(g_d))$ for $(g_1,\dots,g_d)\in(\mathrm{Spin}_7)^d$
and $\phi_2(1,\dots,1,\tau)=\mathbb{P}_{\tau,8,d}$ for $\tau\in\mathrm{Gal}(F'/\mathbb{Q})$.

Then corresponding to these two representations ϕ_1,ϕ_2 we have two Langlands L-functions associated to an automorphic representation $\pi=\otimes'_{p\notin S}\pi_p$ of $\overline{G}(\mathbb{A}_{\mathbb{Q}})$. They are respectively $L^S(s,\pi,\phi_1)$ and $L^S(s,\pi,\phi_2)$. These two L-functions are defined in the same way as in (2.7.1).

However, it remains to define such local L-functions for the remaining places, i.e., for all $p\in S$. We are dealing with level 1 Siegel-Hilbert automorphic cusp forms and at level 1 case finite primes are all such that π_p are unramified. Now if not level 1 then there are finite number of ramified primes. To define the completed L-function, we need to define L-factors at those 'bad' primes. It is trickier to define L-function at those bad places though, as we can not define Satake parameters and calculate. The way is to go about it, is to take Rankin-Selberg convolutions (global zeta integrals) which are Eulerian integral representations associated to π. Though this concept is valid when π is generic (because then we can associate a Whittaker model to it). The global Whittaker function then decomposes as a product of local Whittaker functions, the product varies over all places of $\mathbb{Q}$. The zeta integrals are defined with the help of Whittaker functions and having reduced the matters to the local theory, it remains to analyse the L-factors in terms of the local zeta integrals. Then the integrals corresponding to automorphic forms at finite ramified places v form a principal ideal. And that principal ideal is generated by a rational function of the form q_v^{-s}. Hence, the resulting L-factors at the ramified places v are rational function in q_v^{-s}.

Generally, this is how L-factors are defined at ramified places under the condition that π is generic, for example, see [BG92]. Unfortunately, Siegel modular forms (in our case Siegel-Hilbert automorphic forms) are not generic. On the other hand, Piatetski-Shapiro and Rallis [GPSR87] introduced integral representation using doubling method, which represents the standard L-function for any classical group over any number field F. Where the cuspidal automorphic representation of that classical group needs not to be generic. Recently Cai-Friedberg-Ginzburg-Kaplan [CFGK17] generalised the doubling method and provided integral representations for L-functions for arbitrary cuspidal automorphic representations of classical groups twisted by automorphic cuspidal representations of arbitrary rank general linear groups. The authors worked out the case for the symplectic group Sp_{2n} in detail in this paper [CFGK17]. The global integral coming from the generalised doubling construction in [CFGK17] uses the specialised inducing data namely the generalised Speh representations. This global integral converges absolutely in some right half-plane and admits meromorphic continuation to the whole complex plane. Cai-Friedberg-Ginzburg-Kaplan introduced a new generalised model known as Whittaker-Speh-Shalika model and that includes the generic and non-generic automorphic cuspidal representations of $\mathrm{Sp}_{2n}(\mathbb{A})$. Using this model the global integral unfolds to an adelic integral. That adelic integral is almost Eulerian in the sense that every unramified component can be separated ([CFGK17, Section 3, equation 3.1, Theorem 21]). Consequently this integral represents the partial L-function which is a product of local L-functions over all finite places of F for which the local data is unramified. In Section 3 of [CFGK17] the authors computed the local factors with unramified data. In their second paper, Cai-Friedberg-Kaplan [CFK18] have developed the local theory of the doubling integrals over all places of F including ramified and Archimedean ones. Since these theories are applicable for any cuspidal automorphic representaions of the classical groups, hence it shall include the case of Siegel modular forms too. Thus one can recover the standard L-function of GSp_{2n} via the doubling method. Infact the paper [CFK18] covers the complete local and global theory (over all places of F) of tensor product L-functions for any cuspidal automorphic representaions of GSpin group with represenations of GL_k without any condition on genericity on representations of GSpin. Hence, one can recover the spin L-function of GSp_{2n} via the doubling method too. Please see the paper [CFK18] for more details.

The L-functions at the Archimedean places are defined by local Langlands correspondence (cf. [Lan71a]). We attach the Archimedean Euler factor computations in the next section.

2.7.1 Archimedean Euler factors

In this section, we mostly follow Schmidt [Sch02]. This section is devoted to give the formula for the archimedean Euler factors for ϕ_1 and ϕ_2. We need to set the stage by recalling some basic facts about representations of real Weil groups.

Representations of the Weil group

The real Weil group, denoted by $W_{\mathbb{R}}$, is defined as a semidirect product $W_{\mathbb{R}} := \mathbb{C}^* \rtimes < j >$, where j is an element such that $j^2 = -1$ which acts on $\mathbb{C}^*$ by $jzj^{-1} = \bar{z}$ for $z \in \mathbb{C}^*$. Here bar denotes the complex conjugation. We are interested in finite-dimensional complex semisimple representations of $W_{\mathbb{R}}$. A representation of $W_{\mathbb{R}}$ is called **semisimple** if the image of $W_{\mathbb{R}}$ consists of semisimple elements in some finite-dimensional complex vector space. Every such representation is completely reducible. Any irreducible semisimple representation of $W_{\mathbb{R}}$ has dimension 1 or 2. They are listed as follows:

<u>One-dimensional representations:</u>

$$\tau_{+,t} : z \mapsto |z|^t,\ j \mapsto 1, \tag{2.7.2}$$

$$\tau_{-,t} : z \mapsto |z|^t,\ j \mapsto -1. \tag{2.7.3}$$

Where $t \in \mathbb{C}$ and $|\cdot|$ is the usual absolute value on $\mathbb{C}$.

<u>Two-dimensional representations:</u>

$$\tau_{u,t} : re^{i\theta} \mapsto \begin{pmatrix} r^{2t}e^{iu\theta} & \\ & r^{2t}e^{-iu\theta} \end{pmatrix},\ j \mapsto \begin{pmatrix} & (-1)^u \\ 1 & \end{pmatrix}. \tag{2.7.4}$$

Where we have $t \in \mathbb{C}$ and u as positive integers. An L-factor is attached to a semisimple representation of $W_{\mathbb{R}}$. For an arbitrary semisimple representation, the associated L-factor is the product of the L-factors of its irreducible components. For the aforementioned irreducible representations the L-factors are the following:

$$L(s, \tau_{+,t}) = \pi^{-\frac{(s+t)}{2}} \Gamma\left(\frac{s+t}{2}\right), \tag{2.7.5}$$

$$L(s, \tau_{-,t}) = \pi^{-\frac{(s+t+1)}{2}} \Gamma\left(\frac{s+t+1}{2}\right), \tag{2.7.6}$$

$$L(s, \tau_{u,t}) = 2(2\pi)^{-(s+t+u/2)} \Gamma\left(s+t+\frac{u}{2}\right). \tag{2.7.7}$$

The local Langlands correspondence (**LLC**) is a parametrization of the infinitesimal equivalence classes of irreducible admissible representations of a real reductive group $\mathsf{G}(\mathbb{R})$ by admissible homomorphisms $W_{\mathbb{R}} \to {}^L\mathsf{G}$ into the L-group of G. If G is split over $\mathbb{R}$ then instead of ${}^L\mathsf{G}$ we can work with the identity component of ${}^L\mathsf{G}$, i.e., the complex group ${}^L\mathsf{G}^0$. Let π be an irreducible admissible representation of G with archimedean component as π_∞ and ρ be a finite-dimensional representation of ${}^L\mathsf{G}$. Let $\varphi : W_{\mathbb{R}} \to {}^L\mathsf{G}$ be the local parameter attached to the representation π_∞. If we define a semisimple representation of $W_{\mathbb{R}}$ by $\tau := \rho \circ \varphi$, then the L-factor associated to π_∞ and ρ is defined by

$$L(s, \pi_\infty, \rho) := L(s, \tau).$$

We will be interested in the following situation. When G is $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{PGSp}_6)$ (already denoted by $\overline{G}$ in the beginning of Section 2.1) and π_∞ being the archimedean component of the automorphic representation of $\mathrm{PGSp}_6(\mathbb{A}_F)$ corresponding to a Siegel-Hilbert automorphic form Φ of weight $k = (k_1, k_2, \ldots, k_d)$, where each k_i is an integer and $k_i > 3$ for $i = 1, \ldots, d$. We write

$$\pi_\infty = \prod_{\sigma \in S_\infty} \pi_\sigma = \pi_{\sigma_1} \otimes \cdots \otimes \pi_{\sigma_d},$$

where $S_\infty = \{\sigma_1, \ldots, \sigma_d\}$ is the finite set of all archimedean places. In this case, ${}^L\overline{G}$ is $(\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q})$. We are concerned with two types of finite-dimensional representations: one is $\phi_1 : (\mathrm{Spin}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q}) \to \mathrm{GL}_{7d}(\mathbb{C})$ and the other one is $\phi_2 : (\mathrm{Spin}_7)^d \rtimes$

$\mathrm{Gal}(F'/\mathbb{Q}) \to \mathrm{GL}_{8d}(\mathbb{C})$. Since $\overline{G}$ splits over F', we can replace ${}^L\overline{G}$ with its identity component ${}^L\overline{G}^0 = (\mathrm{Spin}_7)^d$ and work with that. By abuse of the notations, we will denote the restriction of ϕ_1 and ϕ_2 on ${}^L\overline{G}^0$ by ϕ_1 and ϕ_2 only. So $\phi_1 : (\mathrm{Spin}_7)^d \to \mathrm{GL}_{7d}(\mathbb{C})$ is defined as $\phi_1(g_1, g_2, \ldots, g_d) = \mathrm{diag}(\rho_1(g_1), \rho_1(g_2), \ldots, \rho_1(g_d))$ for $(g_1, g_2, \ldots, g_d) \in (\mathrm{Spin}_7)^d$ and ρ_1 is the projection representation. And $\phi_2 : (\mathrm{Spin}_7)^d \to \mathrm{GL}_{8d}(\mathbb{C})$ is defined as $\phi_2(g_1, \ldots, g_d) = \mathrm{diag}(\rho_2(g_1), \rho_2(g_2), \ldots, \rho_2(g_d))$ for $(g_1, g_2, \ldots, g_d) \in (\mathrm{Spin}_7)^d$ and ρ_2 is the spin representation.

In this set up, we want to calculate Archimedean Euler factors $L(s, \pi_\infty, \phi_1)$ and $L(s, \pi_\infty, \phi_2)$. Let $X := \{\sum_{i=1}^3 c_i e_i \mid \sum c_i \in 2\mathbb{Z}\}, P := \langle e_1, e_2, e_3 \rangle$ and $Q := \langle e_1 - e_2, e_2 - e_3, 2e_3 \rangle$ denote the character lattice, weight lattice and root lattice of the group PGSp_6, respectively. Then $X^\vee, P^\vee, Q^\vee$ denote the co-character lattice, co-weight lattice and co-root lattice, respectively, where $\{e_1, e_2, e_3\}$ is a basis of $X \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\{f_1, f_2, f_3\}$ is a basis of $X^\vee \otimes_{\mathbb{Z}} \mathbb{Q}$ dual to each other in a sense that, $e_i(f_i)(x) = x$ and $e_i(f_j)(x) = 1$ for $i \neq j$. This implies $\langle f_1, f_2, f_3 \rangle$ and $\langle e_1, e_2, e_3 \rangle$ denote character lattice and co-character lattice for Spin_7, respectively. The element $\nu_l = \sum_{m=1}^3 (k_l - m) e_m$ is the Harish Chandra parameter for representation π_{σ_l} ($l = 1, \ldots, d$) of PGSp_6. For $z \in \mathbb{C}^*$, we have $z^{\nu_l} = z^{(k_l-1)e_1 + (k_l-2)e_2 + (k_l-3)e_3} = \prod_{m=1}^{3} e_m(z)^{k_l - m}$. Writing $z = re^{i\theta}$, we get $z^{\nu_l} = \prod_{m=1}^{3} e_m(re^{i\theta})^{k_l - m} = \prod_{m=1}^{3} e_m\left(r^{k_l - m} e^{i(k_l - m)\theta}\right)$. Similarly, we get $\bar{z}^{-\nu_l} = \prod_{m=1}^{3} e_m\left(r^{-(k_l - m)} e^{i(k_l - m)\theta}\right)$. We define the local parameter $\phi : W_{\mathbb{R}} \to (\mathrm{Spin}_7)^d$ attached to π_∞ as follows:

$$\phi(z) = \prod_{l=1}^{d} \phi_l(z) = (z^{\nu_1}\bar{z}^{-\nu_1}, z^{\nu_2}\bar{z}^{-\nu_2}, \ldots, z^{\nu_d}\bar{z}^{-\nu_d}) \in (\mathrm{Spin}_7)^d$$

for $z \in \mathbb{C}^*$ and $\phi_l(z) = z^{\nu_l}\bar{z}^{-\nu_l}$ denoting local parameters attached to π_{σ_l} for $l = 1, 2, \ldots, d$.

$$\phi(j) = (w, w, \ldots, w),$$

where $\phi_l(j) = w$ is a representative of the longest Weyl group element (meaning it sends e_m to $-e_m$ for each $m \in \{1, 2, 3\}$).

Writing $z = re^{i\theta}$, we get $\phi_l(re^{i\theta}) = \prod_{m=1}^{3} e_m\left(e^{2i(k_l - m)\theta}\right) = \left(e^{i\theta}\right)^{2\sum_{m=1}^3 (k_l - m)e_m} = \left(e^{i\theta}\right)^{2\nu_l}$.

Archimedean Euler factors for ϕ_1:

Now, we have to consider the semisimple representation

$$\tau_1 := \phi_1 \circ \phi : W_{\mathbb{R}} \to \mathrm{GL}(\mathbb{C}^7 \oplus \cdots \oplus \mathbb{C}^7)$$

into irreducible representations. The weights of the projection representation ρ_1 are well-known and they are: $f_1, f_2, f_3, 0, -f_1, -f_2, -f_3$. Each weight space is one-dimensional (for details please see Asgari-Schmidt [AS01, p. 181, Section 3.4]. Let $v_{\varepsilon_n n}$ be the spanning vectors of one-dimensional weight spaces corresponding to the weights $\varepsilon_n f_n$ ($n = 1,2,3$ and $\varepsilon_n \in \{\pm 1\}$) and v_0 is the weight vector corresponding to the the weight 0. Let $v^l_{\varepsilon_n n} := (0,\ldots,0,v_{\varepsilon_n n},0\ldots,0)$ denote the vector in $\mathbb{C}^7 \oplus \cdots \oplus \mathbb{C}^7$, where l^{th} entry is $v_{\varepsilon_n n}$ and other entries are zero. Therefore, for $n = 1,2,3$, we have,

$$\begin{aligned}
\tau_1(z)(v^l_{\varepsilon_n n}) &= (\phi_1 \circ \phi)(z)(v^l_{\varepsilon_n n}) \\
&= \phi_1(z^{\nu_1}\bar{z}^{-\nu_1}, z^{\nu_2}\bar{z}^{-\nu_2}, \ldots, z^{\nu_d}\bar{z}^{-\nu_d})(v^l_{\varepsilon_n n}) \\
&= \phi_1\left(\left(e^{i\theta}\right)^{2\nu_1}, \ldots, \left(e^{i\theta}\right)^{2\nu_d}\right)(v^l_{\varepsilon_n n}) \\
&= \mathrm{diag}\left(\rho_1\left(e^{i\theta}\right)^{2\nu_1}, \ldots, \rho_1\left(e^{i\theta}\right)^{2\nu_d}\right)(v^l_{\varepsilon_n n}) \\
&= \left(0,\ldots,0,\rho_1\left((e^{i\theta})^{2\nu_l}\right)v_{\varepsilon_n n},\ldots,0\right) \\
&= \varepsilon_n f_n\left((e^{i\theta})^{2\nu_l}\right)v^l_{\varepsilon_n n} \\
&= \varepsilon_n f_n\left(\prod_{m=1}^{3} e_m(e^{i\theta})^{2(k_l-m)}\right)v^l_{\varepsilon_n n} \\
&= e^{2i\varepsilon_n(k_l-n)\theta}v^l_{\varepsilon_n n} \\
&= e^{iu^l\theta}v^l_{\varepsilon_n n} \quad (\text{where } u^l := 2\varepsilon_n(k_l-n)).
\end{aligned}$$

Similarly, let $v^l_0 := (0,\ldots,0,v_0,0\ldots,0)$ denote the vector in $\mathbb{C}^7 \oplus \cdots \oplus \mathbb{C}^7$, where l^{th} entry is v_0 and other entries are zero. Now $\tau_1(z)v^l_0 = \rho_1(e^{i\theta})^{2\nu_l}v^l_0 = v^l_0$. For the action of j, observe that, $\tau_1(j) = (\phi_1 \circ \phi)(j) = \phi_1(w,w,\ldots,w) = \mathrm{diag}\left(\rho_1(w),\rho_1(w),\ldots,\rho_1(w)\right)$. Define, $w_0 :=$

$\rho_1(w)$. Here w_0 is a representative of the longest Weyl group element in $\mathrm{SO}_7(\mathbb{C})$. We choose the representative to be $\begin{pmatrix} 0 & I_3 & 0 \\ I_3 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = w_0$. I_3 denotes 3×3 identity matrix.

