Hawking Radiation in Analogue Gravity

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Hawking Radiation in Analogue Gravity" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Chaitra A at the Indian Institute of Science Education and Research, Pune under the supervision of Dr. Arijit Bhattacharyay during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled "Hawking Radiation in Analogue Gravity" are the results of the investigations carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Arijit Bhattacharyay and the same has not been submitted elsewhere for any other degree.

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Abstract

The phenomenon of Hawking Radiation is explored in analogue gravity using Bose-Einstein Condensates. The process of spontaneous pair production that occurs at the event horizon is treated as a scattering problem, and the method of establishing Hawking Radiation using density correlations is reviewed. Based on the results obtained, the existing formalism is modified to incorporate the changes to the asymptotic modes due to the presence of the horizon.

Contents

Chapter 1

Introduction

Black holes are a solution to Einstein's equation. These are extremely massive objects - with a singularity surrounded by an event horizon (the null surface interior boundary of the space-time from which light can escape to infinity). Black hole physics has a striking similarity to the laws of thermodynamics. Entropy, a thermodynamical quantity that never decreases with time is associated with the area of a black hole under the Generalized Second Law of thermodynamics [1]. Classically, black holes were thought to be perfect absorbers, i.e. they were thought to have a temperature of absolute zero. However, Hawking, with the techniques of quantum field theory in curved space-times showed that black-holes do radiate thermally, thus associating a non-zero temperature to black holes [2]. This is called Hawking radiation, and the temperature is $T = \frac{\hbar\kappa}{2\pi}$, where κ is the surface gravity (the non-affineness of the Killing generator of the horizon).

1.1 The Trans-Planckian Puzzle

Hawking Radiation is a quantum effect. Vacuum fluctuations near the event horizon spontaneously create particle - antiparticle pairs. These have positive and negative energies, with the particle (of positive energy) propagating out to future infinity, and the antiparticle (of negative energy) falling into the black hole, thereby reducing its mass. This process is referred to as black hole evaporation. Soon after the publication of Hawking's result, it was realized that the result assumes the validity of quantum field theory in curved space time upto arbitrary energies. Problem arises when a wavepacket of a certain frequency in future infinity is propagated back in time. It will have exponential energy - beyond the Planck scale - wherein current theories break down. This is called the trans-Planckian problem.

The trans-Planckian problem has raised the question of whether Hawking radiation is a spurious result. While there have been efforts to rectify this problem using quantum field theories in curved space-times [3], there is an alternate approach - Analogue Gravity [4].

1.2 Analogue Gravity

Analogue Gravity is an approach to study quantum fields in curved space-time using systems - typically condensed matter systems - whose physics is well understood. In 1980, Unruh demonstrated the analogy between a quantized trans-sonic fluid flow and a black hole [5]. As any moving fluid drags sound waves along with it, when the fluid velocity becomes supersonic, the sound waves can no longer propagate upstream, thus creating a black-hole like configuration.

"Under the approximation of the fluid flow being barotropic, inviscid and irrotational, the equations of motion of the velocity potential governing the acoustic disturbance can be recast as the d'Alembertian equation of motion for a minimally coupled massless scalar field propagating in a (3+1) dimensional Lorentzian geometry" [4].

The fundamental equations of such a fluid are the continuity equation and the Euler equation

$$\partial_t \rho_0 + \nabla.(\rho_0 \mathbf{v}) = 0 \tag{1.1}$$
$$-\partial_t \phi_0 + h(p) + \frac{1}{2}\nabla\phi_0^2 = 0$$

where, ρ_0 is the density of the fluid, h(p) is the specific enthalpy of the barotropic fluid and ϕ_0 is the velocity potential such that $\mathbf{v} = -\nabla\phi_0$. These equations, when linearised give the wave equation, which in turn can be recast as

$$\Delta\phi = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi), \tag{1.2}$$

where the effective acoustic metric $g_{\mu\nu}$ is as follows

$$g_{\mu\nu} = \frac{\rho_0}{c}\begin{pmatrix} -(c^2 - v_0^2) & -\mathbf{v_0}^{\mathbf{T}} \\ -\mathbf{v_0} & \mathbf{I} \end{pmatrix} \tag{1.3}$$

Here, c is the speed of sound with respect to the fluid, $\mathbf{v_0}$ is the velocity of the fluid, $\mathbf{I}$ is the 3 X 3 identity matrix, and ρ_0 is the density of the fluid.

The advantage of analogue gravity is two-fold. One can study relativistic phenomena in analogue systems, thus establishing some results which are otherwise hard to establish experimentally-to the extent that the analogy holds. Secondly, the physics of analogue systems can be used to gain insights into gravity, curved-space quantum field theory or quantum/emergent gravity.

1.3 The acoustic metric for BECs

In section 1.2, the acoustic metric, obtained with the hydrodynamic approximation is valid for a trans-sonic fluid flow. This section provides the acoustic metric, where BECs are the analogue systems. The governing equation of BECs is the Gross-Pitaevskii (GP) equation

$$i\hbar\partial_t\hat{\Psi} = (\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V_{ext} + g\hat{\Psi}^\dagger\hat{\Psi})\hat{\Psi} \tag{1.4}$$

where g is the strength of interaction, m is the mass of the atoms and $\hat{\Psi}$ is the bosonic field operator. Performing a mean field expansion $\hat{\Psi} = \psi + \hat{\phi}$, where ψ is the macroscopic condensate and $\hat{\phi}$ is the fluctuation and adopting the ansatz, $\Psi = \sqrt{n_c(t,x)}e^{-i\theta(t,x)/\hbar}$ for the wavefunction of the condensate, the GP equation can be rewritten as

$$\partial_t n_c + \nabla.(n_c\mathbf{v}) = 0 \tag{1.5}$$

$$\partial_t v + \nabla(\frac{mv^2}{2} + V_{ext}(t,x) + gn_c - \frac{\hbar^2}{2m}\frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}) = 0 \tag{1.6}$$

Here, $n_c = |\psi|^2$ is the density of the condensates, and θ is such that $v = \frac{\nabla\theta}{m}$. The above equations are analogous to the equations 1.1, except for the last term in equation 1.6, which is the quantum pressure term. Neglecting this term, leads to the analogue of the hydrodynamic approximation (gradients in the density of the condensates are small), and as before, these equations can be recast to obtain the acoustic metric.

1.4 Hawking Radiation in BECs

There are many analogue models that can be used to study Hawking Radiation. However, BECs have proven to be the most suited.

1. Hawking Radiation is thermal in nature and the value of the temperature associated with this phenomenon is extremely low. To detect this in analogue systems experimentally, there is a need to reduce background thermal noise to the maximum possible extent. This can be best done with BECs.

2. Effective 1-D calculations have shown the existence of Hawking radiation. In an experimental scenario, it is easy to deal with an effective 1-D system, while the instabilities/vortices can be in the transverse direction, which does not affect the results.

3. The hydrodynamic approximation breaks down at short length scales. However, BECs have a natural short length scale cut-off: the healing length ξ. This is the distance over which the kinetic energy and the interaction energy balance. The short length scale cut-off is the analogue of Planck scale, below which a Lorentz breakdown occurs. The physics of BECs below this length scale is well known, and thus enables one to tackle the trans-Planckian issue.

4. As both partners are accessible to the observer in an analogue system, unlike a relativistic system, density correlations between the Hawking partners can be studied. In addition, one doesn't need to concern with background thermal noise, thereby minimizing experimental errors.

To establish Hawking radiation avoiding the trans- Planckian problem, many numerical and analytical calculations have been done. [6], [7]. Many of these works address the dependency of Hawking Radiation on high frequencies [8], [9], [10], [11]. This brings into question exact Lorentz invariance, thus making a case for using modified dispersion relations. A discussion on this is provided in [12]. One such work, which uses explicit solutions of the dispersion relation in order to study Hawking Radiation is by A. Fabbri and C. Mayoral [13]. In this paper, the linear dispersion for BECs is used (equivalent to the hydrodynamic limit) to study density correlations, but without any scope for a black hole like configuration. This analysis, extended to include dispersion effects allows for a black hole like configuration [14]. In addition, it has been shown that in order to obtain the best correlations, we have to consider a variation in the speed of sound along with a variation in the velocity of the condensate[6]. Therefore, while a discontinuity in the speed of sound in fluids is considered in the above works, in the present work we consider a varying velocity profile within the framework already present, and suitably extend the model. In addition, a continuously varying velocity profile is considered in order to avoid a singular surface gravity, as encountered in [13] and [14].

Chapter 2

Hawking radiation using super-luminal dispersion relation

A Bose gas in the dilute gas approximation is described by a field operator . This obeys the equal time bosonic commutation relation [1]

$$[\hat{\Psi}(t,\vec{x}), \hat{\Psi}^{\dagger}(t,\vec{x'})] = \delta^3(\vec{x}-\vec{x'}) \tag{2.1}$$

The time evolution is given by the Gross-Pitaevskii (GP) equation

$$i\hbar\partial_t\hat{\Psi} = (\frac{-\hbar^2}{2m}\vec{\nabla}^2 + V_{ext} + g\hat{\Psi}^{\dagger}\hat{\Psi})\hat{\Psi} \tag{2.2}$$

where m is the mass of the atoms and g is the strength of the contact interaction, given by $g = \frac{4\pi\hbar^2 a}{m}$, where a is the s-wave scattering length. Performing a mean field approximation for $\hat{\Psi}$, we have

$$\hat{\Psi}(t,x) = \Psi_0(x)(1+\hat{\phi}(t,x)) \tag{2.3}$$

where Ψ_0 is the macroscopic condensate wavefunction and $\hat{\phi}$ is the fluctuation. Using the above ansatz for $\hat{\Psi}$ in equation 2.2, we have a similar equation for the macroscopic condensate, while the fluctuation field obeys the Bogoliubov de Gennes (BdG) equation, which is independent of V_{ext}.

$$i\hbar\partial_t\hat{\phi} = (\frac{-\hbar^2}{2m}\vec{\nabla}^2 - \frac{\hbar^2}{m}\frac{\vec{\nabla}\Psi_0}{\Psi_0}\vec{\nabla})\hat{\phi} + gn(\hat{\phi}+\hat{\phi}^{\dagger}) \tag{2.4}$$

As Ψ_0 obeys the GP equation, $\Psi_0 = \sqrt{n}e^{ik_0x - i\omega_0 t}$ is a solution everywhere, if the GP equation does not change. Using the fact that $v = \frac{\hbar k}{m}$ and that $c = \sqrt{\frac{gn}{m}}$, we have,

$$i\hbar(\partial_t + v\partial_x)\hat{\phi} = \frac{-\hbar^2}{2m}\vec{\nabla}^2\hat{\phi} + mc^2(\hat{\phi}+\hat{\phi}^{\dagger}) \tag{2.5}$$

[1] The work described in this section is based on [14]

2.1 Non-linear dispersion relation

We consider condensates with constant density n. The operator $\hat{\phi}$ is expanded in terms of its particle and antiparticle components

$$\hat{\phi}(t,x) = \sum_j [\hat{a}_j \phi_j(t,x) + \hat{a}_j^\dagger \varphi_j^*(t,x)] \tag{2.6}$$

where $\hat{a}_j$ and $\hat{a}_j^\dagger$ are the creation and annihilation operators. Inserting the above expansion in equation 2.4, we have

$$[i(\partial_t + v\partial_x) + \frac{\xi c}{2}\partial_x^2 - \frac{c}{\xi}]\phi_j = \frac{c}{\xi}\varphi_j \tag{2.7}$$

$$[-i(\partial_t + v\partial_x) + \frac{\xi c}{2}\partial_x^2 - \frac{c}{\xi}]\varphi_j^* = \frac{c}{\xi}\phi_j^* \tag{2.8}$$

where ξ is the healing length of the condensate, given by $\xi = \frac{1}{\sqrt{8\pi a n}}$. We assume the fields ϕ and φ to be plane waves over a flat space such that

$$\begin{aligned} \phi(\omega) &= D(\omega) e^{-i(\omega t - k(\omega)x)} \\ \varphi(\omega) &= E(\omega) e^{-i(\omega t - k(\omega)x)} \end{aligned} \tag{2.9}$$

where D(ω) and E(ω) are normalization constants, to be determined. Equations 2.6 and 2.7 give,

$$\begin{aligned} [(\omega - vk) - \frac{\xi c k^2}{2} - \frac{c}{\xi}]D(\omega) &= \frac{c}{\xi}E(\omega) \\ [-(\omega - vk) - \frac{\xi c k^2}{2} - \frac{c}{\xi}]E(\omega) &= \frac{c}{\xi}D(\omega) \end{aligned} \tag{2.10}$$

