Introduction to Algebraic Geometry

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Introduction to Algebraic Geometry towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Vyshnav V. at Indian Institute of Science Education and Research under the supervision of Dr. Vivek Mohan Mallick, Associate Professor, Department of Mathematics , during the academic year 2019-2020.

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Declaration

I hereby declare that the matter embodied in the report entitled Introduction to Algebraic Geometry are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Vivek Mohan Mallick and the same has not been submitted elsewhere for any other degree.

Vyshnav V.

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Abstract

This one year project is devided into two parts, first part is devoted to learning basic algebraic geometry and the second part is devoted to learning Cohomology and applying it to the settings of algebraic geometry.

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Introduction

The idea of finding geometric solutions to algebraic problems date back to greeks (A typical example is a problem of "doubling a cube") they invented several curves for the same (the most important among them is the conics). Later mathematicians also understood that without the aid of algebra, the geometric approach fails to study complex phenomenon (A typical example is the classification problem of linear transformations). Algebraic geometry explores and exploits the connection between algebraic notions (techniques) and geometric intuitions. Classical algebraic geometry grew out of the study of zeros of a system of polynomials. The subject focuses more on the algebraic and topological aspects of the set of solutions rather than specifically finding a solution. The fundamental object of classical algebraic geometry is "Algebraic varieties", which are a "geometric" manifestation of sets of zeros of polynomials. Algebraic geometry occupies a key position in modern mathematical world and it interacts with a number of fields such as complex analysis, topology and number theory. To learn more about the subject, I closely followed the book "Algebraic Geometry I" by Ulrich Gortz and Torsten Wedhorn (Most of the algebraic geometry discussed in the thesis comes from this source). The thesis can be devided into two parts, the first part deals with sheaves and schemes and in the second part we discuss some cohomology.

The first chapter deals with the two significant drawbacks of affine varieties, by introducing the notion of the spectrum of a ring. One of the major drawbacks of affine variety is that it is "coordinate dependent", that is it depends on how it is embedded in the ambient affine space. To get a "coordinate free" description of affine variety we note that there is a bijection between the collection of maximal ideals of the coordinate ring and the points of the underlying topological space. For a commutative ring A there are only a few maximal ideals, so we construct a topological space using the set of prime ideals which is denoted by "$Spec(A)$" (In some sense the topology defined on theis set mimics the Zarisky topology defined on the affine space). The other reason to use prime ideals is that it gives a contravariant functor from the category of rings to the category of topological spaces (The second drawback was the construction of affine varieties were limited to the case of an algebraically closed field, but this is already rectified by considering a general commutative ring). Given $Spec(A)$ we can not get back the ring A, so we need to define additional structures on $Spec(A)$ which will help us to get back the ring A. We will try to view elements of A as functions defined on $Spec(A)$. To make this notion precise we need a more robust construction. We observe that the key point for working with "functions" is "restricting and glueing of functions" (rather than evaluating them at points of the source). Abstrating this we get the notion of a "Sheaf". A presheaf is defined as a contravariant functor from the "category of open sets" of a topological space to some "concrete locally small category". A Sheaf is a presheaf which satisfies the additional axiom of "glueing". Associated to sheaves, we define the notion of stalks. Sheaves have a very local nature; this is reflected in the "sheaf condition"; the concept of stalks try to capture this local nature. We proceed to define a sheaf structure on $Spec(A)$ (which is a locally ringed space) and call any sheaf isomorphic to this an "Affine Scheme". We obtain an anti-equivalence between the "category of rings" and the "category of affine schemes". Schemes are obtained by glueing affine schemes. Later part of this chapter is devoted to studying the connection between prevarieties and certain schemes. Via soberification, we obtain equivalence between the "category of integral schemes of finite type over k" and the "category of prevarieties over k".

In the second chapter we discuss some properties and charecterization of schemes and morphisms of schemes. One of the most important thing we discuss here is the existance of fiber product in the "category of schemes" and "base change". Later part of the chapter is all about "dimension" of a scheme. We give an outline of a proof of the Bezout's theorem as a first application of the theory so far developed.

From third chapter the second part of the thesis begins where we discuss cohomology. First part of the third chapter discusses basic notions of homology in an "abelian category" set up, a reader who is familiar with commutative algebra immediately senses that most of the notions are parellel to those in the commutative algebra set up. Here we adapt the idea of diagram chasing to a general "abelian category" set up. The second part of the chapter is devoted for developing the notion of "derived category" and "derived functor" which arises from localisation of an abelian category.
In fourth chapter we introduce the notion of "coherent sheaves" so that we can apply the cohomology developed earlier. "Quasi coherent sheaves" and "coherent sheaves" are interesting on their own, they arise as a result of the attempt to charecterize sheaf of ideals which give rise to closed subschemes. This chapter provides basic definitions and properties of "quasi coherent modules".
Fifth chapter is the place where we apply the cohomology theories so far developed to the case of sheaves. We see that the category of coherent sheaves has "enough injectives", but finding an injective resolutionis is inconvenient we overcome this by relying on "flasque resolution". Later we introduce the "Čech cohomology" which is relatively easy to compute and it coincides with sheaf cohomology when the underlying scheme is noetherian and seperated.
A note to the reader:

- Due to restriction on the size most of the proofs and results are omitted.

Chapter 1

Sheaves and Schemes

We assume that the reader is familiar with varieties and Prime spectrum of a ring. Two major drawback with the construction of affine varieties is that

- It depend Completely on the underlying subset of $\mathbb{A}^n(k)$ (affine space over an algebraically closed field k).
- They are only useful in the case of an algebraically closed field and not in the case of a general commutative ring.

