Stochastic Differential Equation and Integration

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Stochastic Differential Equation and Integrationtowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vikas Shuklaat Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami, Professors, IISER Pune, during the academic year 2019-2020.

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This thesis is dedicated to Math finance community

Declaration

I hereby declare that the matter embodied in the report entitled Stochastic Differential Equation and Integration are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anindya Goswami and the same has not been submitted elsewhere for any other degree.

Vikas Shukla

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Abstract

The main Objective of the MS project was to investigate the - Regime Switching Diffusion Model with Semi-Markov Regime. Thorough knowledge of SDE's under different process is necessary. Chapter 3 deals with SDE driven by Brownian motion, [1] lacks the mathematical clarity. Although the approach is simplistic in nature, the proof is not satisfactorily complete. The proof given in this thesis have tried to fill the gaps. Chapter 4 investigates SDE driven by Semi-Martingales and in process we learn Kunita-Watanabe Inequality, Itôs Formula (general version), Existence and Uniqueness of SDE. In the last Chapter we study the SDE with jump i.e SDE driven by Poisson point process.

Contents

Chapter 1

Introduction

In my previous Summer and Semester projects, I had opportunity to learn Multi-variate analysis, Time-series analysis, Market portfolio theory and option pricing model. In other words, I came into this stream by learning financial mathematics. Along this journey I realized deep knowledge of Stochastic Process, SDE, etc is required. In the current chapter I would give some basic motivation for my proceeding MS project.

- **What are Options?:**
  An Option is a contract(of stock) that gives the buyer(holder,owner) of the option the right, but not the obligation to buy or sell an underlying asset or instrument at a specified strike price(K) on a specified date(T), depending on the form of the option.

- **Options Type:**
  - **Call Options:** gives the holder the right—but not the obligation—to buy something at a specific price for a specific time period.
  - **Put Options:** gives the holder the right—but not the obligation—to sell something at a specific price for a specific time period.

- **Why Options exist?**
  Buyer afraid that the price of stock may rise(fall) in future so hedges himself by buying the call(put)option whereas seller of the option fears that the value may fall(rise) so he hedges buy selling the option. So, option exist as it hedges the risk.

- **Problem:**What is a fair price to charge for the option?(PRICING PROBLEM). How much should the holder pay to the writer?(HEDGING PROBLEM) Option valuation is an ongoing research various models are used to solve this problem.

- **BSM Option pricing Theory:**

  - **Assumptions:**

    * No Arbitrage
    * Constant risk-free rate(r) to borrow and lend cash
    * Complete market (Transaction cost 0 and portfolio can be replicable(self-financing strategy exist)
    * Stock price follow GBM with constant μ(drift) and σ(volatility)i.e $dS_t = \mu S_t dt + \sigma S_t dB_t$
    * No dividends
    * Option cannot be exercised before the maturity date(T)

  - **Problem:** What is the option price under BS-Model?

  - **Parabolic PDE:** Under above Assumptions, the call option(C_t) satisfies:

$$\sigma^2 S^2 \frac{\partial^2 C}{2\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0$$

  Where, S_t is the underlying Stock price.

  As measure theory was introduced in 7th Sem and Stochastic process was introduced in 8th Sem. The mathematical rigorousness of

    * What do I mean by Integration w.r.t Brownian motion?
    * What do I mean by Solution of the SDE?
    * What can we tell about existence and uniqueness of the Solution of the SDE?
    * Can I generalize the SDE w.r.t to B.M to some general process i.e For a given integrator(For E.g: B.M.) what is the class of integrand for which the solution exists?

  was lacking.

- **Regime-Switching Diffusion Model:**
  The above models assume that market has no jumps but in reality various factors

affects the behaviour of Financial market. For example; consider, Zomato, the various fake news have let the economic slow down. To model such market, we need atleast two regime before the news and after the news. Economic Crisis(global depression, global financial crisis-2008), various financial market behaviour, communication models, biological models can be studied using regime switching diffusion process.

- **Definition :**

  For a given State Space $\mathbf{S} = \{1, 2, ..., n, ..\}$ and $(\Omega, \mathscr{F}, P)$-complete probability space, we say $\{(X_t, \Lambda_t)\}_{t\geq 0}$ (two-component process) a **regime-switching diffusion process** if

$$dX_t = b(X_t, \Lambda_t, t)dt + \sigma(X_t, \Lambda_t, t)dW_t \tag{1.1}$$

$$P(\Lambda_{t+\Delta t} = j | \Lambda_t = i, (X_s, \Lambda_s), s \leq t) = \begin{cases} q_{ij}(X_t)\Delta t + o(\Delta t), & \text{if } i \neq j \\ 1 + q_{ii}(X_t)\Delta t + o(\Delta t), & \text{if i = j} \end{cases} \tag{1.2}$$

Where, $\{W_t\}_{t\geq 0}$ is a Brownian motion on $\mathbf{R}^d$ w.r,t right continuous filtration $\{\mathscr{F}_t\}_{t\geq 0}$, $Q(x) := (q_{ij}(x))$ be a Q-matrix and $b : \mathbf{R}^d \times \mathbf{S} \times [0, \infty) \longrightarrow \mathbf{R}^d$, $\sigma : \mathbf{R}^d \times \mathbf{S} \times [0, \infty) \longrightarrow \mathbf{R}^d \otimes \mathbf{R}^d$, $q_{ij} : \mathbf{R}^d \longrightarrow \mathbf{R}$ are measurable functions and $\mathbf{R}^d \otimes \mathbf{R}^d$ denotes the $d \times d$ **R**-valued matrix.

My aim was to study such model in totality but time restrained me only to study jump SDE, which I have presented in the last chapter of thesis.

Chapter 2

Preliminary

- **Definition 1:**
  Let, X and Y be stochastic processes on Probability Space$(\Omega, \mathscr{F}, P)$,
  We say that X is a **modification or version** of Y if, P$\{Y_t = X_t\}$=1 $\forall$ t and
  X and Y are **indistinguishable** if P$\{Y_t = X_t \ \forall$ t$\}$=1.
  Result: Any two RCLL process X, Y which are modification of each other are indistinguishable.

- **Definition 2:**
  The stochastic process X is **Progressively measurable** w.r.t filtration$\{\mathscr{F}_t\}$ if
  $\forall B \in \mathscr{B}(\mathbb{R}^n)$ and $t \geq 0$, the map

$$X : ([0,t] \times \Omega, \mathscr{B}([0,t]) \otimes \mathscr{F}_t) \longmapsto (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)) \tag{2.1}$$

  is a measurable map.

- **Definition 3:**
  The stochastic process $\{X_n\}_{n\geq 0}$ is said to be **discrete time Markov process** if $\forall B \in S$(any state space),

$$P(\{X_{n+1} \in B\}|\mathscr{F}_n) = P(\{X_{n+1} \in B\}|\sigma(X_n)), \forall n \geq 0 \tag{2.2}$$

  with probability 1.
  Where, $\sigma(X_n)$ denotes the sigma algebra generated by X_n and $\mathscr{F}_n = \sigma(X_k : 0 \leq k \leq n)$ and $X_n : (\Omega, \mathscr{F}, P) \mapsto (S, \mathscr{S}) \ \ \forall n \geq 0$

And if we replace n (everywhere) by any t and n+1 by t+s (t, s$\in [0,\infty)$) then we say it to be **continuous time markov chain**.
And if the probability is independent of n (and t for continuous process) then we Homogeneous markov chain (i.e distribution remains the same for any n) .

- **Definition 4:**
  A **Semigroup of bounded linear operator** is family$\{S_t\}_{t\geq 0}$ of one parameter bounded linear operator from a banach space X to itself and satisfies:

$$S_0(x) = I(x) = x\ (Identity\ Operator)\ \ \forall x \in X \tag{2.3}$$

$$S_{t+s}(x) = S_t(S_s(x))\ \ \forall t, s \in [0,\infty)\ \ \forall x \in X \tag{2.4}$$

We say that a semigroup is **(strongly continuous or) C_0-Semigroup** if for every $x \in X$

$$\|S_t x - x\| \longmapsto 0\ \ as\ \ t \downarrow 0 \quad (\|\cdot\|\ \ is\ \ a\ \ norm\ \ from\ \ banach\ \ space\ \ X) \tag{2.5}$$

i.e if for every x $\in$X (Banach Space) $S_{(\cdot)} : [0,\infty) \mapsto X$is continuous

And if

$$\|S_t - I\|_{op} \longmapsto 0\ \ as\ \ t \downarrow 0 \quad (\|\cdot\|_{op}\ \ is\ \ a\ \ operator\ \ norm\ \ ) \tag{2.6}$$

then we say that the **semigroup is *uniformly continuous***
We say $\{S_t\}_{t\geq 0}$ is a **Contraction Semigroup** if it is a semigroup and

$$\|S_t\|_{op} \leq 1 \quad (t \geq 0) \tag{2.7}$$

Note: Clearly, (6) implies (5)

- **Definition 5:**
  The **Infinitesimal Generator A of C_0-Semigroup** $\{S_t\}_{t\geq 0}$ is map
  $A : D(A) \mapsto X$

$$Ax := \lim_{t\downarrow 0} \frac{S_t(x) - x}{t} \tag{2.8}$$

Here, D(A) is the **domain of the generator** A i.e where the limit exist in RHS.

$$D(A) := \{x \mid \lim_{t \downarrow 0} \frac{S_t(x) - x}{t} \ exists\} \tag{2.9}$$

Remark: For uniformly continuous semigroup D(A)= X and in C_0-semigroup its dense in X.

- Example 2.1.1
  $S_t f(a) = f(a + ct)$ for some constant c and X=(C(K),supnorm). Clearly its C_0-semigroup with generator $c\frac{d}{dx}$ with domain $C^1(K)$

- Example 2.1.2
  For *continuous time homogeneous Markov chain* $\{X_t\}_{t \geq 0}$ with finite state space S={1,2,..,n}

$$P(X_{t+h} = j | X_t = i) = \lambda_{ij} h + o(h) \ (for \ j \neq i) \ \ i.e$$

$$E[1_{\{j\}} X_{t+h} | X_t = i] = h \Lambda 1^i_{\{j\}}(X_{t+h}) + o(h) \ \ (for \ j \neq i) \tag{2.10}$$

Notation: $\Lambda 1^i_{\{j\}} = \lambda_{ij}$ and $1_{\{j\}}$ is an indicator function.

$$P(X_{t+h} = i | X_t = i) = 1 - \textstyle\sum_{j : j \neq i} \lambda_{ij} h + o(h)$$

$$\lambda_{ii} := \textstyle\sum_{j : j \neq i} \lambda_{ij} \text{ and therefore,}$$

$$E[1_{\{j\}} X_{t+h} | X_t = i] - 1^i_i(X_{t+h}) = h \Lambda 1^i_{\{j\}}(X_{t+h}) + o(h) \ \ (\forall \ j, i) \tag{2.11}$$

So we conclude ,

$$\lim_{h \to 0} \frac{E[1_{\{j\}} X_{t+h} | X_t = i - 1^i_i(X_{t+h})]}{h} = \Lambda 1^i_{\{j\}}(X_{t+h}) \ \ (\forall \ j, i) \tag{2.12}$$

Now, define

$$S_h(f(i)) := E[f(X_{t+h}) | X_t = i] \tag{2.13}$$

where, X= BL(S) As infinitesimal generator function A agrees on $\{1_{\{j\}}\}$ standard basis implies $A = \Lambda$.

In general state space S (usually **R**) of a homogeneous continuous time Markov chain

the semigroup is given as
$S_t : BL(\mathbf{R}) \mapsto BL(\mathbf{R})$

$$S_t f(x) := E[f(X_t)|X_0 = x] \tag{2.14}$$

Using Markov property and definition of conditional expectation we can show that its a semigroup.

- **Definition 6:** $(\Omega, \mathscr{F}, \mathscr{F}_t(\mathscr{F}_n), P)$
  Let, A be an **increasing process** (adapted process with $A_0(\omega) = 0$ and right continuous and non-decreasing with t (or n) a.s.and $EA_t < \infty$ (or $EA_n < \infty$) for every $t \geq 0$ (or $n \geq 0$)
  We say that A is **natural** if $\forall$ right continuous(r.c) martingale $\{M_t, \mathscr{F}_t : t \geq 0\}(\{M_n, \mathscr{F}_n : n \geq 0\})$ the following holds:

$$E \int_{(0,t]} M_s dA_s = E \int_{(0,t]} M_{s\_} dA_s \ \ (for\ every\ 0 < t < \infty) \tag{2.15}$$

  For discrete time,

$$EM_n A_n = E \sum_{i=1}^{n} M_{i-1}(A_i - A_{i-1}) \ \ (for\ every\ n \geq 1) \tag{2.16}$$

- **Definition 7:**
  we say **usual conditions/ usual hypothesis** are satisfied by a filtration$\{\mathscr{F}_t\}_{t\geq 0}$ if $\mathscr{F}_0$ contains all 0-P measure set and the filtration is right continuous (i.e $\mathscr{F}_{t+} = \mathscr{F}_t$)

- **Definition 8:**
  $\mathscr{S}_\infty := \{\tau \mid \tau\ is\ F_t\ measurable\ stopping\ time\ and\ P(\tau < \infty) = 1\}$
  $\mathscr{S}_a := \{\tau \mid \tau\ is\ F_t\ measurable\ stopping\ time\ and\ P(\tau < a) = 1\}$
  $D := \{X_\tau \mid \tau \in \mathscr{S}_\infty\ and\ X\ is\ r.c\ uniformly\ integrable\ random\ variable\}$
  $DL := \{X_\tau \mid \tau \in \mathscr{S}_a\ and\ X\ is\ r.c\ uniformly\ integrable\ random\ variable\ \forall a \in (0, \infty)\}$

  Remark: $\mathscr{S}_a \subset \ \mathscr{S}_\infty\ \forall\ a$ and D $\subset$ DL

Theorem 1 (Doob-Meyer Decomposition:)**.** *(discrete version) Any Submartingale, can be written uniquely as sum of martingale and a predictable increasing sequence.*

(General): For every r.c sub-martingale $X = \{X_t, \mathscr{F}_t | t \geq 0\}$ in class of DL and $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying usual condition can be written as sum of a r.c martingale and an increasing process and the decomposition is unique up to indistinguishability if increasing process is chosen to be natural i.e

$$X_t = M_t + A_t \tag{2.17}$$

Where, $M = \{M_t, \mathscr{F}_t\}_{t\geq 0}$ and $A = \{A_t, \mathscr{F}_t\}_{t\geq 0}$ are r.c martingale and an increasing process respectively. Also if X is from class D then M is u.i and A is integrable.

- **Definition 9:**
  We notate X$\in \mathscr{M}_2$ if X=$\{X_t, \mathscr{F}_t\}_{t\geq 0}$ is **square-integrable** ($EX_t^2 < \infty \ \ \forall\ t \geq 0$) r.c martingale process.
  And X$\in \mathscr{M}_2^c$ if X$\in \mathscr{M}_2$ and is a continuous process

- **Definition 10:** For X$\in \mathscr{M}_2$, the **quadratic variation process** $\langle X \rangle := \{\langle X \rangle_t\}_{t\geq 0}$ and $\langle X \rangle_t := A_t$, where, A is an unique natural increasing process of Doob-Meyer decomposition of X^2.
  Note: X^2 is in DL as X^2 is a non-negative r.c sub-martingale.

- **Definition 11:**
  let, M$\in \mathscr{M}_2$ and $\mu_M(\cdot) : \mathscr{B}([0,\infty)) \otimes \mathscr{F} \mapsto [0,\infty)$ be a measure defined as

$$\mu_M(A) := E\int_0^\infty 1_A(t,\omega) d\langle M \rangle_t(\omega) \tag{2.18}$$

We say two $\{\mathscr{F}_t\}$-adapted, measurable Process X and Y are equivalent if

$$X_t(\omega) = Y_t(\omega) \qquad \mu_M \ a.e\ (t,\omega) \tag{2.19}$$

For measurable $\{\mathscr{F}_t\}$-adapted process X,

$$([X]_T)^2 := E\int_0^T X_t^2(\omega) d\langle M \rangle_t(\omega) \tag{2.20}$$

assuming RHS exists.
Observe: The above $[X]_T$ is just the L_2-norm restricted to space $\Omega \times [0,T]$ w.r.t μ_M-measure.

Result: Two adapted, measurable Process X and Y are equivalent iff $[X-Y]_T$=0 $\forall\ T>0$.

- **Definition 12:**
$\mathscr{L}(M)$ be the set of all equivalence class of $\{\mathscr{F}_t\}$-adapted, measurable Process s.t $[X]_T < \infty\ \forall\ T>0$
Define,

$$[X] := \sum_{n=1}^{\infty} \frac{[X]_n \wedge 1}{2^n} \tag{2.21}$$

then, ($\mathscr{L}(M)$,[ ]) is a normed linear space.
Similarly, we define, $\mathscr{L}^*(M)$=$\{X \in \mathscr{L}|$ X is progressively measurable$\}$ with same norm ([ ]) and $M \in \mathscr{M}_2^c$ is a Banach space.

STOCHASTIC INTEGRALS:

We are interested to define the Stochastic integral, similar to Lebesgue integral. we begin by defining class of simple process (analog to class of simple function but with randomness)

- **Definition 13:**
$\mathscr{L}_0$=$\{X|\ \ i)X_t = \xi_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{\infty} \xi_i(\omega)1_{(t_i,t_{i+1}]}(t)\ \forall\ \omega \in \Omega$ and $t \in [0,\infty)$, ii) ξ_i is $\mathscr{F}_{t_i}$-measurable random variable and iii) $\sup_\omega \sup_i |\xi| < \infty\}$.
Remark: $\mathscr{L}_0 \subset \mathscr{L}^*(M) \subset \mathscr{L}(M)$
Result:1 For a bounded measurable adapted process X $\exists$ sequence of simple process $\{X^{(n)}\}_{n=1}^{\infty}$ s.t.

$$\sup_{T>0} \lim_{n\to\infty} E\int_0^T |X_t(n) - X_t|^2 dt = 0 \tag{2.22}$$

Result:2 If the map $t \mapsto \langle M \rangle_t$ is absolutely continuous a.s then $\mathscr{L}_0$ is dense in $\mathscr{L}(M)$ wrt metric [ ] and is dense in $\mathscr{L}^*(M)$ if the map is continuous wrt to same metric [ ].

- **Definition 14:**
For $X \in \mathscr{L}_0$ and $m \in \mathscr{M}_2^c$, we define the the stochastic Integral I(X) as

$$I_t(X) := \int_0^t X_s dM_s := \sum_{i=0}^{\infty} \xi_i(M_{t_{i+1}\wedge t} - M_{t_i \wedge t}) \tag{2.23}$$

The definition extends for $X \in \mathscr{L}^*(M)$ and M$\in \mathscr{M}_2^c$

$$I_t(X) := \int_0^t X_s dM_s \tag{2.24}$$

is the unique, square integral martingale with $\lim_{n\to\infty} \|I(X^n) - I(X)\| = 0$ for any sequence$\{X^n\}$ in $\mathscr{L}_0$, also $\lim_{n\to\infty}[I(X^n) - I(X)] = 0$, where,

$$\|X\| := \sum_{n=1}^{\infty} \frac{1 \wedge \|X\|_n \,(:= \sqrt{EX_n^2})}{2^n} \tag{2.25}$$

As $X \in \mathscr{M}_2 \implies \|X\|_t \,(:= \sqrt{EX_t^2}) \ \uparrow as\ t \uparrow$ (by Jensen's Inequality for conditional expectation and as X is a martingale)

Result:1. $\mathscr{M}_2$ is a Banach space and $\mathscr{M}_2^c$ is a closed subspace of $\mathscr{M}_2$ under (25) metric.

Result:2. If X,Y$\in \mathscr{M}_2$ with $\|X - Y\| = 0$ then X and Y are indistinguishable.