Therefore, for $n = 1,2,3$, we have,

$$\tau_1(j)v^l_{\varepsilon_n n} = v^l_{-\varepsilon_n n},$$
$$\tau_1(j)v^l_0 = -v^l_0.$$

It follows that for $n = 1,2,3$ and $l \in \{1,2,\dots,d\}$ the two-dimensional spaces $\langle v^l_{\varepsilon_n n}, v^l_{-\varepsilon_n n}\rangle$ and one-dimensional subspaces $\langle v^l_0\rangle$ are invariant for the action of $W_{\mathbb{R}}$. For each l in $\{1,2,\dots,d\}$ let $\tau^l_{\varepsilon_n n}$ (for $n = 1,2,3$) and τ^l_0 be the representations on these two-dimensional and one-dimensional spaces, respectively. Therefore

$$\tau^l_{\varepsilon_n n} = \tau_{|u^l|,0}\ (n = 1,2,3;\ u^l = 2\varepsilon_n(k_l - n))$$
$$\tau^l_0 = \tau_{-,0}$$

where $\tau_{u,t}$ and $\tau_{-,t}$ are defined in equations 2.7.4 and 2.7.3, respectively. Hence, $\tau_1 = \oplus_{l=1}^d \left(\oplus_{n=1}^3 \tau^l_{\varepsilon_n n} \oplus \tau^l_0\right)$. The archimedean L-factor associated to π_∞ is then given by

$$\begin{aligned} L(s,\pi_\infty,\phi_1) = L(s,\tau_1) &= \prod_{l=1}^d \left(L(s,\tau^l_0) \prod_{n=1}^3 L(s,\tau^l_{\varepsilon_n n}) \right) = \prod_{l=1}^d \left(L(s,\tau_{-,0}) \prod_{n=1}^3 L(s,\tau_{|u^l|,0}) \right) \\ &= \prod_{l=1}^d \pi^{-(\frac{s+1}{2})}\Gamma\left(\frac{s+1}{2}\right) 2(2\pi)^{-(s+k_l-1)}\Gamma(s+k_l-1)2(2\pi)^{-(s+k_l-2)} \\ &\quad \Gamma(s+k_l-2)2(2\pi)^{-(s+k_l-3)}\Gamma(s+k_l-3). \end{aligned}$$

Archimedean Euler factors for ϕ_2:

Similarly we have to write the decomposition of semisimple representation

$$\tau_2 := \phi_2 \circ \phi : W_{\mathbb{R}} \to \mathrm{GL}(\mathbb{C}^8 \oplus \cdots \oplus \mathbb{C}^8)$$

into irreducible representations. The weights of the spin representation ρ_2 are:

$$\frac{\varepsilon_1 f_1 + \varepsilon_2 f_2 + \varepsilon_3 f_3}{2}, \varepsilon_n \in \{\pm 1\}$$

and each weight space is one-dimensional. Let $v_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ be the corresponding weight vectors spanning the weight spaces. For each l in $\{1,2,\ldots,d\}$ let $v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} := (0,\ldots,0,v_{\varepsilon_1,\varepsilon_2,\varepsilon_3},0\ldots,0)$ denote the vector in $\mathbb{C}^8 \oplus \cdots \oplus \mathbb{C}^8$, where the l^{th} entry is $v_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ and other entries are zero. Therefore, for $z \in \mathbb{C}^*$, we have,

$$\begin{aligned}
\tau_2(z)(v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}) &= (\phi_2 \circ \phi)(z)(v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}) \\
&= \phi_2(z^{\nu_1}\bar{z}^{-\nu_1}, z^{\nu_2}\bar{z}^{-\nu_2},\ldots,z^{\nu_d}\bar{z}^{-\nu_d})(v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}) \\
&= \operatorname{diag}\left(\rho_2\left(e^{i\theta}\right)^{2\nu_1},\ldots,\rho_2\left(e^{i\theta}\right)^{2\nu_d}\right)(v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}) \\
&= \left(0,\ldots,0,\rho_2\left((e^{i\theta})^{2\nu_l}\right)v_{\varepsilon_1,\varepsilon_2,\varepsilon_3},\ldots,0\right) \\
&= \left(\frac{\varepsilon_1 f_1+\varepsilon_2 f_2+\varepsilon_3 f_3}{2}\right)\left(e^{i\theta}\right)^{2\nu_l} v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \\
&= \left(\frac{\varepsilon_1 f_1+\varepsilon_2 f_2+\varepsilon_3 f_3}{2}\right)\left(\prod_{m=1}^{3} e_m(e^{i\theta})^{2(k_l-m)}\right) v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \\
&= e^{i(\varepsilon_1(k_l-1)+\varepsilon_2(k_l-2)+\varepsilon_3(k_l-3))\theta} v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \\
&= e^{i\bar{u}^l\theta} v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} \text{ (where } \bar{u}^l := \varepsilon_1(k_l-1)+\varepsilon_2(k_l-2)+\varepsilon_3(k_l-3)).
\end{aligned}$$

Since $\rho_2(w)$ is a representative of the longest Weyl group element in $\mathrm{SO}_8(\mathbb{C})$, the action of j gives the following,

$$\tau_2(j)v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \operatorname{diag}(\rho_2(w),\ldots,\rho_2(w))v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = (0,0,\ldots,\rho_2(w)\cdot v_{\varepsilon_1,\varepsilon_2,\varepsilon_3},\ldots,0) = v^l_{-\varepsilon_1,-\varepsilon_2,-\varepsilon_3}.$$

These calculations imply that for each $l \in \{1,2,\ldots,d\}$ the two-dimensional subspaces $\langle v^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}, v^l_{-\varepsilon_1,-\varepsilon_2,-\varepsilon_3}\rangle$ are invariant for the action of $W_{\mathbb{R}}$. For each l in $\{1,2,\ldots,d\}$ let $\tau^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ be the representations on these two-dimensional spaces. Therefore

$$\tau^l_{\varepsilon_1,\varepsilon_2,\varepsilon_3} = \begin{cases} \tau_{|\bar{u}^l|,0} & \text{if } \bar{u}^l \neq 0, \\ \tau_{+,0} \oplus \tau_{-,0} & \text{otherwise} \end{cases}$$

By Legendre's formula for the Γ-function $L(s,\tau_{|\bar{u}^l|,0})$ and $L(s,\tau_{+,0}\oplus\tau_{-,0})$ are the same factors [Sch02, p. 8]. So, the archimedean Euler factor associated to π_∞ and ϕ_2 is then given by,

$$
\begin{aligned}
L(s,\pi_\infty,\phi_2) = L(s,\tau_2) = & \prod_{\substack{l=1 \\ |\bar{u}^l|\neq 0}}^{d} 2(2\pi)^{-s}(2\pi)^{-(\frac{|\bar{u}^l|}{2})}\Gamma\left(s+\frac{|\bar{u}^l|}{2}\right) \\
= & \prod_{l=1}^{d} 2(2\pi)^{-s}(2\pi)^{-(\frac{3k_l-6}{2})}\Gamma\left(s+\frac{3k_l-6}{2}\right) 2(2\pi)^{-s}(2\pi)^{-(\frac{k_l}{2})}\Gamma\left(s+\frac{k_l}{2}\right) \\
& 2(2\pi)^{-s}(2\pi)^{-(\frac{k_l-2}{2})}\Gamma\left(s+\frac{k_l-2}{2}\right) 2(2\pi)^{-s}(2\pi)^{-(\frac{|k_l-4|}{2})}\Gamma\left(s+\frac{|k_l-4|}{2}\right).
\end{aligned}
$$

Chapter 3

Meromorphic continuation of the L-functions

In this chapter, we are going to prove the meromorphic continuation of the standard L-function $L(s,\pi,\phi_1)$ and spin L-function $L(s,\pi,\phi_2)$ defined in the previous chapter using Langlands' theory of Euler products. Let us briefly recall Langlands' theory. We will be using the notations and recalling this theory from Shahidi [Sha88] and Asgari [AS01].

3.1 Langlands theory

Let $\mathbf{G}$ be a connected quasi-split reductive algebraic group over a number field k. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ over k with $\mathbf{B} = \mathbf{TU}$ where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$ and $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ over k. Let $\mathbf{M}$ be a maximal standard Levi subgroup in $\mathbf{G}$. Let $\mathbf{P} = \mathbf{MN}$ be a standard parabolic subgroup in $\mathbf{G}$. We take $\mathbf{B} \subset \mathbf{P}$. The L-group of $\mathbf{P}$ is then ${}^L\mathbf{P} = {}^L\mathbf{M}{}^L\mathbf{N}$ in ${}^L\mathbf{G}$. Let ${}^Lr$ denote the adjoint action of ${}^L\mathbf{M}$ on ${}^L\mathfrak{n}$, Lie algebra of ${}^L\mathbf{N}$. Since ${}^L\mathbf{M}$ is a reductive group itself, by complete reducibility theorem, we can write, ${}^Lr = \oplus_{i=1}^{l}{}^Lr_i$ with ${}^Lr_i$'s being the irreducible constituents of ${}^Lr$. For every place v of k, let $G_v := \mathbf{G}(k_v)$. Similarly, we will write P_v, M_v, N_v. For the places v where $\mathbf{G}$ is unramified over v, we define $K_v = \mathbf{G}(O_v)$ and $K = \otimes_v K_v$. Let $\pi = \otimes_v \pi_v$ be a cusp form on $M = \mathbf{M}(\mathbb{A}_k)$, where $\mathbb{A}_k$ denotes the ring of adeles of k. Let $\mathbf{A}$ be the split torus in the center of $\mathbf{M}$. For each v, there exists a homomorphism H_{P_v} from M_v into the real Lie algebra of $\mathbf{A}$ as a group over k_v. Let

$$I(s,\pi_v) = \mathrm{Ind}_{M_vN_v\uparrow G_v}\pi_v \otimes q_v^{\langle s,H_{p_v}(.)\rangle} \otimes 1$$

be the corresponding induced representation of G_v for $v < \infty$. Here $s \in \mathbb{C}$ and q_v denotes the cardinality of the residue field k_v. If $v = \infty$, q_v is replaced by $\exp\langle s, H_{p_v}(.)\rangle$. Note that for $s \in \mathbb{C}$ we have representation $I(s,\pi) = \mathrm{Ind}_{\mathbf{P}\uparrow\mathbf{G}}\pi \otimes \exp\langle s, H_{p_v}(.)\rangle \otimes 1$ of $\mathbf{G}$, where $I(s,\pi) = \otimes_v I(s,\pi_v)$ and $I(s,\pi_v)$ is defined as above. The 1 in the formula implies that $\pi \otimes \exp\langle s, H_{p_v}(.)\rangle$ is extended trivially across $\mathbf{N}$. Let $\mathbf{A}_0$ be the maximal k-split torus in $\mathbf{T}$. Let W be the Weyl group of $\mathbf{A}_0$ in $\mathbf{G}$. Let Δ denote the set of simple roots and the unique reduced root of $\mathbf{A}$ in $\mathbf{N}$ be identified by a simple root α. The complement set of α in Δ generates $\mathbf{M}$. We denote this set by θ. Now given a K-finite function ϕ in the space of π, we get a function $\tilde{\phi}$ extending ϕ to $\mathbf{G}$ and we set

$$\Phi_s(g) = \tilde{\phi}(g)\exp\langle s + \rho_{\mathbf{P}}, H_P(g)\rangle.$$

The associated Eisenstein series is then given as

$$E(s,\tilde{\phi},g,P) = \sum_{\gamma\in\mathbf{P}(F)\backslash\mathbf{G}(F)} \Phi_s(\gamma g). \tag{3.1.1}$$

Here $\rho_{\mathbf{P}}$ denotes half the sum of k-roots generating $\mathbf{N}$. The constant term of $E(s,\tilde{\phi},g,P)$ along with a parabolic subgroup $\mathbf{Q} = \mathbf{M}_Q\mathbf{N}_Q$ is then given by

$$E_Q(s,\tilde{\phi},g,P) = \int_{\mathbf{N}_Q(k)\backslash\mathbf{N}_Q(\mathbb{A}_k)} E(s,\tilde{\phi},ng,P)dn. \tag{3.1.2}$$

Unless Q is P or conjugate parabolic of P, $E_Q(s,\tilde{\phi},g,P)$ is zero. For details please see Kim [Kim04, Chapter 5, Section 2] and Langlands-Shahidi [Lan71b, Sha78]. There exists a unique element $\tilde{w} \in W$ such that $\tilde{w}$ takes α to a negative root and the remaining simple roots θ into Δ. Fix a representative of $\tilde{w}$ as w. We let $\mathbf{M}'$ denote the subgroup of $\mathbf{G}$ generated by $\tilde{w}(\theta)$. Then there exists a parabolic subgroup $\mathbf{P}' \supset \mathbf{B}$ which contains $\mathbf{M}'$ as its Levi factor. Let $\mathbf{N}'$ be the corresponding unipotent radical. Given $f \in I(s,\pi)$ and $\mathrm{Re}(s)$ sufficiently large, set

$$M(s,\pi)f(g) = \int_{N'} f(w^{-1}ng)dn \quad (g \in G_v). \tag{3.1.3}$$

Observe that if $f = \otimes_v f_v$ then for almost all v, f_v is the unique K_v-fixed function normalized by $f_v(e_v) = 1$. Finally, if at each v we define a local intertwining operator attached to $I(s, \pi_v)$ by

$$A(s, \pi_v, w) f_v(g) = \int_{N'_v} f_v(w^{-1} n g) dn \quad (g \in G_v) \tag{3.1.4}$$

then

$$M(s, \pi) = \otimes_v A(s, \pi_v, w). \tag{3.1.5}$$

$M(s, \pi)$ denotes the nontrivial part of the constant term of the Eisenstein series 3.1.1. Let S denote a finite set of places of k containing the archimedean places too such that for $v \notin S, \mathbf{G}, \pi_v$ are all unramified. Then for every finite place $v \notin S$ we can attach a L-function $L(s, \pi_v, {}^L r_v)$, where ${}^L r_v = {}^L r|_{{}^L M_v}$ and $s \in \mathbb{C}$.