One of the major drawbacks of affine variety is that it is co-ordinate dependent that is it depends on how it is embedded in the ambient affine space. To get a co-ordinate free description of affine variety we note that there is a bijection between maximal ideals of the co-ordinate ring and the points of the underlying topological space. For a commutative ring B there are only few maximal ideals, so we construct a topological space using the set of prime ideals which is denoted by $Spec(B)$ (In some sense the topology defined on this set mimics the Zarisky topology defined on the affine space).

1.1 Sheaves

Given $Spec(B)$ we can not get back the ring B, so we need to define additional structures on $Spec(B)$ which will enable us to retrieve back the ring B. We proceed as in the case of Prevarieties, we will try to view elements of B as maps defined on $Spec(B)$. But in reality the elements of B are not maps defined on $Spec(B)$, hence its not appropriate to use the notion of "system of functions". We take an extensile approach keeping in mind the analogy of "system of functions", and it turns out that the key point for working with "functions" is "restricting and gluing" of functions. This leads to the notion of sheaf.

1.1.1 Presheaves and Sheaves

We consider only ("concrete category") the categories whose objects are "small sets" (which have additional properties).

Definition 1.1.1. *Consider a topological space W. A **Presheaf** $\mathcal{F}$ on W consist of the following data*

- *Given any open set U of W a set $\mathcal{F}(U)$,*
- *For each pair of open set $U \subseteq V$ a map*

$$res^V_U : \mathcal{F}(V) \to \mathcal{F}(U)$$

it is known as the restriction map, and it satisfies the following conditions.

1. *Given any open set $U \subseteq W$, $res_U^U = id_{\mathcal{F}(U)}$.*
2. *Consider the open sets $U \subseteq V \subseteq X$ of W then*

$$res_U^X = res_U^V \circ res_V^X$$

Let $\mathcal{F}$ be presheaf on W and $U \subseteq_{open} W$, elemets of $\mathcal{F}(U)$ are known as sections (defined on U). We can also describe the concept of presheaf using categories and functors this description will be very useful at times.

Definition 1.1.2. *Consider the topological space W, we define a category $Open_W$ as below*

- *Objects of $Open_W$ are open sets of W*
- *Consider open subsets U, V of W. If $U \nsubseteq V$ then $Hom(U,V) = \phi$ and if $U \subseteq V$ then $Hom(U,V)$ consist of a single element the inclusion map from U to V*

We define a presheaf as follows.

Definition 1.1.3. *A presheaf $\mathcal{F}$ on a topological space W is a contravariant functor from the category $Open_W$ to the category of sets.*

Similarly a presheaf on a topological space W that takes values in a category $\mathcal{C}$ is a contravariant functor from the category $Open_W$ to the category $\mathcal{C}$. That is given any open set $U \subseteq W$, $\mathcal{F}(U)$ is an object in $\mathcal{C}$ and the restriction maps are morphisms in $\mathcal{C}$. Let $\mathcal{F}_1, \mathcal{F}_2$ be presheaves that takes values in $\mathcal{C}$ then a morphism $\mathcal{F}_1 \to \mathcal{F}_2$ is simply a morphism of functors.

Definition 1.1.4. *Consider the topological space W, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two presheaves on W. A morphism of presheaves $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ is a family of maps $\psi_U : \mathcal{F}_1(U) \to \mathcal{F}_2(U)$ (for all open sets U of W) such that for any open sets $U \subseteq V \subseteq W$ the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{F}_1(V) & \xrightarrow{\psi_V} & \mathcal{F}_2(V) \\ \downarrow{\scriptstyle res_U^V} & & \downarrow{\scriptstyle res_U^V} \\ \mathcal{F}_1(U) & \xrightarrow{\psi_U} & \mathcal{F}_2(U) \end{array}$$

Notation: If $U \subseteq V$ are open sets of W and $\mathcal{F}$ be a presheaf on W . If $f \in \mathcal{F}(V)$, $f|_U$ denotes $res_U^V(f)$.

Definition 1.1.5. *Let $\mathcal{F}$ be a presheaf on W it is called a **Sheaf** if it satisfies for every open subset U and for every open covering $(U_i)_i$ of U the following conditions*

- *(Sh1) Let $s, s' \in \mathcal{F}(U)$, if $s|_{U_i} = s'|_{U_i} \quad \forall i$ then $s = s'$.*
- *(Sh2)Given $s_i \in \mathcal{F}(U_i) \quad \forall i$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \forall i, j$ then there exist $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i \quad \forall i$ (Uniqueness follows from Sh1).*

If we consider our analogy of "Space with functions", intutively these conditions says that functions are completely determined by their values on an open cover (determined by local information) and compatible functions can be glued. A morphism between sheaves is same as the morphism between the underlying presheaves. Via this definition of morphism we get the category of "Sheaves" on the topological space W, we denote it by $(Sh(W))$. In the case of a category $\mathcal{C}$ we may not be able to refer to elements of the object $\mathcal{F}(U)$ (for some open subset U of W and a presheaf $\mathcal{F}$ on W which takes value in $\mathcal{C}$) so it is useful to define the concept of a sheaf using categorical notions for this purpose we define the following maps.
Consider a topological space W, let U be an open subset of W and $\mathcal{U} := (U_i)_{i \in I}$ (for somr indexing set I) be an open covering of U. For a given presheaf $\mathcal{F}$ on W and an open cover $\mathcal{U}$ of U we define the following maps

$$\rho : \mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i), \quad s \mapsto (s|_{U_i})_i =: (s_i)_i$$

$$\sigma : \prod_{i\in I} \mathcal{F}(U_i) \longrightarrow \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j), \quad (s_i)_i \mapsto (s_i|_{U_i\cap U_j})_{(ij)}$$

$$\sigma' : \prod_{i\in I} \mathcal{F}(U_i) \longrightarrow \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j), \quad (s_i)_i \mapsto (s_j|_{U_i\cap U_j})_{(ij)}.$$