Demanding that a non-trivial solution exists, implies the determinant is zero. This gives the second order dispersion relation

$$(\omega - vk)^2 = c^2k^2 + \frac{c^2\xi^2k^4}{4} \tag{2.11}$$

2.2 The model

In this section, a simple model to study Hawking radiation is described [14]. A discontinuity in the speed of sound is considered, such that, $c(x) = c_r(\Theta(x)) + c_l(\Theta(-x))$, where Θ is the Heavyside function. As the speed of sound changes in each sector, the healing length ξ of BECs also changes correspondingly, as it is defined as $\xi = \frac{\hbar}{mc}$. At $x = 0$ there is a step-discontinuity in the speed of sound. The velocity of the condensate, v is taken to be negative. This implies that, when $|v| < c_r$ and $|v| < c_l$, we have a subsonic - subsonic configuration, whereas when $|v| < c_r$, but $|v| > c_l$, we have a subsonic - supersonic configuration. In this case, the point $x = 0$ acts as the event horizon. However, the surface gravity, defined as $\kappa = \frac{1}{2c}\frac{d(c^2-v^2)}{dn}|_{hor}$ where n is the coordinate normal to the horizon, is singular because of the step-discontinuity in the speed of sound.

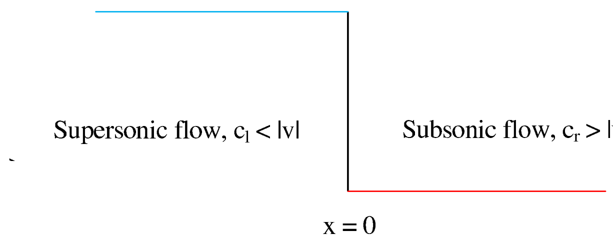


Fig 2.1: Model for subsonic - supersonic configuration

The idea of this formulation is to look at Hawking radiation as a scattering phenomenon at the horizon. The variation in the speed of sound is brought about by changing the scattering length. However, the external potential is also simultaneously changed, such that equation 2.2 remains unchanged in both sectors.

$$V^r_{ext} + g^r n = V^l_{ext} + g^l n \tag{2.12}$$

This allows for a matching at the point $x = 0$, and as the total energy is conserved, probability current conservation can be studied. The procedure followed here is as follows:

• **Solving the dispersion relation:** This gives the modes that get scattered at the horizon. The nature of the modes determine the configuration for the scattering. There will be incoming modes (which propagate with the fluid flow, and hence towards the horizon on the right and away from the horizon on the left) and out-going modes (which propagate against the fluid flow, and hence away from the horizon on the right and towards the horizon on the left), with positive or negative energies.

- **Matching Conditions:** We expand the fluctuation field in terms of the modes. Each mode is weighted with an amplitude. As the field obeys a second order differential equation, the field and its first derivatives can be matched at the boundary, $x = 0$, giving a matching matrix, M.

- **Probability Currents:** The BdG equation can be used to derive the probability currents for the system. As this is a scattering problem, the amplitudes must be such that this current is conserved for each scattering configuration considered.

- **Determining Amplitudes:** The matrix M is completely known, and hence determines the amplitudes for the modes getting scattered. Although at this point it is not clear that this equation is completely determinate, when the nature of the modes are examined, it will be seen that it is indeed so.

- **Bogoliubov transformation:** At a time, only one incoming mode is considered, setting the amplitudes of all the other incoming modes to zero. This allows us to write each incoming mode as a linear combination of the out-going modes, hence providing the Bogoliubov transformation. This defines the in basis and similarly considering out-going modes, one can define the out basis. The conservation of probability currents implies that the transformation is unitary, which is essential to the result.

- **Mode mixing:** The same transformation gives the creation and annihilation operators in a basis in terms of these operators in the other basis. If there is mode mixing, it becomes apparent, as the creation operator in one basis will not be a linear combination of just the creation operators in the other basis, but will have a contribution from the annihilation operator in the other basis.

- **Correlation function:** Using this, fields are written in one basis, and the two point function in fluctuations is computed, the result of which will determine whether or not Hawking radiation is observed. If a non-trivial correlation is obtained, the Hawking temperature can be read off, and thus, a surface gravity can be associated to the black hole like configuration.

2.2.1 Solving the dispersion relation

The above dispersion is solved in two regimes: low momentum regime, in which the hydrodynamic approximation, and a high momentum regime, which gives two complex conjugate modes, if the flow is sub-sonic and two real modes if the flow is supersonic.

Low momentum or hydrodynamic limit: In this limit, the quantity $k\xi$ is small. Hence, the Lorentz breaking term in equation 2.9 can be neglected, giving back the linear dispersion relation

$$\omega - vk = \pm ck \tag{2.13}$$

Using equation 2.9 to solve for k perturbatively with ξ as the small parameter,

$$k_u = \frac{\omega}{v+c}(1 - \frac{c\xi^2\omega^2}{8(v+c)^3}) \tag{2.14}$$

$$k_v = \frac{\omega}{v-c}(1 - \frac{c\xi^2\omega^2}{8(v-c)^3}) \tag{2.15}$$

High momentum or small ω limit: In this regime, ω is treated to be the small parameter. Then, perturbatively obtaining the roots we have,

$$k_{d,(g)} = \frac{-\omega|v|}{c^2-v^2}(1 - \frac{(c^2+v^2)c^2\xi^2\omega^2}{2(c^2-v^2)^3}) \pm 2i\frac{\sqrt{c^2-v^2}}{c\xi}(1 + \frac{(c^2+2v^2)c^2\xi^2\omega^2}{8(c^2-v^2)^3}) \tag{2.16}$$

in the sub-sonic region. The mode with the positive imaginary part is the decaying mode k_d when $x > 0$ region, and it becomes the growing mode k_g when $x < 0$. The analytic continuation of the complex modes in the supersonic region gives real modes:

$$k_{3,(4)} = \frac{-\omega|v|}{c^2-v^2}(1 - \frac{(c^2+v^2)c^2\xi^2\omega^2}{2(c^2-v^2)^3}) \pm 2\frac{\sqrt{v^2-c^2}}{c\xi}(1 + \frac{(c^2+2v^2)c^2\xi^2\omega^2}{8(c^2-v^2)^3}) \tag{2.17}$$

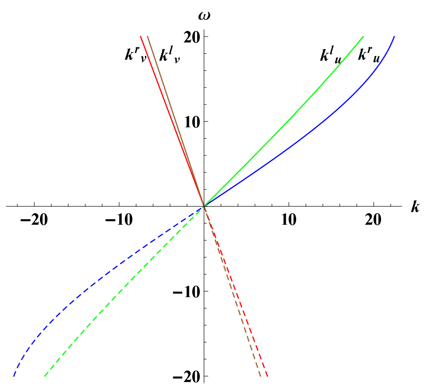


Fig. 2.2: Dispersion relation for subsonic - subsonic case. The solid (dashed) line corresponds to the positive (negative) norm branch

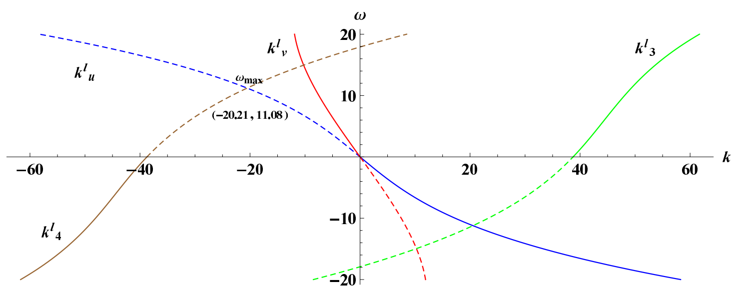


Fig. 2.3: Dispersion relation for subsonic - supersonic case. The solid (dashed) line corresponds to the positive (negative) norm branch

As seen from the graph above, there exists an ω_{max}, corresponding to a k_{max}. This can be calculated by setting $\frac{d\omega}{dk} = 0|_{k_{max}}$. Beyond this ω_{max}, in the supersonic region, the two real modes from the small ω limit become complex modes, giving back the subsonic scenario.

The nature of these modes decide the configuration that can exist. The group velocity of a mode (defined as $\frac{d\omega}{dk}$) determines whether it is an incoming or an out-going one, and the norm of a mode (defined by the co-moving frequency $\omega - vk$) determines whether it has positive or negative energy. Each positive norm mode has a negative counterpart, however, the norm should not be attributed based on the sign of the mode. The co-moving frequency could turn out to be positive even for a negative mode.

Mode	Norm	Group velocity
k_v^r	Positive	Negative, Incoming mode
k_u^r	Positive	Positive, Out-going mode
k_v^l if $(\lvert v_l\rvert > c)$	Positive	Negative, Out-going mode
k_u^l if $(\lvert v_l\rvert > c)$	Negative	Negative, Out-going mode
k_3^l	Positive	Positive, Incoming mode
k_4^l	Negative	Positive, Incoming mode

2.2.2 Normalization

Equal time commutator for $\hat{\phi}$ as obtained from equation 2.1 is

$$[\hat{\phi}_j(t,x), \hat{\phi}^\dagger_{j'}(t,x')] = \frac{\delta(x-x')}{n}\delta_{jj'} \tag{2.18}$$

Here, $j \equiv \omega$. This, on integration yields,

$$\int dx[\phi_j\phi^*_{j'} - \varphi^*_j\varphi_{j'}] = \pm\frac{\delta_{jj'}}{n} \tag{2.19}$$

For the real solutions of the dispersion relation (k_u, k_v, k_3 and k_4), the normalization condition 2.18 along with 2.7 gives

$$|D(\omega)|^2 - |E(\omega)|^2 = \frac{1}{2\pi n}\left|\frac{dk}{d\omega}\right| \tag{2.20}$$

which, along with equation 2.8 implies

$$\begin{aligned} D(\omega) &= \frac{(\omega - vk) + c\xi k^2/2}{\sqrt{4\pi nc\xi k^2\left|(\omega - vk)(\frac{dk}{d\omega})^{-1}\right|}} \\ E(\omega) &= -\frac{(\omega - vk) - c\xi k^2/2}{\sqrt{4\pi nc\xi k^2\left|(\omega - vk)(\frac{dk}{d\omega})^{-1}\right|}} \end{aligned} \tag{2.21}$$

For the complex modes (k_d and k_g) where B is the imaginary part,

$$|D(\omega)|^2 - |E(\omega)|^2 = \pm\frac{2B}{n} \tag{2.22}$$

Now that the modes and the normalization constants are known, the fields can be written down.

Matching conditions: As ϕ and φ satisfy a second order differential equation, there are two matching conditions for each of them. The fields are written as:

$$\phi^r = e^{-\imath\omega t}[A^r_v D^r_v e^{\imath k^r_v x} + A^r_u D^r_u e^{\imath k^r_u x} + A^r_d D^r_d e^{\imath k^r_d x} + A^r_g D^r_g e^{\imath k^r_g x}] \quad (2.23)$$

$$\phi^l = e^{-\imath\omega t}[A^l_v D^l_v e^{\imath k^l_v x} + A^l_u D^l_u e^{\imath k^l_u x} + A^l_g D^l_g e^{\imath k^l_g x} + A^l_d D^l_d e^{\imath k^l_d x}] \quad (2.24)$$

The equations for the field φ are similar, with D replaced by E in the above equations. Here, the A's are the amplitudes for the modes. ω is a constant throughout the scattering, as it is the Killing frequency, and hence is conserved.