The Euler product

$$L^S(s, \pi, {}^L r) = \prod_{v \notin S} L(s, \pi_v, {}^L r_v)$$

always converges absolutely for $\mathrm{Re}(s) >> 0$ (cf. [Bor79], [Lan70]). The theory of Euler products developed by Langlands gives us the following,

$$M(s, \pi) f = (\otimes_{v \in S} A(s, \pi_v, w) f_v) \otimes (\otimes_{v \notin S} \tilde{f}_v) \cdot \prod_{i=1}^{l} \frac{L^S(is, \pi, {}^L \tilde{r}_i)}{L^S(1 + is, \pi, {}^L \tilde{r}_i)} \tag{3.1.6}$$

where $f = \otimes_v f_v, f_v \in I(s, \pi_v), f \in I(s, \pi)$ and for every $v \notin S, f_v$ and $\tilde{f}_v$ are the unique normalised fixed functions in $I(s, \pi_v)$ and $I(-s, \tilde{w}(\pi_v))$ respectively. For $i = 1, 2, \ldots, l, {}^L \tilde{r}_i$ denotes the contragredient of ${}^L r_i$. Each of these representations ${}^L r_i$ is irreducible (cf. [Sha88]). This method deals with this specific type of representations ${}^L r$, so that with the appropriate choices of $\mathbf{M}$ and $\mathbf{G}$, they cover the most important examples of L-functions. The function $M(s, \pi)$ extends to a meromorphic function of $s \in \mathbb{C}$. The intertwining operators $A(s, \pi_v, w)$ for $v \in S$ is non-vanishing and has a meromorphic continuation to all of $\mathbb{C}$. This result is due to Shahidi [Sha88].

Now assume, $l = 1$, then by expression (3.1.6) and discussion followed by expression (3.1.6), we get

$$\mathbf{F}(s) = \frac{L^S(s, \pi, {}^L \tilde{r}_i)}{L^S(s + 1, \pi, {}^L \tilde{r}_i)}$$

is meromorphic. Writing this above expression as

$$\mathbf{F}(s)L^S(s+1,\pi,{}^L\tilde{r}_i) = L^S(s,\pi,{}^L\tilde{r}_i) \tag{3.1.7}$$

and noting the fact that $L^S(s,\pi,{}^L\tilde{r}_i)$ is analytic for sufficiently large $\mathrm{Re}(s)$ (cf.[Sha88]), we can apply induction on $L^S(s,\pi,{}^L\tilde{r}_i)$ and conclude that $L^S(s,\pi,{}^L\tilde{r}_i)$ is meromorphic to all of $\mathbb{C}$.

3.2 Meromorphic continuation

As mentioned in the beginning of the section our main goal is to deduce the meromorphic continuation of the standard L-function $L(s,\pi,\phi_1)$ and spin L-function $L(s,\pi,\phi_2)$ associated with an automorphic representation π of $\mathrm{PGSp}_6(\mathbb{A}_F)$ and with the standard representation ϕ_1 and the spin representation ϕ_2 (introduced in Section 2.7) to all of $\mathbb{C}$. We will use the Langlands theory and notations from the above discussion.

Proposition 3.2.1. *The function $L(s,\pi,\phi_1)$ has a meromorphic continuation to all of $\mathbb{C}$.*

Proof. Let us first observe that, $\mathrm{GL}_1 \times \mathrm{Sp}_6$ sits as a standard Levi subgroup in the symplectic group Sp_8. Let the corresponding Levi decomposition of parabolic P in Sp_8 be $P = (\mathrm{GL}_1 \times \mathrm{Sp}_6)N$. Now consider, $\mathbf{M}$ as $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1 \times \mathrm{Sp}_6)$ and $\mathbf{G}$ as $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{Sp}_8)$ from 3.1. $\mathbf{M}$ is a Levi-subgroup of $\mathbf{G}$ over $\mathbb{Q}$ (by [Bor79, Section 5.2, p. 35]). Let corresponding Levi decomposition be $\mathbf{P} = \mathbf{MN}$ in $\mathbf{G}$. Now ${}^L\mathbf{M} = {}^L(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1 \times \mathrm{Sp}_6)) = (\mathbb{C}^\times \times \mathrm{SO}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q})$, where $\mathbb{C}^\times \times \mathrm{SO}_7$ is the complex dual of $\mathrm{GL}_1 \times \mathrm{Sp}_6$. ${}^L\mathbf{M}$ is a Levi-subgroup of a parabolic subgroup ${}^L\mathbf{P}$ in ${}^L(\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{Sp}_8))$;${}^L\mathbf{P} = {}^L\mathbf{M}{}^L\mathbf{N}$, where ${}^L\mathbf{N}$ is the unipotent radical of ${}^L\mathbf{P}$ and ${}^L\mathbf{N} = {}^L\mathbf{N}^0$ (see Section 2.5), ${}^L\mathfrak{n}$ =Lie algebra of ${}^L\mathbf{N}^0 = \hat{\mathfrak{n}} \oplus \cdots \oplus \hat{\mathfrak{n}}$ (d copies). Here $\hat{\mathfrak{n}}$ = Lie algebra of $\hat{N}$ ($\hat{N}$ is the complex dual of N). Let ${}^Lr$ be the adjoint action of ${}^L\mathbf{M}$ on ${}^L\mathfrak{n}$ and r be the adjoint action of $\mathbb{C}^\times \times \mathrm{SO}_7$ on $\hat{\mathfrak{n}}$, i.e., $r : \mathbb{C}^\times \times \mathrm{SO}_7 \to \mathrm{GL}(\hat{\mathfrak{n}})$. Using Shahidi [Sha88, Case (C_n), Section 4], Asgari and Schmidt have proved that r is an irreducible self-dual 7-dimensional representation of $\mathbb{C}^\times \times \mathrm{SO}_7$ (cf.

[AS01, Theorem 4]). This implies $\dim(\hat{\mathfrak{n}}) = 7$. The representation

$$^L r : {}^L\mathbf{M} \to \mathrm{GL}(\hat{\mathfrak{n}} \oplus \cdots \oplus \hat{\mathfrak{n}})$$

is defined as

$$\begin{aligned} {}^L r(m_1, m_2, \ldots, m_d, 1)(n_1, n_2, \ldots, n_d) &= (m_1, m_2, \ldots, m_d, 1)(n_1, n_2, \ldots, n_d)(m_1, m_2, \ldots, m_d, 1)^{-1} \\ &= (m_1 n_1 m_1^{-1}, m_2 n_2 m_2^{-1}, \ldots, m_d n_d m_d^{-1}) \\ &= (r(m_1)(n_1), r(m_2)(n_2), \ldots, r(m_d)(n_d)) \end{aligned}$$

for $(m_1, m_2, \ldots, m_d, 1) \in (\mathbb{C}^\times \times \mathrm{SO}_7)^d \rtimes \mathrm{Gal}(F'/\mathbb{Q})$ and $(n_1, n_2, \ldots, n_d) \in \hat{\mathfrak{n}} \oplus \cdots \oplus \hat{\mathfrak{n}}$. For $(1, 1, \ldots, 1, \tau) \in {}^L M$, where $\tau \in \mathrm{Gal}(F'/\mathbb{Q})$, we have ${}^L r(1, 1, \ldots, 1, \tau) = \mathbb{P}_{\tau,7,d}$ (since $\dim(\hat{\mathfrak{n}}) = 7$).

Claim: ${}^L r$ is irreducible.

Proof. Let W be a non-zero ${}^L\mathbf{M}$-invariant subspace of $\hat{\mathfrak{n}} \oplus \cdots \oplus \hat{\mathfrak{n}}$. For any $w = (w_1, w_2, \ldots, w_d) \in W$, w will have at least one component $w_i \neq 0$. Without loss of generality, let $w_1 \neq 0$. Now, $\hat{\mathfrak{n}} \oplus \cdots \oplus 0 \cong \hat{\mathfrak{n}}$ and $(r, \hat{\mathfrak{n}})$ is an irreducible representation of $\mathbb{C}^\times \times \mathrm{SO}_7$. The space

$$\hat{\mathfrak{n}}(w_1) := \{w \mid w = c_1 g_1 w_1 + c_2 g_2 w_1 + \cdots + c_n g_n w_1 \text{ for some } c_i \in \mathbb{C}, g_i \in (\mathbb{C}^\times \times \mathrm{SO}_7)\}$$

is a $\mathbb{C}^\times \times \mathrm{SO}_7$-invariant subspace of $\hat{\mathfrak{n}}$. By the irreducibility of $\hat{\mathfrak{n}}$, we have $\hat{\mathfrak{n}}(w_1) = \hat{\mathfrak{n}}$. This implies $\hat{\mathfrak{n}} \oplus \cdots \oplus 0 \subset W$, i.e., $\hat{\mathfrak{n}} \subset W$. Now by the action of $\mathrm{Gal}(F'/\mathbb{Q})$ different copies of $\hat{\mathfrak{n}}$ gets permuted. This means $\mathrm{Gal}(F'/\mathbb{Q})$ is a transitive subgroup of the symmetric group S_d. So for any $i \in \{1, 2, \ldots, d\}$ there will always exist $\sigma \in \mathrm{Gal}(F'/\mathbb{Q})$ such that $w_i = w_{\sigma(1)} \neq 0$. Hence $(\hat{\mathfrak{n}} \oplus \cdots \oplus \hat{\mathfrak{n}}) \subset W$. This completes the proof of irreducibility of ${}^L r$. □

Since adjoint representations are self-dual, so ${}^L r$ is self-dual too. Now, let π' be an automorphic representation of $\mathrm{GSp}_6(\mathbb{A}_F)$. We restrict π' to the derived subgroup $\mathrm{Sp}_6(\mathbb{A}_F)$ of $\mathrm{GSp}_6(\mathbb{A}_F)$. We further denote by π, the irreducible constituent of $\pi'|_{\mathrm{Sp}_6}$. Put the cusp form $\sigma = 1 \otimes \pi$ on $\mathbf{M}(\mathbb{A}_\mathbb{Q})$. Then

$$L(s, \sigma, {}^L\tilde{r}) = L(s, \sigma, {}^L r_1) = L(s, \pi, \phi_1)$$

(since ${}^L r$ and ϕ_1 are constructed out of r and ρ_1 and for r and ρ_1 this equality holds by [AS01, Theorem 4]) . Now by the same argument as in (3.1.7), $L(s,\sigma,{}^L r)$ is meromorphic to all of $\mathbb{C}$. Hence the standard L-function $L(s,\pi,\phi_1)$ has meromorphic continuation to all of $\mathbb{C}$. $\square$

Theorem 3.2.2. *The spin L-function $L(s,\pi,\phi_2)$ has a meromorphic continuation to all of $\mathbb{C}$.*

Proof. Let H be a Chevalley group of type F_4. This is a split as well as adjoint simply connected simple algebraic group. The complex dual of H is again of type F_4. Let $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ be a system of simple roots of H, where α_1, α_2 are long roots and α_3, α_4 denote the short ones. Consider the standard Levi subgroup M corresponding to $\{\alpha_2, \alpha_3, \alpha_4\}$. Then by [Asg00, Proposition 4.1.1] one is able to show $M \cong \mathrm{GSp}_6$. The complex dual of M is $\mathrm{GSpin}_7(\mathbb{C})$. Let $\hat{\mathbf{P}} = \hat{\mathbf{M}}\hat{\mathbf{N}}$ be the corresponding Levi decomposition in the dual of H. Let r be the adjoint action of $\hat{M}$ on $\hat{\mathfrak{n}} = \mathrm{Lie}(\hat{N})$. Asgari and Schmidt showed that $r = r_1 \oplus r_2$ with r_1 an irreducible self-dual 8-dimensional and r_2 an irreducible self-dual 7-dimensional representation of $\hat{M}$ (cf. [AS01, Theorem 4]), i.e., $\hat{\mathfrak{n}} = \hat{\mathfrak{n}}_1 \oplus \hat{\mathfrak{n}}_2$; $\dim(\hat{\mathfrak{n}}_1) = 8$ and $\dim(\hat{\mathfrak{n}}_2) = 7$. Moreover, $r|_{\mathrm{Spin}_7} = r_1|_{\mathrm{Spin}_7} \oplus r_2|_{\mathrm{Spin}_7} = \rho_2 \oplus \rho_1$. Let $\mathbf{M} = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSp}_6)$ and $\mathbf{G} = \mathrm{Res}_{F/\mathbb{Q}}(H)$ in our case from Section (3.1), where GSp_6 and H are defined over F now. That means our group G sits as a Levi in $\mathrm{Res}_{F/\mathbb{Q}}(H)$.

Consider the corresponding Levi-decomposition of a standard parabolic ${}^L P = {}^L G {}^L N$ and the adjoint action

$$ {}^L r : {}^L G \to \mathrm{GL}({}^L \mathfrak{n}) $$

by

$$
\begin{aligned}
{}^L r(g_1,g_2,\dots,g_d,1)(n_1,n_2,\dots,n_d) &= (g_1,g_2,\dots,g_d,1)(n_1,n_2,\dots,n_d)(g_1,g_2,\dots,g_d,1)^{-1} \\
&= (g_1 n_1 g_1^{-1}, g_2 n_2 g_2^{-1}, \dots, g_d n_d g_d^{-1}) \\
&= (r(g_1)(n_1), r(g_2)(n_2), \dots, r(g_d)(n_d)) \\
&= ((r_1 \oplus r_2)(g_1)(n_1), \dots, (r_1 \oplus r_2)(g_d)(n_d)),
\end{aligned}
$$

where ${}^L\mathfrak{n} = {}^L\mathfrak{n}^0 = \mathrm{Lie}({}^LN) = \hat{\mathfrak{n}} \oplus \cdots \oplus \hat{\mathfrak{n}}$ and ${}^Lr(1,\ldots,1,\tau) = \mathbb{P}_{\tau,15,d}$.

Our claim is to show that ${}^Lr = {}^Lr_1 \oplus {}^Lr_2$, i.e., ${}^Lr$ decomposes into two irreducible representations of ${}^LG$. Now

$$ {}^Lr_1 : {}^LG \to \mathrm{GL}(\hat{\mathfrak{n}}_1 \oplus \cdots \oplus \hat{\mathfrak{n}}_1) $$

and

$$ {}^Lr_2 : {}^LG \to \mathrm{GL}(\hat{\mathfrak{n}}_2 \oplus \cdots \oplus \hat{\mathfrak{n}}_2). $$

We prove that claim by giving the same argument as we gave for the previous case. We can further observe that both ${}^Lr_1$ and ${}^Lr_2$ are self-dual (since from the observation, ${}^Lr_1 \oplus {}^Lr_2 = {}^Lr = {}^L\tilde{r} = {}^L\tilde{r}_1 \oplus {}^L\tilde{r}_2$ and by the dimension calculations of theses representations). Also, note that restrictions of ${}^Lr_1$ and ${}^Lr_2$ on ${}^L\overline{G}$ give, ${}^Lr_1|_{{}^L\overline{G}} = \phi_2$ and ${}^Lr_2|_{{}^L\overline{G}} = \phi_1$. Let π be the representation on $\mathbf{M}(\mathbb{A}_{\mathbb{Q}}) = G(\mathbb{A}_{\mathbb{Q}})$ lifted from the representation $\overline{\pi}$ of $\overline{G}(\mathbb{A}_{\mathbb{Q}})$. Then expression (3.1.6) and its subsequent argument imply

$$ \mathscr{M}(s) = \frac{L^S(s,\pi,{}^Lr_1)}{L^S(s+1,\pi,{}^Lr_1)} \cdot \frac{L^S(2s,\pi,{}^Lr_2)}{L^S(2s+1,\pi,{}^Lr_2)} \tag{3.2.1} $$

has a meromorphic continuation to all of $\mathbb{C}$. The standard L-function $L(s,\bar{\pi},\phi_1) = L(s,\pi,{}^Lr_2)$ has a meromorphic continuation to all of $\mathbb{C}$ (by Proposition (3.2.1)). Writing the expression (3.2.1) as $L^S(s,\pi,{}^Lr_1) = \mathscr{M}(s) \cdot \frac{L^S(2s+1,\pi,{}^Lr_2)}{L^S(2s,\pi,{}^Lr_2)} \cdot L^S(s+1,\pi,{}^Lr_1)$ and applying induction argument as in (3.1.7) we get the meromorphic continuation of the spin function to all of $\mathbb{C}$.