These maps can also be described in terms of the restriction maps without refering to the elemets of $\mathcal{F}(U)$

Definition 1.1.6. *Let $\mathcal{F}$ be a presheaf on W it is called a* ***Sheaf*** *if it satisfies for every open subset U and for every open covering $(U_i)_i$ of U the following conditions:*
The diagram

$$\mathcal{F}(U) \xrightarrow{\rho} \prod_{i\in I} \mathcal{F}(U_i) \underset{\sigma}{\overset{\sigma'}{\rightrightarrows}} \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j)$$

is exact. That is the map ρ is injective and its image is the set of elements $(s_i)_{i\in I} \in \prod_{i\in I} \mathcal{F}(U_i)$ such that $\sigma((s_i)_i) = \sigma'((s_i)_i)$.

.

Examples:

1. Consider a topological space W and $U \subseteq W$ be an open set of W. For any presheaf $\mathcal{F}$ on W we can define a presheaf $\mathcal{F}|_U$ on U by setting $\mathcal{F}|_U(V) = \mathcal{F}(V)$ for any open set V of U, moreover if $\mathcal{F}$ is a sheaf then the presheaf defined on U is also a sheaf. $\mathcal{F}|_U$ is known as the restriction of $\mathcal{F}$ on U.

2. Consider the topological space W, Z. Define a presheaf $\mathcal{F}$ on W by setting $\mathcal{F}(U)$ as the collection of continuous functions from U to Z for every open set U of W. Define the restriction maps using the idea of restriction of a function. The above defined presheaf is sheaf.

3. Consider a field k and the "space with functions" $W, \mathcal{O}_W$ over k then $\mathcal{O}_W$ is a "sheaf of $k-$algebras" on W.

4. Consider a topological space W and set

$$\mathcal{F}(U) = \{f : U \to \mathbb{R} \quad continuous : f(U) \subseteq \mathbb{R} \quad bounded\}$$

   for all $U \subseteq_{open} W$, then $\mathcal{F}$ is a presheaf but its not a sheaf in general (Let $W = \mathbb{R}$ then define $f(x) = x$, $f \in \mathcal{F}(-a, a)$ for all $a \in \mathbb{R}$ but it does not belong to $\mathcal{F}(\mathbb{R})$).

Consider a topological space W and $\mathcal{B}$ be its basis, let $\mathcal{F}$ be a sheaf on W. From the axiom of gluing we get that if we know $\mathcal{F}(U)$ $\forall U \in \mathcal{B}$ then we can determine $\mathcal{F}(V)$ for an arbitary open set V by gluing appropriate elements of $\mathcal{F}(V_i)$ where $V_i \in \mathcal{B}$ and $V = \bigcup_{i\in I} V_i$
$\mathcal{F}(V) = \{s = (s_i)_i \in \prod_{i\in I} \mathcal{F}(V_i) : s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j} \quad \forall i, j \in I\} = \{s = (s_u)_u \in \prod_{u\subseteq V, u\in\mathcal{B}} \mathcal{F}(u) : \forall u' \subseteq u; \quad u, u' \in \mathcal{B} \quad s_u|_{u'} = s_{u'}\}$
All these equalities has to be understood as bijective correspondences. Let $\mathcal{U}$ be the collection of $u \in \mathcal{B}$ such that $u \subseteq V$. $\mathcal{U}$ (is a poset) is a subcategory of $Open_W$ and it forms an inverse system.

$$\mathcal{F}(V) = \varprojlim_{u\in\mathcal{U}} \mathcal{F}(u)$$

Hence it is enough to specify a sheaf on some basis of a given topological space.

Definition 1.1.7. *Consider a topological space W with a basis $\mathcal{B}$. Treat $\mathcal{B}$ as a full subcategory of $Open_W$. We define a presheaf on $\mathcal{B}$ as a contravariant functor from the category $\mathcal{B}$ to the category of sets.*

A morphism of presheaves on $\mathcal{B}$ is again defined as a morphism of functors. Any such presheaf $\mathcal{F}$ on $\mathcal{B}$ can be extended to a presheaf $\mathcal{F}'$ on W by setting,

$$\mathcal{F}'(V) = \varprojlim_{U \in \mathcal{B}, U \subseteq V} \mathcal{F}(U)$$

for any open set V of W.

Proposition 1.1.8. *Consider a topological space W and its basis $\mathcal{B}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{B}$ and $\mathcal{F}'$ be the extension of $\mathcal{F}$ to W .The presheaf $\mathcal{F}'$ on W is a sheaf if and only if $\mathcal{F}$ satisfies the sheaf condition((Sh1) and (Sh2)) for every $U \in \mathcal{B}$ and for every open covering of U by basis elements.*

We get an equivalence between the "category of sheaves on $\mathcal{B}$" and the category of sheaves on W.

1.1.2 Stalks of Sheaves

Sheaves have a very local nature; this is reflected in the sheaf condition; the concept of stalks try to capture this local nature. Most of the times statement s about sheaves and morphism of sheaves can be verified by considering the analogues statement at the level of stalks.