Result:3. For X$\in \mathscr{L}_0 \implies$ I(X) $\in \mathscr{M}_2^c$.

Result:4. For M$\in \mathscr{M}_2^2$ and X$\in \mathscr{L}^*(M)$, then $I(X) := \{I_t(X)\}_{t\geq 0}$ satisfies

$$I_0(X) = 0\ a.s\ P \tag{2.26}$$

$$E([I_t(X)]^2) = ([X]_t)^2 = E\int_0^t X_u^2 d\big\langle M\big\rangle_u(\omega) \tag{2.27}$$

$$[X] = \|I(X)\| \tag{2.28}$$

$$I(aX + bY) = aI(X) + bI(Y) \tag{2.29}$$

$$E[(I_t - I_s)^2|\mathscr{F}_s] = E[\int_s^t X_u^2 d\big\langle M\big\rangle_t(\omega)|\mathscr{F}_s] \tag{2.30}$$

$$\big\langle I(X)\big\rangle_t = E\int_0^t X_u^2 d\big\langle M\big\rangle_u(\omega) \tag{2.31}$$

(We will be using these results in Existence and Uniqueness of SDE).

- **Definition 15:**

  Let, $X = \{X_t\}_{t\geq 0}$ and $Y = \{Y_t\}_{t\geq 0}$ be two $\mathscr{M}_2^c$ $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process then we define **Cross-Variation** $A = \{A_t = \langle X, Y\rangle_t\}_{t\geq 0}$

$$\langle X, Y\rangle_t \ := \frac{\langle X+Y\rangle_t - \langle X-Y\rangle_t}{4} \tag{2.32}$$

is the unique (up-to indistinguisability), $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process, continuous BV process with $A_0 = \langle X, Y\rangle_0 = 0$ and $XY - A$ Martingale process.

- **Definition 16:**
  We say $X = \{X_t\}_{t\geq 0}$ is a **Local Martingale**(i.e X $\in \mathscr{M}^{loc}$) if it is a $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process s.t. $X_0 = 0$ a.s. and if $\exists$ a non-decreasing sequence of stopping time $\{\tau_n\}_{n=1}^{\infty}$ with $P(\lim_{n\to\infty} \tau_n = \infty) = 1$ (i.e $\{\tau_n\}_{n=1}^{\infty}$ diverges a.s) s.t $X_t^{(n)} := X_{t\wedge\tau_n}$ is a $\{\mathscr{F}_t\}_{t\geq 0}$ Martingale for each n$\geq$ 1. We Notate *Continuous local martingale* process X as X $\in \mathscr{M}^{c,loc}$.

- **Definition 17:**
  A **Semi-Martingale process** $X = \{X_t\}_{t\geq 0}$ is a $\{\mathscr{F}_t\}_{t\geq 0}$-adapted process s.t it can be decomposed into a X_0, local martingale M_t and a right-continuous process A_t of bounded variation on compacts for all t a.s. i.e

$$X_t = X_0 + M_t + A_t \tag{2.33}$$

And in *Continuous Semi-Martingale* the decomposition processes are continuous.

Result: The Decomposition of Continuous Semi-Martingale process is unique a.s. i.e if $X_t = X_0 + M_t + A_t = X_0 + \bar{M}_t + \bar{A}_t$ then $M_t = \bar{M}_t$ a.s and $A_t = \bar{A}_t$ a.s for all t

Chapter 3

Existence and Uniqueness of SDE wrt Brownian Motion:

Theorem: For t$\in [0,T]$, consider SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{3.1}$$

$$X_0 = Z \tag{3.2}$$

such that, $T > 0$, $b(\cdot,\cdot) : [0,T] \times \mathbf{R}^n \to \mathbf{R}^n$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbf{R}^n \to \mathbf{R}^{n\times m}$ be the measurable function satisfying

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|) \qquad x \in \mathbf{R}^n,\ t \in [0,T] \tag{3.3}$$

where, $|\sigma|^2 := \sum_{i=1,j=1}^{m,n} |\sigma_{ij}|^2$, $|x|$ is a Euclidean norm and C is some constant. Also,

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D(|x - y|) \qquad x, y \in \mathbf{R}^n,\ t \in [0,T] \tag{3.4}$$

Finally, let Z be independent of $\sigma(\{W_t\}_{t\geq 0})$ and $E|Z|^2 < \infty$.
Then the SDE has unique continuous(wrt t) solution $X_t(\omega)$ s.t. it is $\{\mathscr{F}_t \wedge \sigma(Z)\}_{t\geq 0}$-adapted($\mathscr{F}_t := \sigma(\{W_s : s \in [0,t]\})$ and $E\int_0^T |X|^2 dt < \infty$.

Proof. **Definition 1:** For a given probability Space$(\Omega, \mathscr{F}, P)$, We say $X = \{X_t\}_{t\geq 0}$ is a

strong solution of the SDE (1), (2) w.r.t fixed Brownian motion $W = \{W_t\}_{t\geq 0}$ and has a continuous sample paths, satisfying following properties:

1) X_t is $\mathscr{F}_t$-measurable

2) $X_0 =$ Z a.s(P)

3) $\int_0^\infty \{|b_i(s, X_s)| + \sigma_{ij}^2(s, X_s)\}ds < \infty$ a.s holds for all $1 \leq i \leq n$ and $1 \leq j \leq n$ and $0 \leq t < \infty$

4)P$[X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,\ 0 \leq t < \infty\ ]$=1 i.e the solution satisfy the SDE indistinguishably.

Result1: If$\{a_i\}_{i=1}^n$ is a real-valued sequence, we have $(a_1 + a_2 + a_3 + ... + a_n)^2 \leq n(a_1^2 + a_2^2 + a_3^2 + ... + a_n^2)$ (true for any n)

Result2(Grönwall's Inequality): For an interval I(here=[0,T]);a(t),b(t),w(t) be real-valued function on I s.t b(t), w(t) are continuous and if $a = a_+ + a_-$ then a_n is integrable on every compact sub-interval in I (closed and bounded sub-interval), then
if $b \geq 0$, a non-decreasing and if w satisfies

$$w(t) \leq a(t) + \int_0^t b(s)w(s)ds, \quad \forall t \in I, \tag{3.5}$$

then,

$$w(t) \leq a(t)e^{\int_0^t b(s)ds}, \ \ t \in I. \tag{3.6}$$

UNIQUENESS:

Let $X_t^{Z_1}$ and $\bar{X}_t^{Z_2}$ be the Solutions of above SDE with Z_1 and Z_2 be the initial condition respectively. (Here, Z_1=Z_2 $= Z$ but we will do in general), We use Itô-isometry, Hölder inequality, Fubini's theorem and (4)

Note:(4) $\implies |\sigma(t,x)-\sigma(t,x)| \leq D(|x-y|)$

$$
\begin{aligned}
& E[|X_t^{Z_1} - \bar{X}_t^{Z_2}|^2] \\
= \;& E[\{Z_1 - Z_2 + \int_0^t (b(s, X_s^{Z_1}) - b(s, \bar{X}_s^{Z_2}))ds + \int_0^t (\sigma(s, X_s^{Z_1}) - \sigma(s, \bar{X}_s^{Z_2}))dW_s\}^2] \\
\leq \;& 3E[|Z_1 - Z_2|^2] + 3E[(\int_0^t (b(s, X_s^{Z_1}) - b(s, \bar{X}_s^{Z_2}))ds)^2] + \\
& 3E[\{\int_0^t (\sigma(s, X_s^{Z_1}) - \sigma(s, \bar{X}_s^{Z_2}))dW_s\}^2] \\
\leq \;& 3E[|Z_1 - Z_2|^2] + 3E[(\int_0^t (b(s, X_s^{Z_1}) - b(s, \bar{X}_s^{Z_2}))^2 ds)(\int_0^t 1^2 ds)] + \\
& 3E[\{\int_0^t (\sigma(s, X_s^{Z_1}) - \sigma(s, \bar{X}_s^{Z_2}))^2 dW_s\}] \\
\leq \;& 3E[|Z_1 - Z_2|^2] + 3(1+t)D^2 \int_0^t E[|X_s^{Z_1} - \bar{X}_s^{Z_2}|^2]ds. \qquad (3.7)
\end{aligned}
$$

Define: $w(t) := E[|X_t^{Z_1} - \bar{X}_t^{Z_2}|^2]$ for $t \in [0,T]$
The map $t \mapsto w(t)$ satisfies

$$w(t) \leq a + b\int_0^t w(s)ds \qquad (3.8)$$

where, $a = 3E[|Z_1 - Z_2|^2]$ and $b = 3(1+T)D^2$
By Gronwall's Inequality, we get
$w(t) \leq a * exp(bt)$
As $Z_1 = Z_2 \implies$ a=0 $\forall\; t \geq 0$ Therefore, $P\{X_t^{Z_1} = \bar{X}_t^{Z_2}\} = 1 \quad \forall t \geq 0$
We know that if X is a version of Y and X and Y are right continuous process then they are indistinguishable. So,

$$P\{X_t^{Z_1} = \bar{X}_t^{Z_2} \quad \forall t \geq 0\} = 1 \qquad (3.9)$$

Therefore, the SDE has a unique solution.

EXISTENCE:

We use *Picard's iteration* technique to prove the existence of the solution

$$Y_t^0 := X_0 \qquad (3.10)$$

$$Y_t^{k+1} := X_0 + \int_0^t b(s, Y_s^k)ds + \int_0^t \sigma(s, Y_s^k)dW_s \qquad (3.11)$$

claim: $\{Y_t^k\}_{t\geq 0}$ is Cauchy sequence in $L^2(\mu \times P)$(Banach Space), where, μ denotes the Lebesgue measure.

Proof of the claim: If we follow Similar to (7), we get,

$$E[|Y_t^{k+1} - Y_t^k|^2] \leq (1+T)3D^2 \int_0^t E[|Y_t^k - Y_t^{k-1}|^2]ds \tag{3.12}$$

Now, for $k \geq 1, t \leq T$,

$$\begin{aligned}
E[|Y_t^1 - Y_t^0|^2] &= E[(\int_0^t b(s, X_s^0)ds + \int_0^t \sigma(s, X_s^0)dW_s)^2] \\
&\leq E[(\int_0^t |b(s, X_s^0)|ds + \int_0^t \sigma(s, X_s^0)dW_s)^2] \\
&\leq 2E[(\int_0^t |b(s, X_s^0)|ds)^2] + 2E[(\int_0^t \sigma(s, X_s^0)dW_s)^2] \\
&We\ use\ It\hat{o} - isometry\ for\ the\ second\ integral\ to\ get, \\
&\leq 2E[(\int_0^t |b(s, X_s^0)|ds)^2] + 2E[(\int_0^t |\sigma(s, X_s^0)|^2 ds) \qquad (3.13) \\
&From\ (3)\ we\ get, \\
&\leq 2E[(\int_0^t C(1+|X_0^s|)ds)^2] + 2E[(\int_0^t (C[(1+|X_0^s|)ds))^2 ds)] \\
&\leq 2C^2(t^2+t)E[(1+|X_0^s|)^2] \\
&\leq 2C^2 E[(1+|X_0^s|)^2](1+T)t \\
&:= A_1 t
\end{aligned}$$

From(12) and (13), We get,

$$\begin{aligned}
E[|Y_t^2 - Y_t^1|^2] &\leq (1+T)3D^2 \int_0^t E[|Y_t^1 - Y_t^0|^2]ds \\
&\leq 6D^2C^2E[(1+|X_0^s|)^2](1+T)^2t^2/2
\end{aligned} \tag{3.14}$$

So, If we take $A_2 = 3(1+T)max\{C^2, D^2\}E[(1+|X_0^s|)^2]$ then we have,

$$\begin{aligned}
E[|Y_t^{k+1} - Y_t^k|^2] &\leq A_2 \int_0^t E[|Y_t^k - Y_t^{k-1}|^2]ds \\
&By\ induction\ we\ get, \\
&\leq \frac{A_2^{k+1}t^{k+1}}{(k+1)!} \quad k \geq 0,\ t \in [0, T]
\end{aligned} \tag{3.15}$$

$$\begin{aligned}
\|Y_t^m - Y_t^n\|_{L^2(\mu\times P)} &\leq \sum_{k=n}^{m-1} \left\|Y_t^{k+1} - Y_t^k\right\|_{L^2(\mu\times P)} \\
&\leq \sum_{k=n}^{m-1} (\int_0^T E[|Y_t^{k+1} - Y_t^k|^2])^{\frac{1}{2}} \\
&\leq \sum_{k=n}^{m-1} (\int_0^T \frac{A_2^{k+1} t^{k+1}}{(k+1)!})^{\frac{1}{2}} \\
&= \sum_{k=n}^{m-1} (\frac{A_2^{k+1} T^{k+2}}{(k+2)!})^{\frac{1}{2}} \to 0 \; as \; m, n \to \infty
\end{aligned} \tag{3.16}$$

Now, as $L^2(\mu \times P)$ is a Banach Space, implies (upto a.s) $\exists! \, \{X_t(\omega)\}_{t\geq 0} \in L^2(\mu \times P)$ s.t the sequence converges to X_t in L^2 -sense i.e we can define,

$$X := \lim_{n\to\infty} Y^n \quad (The\ limit\ is\ in\ L^2(\mu \times P)) \tag{3.17}$$

As $\forall t$, Y_t^k is $\mathscr{F}_t^W$-measurable $\implies$ $\forall t$ X_t is also $\mathscr{F}_t^W$-measurable.(Why?)
Also, if we take $\lim_{m\to\infty}$ and $n = 0$, the RHS form a geometric sum which is finite $\implies$ $E\int_0^T |X|^2 dt < \infty$

Question: Does X satisfy the SDE?

Claim: $X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$ a.s. $\forall t < \infty$ (Modification or Version)

Proof of the claim: We know that if a sequence converges to X and Y both in $L^2 - limit$, then X=Y a.s.
From (11) we have,
$Y_t^{k+1} := X_0 + \int_0^t b(s, Y_s^k)ds + \int_0^t \sigma(s, Y_s^k)dW_s$ holds $\forall k$ and $\forall t \in [0, T]$

If we show, in $L^2(P)$, that as n$\to \infty$
$\int_0^t b(s, Y_s^n)ds \to \int_0^t b(s, X_s)ds$ and $\int_0^t \sigma(s, Y_s^n)dW_s \to \int_0^t \sigma(s, X_s)dW_s$
$\implies \forall t \;\; X_t := \lim_{n\to\infty} Y_t^n = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \quad (The\ limit\ is\ in\ L^2(P))$

Then, we are done by a.s uniqueness of L^2-limit.

We Hölder-inequality and (4),

$$E[|\int_0^t b(s,Y_s^n)-b(s,X_s)ds|^2] \leq E[(\int_0^t |b(s,Y_s^n)-b(s,X_s)|*1ds)^2]$$
$$\leq E[(\int_0^t |b(s,Y_s^n)-b(s,X_s)|^2 ds)^{\frac{2}{2}}(\int_0^t 1^2 ds)^{\frac{2}{2}}]$$
$$\leq tE[(\int_0^t D^2|Y_s^n - X_s|^2 ds)]$$
$$\leq TD^2E[(\int_0^t |Y_s^n - X_s|^2 ds)] \to 0 \quad (By\ (17)\ and\ on\ taking\ limit) \tag{3.18}$$

Next, we use Itô-Isometry and (4),

$$E[(\int_0^t \sigma(s,Y_s^n)-\sigma(s,X_s)dW_s)^2] = E[\int_0^t (\sigma(s,Y_s^n)-\sigma(s,X_s))^2 ds]$$
$$by\ steps\ similar\ to\ above\ we\ get$$
$$\leq D^2E[(\int_0^t |Y_s^n - X_s|^2 ds)] \to 0 \quad (By\ (17)\ and\ on\ taking\ limit) \tag{3.19}$$

As (18) and (19) holds for all t. Hence proved.

Note: We have only shown that there exist a modification which satisfies SDE and not the Indistinguishable solution, but once we show that there exist a continuous version of solution implies the continuous process is the indistinguishable solution satisfying the SDE.

Left to show: ∃! t-continuous square-integrable $\mathscr{F}_t^Z$-adapted solution.
proof of the claim: From (17), we get that ∃ a sub-sequence which converges a.s.and let that sub-sequence be the sequence then still the sequence converges to same process X, but this time a.s ∃ a set N s.t X = limit of sequence in N^c and $\mu \times \lambda(N) = 0$
Let us define,
$\tilde{X}_t := X_t$ if P($N|_t$)=0
$\tilde{X}_t := X_0 + \int_0^\infty bds + \int_0^\infty \sigma dW_s$ Otherwise
As, $\lambda\{t|\ P(N|_t) = 0\} = 0$, then for each $t \geq 0$ limit of Y_t^k produces,
$\tilde{X}_t = \tilde{X}_0 + \int_0^\infty b(s,\tilde{X}_s)ds + \int_0^\infty \sigma(s,\tilde{X}_s)dW_s$

We will show that, $\tilde{X}_t$ is a.s continuous.

Consider,

$$E[\int_0^T \sigma^2(s, \tilde{X}_s)ds] \leq 2c^2 E(\int_0^T 1 + |X|^2 ds)$$
$$\leq 2c^2(T + \|X\|_{L^2(\lambda \times P)})(< \infty)$$

Hence, $\{\sigma(t, \tilde{X}_t)\}_{t \geq 0} \in \mathscr{L}^*(W) \implies I_t(\sigma) \in \mathscr{M}_c^2$ i.e $t \mapsto \int_0^t \sigma dW_s$ is continuous a.s

Let, $c^{'}(\omega) := \sup_{t \in [0,T]} \left\| \int_0^t \sigma dW_s \right\| < \infty$
Due to a.s continuity of σ, P($c^{'}(\omega) < \infty$)=1.
Now,

$$\left\| \tilde{X}_t \right\| \leq \|X_0\| + \int_0^t \left\| b(s, \tilde{X}_s) \right\| ds + c^{'}$$
$$\leq c^{''} + c \int_0^t \left\| \tilde{X}_s \right\| ds$$

Where, $c^{''} = c^{'} + \leq \|X_0\| + cT$. Now, we use Grönwall's inequality for t∈[0,T]

$$\sup_{t \in [0,T]} \left\| \tilde{X}_t \right\| \leq c^{"} e^{cT}(< \infty)$$

Thus, $s \mapsto b(s, \tilde{X}_s)$ is bounded $\forall t \in [0,T]$ with probability 1 $\implies$ $t \mapsto$ $\int_0^t b(s, \tilde{X}_s)ds$ is continuous a.s.
Therefore, $\tilde{X}_s$ is continuous □

Chapter 4

Integration Theory for Semi-Martingales:

In this chapter we aim to give a meaning to the idea of a SDE to a wider class of stochastic process. We begin by laying the foundation of Semi-Martingales, discuss different properties, define stochastic integral driven by Semi-Martingale and try to widen the class of integrands, talk about general Itô's Formula and end the chapter by discussing the existence and uniqueness of SDE wrt Semi-Martingale.

Note: We will assume that the filtered, complete probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$ satisfy the usual hypothesis.

4.1 Semi-Martingale and its properties:

- **Definition 1:**

  For any finite sequences of stopping times $0 = T_0 \leq T_1 \leq ... \leq T_n + 1 < \infty$, $H_i \in \mathscr{F}_{T_i}$ with $|H_i| < \infty$ a.s, $0 \leq i \leq n$, if H has a representation

  $$H_t = H_0 1_{\{0\}}(t) + \sum_{i=1}^{n} H_i 1_{(T_i, T_{i+1}]}(t)$$

  then we say H is **Semi-predictable process**.

  let, $\mathbf{S}$ denote the collection of semi-predictable process and $\mathbf{S_u}$ denote collection that of endowed with uniform convergence topology. Also, let $\mathbf{L^0}$ denote the space of finite-valued random variables topologized by convergence in probability

- **Definition 2:** For a given process X, define a linear map $I_X : \mathbf{S} \to \mathbf{L^0}$

$$I_X(H) = H_0 X_0 + \sum_{i=1}^{n} H_i (X_{T_{i+1}} - X_{T_i})$$

Where, H is a Semi-predictable process.
The Linear Operator $I_X(\cdot)$ looks like the definite integral of H wrt X.