$\square$

Chapter 4

Algebraic theory of automorphic forms

It is an interesting problem to compute the dimensions of the space of genus 3 Siegel automorphic forms for various small weights for the group GSp_6 over $\mathbb{Q}$. This space is not computable directly. So we compute the dimensions of the space of automorphic forms for rank 3 unitary groups, where the entries of the group are from a definite quaternion algebra over $\mathbb{Q}$. This group is an inner form of $\mathrm{GSp}_6/\mathbb{Q}$. The theory of algebraic modular forms on quaternion algebras are set up by Gross [Gro99]. In this chapter, we include all the preliminaries to carry out his theory for our computations. This theory deals with a connected reductive algebraic group over $\mathbb{Q}$, where the group satisfies the condition that all its arithmetic subgroups are finite. Then the conjectural Jacquet-Langlands allows us to go from the algebraic theory of modular forms to automorphic forms for $\mathrm{GSp}_6/\mathbb{Q}$. Let us set the stage by recalling some facts on quaternion algebra.

4.1 Quaternion algebras

Definition 4.1.1. *Let $\mathtt{F}$ be any field of characteristic $\neq 2$ and $a, b \in \mathtt{F}^{\times}$. A **quaternion algebra** is an associative $\mathtt{F}$-algebra of dimension 4 with basis $1, i, j, k$ denoted by $\left(\frac{a,b}{\mathtt{F}}\right)$, where $i^2 = a$, $j^2 = b$ and $ij = -ji = k$.*

Fact 4.1.2.

(1) The matrix algebra $\mathrm{M}_2(\mathtt{F})$ is called *trivial* or *split* quaternion algebra. In particular, if $\mathtt{F} = \mathbb{C}$, this is a unique quaternion algebra over $\mathbb{C}$.

(2) A quaternion algebra is either a division algebra or a matrix algebra.

(3) There are exactly two real quaternion algebras: $\mathbb{H} = \left(\frac{-1,-1}{\mathbb{R}}\right)$ (Hamiltonian algebra) and $\mathrm{M}_2(\mathbb{R})$.

If $\mathtt{F}'/\mathtt{F}$ is a field extension, then we have,

$$\left(\frac{a,b}{\mathtt{F}'}\right) \cong \left(\frac{a,b}{\mathtt{F}} \otimes_{\mathtt{F}} \mathtt{F}'\right).$$

So, $B \otimes_{\mathtt{F}} \overline{\mathtt{F}} \cong \mathrm{M}_2(\overline{\mathtt{F}})$ for any quaternion algebra B, where $\overline{\mathtt{F}}$ denotes the algebraic closure of $\mathtt{F}$.

Definition 4.1.3. *The anti-involution map of B defined as $x = \alpha + \beta i + \gamma j + \delta k \mapsto \overline{x} := \alpha - \beta i - \gamma j - \delta k$, defines the norm structure on B. So, the norm of any element x in B is defined as $N(x) := x\overline{x}$, i.e., $N(\alpha + \beta i + \gamma j + \delta k) = \alpha^2 - a\beta^2 - b\gamma^2 + ab\delta^2$.*

In our case, we will deal with the rational field $\mathbb{Q}$. Let $B := \left(\frac{a,b}{\mathbb{Q}}\right)$ and let v be a place of $\mathbb{Q}$ with completion $\mathbb{Q}_v$ (so it is either $\mathbb{Q}_p$ for some prime p or $\mathbb{R}$). Define $B_v := B \otimes_{\mathbb{Q}} \mathbb{Q}_v$, i.e.,

$$\left(\frac{a,b}{\mathbb{Q}_v}\right) \cong \left(\frac{a,b}{\mathbb{Q}}\right) \otimes_{\mathbb{Q}} \mathbb{Q}_v,$$

which is a quaternion algebra over $\mathbb{Q}_v$. We say that B is *split* or *unramified* at v if $B_v \cong \mathrm{M}_2(\mathbb{Q}_v)$ and B is *non-split* or *ramified* at v if B_v is the quaternion division algebra over $\mathbb{Q}_v$.

Remark 4.1.4.

(1) The number of places where a quaternion algebra over $\mathbb{Q}$ ramifies is always even, and this is equivalent to quadratic reciprocity law over $\mathbb{Q}$. For any finite set $\mathtt{S}$ with even cardinality there is a unique quaternion algebra over $\mathtt{F}$ such that the set of places v, where B is ramified is exactly the set $\mathtt{S}$.

(2) The product of the primes at which B ramifies is called the *discriminant* of B.

Definition 4.1.5. *A quaternion algebra over $\mathbb{Q}$ is called* **definite** *if B_∞ is not split. It is* **indefinite** *otherwise.*

Remark 4.1.6. (1) Note that $\left(\frac{a,b}{\mathbb{Q}}\right)$ is definite if and only if $a, b < 0$.

(2) For each prime p of $\mathbb{Q}$, there is a unique (up to isomorphism) definite quaternion algebra B over $\mathbb{Q}$ ramified exactly at p and ∞.

(3) An order of a quaternion algebra $\left(\frac{a,b}{F}\right)$ over F is a subring $\mathscr{O} \subset \left(\frac{a,b}{F}\right)$, which is a $\mathscr{O}_F$-lattice in $\left(\frac{a,b}{F}\right)$ and each order is contained in a maximal order.

For our purpose, we fix the quaternion algebra $B := \left(\frac{-1,-1}{\mathbb{Q}}\right)$ and a maximal order $\mathscr{O}_B := \mathbb{Z} \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}\left(\frac{1+i+j+ij}{2}\right)$ of B throughout this chapter. Note that B is the unique definite quaternion algebra over $\mathbb{Q}$, ramified exactly at 2 and ∞ and unramified at all odd primes. Therefore the discriminant of B is 2. We need to choose a finite Galois extension $E/\mathbb{Q}$, contained in $\mathbb{C}$ such that E splits B. In our case we fix E to be $\mathbb{Q}(\mathrm{I})$, the imaginary quadratic field where I is the imaginary unit. Note that we have a splitting isomorphism $B \otimes_{\mathbb{Q}} E \overset{\iota}{\cong} \left(\frac{-1,-1}{E}\right) \cong \mathrm{M}_2(E)$. For any $g \in \mathrm{M}_3(B)$, $\mathrm{M}_3(B) \hookrightarrow \mathrm{M}_3(B \otimes_{\mathbb{Q}} E) \cong \mathrm{M}_6(E)$ maps $g \mapsto g \otimes 1$. Define, $\det(g) := \det(g \otimes 1)$. For each prime p $(\neq 2)$ in $\mathbb{Q}$, we fix a local isomorphism $(\mathscr{O}_B)_p = \mathscr{O}_{B,p} \cong \mathrm{M}_2(\mathbb{Z}_p)$ and extend it to $B_p \cong \mathrm{M}_2(\mathbb{Q}_p)$.

4.2 Theoretical background

In this section, we are going to discuss briefly the theory of modular forms, which B. H. Gross developed totally algebraically for connected reductive algebraic groups over $\mathbb{Q}$ in his paper [Gro99]. Let $\mathtt{G}$ be such a connected reductive group over $\mathbb{Q}$. We are going to follow the notations, set up by Gross himself from his paper [Gro99]. Let $\mathtt{G}(\mathbb{Q})$ denote the $\mathbb{Q}$-rational points of $\mathtt{G}$ and more generally let $\mathtt{G}(\mathbb{A})$ denote the group of adèlic points. $\mathtt{G}(\mathbb{R})_+$ will denote the connected component of the identity in the Lie group $\mathtt{G}(\mathbb{R})$. That means the group $\mathtt{G}(\mathbb{R})_+$ will contain matrices having positive similitudes. Let S' be the maximal quotient of $\mathtt{G}$ which is a split torus. After fixing an isomorphism $S' \cong \mathtt{G}_m^n$, we get a continuous homomorphism

$$\mathtt{G}(\mathbb{A}) \longrightarrow S'(\mathbb{A}) \cong (\mathbb{A}^\times)^n \xrightarrow{||\cdot||} (\mathbb{R}_+^\times)^n,$$

where the kernel of this composition map is denoted by $\mathtt{G}(\mathbb{A})_1$. The subgroup $\mathtt{G}(\mathbb{Q})$ is discrete in the locally compact group $\mathtt{G}(\mathbb{A})_1$ due to a result by Borel and Harish-Chandra [BHC62].

One of the main results in [Gro99, p. 63, Proposition 1.4] gives a series of equivalent conditions.

Proposition 4.2.1. [Gro99, Proposition 1.4] *The following conditions are equivalent:*

(1) Every arithmetic subgroup Γ of $\mathtt{G}(\mathbb{Q})$ is finite.

(2) $\Gamma = \{e\}$ is an arithmetic subgroup of $\mathtt{G}(\mathbb{Q})$.

(3) $\mathtt{G}(\mathbb{Q})$ is a discrete subgroup of the locally compact group $\mathtt{G}(\widehat{\mathbb{Q}})$.

(4) $\mathtt{G}(\mathbb{Q})$ is a discrete subgroup of the locally compact group $\mathtt{G}(\widehat{\mathbb{Q}})$ and the quotient space $\mathtt{G}(\mathbb{Q})\backslash\mathtt{G}(\widehat{\mathbb{Q}})$ is compact.

(5) $\mathscr{S}$ is a maximal split torus in $\mathtt{G}$ over $\mathbb{R}$.

(6) The Lie group $\mathtt{G}(\mathbb{R})_1 = \mathtt{G}(\mathbb{R}) \cap \mathtt{G}(\mathbb{A})_1$ is a maximal compact subgroup of $\mathtt{G}(\mathbb{R})$.

(7) For every irreducible representation V of $\mathtt{G}$ there is a character $\mu : \mathtt{G} \to \mathbb{G}_m$ and a positive definite symmetric bilinear form $\langle , \rangle : V \times V \to \mathbb{Q}$ which satisfy

$$\langle gv, gv' \rangle = \mu(g)\langle v, v' \rangle$$

for all $g \in \mathtt{G}(\mathbb{Q})$ and $v, v' \in V$.

The proof of this proposition can be found in [Gro99, p. 63].

If a connected reductive group $\mathtt{G}/\mathbb{Q}$ satisfies one of the equivalent conditions of Proposition 4.2.1 with K a compact open subgroup of $\mathtt{G}(\widehat{\mathbb{Q}})$ and V an irreducible representation of $\mathtt{G}$ over $\mathbb{Q}$, then Gross defined the space of algebraic modular forms of weight V and level K by the following space of functions:

$$M(V,K) = \{f : \mathtt{G}(\mathbb{A})/(\mathtt{G}(\mathbb{R})_+ \times K) \to V \mid f(\gamma g) = \gamma f(g) \text{ for } \gamma \in \mathtt{G}(\mathbb{Q})\}.$$

He proved another two propositions which we will be going to use for our purpose. We include them here.

Proposition 4.2.2. [Gro99, Proposition 4.3]

(1) The double coset space $\mathtt{G}(\mathbb{Q})\backslash\mathtt{G}(\mathbb{A})/(\mathtt{G}(\mathbb{R})_+ \times K)$ *is finite. The cardinality of this double coset space is called the class number of* $\mathtt{G}$*.*

(2) $M(V,K)$ *is a finite-dimensional D-vector space, where* $D = \mathrm{End}_{\mathtt{G}}(V)$ *is a division algebra of finite dimension over* $\mathbb{Q}$*.*

Proposition 4.2.3. [Gro99, Proposition 4.5] *If we fix representatives of the classes in the above mentioned set of double cosets by* $\{g_\alpha\}$ *then denoting* Γ_α *by* $\mathtt{G}(\mathbb{Q}) \cap g_\alpha(\mathtt{G}(\mathbb{R})_+ \times K)g_\alpha^{-1}$*, each function* f *in* $M(V,K)$ *is completely determined by the values* $f(g_\alpha)$ *in* V^{Γ_α}*, where* V^{Γ_α} *is the* Γ_α*-invariant subspaces of* V *and furthermore,*

$$M(V,K) \cong \oplus V^{\Gamma_\alpha}.$$

Now, the broad steps of the algorithm to compute $\dim M(V,K)$ is as follows,

1. Compute the class number of $\mathtt{G}$.

2. Compute Γ_α explicitly.

3. Calculate the invariant subspaces V^{Γ_α}.

4.3 Space of algebraic automorphic forms

In our case, first we will define the group which will play the role of $\mathtt{G}$. Then we will talk about the space of algebraic automorphic forms on that group defined in the sense of Gross 4.2.

4.3.1 Similitude groups

Let X be a free left B-module of rank n equipped with a positive definite Hermitian form

$$\varphi : X \times X \to B$$

such that

(1) $\overline{\varphi(x,y)} = \varphi(y,x)$ and

(2) $\varphi(\alpha x, \beta y) = \alpha\varphi(x,y)\overline{\beta}$.

for all $x, y \in X$ and for all $\alpha, \beta \in B$. Here $\bar{\ }$ denotes the anti-involution map in B. Then the group of similitudes G^B over $\mathbb{Q}$ is defined as the following:

$$G^B = \{T \in \mathrm{End}(X, \varphi) \mid \varphi(Tx, Ty) = \mu(T)\varphi(x,y)\ \forall x, y \in X, \quad \mu(T) \in \mathbb{Q}^\times\},$$

where $\mathrm{End}(X, \varphi)$ is the ring of all B-linear endomorphisms of X. By fixing a basis of X we can associate a matrix to φ. Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for X, set $\varphi_{ij} := \varphi(e_i, e_j)$ for all $1 \leq i, j \leq n$. Then $[\varphi] := (\varphi_{ij})$ is called the matrix of φ relative to $\{e_1, e_2, \ldots, e_n\}$. If $x, y \in X$, write $x = \sum_i x_i e_i$, and $y = \sum_j y_j e_j$, so that x and y are represented by row vectors $\mathbf{x} = (x_1 \cdots x_n)$ and $\mathbf{y} = (y_1 \cdots y_n)$. Then $\varphi(x,y) = \mathbf{x}[\varphi]\bar{\mathbf{y}}^t$ for all $x, y \in X$, where $\mathbf{x}, \mathbf{y}$ are the row vectors with the entries being the components of x, y with respect to the given basis $\{e_1, e_2, \ldots, e_n\}$ of X. In our situation, we will work with the following B-Hermitian form

$$\varphi(x,y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n,$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X$. Therefore the matrix representation of φ with respect to the standard basis $\{e_1, \ldots, e_n\}$ of X is $[\varphi] = I_n$. In matrix terminology, we have

$$G^B = \{g \in \mathrm{GL}_n(B) \mid g\bar{g}^t = \mu(g)I_n, \quad \mu(g) \in \mathbb{Q}^\times\}.$$

For the rest of the following chapters, we fix and deal with $n = 3$ situation. So the group of similitudes G^B over $\mathbb{Q}$ is

$$G^B = \{g \in \mathrm{GL}_3(B) \mid g\bar{g}^t = \mu(g)I_3, \quad \mu(g) \in \mathbb{Q}^\times\}.$$

For any $\mathbb{Q}$-algebra A, the set of A-rational points of G^B is given by

$$G^B(A) = \{g \in \mathrm{GL}_3(B \otimes_{\mathbb{Q}} A) \mid g\bar{g}^t = \mu(g)I_3, \mu(g) \in A^\times\}.$$

Note that $G^B/\mathbb{Q}$ is the algebraic group whose $\mathbb{Q}-$rational points are given by unitary similitude group $\mathrm{GU}_3(B)$, which is an inner form of $\mathrm{GSp}_6/\mathbb{Q}$ such that $G^B(\mathbb{R})$ is compact modulo its center. Also, G^B admits an integral model $\underline{G}^B/\mathbb{Z}$ in the sense of Gross [Gro96]. Maximal order $\mathscr{O}_B$ determines the following integral structure on G^B.