The first matching condition is that $\phi^r = \phi^l$. This gives,

$$A^l_v D^l_v + A^l_u D^l_u + A^l_d D^l_d + A^l_g D^l_g = A^r_v D^r_v + A^r_u D^r_u + A^r_d D^r_d + A^r_g D^r_g \quad (2.25)$$

The matching of first spatial derivatives, $\phi'^l = \phi'^r$ gives,

$$A^l_v D^l_v k^l_v + A^l_u D^l_u k^l_u + A^l_d D^l_d k^l_d + A^l_g D^l_g k^l_g = A^r_v D^r_v k^r_v + A^r_u D^r_u k^r_u + A^r_d D^r_d k^r_d + A^r_g D^r_g k^r_g \quad (2.26)$$

We get two other equations by matching φ and it's first spatial derivative. Writing this in matrix form, we have,

$$\begin{pmatrix} D^l_v & D^l_u & D^l_d & D^l_g \\ D^l_v k^l_v & D^l_u k^l_u & D^l_d k^l_d & D^l_g k^l_g \\ E^l_v & E^l_u & E^l_d & E^l_g \\ E^l_v k^l_v & E^l_u k^l_u & E^l_d k^l_d & E^l_g k^l_g \end{pmatrix} \begin{pmatrix} A^l_v \\ A^l_u \\ A^l_d \\ A^l_g \end{pmatrix} = \begin{pmatrix} D^r_v & D^r_u & D^r_d & D^r_g \\ D^r_v k^r_v & D^r_u k^r_u & D^r_d k^r_d & D^r_g k^r_g \\ E^r_v & E^r_u & E^r_d & E^r_g \\ E^r_v k^r_v & E^r_u k^r_u & E^r_d k^r_d & E^r_g k^r_g \end{pmatrix} \begin{pmatrix} A^r_v \\ A^r_u \\ A^r_d \\ A^r_g \end{pmatrix}$$

Probability Currents: The BdG equation, or equivalently the coupled equations 2.6 and 2.7 can be used to obtain the probability current. From equation 2.6 we have,

$$\partial_t \rho_\phi + \frac{\xi c}{2i}\partial_x(\phi^*\partial_x\phi - \phi\partial_x\phi^*) = \frac{c}{\xi i}(\phi^*\varphi - \phi\varphi^*) \quad (2.27)$$

From equation 2.7 we have,

$$\partial_t \rho_\varphi - \frac{\xi c}{2i}\partial_x(\varphi^*\partial_x\varphi - \varphi\partial_x\varphi^*) = -\frac{c}{\xi i}(\varphi^*\phi - \phi^*\varphi) \quad (2.28)$$

Adding the two expressions gives,

$$\partial_t(\rho_\phi + \rho_\varphi) + \frac{\xi c}{2i}\partial_x(\phi^*\partial_x\phi - \phi\partial_x\phi^*) - \frac{\xi c}{2i}\partial_x(\varphi^*\partial_x\varphi - \varphi\partial_x\varphi^*) = -2\frac{c}{\xi i}(\varphi^*\phi - \phi^*\varphi) \quad (2.29)$$

From the above expression it is evident that there is no source term for real modes, whereas there exists a source term in case of complex modes.

Complex modes and the source term: We see that the source term and the contribution from the complex modes should equal one another, irrespective of the amplitudes. Hence, when looking at the conservation of probability currents in the scattering configurations, we need to check only for real modes. When the modes are complex we have,

$$\frac{\xi c}{2i}[(ik))^2|D|^2 - (-ik^*)^2|D|^2 - (ik)^2|E|^2 + (-ik^*)^2|E|^2] = \frac{2c}{\xi i}(D^*E - E^*D) \tag{2.30}$$

If k=a+ib, then we have

$$\frac{\xi c}{2i}[(|D|^2 - |E|^2)((a - ib)^2 - (a + ib)^2)] = \frac{2c}{\xi i}(D^*E - E^*D) \tag{2.31}$$

giving

$$-2ab\xi c(|D|^2 - |E|^2) = \frac{2c}{\xi i}(D^*E - E^*D) \tag{2.32}$$

As the expressions for D and E are cumbersome analytically, the equality of this equation has been checked numerically for a range of ω less than ω_{max}, and are seen to not obey the equality. The results are attached in the appendix.

Determining amplitudes: For a subsonic - subsonic configuration, there are two incoming modes, and two out-going modes, and hence four possible configurations. We try to determine the scattering of incoming modes, in an attempt to write them in the out basis (the same procedure holds if one wants to write the out-going modes in the in basis). The matrix equation being used to solve for amplitudes is a system of four equations. Considering one incoming mode at a time implies there are three other real modes and four complex modes, whose amplitudes are not determined. However, in each sector, the amplitude of the unphysical growing mode is set to zero. This leaves five amplitudes to be determined with four equations. However, without loss of generality, the incoming amplitude can be set to unity. With this normalization, there are four equations, with four unknown amplitudes, thus making the system unambiguous.

2.3 Subsonic - Subsonic Configuration

In this configuration there will be two real roots and two complex roots. As each region is subsonic, the nature of the modes in both regions is the same. However, the decaying mode in the right becomes the growing mode on the left and vice versa. In the following configurations, A_u^r is the reflected amplitude, and A_v^l is the transmitted amplitude.

- **Mode $u_\omega^{v,in}$:** The incoming mode is k_v^r. This implies, $|A_v^r| = 1$, and $|A_u^l| = 0$.

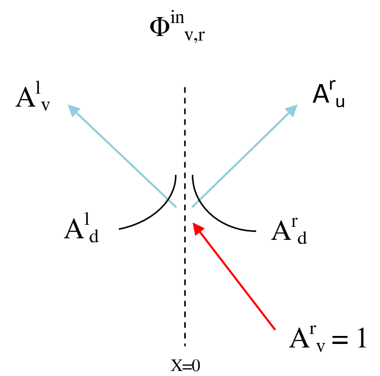


Fig. 2.4: Configuration 1

The matrix equation for this configuration becomes

$$\begin{pmatrix} A_v^l \\ 0 \\ A_d^l \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ A_u^r \\ A_d^r \\ 0 \end{pmatrix} \tag{2.33}$$

Probability Currents: Using equation 2.29, the probability current for the incoming mode is $\xi c(|D_v^r|^2 - |E_v^r|^2)k_v^r)$. The reflected and the transmitted currents give the conservation equation:

$$\xi c(|D_v^r|^2 - |E_v^r|^2)k_v^r) = \xi c|A_u^r|^2(|D_u^r|^2 - |E_u^r|^2)k_u^r + \xi c|A_v^l|^2|(D_v^l|^2 - |E_v^l|^2)k_v^l$$

Probability current conservation equation is checked numerically, and it is seen that the currents are not conserved. The results are provided in Appendix A.

• **Mode $u_\omega^{u,in}$:** The incoming mode is k_u^l. This implies, $|A_v^r| = 0$, and $|A_u^l| = 1$.

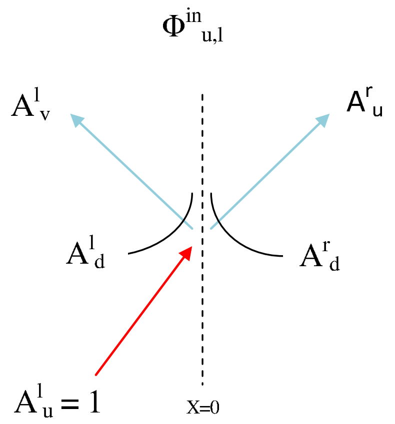


Fig. 2.5: Configuration 2

The matrix equation for this configuration becomes

$$\begin{pmatrix} A_v^l \\ 1 \\ A_d^l \\ 0 \end{pmatrix} = M \begin{pmatrix} 0 \\ A_u^r \\ A_d^r \\ 0 \end{pmatrix} \tag{2.34}$$

Probability Currents: Using equation 2.29, the probability current for the incoming mode is (for the field ϕ) is $\xi c(|D_u^l|^2 - |E_u^l|^2)k_u^l$. The reflected and the transmitted currents give the conservation equation:

$$\xi c(|D_u^l|^2 - |E_u^l|^2)k_u^l = \xi c|A_u^r|^2(|D_u^r|^2 - |E_u^r|^2)k_u^r + \xi c|A_v^l|^2(|D_v^l|^2 - |E_v^l|^2)k_v^l$$

Probability current conservation equation is checked numerically, and it is seen that the currents are not conserved. The results are provided in Appendix A.

2.4 Subsonic - Supersonic Configuration

In this configuration, there will be two real modes from the hydrodynamic approximation, and two complex roots from the small ω approximation in the subsonic region. In the supersonic region, there are four real modes.

$$\phi^r = e^{-\imath\omega t}[A_v^r D_v^r e^{\imath k_v^r x} + A_u^r D_u^r e^{\imath k_u^r x} + A_d^r D_d^r e^{\imath k_d^r x} + A_g^r D_g^r e^{\imath k_g^r x}] \quad (2.35)$$

$$\phi^l = e^{-\imath\omega t}[A_v^l D_v^l e^{\imath k_v^l x} + A_u^l D_u^l e^{\imath k_u^l x} + A_3^l D_3^l e^{\imath k_3^l x} + A_4^l D_4^l e^{\imath k_4^l x}] \quad (2.36)$$

The equations for the field φ are similar, with D replaced by E in the above equations.

The first matching condition is that $\phi^r = \phi^l$. This gives,

$$A_v^l D_v^l + A_u^l D_u^l + A_3^l D_3^l + A_4^l D_4^l = A_v^r D_v^r + A_u^r D_u^r + A_d^r D_d^r + A_g^r D_g^r \quad (2.37)$$

The matching of first spatial derivatives, $\phi'^l = \phi'^r$ gives,

$$A_v^l D_v^l k_v^l + A_u^l D_u^l k_u^l + A_3^l D_3^l k_3^l + A_4^l D_4^l k_4^l = A_v^r D_v^r k_v^r + A_u^r D_u^r k_u^r + A_d^r D_d^r k_d^r + A_g^r D_g^r k_g^r \quad (2.38)$$

We get two other equations by matching φ and it's first spatial derivative. Writing this in matrix form, we have,

$$\begin{pmatrix} D_v^l & D_u^l & D_3^l & D_4^l \\ D_v^l k_v^l & D_u^l k_u^l & D_3^l k_3^l & D_4^l k_4^l \\ E_v^l & E_u^l & E_3^l & E_4^l \\ E_v^l k_v^l & E_u^l k_u^l & E_3^l k_3^l & E_4^l k_4^l \end{pmatrix} \begin{pmatrix} A_v^l \\ A_u^l \\ A_3^l \\ A_4^l \end{pmatrix} = \begin{pmatrix} D_v^r & D_u^r & D_d^r & D_g^r \\ D_v^r k_v^r & D_u^r k_u^r & D_d^r k_d^r & D_g^r k_g^r \\ E_v^r & E_u^r & E_d^r & E_g^r \\ E_v^r k_v^r & E_u^r k_u^r & E_d^r k_d^r & E_g^r k_g^r \end{pmatrix} \begin{pmatrix} A_v^r \\ A_u^r \\ A_d^r \\ A_g^r \end{pmatrix}$$

Determining amplitudes: As there are six real modes, instead of four, we will have six configurations, three incoming and three out-going modes. As in the previous section, we consider only the scattering of the incoming modes, and hence three configurations. In each configuration, A_u^r is the reflected amplitude, with A_v^l and A_u^l being the transmitted amplitudes.

- **Mode $u_\omega^{v,in}$:** The incoming mode is k_v^r. This implies, $|A^r_{v,1}| = 1$, $|A^l_{3,1}| = 0$ and $|A^l_{4,1}| = 0$. Here, $A^r_{v,1}$ implies the amplitude of the mode k_v^r in configuration 1, and so on.