- **Definition 3:** A process X is a **total semi-martingale** if X is RCLL, adapted and $I_X : \mathbf{S_u} \to \mathbf{L^0}$ is continuous

- **Definition 4:** A process X is **semi-martingale** if for each $t \in [0, \infty)$, X^t is a total semi-martingale

Theorem 2. *The Collection of Semi-Martingale (total Semi-Martingale) is a vector space.*

Proof. We will prove that if X and Y are total Semi-Martingale then $X + aY$ is a total Semi-Martingale. The other part follows from it. As X and Y are total Semi-Martingale, clearly, $X + aY$ is a RCLL, adapted process and Operator I_X, I_Y are continuous. As, $I_{X+aY} = I_X + aI_Y$ for any constant $\implies$ $X + aY$ is a total Semi-Martingale. □

Theorem 3. *If $Q \ll P$ (Probability measure), then every P-Semi-Martingale(total Semi-Martingale) X is also Q-Semi-Martingale(total Semi-Martingale).*

Proof. We know that as X is P-Semi-Martingale(total Semi-Martingale), X is an adapted, RCLL process. Left to show that I_X is continuous wrt Probability measure-Q. From [14] Section 18 proposition 19, $Q \ll P$ iff for each $\epsilon > 0$, $\exists \delta > 0$ s.t for any set $B \in \mathscr{F}$ if $P(B) < \delta \implies Q(B) < \epsilon$. Let, $B_n = \{\omega : |I_X(H_n - H)| \geq \alpha$ s.t $H_n \longrightarrow H$ uniformly$\}$, We choose n large enough s.t for small ϵ, $Q(B_n) < \epsilon$ and clearly the $\exists \delta$ small s.t $P(B_n) < \delta$ i.e in words Convergence in P implies convergence in Q and their for X is also a Q-Semi Martingale(total Semi Martingale). □

Theorem 4. *Let, $(P_i)_{i \geq 1}$ a sequence of Probability measures s.t, X is P_i Semi-Martingale for each i. Then X is $O(:= \sum_{i=1}^{\infty} \alpha_i P_i)$ Semi-Martingale, where $\sum_{i=1}^{\infty} \alpha_i = 1$.*

Proof. First Observe that O is a probability measure. We have to just prove the Continuity of I_X-Operator wrt O. Note: $\sum_{i=1}^{\infty} \alpha_i P_i$ is dominated by constant 1, therefore, by Dominated convergence Theorem $\lim \sum_{i=1}^{\infty} \alpha_i P_i = \sum_{i=1}^{\infty} \alpha_i \lim P_i$ and therefore I_X is continuous wrt O. □

Theorem 5 (Stricker's Theorem)**.** *Let, Y be a Semi Martingale for the filtration* $\mathbf{F}$ *and Let, Y be adapted to a* $\mathbf{G}$*(subfiltration of* $\mathbf{F}$*). Then X is a* $\mathbf{G}$ *Semi-Martingale.*

Proof. In this Theorem we don't have issue with adaptability and RCLL. Note: as $I_X : \mathbf{S_u} \to \mathbf{L_0}$, we need to understand the underlying Space $\mathbf{S}$. For the filtration $\mathbf{F}$, $\mathbf{S}(\mathbf{F})$ denote the space of Simple-Predictable Process for the filtration $\mathbf{F} = (\mathscr{F}_t)_{t\geq 0}$. Clearly, $\mathbf{S}(\mathbf{G}) \subset \mathbf{S}(\mathbf{F})$ and hence Continuity follows. □

In the previous Theorem we try to shrink the filtration and try to preserve the Semi-Martingale's property. what about if we expand the filtration?(adaptability and RCLL is preserved only we need to check the continuity of I_X-operator). We present an rudimentary result.

Theorem 6. *Let,* $\mathcal{D}$ *be a collection of events in* $\mathscr{F}$ *s.t if* $D_\alpha,\ D_\beta \in \mathcal{D}$*, then* $D_\alpha \cap D_\beta = \phi,\ \alpha \neq \beta$*. Let,* $\mathbf{G} = \{\mathscr{G}_t := \sigma(\mathscr{F}_t \vee \mathcal{D})\}_{t\geq 0}$*. Then, every* $(\mathbf{F}, P)$ *Semi-Matrtingale is an* $(\mathbf{G}, P)$ *Semi-Martingale also.*

Proof. Note: For $D_\alpha \in \mathcal{D}$ s.t $P(D_\alpha) = 0,\ D_\alpha, D_\alpha^c \in \mathscr{F}$ (by usual hypothesis). So. WLOG we assume that $P(D_\alpha) > 0$ for every $D_\alpha \in \mathcal{D}$(0-measure set won't contribute to any property while expansion of filtration as they already belongs to earlier filtration). Also, observe that only countable sets belongs to $\mathcal{D}$ with positive measure. [As, $\#K_n(:= \{D_\alpha \in \mathcal{D} : P(D_\alpha) = \frac{1}{n}\}) \leq n$ and $\cup_n K_n = \mathcal{D} \implies \mathcal{D}$ is countable.]
Remark: When we check whether X is a Semi-Martingale or not, we only see the underlying filtration (Here, $\mathbf{G}$) and not the collection $\mathcal{D}$. So, we also assume WLOG that if $D_n \in \mathcal{D}$, $\Lambda := \cup_n D_n$, then $\Lambda^c \in \mathcal{D}$i.e we assume that the collection $\mathcal{D}$ is a countable disjoint partition of Ω with each element having positive measure.
For every fixed $D_n \in \mathcal{D}$, we define

$$Q_n := P(\cdot|D_n)$$

Then, $Q_n \ll P$ and therefore, X is an $(\mathbf{F}, Q_n)$ Semi-Martingale. Let, $\mathbf{J^n} = (\mathscr{J}_t^n)_{t\geq 0}$ where, $\mathscr{J}_t^n := \sigma(\mathscr{F}_t \vee \{A \in \mathscr{F} : Q_n(A) = 0 \text{ or } 1\})\}$.

Note: Adding 0 or 1 measure set doesn't disturb the continuity of I_X. Therefore, X is $(\mathbf{J}, Q_n)$ Semi-Martingale. Also, as $Q_n(D_m) = 0$ or 1 for $m \neq n$ we have $\mathscr{F}_t \subset \mathscr{G}_t \subset \mathscr{J}_t^n$ for $t \geq 0$ and $\forall n$. From theorem we conclude that, X is an $(\mathbf{G}, \mathbf{Q_n})$ semi-martingale for each n.
As $dP = \sum_{n\geq 1} P(D_n) dQ_n$, we conclude that X is a $(\mathbf{G}, P)$ Semi-Martingale. □

Let, $\mathcal{D}$ be a finite collection of events in $\mathscr{F}$ and let, $\mathbf{G} = (\mathscr{G}_t)_{t\geq 0}$ s.t $\mathscr{G}_t := \sigma(\mathscr{F}_t \vee \mathcal{D})$. Then, every $(\mathbf{F}, P)$ Semi-Matrtingale is an $(\mathbf{G}, P)$ Semi-Martingale also.

Theorem 7. *Let, Y be a RCLL, adapted process. Let, τ_n be a sequence of positive random variable increasing to ∞ and (Y^n) be a sequence of semi-martingales s.t for each n, $Y^{\tau_n -} = (Y^n)^{\tau_n -}$. Then Y is a semi-martingale.*

Proof. Need to show: X^t is a total Semi-Martingale, for each $t > 0$. Define, $R_n := \tau_n 1_{\{\tau_n \leq n\}} + \infty 1_{\{\tau_n > n\}}$. Then, for some constant c,

$$\begin{aligned} & P(\{|I_{Y^t}(H)| \geq c\}) \\ = \;& P(\{|I_{Y^t}(H)| \geq c, R_n = \infty\} \cup \{|I_{Y^t}(H)| \geq c, R_n < \infty\}) \\ = \;& P(\{|I_{Y^t}(H)| \geq c, t < \tau_n\} \cup \{|I_{Y^t}(H)| \geq c, R_n < \infty\}) \\ = \;& P(\{|I_{(Y^n)^{\tau_n -}}(H)| \geq c\} \cup \{|I_{Y^t}(H)| \geq c, R_n < \infty\}) \\ \leq \;& P(\{|I_{(Y^n)^{\tau_n -}}(H)| \geq c\}) + P(\{R_n < \infty\}) \\ = \;& P(\{|I_{(Y^n)^{\tau_n -}}(H)| \geq c\}) + P(\{\tau_n \leq t\}) \end{aligned} \tag{4.1}$$

Since, $\tau_n \to \infty$ a.s as $n \to \infty \implies$ for large n $P(\{\tau_n \leq t\}) < \frac{\epsilon}{2}$ and as Y^n is a Semi-martingale if $H^i \to 0$ in $\mathbf{S_u} \implies$ for a given large n choose i s.t $P(\{|I_{(Y^n)^{\tau_n -}}(H^i)| \geq c\}) < \frac{\epsilon}{2}$ and hence Y is a Semi-Martingale. □

For a given process Y and let, τ_n be a sequence of positive random variable increasing to ∞ s.t for each n, Y^{τ_n} is a semi martingale. Then Y is a semi-martingale.
Examples of Semi-Martingales:

- Each adapted with RCLL paths of FV on compacts is a Semi-Martingale.
- Each $\mathscr{M}^2$ process with RCLL paths is a Semi-Martingale.

– A decomposable process X($X_t = X_0 + M_t + A_t$, where $M \in \mathscr{M}^2_{loc}$ and A is a RCLL, adapted with path of FV on compacts.) is Semi-Martingale. (For example, Levý process)

4.2 Stochastic Integrals and its Properties:

In previous section we saw that Semi-Martingales are good integrators on **S**. In this section we will try to increase the domain of integrands and extend continuously the domain of definition of I_X and study some properties of integrals.
Let, **D** be the space of RCLL, adapted process. Let, **L** be the space of LCRL, adapted process and **bL** be the space of bounded LCRL, adapted process.

- **Definition 5:** We say sequence of processes $(H^n)_{n\geq 1}$ convereges in ucp (uniformly on compacts in probability) if, for each $t > 0$, $\sup\limits_{0\leq s\leq t} |H^n_s - H_s| \to 0$ in probability i.e $\lim_{n\to\infty} P(\sup\limits_{0\leq s\leq t} |H^n_s - H_s| \geq \epsilon) = 0$.
Note: The above definition doesn't make sense for any general stochastic process as the supremum is taken over uncountable index $[0, t]$ and the set need not be measurable. However as we are dealing with Right continuous or Left continuous process, the supremum can only be restricted to rational point in the interval, which is clearly measurable.
Let,$\mathbf{S_{ucp}}, \mathbf{D_{ucp}}, \mathbf{L_{ucp}}$, be the respective space endowed with ucp topology.

Theorem 8. *The Space* **S** *is dense in* **L** *under ucp topology.*

Proof. Let, $X \in \mathbf{L}$, define $R_n := \inf\{t : |X_t| > n\}$, $X^n := X^{R_n} 1_{\{R_n>0\}}$. Clearly, R_n is a stopping time (Hitting time of open set) increasing to ∞ a.s and $X^n \in \mathbf{bL}$.
Claim: $X^n \to X$ in ucp i.e **bL** is dense in **L**
proof of the claim:

$$\begin{aligned} &\{\omega : \sup_{s\leq t} |X^n_s - X_s| \geq \epsilon\} \\ = \quad &\{\omega : \sup_{s\leq t} |X^{R^n}_s 1_{\{R_n>0\}} - X_s| \geq \epsilon\} \end{aligned}$$

For each fixed t, $\exists\ n_t \in \mathbf{N}$ s.t $R_n \geq t$ a.s $\implies$ for each t, $n \geq n_t$ the above set is measure 0 set and hence the claim.

WLOG we assume that $X \in \mathbf{bL}$.

Define, $Z = (Z_t)_{t\geq 0}$, where $Z_t := \lim\limits_{\substack{u\to t\\ u>t}} X_t$. Observe $Z \in \mathbf{D}$ (As, $Z_{t+} = X_{t+} = Z_t$ and $Z_{t-} = X_t$ exists). For, $\epsilon > 0$, define a sequence of stopping time,

$$\tau_0^\epsilon = 0$$

$$\tau_{n+1}^\epsilon = \inf\{t : t > \tau_n^\epsilon \textit{ and } |Z_t - Z_{\tau_n^\epsilon}| > \epsilon\}$$

Claim: For a given $\epsilon > 0$, $\{\tau_n^\epsilon\}_{n\geq 0}$ is a $\mathscr{F}_t$-stopping time $\uparrow \infty$ a.s. as $n \uparrow \infty$.
proof of claim: As Z is a RCLL process, $\tau_n^\epsilon \uparrow \infty$ a.s. as $n \uparrow \infty$. clearly, τ_0^ϵ is a stopping time .

$$\begin{aligned}
&\{\tau_1^\epsilon \leq t\} \\
&= \{\omega : \inf\{s : s > 0 \textit{ and } |Z_s - Z_0| > \epsilon\} \leq t\} \\
&= \{\omega : |Z_s - Z_0| > \epsilon, s \leq t\} \in \mathscr{F}_t \ (\textit{as } Z \in \mathbf{D})
\end{aligned}$$

We prove by induction, so we assume that τ_n^ϵ is $\mathscr{F}_t$-stopping time and want to prove that τ_{n+1}^ϵ is $\mathscr{F}_t$-stopping time

$$\begin{aligned}
&\{\tau_{n+1}^\epsilon \leq t\} \\
&= \{\omega : \inf\{s : s > \tau_n^\epsilon \textit{ and } |Z_s - Z_{\tau_n^\epsilon}| > \epsilon\} \leq t\} \\
&= \{\omega : \tau_n^\epsilon < t \ \} \cap \{\omega : |Z_s - Z_{\tau_n^\epsilon}| > \epsilon, s \leq t\} \in \mathscr{F}_t \cap \mathscr{F}_{t\wedge\tau_n^\epsilon} (\subset \mathscr{F}_t) \ (\textit{as } Z \in \mathbf{D})
\end{aligned}$$

Define, $Z^\epsilon := \sum_k Z_{\tau_k^\epsilon} 1_{[\tau_k^\epsilon, \tau_{k+1}^\epsilon)}$ for each $\epsilon > 0$. As $X \in \mathbf{bL} \implies Z^\epsilon$ is bounded. Also, $Z^\epsilon \to Z$ uniformly as $\epsilon \to 0$ since for a given t, $t \in [\tau_{k_0}^\epsilon, \tau_{k_0+1}^\epsilon)$ for some k_0(say) and $|Z_{\tau_{k_0}^\epsilon} - Z_t| < \epsilon \implies$ for $s \leq t$, $s \in [\tau_k^\epsilon, \tau_{k+1}^\epsilon)$ for $k \leq k_0$ and $|Z_{\tau_k^\epsilon} - Z_s| < \epsilon$(by definition of τ_k^ϵ) and hence uniform convergence.

As the theorem talks about space $\mathbf{L}(\mathbf{bL})$, we define $V^\epsilon := X_0 1\{0\} + \sum_k Z_{\tau_k^\epsilon} 1_{(\tau_k^\epsilon, \tau_{k+1}^\epsilon]}$. One can prove that $V^\epsilon \to X_0 1\{0\} + Z_- = X$ uniformly in compacts. So, we define,

$$X^{n,\epsilon} := X_0 1\{0\} + \sum_{k=1}^{n} Z_{\tau_k^\epsilon} 1_{(\tau_k^\epsilon, \tau_{k+1}^\epsilon]}$$

we choose small enough ϵ and large enough n s.t $X^{n,\epsilon}$ is arbitrarily close to X. $\square$

- **Definition 6:** Define a linear map $J_X : \mathbf{S} \to \mathbf{D}$ as

$$J_X(H) := H_0 X_0 + \sum_{k=1}^{n} H_k (X^{T_{k+1}} - X^{T_k})$$

For $H \in \mathbf{S}$ and X a RCLL, adapted process. We call $J_X(H)(\int H_s dX_s,\ H \cdot X$ other notations) as **Stochastic integrals of H wrt X**.

Note: We have continuously expanded the domain of definition of integrals from definite integral $I_X(H) = J_X(H)_\infty,\ I_{X^t}(H) = (J_X(H))_t$ to indefinite integral $J_X(H)$ which is also a function of time.

Theorem 9. *For a Semi-Martingale X, the map $J_X : \mathbf{S_{ucp}} \to \mathbf{D_{ucp}}$ is continuous.*

Proof. As we are interested only in convergence in compacts set, WLOG we take X to be a total Semi-Martingale.

- **Step 1:** We prove that for bounded $H^i \in \mathbf{S}$ if $H^i \to 0$ uniformly then $J_X(H^i) \to 0$ in ucp i.e the map $J_X : \mathbf{S_u} \to \mathbf{D_{ucp}}$ is continuous
- **Step 2:** We use step 1 to prove (the desired result) $J_X : \mathbf{S_{ucp}} \to \mathbf{D_{ucp}}$ is continuous.
- **Proof of Step 1:** Let, $\delta > 0$ be given and define sequence of stopping times $\tau_i := \inf\{s : |(H^i \cdot X)_s \geq \delta|\}$ (It's a stopping time as $H \cdot X$ is a RCLL process). Clearly, $H^i 1_{[0,\tau_i]} \in \mathbf{S} \to 0$ uniformly as $i \to \infty$.(**Notation:** $(\cdot)^*_t = \sup_{s \leq t}(\cdot)$.)Thus for every t,

$$\begin{aligned} & P(\{\omega : (H^i \cdot X)^*_t > \delta\}) \\ \leq \quad & P(\{\omega : |(H^i 1_{[0,\tau_i]} \cdot X)_t| \geq \delta\}) \\ = \quad & P(\{\omega : |I_X(H^i 1_{[0,\tau_i]})_t| \geq \delta\}) \to 0 \end{aligned}$$

  (By definition of Semi-Martingale.)

- **proof of Step 2:** Now, suppose that $H^i \xrightarrow{\text{ucp}} 0$. Let, $\delta > 0,\ \epsilon > 0,\ t > 0$ from step 1 we know that $\exists\ \eta > 0$ s.t $|H|_u \leq \eta \implies P(J_X(H)^*_t > \delta) < \frac{\epsilon}{2}$. Define a sequence of stopping time $T_i := \inf\{s : |H^i_s| > \eta\}$ and let $\hat{H}^i := H^i 1_{[0,T_i]} 1_{\{T_i > 0\}}$.

Then, $\hat{H}^i \in \mathbf{S}$ and $|\hat{H}^i|_u \leq \eta$ by left continuity (i.e $\hat{H}^i \xrightarrow{uniformly} 0$ [we will use step 1]).

$$\begin{aligned} & P(\{\omega : (|H^i \cdot X)^*_t| > \delta\}) \\ &= P(\{\omega : |(H^i \cdot X)^*_t| \geq \delta, T_i \leq t\}) + P(\{\omega : |(H^i \cdot X)^*_t| \geq \delta, T_i > t\}) \\ &\leq P(\{\omega : |(H^i \cdot X)_t|^* \geq \delta, T_i \leq t\}) + P(\{\omega : T_i > t\}) \\ &= P(\{\omega : |(\hat{H}^i \cdot X)^*_t| \geq \delta\}) + P(\{\omega : T_i > t\}) \\ &\leq \frac{\epsilon}{2} + P(\{(H^i)^*_t > \eta\}) \\ &< \epsilon \end{aligned}$$

for i large enough as $H^i \xrightarrow{\text{ucp}} 0$.