For every $\mathbb{Z}$-algebra A, the group of A-rational points is given by

$$\underline{G}^B(A) = \{g \in \mathrm{GL}_3(\mathscr{O}_B \otimes_{\mathbb{Z}} A) \mid g\bar{g}^t = \mu(g)I_3, \mu(g) \in A^\times\}.$$

From now on, we simply denote the group of similitudes over B by G^B and its integral model associated with the maximal order $\mathscr{O}_B$ by $\underline{G}^B$. For the sake of completeness, we include the definitions of 'inner forms' and 'arithmetic groups' here.

Definition 4.3.1. *A* **form of an algebraic group** *G/F is another algebraic group G'/F, which is isomorphic to G over some extension F'/F, i.e.,$G/F \cong G'/F$ over F'. In this case, G' is said to be an F'/F form of G.*

Remark 4.3.2. Two algebraic groups G and G' would be *inner forms* if they are Galois twists of each other, with the twists lying in $\mathrm{Inn}(G)$.

Remark 4.3.3. Given a connected, reductive linear algebraic F-group G, there is always a unique quasi-split F-group G', which is an inner form of G. For example, $\mathrm{SU}(2,1)$ and $\mathrm{SU}(3)$ are inner forms.

Definition 4.3.4. *A group is said to be an* ***arithmetic group*** *if it is obtained as the integer points of an algebraic group.*

Example 4.3.5. $\mathrm{SL}(n,\mathbb{Z})$, $\mathrm{Sp}(2n,\mathbb{Z})$.

Remark 4.3.6. Let G be an algebraic subgroup of $\mathrm{GL}_n(\mathbb{Q})$ for some n, then $\Gamma := \mathrm{GL}_n(\mathbb{Z}) \cap G(\mathbb{Q})$ is a group of integer points, Γ is an arithmetic subgroup of G.

Observation 1: Now observe that our group G^B satisfies the equivalent conditions as in Proposition 4.2.1. To prove that, let us consider $\mathscr{S}$ to be the maximal split torus in the center of G^B, Now the center of $G^B \cong \mathbb{Q}^\times$. This implies $\dim(\mathscr{S}) = 1$. If S' is the maximal quotient of G^B, which is a split torus, then the composite map

$$\mathscr{S} \to G^B \to S'$$

is an isogeny of tori (see Gross [Gro99, p. 62]). Again this implies that $\dim(S') = \dim(\mathscr{S}) = 1$. Once we fix an isomorphism, $S' \cong \mathbb{G}_m$, we get a continuous homomorphism

$$G^B(\mathbb{A}) \xrightarrow{\mu} \mathbb{A}^\times \xrightarrow{||\cdot||} \mathbb{R}_+^\times,$$

where μ denotes the similitude character. Define $G^B(\mathbb{A})_1 := \ker(||\cdot|| \circ \mu)$. Hence the Lie group $G^B(\mathbb{R})_1$ defined by $G^B(\mathbb{R})_1 := G^B(\mathbb{R}) \cap G^B(\mathbb{A})_1$ turns out to be $\mathrm{U}_3(\mathbb{H})$, which is a maximal compact subgroup of $G^B(\mathbb{R})(= \mathrm{GU}_3(\mathbb{H}))$. $\square$

4.3.2 Class number and mass formula

We consider $\mathscr{O}_B^{\oplus 3}$ as a lattice in X ($\mathscr{O}_B$ and X as in Section 4.1 and 4.3.1). The principal genus of G^B is denoted by $\mathscr{L}(\mathscr{O}_B)$ and defined as the collection of $\mathscr{O}_B$-lattices in X containing $\mathscr{O}_B^{\oplus 3}$. For each finite prime p, let $\mathscr{O}_{B,p} = \mathscr{O}_B \otimes_{\mathbb{Z}} \mathbb{Z}_p$, $L_p = L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $G_p^B = G^B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ be p-adic completions of $\mathscr{O}_B, L$ and G^B respectively. Then by definition, an $\mathscr{O}_B$-lattice L in X belongs to $\mathscr{L}(\mathscr{O}_B)$ if and only if $L_p = (\mathscr{O}_{B,p}^{\oplus 3})g_p$, where $g_p \in G_p^B$ for all prime p. The adèlic group $G^B(\mathbb{A})$ of G^B acts transitively on $\mathscr{L}(\mathscr{O}_B)$ by $Lg = \cap_p(L_p g_p \cap X)$ and then the stabiliser $\mathscr{K} := \mathrm{Stab}_{G^B(\mathbb{A})}(\mathscr{O}_B^{\oplus 3})$ is given by $\mathscr{K} = G_\infty^B \times K$ where $K = \prod_p U_p$ and $U_p = G_p^B \cap \mathrm{GL}_3(\mathscr{O}_{B,p})$, $\mathrm{Stab}_{G^B(\mathbb{Q})}(\mathscr{O}_B^{\oplus 3}) = G^B(\mathbb{Q}) \cap (G_\infty^B \times K)$

Remark 4.3.7. The notations are borrowed from Hashimoto [Has83] and this definition works in much more generality, for example, see [Has83]. But we have restricted ourselves to $n = 3$ case.

The number of the G^B-orbits in $\mathscr{L}(\mathscr{O}_B)$ is called the *class number*. There is a well-known fact which says this class number is equal to the number of $(G^B, \mathscr{K})$ double cosets in $G^B \backslash G^B_{\mathbb{A}} / \mathscr{K}$. These double cosets are called $\mathscr{K}$-classes. Hashimoto and Ibukiyama studied the class numbers of positive definite quaternary Hermitian forms in their papers (for details, see [HI80], [HI81]). There they classified the conjugacy classes of the group of similitudes for different forms and for arbitrary rank n. Using the traces of Brandt matrices associated with such forms, they explicitly worked out formulas in the binary case ($n = 2$) (cf.[Has80]) and in the ternary case ($n = 3$) (See [Has83]) under the condition that the discriminant of B is a prime p.

In our case, B has discriminant prime 2. So, from the table ([Has83, p. 493]) the group of similitudes G^B has class number 1 in the principal genus. Hence, we can choose the identity element as a representative of our $\mathscr{K}$-class.

Then the mass of genus $\mathscr{L}(\mathscr{O}_B)$ (cf. [Shi99, p. 67]) is defined to be the real number given by,

$$\mathrm{Mass}(\mathscr{O}_B^{\oplus 3}) = [\Gamma : 1]^{-1},$$

where $\Gamma := \mathrm{Stab}_{G^B(\mathbb{Q})}(\mathscr{O}_B^{\oplus 3})$. This formula (for details, see [Shi99, p. 68]) could be further simplified in our case and can be written as

$$\mathrm{Mass}(\mathscr{O}_B^{\oplus 3}) = \prod_{k=1}^{3} \frac{|B_{2k}|}{4k} \prod_{\substack{p \text{ prime} \\ p|\mathrm{disc(B)}}} \left(\prod_{k=1}^{3} (p^k + (-1)^k) \right),$$

where B_{2k} denote the Bernoulli numbers. Putting the values for B_{2k}, the mass formula gives

$$\begin{aligned}
\mathrm{Mass}(\mathscr{O}_B^{\oplus 3}) &= \prod_{k=1}^{3} \frac{|B_{2k}|}{4k} \prod_{\substack{p \text{ prime} \\ p|2}} \left(\prod_{k=1}^{3} (p^k + (-1)^k) \right) \\
&= \frac{|B_2||B_4||B_6|}{4 \cdot 8 \cdot 12} (2-1)(2^2+1)(2^3-1) \\
&= \frac{1}{82,944},
\end{aligned}$$

where $B_2 = 1/6, B_4 = -1/30, B_6 = 1/42$. Therefore $[\Gamma : 1]^{-1} = \frac{1}{82,944}$. Hence the cardinality of Γ is $82,944$. We will come back to the cardinality of Γ later in Chapter 5, where we will give an algorithm to compute it using MAGMA computational software system.

4.3.3 Algebraic automorphic forms of genus-3

Now, we fix an irreducible algebraic representation (ρ, V) of $G^B(\mathbb{Q})$ where V is a $\mathbb{Q}$-vector space. For any finite prime $p \neq 2$, we choose an isomorphism, $\mathrm{GU}_3(\mathscr{O}_{B,p}) \cong \mathrm{GSp}_6(\mathbb{Z}_p)$ which is compatible with the splitting isomorphism ι we fixed earlier in Section 4.1. Let us choose the maximal compact open subgroup of $G^B(\widehat{\mathbb{Q}})$ as

$$\underline{K} := \underline{G}^B(\widehat{\mathbb{Z}}) = \prod_{\substack{p<\infty \\ p\neq 2}} \mathrm{GSp}_6(\mathbb{Z}_p) \times \mathrm{GU}_3(\mathscr{O}_{B,2}) = \prod_{p<\infty} \underline{G}^B(\mathbb{Z}_p).$$

We want to take this compact open subgroup so that, we get automorphic forms of '*level* 1' in some sense. The space of algebraic automorphic forms of weight V, genus 3 and level $\underline{K}$ is then defined by

$$M_{G^B}(V) = \{f : G^B(\mathbb{A})/(G^B(\mathbb{R})_+ \times \underline{G}^B(\widehat{\mathbb{Z}})) \to V \mid f(\gamma g) = \gamma f(g) \text{ for } \gamma \in G^B(\mathbb{Q})\}. \tag{4.3.1}$$

In our case under the assumption of the existence of a conjectural Jacquet-Langlands correspondence between G^B and $\mathrm{GSp}_6/\mathbb{Q}$ our goal is to compute the dim $(M_{G^B}(V))$. We refer the readers to see Section 4.4 to know about conjectural Jacquet-Langlands correspondence. Now by Proposition 4.2.2 the double coset space

$$G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/(G^B(\mathbb{R})_+ \times \underline{K})$$

is finite, where $G^B(\mathbb{R})_+$ is the connected component of the identity in the Lie group $G^B(\mathbb{R})$. By definition, any algebraic automorphic form f in $M_{G^B}(V)$ is completely determined by its values on this double coset space. In fact, we could prove the following.

Observation 2: The cardinality of $G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/(G^B(\mathbb{R})_+ \times \underline{K})$ is 1.

We already know that in our case, B has discriminant prime 2. So, from the table ([Has83, p. 493]) the group of similitudes G^B has class number 1 in the principal genus. This implies the cardinality of $G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/(G^B(\mathbb{R}) \times \underline{K})$ is 1. Now, let us consider the following two exact sequences

$$1 \to G_1^B(\mathbb{Q})\backslash G_1^B(\mathbb{A})/G_1^B(\mathbb{R}) \times \underline{K}_1 \to G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/G^B(\mathbb{R}) \times \underline{K} \xrightarrow{\mu} \mathbb{Q}^\times\backslash\mathbb{A}^\times/(\mathbb{R}^\times \times \widehat{\mathbb{Z}}) \to 1$$

and

$$1 \to G_1^B(\mathbb{Q})\backslash G_1^B(\mathbb{A})/G_1^B(\mathbb{R}) \times \underline{K}_1 \to G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/G^B(\mathbb{R})_+ \times \underline{K} \xrightarrow{\mu} \mathbb{Q}^\times\backslash\mathbb{A}^\times/\mathbb{R}_{>0}^\times \times \widehat{\mathbb{Z}} \to 1$$

where μ is the similitude character. The groups G_1^B, $\underline{K}_1$ are the collection of matrices from G^B and $\underline{K}$ respectively, where the matrices have similitude 1. Since,

(1) $|\mathbb{Q}^\times\backslash\mathbb{A}^\times/(\mathbb{R}^\times \times \widehat{\mathbb{Z}})| = |\mathbb{Q}^\times\backslash\mathbb{A}^\times/(\mathbb{R}_{>0}^\times \times \widehat{\mathbb{Z}})| = 1$ (because the field $\mathbb{Q}$ has the narrow class number 1).

(2) Both the double coset sets

$$G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/(G^B(\mathbb{R}) \times \underline{K}) \text{ and } G^B(\mathbb{Q})\backslash G^B(\mathbb{A})/(G^B(\mathbb{R})_+ \times \underline{K})$$

are finite sets and they have the same kernel and image space under the map μ, hence the cardinality of both the double coset sets are same and that is 1.

This observation implies there is only one class in the set of double cosets. So we can take identity element I_3 as a representative of that class. Then by Proposition 4.2.3, the space of automorphic forms for G^B of full level and weight V is isomorphic to the subspace of Γ-invariants V^Γ via the map $f \to f(I_3)$ and we have

$$M_{G^B}(V) \cong V^\Gamma, \tag{4.3.2}$$

where

$$\Gamma = G^B(\mathbb{Q}) \cap (G^B(\mathbb{R})_+ \times \underline{G}^B(\widehat{\mathbb{Z}})),$$
$$V^\Gamma = \{v \in V \mid \rho(\gamma)v = v\ \forall \gamma \in \Gamma\}.$$

Note that Γ being the arithmetic subgroup of G^B (by Proposition 4.2.1) is finite. In fact, we have already calculated the cardinality of Γ using the theory of mass formula by Shimura [Shi99] in the previous section.

We make the following observation about Γ.

Observation 3: We already know

$$\Gamma = \mathrm{Stab}_{G^B(\mathbb{Q})}(\mathscr{O}_B^{\oplus 3}) = G^B(\mathbb{Q}) \cap (G^B(\mathbb{R})_+ \times \underline{K}).$$

Our next claim is

$$\Gamma = \mathrm{Stab}_{G_1^B(\mathbb{Q})}(\mathscr{O}_B^{\oplus 3}).$$

To prove that, let us take an element $\gamma \in \mathrm{Stab}_{G^B(\mathbb{Q})}(\mathscr{O}_B^{\oplus 3})$. Then we get $\mu(\gamma) \in \mathbb{Q}_+^\times$. We have $\mu(\gamma) \in \widehat{\mathbb{Z}}^\times$ too, since γ is in $\underline{K}$. This implies $\mu(\gamma) = 1$. Therefore, $\mathrm{Stab}_{G_1^B(\mathbb{Q})}(\mathscr{O}_B^{\oplus 3}) = \Gamma$. So, we could write Γ more explicitly as,

$$\Gamma = \{\gamma \in \mathrm{GL}_3(\mathscr{O}_B) \mid \gamma \bar{\gamma}^t = I_3\}. \tag{4.3.3}$$

We will work with this expression of Γ later on.

4.4 Conjectural Jacquet-Langlands correspondence

This section is dedicated to a brief discussion about the conjectural Jacquet-Langlands correspondence in the case of symplectic similitude groups GSp_6 and its inner forms. This correspondence is a theorem for the case of GL_2, proved by Jacquet-Langlands [JL70]. This relates the automorphic representations of the multiplicative group of quaternion algebra with certain automorphic representation of GL_2. We refer to Ihara [Iha64], Hashimoto and Ibukiyama [HI81], [Ibu84] for the analogue of conjectural J-L correspondence in the case of GSp_4 over $\mathbb{Q}$. In particular Sorensen proved this conjecture to be true for GSp_4 over $\mathbb{Q}$ in paper [Sor09a] and for GSp_4 over a totally real field F of even degree in paper [Sor09b].