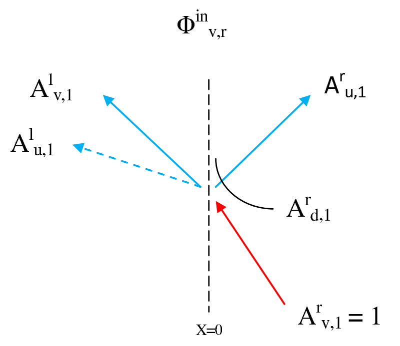


Fig. 2.6: Configuration 1

The matrix equation for this configuration becomes

$$\begin{pmatrix} A^l_{v,1} \\ A^l_{u,1} \\ 0 \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ A^r_{u,1} \\ A^r_{d,1} \\ 0 \end{pmatrix} \tag{2.39}$$

Probability Currents: Using equation 2.27, the incoming current (I), the reflected current (R), and the transmitted current (T) are calculated using equation 2.29. The probability current conservation equation is $I = R + T$.

$$I = \xi c(|D_v^r|^2 k_v^r) \qquad R = \xi c|A^r_{u,1}|^2(|D_u^r|^2 - |E_u^r|^2)k_u^r$$

$$T = \xi c|A^l_{v,1}|^2(|D_v^l|^2 - |E_v^l|^2)k_v^l - \xi c|A^l_{u,1}|^2(|D_u^l|^2 - |E_u^l|^2)k_u^l$$

Probability current conservation is checked numerically, and it is seen that the currents are not conserved. The results are provided in Appendix B.

- **Mode $u_\omega^{3,in}$:** The incoming mode is k_3^l. This implies, $|A_{3,2}^l| = 1$, $|A_{v,2}^r| = 0$ and $|A_{4,2}^l| = 0$.

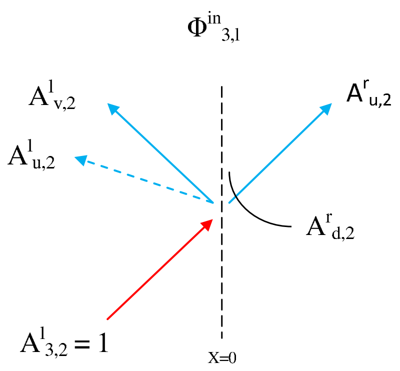


Fig. 2.7: Configuration 2

The matrix equation for this configuration becomes

$$\begin{pmatrix} A_{v,2}^l \\ A_{u,2}^l \\ 1 \\ 0 \end{pmatrix} = M \begin{pmatrix} 0 \\ A_{u,2}^r \\ A_{d,2}^r \\ 0 \end{pmatrix} \tag{2.40}$$

Probability Currents: Using equation 2.27, the incoming current (I), the reflected current (R) and the transmitted current (T) are calculated using equation 2.29. The probability current conservation relation is $I = R + T$

$$I = \xi c(|D_{3,2}^l|^2 - |E_3^l|^2)k_3^l \qquad R = \xi c|A_{u,2}^r|^2(|D_u^r|^2 - |E_u^r|^2)k_u^r$$

$$T = \xi c|A_{v,2}^l|^2(|D_v^l|^2 - |E_v^l|^2)k_v^l - \xi c|A_{u,2}^l|^2(|D_u^l|^2 - |E_u^l|^2)k_u^l$$

Probability current conservation is checked numerically, and it is seen that currents are not conserved. The results are provided in Appendix B.

• **Mode $u_\omega^{4*,in}$:** The incoming mode is k_4^l. This implies, $|A_{4,3}^l| = 1$, and $|A_{v,3}^r| = |A_{3,3}^l| = 0$.

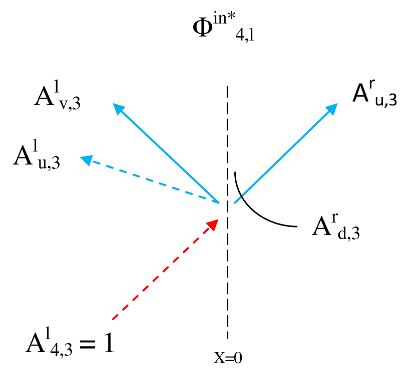


Fig. 2.8: Configuration 3

The matrix equation for this configuration becomes

$$\begin{pmatrix} A_{v,3}^l \\ A_{u,3}^l \\ 0 \\ 1 \end{pmatrix} = M \begin{pmatrix} 0 \\ A_{u,3}^r \\ A_{d,3}^r \\ 0 \end{pmatrix} \tag{2.41}$$

Probability Currents: Using equation 2.27, the incoming current (I), the reflected current (R) and the transmitted current (T) are calculated using equation 2.29. The probability current for this configuration is $I = R + T$

$$I = \xi c(|D_4^l|^2 - |E_4^l|^2)k_4^l \qquad R = \xi c|A_{u,3}^r|^2(|D_u^r|^2 - |E_u^r|^2)k_u^r$$

$$T = \xi c|A_{v,3}^l|^2(|D_v^l|^2 - |E_v^l|^2)k_v^l - \xi c|A_{u,3}^l|^2(|D_u^l|^2 - |E_u^l|^2)k_u^l$$

Probability current conservation equation is checked numerically, and it is seen that the currents are not conserved. The results are provided in the appendix. The fact that a negative norm mode is involved in this configuration is crucial for mode mixing to occur. While writing the annihilation operators in terms of the out basis it is this mode that comes with a creation operator, thereby indicating a spontaneous pair production at the horizon.

2.5 Results and Discussion

In [14], the unitarity of the amplitudes in each configuration, allowed to proceed to calculate the Bogoliubov transformation, and from there the two point correlation function. It has been shown that, in the subsonic - subsonic case, the correlation is just the repulsive interaction between the particles. As there is no black hole like configuration, no Hawking like radiation was expected to be observed.

In the same paper [14], for the case of a subsonic - supersonic configuration, Hawking like radiation was observed, and the temperature read off to be the coefficient of the $\frac{1}{\omega}$ term, as in a thermal Bose distribution.

Due to the step-discontinuous nature of the transition in the speed of sound from the $x > 0$ region to the $x < 0$ region, the surface gravity is infinite, although there is a finite temperature.

However, the same calculations, as shown in the appendix did not conserve probability currents, and when analytical calculations were performed, unitarity, as considered by [14] was not reproduced. Further,

In solving for the modes, the small ω approximation is made. While calculating the two point correlation function, there exists an integral over ω.

$$\hat{\phi} = \int_0^{\omega_{max}} d\omega[a_\omega^{v,in} u_{\omega,\phi}^{v,in} + a_\omega^{u,in} u_{\omega,\phi}^{u,in} + a_\omega^{v,in\dagger} u_{\omega,\varphi}^{v,in*} + a_\omega^{u,in\dagger} u_{\omega,\varphi}^{u,in*}] \qquad (2.42)$$

Here, u is $ANe^{-i\omega t+ikx}$, where A is the amplitude of the mode, and N is the corresponding normalization constant. This expansion is then written explicitly in terms of the in or the out basis, using the transformation obtained to further study density correlations. This integration is inconsistent with the small ω approximation.

In plotting the graphs of Fig. 2.2 and Fig. 2.3, the perturbative expressions for the modes are used. From the graph it is seen that ω_{max} corresponds to a value of 11.08. However, numerical calculations done using the exact solutions to the dispersion relation, put the value of ω_{max} closer to 8.8. This obvious discrepancy brings into question the perturbative scheme.

The complex modes are not considered in balancing probability currents. However, the Poynting vector for these modes is non-zero, which indicates that they need to be considered for this balance. Even with this inclusion, however, probability currents are not conserved, which led us to reconsider the assumptions on which this model is based.

In [13] and [14] a discontinuity in the speed of sound is considered to obtain the scattering matrix. However, this changes the Compton wavelength ($\lambda_c = \frac{\hbar}{mc}$) in the analogue system, which is a constant in the physical world. This is equivalent to looking at the same scattering process from two very different length scales, and equating the results. Hence, in the following section, which presents the modification to this model, a discontinuity in the velocity of the condensates is considered.

Chapter 3

Hawking radiation using linear dispersion relation

The basic theory of BECs, as described in Chapter 2 lead to the BdG equation for the fluctuations [1]. Writing $\hat{\phi}$ as in equation 2.5 and using the BdG gives,

$$i\hbar(\partial_t + v\partial_x)\phi_j + \frac{\hbar^2}{2m}\partial_x^2\phi_j - mc^2\phi_j = mc^2\varphi_j \tag{3.1}$$

$$-i\hbar(\partial_t + v\partial_x)\varphi_j^* + \frac{\hbar^2}{2m}\partial_x^2\varphi_j^* - mc^2\varphi_j^* = mc^2\phi_j^* \tag{3.2}$$

In the hydrodynamic limit, only those length scales which are greater than the healing length ξ are studied. Hence, ignoring the order ξ^2 term $\frac{\hbar^2}{2m}$ in equation 3.1,

$$\varphi_j = \frac{i\hbar}{mc^2}(\partial_t + v\partial_x)\phi_j - \phi_j \tag{3.3}$$

Using this in equation 3.2 and again, ignoring terms of order ξ^2 gives a trivial result. Hence, the phase-density representation for the fluctuation field, which can reproduce the d'Alembertian equation of motion in the hydrodynamic limit is used.

$$\hat{\phi} = \frac{\hat{n}^1}{2n} + i\frac{\hat{\theta}^1}{\hbar} \tag{3.4}$$

where, n^1 is the fluctuation in the density and θ^1 is the fluctuation in the phase. Further, expanding these as

$$\hat{n}^1(t,x) = \sum_j [\hat{a}_j n_j^1(t,x) + \hat{a}^\dagger n_j^{1*}(t,x)] \tag{3.5}$$

$$\hat{\theta}^1(t,x) = \sum_j [\hat{a}_j \theta_j^1(t,x) + \hat{a}^\dagger \theta_j^{1*}(t,x)]$$

[1] This section is based on [13]

and using this expansion in the BdG equation gives

$$(\partial_t + v\partial_x)n^1 + \frac{n}{m}\partial_x^2\theta^1 = 0 \tag{3.6}$$

$$(\partial_t + v\partial_x)\theta^1 + \frac{mc^2}{n}n^1 - \frac{\hbar^2}{4mn}\partial_x^2\theta^1 = 0 \tag{3.7}$$

The last term in equation 3.4 translates to $\frac{mc^2\xi^2}{4n}$. In the hydrodynamic limit, this term can be neglected, giving n^1 in terms of θ^1.

$$n^1 = -\frac{n}{mc^2}(\partial_t + v\partial_x)\theta^1 \tag{3.8}$$

This, when used in equation 3.3 gives a second order differential equation (θ^1 satisfies the Klein Gordon equation) which can be recast as

$$-\partial_t[\frac{\partial_t\theta^1}{c^2} + \frac{v}{c^2}\partial_x\theta^1] + \partial_x[(1 - \frac{v^2}{c^2})\partial_x\theta^1 - \frac{v}{c^2}\partial_t\theta^1] = 0 \tag{3.9}$$

3.1 Linear dispersion relation

The linear dispersion relation for BECs is

$$\omega - vk = \pm ck \tag{3.10}$$

Solving for k in terms of ω we have two real modes

$$k_u = \frac{\omega}{v + c} \tag{3.11}$$

$$k_v = \frac{\omega}{v - c} \tag{3.12}$$

θ^1 can be expanded in terms of these modes as

$$\theta^1 = \int_0^\infty d\omega[\hat{a}^u_\omega u^u_\omega(t,x) + \hat{a}^v_\omega u^v_\omega(t,x) + h.c] \tag{3.13}$$

Here,

$$u^u_\omega = B_u e^{-i\omega t + ik_u x} \tag{3.14}$$

$$u^v_\omega = B_v e^{-i\omega t + ik_v x} \tag{3.15}$$

B_u and B_v are normalization constants to be determined.