□

- **Definition 7:** We know that for Semi-Martingale X, J_X is continuous linear map on $\mathbf{S_{ucp}}$ and as $\mathbf{S_{ucp}}$ is dense in $\mathbf{L_{ucp}}$, we extend continuously domain of definition of$J_X : \mathbf{S_{ucp}} \to \mathbf{D}$ to $J_X : \mathbf{L_{ucp}} \to \mathbf{D}$. We will call this extend continuous linear map **Stochastic Integrals**.

- **Properties of stochastic integrals:** Here, we denote Y- Semi Martingale and $H \in \mathbf{L}$ for $(\Omega, \mathscr{F}, P)$

Theorem 10. *Let T be a stopping time. Then $(H \cdot Y)^T = H1_{[0,T]} \cdot Y = H \cdot (Y^T)$*

Proof. As (by definition/notation) $(H \cdot Y)^t = H1_{[0,t]} \cdot Y = H \cdot (Y^t)$ for any $t > 0 \implies$ $(H \cdot Y)^{t \wedge T} = H1_{[0,t\wedge T]} \cdot Y = H \cdot (Y^{t\wedge T})$ taking limit $t \to \infty$ we get the result. □

For, $Q \ll P$, $H_Q \cdot Y$ is Q indistinguishable for $H_P \cdot Y$

Theorem 11. *The jump process $\Delta(H \cdot Y)_s$is indistinguishable from $H_s \cdot \Delta Y_s$ ($\Delta X_s := X_s - X_{s-}$).*

Proof. For $H \in \mathbf{S}$ the above result holds. Now, using right continuity of $H \cdot X$ for $H \in \mathbf{L}$ and denseness of $\mathbf{S}$ in $\mathbf{L}$. □

Theorem 12. *Let, $Q \ll P$. Then $H_Q \cdot Y$ is Q indistinguishable from $H_P \cdot Y$*

Let, Q_i be a sequence of probabilities s.t Y is a Q_k Semi-Martingale for each i. Let, $P := \sum_{i=1}^{\infty} \alpha_i Q_i$, where $\alpha_i \geq 0$ and $\sum_{i=1}^{\infty} \alpha_i = 1$. Then $H_{Q_i} \cdot Y$ is Q_i indistinguishable from $H_P \cdot Y$ for all k for $\alpha_i > 0$.

Proof. We know that Y is also a P- Semi-Martingale and $Q_i \ll P$. The result follows from the previous theorem $\square$

Theorem 13. *Let, $\mathbf{G}$ be another filtration s.t $H \in \mathbf{L}(\mathbf{G}) \cap \mathbf{L}(\mathbf{F})$ and X is also $\mathbf{G}$-Semi-Martingale. Then $H_{\mathbf{G}} \cdot Y = H_{\mathbf{F}} \cdot Y$. We denote $\mathbf{L}(\mathbf{F})$ as a space of LCRL process adapted to $\mathbf{F}$-filtration*

Proof. Note: $\mathbf{S}(\mathbf{G}) \cap \mathbf{S}(\mathbf{F})$ is dense in $\mathbf{L}(\mathbf{G}) \cap \mathbf{L}(\mathbf{F})$. For $H \in \mathbf{S}(\mathbf{G}) \cap \mathbf{S}(\mathbf{F})$ the result is vacuously true, the result follows by ucp convergence. $\square$

We have defined a stochastic integral for a large class, the next theorem correlates it with the Lebesgue-Stieltjes integral.

Theorem 14. *If we assume that Y has a finite variation on compacts, then $H \cdot Y$ is indistinguishable from the Lebesgue-Stieltjes integral (enumerated path by path).*

Proof. For $H \in \mathbf{S}$ the integral is similar to integration of simple function wrt Y. For, $H \in \mathbf{L}$, $\exists\, H^n \in \mathbf{S} \xrightarrow{ucp} H$. Then $\exists$ subsequence n_l s.t $\exists\, H^{n_l} \to H$ uniformly in compacts a.s. and as interchange of limit and integral is true for uniform convergence, the result follows. $\square$

Theorem 15. *Let, Y, $\tilde{Y}$ be semi-martingales and H, $\tilde{H} \in \mathbf{L}$. Let, $A_1 = \{\omega : H_.(\omega) = \tilde{H}_.(\omega), \omega : Y_.(\omega) = \tilde{Y}_.(\omega)\}$ and $A_2 = \{\omega : s \mapsto Y_s(\omega)$ is of finite variation $\}$. Then $H \cdot Y = \tilde{H} \cdot \tilde{Y}$ on A_1 and $H \cdot Y =$ path-by path Lebesgue-Stieltjes integral on A_2.*

Theorem 16 (Associativity)**.** *The stochastic integral process $Y_1 = H \cdot Y$ is a Semi-Martingale and for $K \in \mathbf{L}$,*

$$K \cdot Y_1 = K \cdot (H \cdot Y) = (KH) \cdot Y$$

Proof. Suppose that Y_1 is a Semi-Martingale. If $K,\ H \in \mathbf{S}$ and $H = H_0 1_{\{0\}} + \sum_{i=1}^{n} H_i 1_{(T_i,T_{i+1}]}$ and $K = K_0 1_{\{0\}} + \sum_{j=1}^{m} K_j 1_{(\tau_j,\tau_{i+1}]}$ be the representation of H and K, then $KH(t) = K_0 H_0 + \sum_{i,j} K_j H_i 1_{\{t\in(T_i,T_{i+1}]\cap(\tau_j,\tau_{j+1}]\}}$.

$$\begin{aligned}
&H \cdot Y_1\\
&= \sum_{j=1}^{n} K_j((H \cdot Y)^{\tau_{j+1}} - (H \cdot Y)^{\tau_j})\\
&= \sum_{j=1}^{n} K_j((H^{\tau_{j+1}} \cdot Y) - (H^{\tau_i} \cdot Y))\\
&= \sum_{j=1}^{n} K_j(H 1_{\tau_i,\tau_{i+1}]}) \cdot Y\\
&= \sum_{j=1}^{n}(\sum_{i=1}^{n} 1_{(T_i,T_{i+1}]\cap(\tau_j,\tau_{j+1}]} K_j H_i) \cdot Y\\
&= (KH) \cdot Y
\end{aligned}$$

Therefore, the associativity of stochastic integrals holds for Semi-predictable process and can be extended to $\mathbf{L}$ by continuity.

Left to show: $Y_1 = H \cdot Y$ is a Semi-Martingale (i.e for any $K^n \to K$ in $\mathbf{S_u} \implies J_{Y_1^t}(K^n) \xrightarrow{P} J_{Y_1^t}(K)$ for each $t > 0$).

Observe: For $K, H \in \mathbf{S}$, $I_{Y_1}(K) = I_Y(GH)$ and we know that $KH \in \mathbf{S}$. Therefore, the continuity of I_{Y_1} is guaranteed by continuity of I_Y and as Y is a Semi-Martingale, so is Y_1 (for $H \in \mathbf{S}$).

Let, $H^i(\in \mathbf{S}) \xrightarrow{ucp} H(\in \mathbf{L})$. Then by definition, $H^i \cdot Y \xrightarrow{ucp} H \cdot Y \implies \exists\ (i_n)_{n\geq 0}$ s.t $H^{i_n} \cdot Y \to H \cdot Y$ a.s and uniformly on compacts. Let, $K \in \mathbf{S}$ and let, $Y_1^{i_n} = H^{i_n} \cdot Y$ and we know that Y^{i_n} are semi martingale converging pointwise to Y_1 ($\implies Y_1 \in \mathbf{D}$).

As J_{Y_1} is defined for any $Y_1 \in \mathbf{D}$, we have

$$\begin{aligned} & J_{Y_1}(K) \\ &= K \cdot Y_1 \\ &= \lim_{i_n \to \infty} K \cdot Y_1^{i_n} \\ &= \lim_{i_n \to \infty} K \cdot (H^{i_n} \cdot Y) \\ &= \lim_{i_n \to \infty} (KH^{i_n}) \cdot Y \\ &= \lim_{i_n \to \infty} J_Y(KH^{i_n}) \\ (As\ Y\ is\ Semi-Martingale) & \\ &= J_Y(KH) \end{aligned}$$

Therefore, for $H \in \mathbf{S}$, $J_{Y_1}(K) = J_Y(KH)$.

Now, let $K^n \to K$ in $\mathbf{S_u}$ which means that $K^n H \to KH$ in $\mathbf{L_{ucp}}$

$$\begin{aligned} & \lim_{n \to \infty} J_{Y_1}(K^n) \\ &= \lim_{n \to \infty} J_Y(K^n H) \\ (As\ Y\ is\ Semi-Martingale) & \\ &= J_Y(KH) \\ &= J_{Y_1}(K) \end{aligned} \tag{4.2}$$

Note: As convergence is in ucp for every $t > 0$, Y_1^t is a total Semi-Martingale and hence Y_1 is a Semi-Martingale. □

- **Definition 8:** A **random partition** is a finite collection of finite stopping time $\Lambda = (\tau_i)_{i=0}^k$ s.t. $0 = \tau_0 \leq \tau_1 \leq ... \leq \tau_i \leq ... \leq \tau_k < \infty$. And a sequence of such random partition $\Lambda^n = (\tau_i^n)_{i=0}^{k_n}$, $0 = \tau_0^n \leq \tau_1^n \leq ... \leq \tau_i^n \leq ... \leq \tau_{k_n}^n < \infty$ is said to **tend to identity** if
  - $\lim_n \sup_j \tau_j^n = \infty$ a.s
  - $\|\Lambda^n\| := \sup_i |\tau_{i+1}^n - \tau_i^n| \to 0$ a.s

- **Definition 9:** Let, Λ be a random partition and X be any process. Then the process X sampled at Λ is defined as

$$X^{\Lambda} := X_0 1_{\{0\}} + \sum_i X_{\tau_i} 1_{(\tau_{i+1}, \tau_i]}$$

Theorem 17. *Let, $X \in \mathbf{D}$ or $\mathbf{L}$ and Y a Semi-Martingale and Λ^n tends to identity. Then, $\int_{0+}^t X_s^{\Lambda^n} dY_s (= \sum_i X_{\tau_i^n}(Y^{\tau_{i+1}^n} - Y^{\tau_i^n})) \xrightarrow{ucp} (X_-) \cdot Y$*

4.3 Quadratic variation and its properties:

- **Definition 10:** Let, A, B be Semi-Martingales. The **Quadratic Variation process** of A is defined as

$$[A, A] := A^2 - 2 \int A_- dA$$

The **Quadratic Covariation** of A,B is defined as

$$[A, B] := AB - \int A_- dB - \int B_- dA$$

Observe: $(A, B \to [A, B])$ is bilinear and symmetric and $[A, B] = \frac{1}{2}([A+B, A+B] - [A, A] - [B, B])$- Polarization Identity.

Theorem 18. *The Quadratic variation of A is a RCLL, adapted, increasing process satisfying the following properties:*

  - *1.* $[A, A]_0 = A_0^2, \ \Delta[A, A] = (\Delta A)^2$
  - *2. If Λ^n tends to identity, then*

$$A_0^2 + \sum_i (A^{\tau_{i+1}^k} - A^{\tau_i^k})^2 \xrightarrow[k \to \infty]{ucp} [A, A]$$

  - *3. For any stopping time T,* $[A^T, A] = [A, A^T] = [A^T, A^T] = [A, A]^T$

Proof. As we have Quadratic Variation is defined only for Semi-Martingales and stochatic integral is Semi-Martingale and product of RCLL, adapted process is also RCLL, adapted process implies that [A,A] is a RCLL, adapted process. If we assume the

that property 2 holds then for $s \leq t$ the LHS has more positive terms in the summation for t and hence $[A,A]_s \leq [A,A]_t$ a.s
By definition, $[A,A]_0 = A_0^2$ and we know that $\Delta(H \cdot Y)_s = H_s \cdot (\Delta Y)_s$. So,

$$\begin{aligned}
&(\Delta A)_s^2 \\
&= (A_s - A_{s-})^2 \\
&= A_s^2 - 2A_{s-}A_s + A_{s-}^2 \\
&= A_s^2 - 2A_{s-}A_s + A_{s-}^2 \\
&= A_s^2 - A_{s-}^2 - 2A_{s-}(A_s - A_{s-}) \\
&= \Delta(A^2)_s - 2A_{s-}(\Delta A_s) \\
&= \Delta[A,A]_s
\end{aligned}$$

Note: If We prove (2), (3) follows.

For (2), WLOG we assume $A_0 = 0$ (as we can take $\bar{A} = A - A_0$). Let, $T_n = \sup_i \tau_i^n$ ($< \infty$ by definition of random partition) and $\lim_{n\to\infty} T_n = \infty$ a.s. Thus, $(A^2)^{T_n} = \sum_i [A^{\tau_{i+1}^n} - A^{\tau_i^n}] \xrightarrow{ucp} A^2$ And by theorem 17, $\sum_i A_{\tau_i^n}(A^{\tau_{i+1}^n} - A^{\tau_i^n}) \xrightarrow{ucp} \int A_- dA$.

$$\begin{aligned}
&\sum_i (A^{\tau_{i+1}^k} - A^{\tau_i^k})^2 \\
&= \sum_i ((A^{\tau_{i+1}^k})^2 - (A^{\tau_i^k})^2 - 2A^{\tau_i^k}(A^{\tau_{i+1}^k} - A^{\tau_i^k})) \\
&\xrightarrow{ucp} A^2 - 2\int A_- dA \\
&= [A,A]
\end{aligned}$$

□

The Quadratic variation process of two Semi Martingale has a paths of finite variation(FV) and hence its also a Semi-Martingale

Proof. From Polarization Identity, the co-varation process can be return as diffrence of two increasing process, hence it has a path of finite variation process. Since, its also a RCLL, adapted process implies Semi-Martingality. □

[Integration By Parts] Let, A,B be Two Semi-Martingales. Then,

$$AB = \int A_{-}dB + \int B_{-}dA + [A, B]$$

Hence, AB is a Semi-Martingale.

Proof. Follows from definition of [A,B] and space of Semi-Martingale is a vector space. □

Space of Semi-Martingale is an Algebra for given $(\Omega, \mathscr{F}, \mathbf{F}, P)$. From the previous theorem, one can conclude the following theorem.

Theorem 19. *The Quadratic co-variation of A, B 0(Semi-Martingales) is a RCLL, adapted, increasing process satisfying the following properties:*

- *1.* $[A, B]_0 = A_0 B_0,\ \Delta[A, A] = \Delta A \Delta B$
- *2. If* Λ^n *tends to identity, then*

$$A_0^2 + \sum_i (A^{\tau_{i+1}^k} - A^{\tau_i^k})(B^{\tau_{i+1}^k} - B^{\tau_i^k}) \xrightarrow[k\to\infty]{ucp} [A, B]$$

- *3. For any stopping time T,* $[A^T, B] = [A, B^T] = [A^T, B^T] = [A, B]^T$

Theorem 20. *Let,* $\mu,\ \nu, \eta$ *be right continuous functions from* $[0, \infty)$ *to* $\mathbf{R}$ *s.t* $\mu(0) = \nu(0) = \eta(0) = 0$ *and* μ *be of FV,* ν, η *increasing function. If for all* $s,\ t$ *with* $s \leq t$*,*

$$|\int_s^t d\mu_u| \leq (\int_s^t d\nu_u)^{\frac{1}{2}} (\int_s^t d\eta_u)^{\frac{1}{2}}$$

Then for any measurable functions $f,\ g$

$$\int_s^t |f||g| d|\mu_u| \leq (\int_s^t f^2 d\nu_u)^{\frac{1}{2}} (\int_s^t g^2 d\eta_u)^{\frac{1}{2}}$$

Where, $d|\mu_u|$ *is a total variation measure measure wrt* $d\mu_u$*.*

The proof of the theorem is similar to the theorem to follow.

Theorem 21 (Kunita-Watanabe Inequality)**.** *Let, A, B be two Semi-Matingales and H, G be two measurable process. Then,*

$$\int_0^\infty |H_s||G_s|d|[A,B]|_s \leq (\int_0^\infty H_s^2 d[A,A]_s)^{\frac{1}{2}} (\int_0^\infty K_s^2 d[B,B]_s)^{\frac{1}{2}} \ a.s \tag{4.3}$$

Where, $d|\cdot|$ is a total variation measure of $d(\cdot)$.

Proof. Define, $[A,B]_s^t := [A,B]_t - [A,B]_s$. From observing Theorem 20, we begin by showing for any $s,\ t\ (s \leq t)$

$$|\int_s^t d[A,B]_u| \leq (\int_s^t d[A,A]_u)^{\frac{1}{2}} (\int_s^t d[B,B]_u)^{\frac{1}{2}} \ a.s$$

i.e

$$[A,B]_s^t \leq ([A,A]_s^t)^{\frac{1}{2}} ([B,B]_s^t)^{\frac{1}{2}} \ a.s$$

$$0 \leq [A+xB, A+xB]_s^t = x^2[A,A]_s^t + 2x[A,B]_s^t + [B,B]_s^t$$

The above equation is a Quadratic equation and the minimum value is attained at

$$x = \frac{-[A,B]_s^t}{[A,A]_s^t}$$

And the minimum value is $-\frac{([A,B]_s^t)^2}{[A,A]_s^t} + [B,B]_s^t$ but the Quadratic equation is postive for any value of x. Therefore, we get

$$([A,B]_s^t)^2 \leq [A,A]_s^t \cdot [B,B]_s^t$$

Now, firstly let, $H,\ G$ be simple integrands(process) with representation (for simplicity)

$$G = \sum_{i=1}^m 1_{(t_{i-1},t_i]}(u) G_j$$

and

$$H = \sum_{i=1}^m 1_{(t_{i-1},t_i]}(u) H_j$$

Where, $H_j,\ G_j$ are $\mathscr{F}_{t_{j-1}}$-measurable.

$$
\begin{aligned}
& |\int_0^t H_s G_s d[A,B]_s| \\
\leq & \sum_{i=1}^m |H_j||G_j||[A,B]_{t_{i-1}}^{t_i}| \\
\leq & \sum_{i=1}^m |H_j||G_j|([A,A]_{t_{i-1}}^{t_i})^{\frac{1}{2}}([B,B]_{t_{i-1}}^{t_i})^{\frac{1}{2}} \\
\leq & (\sum_{i=1}^m |H_j|^2[A,A]_{t_{i-1}}^{t_i})^{\frac{1}{2}}(\sum_{i=1}^m |G_j|^2[B,B]_{t_{i-1}}^{t_i})^{\frac{1}{2}} \\
= & (\int_0^t H^2 d[A,A])^{\frac{1}{2}}(\int_0^t G^2 d[B,B])^{\frac{1}{2}} \qquad (4.4)
\end{aligned}
$$

The last inequality is due to Hölder's inequality on counting measure.

Now, if the $H,\ G$ are bounded measurable process then $\exists\ H^n,\ G^n$-simple processes converging to $H,\ G$ a.s. respectively. Also, as we are integrating over $[s,t]$ a compact set implies that $[A,A]_u,\ [B,B]_u$ are finite measure a.s. So, we apply Bounded Convergence Theorem so that eqn(4.4) holds for bounded measurable functions.As the LHS of the equation is not total variation measure we will use the following proposition.

Proposition: Let, ν be a complex measurable on $(\mathscr{F},X)$ space. Then $\exists$ a measurable function f s.t $|f(x)|=1\ \forall x\in X$ and $d\nu = fd|\nu|$, where $d|\nu|$ is total variation measure of ν.