Conjecturally, Jacquet-Langlands correspondence is a bijection between smooth automorphic representations on the compact side, i.e., of G^B to cuspidal automorphic representations of GSp_6 that are square integrable representations at each place where B is ramified. In our case, B is the unique (upto isomorphism) definite quaternion algebra over $\mathbb{Q}$ with ramification at 2 and ∞. Let $\mathtt{S} = \{2, \infty\}$. We now describe analogue of JL correspondence for the group GSp_4 as described in Sorensen (cf. [Sor09a]). Let π be an automorphic representation of $G^B(\mathbb{A})$, with trivial central character and π_∞ some finite-dimensional representation. Then conjecturally there exists a cuspidal automorphic representation Π of $G'(\mathbb{A})$, with trivial central character such that $\Pi_p = \pi_p$ for all $p \notin \mathtt{S}$, and Π_∞ is a cohomological discrete series representation. Moreover, it is further expected that if π_2 is paraspherical (i.e., has fixed vectors by a paramodular group) then Π_2 is paraspherical too. Perhaps, an additional local assumption is needed as in Sorensen (cf. [Sor09a, Section 4.2.2]) to get the JL correspondence, otherwise we might get a weak version of it.

In our case, $\mathrm{JL}(\pi) = \Pi$ is ramified at 2. Because at 2, we can't have principal series representation, as principal series representations are never square integrable. Because of the same reason, we can not have level 1 either, since at level 1 we only get principal series representations.

The heart of the proof is the 'character identity' which comes from Arthur's trace formula. We are interested in the spectral side of the trace formula for both the groups G^B and G'. There is a distribution, denoted by $I^{G'}_{\mathrm{disc}}$ which is supported on automorphic representations occurring discretely in the trace formula. The distribution has an expansion of the following form

$$I^{G'}_{\mathrm{disc}}(f') = \sum_{\Pi} a^{G'}_{\mathrm{disc}}(\pi)\mathrm{tr}\Pi(f')$$

for a smooth function f' on $G'(\mathbb{A})$. Here $a^{G'}_{\mathrm{disc}}$ denotes a complex number attached to an automorphic representation Π. However, the distribution formed this way is unstable. To make it stable, a certain suitable error term needs to be subtracted (see Arthur [Art98]). There is a similar formula for the group G^B. But since G^B is anisotropic modulo center so all the term occurs discretely and $a^{G}_{\mathrm{disc}}(\pi)$ always denotes the multiplicity of π (see Sorensen [Sor09a, Section 4.2]).

A standard argument based on the spectral identity connecting the stable trace formula of G^B and G' says that there exists an irreducible representation Π of $G'(\mathbb{A})$ such that $\Pi^{\mathrm{S}} = \pi^{\mathrm{S}}$. Now to talk about the infinity component Π_∞ of the representation, we recall that by Langlands classification, the irreducible admissible representations of $G'(\mathbb{R})$ are partitioned into finite L-packets Π_ϕ parametrized by admissible homomorphisms $\phi : W_{\mathbb{R}} \to {}^{L}G^B$. Since $G^B(\mathbb{R})$ is compact modulo center, the L-packets are singletons $\{\pi_\phi\}$. Now, pick a cohomological discrete series representation (see Sorensen [Sor09a, Section 4.2.3]) Π_∞ of $G'(\mathbb{R})$ from the L-packet Π_ϕ with the same central and infinitesimal characters as π_ϕ, then the key identity is as follows:

$$\sum_{\pi_2} a^{G}_{\mathrm{disc}}(\pi_\infty \otimes \pi_2 \otimes \pi^{\mathrm{S}})\mathrm{tr}\pi_2(f_2) = \sum_{\Pi_2} a^{G'}_{\mathrm{disc}}(\Pi_\infty \otimes \Pi_2 \otimes \Pi^{\mathrm{S}})\mathrm{tr}\Pi_2(f_2') \tag{4.4.1}$$

which is valid for any discrete L-parameter ϕ, and $\Pi_\infty \in \Pi_\phi$ and any matching pair of smooth functions f_2 and f_2' on $G(\mathbb{Q}_2)$ and $G'(\mathbb{Q}_2)$ respectively. Now to get information at prime 2, we use argument on linear independence of characters for $G^B(\mathbb{Q}_2)$. There exists a function f_2 and an automorphic representation π_2 such that the left hand side of the key identity Equation (4.4.1) is non-zero. Then the right hand side of Equation (4.4.1) is non-zero too. This implies there exists at least one matching function f_2' and correspondingly one representation Π_2 with $\mathrm{tr}\Pi_2(f_2') \neq 0$.

Chapter 5

On the computation of genus-3 algebraic automorphic forms over $\mathbb{Q}$

In this chapter, we will discuss about the algorithm in computing the dimension of the space $M_{G^B}(V)$ of automorphic forms on B of weight V and full level. At the end of this chapter we will give a Table 5.1 of dimensions for various small weights V computed using the algorithm, which we have implemented in MAGMA using the packages of the Magma computational Algebra system (version V2.24). Table 5.1 is the main result of this chapter. We will discuss some implementation issues related to the algorithm as well. But at first, we will briefly recall some necessary facts on the highest weight theory for symplectic Lie algebras.

5.1 Background on the highest weight theory

According to the basic results of Fulton-Harris ([FH91, Lecture 7]) representations of a complex Lie algebra $\mathfrak{g}$ will correspond exactly to the representations of the associated simply connected Lie group $\widetilde{G}$. For any other group given as $G = \widetilde{G}/C$, where $C \subset Z(\widetilde{G})$ with Lie algebra $\mathfrak{g}$, representations of G are simply the representations of $\widetilde{G}$ trivial on C (cf.[FH91, p. 369, Lecture 23]). Let $\mathfrak{h}$ be the Cartan subalgebra and the root space $\mathfrak{h}^*$ be spanned by weights $L_1, L_2, \ldots, L_n$. Then any weight can be written uniquely as an integral linear combination $\lambda_1 L_1 + \lambda_2 L_2 + \cdots + \lambda_n L_n$.

Fact 5.1.1. The following facts with all the notations intact are borrowed from different sections of Fulton-Harris [FH91].

(1) Let V_λ be the irreducible representation of $\mathfrak{sp}_{2n}$ with highest weight $\lambda = (\lambda_1 + \lambda_2 + \cdots + \lambda_n)L_1 + (\lambda_2 + \cdots + \lambda_n)L_2 + \cdots + \lambda_n L_n$. Then V_λ will be a representation of $\mathrm{Sp}_{2n}(\mathbb{C})/\{\pm 1\}$ if $\sum \lambda_j$ is even [FH91, p. 371, Proposition 23.13].

(2) $V^{(k)} = V_{0,\ldots,1,\ldots,0}$ is the irreducible representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ with highest weight $L_1 + \cdots + L_k$ ([FH91, p. 262]).

(3) Any other representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ will occur in a tensor product of these $V^{(k)}$. Specifically, the irreducible representation $V_{a_1,a_2,\ldots,a_n}$ with highest weight $\lambda = (a_1 + \cdots + a_n)L_1 + \cdots + a_n L_n = a_1 L_1 + a_2(L_1 + L_2) + \cdots + a_n(L_1 + \cdots + L_n)$ will occur inside the space
$$\mathrm{Sym}^{a_1} V^{(1)} \otimes \mathrm{Sym}^{a_2} V^{(2)} \otimes \cdots \otimes \mathrm{Sym}^{a_n} V^{(n)},$$
where $V^{(1)}$ is the standard representation of $\mathfrak{sp}_{2n}(\mathbb{C})$ on $\mathbb{C}^{2n}$ ([FH91, p. 262]).

(4) The k^{th} symmetric powers $\mathrm{Sym}^k(\mathbb{C}^{2n})$ of the standard representation are all irreducible in both the cases for Lie algebra and for group $\mathrm{Sp}_{2n}(\mathbb{C})$ ([FH91, p. 265, p. 406]).

5.2 Dual spaces, contractions, and exterior powers

This section contains some necessary background theory about duals, contraction maps and exterior powers. We have followed the exact notations from [FH91].

Fact 5.2.1. (1) If $\{e_i\}$ is a basis for V, then $\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} \mid i_1 < i_2 < \cdots < i_n\}$ is a basis for the exterior power $\bigwedge^n V$ of V.

(2) If V^* denotes the dual space of V, then $\bigwedge^n(V^*) \cong (\bigwedge^n V)^*$.

(3) The dual basis for $\bigwedge^n(V^*)$ is $\{e_{i_1}^* \wedge e_{i_2}^* \wedge \cdots \wedge e_{i_n}^* \mid i_1 < i_2 < \cdots < i_n\}$.

(4) The contraction maps $C^i_j : V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes (p-1)} \otimes (V^*)^{\otimes (q-1)}$ for any $1 \leq i \leq p$ and $1 \leq j \leq q$, are determined by, evaluating the j^{th} co-ordinates of $(V^*)^{\otimes q}$ on the i^{th} co-ordinate of $V^{\otimes p}$. i.e., $C^i_j : V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes (p-1)} \otimes (V^*)^{\otimes (q-1)}$ is given by $C^i_j(v_1 \otimes \cdots \otimes v_p \otimes \phi_1 \otimes \cdots \otimes \phi_j \otimes \cdots \otimes \phi_q) = \phi_j(v_i) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots v_p \otimes \phi_1 \otimes \cdots \otimes \hat{\phi}_j \otimes \cdots \otimes \phi_q$.

(5) There are related contractions between exterior powers and dual spaces of exterior powers. They are known as internal products. The contraction maps for the exterior powers are denoted by $\lrcorner$ and $\llcorner$ respectively, and they are given as:

$$\bigwedge^p V \otimes \bigwedge^{p+q}(V^*) \to \bigwedge^q (V^*),\ x \otimes \alpha \mapsto x \lrcorner \alpha$$

and

$$\bigwedge^{p+q} V \otimes \bigwedge^p (V^*) \to \bigwedge^q (V^*),\ x \otimes \alpha \mapsto x \llcorner \alpha,$$

where $x \lrcorner \alpha$ and $x \llcorner \alpha$ are defined as follows:

(a) If $x = v_1 \wedge \cdots \wedge v_p$ and $\alpha = \phi_1 \wedge \cdots \wedge \phi_{p+q}$ with $v_i \in V$ and $\phi_j \in V^*$, then

$$x \lrcorner \alpha = \sum \operatorname{sgn}(\sigma) \phi_{\sigma(q+1)}(v_1) \cdot \ldots \cdot \phi_{\sigma(q+p)}(v_p) \cdot \phi_{\sigma(1)} \wedge \cdots \wedge \phi_{\sigma(q)},$$

the sum over all permutations σ of $\{1, \ldots, p+q\}$ that preserve the order of $\{1, \ldots, q\}$.

(b) If $x = v_1 \wedge \cdots \wedge v_{p+q}$ and $\alpha = \phi_1 \wedge \cdots \wedge \phi_p$ with $v_i \in V$ and $\phi_j \in V^*$, then

$$x \llcorner \alpha = \sum \operatorname{sgn}(\sigma) \phi_1(v_{\sigma(1)}) \cdot \ldots \cdot \phi_p(v_{\sigma(p)}) \cdot v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(p+q)},$$

the sum over all permutations that preserve the order of $\{p+1, \ldots, p+q\}$. For details, we refer to the reader [FH91, Exercise B.15].

There are analogous formulas for symmetric powers too [FH91, Appendices B.3].

5.3 Outline of the algorithm

Our primary goal is to compute the dimension of $M_{G^B}(V)$. But by the isomorphism as in expression (4.3.2) in Section (4.3.3) calculating this dimension is same as calculating the dimension of V^{Γ}. The first and foremost step towards this is to calculate the group Γ explicitly.

Cardinality of Γ:

We calculate Γ explicitly by writing a program using MAGMA software. For this calculation, let us look at the description of Γ as in Equation (4.3.3) (Observation 3), i.e.,

$$\Gamma = \{\gamma \in \mathrm{GL}_3(\mathscr{O}_B) \mid \gamma\bar{\gamma}^t = I_3\}.$$

Let $\gamma := \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & m \end{pmatrix}$ be an arbitrary 3×3 matrix from Γ where a,b,c,d,e,f,g,h,m are arbitrary elements from the quaternion algebra B. We could further write a,b,c in the following ways $a := a_1 + a_2 i + a_3 j + a_4 k$; $b := b_1 + b_2 i + b_3 j + b_4 k$; $c := c_1 + c_2 i + c_3 j + c_4 k$, where $a_1,\ldots,a_4,b_1,\ldots,b_4,c_1,\ldots,c_4$ are rational numbers. Similar expressions hold for b,c,d,e,f,g,h,m also. Since the matrix γ is from Γ so we could express the matrix entries as elements from $\mathscr{O}_B$. Hence each a,b,c,d,e,f,g,h,m has another set of expressions, such as $a := \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 l$, $b := \beta_1 + \beta_2 i + \beta_3 j + \beta_4 l$, $c := \gamma_1 + \gamma_2 i + \gamma_3 j + \gamma_4 l$, where $\alpha_1 \ldots \alpha_4, \beta_1 \ldots \beta_4, \gamma_1 \ldots \gamma_4$ are integers. Now, if we equate these two expressions of a, we get,

$$\begin{aligned} a_1 + a_2 i + a_3 j + a_4 k &= \alpha_1 + \alpha_2 i + \alpha_3 j + \alpha_4 \left(\frac{1+i+j+ij}{2}\right) \\ &= \left(\frac{2\alpha_1 + \alpha_4}{2}\right) + \left(\frac{2\alpha_2 + \alpha_4}{2}\right) i + \left(\frac{2\alpha_3 + \alpha_4}{2}\right) j + \left(\frac{\alpha_4}{2}\right) k. \end{aligned}$$

If we compare the constant terms and coefficients of i, j, k we get a_1,a_2, a_3,a_4 as half integers. We could similarly do for the rest of the matrix entries. Now, γ being the member of Γ gives us,

$$a\bar{a} + b\bar{b} + c\bar{c} = 1.$$

This implies,

$$a_1^2+a_2^2+a_3^2+a_4^2+b_1^2+b_2^2+b_3^2+b_4^2+c_1^2+c_2^2+c_3^2+c_4^2=1.$$

Furthermore, if we expand these expressions putting the values of $a_1,\dots,c_4$, we get,

$$(2\alpha_1+\alpha_4)^2+(2\alpha_2+\alpha_4)^2+(2\alpha_3+\alpha_4)^2+\alpha_4^2+(2\beta_1+\beta_4)^2+(2\beta_2+\beta_4)^2+(2\beta_3+\beta_4)^2$$

$$+\beta_4^2+(2\gamma_1+\gamma_4)^2+(2\gamma_2+\gamma_4)^2+(2\gamma_3+\gamma_4)^2+\gamma_4^2=4.$$

This implies, $\alpha_4,\beta_4,\gamma_4$ are integers in $[-2,2]$ and for different choices of α_4, we could show that $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3,\gamma_1,\gamma_2,\gamma_3$, are integers in $[-1,1]$. By this calculation, we are able to give finite bounds for entries of an arbitrary matrix in Γ. Since the computation is now over a finite discrete set, so the cardinality of Γ is finite. We wrote one MAGMA program and got the cardinality 82944.

Remark 5.3.1. This cardinality matches with the mass formula calculation as we did in Chapter 4 (cf. Section 4.3.2).