3.1.1 Normalization

The equal time commutator from equation 2.1, under the phase-density representation becomes,

$$[\hat{n}^1(t,x), \hat{\theta}^1(t,x')] = \delta(x - x') \tag{3.16}$$

Integration of this, gives

$$\int dx [n^1_j \theta'^{1*}_j - n^{1*}_j \theta^{1\prime}_j] = \delta_{jj'} \tag{3.17}$$

Using the definition of n^1 in terms of θ^1, and using the above normalization, we have

$$(\theta^1_j, \theta'^1_j) = i \int dx \frac{n}{mc^2} [\theta^{1*}_j (\partial_t + v\partial_x)\theta'^1_j - \theta'^1_j (\partial_t + v\partial_x)\theta^{1*}_j] = \delta_{jj'} \tag{3.18}$$

Equivalently,

$$(u_\omega, u_{\omega'}) = i \int dx \frac{n}{mc^2} [u^*_\omega (\partial_t + v\partial_x) u_{\omega'} - u_{\omega'} (\partial_t + v\partial_x) u^*_\omega] = \delta(\omega - \omega') \tag{3.19}$$

Using equations 3.11 and 3.12, we have the normalization constants

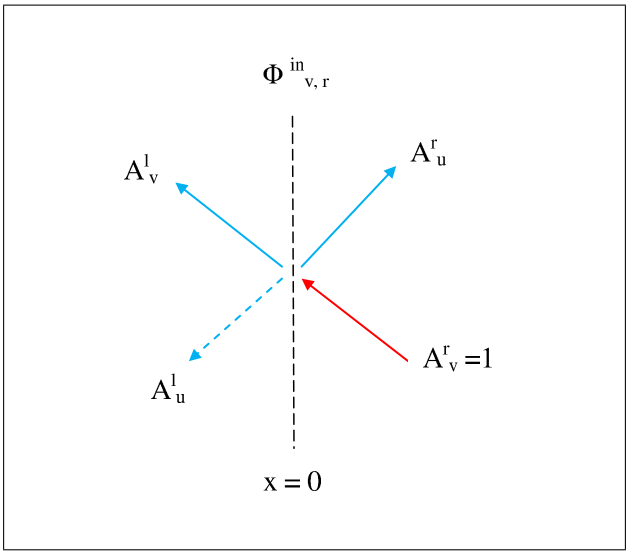
$$|B_u|^2 = |B_v|^2 = \frac{mc}{4\pi n \omega} \tag{3.20}$$

Subsonic - Supersonic configurations: As in the previous section, the nature of the modes decide the basis.

Mode	**Norm**	**Phase velocity**
k^r_v	Positive	Negative, Incoming mode
k^r_u	Positive	Positive, Out-going mode
k^l_v if $(\lvert v\rvert > c)$	Positive	Negative, Out-going mode
k^l_u if $(\lvert v\rvert > c)$	Negative	Negative, Out-going mode

As is evident from the table above, there are no incoming modes on the supersonic side. This is because, due to the non-superluminal nature of the dispersion relation, the modes in the supersonic region are dragged with the fluid. This makes it impossible to consider a black hole like scenario, as the matrix equation becomes indeterminate.

Consider the incoming mode k_v^r for scattering.



Then the matrix equation becomes

$$\begin{pmatrix} A_v^l \\ A_u^l \end{pmatrix} = M \begin{pmatrix} 1 \\ A_u^r \end{pmatrix} \tag{3.21}$$

This leaves three amplitudes unknown, with two equations, hence making the system indeterminate. Similarly, when an outgoing mode is considered, a trivial result is obtained. However, a subsonic - subsonic scenario can be studied.

3.2 Scattering matrix: c - discontinuity

The model described in the previous section, with two semi-infinite regions is considered. Here, $c(x) = c_r\Theta(x) + c_l\Theta(-x)$. Then,

$$\theta_l^1 = e^{-i\omega t}\sqrt{\frac{mc_l}{4\pi\omega n}}[A_v^l e^{ik_v^l x} + A_u^l e^{ik_u^l x}] \tag{3.22}$$

$$\theta_r^1 = e^{-i\omega t}\sqrt{\frac{mc_r}{4\pi\omega n}}[A_v^r e^{ik_v^r x} + A_u^r e^{ik_u^r x}] \tag{3.23}$$

Matching conditions are as follows:

$$\theta_l^1 - \theta_r = 0 \tag{3.24}$$

$$(1 - \frac{v^2}{c_l^2})\partial_x\theta_l^1 - \frac{v}{c_l^2}\partial_t\theta_l^1 = (1 - \frac{v^2}{c_r^2})\partial_x\theta_r^1 - \frac{v}{c_r^2}\partial_t\theta_r^1 \tag{3.25}$$

Equation 3.23 gives,

$$\sqrt{c_l}[A_v^l + A_u^l] = \sqrt{c_r}[A_v^r + A_u^r] \tag{3.26}$$

Equation 3.24 gives,

$$((1-\tfrac{v^2}{c_l^2})(ik_v^l) + \tfrac{i\omega v}{c_l^2})A_v^l B_v^l e^{-i\omega t + ik_v^l x} + ((1-\tfrac{v^2}{c_l^2})(ik_u^l) + \tfrac{i\omega v}{c_l^2})A_u^l B_u^l e^{-i\omega t + ik_u^l x}$$
$$= ((1-\tfrac{v^2}{c_r^2})(ik_v^r) + \tfrac{i\omega v}{c_r^2})A_v^r B_v^r e^{-i\omega t + ik_v^r x} + ((1-\tfrac{v^2}{c_r^2})(ik_u^r) + \tfrac{i\omega v}{c_r^2})A_u^r B_u^r e^{-i\omega t + ik_u^r x}$$

Using the expressions for the normalization constants and the modes k_v^l, k_u^l, k_v^r and k_u^r, and simplifying, we have

$$\frac{-A_v^l}{\sqrt{c_l}} + \frac{A_u^l}{\sqrt{c_l}} = \frac{-A_v^r}{\sqrt{c_r}} + \frac{A_u^r}{\sqrt{c_r}} \tag{3.27}$$

Writing as a matrix equation, $W_l A^l = W_r A^r$, inverting W_l and multiplying with W_r gives the scattering matrix, M.

$$\begin{pmatrix} A_v^l \\ A_u^l \end{pmatrix} = \frac{1}{2\sqrt{c_l c_r}} \begin{pmatrix} c_r + c_l & c_r - c_l \\ c_r - c_l & c_r + c_l \end{pmatrix} \begin{pmatrix} A_v^r \\ A_u^r \end{pmatrix} \tag{3.28}$$

Subsonic - subsonic configuration: The nature of the modes is as follows.

Mode	**Norm**	**Phase velocity**
k_v^r	Positive	Negative, Incoming mode
k_u^r	Positive	Positive, Out-going mode
k_v^l if $(\lvert v\rvert < c_l)$	Positive	Negative, Out-going mode
k_u^l if $(\lvert v\rvert < c_l)$	Positive	Positive, Incoming mode

Mode u_{in}^v:

$$\begin{pmatrix} A_v^l \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{c_l c_r}} \begin{pmatrix} c_r + c_l & c_r - c_l \\ c_r - c_l & c_r + c_l \end{pmatrix} \begin{pmatrix} 1 \\ A_u^r \end{pmatrix} \tag{3.29}$$

This implies,

$$A_u^r = -\frac{c_r - c_l}{c_r + c_l} \equiv R \tag{3.30}$$

$$A_v^l = \frac{2\sqrt{c_r c_l}}{c_r + c_l} \equiv T \tag{3.31}$$

Here, R is theamplitude of the reflected mode and T is the amplitude of the transmitted mode. As is evident, $R^2 + T^2 = 1$. Similarly,

Mode u^u_{in}:

$$\begin{pmatrix} A^l_v \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{c_l c_r}} \begin{pmatrix} c_r + c_l & c_r - c_l \\ c_r - c_l & c_r + c_l \end{pmatrix} \begin{pmatrix} 0 \\ A^r_u \end{pmatrix} \tag{3.32}$$

This implies,

$$A^r_u = \frac{c_r - c_l}{c_r + c_l} \equiv -R \tag{3.33}$$

$$A^l_v = \frac{2\sqrt{c_r c_l}}{c_r + c_l} \equiv T \tag{3.34}$$

In this configuration, the relation $R^2 + T^2 = 1$ holds. These configurations provide the transformation matrix from the in basis to the out basis, which is $\begin{pmatrix} T & R \\ -R & T \end{pmatrix}$

Probability currents The probability current for this system is derived using the BdG equation and the phase-density representation.

$$j = \frac{-\hbar}{2mi}(\phi^* \partial_x \phi - \phi \partial_x \phi^*) \tag{3.35}$$

using the definitions of ϕ and the expression for the normalization constants,

$$j = \frac{-mcv|A^2|}{4\pi n\omega}(\frac{(\omega - vk)^2}{4m^2c^4} + \frac{(\omega - vk)}{mc^2\hbar} + \frac{1}{\hbar^2}) \tag{3.36}$$

If the expressions for k and the amplitudes are used to balance the currents, it can be easily seen that this fails to be conserved for any given configuration. In [13], the correlation function has been calculated for this configuration, giving a result which encodes the repulsive interactions between atoms in the system, as is expected.

In the above configuration, the scattering matrix M was non-trivial, only because the speeds of sound in different regions were different. However, this non-trivial scattering may be an artefact of looking at the modes from different length scales. To avoid this, we need to write equation 3.9 with a dimensionless parameter. Another check is to consider a discontinuity in the velocity of the condensate and see how the scattering matrix changes.

3.3 Scattering matrix: v - discontinuity

The model described in the previous section, with two semi-infinite regions is considered, with the distinction that a discontinuity in the velocity of the condensates replacing the discontinuity in the speed of sound: $v(x) = v_r\Theta(x) + v_l\Theta(-x)$. In case of c - discontinuity, the GP equation remains unchanged in both sectors, and when $\hat{\Psi}$ is matched at the boundary, due to the constant velocity the matching condition turns out to be just the matching of the fluctuation fields. In case of discontinuity in the velocity of the condensate, though the GP equation remains unchanged, the matching of $\hat{\Psi}$ at the boundary will have an e^{ik_0x} factor. The fact that the macroscopic condensate does not experience any curvature implies that, if Hawking radiation can be looked at as a scattering problem, the matching at the boundary has to apply only for the fluctuation fields, and still preserve probability currents.

$$\theta_l^1 = e^{-i\omega t}\sqrt{\frac{mc}{4\pi\omega n}}[A_v^l e^{ik_v^l x} + A_u^l e^{ik_u^l x}] \tag{3.37}$$

$$\theta_r^1 = e^{-i\omega t}\sqrt{\frac{mc}{4\pi\omega n}}[A_v^r e^{ik_v^r x} + A_u^r e^{ik_u^r x}] \tag{3.38}$$

Matching the fields and their derivatives at $x = 0$,

$$\theta_l^1 = \theta_r^1 \tag{3.39}$$

As the normalization constants for both the fields are the same, we have

$$[A_v^l + A_u^l] = [A_v^r + A_u^r] \tag{3.40}$$

$$[(1 - \frac{v_l^2}{c^2})\partial_x\theta_l^1 - \frac{v_l}{c^2}\partial_t\theta_l^1] = [(1 - \frac{v_r^2}{c^2})\partial_x\theta_r^1 - \frac{v_r}{c^2}\partial_t\theta_r^1] \tag{3.41}$$

Simplification using the expressions for the modes gives,

$$[-\frac{A_v^l}{c} + \frac{A_u^l}{c}] = [-\frac{A_v^r}{c} + \frac{A_u^r}{c}] \tag{3.42}$$

Hence, we have

$$\begin{pmatrix} 1 & 1 \\ \frac{-1}{c} & \frac{1}{c} \end{pmatrix}\begin{pmatrix} A_v^l \\ A_u^l \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{-1}{c} & \frac{1}{c} \end{pmatrix}\begin{pmatrix} A_v^r \\ A_u^r \end{pmatrix} \tag{3.43}$$

As the matrices multiplying the vectors on either side are the same, the scattering matrix becomes identity, i.e.

$$\begin{pmatrix} A_v^l \\ A_u^l \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_v^r \\ A_u^r \end{pmatrix} \tag{3.44}$$

As shown by the calculation above, the scattering problem does not get defined when a discontinuity in the velocity of the condensate is considered. The unstated assumption underlying the calculations till now is that the modes scattered are the asymptotic modes, which retain their plane wave nature. In all the calculations above, a singular surface gravity is obtained due to the step-discontinuous nature of the change in speeds. The non-conservation of probability currents, combined with the triviality of the scattering matrix in the case above and a singular surface gravity motivates the questioning some of the basic assumptions in these models.