The above proposition implies that for almost all $\omega\in\Omega\ \exists,\ f_s(\omega)\in\{-1,1\}$ s.t. $d[A,B]_s(\omega) = f_s(\omega)d|[A,B]_s|$. So, in LHS eqn(4.4) we get the desired. And as any arbitary measurable process can be approximated by increasing sequence of bounded measurable process we get the reult by applyoing the Motone Convergence Theorem. □

For, A,B Semi-Martingales and H,K- measurable processes,

$$
E[\int_0^\infty |H_s||G_s|d|[A,B]|_s \leq \left\|(\int_0^\infty H_s^2 d[A,A]_s)^{\frac{1}{2}}\right\|_p \left\|(\int_0^\infty K_s^2 d[B,B]_s)^{\frac{1}{2}}\right\|_q
$$

s.t. $\frac{1}{p}+\frac{1}{q}=1$ For a Semi-Martingale Z, as $\Delta[Z,Z]_t = (\Delta Z_t)^2$ and as $[Z,Z]$ is a non-decreasing RCLL process We decompose $[Z, Z]$ into path-by path **continuous part of** $[Z, Z]$ (Notation:$[Z, Z]^c$) and jump part i.e for any $t \geq 0$,

$$[Z, Z]_t = [Z, Z]_t^c + \sum_{0 \leq s \leq t} (\Delta Z_s)^2$$

We say that Z will be a **Quadratic pure jump process** if $[Z, Z]^c = 0$. For example, the Poisson process is a Quadratic pure jump process.

Theorem 22. *If Z is a RCLL, adapted with paths of FV on compacts, then Z is a quadratic pure jump semi-martingale.*

Proof. We already know that Z is a Semi-Martingale and Stochastic integral is path-by path Lebesgue Stieljes Integral. We use path-by path $d(X_s)^2 = (X_s)^2 - (X_{s-})^2 = X_s(X_s - X_{s-}) + X_{s-}(X_s - X_{s-})$, therefore

$$\begin{aligned} & X^2 - 2\int X_- dX \\ = & \int X dX + \int X_- dX - 2\int X_- dX \\ = & \int X_- + \Delta X dX + \int X_- dX - 2\int X_- dX \\ = & \int \Delta X dX + 2\int X_- dX - 2\int X_- dX \\ = & \sum_{s \leq t} (\Delta X_s)^2 \end{aligned}$$

□

Theorem 23. *Let, Z be a local-Martingale with non-constant continuous paths. Then $[Z, Z]$ is not constant process and $Z^2 - [Z, Z]$ is a continuous local Martingale.*

Theorem 24. *If $Y,\ Z$ Semi-Martingales and $H,\ G \in \mathbf{L}$ then*

$$[H \cdot Y, G \cdot Z]_t = \int_0^t H_s G_s d[Y, Z]_s$$

Theorem 25. *Let, $H \in \mathbf{D}$, $Y,\ Z$ be Semi-Martingales and Λ^n be random partitions tending to identity. Then*

$$\sum H_{\tau_i^n}(Y^{\tau_{i+1}^n} - Y^{\tau_i^n})(Z^{\tau_{i+1}^n} - Z^{\tau_i^n})] \xrightarrow{ucp} \int H_{s-} d[Y, Z]_s$$

Proof. By defintion, $[Y,Z] = YZ - Y_- \cdot Z - Z_- \cdot Y$. Consider, the RHS of the result i.e

$$\begin{aligned} & H_- \cdot [Y,Z] \\ = \ & H_- \cdot (YZ) - H_- \cdot (Y_- \cdot Z) - H_- \cdot (Z_- \cdot Y) \\ = \ & H_- \cdot (YZ) - (HY)_- \cdot Z) - (HZ)_- \cdot Y) \end{aligned}$$

By Theorem 17, we know that the sum is a limit (in ucp) of

$$\begin{aligned} & \sum_i \{H_{\tau_i^n}((YZ)^{\tau_{i+1}^n} - (YZ)^{\tau_i^n}) - (HY)_{\tau_i^n}(Z^{\tau_{i+1}^n} - Z^{\tau_i^n}) \\ - \ & (HZ)_{\tau_i^n}(Y^{\tau_{i+1}^n} - Y^{\tau_i^n})\} \\ = \ & \sum_i H_{\tau_i^n}\{Y^{\tau_{i+1}^n}Z^{\tau_{i+1}^n} - Y^{\tau_i^n}Z^{\tau_i^n} - Y^{\tau_i^n}(Z^{\tau_{i+1}^n} - Z^{\tau_i^n}) - Z^{\tau_i^n}(Y^{\tau_{i+1}^n} - Y^{\tau_i^n})\} \\ = \ & \sum_i H_{\tau_i^n}\{Y^{\tau_{i+1}^n}Z^{\tau_{i+1}^n} - Y^{\tau_i^n}Z^{\tau_{i+1}^n} - Z^{\tau_i^n}(Y^{\tau_{i+1}^n} - Y^{\tau_i^n})\} \\ = \ & \sum_i H_{\tau_i^n}\{(Z^{\tau_{i+1}^n} - Z^{\tau_i^n})(Y^{\tau_{i+1}^n} - Y^{\tau_i^n})\} \end{aligned}$$

Note: We have used that $(YZ)^T = Y^T Z^T$ and also, $H_T = H^T$ in and interval $[T, \cdot]$. □

4.4 Itô's Formula and It's Applications:

Theorem 26 (Taylor's Theorem:). *Let, $f \in \mathscr{C}^n(\mathbf{I})$ for some fixed n and I be an Open interval around a, then for each $x \in I$ $(\neq a)$ $\exists$ z in between z and a s.t*

$$f(x) - f(a) = \sum_{i=1}^{n-1} \frac{f^i(a)}{i!}(x-a)^i + f^n(z)(x-a)^n \tag{4.5}$$

which can be further simplified as

$$f(x) - f(a) = \sum_{i=1}^{n} \frac{f^i(a)}{i!}(x-a)^i + R(x,a) \tag{4.6}$$

where, f^i denotes the i^{th} derivative of f, $|R(x,a)| \leq r(|x-a|)(|x-a|)^n$ and here, r is an increasing positive valued function with $\lim_{\alpha \downarrow 0} r(\alpha) = 0$.

We will use explicitly Taylor's Theorem in the Itô's Formula for n=2.

Theorem 27 (Itô's Formula)**.** *Let, Z be a Semi-Martingale and f be a $\mathscr{C}^2(\mathbf{R})$-valued function. Then $f(Z)$ is a Semi-Martingale and*

$$f(Z_t) - f(Z_0) = \int_{0+}^{t} f'(Z_{s-})dZ_s + \frac{1}{2}\int_{0+}^{t} f''(Z_{s-})d[Z,Z]_s^c + \sum_{0<s\leq t}\{f(Z_s) - f(Z_{s-}) - f'(Z_{s-})\Delta Z_s\} \tag{4.7}$$

Proof. Observe: The jump part of the $\int_{0+}^{t} f''(Z_{s-})d[Z,Z]_s$ (finite value) is $\sum_{s\leq t} f''(Z_{s-})(\Delta Z_s)^2$ is a convergent series. We add and subtract $\frac{1}{2}$ times this series to the RHS of (4.7) to get,

$$\begin{aligned} & f(Z_t) - f(Z_0) \\ = & \int_{0+}^{t} f'(Z_{s-})dZ_s + \frac{1}{2}\int_{0+}^{t} f''(Z_{s-})d[Z,Z]_s \\ + & \sum_{0<s\leq t}\{f(Z_s) - f(Z_{s-}) - f'(Z_{s-})\Delta Z_s - \frac{1}{2}\sum_{s\leq t} f''(Z_{s-})(\Delta Z_s)^2\} \end{aligned} \tag{4.8}$$

We aim to prove eqn(4.8).

Step 1: Z be a continuous Semi-Martingale
Proof of Step 1: WLOG assume $Z_0 = 0$. Define,

$$T_m := \inf\{t : |Z_t| \geq m\}$$

are sequence of increasing stopping time $\uparrow \infty$. Z^{T_m} are bounded stopped process ($\leq m$) and if Itô's formula holds then it will be valid for Z^{T_m} for each m. Therefore, we assume that Z takes values in a compact set. For a fixed $t > 0$ let, Λ^n be a sequence of random partition (i.e $\Lambda^n = (0 = \tau_0^n \leq \tau_1^n \leq ... \leq \tau_{k_n}^n = t)$) tending to identity. Then, by Taylor's Theorem

$$\begin{aligned} & f(Z_t) - f(Z_0) \\ = & \sum_{i=0}^{k_n}\{f(Z_{\tau_{i+1}^n}) - f(Z_{\tau_i^n})\} \\ = & \sum_{i=0}^{k_n}\{f'(Z_{\tau_i^n})(Z_{\tau_{i+1}^n} - Z_{\tau_i^n})\} + \frac{1}{2}\sum_{i=0}^{k_n}\{f''(Z_{\tau_i^n})(Z_{\tau_{i+1}^n} - Z_{\tau_i^n})^2\} + \sum_{i=0}^{k_n}\{R(Z_{\tau_i^n}, Z_{\tau_{i+1}^n})\} \end{aligned}$$

The first and Second series converges in ucp(in probability as we are talking only in compact set) to $\int_0^t f'(Z_{s-})dZ_s$ and $\frac{1}{2}\int_0^t f'(Z_s)d[Z,Z]_s$ by Theorem 17 and 25 respectively. The third term,

$$\begin{aligned}
&\sum_{i=0}^{k_n}\{R(Z_{\tau_i^n}, Z_{\tau_{i+1}^n})\} \\
&\leq \sup_i r(|Z_{\tau_{i+1}^n} - Z_{\tau_i^n}|)\{\sum_{i=0}^{k_n}(Z_{\tau_{i+1}^n} - Z_{\tau_i^n})^2\}
\end{aligned}$$

By Theorem 18, $\sum_{i=0}^{k_n}(Z_{\tau_{i+1}^n} - Z_{\tau_i^n})^2 \xrightarrow{ucp} [Z,Z]_t$, by assumption for each fixed $\omega \in \Omega$ $u \mapsto Z_u(\omega)$ is continuous and as $[0,t]$ is a compact set, the map is uniformly continuous. But, as $\lim_{n\to\infty}\sup_i |\tau_{i+1}^n - \tau_i^n| = 0$ (tends to identity) $\implies$ the third tem tends to 0. Hence, we have proved for the continuous case.

Step 2: Z be any Semimartingale

Proof of Step 2: For any $t > 0$ we know that, $\sum_{0<s\leq t}(\Delta Z_s)^2 \leq [Z,Z]_t$ (by decomposition of $[Z,Z]$). Therefore the series is convergent and as Z is RCLL process only countably many jump time $\implies$ for every $\epsilon > 0$ only finitely many jump times are in A a.s where, $B = B(\epsilon, t) := \{s \in (0,t] : \sum_{0<s\leq t}(\Delta Z_s)^2 \leq \epsilon^2\}$ and $A := B^c$ (A, B contains all jump points). Now, we use similar methodology used in step 1.

$$\begin{aligned}
&f(Z_t) - f(Z_0) \\
&= \sum_i \{f(Z_{\tau_{i+1}^n}) - f(Z_{\tau_i^n})\} \\
&= \sum_{i,A}\{f(Z_{\tau_{i+1}^n}) - f(Z_{\tau_i^n})\} + \sum_{i,B}\{f(Z_{\tau_{i+1}^n}) - f(Z_{\tau_i^n})\}
\end{aligned}$$

Where, $\sum_{i,B}$ denotes $\sum_i 1_{\{B\cap(\tau_{i+1}^n, \tau_i^n]\neq\phi\}}$ and we choose large n enough such that $\sum_i = \sum_{i,A} + \sum_{i,B}$. So,

$$\lim_n \sum_{i,A}\{f(Z_{\tau_{i+1}^n}) - f(Z_{\tau_i^n}) = \sum_{s\in A}\{f(Z_s) - f(Z_{s-})\}$$

Now, as $f \in \mathscr{C}^2\mathbf{R}$, we use Taylor's Theorem,

$$\begin{aligned}
&\sum_{i,B}\{f(Z_{\tau^n_{i+1}}) - f(Z_{\tau^n_i})\} \\
&= \sum_i \{f'(Z_{\tau^n_i})(Z_{\tau^n_{i+1}} - Z_{\tau^n_i})\} + \frac{1}{2}\sum_i \{f''(Z_{\tau^n_i})(Z_{\tau^n_{i+1}} - Z_{\tau^n_i})^2\} \\
&- \sum_{i\in A}\{f'(Z_{\tau^n_i})(Z_{\tau^n_{i+1}} - Z_{\tau^n_i}) + \frac{1}{2}f''(Z_{\tau^n_i})(Z_{\tau^n_{i+1}} - Z_{\tau^n_i})^2\} + \sum_{i\in B}\{R(Z_{\tau^n_i}, Z_{\tau^n_{i+1}})\} \quad (4.9)
\end{aligned}$$

Similar to continuous case the first two terms converges to to $\int_{0+}^t f'(Z_{s-})dZ_s$ and $\frac{1}{2}\int_{0+}^t f'(Z_{s-})d[Z,Z]_s$ respectively. The Third term converges to

$$-\sum_{s\in A}\{f(Z_{s-})\Delta Z_s + \frac{1}{2}f''(Z_{s-})(\Delta Z_s)^2)\}$$

□

We know that for a absolutely convergent series limit and Sum are interchangeable. So, if we prove that for $\tilde{Z} := Z1_{[0,U_k)}$, where, $U_k := \inf\{u > 0 : |Z_u| \geq k\}$ with $Z_0 = 0$, the above series converges absolutely to the last term of (4.8) for Z replaced by $\tilde{Z}$. Then, we take limit $\epsilon \to 0$ and then $k \to \infty$, we get the desired result as $|\tilde{Z}_s| \leq k$ $\forall s \leq t \implies f''$ is uniformly continuous (Bounded continuous function on compact is uniformly continuous) and as $\tilde{Z}$ is RCLL process, we have

$$\limsup_n \sum_{i\in B}\{R(\tilde{Z}_{\tau^n_i}, \tilde{Z}_{\tau^n_{i+1}})\} \leq r(\epsilon+)[\tilde{Z},\tilde{Z}]_t \xrightarrow{\epsilon\to 0} 0$$

Note: $\tilde{Z}$ is a Semi-Martingale(Product of two Semi-Martingale).

Left to Show: Absolute convergence of the series when Z is replaced by $\tilde{Z}$.
As $|\tilde{Z}| \leq k$, the map f is restricted to $[-k,k]$, we use (4.5) for n=2 (take $C = \sup_{z\in[-k,k]} f''(z)$) to conclude that

$$|f(y) - f(x) - (y-x)f'(x)| \leq C(y-x)^2$$

.

$$\begin{aligned}
&\sum_{s\in A}\{|f(\tilde{Z}_s)-f(\tilde{Z}_{s-})-f'(\tilde{Z}_{s-})\Delta\tilde{Z}_s|\}\\
&\le \sum_{0<s\le t}\{|f(\tilde{Z}_s)-f(\tilde{Z}_{s-})-f'(\tilde{Z}_{s-})\Delta\tilde{Z}_s|\}\\
&\le C\sum_{0<s\le t}(\Delta Z_s)^2\\
&\le C[\tilde{Z},\tilde{Z}]_t<\infty
\end{aligned}$$

Also,

$$\begin{aligned}
&\sum_{s\in A}\{|f''(\tilde{Z}_{s-})(\Delta\tilde{Z}_s)^2|\}\\
&\le C\sum_{0<s\le t}(\Delta\tilde{Z}_s)^2\\
&\le C[\tilde{Z},\tilde{Z}]_t<\infty \ a.s
\end{aligned}$$

Thus, the given series is absolutely convergent.

Theorem 28. *Let, $Z=(Z^1,Z^2,..Z^m)$ be a m-tuple Semi-Martingale and $f\in\mathscr{C}^2(\mathbf{R^n},\mathbf{R})$. Then, $f(Z)$ is a Semi-Martingale and*

$$\begin{aligned}
&f(Z_t)-f(Z_0) \qquad (4.10)\\
&= \sum_{i=1}^{m}\int_{0+}^{t}\frac{\partial f}{\partial z_i}(Z_{s-})dZ_s^i+\sum_{1\le i,j\le m}\frac{1}{2}\int_{0+}^{t}\frac{\partial^2 f}{\partial z_i\partial z_j}(Z_{s-})d[Z^i,Z^j]_s^c\\
&+ \sum_{0<s\le t}\{f(Z_s)-f(Z_{s-})-\sum_{i=1}^{m}\frac{\partial f}{\partial z_i}(Z_{s-})\Delta Z_s^i\}
\end{aligned}$$

- **Definition 11:** For Y, Z Semi-Martingales, the **Fisk-Stratonovich integral** of Y wrt Z is defined as
$$\int_0^t Y_{s-}\circ dZ_s:=\int_0^t Y_{s-}dZ_s+\frac{1}{2}[Y,Z]_t^c$$

Theorem 29. *For a Semi-Martingale Z and $f\in\mathscr{C}^3$, the following holds*

$$f(Z_t)-f(Z_0)=\int_{0+}^{t}f'(Z_{s-})\circ dZ_s+\sum_{0<s\le t}\{f(Z_s)-f(Z_{s-})-f'(Z_{s-})\Delta Z_s\}$$

Theorem 30. *For continuous Semi-Martingales X, Y, let $Z = X + iY$ and h be analytic function. Then*

$$f(Z_t) - f(Z_0) = \int_{0+}^{t} f'(Z_{s-})dZ_s + \frac{1}{2}\int_{0+}^{t} f''(Z_{s-})d[Z,Z]_s$$

Proof. We use Itô's formula for $Z = (X, iY)$ and Cauchy-Riemann equations to get the desired result. □

Theorem 31. *For Semi-Martingales X, Y, let $Z = X + iY$ and h be analytic function. Then*

$$\begin{aligned} &f(Z_t) - f(Z_0) \\ &= \int_{0+}^{t} f'(Z_{s-})dZ_s + \frac{1}{2}\int_{0+}^{t} f''(Z_{s-})d[Z,Z]_s + \sum_{0<s\leq t}\{f(Z_s) - f(Z_{s-}) - f'(Z_{s-})\Delta Z_s\}\end{aligned}$$

4.5 Existence and Uniqueness of SDE wrt Semi-Martingales

- **Definition 12:** We say $Y \in \mathbf{D}$ is **decomposable** if $\exists\ N,\ A$ s.t

  $$Y_t = Y_0 + N_t + A_t$$

  Where, N-locally square integrable local martingale ($N_0 = 0$) and A is FV process ($A_0 = 0$).

- **Definition 13:** We say $Y \in \mathbf{D}$ is **classical Semi-Martingale** if $\exists\ N,\ A$ s.t

  $$Y_t = Y_0 + N_t + A_t$$

  Where, N-local martingale ($N_0 = 0$) and A is FV process ($A_0 = 0$).

- **Definition 14:** The **Predicatable σ-algebra** $\mathscr{P}$ is the smallest σ-algebra making all $\mathbf{L}$-processes measurable. We will also notate $\mathscr{P}$ for the space of processes that are predictably measurable.

Theorem 32. *For $Y \in \mathbf{D}$. The following are equivalent:*

- *Y is a Semi-Martingale*
- *Y is decomposable*
- *given $\alpha > 0$, $\exists\ N,\ A$ with $N_0 = A_0 = 0$ and N-local Martingale with jumps bounded by α, A- FV process, s.t $Y_t = Y_0 + N_t + A_t$*
- *Y is a classical Semi-Martingale The decomposition is unique if N is local martingale and A a predictably measurable FV process.*

We will use the above result explicitly in this section.

- **Definition 14:**
  For $H \in \mathbf{D}$, define

$$H^* = sup_t |H_t|$$

$$\|H\|_{\underline{\underline{\mathbf{S}}}^p} := \|H^*\|_{L_p}$$

The above norm can similarly be defined for $\mathbf{L}$

- **Definition 15:** For a Semi martingale Z with $Z_0 = 0$, let $Z = N + A$ be one of the decomposition, where N is a local martingale and A an adapted, RCLL process with FV($N_0 = 0 = A_0$) , for $0 \le p \le \infty$ we define,

$$j_p(N, A) := \|[N, N]_\infty^{\frac{1}{2}} + \int_0^\infty |dA_s|\|_{L_p}$$

$$\|Z\|_{\underline{\underline{\mathbf{H}}}^p} := \inf_{Z=N+A} j_p(N, A)$$

the infimum is taken overall possible decompositions.