Definition 1.1.9. *Consider a topological space W and a presheaf $\mathcal{F}$ defined on W then the stalk of $\mathcal{F}$ in $w \in W$*

$$\mathcal{F}_w = \varinjlim_{w \in U} \mathcal{F}(U)$$

- Let $w \in W$ then we can view $\mathcal{F}_w$ the stalk of $\mathcal{F}$ in w as the set of equivalence classes of pairs (U, s) where U is an open set containing w and $s \in \mathcal{F}(U)$. $(U_1, s_1) \sim (U_2, s_2)$ if there exist an open set V containing w such that $V \subseteq U_1 \cap U_2$ and $s_1|_V = s_2|_V$.
$$\mathcal{F}_w = (\coprod_U \mathcal{F}(U))/ \sim$$
where U runs through the open sets of W containing w and if $s_1 \in \mathcal{F}(U_1)$ and $s_2 \in \mathcal{F}(U_2)$ (U_1, U_2 are open sets of W) then $s_1 \sim s_2$ iff there exist an open set V containing x such that $V \subseteq U_1 \cap U_2$ of W and $res_V^{U_1}(s_1) = res_V^{U_2}(s_2)$
- Let $w, z \in W$. If every open set containing w contains z and every open set containing z contains w then $\mathcal{F}_w = \mathcal{F}_z$
- For any $U, \subseteq_{open} W$ containing w we have a canonical map (from the definition of direct limit)
$$\theta_w : \mathcal{F}(U) \to \mathcal{F}_w \quad s \mapsto s_w$$
which sends $s \in \mathcal{F}(U)$ to the class of (U, s) in $\mathcal{F}_w$. We call s_w the germ of s in w.
- Let $U \subseteq_{open} W$ and $w \in W$ then $\mathcal{F}_w = (\mathcal{F}|_U)_w$. This follows immediatly from the universality of the direct limit.

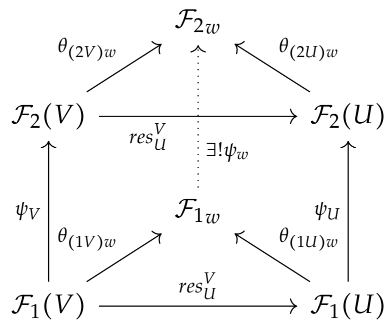
Now we will discuss how the presheaf $\mathcal{F}$ is related to $\mathcal{F}_w$.

- If $\psi : \mathcal{F}_1 \to \mathcal{F}_2$ is a morphism on preseaves on W then we have an induced map $\psi_w : \mathcal{F}_{1w} \to \mathcal{F}_{2w}$. Let $s_w \in \mathcal{F}_{1w}$ and (U, s) be a representative of s_w define $\psi_w(s_w) = (\psi_U(s))_w$ (equivalence class of $(U, \psi_U(s))$ in $\mathcal{F}_{2w}$). Let $(U, s) \sim (U', s')$ in $\mathcal{F}_{1w}$ then there exist $V \subseteq U \cap U'$ such that $s|_V = s'|_V$. The following diagram shows that if $s|_V = s'|_V$ then $\psi_U(s)|_V = \psi_{U'}(s')|_V$.

$$\begin{array}{ccc}
\mathcal{F}_1(U) & \xrightarrow{\psi_U} & \mathcal{F}_2(U) \\
\big\downarrow res^U_V & & \big\downarrow res^U_V \\
\mathcal{F}_1(V) & \xrightarrow{\psi_V} & \mathcal{F}_2(V) \\
\big\uparrow res^{U'}_V & & \big\uparrow res^{U'}_V \\
\mathcal{F}_1(U') & \xrightarrow{\psi_{U'}} & \mathcal{F}_2(U')
\end{array}$$

that is if $(U, s) \sim (U', s')$ then $(U, \psi_U(s)) \sim (U', \psi_{U'}(s'))$, hence the map ψ_w is well defined.

- We can have an alternate definition of the map ψ_w without refering to the elements of $\mathcal{F}_w$. Let $w \in U \subseteq V$, consider the following commuatative diagram (which follows from the universal property of the direct limit)



The unique map given by the universality of $\mathcal{F}_{1w}$ is the desired map ψ_w. Form the relation $u \circ \theta_{(1U)w} = \theta_{(2U)w} \circ \psi_U$ for all open set U containing w, it follows immediately that this map is the same as the one defined previously. (We will use this as the definition of the induced map from now onwards.)

- Let A, B, P, Q be objects of a category $\mathcal{C}$. Let $\psi : A \to B$ and $\phi : P \to Q$ be some morphism between these objects. We can define another category $\mathcal{C}'$ whose objects are the morphisms of the category $\mathcal{C}$ and given $\psi, \phi \in \mathcal{C}'$ define morphism from ψ to ϕ if the following commutative diagram (in $\mathcal{C}$) exists.

$$\begin{array}{ccc}
A & \xrightarrow{\psi} & B \\
\big\downarrow a & & \big\downarrow b \\
P & \xrightarrow{\phi} & Q
\end{array}$$

(morphisms of $\mathcal{C}'$ are commutative diagrams). Let $\mathcal{F}_1, \mathcal{F}_2$ be presheafs from W to the category $\mathcal{C}$, then all the morphisms given in the previous commutative diagram are objects of $\mathcal{C}'$ and it is clear that if $U \subseteq V$ $(V \leq U)$ then there is a morphism (f_{VU}) from ψ_V to ψ_U hence they form a direct system. From every ψ_U there is a morphism to ψ_w. it is easy to check that

$$\psi_w = \varinjlim_{w \in U} \psi_U$$

Consider a category $\mathcal{C}$ in which filtered inductive limit exist, let $\mathcal{F}$ be a presheaf which takes values in $\mathcal{C}$ then $\mathcal{F}_w$ is an object of $\mathcal{C}$ this give rise to a functor from the category of presheaves on W which takes value in $\mathcal{C}$ to $\mathcal{C}$.