To avoid a singular surface gravity, we consider a continuously varying velocity profile, such that the asymptotic value is a constant. However, near the horizon, the continuously varying velocity guarantees a smooth transition into the supersonic region. The next modification to the model is to challenge the assumption that the asymptotic modes are unchanged by the horizon. To this effect, we introduce a position and velocity dependent (mode dependent) phase in the modes. The phases are such that the modes are unchanged in the asymptotic regions, however, they are no longer plane waves near the horizon. This change is consistent with the discussions in [12] and [15] about the Lorentz non-invariant dispersion relations. However, the nature of the Lorentz breaking is not pre-determined by BEC theory, but allows for the constraints due to the presence of the horizon.

In the hydrodynamic limit, it is not possible to consider a black hole like configuration, when a system like the one described in the previous section is considered. This is because, even though there is an outgoing mode on the super-sonic side, as it has no superluminal component, it gets dragged by the fluid, thereby giving an indeterminate system. In considering different phases for different modes we hope to achieve a black hole like configuration using a modified linear dispersion.

Chapter 4

A possible solution

As discussed in the previous section, a phase is introduced, such that the asymptotic wavenumbers change as they approach the horizon. The profile of velocity is tan-hyperbolic or a close approximation of it, such that the asymptotic modes are recovered at infinity. The introduction of a phase changes u_ω.

$$u_\omega = Be^{-i\omega t+ikx+i\eta(v,\omega)} \tag{4.1}$$

where η is the velocity/mode dependent phase and k are the asymptotic modes, which depend on the asymptotic velocities v_{0l} and v_{0r}. The introduction of the phase modifies the dispersion relation. η is assumed to be real such that it tends to a constant at infinity. The constraints on these phases will be examined based on the configuration being studied.

Normalization: From equation 3.16 we have,

$$(u_\omega, u_{\omega'}) = i\int dx \frac{n}{mc^2}[u^*_\omega(\partial_t + v\partial_x)u_{\omega'} - u_{\omega'}(\partial_t + v\partial_x)u^*_\omega] = \delta(\omega - \omega') \tag{4.2}$$

Using the modified form for u_ω,

$$(u_\omega, u_{\omega'}) = \frac{n}{mc^2}\int dx[B_\omega B_{\omega'} e^{-i(\omega-\omega')t+i(k_{\omega'}-k_\omega)x+i(\eta(\omega')-\eta(\omega))}(\omega+\omega'-v_0(k_{\omega'}k_\omega+\eta'_{\omega'}+\eta'_\omega))] \tag{4.3}$$

Integrating this expression for different modes, with the corresponding phases will give the normalization constant. At a fixed frequency,

$$\theta^1_l = B^l_v e^{-i\omega t+ik^l_v x+i\eta^l_v} A^l_v + B^l_u e^{-i\omega t+ik^l_u x+i\eta^l_u} \tag{4.4}$$

$$\theta^1_r = B^r_v e^{-i\omega t+ik^r_v x+i\eta^r_v} A^r_v + B^r_u e^{-i\omega t+ik^r_u x+i\eta^r_u} \tag{4.5}$$

Matching condittions: The ansatz for $\hat{\phi}$ is, as in the previous section, the phase density representation. As this obeys a second order equation, the matching conditions are similar to equations 3.23 and 3.24. The presence of the position dependent modes gives an extra factor in the condition 3.24. The first matching condition is that $\theta_l^1 = \theta_r^1$. This gives

$$A_v^l B_v^l e^{i\eta_v^l} + A_u^l B_u^l e^{i\eta_u^l} = A_v^r B_v^r e^{i\eta_v^r} + A_u^r B_u^r e^{i\eta_u^r}$$

Matching the first spatial-derivatives gives

$$A_v^l B_v^l [(1 - \tfrac{v_{0l}^2}{c^2})(\eta_v'^l) - \tfrac{\omega}{c}] e^{i\eta_v^l} + A_u^l B_u^l [(1 - \tfrac{v_{0l}^2}{c^2})(\eta_u'^l)] e^{i\eta_u^l} =$$
$$A_v^r B_v^r [(1 - \tfrac{v_{0r}^2}{c^2})(\eta_v'^r) - \tfrac{\omega}{c}] e^{i\eta_v^r} + A_u^r B_u^r [(1 - \tfrac{v_{0r}^2}{c^2})(\eta_u'^r)] e^{i\eta_u^r}$$

Here, η' is the derivative with respect to the position x. Writing the above two conditions in matrix form, we have

$$W_l \begin{pmatrix} A_v^l \\ A_u^l \end{pmatrix} = W_r \begin{pmatrix} A_v^r \\ A_u^r \end{pmatrix} \tag{4.6}$$

Inverting W_l and multiplying with W_r gives the matching matrix M

$$\begin{pmatrix} A_v^l \\ A_u^l \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A_v^r \\ A_u^r \end{pmatrix} \tag{4.7}$$

where,

$$A = \frac{B_v^r(2\omega c + (c^2 - v_{0l}^2)\eta_u'^l - (c^2 - v_{0r}^2)\eta_v'^r)}{B_v^l(2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l))} e^{i(\eta_v^r - \eta_v^l)}$$

$$B = \frac{B_u^r((c^2 - v_{0l}^2)\eta_u'^l - (c^2 - v_{0r}^2)\eta_u'^r)}{B_v^l(2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l))} e^{i(\eta_u^r - \eta_v^l)}$$

$$C = \frac{B_v^r((c^2 - v_{0r}^2)\eta_v'^r - (c^2 - v_{0l}^2)\eta_v'^l)}{B_u^l(2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l))} e^{i(\eta_v^r - \eta_u^l)}$$

$$D = \frac{B_u^r(2\omega c + (c^2 - v_{0r}^2)\eta_u'^r - (c^2 - v_{0l}^2)\eta_v'^l)}{B_u^l(2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l))} e^{i(\eta_u^r - {}^l_u)}$$

Probability currents: The currents are derived using the time evolution equation. The currents for this model will have an additional contribution from the phases, as compared to equation 3.36. The introduction of phase implies that the amplitudes of the modes, in general depend on the position x. However, as a special case, we consider constant amplitudes, to see if a black hole like scenario exists. Once a particular profile for velocity is considered, η is fixed automatically, under the condition that it must tend to constant asymptotically. Given this, the amplitudes should turn out such that there is probability current conservation.

$$j = |B|^2(v + \xi ck + \eta')|A|^2[\frac{(\omega - vk + v\eta')^2}{4m^2c^4} + \frac{(\omega - vk + v\eta')}{mc^2\hbar} + \frac{1}{\hbar^2}] \quad (4.8)$$

4.0.1 Constraints on the phases

Now that a scattering matrix has been obtained, there needs to be a check on whether the phases can vary in a way to form a black-hole like configuration, without any inconsistencies. Such a configuration requires at least one negative norm mode, and two incoming and two out-going modes. Once a velocity profile is fixed, phases can be solved for, using equations 3.1 and 3.2. The constraints on the phases of different modes to obtain a black hole configuration is listed here.

Subsonic - subsonic configuration

Mode	Norm	Group velocity
k_v^r	Positive, $\eta_v'^r < -\frac{\omega c}{v_{0r}(v_{0r}-c)}$	Negative
k_u^r	Positive, $\eta_u'^r < \left|-\frac{\omega c}{v_{0r}(v_{0r}+c)}\right|$	Positive
k_v^l	Negative, $\eta_v'^l < -\frac{\omega c}{v_{0l}(v_{0l}-c)}$	Negative, $\frac{d\eta_v^l}{d\omega} < 0$ or $\left|\frac{d\eta_v^l}{d\omega}\right| < \frac{1}{v_{0l}-c}$
k_u^l	Positive, $\eta_u'^l < \left|-\frac{\omega c}{v_{0l}(v_{0l}+c)}\right|$	Positive, $\frac{d\eta_u^l}{d\omega} > \left|\frac{1}{v_{0l}+c}\right|$

Subsonic - supersonic configuration

Mode	Norm	Group velocity
k_v^r	Positive, $\eta_v'^r < -\frac{\omega c}{v_{0r}(v_{0r}-c)}$	Negative
k_u^r	Positive, $\eta_u'^r < \left|-\frac{\omega c}{v_{0r}(v_{0r}+c)}\right|$	Positive
k_v^l	Negative, $\eta_v'^l > \left|-\frac{\omega c}{v_{0l}(v_{0l}-c)}\right|$	Negative, $\frac{d\eta_v^l}{d\omega} < 0$ or $\left|\frac{d\eta_v^l}{d\omega}\right| < \frac{1}{v_{0l}-c}$
k_u^l	Positive, $\left|\eta_u'^l\right| > \frac{\omega c}{v_{0l}(v_{0l}+c)}$	Positive, $\frac{d\eta_u^l}{d\omega} > \left|\frac{1}{v_{0l}+c}\right|$

4.1 Constructing the basis

Configuration 1: k_v^r is the incoming mode. This implies, $|A_v^r| = 1$ and $|A_u^l| = 0$. The matrix equation becomes

$$\begin{pmatrix} A_{v1}^l \\ 0 \end{pmatrix} = M \begin{pmatrix} 1 \\ A_{u1}^r \end{pmatrix}$$

$$A_{v1}^l = \frac{B_v^r}{B_v^l}[\frac{2\omega c + (c^2 - v_{0l}^2)\eta_u'^l - (c^2 - v_{0r}^2)\eta_v'^r}{2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l)}$$

$$+\frac{(c^2 - v_{0l}^2)\eta_u'^l - (c^2 - v_{0r}^2)\eta_u'^r}{2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l)} \frac{(c^2 - v_{0r}^2)\eta_v'^r - (c^2 - v_{0l}^2)\eta_v'^l}{2\omega c + (c^2 - v_{0r}^2)\eta_u'^r - (c^2 - v_{0l}^2)\eta_v'^l}]e^{i(\eta_v^r - \eta_v^l)}$$

$$A_{u1}^r = \frac{B_v^r}{B_u^r} \frac{(c^2 - v_{0r}^2)\eta_v'^r - (c^2 - v_{0l}^2)\eta_v'^l}{2\omega c + (c^2 - v_{0r}^2)\eta_u'^r - (c^2 - v_{0l}^2)\eta_v'^l} e^{i(\eta_v^r - \eta_u^r)}$$

The structure of the amplitudes points to the fact that the ratio of the reflected current (or the transmitted current) to the incoming current should be independent of the normalization constants.

Configuration 2: k_u^l is the incoming mode. This implies, $|A_v^r| = 0$ and $|A_u^l| = 1$. The matrix equation becomes

$$\begin{pmatrix} A_{v2}^l \\ 1 \end{pmatrix} = M \begin{pmatrix} 0 \\ A_{u2}^r \end{pmatrix}$$

$$A_{u2}^r = \frac{B_u^l}{B_u^r} \frac{2\omega c + (c^2 - v_{0l}^2)(\eta_u'^l - \eta_v'^l)}{2\omega c + (c^2 - v_{0r}^2)\eta_u'^r - (c^2 - v_{0l}^2)\eta_v'^l} e^{i(\eta_u^l - \eta_u^r)}$$

$$A_{v2}^l = \frac{B_u^l}{B_v^l} \frac{(c^2 - v_{0l}^2)\eta_u'^l - (c^2 - v_{0r}^2)\eta_u'^r}{2\omega c + (c^2 - v_{0r}^2)\eta_u'^r - (c^2 - v_{0l}^2)\eta_v'^l} e^{i(\eta_u^l - \eta_v^l)}$$

Probability current conservation is checked using equation 4.8 appropriately. On adopting a form for $v(x)$, or equivalently $\eta(x, k)$, the constraints and the current conservation can be explicitly checked.