Theorem 33. *For any Semi-Martingale Y with $Y_0 = 0$,*

$$\|[Y, Y]_\infty^{\frac{1}{2}}\|_{L_p} \le \|Y\|_{\underline{\underline{\mathbf{H}}}^p}$$

holds.($1 \le p \le \infty$)

Proof. In Kunita-Watanabe inequality (21) put $H = G = 1$ we get

$$[A, B]_\infty \le ([A, A]_\infty^{\frac{1}{2}})([B, B]_\infty^{\frac{1}{2}})$$

Let, $Y = N + A$ be the decomposition, then (Observe: A is a Quadratic pure jum process)

$$\begin{aligned}
&[Y,Y]_\infty^{1/2} \\
&= [N+A,N+A]_\infty^{1/2} \\
&= ([N,N]_\infty + 2[N,A]_\infty + [A,A]_\infty)^{1/2} \\
&\leq ([N,N]_\infty + 2([N,N]_\infty^{1/2})([A,A]_\infty^{1/2}) + [A,A]_\infty)^{1/2} \\
&= (([N,N]_\infty^{1/2} + [A,A]_\infty^{1/2})^2)^{1/2} \\
&= [N,N]_\infty^{1/2} + [A,A]_\infty^{1/2} \\
&= [N,N]_\infty^{1/2} + (\sum_s (\Delta A_s)^2)^{1/2} \\
&\leq [N,N]_\infty^{1/2} + \sum_s |\Delta A_s| \ (H\ddot{o}lder's\ Inequality) \\
&\leq [N,N]_\infty^{1/2} + \int_0^t |dA_s|
\end{aligned}$$

Where, $|dA_s|$ denotes the total-variation measure. Taking L_p norms followed by inf over all decomposition's give the results. □

If Y is a local Martingale then above equality holds.

Theorem 34. $(1 \leq p < \infty)$, $\exists$ *a constant* c_p *s.t for any* Z*- semi-martingale with* $Z_0 = 0$*, we have*

$$\|Z\|_{\underline{\underline{\mathbf{S}^p}}} \leq c_p \|Z\|_{\underline{\underline{\mathbf{H}^p}}}$$

Proof. As, $Z \in \mathbf{D}$ implies both the norm make sense.

$$
\begin{aligned}
&\|Z\|^p_{\underline{\underline{\mathbf{S}^p}}}\\
&= E[(Z^*_\infty)^p]\\
&\leq E[(M^*_\infty + \int_0^\infty |dA_s|)^p]\\
&\leq 2^{p-1}E[(M^*_\infty)^p + (\int_0^\infty |dA_s|)^p]\\
&\leq 2^{p-1}E[\hat{c}_p[M,M]^{p/2}_\infty + (\int_0^\infty |dA_s|)^p] \ (Burkholder's inequality)\\
&\leq 2^{p-1}\tilde{c}_pE[[M,M]^{p/2}_\infty + (\int_0^\infty |dA_s|)^p]\\
&\leq c_p(j_p(N,A))^p
\end{aligned}
$$

where, $\tilde{c}_p = 1 \vee \hat{c}_p$ and $c_p = 2^{p-1}\tilde{c}_p$. Now, take the pth root followe by inf over all decompositions. □

Theorem 35 (Emery's Inequality). *For a Semi-Martingale Y, $H \in \mathbf{L}$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$,*

$$
\|\int_0^\infty H_s dZ_s\|_{\underline{\underline{\mathbf{H}^r}}} \leq \|Z_s\|_{\underline{\underline{\mathbf{H}^p}}}\|H_s\|_{\underline{\underline{\mathbf{S}^p}}} \tag{4.11}
$$

Holds $(1 \leq p \leq \infty, 1 \leq q \leq \infty)$.

Proof. We assume $Z_0 = 0$ a.s and Let, $Y = N + A$ ($N_0 = A_0 = 0$ $a.s$) be the decomposition. Then, $H \cdot Y = H \cdot N + H \cdot A$ and is a decomposition of $H \cdot Y$. Therefore,

$$
\begin{aligned}
&\|H \cdot Z\|_{\underline{\underline{\mathbf{H}^r}}}\\
&\leq j_r(H \cdot N, H \cdot A)\\
&= \|[H \cdot N, H \cdot N]^{\frac{1}{2}}_\infty + \int_0^\infty |H_s||dA_s|\|_{L_r}\\
&= \|(\int_0^\infty H_s^2 d[N,N]_s)^{\frac{1}{2}} + \int_0^\infty |H_s||dA_s|\|_{L_r}\\
&\leq \|H^*_\infty([N,N]^{\frac{1}{2}}_\infty + \int_0^\infty |dA_s|)\|_{L_r}\\
&\leq \|H^*_\infty\|_{L_p}\|([N,N]^{\frac{1}{2}}_\infty + \int_0^\infty |dA_s|)\|_{L_q} \ (H\ddot{o}lder's inequality)\\
&\leq \|H\|_{\underline{\underline{\mathbf{S}^p}}} j_q(N,A)
\end{aligned}
$$

As, the last inequality holds for every decomposition of Y, the result follows by taking inf over such decompositions. □

Recall: For, $Z \in \mathbf{D}$, and stopping time τ, we denote

$$Z^{\tau} = Z_t 1_{[0,\tau)} + Z_{\tau} 1_{[\tau,\infty)}$$

$$Z^{\tau-} = Z_t 1_{[0,\tau)} + Z_{\tau-} 1_{[\tau,\infty)}$$

- **Definition 16:** A process Z is **locally** in $\underline{\underline{\mathbf{S}^p}}$(resp. $\underline{\underline{\mathbf{H}^p}}$) if $\exists$ stopping times $(\tau^n)_{(n\geq 1)} \uparrow \infty$ s.t $Z^{\tau^n} 1_{\{\tau^n>0\}} \in \underline{\underline{\mathbf{S}^p}}$ (resp. $\underline{\underline{\mathbf{H}^p}}$) for each n and a process Z is **pre locally** in $\underline{\underline{\mathbf{S}^p}}$(resp. $\underline{\underline{\mathbf{H}^p}}$) if $Z^{\tau^n -} 1_{\{\tau^n>0\}} \in \underline{\underline{\mathbf{S}^p}}$ (resp. $\underline{\underline{\mathbf{H}^p}}$) for each n.

Theorem 36. *Any Semi-martingale (Z with $Z_0 = 0$) is prelocally in $\underline{\underline{\mathbf{H}^p}}$, $1 \leq p \leq \infty$*

- **Definition 17:** For a given Semi-Martingale $Y \in \underline{\underline{\mathbf{H}^\infty}}$, $\alpha > 0$, we say Y is **α-Sliceable**(notate: $Y \in \mathcal{S}(\alpha)$) if $\exists$ a finite sequence of stopping times $0 = \tau_0 \leq \tau_1 \leq ...\tau_l$ s.t

  - $Z = Z^{\tau_l -}$
  - $\|(Z - Z^{\tau_i})\|^{\tau_{i+1}} \leq \alpha$

Theorem 37. *For a Semi-Martingale with $Z_0 = 0$ a.s*

- *if $Z \in \mathcal{S}(\alpha)$ then $Z^T \in \mathcal{S}(\alpha)$ and $Z^{T-} \in \mathcal{S}(2\alpha)$*
- *For each $\alpha > 0$, $\exists$ an arbitrarily large stopping time s.t $Z^{T-} \in \mathcal{S}(\alpha)$*

Proof. □

- **Definition 17:** A function $F : \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}$ is **Lipschitz** if $\exists$ constant K s.t

  - $|F(s,a) - F(s,b)| \leq K|a-b|$, each $s \in \mathbf{R}_+$
  - $s \mapsto F(s,a)$ is RCLL, for each $a \in \mathbf{R}^n$

- **Definition 18:** A function $F : \mathbf{R}_+ \times \Omega \times \mathbf{R}^n \to \mathbf{R}$ is **Random Lipschitz** if

  - $(s,\omega) \mapsto F(s,\omega,a) \in \mathbf{L}$, for each fixed $a \in \mathbf{R}^n$
  - For each $(s,\omega) \in \mathbf{R}_+ \times \Omega$ $|F(s,\omega,a) - F(s,\omega,b)| \leq K(\omega)|a-b|$, Where K is a finite random variable.

- **Definition 18:** An Operator $F : \mathbf{D}^n \to \mathbf{D}$ is **process Lipschitz** if for any $Y, Z \in \mathbf{D}^n$

  – given any stopping time τ, $Y^{\tau-} = Z^{\tau-} \implies F(Y)^{\tau-} = F(Z)^{\tau-}$

  – $\exists\ K \in \mathbf{L}$ s.t $|F(Y)_t - F(Z)_t| \leq K_t \|Y_t - Z_t\|$, Where the norm is Euclidean norm.

- **Definition 18:** An Operator $G : \mathbf{D}^n \to \mathbf{D}$ is **functional Lipschitz** if for any $Y, Z \in \mathbf{D}^n$

  – given any stopping time τ, $Y^{\tau-} = Z^{\tau-} \implies G(Y)^{\tau-} = G(Z)^{\tau-}$

  – $\exists\ K = (K_t)_{(t \geq 0)}$- increasing finite process s.t $|F(Y)_t - F(Z)_t| \leq K_t \|Y - Z\|_t^*$, a.s, each $t \geq 0$.

Lemma 38. *Let,* $1 \leq p < \infty$*,* $J \in \underline{\underline{\mathbf{S}}}^p$*, F be a functional lipschitz with F(0)=0 and assume* $\sup_t |K_t(\omega)| \leq k < \infty$ *a.s. Let, Y be a Semi-martingale s.t* $\|Y\|_{\underline{\underline{\mathbf{H}}}^\infty} \leq \frac{1}{2c_p k}$*. Then the equation,*

$$X_t = J_t + \int_0^t F(X)_{s-} dY_s$$

has a unique solution in $\underline{\underline{\mathbf{S}}}^p$ *and*

$$\|X\|_{\underline{\underline{\mathbf{S}}}^p} \leq 2\|J\|_{\underline{\underline{\mathbf{S}}}^p}$$

Proof. Define an operator, $\Gamma : \underline{\underline{\mathbf{S}}}^p \to \underline{\underline{\mathbf{S}}}^p$ as

$$\Gamma(X)_t = J_t + \int_0^t F(X)_{s-} dY_s$$

Claim: The operator is Lipschitz

proof of claim: We will use Theorem 34 and 35

$$
\begin{aligned}
&\|\Gamma(X)_t - \Gamma(\hat{X})_t\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq \|\int_0^t (F(X)_{s-} - F(\hat{X})_{s-})dY_s\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq c_p\|\int_0^t (F(X)_{s-} - F(\hat{X})_{s-})dY_s\|_{\underline{\underline{\mathbf{H}^\infty}}} \\
&\leq c_p\|F(X) - F(\hat{X})\|_{\underline{\underline{\mathbf{S}^p}}}\|Y\|_{\underline{\underline{\mathbf{H}^\infty}}} \\
&\leq c_p\|F(X) - F(\hat{X})\|_{\underline{\underline{\mathbf{S}^p}}} \cdot \frac{1}{2c_pk} \\
&\leq c_pk\|X - \hat{X}\|_{\underline{\underline{\mathbf{S}^p}}} \cdot \frac{1}{2c_pk} \\
&\leq \frac{1}{2} \cdot \|X - \hat{X}\|_{\underline{\underline{\mathbf{S}^p}}}
\end{aligned}
$$

By fixed point Theorem, the existence and uniqueness of the solution is guaranteed.

$$
\begin{aligned}
&\|X\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq \|J\|_{\underline{\underline{\mathbf{S}^p}}} + \|\int_0^t F(X)_{s-}dY_s\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq \|J\|_{\underline{\underline{\mathbf{S}^p}}} + c_p\|\int_0^t F(X)_{s-}dY_s\|_{\underline{\underline{\mathbf{H}^\infty}}} \\
&\leq \|J\|_{\underline{\underline{\mathbf{S}^p}}} + c_p\|F(X)\|_{\underline{\underline{\mathbf{S}^p}}}\|Y\|_{\underline{\underline{\mathbf{H}^\infty}}} \\
&\leq \|J\|_{\underline{\underline{\mathbf{S}^p}}} + c_p\|F(X) - F(0)\|_{\underline{\underline{\mathbf{S}^p}}} \cdot \frac{1}{2c_pk} \\
&\leq \|J\|_{\underline{\underline{\mathbf{S}^p}}} + c_pk\|X\|_{\underline{\underline{\mathbf{S}^p}}} \cdot \frac{1}{2c_pk} \\
&\leq \|J\|_{\underline{\underline{\mathbf{S}^p}}} + \frac{1}{2} \cdot \|X\|\|_{\underline{\underline{\mathbf{S}^p}}}
\end{aligned}
$$

$$\implies \frac{1}{2}\|X\|_{\underline{\underline{\mathbf{S}^p}}} \leq \|J\|_{\underline{\underline{\mathbf{S}^p}}}$$

□

Lemma 39. *Let,* $1 \leq p < \infty$*,* $J \in \underline{\underline{\mathbf{S}^p}}$*,* F *be a functional lipschitz with* $F(0) = 0$ *and assume* $\sup_t |K_t(\omega)| \leq k < \infty$ *a.s. Let,* Y *be a Semi-martingale s.t* $Y \in \mathcal{S}(\frac{1}{2c_pk})$*. Then the equation,*

$$X_t = J_t + \int_0^t F(X)_{s-}dYs \tag{*}$$

has a unique solution in $\underline{\underline{\mathbf{S}^p}}$ *and*

$$\|X\|_{\underline{\underline{\mathbf{S}^p}}} \leq C(k,Y)\|J\|_{\underline{\underline{\mathbf{S}^p}}}$$

where, $C(k,Y)$ *is a constant which depends only on* k *and* Y*.*

Proof. Let, $y = \|Y\|_{\underline{\underline{\mathbf{H}^\infty}}}$ and $j = \|J\|_{\underline{\underline{\mathbf{S}^p}}}$. Let, $0 = \tau_0, \tau_1, ..., \tau_l$ be the slicing time times for Y. Consider,

$$X = J^{\tau_i -} + \int_0^t F(X)_{s-} dY_s^{\tau_i -} \quad i = 0, 1, .., l \tag{i}$$

If we can show that $\exists$ X^i-unique solution of (i) indexed equation for each i, then the solution of (*) is given as $X = X^l + J - J^{\tau_l -}$.

As, $J^{0-} = Y^{0-} = 0 \; \forall t \implies X \equiv 0$ for eqn (0). By induction, assume that eqn (i) has a unique solution X^i and let $x^i = \|X^i\|_{\underline{\underline{\mathbf{S}^p}}}$. Consider,

$$\tilde{X}^i = J^{\tau_i} + \int_0^t F(X)_{s-} dY^{\tau_i} s$$

Clearly, the above eqn has a unique solution(denote-$\tilde{X}^i$) as $\tilde{X}^i = X^i + \{\Delta J_{\tau_i} + F(X^i)_{\tau_i -} \Delta Y_{\tau_i}\} 1_{[\tau_i, \infty)}$. Also,

$$\begin{aligned}
&\|\tilde{X}^i\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq \|X^i + \{\Delta J_{\tau_i} + F(X^i)_{\tau_i -}\Delta Y_{\tau_i}\}1_{[\tau_i,\infty)}\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq \|X^i\|_{\underline{\underline{\mathbf{S}^p}}} + \|\Delta J_{\tau_i} 1_{[\tau_i,\infty)}\|_{\underline{\underline{\mathbf{S}^p}}} + \|F(X^i)_{\tau_i -}\Delta Y_{\tau_i} 1_{[\tau_i,\infty)}\|_{\underline{\underline{\mathbf{S}^p}}} \\
&= \|X^i\|_{\underline{\underline{\mathbf{S}^p}}} + \|\{J^{\tau_i} - J^{\tau_i -}\}1_{[\tau_i,\infty)}\|_{\underline{\underline{\mathbf{S}^p}}} + \|(\int_{[\tau_i -, \tau_i]} F(X^i)_{s-} dY_s) 1_{[\tau_i,\infty)}\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\leq \|X^i\|_{\underline{\underline{\mathbf{S}^p}}} + 2\|J\|_{\underline{\underline{\mathbf{S}^p}}} + \|F(X^i)\|_{\underline{\underline{\mathbf{S}^p}}}\|Y\|_{\underline{\underline{\mathbf{H}^\infty}}} \\
&\leq x^i + 2j + kx^i y \\
&= x^i(1+ky) + 2j
\end{aligned}$$

i.e $\tilde{x}^i \leq x^i(1+ky) + 2j$

For any $V \in \mathbf{D}$, define

$$D_i V := (V - V^{\tau_i})^{\tau_{i+1} -}$$

Observe, for any solution X of eqn $(i+1)$,

$$X^{\tau_i} = \tilde{X}^i \quad , \; t \in [0, \tau_{i+1})$$

Set, $V = X - (\tilde{X}^i)^{\tau_{i+1}-}$

$$\begin{aligned}
V = X - (X^{\tau_i})^{\tau_{i+1}-} \\
&= X - X^{\tau_i} \\
&= J^{\tau_{i+1}-} + \int_0^t F(X)_{s-} dY_s^{\tau_{i+1}-} - (J^{\tau_{i+1}-} + \int_0^t F(X)_{s-} dY_s^{\tau_{i+1}-})^{\tau_i} \\
&= J^{\tau_{i+1}-} + \int_0^t F(X)_{s-} dY_s^{\tau_{i+1}-} - (J^{\tau_{i+1}-})^{\tau_i} - (\int_0^t F(X)_{s-} dY_s^{\tau_{i+1}-})^{\tau_i} \\
&= J^{\tau_{i+1}-} + \int_0^t F(X)_{s-} dY_s^{\tau_{i+1}-} - J^{\tau_i} - \int_0^t F(X)_{s-} dY_s^{\tau_i} \\
&= (J - J^{\tau_i})^{\tau_{i+1}-} + \int_0^t F(V + \tilde{X}^i)_{s-} d(Y - Y^{\tau_i})_s^{\tau_{i+1}-} \\
&= D_i J + \int_0^t F(V + \tilde{X}^i)_{s-} dD_i Y_s
\end{aligned}$$

As, $F(\tilde{X}^i + 0)$ may not be 0, define

$$G_i := F(\tilde{X}^i + \cdot) - F(\tilde{X}^i)$$

Thus,

$$V = (D_i J + \int_0^t F(\tilde{X}^i)_{s-} dD_i Y_s) + \int_0^t G_i(V)_{s-} dD_i Y_s$$

Clearly, $\tilde{J} = D_i J + \int_0^t F(\tilde{X}^i)_{s-} dD_i Y_s$ is in $\underline{\underline{\mathbf{S}}}^p$, $D_i Y$ is a Semi-Martingale with $\|D_i Y\|_{\underline{\underline{\mathbf{H}}}^\infty} \leq \frac{1}{2c_p k}$ as $Y \in \mathcal{S}(\frac{1}{2c_p k})$. Therefore by previous lemma this equation has a unique solution in $\underline{\underline{\mathbf{S}}}^p$ and

$$\begin{aligned}
v^i \leq 2\|D_i J + \int_0^t F(\tilde{X}^i)_{s-} dD_i Y_s\|_{\underline{\underline{\mathbf{S}}}^p} \\
&\leq 2(\|D_i J\|_{\underline{\underline{\mathbf{S}}}^p} + \|\int_0^t F(\tilde{X}^i)_{s-} dD_i Y_s\|_{\underline{\underline{\mathbf{S}}}^p}) \\
&\leq 2(2j + c_p k \tilde{x}^i \cdot \frac{1}{2c_p k}) \\
&= 4j + \tilde{x}^i
\end{aligned} \tag{4.12}$$

As $X = V + (\tilde{X}^i)^{\tau_{i+1}-}$, existence and uniqueness of V, $\tilde{X}^i$, yields unique solution of eqn$(i+1)$ in $\underline{\underline{\mathbf{S}^p}}$ (By induction unique solution exists for each eqn(i)) with

$$\begin{aligned}
x^{i+1} &= \|X^{i+1}\|_{\underline{\underline{\mathbf{S}^p}}} \\
&\le \|V^i\|_{\underline{\underline{\mathbf{S}^p}}} + \|\tilde{X}^i\|_{\underline{\underline{\mathbf{S}^p}}} \\
&= 4j + 2\tilde{x}^i \\
&\le 4j + 2(x^i(1+ky) + 2j) \\
&= 8j + 2(1+ky)x^i \\
&= 8j + 2(1+ky)(8j + 2(1+ky)x^{i-1}) = 8j + 8j(2(1+ky)) + (2(1+ky))^2 x^{i-1} \\
&= 8j + 8j(2(1+ky)) + (2(1+ky))^2(8j + 2(1+ky)x^{i-2}) \\
&= 8j + 8j(2(1+ky)) + 8j(2(1+ky))^2 + (2(1+ky))^3 x^{i-2} \\
&= 8j \sum_{\alpha=0}^{i} (2(1+ky))^\alpha \quad (as\ x_0 = 0) \\
&= 8j \cdot \frac{(2(1+ky))^i - 1}{1+2ky}
\end{aligned}$$

As, the solution of eqn(*) is $X = X^l + J - J^{\tau_l -}$, therefore, $\|X\|_{\underline{\underline{\mathbf{S}^p}}} \le x^l + 2j$ and hence we conclude that $C(k, Y) = 2 + 8[\frac{(2(1+ky))^l - 1}{1+2ky}]$. □

Theorem 40. *Let, $Y = (Y^1, ..., Y^m)$ with $Y_0 = \bar{0}$ be a vector of Semi-Martingales, $J^i \in \mathbf{D}$, $1 \le i \le n$ and let, F^i_j be functional Lipschitz operator, then the system of SDE's,*

$$X^i_t = J^i_t + \sum_{j=1}^{m} \int_0^t F^i_j(X)_{s-} dY^j_s \qquad (**)$$

has a unique solution in $\mathbf{D}^n$. Moreover, if $(J^i)_{i=1}^n$ is a vector of Semi-Martingales, then $(X^i)_{i=1}^n$ is too.