Proposition 1.1.10. *If $\mathcal{F}$ is a sheaf and s, t are two sections in $\mathcal{F}(U)$, then $s = t$ if and only if $s_w = t_w$ for all $w \in U$.*

Proof. If $s = t$ then $s_w = t_w$ for all $w \in W$. Now assume $s_w = t_w$ for all $w \in W$ then for every $w \in W$ there exist open set U_w such that $s|_{U_w} = t|_{U_w}$, evidently U_w forms an open cover of U, then the map

$$\rho : \mathcal{F}(U) \to \prod_{w \in U} \mathcal{F}(U_w)$$

should be injective by definition of sheaf, hence $s = t$ (Since $\rho(s) = \rho(t)$).

■

Proposition 1.1.11. *Consider a topological space W, let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on W, and $\phi, \psi : \mathcal{F} \longrightarrow \mathcal{G}$ be morphisms of presheaves.*

1. *If $\mathcal{F}$ is a sheaf, then the induced maps on stalks $\psi_w : \mathcal{F}_w \to \mathcal{G}_w$ are injective for all $w \in W$ if and only of $\psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective for all open subsets $U \subseteq W$.*

2. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on W. The map ψ_w are bijective for all $w \in W$ if and only if ψ_U is bijective for all open sets $U \subseteq W$*

3. *If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on W. ψ and ϕ are equal if and only if $\psi_w = \phi_w$ for all $w \in W$.*

A morphism $\psi : \mathcal{F} \to \mathcal{G}$ of sheaves is called injective (resp. bijective,resp. surjective) if $\psi_w : \mathcal{F}_w \to \mathcal{G}_x$ is injective (resp. bijective,resp. surjective) for all $w \in W$.
If $\psi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, ψ is surjective if and only if for all open sets $U \subseteq W$ and every $t \in \mathcal{G}(U)$ there exist an open cover $U = \bigcup_i U_i$ (depending on t) and sections $s_i \in \mathcal{F}(U_i)$ such that $\psi_{U_i}(s_i) = t_{U_i}$. i.e. locally we can find a preimage of t. But the surjectivity of ψ does not imply that $\psi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is surjective for all open sets U of W

1.1.3 Sheaves assosciated to presheaves

Let $\mathcal{F}$ be a presheaf defined on W, there are two ways a presheaf can fail to be a sheaf

- it has local sections which are compatible that does not patch together to give a global section.
- it has sections which agrees locally that does not agree globally.

If $\mathcal{F}$ is not a sheaf we want to "modify" $\mathcal{F}$ into a sheaf $\overline{\mathcal{F}}$ without "disturbing" the pre existing presheaf structure.

Definition 1.1.12. *Let $\mathcal{F}$ be a presheaf defined on the topological space W then for any open subset $U \subseteq W$ define*
$\overline{\mathcal{F}}(U) := \{s \in \prod_{w \in U} \mathcal{F}_w :$ *for every* $w \in U\ \exists U^w \ni w$ *and* $t \in \mathcal{F}(U^w)$ *such that* $s_u = t_u \quad \forall u \in U^w\}$.

$$i_{\mathcal{F}_U} : \mathcal{F}(U) \to \overline{\mathcal{F}}(U) \quad s \mapsto (s_w)_{w \in U}$$

Proposition 1.1.13. *Let $\mathcal{F}$ be a presheaf on W then $\overline{\mathcal{F}}$ is a sheaf (called the sheafification of $\mathcal{F}$) and $i_{\mathcal{F}_U}$ is a morphism of presheaves such that the induced map on stalks is a bijections for all $w \in W$. Moreover for any presheaf $\mathcal{G}$ on W and any morphism $\psi : \mathcal{F} \to \mathcal{G}$ of presheaves there exist a unique morphism $\overline{\psi} : \overline{\mathcal{F}} \to \overline{\mathcal{G}}$ such that the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i_{\mathcal{F}}} & \overline{\mathcal{F}} \\ \downarrow{\psi} & & \downarrow{\overline{\psi}} \\ \mathcal{G} & \xrightarrow{i_{\mathcal{G}}} & \overline{\mathcal{G}} \end{array}$$

Hence sheafification is a functor from the category of presheaves to the category of sheaves.

Corollary 1.1.14. *If $\mathcal{F}$ is a presheaf and $\mathcal{G}$ is a sheaf on W then for every morphism $\psi : \mathcal{F} \to \mathcal{G}$ there exist a unique morphism $\overline{\psi}$ such that the diagram commutes*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{i_{\mathcal{F}}} & \overline{\mathcal{F}} \\ \downarrow \psi & \swarrow \overline{\psi} & \\ \mathcal{G} & & \end{array}$$

Example: Consider a set E, define a presheaf $\mathcal{F}$ by setting $\mathcal{F}(U) = E$ for every open set U of W (the restriction maps being the identity). Let $a, b \in \mathcal{F}(U)$, if $a|_V = b_V$ for some open subsete V of U then by construction $a = b$. So if $\mathcal{F}$ is not a sheaf it is beacause of the absence of global sections corresponds to local sections which can be patch together. Let $U, V \subseteq W$ be open sets with non empty intersection if there exist $a \in \mathcal{O}_W(U), b \in \mathcal{O}_W(V)$ such that $a|_{U\cap V} = b|_{U\cap V}$ then $a = b$ and we have a corresponding global section of $U \cup V$. Hence if W is irreducible $\mathcal{F}$ is a sheaf otherwise it is not because if $\mathcal{F}$ is a sheaf and $U \cap V = \phi$ then $\mathcal{O}_W(U \cup V) = \mathcal{O}_W(U) \times \mathcal{O}_W(V)$ which is not true. Intutively its clear that the sheafification $\overline{\mathcal{F}}$ is such that $\overline{\mathcal{F}}(U)$ is the collection of locally constant maps from U to E. It evident that $O_{W,w} = E \quad \forall w \in W$ hence from definition of sheafification we get that $\overline{\mathcal{F}}(U)$ is the collection of locally constant functions from U to E.
It is evident that given a presheaf $\mathcal{F}$ of rings, of R-modules, or of R algebras, its associated sheaf is a sheaf of rings, of R-modules, or of R-algebras.