The transformation matrix from one basis to the other will be in terms of the amplitudes calculated. Probability current conservation will ensure that this transformation is unitary. Using these relations, the it should be checked that the conditions for the phase for each mode to obtain a black-hole like configuration is not inconsistent with the solution obtained for the phases by fixing the velocity profile. If, in the scenario described above there is no condition under which a black hole like scenario can exist, then the more general case of amplitudes with position dependence have to be considered. As the change in velocity translates into the change in the acoustic metric (eqn. 1.3), the change in the volume element as the modes approach the horizon should be off set by the amplitudes to maintain a constant probability current. Then, if mode mixing occurs, it is an indication of Hawking radiation, and the temperature can be obtained by using density correlations. As the transition from the subsonic to the supersonic region is continuous in the model described in this section, a singular surface gravity will be avoided.

Chapter 5

Discussions and future directions

Analogue gravity mainly studies the kinematics of gravitational systems, one of which is the phenomenon of Hawking radiation. The models used to study this, as described in chapters 2 and 3 stop short of mimicking Hawking radiation in analogue systems. The most important deviation from a relativistic scenario in these models is the singular surface gravity for a black hole with finite temperature. However, this spurious result can be done away with, using the model described in chapter 4.

Although analogue models are yet to establish Hawking-like radiation, the possibility of avoiding the trans-Planckian issue in these models is lucrative. Jacobson, in his paper [12] has addressed the issue via the process of mode conversion. This is a process which does not assume a trans-Planckian reservoir just outside the event horizon, but provides a mechanism for the existence of out-going modes. In this paper, the question of Lorentz non-invariance is also addressed. The argument is that to have locality and Lorentz invariance one must conclude that there are infinitely many degrees of freedom in any given volume, irrespective of how small that volume may be. To avoid this, one way out is to have a Lorentz breaking term in the dispersion relation. The process of mode conversion is achieved with this assumption. In the model described in chapter 4, although the dispersion relation is Lorentz invariant asymptotically, the modification to the modes due to the horizon may give rise to a Lorentz breaking term. In this case, the results of [12] can be translated to this model.

If successful, this model can establish Hawking radiation, but only at the kinematic level. The effect of back-reaction has not been incorporated into this model, the dynamics of which could very well alter the physics of the phenomenon. A particularly interesting work on the effect of back-reaction on Bogoliubov co-efficients is by 't Hooft, [16] where it is shown that in relativistic black holes, when an incoming state is changed, the out-going state changes by a phase factor.

Further, even if Hawking radiation were established in analogue models, the dynamics of these systems are different from the dynamics of relativistic systems. Analogues of black holes are created at a specific time, and hence cannot reproduce eternal black hole configurations. The event horizons are highly susceptible to the creation process - so much so that an inappropriate process might eliminate the phenomenon of Hawking radiation. In studying Hawking radiation in analogue systems with modified dispersion relations, there are three main assumptions [4]:

(i) The preferred frame selected by the Lorentz non-invariance should be that of the freely falling observer, and not that of the observer at infinity.

(ii) The excitations must start off in a ground state with respect to the freely falling observer.

(iii) The excitations must evolve in an adiabatic way, meaning that the Planckian dynamics must be faster than the sub-Planckian dynamics.

It is possible that one or more of the above assumptions do not extend to physical black holes. In addition, the calculations are based on semi-classical theories, whose validity is questionable. Despite these assumptions, analogue models offer a tractable way of studying systems that are otherwise not easily accessible. This, and the fact that relativistic techniques might lead to insights into condensed matter physics makes this an area of great interest.

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Appendix A

Subsonic - subsonic configurations

The results of the subsonic-subsonic case, which has two configurations are presented. The results were obtained using Mathematica - 9. The configuration has a discontinuity in the speed of sound. The values used are: $\hbar = 1$, $v = -1$, $c_r = -5v/3$, $c_l = -2v$, $m = 20$, $\xi_r = \frac{1}{c_r m}$, $\xi_l = \frac{1}{c_l m}$. The values of ω used are of the same order as ω_{max}. Configuration 1 has k_v^r as the incoming mode and Configuration 2 has k_u^l as the incoming mode. $A_{u,n}^r$ refers to the amplitude of the mode k_u^r in configuration n. G and F are respectively the normalization constants of the growing modes for fields ϕ and φ.

$\omega = 0.2$

Modes	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -0.075$	-0.375	$\frac{3.64}{\sqrt{n}}$	$\frac{-3.64}{\sqrt{n}}$
$k_u^r = 0.33$	1.49	$\frac{3.43}{\sqrt{n}}$	$\frac{-3.39}{\sqrt{n}}$
$k_d^r = -0.11 + i53.34$	-	$-\frac{92.51+i27}{\sqrt{n}}$	$-\frac{0.01+i96.93}{\sqrt{n}}$
$k_v^l = -0.067$	-0.33	$\frac{3.95}{\sqrt{n}}$	$\frac{-3.94}{\sqrt{n}}$
$k_u^l = 0.12$	0.99	$\frac{5.92}{\sqrt{n}}$	$\frac{-5.91}{\sqrt{n}}$
$k_d^l = -0.067 - i69.28$	-	$\frac{125.82`-i72.64}{\sqrt{n}}$	$\frac{-0.00657+i145.758}{\sqrt{n}}$

$$\mathrm{A}_{u,1}^r = \frac{(-0.21+i0.0006)F-(0.21+i0.031)G}{F+(0.98+i0.15)G}$$

$$\mathrm{A}_{v,1}^l = \frac{((1.2-i0.077)F^4+(0.87-i1.66)F^3G-(1.6+i2.68)F^2G^2-(1.88-i0.007)FG^3-(0.64-i1.03)G^4)}{((1.2-i0.077)F^4+(0.87-1.66)F^3G-(1.6+i2.68)F^2G^2-(1.88-i0.007)FG^3-(0.64-i1.03)G^4)}$$

$$\mathrm{A}_{u,2}^r = \frac{(1.65-i0.002)F+(1.63+i0.24)G}{F+(0.98+i0.148)G}$$

$$\mathrm{A}_{v,2}^l = \frac{(-0.065-i0.0008)F-(0.064+i0.01)G}{F+(0.98+i0.15)G}$$

$\omega = 3$

Modes	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -1.12$	-0.375	$\frac{0.96}{\sqrt{n}}$	$\frac{-0.93}{\sqrt{n}}$
$k_u^r = 4.47$	1.47	$\frac{0.99}{\sqrt{n}}$	$\frac{-0.87}{\sqrt{n}}$
$k_d^r = -1.67 + i53.46$	-	$-\frac{-23.21+i6.87}{\sqrt{n}}$	$-\frac{-0.0015+i26.32}{\sqrt{n}}$
$k_v^l = -0.99$	-0.33	$\frac{1.05}{\sqrt{n}}$	$\frac{-1.02}{\sqrt{n}}$
$k_u^l = 2.99$	0.99	$\frac{1.06}{\sqrt{n}}$	$\frac{-0.98}{\sqrt{n}}$
$k_d^l = -0.99 - 69.32$	-	$\frac{31.57-i18.28}{\sqrt{n}}$	$\frac{0.00001-i38.34}{\sqrt{n}}$

$$\mathrm{A}_{u,1}^r = \frac{(-0.21+i0.0007)F-(0.21+i0.031)G}{F+(0.98+i0.15)G}$$

$$\mathrm{A}_{v,1}^l = \frac{((1.2-i0.08)F^4+(0.87-i1.66)F^3G-(1.6+i2.68)F^2G^2-(1.88-i0.007)FG^3-(0.64-i1.03)G^4)}{(F^4+(0.72-i1.2)F^3G-(1.27+i2.12)F^2G^2-(1.43-i0.06)FG^3-(0.48-i0.88)G^4)}$$

$$\mathrm{A}_{u,2}^r = \frac{(1.65-i0.002)F+(1.63+i0.24)G}{F+(0.98+i0.15)G}$$

$$\mathrm{A}_{v,2}^l = \frac{(-0.065-i0.0008)F-(0.06+i0.01)G}{F+(0.98+i0.15)G}$$

$\omega = 5$

Modes	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -1.87$	-0.37	$\frac{0.74}{\sqrt{n}}$	$\frac{-0.7}{\sqrt{n}}$
$k_u^r = 7.38$	1.43	$\frac{0.79}{\sqrt{n}}$	$\frac{-0.64}{\sqrt{n}}$
$k_d^r = -2.75 + i53.67$	-	$-\frac{17.56+5.33I}{\sqrt{n}}$	$\frac{0.0009-21.07I}{\sqrt{n}}$
$k_v^l = -1.66$	-0.33	$\frac{0.81}{\sqrt{n}}$	$\frac{-0.78}{\sqrt{n}}$
$k_u^l = 4.98$	0.99	$\frac{0.84}{\sqrt{n}}$	$\frac{-0.74}{\sqrt{n}}$
$k_d^l = -1.65 - i69.4$	-	$\frac{23.99-i13.95}{\sqrt{n}}$	$\frac{0.0018-i30.16}{\sqrt{n}}$

$$\mathrm{A}_{u,1}^r = \frac{(0.099-i0.004)F+(0.07-i0.05)G}{F+(0.8-i0.46)G}$$

$$\mathrm{A}_{v,1}^l = \frac{((0.8+i0.05)F^4+(0.21-i1.92)F^3G-(2.62+i0.58)F^2G^2-(0.54-i1.87)FG^3+(0.73+i0.42)G^4)}{(F^4+(0.15-i2.5)F^3G-(3.46+i0.72)F^2G^2-(0.79-i2.44)FG^3+(0.88+i0.57)G^4)}$$

$$\mathrm{A}_{u,2}^r = \frac{(0.95-i0.006)F+(0.76-i0.45)G}{F+(0.81-i0.46)G}$$

$$\mathrm{A}_{v,2}^l = \frac{(-0.13-i0.002)F-(0.1-i0.06)G}{F+(0.8-i0.46)G}$$

$\omega = 8$

Modes	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -2.99$	-0.37	$\frac{0.6}{\sqrt{n}}$	$\frac{-0.54}{\sqrt{n}}$
$k_u^r = 11.56$	1.35	$\frac{0.66}{\sqrt{n}}$	$\frac{-0.46}{\sqrt{n}}$
$k_d^r = -4.28 + i54.17$	-	$-\frac{14.47+3.72I}{\sqrt{n}}$	$\frac{0.22-18.21I}{\sqrt{n}}$
$k_v^l = -2.66$	-0.33	$\frac{0.65}{\sqrt{n}}$	$\frac{-0.61}{\sqrt{n}}$
$k_u^l = 7.92$	0.97	$\frac{0.68}{\sqrt{n}}$	$\frac{-0.56}{\sqrt{n}}$
$k_d^l = -2.63 - i69.58$	-	$\frac{18.5-i10.86}{\sqrt{n}}$	$-\frac{0.0017+i24.5}{\sqrt{n}}$

$$\mathrm{A}_{u,1}^r = \frac{((0.14-i0.008)F+(0.1-i0.06)G)}{(F+(0.79-0.41)G)}$$

$$\mathrm{A}_{v,1}^l = \frac{((0.76+i0.06)F^4+(0.27-i1.93)F^3G-(2.57+i0.87)F^2G^2-(0.91-i1.78)FG^3+(0.55+i0.61)G^4)}{(F^4+(0.22-i2.66)F^3G-(3.61+i1.14)F^2G^2-(1.34-i2.44)FG^3+(0.69+i0.84)G^4)}$$