Proof. We will give the proof for $n = m = 1$ i.e consider

$$X_t = J_t + \int_0^t F(X)_{s-} dY_s$$

WLOG assume that $F(0) = 0$ as $X_t = (J_t + \int_0^t F(0)_{s-} dY_s) + \int_0^t G_{s-} dY_s$, where $G(X) := F(X) - F(0)$ will also be functional Lipschitz and we can redefine J.

Step 1: Assume $\sup_t K_t(\omega) \le a < \infty$.

We will use the above two lemma explicitly with $p = 2$ ($c_2 = \sqrt{8}$). By Theorem 37 $\exists\, \mathcal{T}$

arbitrary large stopping time s.t $J^{\mathcal{T}-} \in \underline{\underline{\mathbf{S}}}^2$ anmd $Y^{\mathcal{T}-} \in \mathcal{S}(\frac{1}{4\sqrt{8a}})$. Then, by Lemma 39, $\exists$ a unique solution in $\underline{\underline{\mathbf{S}}}^2$

$$X(\mathcal{T})_t = J^{\mathcal{T}-} + \int_0^t F(X(\mathcal{T}))_{s-} dY_s^{\mathcal{T}-}$$

For, $\mathcal{R} > \mathcal{T}$, $X(\mathcal{T})^{\mathcal{T}-} = X(\mathcal{R})^{\mathcal{T}-}$. So, we define, a process $X : \Omega \times [0, \infty) \to \mathbf{R}$ by $X = X(\mathcal{T})$ on $[0, \mathcal{T})$. By definition of arbitrary large stopping time for each t $\exists \mathcal{T} > t$ and hence the solution exist.

Uniqueness of solution: Let, $\tilde{X}$ be another solution.

Let, $\mathcal{S}$ be arbitrary large stopping time s.t. $(X - \tilde{X})^{\mathcal{S}-}$ is bounded and let, $\mathcal{R} = \min(\mathcal{S}, \mathcal{T})$ which can also be made arbitrary large. Then, $X^{\mathcal{R}-}$, $\tilde{X}^{\mathcal{R}-}$ satisfy

$$Z = J^{\mathcal{R}-} + \int_0^t F(X(\mathcal{R}))_{s-} dY_s^{\mathcal{R}-}$$

As $\tilde{X}^{\mathcal{R}-} \in \mathcal{S}(\alpha)$, the solution is unique i.e $X^{\mathcal{R}-} = \tilde{X}^{\mathcal{R}-}$. As this is true for any arbitrary large stopping time, $X = \tilde{X}$ and hence the solution is unique.

Step 2: We prove existence and uniqueness for a fixed t_0 in $[0, t_0]$.

For fixed t_0, the Lipschitz constant constant reduces to $K(\omega) = K_{t_0}(\omega)$ a random process. WLOG we can assume $K(\omega) < \infty$ a.s. We choose k s.t $P(K(\omega) \leq k) > 0$ and let $\Omega_i := \{\omega : K(\omega) \leq k + i\}$ for each $i = 1, 2, ...$ Observe, $\Omega_n \subset \Omega_m$ for each $m \geq n$. Define, for each i

$$Q_i(\cdot) = \frac{P(\cdot \cap \Omega_i)}{P(\Omega_i)}$$

Clearly, $Q_i \ll P$ and $Q_n \ll Q_m$ for $m > n$ implies every P Semi-Martingale and Q_m Semi-Martingale are Q_n Semi-Martingale. By Theorem 12 every stochastic integral under Q_n is indistinguishable under Q_m also and here is equal in the space Ω_n. Let, X^m be the unique solution wrt Q_m (true by step 1). By uniqueness of the solution, we conclude that $X^m = X^n$ a.s Q_n on Ω_n (indistinguishability). Define,

$$X_t = \sum_{i=1}^{\infty} X_t^i 1_{\{\Omega_i \setminus \Omega_{i-1}\}}$$

Clearly, X, X^i is indistinguishable ($X = X^i$ a.s) wrt Q_i on Ω_i i.e except N_i- Zero measure wrt Q_i set $X = X^i$ on Ω_i and by definition of Q_i, N_i is also zero measure wrt P, $N := \cup_{i=1}^{\infty} N_i$ is also zero-measure and finally by assumption that $K(\omega) < \infty$ a.s $\implies \Omega = \cup_{i=1}^{\infty} \Omega_i \setminus \Omega_{i-1}$ a.s (call N_0 that zero measure set) and hence $N \cup N_0$ is zero-measure set wrt P. From above inference we conclude that on Ω_i

$$\begin{aligned} & X_t \\ = & J_t + \int_0^t F(X^i)_{s-} dY_s \\ = & J_t + \int_0^t F(X)_{s-} dY_s \end{aligned}$$

a.s P for each n and Hence X is a solution. □

Theorem 41. *We assume hypothesis of Theorem 40 and let $(X^1)^i = H^i$ be in $\mathbf{D}$ $(1 \le i \le n)$ and let,*

$$(X_t^k)^i := j_t^i + \sum_{j=1}^{m} \int_0^t F_j^i (X^k)_{s-} dY_s^j \qquad (***)$$

inductively be defined. Let, X be the solution of (**) then $X^k \xrightarrow{ucp} X$.

Chapter 5

Integration wrt Poisson Point Process

We are now interested to understand Stochastic Differential Equations having jump. So we formulate by studying SDE wrt Poisson random measure(PRM). We basically aim to understand the Existence and uniqueness of SDE involving PRM.

5.1 Poisson Random Measure:

- **Definition 1:**
  A random variable X is a said to be **Poisson Distributed with parameter** $\lambda(X \sim$ **Pois**$(\lambda))$ if
  $$\mathbf{P}(\mathbf{X}=\mathbf{k})=\frac{\mathbf{e}^{-\lambda}\lambda^{\mathbf{k}}}{\mathbf{k}!}$$
  for k =0, 1, 2, ...
  Some Basic results:
  - Mean and variance of X$\sim$ $Pois(\lambda)$ is λ
  - MGF of X$\sim$ $Pois(\lambda)$ is $e^{\lambda(e^t-1)}$

  Let $\mathbf{M}$ be a space of all non-negative integer-valued measures on $(\mathcal{U},\mathcal{B}_{\mathcal{U}})$- measurable space s.t $\mathcal{B}_{\mathbf{M}}$ is the smallest $\sigma-$algebra on $\mathbf{M}$ which makes $\mu \in \mathbf{M} \mapsto \mu(B) \in \mathbf{Z}^{+} \cup \{\infty\}$ a measurable map, for all $B \in \mathcal{B}_{\mathcal{U}}$.

- **Definition 2:**

A **Poisson Random Measure(PRM)** μ is a $(\mathbf{M}, \mathcal{B}_{\mathbf{M}})$- valued random variable i.e the map $\mu : \Omega \longrightarrow \mathbf{M}$ defined on $(\Omega, \mathcal{F}, P)$ is $\mathcal{F}/\mathcal{B}_{\mathbf{M}}$ measurable s.t

- for each $B \in \mathcal{B}_{\mathcal{U}}$, $\mu(B) \sim \text{Pois}(\lambda(B))$, for a fixed λ
- if $B_1, B_2, ..., B_n \in \mathcal{B}_{\mathcal{U}}$ are disjoint , then $\mu(B_1), \mu(B_2), ..., \mu(B_n)$ are independent random variable

Observe: A PRM($\mu(\cdot,\cdot)$) can be viewed as a map of two variable $(\omega \in \Omega, B \in \mathcal{B}_{\mathcal{U}})$ where if we fix ω then $\mu(\omega, \cdot)$ is a integer valued measure and if we fix B, $\mu(\cdot\ , B)$ a Poisson distributed random variable s.t properties of above definition hold.

Theorem 42. *For every $\sigma-$finite measure λ on $(\mathcal{U}, \mathcal{B}_{\mathcal{U}})$, $\exists$ a PRM μ with measure $\lambda(B)$ for every $B \in \mathcal{B}_{\mathcal{U}}$.*

Proof. We first assume that λ is a finite measure. We want to construct a PRM s.t. above theorem holds.
Construction:

- For i=1,2,.. define X^i a $\mathcal{U}$-valued random variable s.t $P(X^i \in du) = \frac{\lambda(du)}{\lambda(U)}$
- let p ~**Pois(**$\lambda(\mathcal{U})$**)** (random variable)
- let X^i, p are mutually independent for each i

Note: We assume existence of such random variable.

$$\mu(B) := \sum_{i=1}^{p} I_B(X^i) I_{\{p \geq 1\}}, \quad B \in \mathcal{B}_{\mathcal{U}}$$

Claim:$\mu = \{\mu(B)\}_{B \in \mathcal{B}_{\mathcal{U}}}$ is a required PRM

proof of claim:

$$
\begin{aligned}
& P(\mu(B) = k) \\
= \quad & P(\sum_{i=1}^{p} I_B(X^i) I_{\{p\geq 1\}} = k) \\
= \quad & E[P(\sum_{i=1}^{p} I_B(X^i) = k)|p] \\
& P(I_B(X^i) = 1) = \frac{\lambda(B)}{\lambda(\mathcal{U})} - Bernoulli\ random\ variable. \\
= \quad & E[\binom{p}{k} \cdot \frac{\lambda(B)^k (\lambda(\mathcal{U} \setminus \mathcal{B}))^{p-k}}{\lambda(\mathcal{U})^p}] \\
= \quad & \sum_{p\geq k} \binom{p}{k} \cdot \frac{\lambda(B)^k (\lambda(\mathcal{U} \setminus B))^{p-k} \lambda(\mathcal{U})^p e^{-\lambda(\mathcal{U})}}{\lambda(\mathcal{U})^p p!} \\
= \quad & \frac{e^{-\lambda(\mathcal{U})} \lambda(B)^k}{k!} \sum_{p\geq k} \cdot \frac{\lambda(\mathcal{U} \setminus B)^{p-k}}{(p-k)!} \\
= \quad & \frac{e^{-\lambda(\mathcal{U})} \lambda(B)^k}{k!} \cdot e^{\lambda(\mathcal{U} \setminus B)} \\
= \quad & \frac{e^{-\lambda(\mathcal{B})} \lambda(B)^k}{k!}.
\end{aligned}
$$

Therefore, for every $B \in \mathcal{B}_{\mathcal{U}}$, $\mu(B)$ is a Poisson distributed and *Observe* $\{X^i(\omega), i = 1, 2, ..., p(\omega)\}$is the support of $\mu(\cdot)(\omega)$ - counting measure i.e $\mu(\mathcal{U})(\omega) = p$.
left to show: If $B_1, B_2, .., B_n$ are disjoint then $\mu(B_1, \mu(B_2), ..., \mu(B_n))$ are independent. We will prove for $n = 2$ and can be easily seen that then the result holds for any finite

$n.$

$$\begin{aligned}
&P(\mu(B_1) = k_1, \mu(B_2) = k_2) \\
&= E[P((\sum_{i=1}^{p} I_{B_1}(X^i)) = k_1, \sum_{i=1}^{p} I_{B_2}(X^i)) = k_2)|p] \\
&\quad P(I_{B_i}(X^i) = 1) = \frac{\lambda(B_i)}{\lambda(\mathcal{U})}\ , i = 1, 2. \implies \textit{multinomial distribution} \\
&= \sum_{p \geq k_1 + k_2} \frac{p!}{k_1! k_2! (p - k_1 - k_2)!} \frac{\lambda(B_1)^{k_1} \lambda(B_2)^{k_2} (\lambda(\mathcal{U} \setminus B_1 \cup B_2))^{p-k_1-k_2}}{\lambda(\mathcal{U})^p} \frac{\lambda(\mathcal{U})^p e^{-\lambda(\mathcal{U})}}{p!} \\
&= \frac{e^{-\lambda(\mathcal{U})} \lambda(B_1)^{k_1} \lambda(B_2)^{k_2}}{k_1! k_2!} \sum_{p \geq k_1 + k_2} \cdot \frac{\lambda(\mathcal{U} \setminus B_1 \cup B_2)^{p-k_1-k_2}}{(p - k_1 - k_2)!} \\
&= \frac{e^{-\lambda(B_1)} \lambda(B_1)^{k_1}}{k_1!} \cdot \frac{e^{-\lambda(B_2)} \lambda(B_2)^{k_2}}{k_2!} \\
&= P(\mu(B_1) = k_1) \cdot P(\mu(B_2) = k_2)
\end{aligned}$$

We have used the conditional independence of X^i given p and this is guaranteed by our assumption of $\{X^i\}$ and p are independent. As every X^i are independent this lead to a multinomial distribution having choice to belong to any set $B_1, B_2, \mathcal{U} \setminus (B_1 \cap B_2)$. Also, for fixed ω the $\mu(\omega)(\cdot)$ is a measure. Therefore, we are done.

When λ is σ-finite measure:

The construction is similar to above with some minor changes. As λ is a σ-finite measure $\exists\ B_n \in \mathcal{B}_{\mathcal{U}}$ s.t $\cup_n B_n = \mathcal{U}$ and $0 < \lambda(B_n) < \infty$.

Construction:

- For n=1,2,... i=1,2,.. define $X^i_{(n)}$ a B_n-valued random variable s.t $P(X^i_{(n)} \in du) = \frac{\lambda(du)}{\lambda(B_n)}$
- let p_n ~**Pois(**$\lambda(B_n)$**)**, for n=1,2,.. (random variable)
- let $X^i_{(n)}$, p_n are mutually independent, for each i=1,2, .. and n=1,2,...

Then, observe that $\mu = (\mu(B))$ is a PRM defined as

$$\mu(B) := \sum_{n=1}^{\infty} \sum_{i=1}^{p_n} I_{B \cap B_n}(X^i_{(n)}) I_{\{p_n \geq 1\}}, \quad B \in \mathcal{B}_{\mathcal{U}}$$

□

Note: We say λ as *intensity measure* or *mean measure* of the PRM μ.

5.2 Poisson Point Process:

- **Definition 3:**
For $(\mathcal{U}, \mathcal{B}_{\mathcal{U}})$-measurable space, a map p: $\mathbf{D_p} \longrightarrow \mathcal{U}$, where $\mathbf{D_p}$ is a countable subset of $(0, \infty)$ is said to be **point function** if

$$N_p((0,t] \times B) := \#\{s \in \mathbf{D_p}; s \leq t, p(s) \in B\}$$

where $B \in \mathcal{B}_{\mathcal{U}}, t > 0$ defines a counting measure on $(0, \infty) \times \mathcal{U}$.

A point process can be defined by randomizing the concept of point function. Let, $\Pi_{\mathcal{U}}$ be the space of point functions on $\mathcal{U}$ and $\mathcal{B}_{\Pi_{\mathcal{U}}}$ be the smallest $\sigma-$algebra on $\Pi_{\mathcal{U}}$ s.t. $p \mapsto N_p((0,t] \times B)$ is measurable for each $t > 0, B \in \mathcal{B}_{\mathcal{U}}$.

- **Definition 4:**
A **point process** p is a $(\Pi_{\mathcal{U}}, \mathcal{B}_{\Pi_{\mathcal{U}}})$ valued random variable under probability space $(\Omega, \mathcal{F}, P)$ i.e p$\longrightarrow \Pi_{\mathcal{U}}$ is $\mathcal{F}/\mathcal{B}_{\Pi_{\mathcal{U}}}$-measurable.
We say a point process p **Poisson point process** if the underlying random counting measure is PRM.

- **Definition 5:**
We say that p is a **stationary point process** if for every $t > 0$, p and θ_tp have same probability where $\mathbf{D_{\theta_t p}} := \{s \in (0, \infty); s + t \in \mathbf{D_p}\}$ and $(\theta_t$p)(s):= p(s+t).

Some basic results: p is **stationary-Poisson point process** iff its intensity measure $n_p(dtdx) := E[N_p(dtdx)]$is $= dtn(dx)$ for some measure $n(dx)$ on $(\mathcal{U}, \mathcal{B}_{\mathcal{U}})$(*characteristic measure* on p).

5.3 Stochastic integral wrt to (Poisson) Point process

- **Definition 6:** For a given probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and filtration $\{\mathcal{F}_t\}_{t\geq 0}$, we say that a point process p is $\mathcal{F}_t$**-adapted** if for every $t > 0, B \in \mathcal{B}_{\mathcal{U}}$ the PRM $N_p(t,B) = \sum_{s\in \mathbf{D_p}, s\leq t} I_B(p(s))$ is $\mathcal{F}_t$ measurable.

- **Definition 7:**
A point process p is $\sigma-$**finite** if $\exists\ B_n \in \mathcal{B}_{\mathcal{U}}, n = 1, 2, ...$ s.t $B_n \uparrow \mathcal{U}$ and $E[N_p(t, B_n)] < \infty\ \forall t > 0\ \ n = 1, 2, ...$

- **Definition 8:**
For a given $\mathcal{F}_t$-adapted and $\sigma-$finite point process p we define $\Gamma_p := \{B \in \mathcal{B}_{\mathcal{U}}; E[N_p(t, B_n)] < \infty\ \forall t > 0\}$.
Result: If $B \in \mathcal{B}_{\mathcal{U}}$, then $N_p(t, B)$ is an adapted, integrable, increasing process and satisfies conditions in Doob-Meyer decomposition theorem and therefore $\exists$ a natural increasing process $\hat{N}_p(t, B)$ s.t $\tilde{N}_p(t, B) = N_p(t, B) - \hat{N}_p(t, B)$ is a Martingale. In general $\hat{N}_p(t, B)$ is not a continuous function of t but we will assume continuous function $\hat{N}_p(t, B)$ for every $B \in \mathcal{B}_{\mathcal{U}}$.