Generators of Γ:

The next task is to find the generators of Γ. Now observe that, each element $x+yi+zj+wk$ of B can be associated with a 2×2 matrix over a suitable choice of field $E/\mathbb{Q}$. E is such that E splits B. In our case, we already fixed E to be $\mathbb{Q}(I)$, where $I^2=-1$. Though as some inbuilt packages from Magma works only for complex field $\mathbb{C}$. So, only for programming convenience we consider the complex field $\mathbb{C}$, otherwise, $\mathbb{Q}(I)$ is an adequate choice to work with. Now, using the following bijections:

$$1\leftrightarrow\begin{pmatrix}1&0\\0&1\end{pmatrix}, i\leftrightarrow\begin{pmatrix}I&0\\0&-I\end{pmatrix}, j\leftrightarrow\begin{pmatrix}0&1\\-1&0\end{pmatrix}, k\leftrightarrow\begin{pmatrix}0&I\\I&0\end{pmatrix}.$$

we can observe

$$\begin{pmatrix}I&0\\0&-I\end{pmatrix}^2=-\begin{pmatrix}1&0\\0&1\end{pmatrix};\begin{pmatrix}0&1\\-1&0\end{pmatrix}^2=-\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}I&0\\0&-I\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}=\begin{pmatrix}0&I\\I&0\end{pmatrix}.$$

Hence, an arbitrary element $x+yi+zj+wk$ of B can be associated with a 2×2 matrix in the following way:

$$x+yi+zj+wk \leftrightarrow x\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}+y\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}+z\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}+w\begin{pmatrix}0 & I\\ I & 0\end{pmatrix}=\begin{pmatrix}x+Iy & z+Iw\\ -z+Iw & x-Iy\end{pmatrix}.$$

Using the above observation, an arbitrary element $\gamma := \begin{pmatrix}a & b & c\\ d & e & f\\ g & h & m\end{pmatrix}$ in Γ can be viewed as

$$\begin{pmatrix}\{a_1,a_2,a_3,a_4\} & \{b_1,b_2,b_3,b_4\} & \{c_1,c_2,c_3,c_4\}\\ \{d_1,d_2,d_3,d_4\} & \{e_1,e_2,e_3,e_4\} & \{f_1,f_2,f_3,f_4\}\\ \{g_1,g_2,g_3,g_4\} & \{h_1,h_2,h_3,h_4\} & \{m_1,m_2,m_3,m_4\}\end{pmatrix},$$

here we introduce a new symbol $\{p,q,r,s\} := \frac{2p+s}{2}+\frac{2q+s}{2}i+\frac{2r+s}{2}j+\frac{s}{2}k$ for convenience to write matrix entries.

The above $\gamma\in\Gamma$ can be considered as 6×6 matrix as follows

$$\begin{pmatrix}[a_1,a_2,a_3,a_4] & [b_1,b_2,b_3,b_4] & [c_1,c_2,c_3,c_4]\\ [d_1,d_2,d_3,d_4] & [e_1,e_2,e_3,e_4] & [f_1,f_2,f_3,f_4]\\ [g_1,g_2,g_3,g_4] & [h_1,h_2,h_3,h_4] & [m_1,m_2,m_3,m_4]\end{pmatrix},$$

where each entry of the above 2×2 matrix is defined by

$$[p,q,r,s] := \begin{pmatrix}\frac{2p+s}{2}+I\frac{2q+s}{2} & \frac{2r+s}{2}+I\frac{s}{2}\\ -\frac{2r+s}{2}+I\frac{s}{2} & \frac{2p+s}{2}-I\frac{2q+s}{2}\end{pmatrix}.$$

In fact, we can check that these matrices are from Sp_6 over $\mathbb{Q}$, where

$$\mathrm{Sp}_6 = \{g\in \mathrm{GL}_6 \mid g^tJg = J\}$$

and

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

Remark 5.3.2. For the sake of calculations we have taken this skew-symmetric form J.

Using MAGMA software program we have found the following three matrices generate the group Γ. We fix these three generators of Γ for further computations.

$$M_1 = \begin{pmatrix} \frac{I-1}{2} & \frac{-I+1}{2} & 0 & 0 & 0 & 0 \\ \frac{-I-1}{2} & \frac{-I-1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{I-1}{2} & \frac{-I+1}{2} \\ 0 & 0 & 0 & 0 & \frac{-I-1}{2} & \frac{-I-1}{2} \\ 0 & 0 & \frac{I+1}{2} & \frac{I+1}{2} & 0 & 0 \\ 0 & 0 & \frac{I-1}{2} & \frac{-I+1}{2} & 0 & 0 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{I-1}{2} & \frac{I+1}{2} \\ 0 & 0 & 0 & 0 & \frac{I-1}{2} & \frac{-I-1}{2} \\ 0 & 0 & \frac{I-1}{2} & \frac{-I-1}{2} & 0 & 0 \\ 0 & 0 & \frac{-I+1}{2} & \frac{-I-1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -I \\ 0 & 0 & 0 & 0 & -I & 0 \\ 0 & 0 & \frac{-I-1}{2} & \frac{I-1}{2} & 0 & 0 \\ 0 & 0 & \frac{I+1}{2} & \frac{I-1}{2} & 0 & 0 \\ \frac{I+1}{2} & \frac{-I-1}{2} & 0 & 0 & 0 & 0 \\ \frac{-I+1}{2} & \frac{-I+1}{2} & 0 & 0 & 0 & 0 \end{pmatrix},$$

Remark 5.3.3. Observe also that all of these three matrices M_1, M_2, M_3 satisfy the same characteristic polynomial $x^6+x^5+2x^4+x^3+2x^2+x+1=(x-\omega)^2(x-\bar{\omega})^2(x+\omega)(x+\bar{\omega})$ and the same minimal polynomial $x^4+x^2+1=(x^2+x+1)(x^2-x+1)=(x-\omega)(x-\bar{\omega})(x+\omega)(x+\bar{\omega})$, where ω is the third root of unity. We can see all the roots of the minimal polynomial are distinct. Hence each of the generating matrices is diagonalizable (or semisimple).

As we get these generators of Γ, hence we can write $M_{G^B}(V) \cong V^{\Gamma} = \cap_{i=1}^{3} V^{M_i}$. So, calculating the dimension for the space $M_{G^B}(V)$ is same as calculating the dimension of the intersection of these three subspaces V^{M_i} for $i \in \{1,2,3\}$. Since, we started with an irreducible algebraic representation (ρ, V) of G^B over $\mathbb{Q}$ (cf. Section 4.3.1), it can be parametrized by quadruple of non-negative integers a,b,c,d where the representation $V(=V_{a,b,c,d})$ is the unique highest weight direct summand of

$$\widetilde{V}_{(a,b,c,d)} := \mathrm{Sym}^a(\mathbb{C}^6) \otimes \mathrm{Sym}^b(W) \otimes \mathrm{Sym}^c(U) \otimes \mu^d. \tag{5.3.1}$$

$$\text{Define } \widetilde{V}_{(a,b,c)} := \mathrm{Sym}^a(\mathbb{C}^6) \otimes \mathrm{Sym}^b(W) \otimes \mathrm{Sym}^c(U). \tag{5.3.2}$$

Then we have,

$$V = V_{a,b,c,d} \subseteq \widetilde{V}_{(a,b,c)} \otimes \mu^d. \tag{5.3.3}$$

For details see [FH91, p. 258]. Here μ denotes the similitude factor.

Remark 5.3.4.

1. The spaces W and U are subspaces of exterior powers $\bigwedge^2(\mathbb{C}^6)$ and $\bigwedge^3(\mathbb{C}^6)$ respectively. We will give the full descriptions of W, U below.

2. For the sake of simplicity in our calculations, we will fix $d=0$. If in addition $b=c=0$, then we get, $V_{a,0,0} = \widetilde{V}_{(a,0,0)}$ (by Fact 5.1.1 (4)).

3. The non-negative integers a,b,c have relations among themselves. We need $a+c$ to be even and there is no condition on b (by Fact 5.1.1(1)).

4. Since we do not directly identify our space $V_{a,b,c}$ as a subspace of the vector space $\widetilde{V}_{(a,b,c)}$ so for our purpose, we only compute an upper bound of $V^{\Gamma}_{a,b,c}$. The dimensions of $\widetilde{V}_{(a,b,c)}$ surely give an upper bound for dim $(V^{\Gamma}_{a,b,c})$. Now if we extend the action of Γ on $\widetilde{V}_{(a,b,c)}$ and compute dim $(\widetilde{V}^{\Gamma}_{(a,b,c)})$ then this will give a better bound for dim $(V^{\Gamma}_{a,b,c})$. Hence the next task in our algorithm is to define the action of Γ on $\widetilde{V}_{(a,b,c)}$.

Extension of the action of Γ to $\tilde{V}_{(a,b,c)}$:

We start by considering the standard representation of $\mathfrak{sp}_6(\mathbb{C})$ on $\mathbb{C}^6$. Then the matrices obtained by the action of the generators of Γ on $\mathbb{C}^6$ will be the same M_1, M_2, M_3. Note that the dimension of the space $\mathrm{Sym}^a(\mathbb{C}^6)$ is $\binom{5+a}{a}$. Call this dimension as n_a.

Descriptions of W and U:

The space W:

The vector space $\bigwedge^2(\mathbb{C}^6) = \frac{\mathbb{C}^6\otimes\mathbb{C}^6}{<v\otimes v>}$ has basis $\{e_i \wedge e_j \mid 1 \leq i < j \leq 6\}$. Let $\mathcal{S}$ denote the vector space of all 6×6 skew-symmetric matrices over $\mathbb{C}$. Then we have the following isomorphism of vector spaces $\mathcal{S} \cong (\bigwedge^2 V)^* \cong \bigwedge^2 V^*$ (for the last isomorphism (see (2) in Section 5.2). Under this isomorphism, the skew-form J preserved by the definition of $\mathfrak{sp}_6(\mathbb{C})$, maps to $e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*$, i.e.,

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \mapsto e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*.$$

By the definition of the action of $\mathrm{Sp}_6(\mathbb{C})$ on the standard representation preserves a skew-form so that, the representation on $\bigwedge^2(\mathbb{C}^6)\left(\cong \bigwedge^2(\mathbb{C}^6)^*\right)$ has a trivial summand. This implies that $\bigwedge^2(\mathbb{C}^6)$ has an irreducible subspace $\langle J\rangle$ by the action of $\mathrm{Sp}_6(\mathbb{C})$. The complement of the trivial representation $\langle J\rangle$ in $\bigwedge^2(\mathbb{C}^6)$ is irreducible too (for details, see [FH91, Section 17.1]). For us, W is this irreducible representation such that, $\bigwedge^2(\mathbb{C}^6) \cong W \oplus \mathbb{C}$. Therefore

$\dim(W) = \dim(\bigwedge^2(\mathbb{C}^6)) - 1 = \frac{6(6-1)}{2} - 1 = 14$. In fact, we could explicitly write down all the basis elements of W. Define, a bilinear form on $\mathcal{S}$ by following

$$J_1 : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$$

by $J_1(X,Y) = \mathrm{tr}(JXJY)$.

Claim: The new form J_1 is Sp_6-invariant.

To prove our claim, let A be any symplectic matrix. Then A satisfies $A^tJA = J$. Now Sp_6 acts on $\mathcal{S}$ via $A \cdot X = A^tXA$ (since X is skew-symmetric then A^tXA is also so). Therefore,

$$J_1(A\cdot X, A\cdot Y) = J_1(A^tXA, A^tYA) = \mathrm{tr}(JA^tXA, JA^tYA) = \mathrm{tr}(AJA^tXAJA^tY) = \mathrm{tr}(JXJY).$$

This implies J_1 is Sp_6-invariant. By definition J is Sp_6-invariant. Our required space W is a space perpendicular to J under J_1. Therefore

$$W = J^{\perp} = \{X \in \mathcal{S} \mid J_1(X,J) = 0\} = \{X \in \mathcal{S} \mid \mathrm{tr}(JXJJ) = 0\} = \{X \in \mathcal{S} \mid \mathrm{tr}(JX) = 0\}.$$

Now the trace condition

$$\mathrm{tr}\left[\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} & x_{26} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} & x_{36} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} & x_{46} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 & x_{56} \\ -x_{16} & -x_{26} & -x_{36} & -x_{46} & -x_{56} & 0 \end{pmatrix}\right] = 0.$$

gives, $x_{12} + x_{34} + x_{56} = 0$.

Therefore the space W is $\langle e_{12} - e_{34},\ e_{13},\ e_{14},\ e_{15},\ e_{16}, e_{23}, e_{24}, e_{25}, e_{26}, e_{35}, e_{36}, e_{45}, e_{46},\ e_{34} - e_{56}\rangle$, where e_{ij} denotes the basis element $e_i \wedge e_j$ of $\bigwedge^2(\mathbb{C}^6)$.

Action of Γ on W:

Define the action of Γ on $\bigwedge^2(\mathbb{C}^6)$ by

$$\begin{aligned} g\cdot(e_{i_0}\wedge e_{j_0}) &= g\cdot e_{i_0} \wedge g\cdot e_{j_0} (\text{ for } i_0 < j_0) \\ &= (g_{1i_0}e_1 + g_{2i_0}e_2 + \cdots + g_{6i_0}e_6) \wedge (g_{1j_0}e_1 + g_{2j_0}e_2 + \cdots + g_{6j_0}e_6) \\ &= \sum_{i<j} (g_{ii_0}g_{jj_0} - g_{ij_0}g_{ji_0}) e_i \wedge e_j, \end{aligned}$$

where $g = (g_{ij})$ is 6×6 matrix with $g \cdot e_j = (g_{ij}) \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} g_{1j} \\ g_{2j} \\ \vdots \\ g_{6j}. \end{pmatrix}$

Observation: Γ acts on $\bigwedge^2(\mathbb{C}^6)$ by conjugation, i.e., using the bijection,

$$e^*_{i_0} \wedge e^*_{j_0} \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & * & * \\ * & * & 0 & 1 & * & * \\ * & * & -1 & 0 & * & * \\ * & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where the $[i_0, j_0]^{\text{th}}$ entry is 1 and correspondingly $[j_0, i_0]^{\text{th}}$ entry is -1, we can show

$$g \cdot (e_{i_0} \wedge e_{j_0}) = g \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ * & 0 & * & * & * & * \\ * & * & 0 & 1 & * & * \\ * & * & -1 & 0 & * & * \\ * & * & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} g^t.$$

Since $\Gamma = \langle M_1, M_2, M_3 \rangle$, so it is enough to see how each M_i acts on basis elements of W. Let P_1, P_2, P_3 be the matrices obtained by the action of M_1, M_2, M_3 on W respectively.

Action of Γ on $\mathrm{Sym}^b(W)$:

The vector space $\mathrm{Sym}^b(W)$ has the dimension $\binom{13+b}{13}$. Define, $n_b := \binom{13+b}{13}$. Now to see the action of each M_i (for $i = 1, 2, 3$) on $\mathrm{Sym}^b(W)$, let us recall the following definition of the linear transformation

$$\mathrm{Sym}^b(P_i) : \mathrm{Sym}^b(W) \to \mathrm{Sym}^b(W)$$

$\mathrm{Sym}^b(P_i) \cdot (v_{j_1} \cdot v_{j_2} \cdot \ldots \cdot v_{j_b}) = P_i(v_{j_1}) \cdot P_i(v_{j_2}) \cdot \ldots \cdot P_i(v_{j_b})$, where $1 \le j_1 \le j_2 \le \cdots \le j_b \le 14$, where for each $i = 1, 2, 3$, P_i is a 14×14 matrix and $\{v_{j_i}\}_{1 \le j_i \le 14}$ are arbitrary basis vectors of $W \subset \bigwedge^2(\mathbb{C}^6)$. Precisely the action of each M_i (for $i = 1,2,3$) on $\mathrm{Sym}^b(W)$ are determined by the action of each P_i ($i = 1,2,3$) on W. By abuse of notation, let us denote the matrix obtained by this action as $\mathrm{Sym}^b(P_i)$ too (for each $i = 1,2,3$).

Remark 5.3.5. For each $i \in \{1,2,3\}$, $\mathrm{Sym}^b(P_i)$ is a $n_b \times n_b$ matrix.