$$\mathrm{A}_{u,2}^r = \frac{(1.04-i0.02)F+(0.82-i0.45)G}{F+(0.79-i0.41)G}$$

$$\mathrm{A}_{v,2}^l = \frac{(-0.032-i0.005)F-(0.03-i0.009)G}{F+(0.79-i0.41)G}$$

$\omega = 12$

Modes	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -0.075$	-0.375	$\frac{3.64}{\sqrt{n}}$	$\frac{-3.64}{\sqrt{n}}$
$k_u^r = 0.33$	1.49	$\frac{3.43}{\sqrt{n}}$	$\frac{-3.39}{\sqrt{n}}$
$k_d^r = -0.11 + i53.34$	-	$-\frac{92.51+i27}{\sqrt{n}}$	$-\frac{0.01+i96.93}{\sqrt{n}}$
$k_v^l = -0.067$	-0.33	$\frac{3.95}{\sqrt{n}}$	$\frac{-3.94}{\sqrt{n}}$
$k_u^l = 0.12$	0.99	$^{5}_{.}92\sqrt{n}$	$\frac{-5.91}{\sqrt{n}}$
$k_d^l = -0.067 - i69.28$	-	$\frac{125.82`-i72.64}{\sqrt{n}}$	$\frac{-0.00657+i145.758}{\sqrt{n}}$

$$\mathrm{A}_{u,1}^r = \frac{((-0.21+0.0007)F-(0.21+i0.031)G)}{(F+(0.98+i0.15)G)}$$

$$\mathrm{A}_{v,1}^l = \frac{(1.2-i0.07)F^4+(0.87-i1.66)F^3G-(1.6+i2.68)F^2G^2-(1.88-i0.007)FG^3-(0.64-i1.03)G^4}{F^4+(0.72-i1.2)F^3G-(1.27+i2.12)F^2G^2-(1.43-i0.06)FG^3-(0.48-i0.88)G^4}$$

$$\mathrm{A}_{u,2}^r = \frac{(1.65-i0.002)F+(1.63+i0.24)G}{F+(0.98+i0.15)G}$$

$$\mathrm{A}_{v,2}^l = \frac{(-0.065-0.0008)F-(0.064+i0.01)G}{F+(0.98+i0.15)G}$$

Source terms for complex modes: As shown in chapter 2, there is a source term for complex modes. As the modes do not change with configuration, and the relation 2.31 is independent of amplitudes, this analysis holds for both the configurations above, as well as the subsonic case for the configurations in the next chapter. The two sides of the equation 2.31 are evaluated for different values of ω and presented here.

Right sector

ω	k_d^r	$-2ab\xi c(\|D\|^2 - \|E\|^2)$	$\frac{2c}{\xi i}(D^*E - E^*D)$
0.2	-0.11 + i53.34	$\frac{62.59}{\sqrt{n}}$	$\frac{1.99*10^6}{\sqrt{n}}$
3	-1.67 + i53.46	$\frac{62.59}{\sqrt{n}}$	$\frac{1.99*10^6}{\sqrt{n}}$
5	-2.75 + i53.67	$-\frac{1574.52}{\sqrt{n}}$	$\frac{82220}{\sqrt{n}}$
8	-4.28 + i54.17	$\frac{2473.35}{\sqrt{n}}$	$\frac{58737.1}{\sqrt{n}}$
12	-6.1 + i55	$\frac{62.59}{\sqrt{n}}$	$\frac{1.99*10^6}{\sqrt{n}}$

Left sector

ω	k_d^l	$-2a\xi c(\|D\|^2 - \|E\|^2)$	$\frac{2c}{\xi i}(D^*E - E^*D)$
0.2	-0.067 - i69.28	$\frac{-64.31}{\sqrt{n}}$	$\frac{-5.86*10^6}{\sqrt{n}}$
3	-0.99 - i69.32	$\frac{-64.31}{\sqrt{n}}$	$\frac{-5.86*10^6}{\sqrt{n}}$
5	-1.65 - i69.4	$-\frac{1589.4}{\sqrt{n}}$	$-\frac{231524}{\sqrt{n}}$
8	-2.63 - i69.58	$-\frac{2546.56}{\sqrt{n}}$	$\frac{-145046}{\sqrt{n}}$
12	-3.87 - i69.94	$\frac{-69.28}{\sqrt{n}}$	$\frac{-64.3162}{\sqrt{n}}$

The amplitudes were obtained in terms of F and G, which are respectively the normalization constants of the growing modes for the fields φ and ϕ. As these are not obtained by the theory, the current conservation is not looked at for the subsonic-subsonic scenario.

Appendix B

Subsonic - supersonic configurations

The results of the subsonic-supersonic case are presented. The results were obtained using Mathematica - 9. The configuration has a discontinuity in the speed of sound. The values are obtained by solving the dispersion relation exactly. The values used are: $\hbar = 1$, $v = -1$, $c_r = -5v/3$, $c_l = -v/4$, $m = 20$, $\xi_r = \frac{1}{c_r m}$, $\xi_l = \frac{1}{c_l m}$. $\omega = 12$ is not considered, as ω_{max} is 8.8, and any value of ω greater than that will reproduce the subsonic -subsonic case.

Configuration 1: k_v^r is the incoming mode, with unit amplitude, k_u^r is the reflected mode with amplitude A_u^r, A_v^l and k_u^l are the transmitted modes with amplitudes A_v^l and A_u^l respectively.

Configuration 2: k_3^l is the incoming mode, with unit amplitude, k_u^r is the reflected mode with amplitude A_u^r, A_v^l and k_u^l are the transmitted modes with amplitudes A_v^l and A_u^l respectively.

Configuration 3: k_4^l is the incoming mode, with unit amplitude, k_u^r is the reflected mode with amplitude A_u^r, A_v^l and k_u^l are the transmitted modes with amplitudes A_v^l and A_u^l respectively.

The results showed that the amplitudes of the complex modes are small compared to the amplitudes of the real modes. In the following tables, $\Sigma|A|^2$ signifies the sum of the reflected and transmitted amplitudes, taken with the appropriate signs ($|A_u^l|^2$ will be negative), I indicates the incoming current, R, the reflected current and T the transmitted current (the expressions are as given in section 2.4). If probability currents are to be conserved, then $R + T = I$.

$\omega = 0.2$

Modes	Norm	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -0.075$	Positive	$\frac{-0.375}{\sqrt{n}}$	$\frac{3.64}{\sqrt{n}}$	$\frac{-3.64}{\sqrt{n}}$
$k_u^r = 0.33$	Positive	1.49	$\frac{3.43}{\sqrt{n}}$	$\frac{-3.39}{\sqrt{n}}$
$k_v^l = -0.15$	Negative	-0.79	$\frac{-1.69}{\sqrt{n}}$	$\frac{-1.65}{\sqrt{n}}$
$k_u^l = -0.27$	Negative	-1.33	$\frac{-1.38}{\sqrt{n}}$	$\frac{1.46}{\sqrt{n}}$
$k_3^l = 38.94$	Positive	1.054	$\frac{0.4}{\sqrt{n}}$	$\frac{-0.006}{\sqrt{n}}$
$k_4^l = -38.51$	Negative	1.07	$\frac{-0.006}{\sqrt{n}}$	$\frac{0.4}{\sqrt{n}}$

	A_u^r	A_v^l	A_u^l	$\Sigma\|A\|^2$	I	R	T
C1	-0.34+i0.004	-0.04	-1.7-i0.008	-2.8	0.0002/n	0.0004/n	-0.0007/n
C2	3.99+i2.8	-0.13 - i0.09	-9.36-i6.95	-111.7	0.32/n	0.09/n	0.38/n
C3	3.9-i2.85	-0.13+i0.09	-9.62+i6.85	-116.1	-0.33/n	0.09/n	0.39/n

$\omega = 3$

Modes	Norm	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -1.12$	Positive	-0.375	$\frac{0.96}{\sqrt{n}}$	$\frac{-0.93}{\sqrt{n}}$
$k_u^r = 4.47$	Positive	1.47	$\frac{0.99}{\sqrt{n}}$	$\frac{-0.87}{\sqrt{n}}$
$k_v^l = -2.38$	Negative	-0.78	$\frac{0.45}{\sqrt{n}}$	$\frac{-0.28}{\sqrt{n}}$
$k_u^l = -4.11$	Negative	-1.44	$\frac{-0.24}{\sqrt{n}}$	$\frac{0.53}{\sqrt{n}}$
$k_3^l = 41.69$	Positive	0.92	$\frac{0.38}{\sqrt{n}}$	$\frac{-0.005}{\sqrt{n}}$
$k_4^l = -35.19$	Negative	1.31	$\frac{-0.008}{\sqrt{n}}$	$\frac{0.45}{\sqrt{n}}$

	A_u^r	A_v^l	A_u^l	$\Sigma\|A\|^2$	I	R	T
C1	-0.25+i0.025	1.21 + i0.02	-0.67-i0.037	1.06	0.003/n	0.003/n	-0.011/n
C2	1.28+i0.82	1+i0.84	-1.34 - i1.12	0.79	0.3/n	0.12/n	0.12/n
C3	-0.9-i0.72	1.2-i0.74	-1.87 - i1.05	-1.21	-0.36/n	0.07/n	0.18/n

$\omega = 5$

Modes	Norm	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -1.87$	Positive	-0.37	$\frac{0.74}{\sqrt{n}}$	$\frac{-0.7}{\sqrt{n}}$
$k_u^r = 7.38$	Positive	1.43	$\frac{0.79}{\sqrt{n}}$	$\frac{-0.64}{\sqrt{n}}$
$k_v^l = -3.94$	Negative	-0.76	$\frac{0.39}{\sqrt{n}}$	$\frac{-0.18}{\sqrt{n}}$
$k_u^l = -7.23$	Negative	-1.7	$\frac{-0.14}{\sqrt{n}}$	$\frac{0.53}{\sqrt{n}}$
$k_3^l = 43.46$	Positive	0.85	$\frac{0.37}{\sqrt{n}}$	$\frac{-0.005}{\sqrt{n}}$
$k_4^l = -32.29$	Negative	1.62	$\frac{-0.001}{\sqrt{n}}$	$\frac{0.5}{\sqrt{n}}$

	A_u^r	A_v^l	A_u^l	$\Sigma\|A\|^2$	I	R	T
C1	-0.26 + i0.04	1.13 + i0.04	-0.61 - i0.06	0.98	-0.005/n	0.005/n	0.005/n
C2	1.08 + i0.63	0.67 + i0.61	-0.86 - i0.87	0.9	0.29/n	0.13/n	0.13/n
C3	0.58 - i0.5	0.86 - i0.48	-1.53 + i0.69	-1.2	-0.41/n	0.04/n	-0.25/n

$\omega = 8$

Modes	Norm	v_g	$D(\omega)$	$E(\omega)$
$k_v^r = -2.99$	Positive	-0.37	$\frac{0.6}{\sqrt{n}}$	$\frac{-0.54}{\sqrt{n}}$
$k_u^r = 11.56$	Positive	1.35	$\frac{0.66}{\sqrt{n}}$	$\frac{-0.46}{\sqrt{n}}$
$k_v^l = -6.18$	Negative	-0.72	$\frac{-1.69}{\sqrt{n}}$	$\frac{-1.65}{\sqrt{n}}$
$k_u^l = -14.07$	Negative	-3.55	$\frac{-0.07}{\sqrt{n}}$	$\frac{0.75}{\sqrt{n}}$
$k_3^l = 45.89$	Positive	0.77	$\frac{0.35}{\sqrt{n}}$	$\frac{-0.004}{\sqrt{n}}$
$k_4^l = -25.641$	Negative	3.5	$\frac{-0.026}{\sqrt{n}}$	$\frac{0.75}{\sqrt{n}}$

	A_u^r	A_v^l	A_u^l	$\Sigma\|A\|^2$	I	R	T
C1	-0.28 + i0.08	1.05 + i0.07	-0.36 - i0.07	-1.07 - i0.008	-0.008/n	0.01/n	0.014/n
C2	0.92 + i0.49	0.36 + i0.45	-0.33 - i0.46	1.12	0.28/n	0.14/n	0.12/n
C3	0.31 - i0.26	0.59 - i0.24	-1.19 + i0.24	-0.92	-0.71/n	0.02/n	0.58/n