1.1.4 Direct and inverse images of sheaves .

Given a continuous map $f : W \longrightarrow Z$ of topological spaces one might wonder about how we can trasport sheaves on W to Z or the other way round via the map f. Lets first think about the forward direction, let $\mathcal{F}$ be a presheaf defined on W. $\mathcal{F}$ is a functor from $Open_W$, there is a functor from $Open_Z$ to $Open_W$ induced by the map f. If $V \in Open_Z$ then $V \mapsto f^{-1}(V) \in Open_W$ composing this functor with the functor $\mathcal{F}$ we get a new functor from $Open_Z$. We denote this functor by $f_*\mathcal{F}$. We can describe the presheaf $f_*\mathcal{F}$ as follows ($V \subseteq_{open} Z$)

$$f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

the restriction maps given by the restriction maps for $\mathcal{F}$. $f_*\mathcal{F}$ is called the direct image of $\mathcal{F}$ under f.
If $\psi : \mathcal{F} \to \mathcal{G}$ be morphism of presheaves then the family of maps $f_*(\psi)_V := \psi_{f^{-1}(V)}$ for $V \subseteq_{open} Z$ defines a morphism of presheaves $f_*(\psi) : f_*\mathcal{F} \to f_*\mathcal{G}$. Therefor f_* is a functor from the category of presheaves on W to the category of presheaves on Z.

Proposition 1.1.15. *Let $f : W \to Z$ be a continuous map of topological spaces.*

1. *Consider a sheaf $\mathcal{F}$ on W, then $f_*\mathcal{F}$ is a sheaf on Z . That is f_* give rise to a functor $f_* : (Sh(W)) \to (Sh(Z))$.*
2. *Consider a continuous map $g : Z \to W$, then there exists a relation, $g_*(f_*\mathcal{F}) = (g \circ f)_*\mathcal{F}$ which is functorial in $\mathcal{F}$.*

Similarly we would like to transport a presheaf defined on Z to a presheaf on W using the map f. Consider a presheaf $\mathcal{F}$ on Z then for every V open in W define.

$$\mathcal{G}(V) := \varinjlim_{f(V)\subseteq U} \mathcal{F}(U) \quad U \subseteq_{open} Z$$

We denote the presheaf $\mathcal{G}$ by $f^+\mathcal{F}$, let $f^{-1}\mathcal{F}$ denotes the sheafification of $f^+\mathcal{F}$. $\mathcal{F} \mapsto f^{-1}(\mathcal{F})$ defines a functor from the category of presheaves on Z to the category of sheaves on W. We call $f^{-1}\mathcal{F}$ the inverse image of $\mathcal{G}$ under f.
Consider a cotinuous map $g : Z \longrightarrow Y$ and a presheaf $\mathcal{H}$ on Y then $f^+(g^+(\mathcal{H})) \cong (g \circ f)^+(\mathcal{H})$ this induces an isomorphism

$$f^{-1}(g^{-1}\mathcal{H}) \cong (g \circ f)^{-1}\mathcal{H} \quad (*)$$

which is functorial in $\mathcal{H}$.

Proposition 1.1.16. *Consider the continuous map $f : W \to Z$, and a sheaf $\mathcal{F}$ on W. Then given a presheaf $\mathcal{G}$ on Z there is a bijection*

$$Hom_{(Sh(W))}(f^{-1}\mathcal{G}, \mathcal{F}) \longleftrightarrow Hom_{(preSh(Z))}(\mathcal{G}, f_*\mathcal{F})$$

$$\varphi \to \varphi^{\flat}$$

$$\psi^{\sharp} \leftarrow \psi$$

which is functorial in $\mathcal{F}$ and $\mathcal{G}$.

Given $\psi : \mathcal{G} \to f_*\mathcal{F}$ we get an induced map $\psi^{\sharp} : f^{-1}\mathcal{G} \to \mathcal{F}$, so for every $w \in W$ we have

$$\psi^{\sharp}_w : (f^{-1}\mathcal{G})_W = (\mathcal{G})_{f(w)} \to \mathcal{F}_w$$

We can describe this map in terms of ψ as follows: for every open neighborhood $V \subseteq Z$ of $f(w)$ we have maps

$$\mathcal{G}(V) \xrightarrow{\psi_V} \mathcal{F}(f^{-1}(V)) \longrightarrow \mathcal{F}_w$$

From the universality of the direct limit of $\mathcal{G}(V)$ (V open neighborhood of w) we get the map $\psi^{\sharp}_w : \mathcal{G}_{f(w)} \longrightarrow \mathcal{F}_w$

1.1.5 Locally ringed spaces.

Definition 1.1.17. *Consider a topological space W and a sheaf of commutative rings $\mathcal{O}_W$ on W then the pair $(W, \mathcal{O}_W)$ is called a ringed space. We define the morphism between two ringed spaces $(W, \mathcal{O}_W)$ and $(Z, \mathcal{O}_Z)$ as the pair $(f, f^{\flat})$, where $f : W \longrightarrow Z$ is a continuous map between the topological spaces and $f^{\flat} : \mathcal{O}_Z \to f_*\mathcal{O}_W$ is a morphism of sheaves on Y.*