- **Definition 9:** An $\mathcal{F}_t$-adapted and $\sigma-$finite point process p on $(\Omega, \mathcal{F}, \mathrm{P})$ is said to be in the **class(QL)** if there exists $\hat{N}_p = (\hat{N}_p(t, B))$ s.t
  - for $B \in \Gamma_p, t \mapsto \hat{N}_p(t, B)$ is a continuous $\mathcal{F}_t$-adapted increasing process
  - for each $t > 0$ and a.a $\omega \in \Omega, B \mapsto \hat{N}_p(t, B)$ is a sigma finite measure on $(\mathcal{U}, \mathcal{B}_{\mathcal{U}})$
  - for $B \in \Gamma_p, t \mapsto \tilde{N}_p(t, B) = N_p(t, B) - \hat{N}_p(t, N)$ is a $(\mathcal{F}_t)$-Martingale.

  Note: The $\{\hat{N}_p(t, B)\}$-random measure is called as the *Compensator* of the p.

- **Definition 10:**
An $(\mathcal{F}_t)$ Poisson point process p is an $\mathcal{F}_t$-adapted and $\sigma-$finite Poisson point process s.t $\{N_p(t + h, B) - N_p(t, B)\}_{h>0, B\in\mathcal{B}_{\mathcal{U}}}$ is independent of $\mathcal{F}_t$.
Result: An $(\mathcal{F}_t)$ Poisson point process p is of class(QL) iff $t \mapsto E[N_p(t, B)]$ is continuous for $B \in \Gamma_p$
Proof of the result: Take $\hat{N}_p(t, B) = E[N_p(t, B)]$ then result follows(Uniqueness of $\hat{N}_p$ is guaranteed by Doob-Meyer Decomposition).

Note: Clearly, an $(\mathcal{F}_t)$ stationary Poisson point process p is of class(QL)(as $E[N_p(t,B)] = tn(B)))$

Lemma 43. *For a bounded $(\mathcal{F}_t)$-predictable process f(s)=f(s,ω) and $B \in \Gamma_p$,*

$$X(t) = \int_0^t f(s)d\tilde{N}_p(t,B) := \sum_{s \le t, s \in \mathbf{D_p}} f(s)I_B(p(s)) - \int_0^t f(s)d\hat{N}_p(t,B)$$

then, X(t) is an $(\mathcal{F}_t)$- Martingale.

Proof. Any bounded predictable process can be approximated by bounded, left continuous $(\mathcal{F}_t)$-adapted process f(s,ω). Observe, for every $s \in [0,\infty)$

$$f_n(s) := f(0)I_{s=0}(s) + \sum_{i=0}^{\infty} f(\frac{i}{2^n})I_{(\frac{i}{2^n},\frac{i+1}{2^n}]}(s) \longrightarrow f(s)$$

Therefore, it is enough to prove the lemma for $f_n(s)$.

$$\int_0^t f_n(s)d\tilde{N}_p(t,B) = \sum_{i=0}^{\infty} f(\frac{i}{2^n})[\tilde{N}_p(\frac{i+1}{2^n} \wedge t, B) - \tilde{N}_p(\frac{i}{2^n} \wedge t, B)]$$

But, this is clearly $\mathcal{F}_t$-measurable and hence a $(\mathcal{F}_t)$-Martingale. □

Theorem 44. *If a point process p is in class (QL), then*

- *for every $B \in \Gamma_p, \tilde{N}_p(.,B) \in \mathcal{M}_2$*
- *for $B_1,\ B_2\ \in \Gamma_p,\ \langle \tilde{N}_p(.,B_1)\tilde{N}_p(.,B_2)\rangle(t) = \hat{N}_p(t, B_1 \cap B_2)$.*

Proof. If we show,

$$\tilde{N}_p(t,B_1)\tilde{N}_p(t,B_2) = local - Martingale\ +\ \hat{N}_p(t,B_1 \cap B_2) \tag{5.1}$$

Then, $N_p(t,B)$ will be square-integrable($E[N_p^2(t,B)] = E[\hat{N}_p(t,B)] = E[N_p(t,B)] < \infty$ as $B \in \Gamma_p$) and by uniqueness of quadratic variation we get the desire result. Note: $N_p(\cdot,B)$is right continuous and $\hat{N}_p(\cdot,B)$ is continuous, Therefore, $\tilde{N}_p$ is a right continuous process($\mathcal{F}_t - adapted$). Therefore, ∃ $(\mathcal{F}_t)$-stopping time τ_n s.t $\tau_n \uparrow \infty$ and $\tilde{N}_p^{(n)}(t,B_1) := \tilde{N}_p(t \wedge \tau_n, B_1)$, $\tilde{N}_p^{(n)}(t,B_2) := \tilde{N}_p(t \wedge \tau_n, B_2)$ both are bounded in t. By above equation it is sufficient to show that

$$\tilde{N}_p^{(n)}(t,B_1)\tilde{N}_p^{(n)}(t,B_2) = Martingale\ +\ \hat{N}_p(t \wedge \tau_n, B_1 \cap B_2)$$

We use integration by parts,

$$\begin{aligned}
&\tilde{N}_p^{(n)}(t,B_1)\tilde{N}_p^{(n)}(t,B_2)\\
&= \int_0^\infty \tilde{N}_p^{(n)}(s-,B_1)\tilde{N}_p^{(n)}(ds,B_2) + \int_0^\infty \tilde{N}_p^{(n)}(s-,B_2)\tilde{N}_p^{(n)}(ds,B_1)\\
&+ \ + \int_0^\infty [\tilde{N}_p^{(n)}(s,B_2) - \tilde{N}_p^{(n)}(s-,B_2)]\tilde{N}_p^{(n)}(ds,B_1)
\end{aligned}$$

By the above lemma, first two terms are are martingales. As $\hat{N}_p(t,B)$ is continuous function of t, $\tilde{N}_p^{(n)}(s,B_2) - \tilde{N}_p^{(n)}(s-,B_2) = N_p^{(n)}(s,B_2) - N_p^{(n)}(s-,B_2)$ for every $s \in (0,\infty)$. Also, as $N_p(s,B)$ is a right continuous process, the last term of equation is

$$\begin{aligned}
&\int_0^\infty [\tilde{N}_p^{(n)}(s,B_2) - \tilde{N}_p^{(n)}(s-,B_2)]\tilde{N}_p^{(n)}(ds,B_1)\\
&= \int_0^\infty [N_p^{(n)}(s,B_2) - N_p^{(n)}(s-,B_2)]N_p^{(n)}(ds,B_1)\\
&= \sum_{s\in\mathbf{D_p}, s\le t\wedge\tau_n} IB_1\cap B_2(p(s))\\
&= N_p(t\wedge\tau_n, B_1\cap B_2)\\
&= \hat{N}_p(t\wedge\tau_n, B_1\cap B_2) + \tilde{N}_p(t\wedge\tau_n, B_1\cap B_2)
\end{aligned}$$

The only non-Martingale term is $\hat{N}_p(t\wedge\tau_n, B_1\cap B_2)$ continuity of t and taking limit $\tau_n \uparrow \infty$ gives the desired result. □

5.4 Existence and uniqueness of SDE wrt Poisson point process

Theorem 45. *Let, p be any $(\mathcal{F}_t)$-stationary Poisson point process with characteristic measure n(du) (σ-finite measure) defined on $(\mathcal{U},\mathcal{B}_\mathcal{U})$-measurable space. Let, $B_0 \in \mathcal{B}_\mathcal{U}$ s.t $n(\mathcal{U}\setminus B_0) < \infty$. Let, W_t be any $\mathbf{R}^n$-valued Brownian motion. Let, $\sigma : \mathcal{R}^d \longrightarrow \mathcal{R}^d \otimes \mathcal{R}^n$ and $b : \mathcal{R}^d \longrightarrow \mathcal{R}^d$ be Borel measurable functions and $f : \mathbf{R}^d \times \mathcal{U} \longrightarrow \mathbf{R}^d$ be a $\mathcal{B}(\mathbf{R}^d)\times\mathcal{B}_\mathcal{U}$-measurable function s.t. the following two properties holds:*

$$\|\sigma(x)\|^2 + \|b(x)\|^2 + \int_{B_0} \|f(x,u)\|^2\, n(du) \le K(1+\|x\|^2), \quad x\in\mathbf{R}^d \tag{5.2}$$

$$\|\sigma(x)-\sigma(y)\|^2+\|b(x)-b(y)\|^2+\int_{B_0}\|f(x,u)-f(y,u)\|^2 n(du) \leq K(\|x-y\|^2), \quad x,y \in \mathbf{R}^d \tag{5.3}$$

For some constant K and where, $\|\sigma(x)\|^2 := \sum_{i=1,j=1}^{d,n} (\sigma^i_j(x))^2$ *and others are just Euclidean-norm. Then the SDE*

$$\begin{aligned} X^i(t) = X^i(0) + \sum_{j=1}^{n} \int_0^t (\sigma^i_j(X(s)))dW^j_s + \int_0^t b(X(s))^i ds \\ + \int_0^{t+} \int_{\mathcal{U}} f^i(X(s-),u) I_{B_0} \tilde{N}_p(dsdu) \\ + \int_0^{t+} \int_{\mathcal{U}} f^i(X(s-),u) I_{\mathcal{U}\setminus B_0} N_p(dsdu) \end{aligned} \tag{5.4}$$

i=1,2,...,d.
If $X(0) = \{X^i(0)\}_{i\in\mathbf{N}} \in \mathcal{F}_0$*-random variable and above conditions on* $\sigma(x),\ b(x),\ f(x,u)$ *holds then, there exists unique* $(\mathcal{F}_t)$*-adapted, d-dimensional, RCLL process* $X = \{X(t)\}_{t\geq 0}$ *which satisfies the above SDE.*

Proof. For a given $W = \{W_t\}_{t\geq 0}$, p and $X(0)$, let; $\mathbf{D} := \{s \in \mathbf{D_p} : p(s) \in \mathcal{U} \setminus B_0\}$. Clearly, $\mathbf{D}$ is a dicrete set in $(0,\infty)$. Let, $\tau_1 < \tau_2 < ... < \tau_n < ...$ be the enumeration of all the elements in $\mathbf{D}$. The $\{\tau_n\}$ are $\mathcal{F}_t$-stopping time (as we assumed that p is $(\mathcal{F}_t)$-stationary Poisson point process).
Claim: $\lim_{n\uparrow\infty} \tau_n = \infty$ a.s
proof of the claim: Let, $\lambda = n(\mathcal{U}\setminus B_0) < \infty$. Observe, $t \mapsto N(t,\mathcal{U}\setminus B_0)$ is a Poisson process with intensity λ and τ_n are the n^{th}-transition time of this Poisson process. Let, for a fixed t define, $E_t := \{\tau_n < t, \forall n\}(= \{\lim_{n\uparrow\infty} \tau_n < t\}) = \{N_p(t,\mathcal{U}\setminus B_0) = \infty\} \implies P(E_t) = 0\ \forall\ t$ (as for fixed t, $N_p(t,\mathcal{U}\setminus B_0) \sim Pois(\lambda)$). As $N_p(\cdot,\mathcal{U}\setminus B_0)$ is RCLL process $P(\cup_{t=0}^{\infty} E_t) = 0$. Hence, the claim.
We begin proving existence and uniqueness of solution of SDE step by step i.e Consider the time interval $[0,\tau_1]$ and the SDE:

$$\begin{aligned} Y^i(t) = X^i(0) + \sum_{j=1}^{n} \int_0^t (\sigma^i_j(Y(s)))dW^j_s + \int_0^t b(Y(s))^i ds \\ + \int_0^{t+} \int_{\mathcal{U}} f^i(Y(s-),u) I_{B_0} \tilde{N}_p(dsdu), \ i = 1,2,..,d \end{aligned} \tag{5.5}$$

Now, Observe,

$$
\begin{aligned}
& E[\{\int_0^{t+}\int_{\mathcal{U}} f^i(Y(s-),u) I_{B_0} \tilde{N}_p(dsdu)\}^2] \\
= \;& E[\int_0^{t}\int_{\mathcal{U}} \{f^i(Y(s),u)\}^2 I_{B_0} \langle \tilde{N}_p(dsdu), \tilde{N}_p(dsdu) \rangle] \\
= \;& E[\int_0^{t}\int_{\mathcal{U}} \{f^i(Y(s),u)\}^2 I_{B_0} \hat{N}_p(dsdu)] \\
= \;& E[\int_0^{t}\int_{\mathcal{U}} \{f^i(Y(s),u)\}^2 I_{B_0} E[N_p(dsdu)]] \\
= \;& E[\int_0^{t}\int_{\mathcal{U}} \{f^i(Y(s),u)\}^2 I_{B_0} ds n(du)] \\
= \;& \int_0^{t} ds \int_{B_0} E[\{f^i(Y(s),u)\}^2] n(du) \qquad (5.6)
\end{aligned}
$$

The method of proving the existence and uniqueness of $Y = \{Y^i\}$ is same as that we did when we dealt SDE containing only Brownian motion W_t. We just give the proof of uniqueness of solution and leave reader to prove the existence of solution by Picard'd iteration.

Uniqueness of Y:

Suppose that Y and $\bar{Y}$ be two solution of equation (5.5). Consider,

$$
\begin{aligned}
& E[|Y^i(t) - \bar{Y}^i(t)|^2] \\
\leq \;& 3(E[\{\sum_{j=1}^{n} \int_0^t (\sigma_j^i(Y(s)) - \sigma_j^i(\bar{Y}(s)))dW_s^j\}^2] + E[\int_0^t \{b(Y(s))^i - b(\bar{Y}(s))^i ds\}^2] \\
+ \;& E[\{\int_0^{t+} \int_{\mathcal{U}} f^i(Y(s-),u)I_{B_0} - f^i(\bar{Y}(s-),u)I_{B_0}\tilde{N}_p(dsdu)\}^2]) \\
& \textit{we use } It\hat{o}'s \textit{ isometry, Holder's inequality, conditions on } \sigma,\ b,\ f \textit{ and eqn}(5.6) \\
\leq \;& 3(n \cdot \sum_{j=1}^{n} E[\int_0^t (\sigma_j^i(Y(s)) - \sigma_j^i(\bar{Y}(s)))^2 ds] + t \cdot E[\int_0^t \{b(Y(s))^i - b(\bar{Y}(s))^i\}^2 ds] \\
+ \;& E[\int_0^t ds \int_{B_0} \{f^i(Y(s),u) - f^i(\bar{Y}(s),u)\}^2 n(du))] \\
\leq \;& 3(n \cdot \sum_{j=1}^{n} E[\int_0^t K \cdot |Y(s) - \bar{Y}(s)|^2 ds] + t \cdot E[\int_0^t K \cdot |Y(s) - \bar{Y}(s)|^2 ds] \\
+ \;& E[\int_0^t K \cdot |Y(s) - \bar{Y}(s)|^2 ds]) \\
= \;& 3K((n^2 + 1 + t) \cdot \int_0^t E[|Y(s) - \bar{Y}(s)|^2]ds) \\
& \textit{We use Gronwall's inequality to conclude} \\
= \;& 0
\end{aligned}
$$

Therefore, Y is a modification of $\bar{Y}$ but as they are RCLL process $\implies$ Y, $\bar{Y}$ are indistinguishable. Hence, unique solution exists.
We use the existence and uniqueness of the solution of (5.5) to define the same for SDE (5.4) in an interval $[0, \tau_1]$. Define

$$
X_1(t) = \begin{cases} \mathrm{Y(t)}, & \text{if } 0 \leq \mathrm{t} < \tau_1 \\ \mathrm{Y}(\tau_1-) + \mathrm{f}(\mathrm{Y}(\tau_1-), \mathrm{p}(\tau_1)), & \text{if } \mathrm{t} = \tau_1 \end{cases}
$$

Observe: $\{X_1(t)\}_{t\in[0,\tau_1]}$ is the unique solution of SDE (5.4), as in the interval $[0, \tau_1)$, $N_p([0, \tau_1), \mathcal{U} \setminus B_0) = 0$ and for $t = \tau_1$, $N_p(\{\tau_1\}, \mathcal{U} \setminus B_0) = 1$ implies in the nbhd of τ_1, $N_p(\{\tau_1\}, B_0)$ (no jump in B_0) is continuous i.e $\tilde{N}_p(t, B_0)$ is continuous in that nbhd. So, for $t = \tau_1$ first three terms in the integral is just the left limit due to continuity and the last term gives non-zero term at $\{\tau_1\} \times \{p(\tau_1)\}$.

We now move to the interval $[\tau_1, \tau_2]$. Let, $\tilde{X}_2(0) = X_1(\tau_1)$, $\tilde{W} = \{\tilde{W}_t = W_{t+\tau_1} - W_{\tau_1}\}_{t\geq 0}$, $\tilde{p} = \{\tilde{p}(s) = p(s+\tau_1)\}_{t\geq 0}$, where, $\mathbf{D}_{\tilde{\mathbf{p}}} = \{s : s+\tau_1 \in \mathbf{D}_{\mathbf{p}}\}$. Now, we enumerate similar to done in $X_1(t)$ i.e $\tilde{\mathbf{D}} = \{s \in \mathbf{D}_{\tilde{\mathbf{p}}};\ \tilde{\mathbf{p}}(\mathbf{s}) \in \mathcal{U} \setminus \mathbf{B_0}\}$. Let, $\{\mathcal{F}_{t+\tau_1}\}_{t\geq 0}$ measurable stopping time be named as $\tilde{\tau}_1,\ \tilde{\tau}_2, ..., \tilde{\tau}_m, ...$ and observe that $\tilde{\tau}_k = \tau_k - \tau_1$ in particular $\tilde{\tau}_2 = \tau_2 - \tau_1$. Consider,

$$\tilde{Y}^i(t) = X_1^i(\tau_1) + \sum_{j=1}^{n} \int_0^t (\sigma_j^i(\tilde{Y}(s)))d\tilde{W}_s^j + \int_0^t b(\tilde{Y}(s))^i ds$$
$$+ \int_0^{t+} \int_{\mathcal{U}} f^i(\tilde{Y}(s-), u) I_{B_0} \tilde{N}_{\tilde{p}}(dsdu),\ i = 1, 2, .., d \tag{5.7}$$

By similar argument done in SDE (5.5), we conclude the existence and uniqueness of SDE (5.7) in $[0, \tilde{\tau}_1]$. Define,

$$\tilde{X}_2(t) = \begin{cases} \tilde{Y}(t), & \text{if } 0 \leq \text{t} < \tilde{\tau}_1 \\ \tilde{Y}(\tilde{\tau}_1-) + \text{f}(\tilde{Y}(\tilde{\tau}_1-), \tilde{p}(\tilde{\tau}_1)), & \text{if t} = \tilde{\tau}_1 \end{cases}$$

Now, Define

$$X_2(t) = \begin{cases} X_1(\text{t}), & \text{if t} \in [0, \tau_1] \\ \tilde{X}_2(\text{t} - \tau_1), & \text{if t} \in [\tau_1, \tau_2] \end{cases}$$

By uniqueness in their respective interval and $X_1(\tau_1) = \tilde{X}(0)$, $X_2(t)$ is the unique solution of SDE (5.4) in $[0, \tau_2]$. Continuing this process, we define $X_n(t)$ which satisfies the SDE(5.4) uniquely in the time interval $[0, \tau_n]$ and taking limit $n \uparrow \infty$ the solution of SDE(5.4) is determined and defined globally. □

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