Description of U: We have a contraction map

$$\wedge^3 V \otimes \wedge^2(V^*) \to V$$

defined by $x \otimes \alpha \mapsto x \llcorner \alpha$ (see 5 in Section 5.2), where

$$\begin{aligned} x \llcorner \alpha &= (v_1 \wedge v_2 \wedge v_3) \llcorner (\phi_1 \wedge \phi_2) \\ &= \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma)(\sigma)\phi_1(v_{\sigma(1)})\phi_2(v_{\sigma(2)})v_{\sigma(3)}. \end{aligned}$$

Also, we know the skew-form J preserved by the action of $\mathrm{Sp}_6(\mathbb{C})$ can be identified with the element $e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*$ of $\wedge^2(\mathbb{C}^6)^* (\cong \wedge^2(\mathbb{C}^6))$. Then by [FH91, Section 17.2], the kernel of the contraction map obtained by contracting with $e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*$ is the irreducible representation with highest weight $L_1 + L_2 + L_3$. We call this representation as U. Therefore

$$U = \{v_1 \wedge v_2 \wedge v_3 \in \wedge^3(\mathbb{C}^6) \mid (v_1 \wedge v_2 \wedge v_3) \llcorner (e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*) = 0\}.$$

We can also write down explicitly the basis elements of U by exploring the kernel condition of the contraction map $\llcorner$. But before that let us first show by example, how the formula for '$\llcorner$' works. Choose $e_1 \wedge e_2 \wedge e_3 \in \bigwedge^3(\mathbb{C}^6)$, then

$$\begin{aligned} (e_1 \wedge e_2 \wedge e_3) \llcorner (e_1^* \wedge e_2^*) &= \sum_{\sigma \in S_3} \mathrm{sgn}(\sigma) e_1^*(e_{\sigma(1)}) e_2^*(e_{\sigma(2)}) e_{\sigma(3)} \\ &= e_1^*(e_1)e_2^*(e_2)e_3 - e_1^*(e_1)e_2^*(e_3)e_2 - e_1^*(e_2)e_2^*(e_1)e_3 \\ &+ e_1^*(e_2)e_2^*(e_3)e_1 + e_1^*(e_3)e_2^*(e_1)e_2 - e_1^*(e_3)e_2^*(e_2)e_1 \\ &= e_3 \text{ (since } e_i^*(e_j) = \delta_{ij}, \text{Kronecker delta)} \end{aligned}$$

For $e_1 \wedge e_2 \wedge e_3 \in \bigwedge^3(\mathbb{C}^6)$ and $e_3^* \wedge e_4^* \in \bigwedge^2(\mathbb{C}^6)$, we have $(e_1 \wedge e_2 \wedge e_3) \llcorner (e_3^* \wedge e_4^*) = 0$. Similarly, we can check, $(e_1 \wedge e_2 \wedge e_3) \llcorner (e_5^* \wedge e_6^*) = 0$. Therefore

$$(e_1 \wedge e_2 \wedge e_3) \llcorner (e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*) = e_3.$$

Similarly, we can evaluate this formula for each basis $e_i \wedge e_j \wedge e_k$ of $\bigwedge^3(\mathbb{C}^6)$ and can calculate the kernel of the $\llcorner$ map. Therefore the basis of the kernel or U is $\{e_1 \wedge e_3 \wedge e_5, e_1 \wedge e_3 \wedge e_6, e_1 \wedge e_4 \wedge e_5, e_1 \wedge e_4 \wedge e_6, e_1 \wedge e_5 \wedge e_6 - e_1 \wedge e_3 \wedge e_4, e_2 \wedge e_3 \wedge e_5, e_2 \wedge e_3 \wedge e_6, e_2 \wedge e_4 \wedge e_5, e_2 \wedge e_4 \wedge e_6, e_2 \wedge e_5 \wedge e_6 - e_2 \wedge e_3 \wedge e_4, e_3 \wedge e_4 \wedge e_5 - e_1 \wedge e_2 \wedge e_5, e_3 \wedge e_4 \wedge e_6 - e_1 \wedge e_2 \wedge e_6, e_3 \wedge e_5 \wedge e_6 - e_1 \wedge e_2 \wedge e_3, e_4 \wedge e_5 \wedge e_6 - e_1 \wedge e_2 \wedge e_4\}$. Hence the dimension of U is 14.

Action of Γ on U:

Define the action of Γ on $\bigwedge^3(\mathbb{C}^6)$ by

$$\begin{aligned}
& g \cdot (e_{i_1} \wedge e_{i_2} \wedge e_{i_3}) \text{ (for } i_1 < i_2 < i_3) \\
&= g \cdot e_{i_1} \wedge g \cdot e_{i_2} \wedge g \cdot e_{i_3} \\
&= (g_{1i_1} e_1 + \cdots + g_{6i_1} e_6) \wedge (g_{1i_2} e_1 + \cdots + g_{6i_2} e_6) \wedge (g_{1i_3} e_1 + \cdots + g_{6i_3} e_6) \\
&= \sum_{i_0 < j_0 < k_0} \begin{aligned}[t] &(g_{i_0 i_1} g_{j_0 i_2} g_{k_0 i_3} - g_{j_0 i_1} g_{i_0 i_2} g_{k_0 i_3} + g_{k_0 i_1} g_{i_0 i_2} g_{j_0 i_3} - g_{i_0 i_1} g_{k_0 i_2} g_{j_0 i_3} \\ &\qquad + g_{j_0 i_1} g_{k_0 i_2} g_{i_0 i_3} - g_{k_0 i_1} g_{j_0 i_2} g_{i_0 i_3}) e_{i_0} \wedge e_{j_0} \wedge e_{k_0} \end{aligned}
\end{aligned}$$

for any $g \in \Gamma$. Let Q_1, Q_2, Q_3 be the matrices obtained by restricting the action of generators M_1, M_2, M_3 of Γ on U and the matrices $\mathrm{Sym}^c(Q_1), \mathrm{Sym}^c(Q_2), \mathrm{Sym}^c(Q_3)$ for the action of generators on $\mathrm{Sym}^c(U)$. Now, note that the vector space $\mathrm{Sym}^c(U)$ has the dimension $\binom{13+c}{13}$. Define, $n_c := \binom{13+c}{13}$. Then for each $i \in \{1,2,3\}$, $\mathrm{Sym}^c(Q_i)$ is an $n_c \times n_c$ matrix. The matrix obtained by the action of Γ on the space $\widetilde{V}_{(a,b,c)}$ is $\mathrm{Sym}^a(M_i) \otimes \mathrm{Sym}^b(P_i) \otimes \mathrm{Sym}^c(Q_i)$. Then we have,

$$\dim\left(\widetilde{V}^{\Gamma}_{(a,b,c)}\right) = \dim\left(\bigcap_{i=1}^{3} \widetilde{V}^{M_i}_{(a,b,c)}\right) \tag{5.3.4}$$

$$= \dim\left(\bigcap_{i=1}^{3} \ker\left(\mathrm{Sym}^a(M_i) \otimes \mathrm{Sym}^b(P_i) \otimes \mathrm{Sym}^c(Q_i) - 1\right)\right) \tag{5.3.5}$$

$$= \dim\left(\bigcap_{i=1}^{3} E^i_{a,b,c}\right), \tag{5.3.6}$$

where $E^i_{a,b,c}$ is the eigen space of 1.

Complexity issues: Define $N_{a,b,c} := n_a n_b n_c = \binom{5+a}{a}\binom{13+b}{13}\binom{13+c}{13}$. The matrix $\mathrm{Sym}^a(M_i) \otimes \mathrm{Sym}^b(P_i) \otimes \mathrm{Sym}^c(Q_i)$ is an $N_{a,b,c} \times N_{a,b,c}$ matrix. Now $N_{a,b,c}$ grows rapidly as a,b,c increases. This process calculates three eigenspaces for three big $N_{a,b,c} \times N_{a,b,c}$ matrices and then compute the dimension of their intersections. We could bypass the idea of directly calculating the kernel and can reduce the complexity. So, in our algorithm to reduce the number of operations involving nullspace calculations, we compute for each $i \in \{1,2,3\}$ when all those i_x, j_y, k_l with the condition, $\sum i_x = a; \sum j_x = b; \sum k_x = c$ when the following expression involving eigenvalues is true.

$$\alpha^{i_1}_{1M_i}\alpha^{i_2}_{2M_i}\cdots\alpha^{i_6}_{6M_i}\beta^{j_1}_{1P_i}\beta^{j_2}_{2P_i}\cdots\beta^{j_{14}}_{14P_i}\gamma^{k_1}_{1Q_i}\gamma^{k_2}_{2Q_i}\cdots\gamma^{k_{14}}_{14Q_i} = 1, \tag{5.3.7}$$

Here $\alpha_{1M_i}, \alpha_{2M_i}, \dots, \alpha_{6M_i}, \beta_{1P_i}, \beta_{2P_i}, \dots, \beta_{14P_i}, \gamma_{1Q_i}, \gamma_{2Q_i}, \dots, \gamma_{14Q_i}$ are eigenvalues of M_i, P_i and Q_i respectively for each $i \in \{1,2,3\}$. Correspondingly $\alpha^{i_1}_{1M_i}\alpha^{i_2}_{2M_i}\cdots\alpha^{i_6}_{6M_i}$ with $\sum i_x = a$ are the eigenvalues of $\mathrm{Sym}^a(M_i)$ (for $i \in \{1,2,3\}$). Similarly, we can write the eigenvalues of $\mathrm{Sym}^b(P_i)$ and $\mathrm{Sym}^c(Q_i)$.

Table 5.1: The dimensions of the space for different inputs of a,b,c

a	1	2	2	4	6	8	10	12	14
b	0	0	1	0	0	0	0	0	0
c	1	0	0	0	0	0	0	0	0
$\dim\left(\widetilde{V}^{\Gamma}_{(a,b,c)}\right)$	0	0	0	0	1	1	0	3	2

Observation 5.3.6. Since M_1, M_2, M_3 have the same characteristic polynomial and same minimal polynomial, hence their eigenvalues are same too with the same multiplicity. In fact, we prove that M_1, M_2, M_3 are diagonalisable matrices (see remark 5.3.3). We can also prove that $P_1, P_2, P_3, Q_1, Q_2, Q_3$ are diagonalisable too. Moreover, each of P_1, P_2, P_3 satisfies the same minimal polynomial, characteristic polynomial and each of Q_1, Q_2, Q_3 has the same set of eigenvalues with the same multiplicity.

The above two observations lead us to imply that $E^i_{a,b,c}$ for each $i \in \{1,2,3\}$ will be spanned by eigenvectors and we get,

$$E^i_{a,b,c} = \text{Span}\{v^{i_1}_{1M_i} \cdots v^{i_6}_{6M_i} \otimes w^{j_1}_{1P_i} \cdots w^{j_{14}}_{14P_i} \otimes u^{k_1}_{1Q_i} \cdots u^{k_{14}}_{14Q_i} \mid \text{Expression(5.3.7) is true}\}.$$

Now, there is a general formula to calculate the dimension of the intersection of three subspaces V_1, V_2, V_3 of a vector space V due to [Tia02] given as following,

$$\dim(V_1 \cap V_2 \cap V_3) = \text{rk}(A_1) + \text{rk}(A_2) + \text{rk}(A_3) - \text{rk}\begin{pmatrix} A_1 & A_2 & 0 \\ A_1 & 0 & A_3 \end{pmatrix} \tag{5.3.8}$$

where $V_i =$ column space of the matrix A_i. This means that each column of A_i is actually a basis of V_i. Using this general formula for computing dimension of three vector spaces in our situation, we get

$$\dim\left(\bigcap_{i=1}^{3} E^i_{a,b,c}\right) = \sum_{i=1}^{3} \text{rk}(A_i) - \text{rk}\begin{pmatrix} A_1 & A_2 & 0 \\ A_1 & 0 & A_3 \end{pmatrix},$$

where $E^i_{a,b,c} =$ column space of A_i. Each column of A_i actually gives an eigen basis in our case. We calculate $\text{rk}(A_i)$ for each i, during the process of the program and get $\text{rk}(A_1) = \text{rk}(A_2) = \text{rk}(A_3)$. Denote this rank $\text{rk}(A_1)$ by h. Then, calculating the $\dim\left(\widetilde{V}^{\Gamma}_{(a,b,c)}\right)$ boils down to calculate the rank of the last matrix $\begin{pmatrix} A_1 & A_2 & 0 \\ A_1 & 0 & A_3 \end{pmatrix}$. This matrix has order $2N_{a,b,c} \times 3h$.

Conclusion 5.3.7. The key reason for choosing this rank calculation approach is to reduce the complexity. The matrix $\begin{pmatrix} A_1 & A_2 & 0 \\ A_1 & 0 & A_3 \end{pmatrix}$ has size $2N_{a,b,c} \times 3h$. The time complexity for calculating the rank of $\begin{pmatrix} A_1 & A_2 & 0 \\ A_1 & 0 & A_3 \end{pmatrix}$ involves $\tilde{O}(N_{a,b,c} \cdot h)$ field operations [CKL13]. The notation $\tilde{O}$ is used to hide (small) polylog factors in the time bounds. Whereas if we want to calculate the intersection of three eigenspaces and then calculate the dimension, then this process will involve more operations. In our case, $V_1 = \text{Range}(A_1)$ and $V_2 =$

Range(A_2). A vector $w \in V_1 \cap V_2$ if and only if $A_1 u = A_2 v = w$ for some $u \in V_1$ and $v \in V_2$. Therefore $A_1 u - A_2 v = w - w = 0$. Now consider the matrix $A = [A_1, -A_2]$. Hence $V_1 \cap V_2 = \{w = A_1 u \mid \begin{pmatrix} u \\ v \end{pmatrix} \in \ker(A)\}$. So it is enough to compute the $\ker(A)$. Now kernel calculation involves Gaussian elimination process. If we want to calculate $(V_1 \cap V_2) \cap V3$, then this will be computationally more intensive. So calculating the intersection of three eigenspaces and then calculating the dimension of the intersection involves more operations than calculating the rank of $\begin{pmatrix} A_1 & A_2 & 0 \\ A_1 & 0 & A_3 \end{pmatrix}$. This is the key observation of this approach.

We record this discussion as the following theorem.

Theorem 5.3.8. *The aforementioned algorithm takes non-negative integer values for a, b, c as inputs under the condition that $a + c$ must be even and b can be any non-negative integer. And as an output, it returns* $\dim\left(\widetilde{V}^{\Gamma}_{(a,b,c)}\right)$ *which gives bounds for the dimension of the space of cuspidal algebraic automorphic forms $M_{G^B}(V)$. For the choices $(a,0,0)$, we get the exact dimensions of $M_{G^B}(V_{a,0,0})$. In other cases, we just get bounds of* $\dim M_{G^B}(V_{a,b,c})$. *In particular, from the Table 5.1 the dimensions of the space of cusp forms of weights* $(6,0,0), (8,0,0), (12,0,0), (14,0,0)$ *are* $1, 1, 3, 2$ *respectively. Whereas the space of cusp forms of weights* $(1,0,1), (2,0,0), (2,1,0), (4,0,0)$ *and* $(10,0,0)$ *are trivial.*

Remark 5.3.9. In the above theorem the forms are cuspidal since $G^B(\mathbb{R})$ is compact modulo center hence it does not have any cusps. This precisely means the Table 5.1 gives the bounds for the dimensions of the space of cuspidal algebraic automorphic form $M_{G^B}(V)$.

Future Plans: In future, we want to work for reducing the complexity and if possible getting more values to fill up the table of dimensions of $\widetilde{V}_{(a,b,c)}$. There is a scope of using faster rank calculating programs which involves parallel processing. We intend to work on that. If by using tools of representation theory we could identify the space $V_{a,b,c}$ as a subspace of $\widetilde{V}^{\Gamma}_{(a,b,c)}$ then we can have a table of dimensions for the space $M_{G^B}(V)$ itself.

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