- The datum $f^{\flat}$ encapsulate the datum of a homomorphism of sheaves of rings $f^{\sharp} : f^{-1}\mathcal{O}_Z \longrightarrow \mathcal{O}_W$
- The composition of morphisms of ringed spaces is defined in the obvious way. Let $(f, f^{\flat}) : (W, \mathcal{O}_W) \longrightarrow (Z, \mathcal{O}_Z)$ and $(g, g^{\flat}) : (Z, \mathcal{O}_Z) \longrightarrow (Y, \mathcal{O}_Y)$, then $g \circ f : W \longrightarrow Z$ and if $U \subseteq Y$ open then

$$\mathcal{O}_Y(U) \xrightarrow{g^b} \mathcal{O}_Z(g^{-1}(U)) \xrightarrow{f^b} \mathcal{O}_W(f^{-1}(g^{-1}(U)))$$

  This defines a map $(g \circ f, (g \circ f)^{\flat}) : (W, \mathcal{O}_W) \longrightarrow (Y, \mathcal{O}_Y)$. We obtain the category of ringed spaces.
- Given a ringed space $(W, \mathcal{O}_W)$ we say $\mathcal{O}_W$ is the structure sheaf of this ringed space. Often we denote $(W, \mathcal{O}_W)$ by W .
- If we view the structure sheaf on W as the system of all "permissible" functoins. If we compose f with a "permissible" function on an open set U of Z we should get a "permissible" function on $f^{-1}(U)$ of W this would be a nice property to demand. Since viewing sections of the structure sheaves as functions is only a heuristic, we cannot actually compose sections with the map f . So we put an expicit condition that for every $U \subseteq Z$ there should be a map $\mathcal{O}_Z(U) \to \mathcal{O}_W(f^{-1}(U))$. These maps must be compatible with restrictions, and constitute the sheaf homomorphism $f^{\flat}$.

Given an open set U of W we think $\mathcal{O}_W(U)$ as functions on U. If we move forward with this analogy then the germs at a point w are precisely functions defined on some open neighbouhood around w, if two functions agree locally then they represent the same germ. A reasonable property to ask of such functions is that those which do not vanish at w are invertible in some (small) neighborhood of w. Then all elements of the stalk not contained in the ideal of functions vanishing at w are units of the stalk. This shows that the stalk is indeed a local ring, with maximal ideal the ideal of all functions vanishing at w. Now consider a morphism $(f, f^\flat) : W \longrightarrow Z$ of ringed spaces of this nature(ringed spaces of this nature are called locally ringed spaces). The sheaf homomorphism is our replacement for "composition of functions with f ". Certainly, if some function on Z vanishes at a point $f(w)$, $w \in W$, then its composition with f must vanish at w. In other words, the maximal ideal of $O_{Z,f(w)}$ must be mapped into the maximal ideal of $(f_*\mathcal{O}_W)_{f(w)}$(or the maximal ideal of $(f^{-1}\mathcal{O}_Z)_w$ is mapped in to the maximal ideal of $\mathcal{O}_{W,w}$) . Since viewing sections of the structure sheaves as functions is only a heuristic, we put this as an explicit condition .

Definition 1.1.18. *Consider a ringed space $(W, \mathcal{O}_W)$, if for every $w \in W$ the stalk $\mathcal{O}_{W,w}$ is a local ring then we say that $(W, \mathcal{O}_W)$ is a locally ringed space. A morphism between locally ringed spaces is a morphism of ringed spaces which induces local ring homomorphisms at the level of stalks for every $w \in W$.*

- For any local ring A using $\mathfrak{m}_A$ we denote its only maximal ideal and we set $\kappa(A) : A/\mathfrak{m}_A$, which is called the residue field of A If A is a local ring we denote by $\mathfrak{m}_A$ its maximal ideal and by $\mathcal{K}(\mathcal{A}) = A/\mathfrak{m}_A$ its residue field . A morphism of local rings $\psi : A \longrightarrow B$ is clled local if $\psi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.
- Locally ringed sapeces from a category.
- A morphism of ringed spaces between two locally ringed spaces may not be a morphism of locally ringed spaces, that is if $(W, \mathcal{O}_W)$ and $(Z, \mathcal{O}_Z)$ are locally ringed spaces and if f is a morphism of ringed spaces between them then f need not be a morphism of locally ringed spaces. In short the "category of locally ringed spaces" is not a full subcategory of the "category of ringed spaces".
- Consider the locally ringed space $(W, \mathcal{O}_W)$ and the stalk $\mathcal{O}_{W,w}$ for some $w \in W$. We call the maximal ideal $\mathfrak{m}_A$ of $\mathcal{O}_{W,w}$ the local ring of W in w and we set $\mathcal{K}(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$.
- If U is open neighborhood of w and if $f \in \mathcal{O}_W(U)$ we denote by $f(w) \in \mathcal{K}(w)$ the image of f under the canonical homomorphism $\mathcal{O}_W(U) \longrightarrow \mathcal{O}_{W,w} \longrightarrow \mathcal{K}(w)$

1.2 Locally ringed space and Spectrum of a ring

1.2.1 Structure sheaf on $SpecB$.

Set $W = Spec(B)$. We wish to define a sheaf on the topological space W such that $\mathcal{O}_W(W) = B$. We know that the principal open sets $D(f)$, $f \in B$ form a basis of the toplogy on W (let $\mathcal{B}$ denotes this basis). We define a presheaf on $\mathcal{B}$ which is a sheaf and then we extend this sheaf to the space W.
We try to view elements of B as functions defined on $Spec(B)$, if $f \in B$ define $f(w)$ ($w \in W$ and $\mathfrak{p}_w$ be the corresponding prime ideal) as the image of f under the canonical homomorphism $B \longrightarrow B_{\mathfrak{p}_w}/\mathfrak{p}_w A_{\mathfrak{p}_w}$. It is reasonable to demand that the inverse of a nonzero "permissible" function on the open set $D(f) \subseteq W$ is also a "permissible" function on $D(f)$ (in fact this demand makes the ringed space we are going to define into a locally ringed space). In light of this demand we define $\mathcal{O}_W(D(f))$ as follows

$$\mathcal{O}_W(D(f)) := \{g = g_1/g_2 : g_1, g_2 \in B \quad and \quad g_2(w) \neq 0 (g_2 \notin \mathfrak{p}_w) \quad \forall w \in D(f)\}/\sim$$

where $\dfrac{g_1}{g_2} \sim \dfrac{h_1}{h_2}$ if and only if $(g_1h_2 - g_2h_1)u = 0$ for some $u \in B$ such that $u(w) \neq 0 \quad \forall w \in D(f)$.
Let $f, g, h \in B$

- It is evident that if $D(f) = D(g)$ then $\mathcal{O}_W(D(f)) = \mathcal{O}_W(D(g))$

- Its obvious that $B_f \subseteq \mathcal{O}_W(D(f))$. If $g = g_1/g_2 \in \mathcal{O}_W(D(f))$ then we have $D(f) \subseteq D(g_2)$, it follows that $\exists n \geq 1$ such that $f^n \in (g_2) = Bg_2$. That is $f^n = hg_2$ for some $h \in B$ that is $1/g_2 = h/f^n$ as elements of $\mathcal{O}_W(D(f))$ hence $1/g_2 \in B_f$ (so any element of $\mathcal{O}_W(D(f))$ have a representative in B_f). Hence $B_f = \mathcal{O}_W(D(f))$ (from now onwards we use this as the definition of $\mathcal{O}_W(D(f))$).

- Define $i_f : B \longrightarrow B_f$ as the canonical homomorphism. If $D(f) \subseteq D(g)$ then we have $1/g = h/f^n$ (as elements of B_f) as described previously. Define a homomorphism

$$\rho_{f,g} : B_g \longrightarrow B_f \quad a/g^m \mapsto a.((h/f^n))^m = a.(1/g^m)(\quad in \quad B_f)$$

then $\rho_{f,g} \circ i_g = i_f$ and Whenever $D(f) \subseteq D(g) \subseteq D(h)$, we have $\rho_{f,g} \circ \rho_{g,h} = \rho_{f,h}$

- If $D(f) \subseteq D(g)$ define $res^{D(g)}_{D(f)} = \rho_{f,g}$. This ddefines a presheaf on the basis $\mathcal{B}$.

Theorem 1.2.1. *The presheaf $\mathcal{O}_W$ is a sheaf on $\mathcal{B}$.*

We can extend the sheaf $\mathcal{O}_W$ defined on $\mathcal{B}$ to W as follows (we use the sane notation $\mathcal{O}_W$ to denote this new sheaf as well), for any open set $U \subseteq W$

$$\mathcal{O}_W(U) := \varprojlim_{D(f)} \mathcal{O}_W(D(f))$$

where $D(f) \subseteq U$

Let $w \in W$, define $B_x := \varinjlim \mathcal{O}_W(D(f))$ where $D(f)$ are the principal open set containing w and define $C_x = \varinjlim \mathcal{O}_W(U)$ where U is an open neighborhood of w. Since for every principal open set we have maps from $\mathcal{O}_W(D(f))$ to C_x it is evident that we have a unique map from B_x to C_x, for every open set U containing w we have a restriction map from $\mathcal{O}_W(U)$ to $\mathcal{O}_W(D(f))$ (for some principal set) from which there is a map to B_x hence we have a unique map from C_X to B_x. It is easy to check that $B_x \cong C_x$.

The above discussion shows that

$$O_{W,w} = \varinjlim_{w \in D(f)} \mathcal{O}_W(D(f)) = \varinjlim_{f \notin \mathfrak{p}_w} B_f = B_{\mathfrak{p}_x}$$

$B_{\mathfrak{p}_w}$ is a local ring hence $(W, \mathcal{O}_W)$ is a locally ringed space.

Remark 1.2.2. *An element of B whose image under the canonical morphism $B \longrightarrow B_{\mathfrak{p}_w}/\mathfrak{p}_w B_{\mathfrak{p}_w}$ is zero for all prime ideal $\mathfrak{p}_w$ need not be the zero element of B (it is zero if B is reduced)*

1.2.2 The functor $A \mapsto (Spec(A), \mathcal{O}_{Spec(A)})$

Definition 1.2.3. *Consider a locally ringed space $(W, \mathcal{O}_W)$, if $(W, \mathcal{O}_W)$ isomorphic to $(Spec(B), \mathcal{O}_{Spec(B)})$ for some ring B then $(W, \mathcal{O}_W)$ is called an affine scheme.*

We define morphism between affine schemes as morphism between the respective locally ringed spaces this makes affine schemes into a category which we denote by (Aff).

Given Rings A, B and a ring homomorphism $\psi : A \longrightarrow B$ we can construct the topological spaces $Spec(A), Spec(B)$ and a morphism $Spec(\psi) : Spec(B) \longrightarrow Spec(B)$ (for notational convenience we denote $Spec(B)$ by W, $Spec(A)$ by Z and $Spec(\psi)$ by f). For $Spec$ to be a functor from category of Rings to category of Affine schemes we need to construct a morphism of sheaves $f^\flat : \mathcal{O}_Z \longrightarrow f_*\mathcal{O}_W$ (we will show that $(f, f^\flat)$ is a morphism of locally ringed spaces). We demand that $f^\flat_Z : A = \mathcal{O}_Z(Z) \longrightarrow f_*\mathcal{O}_W(Z) = \mathcal{O}_W(f^{-1}(Z)) = \mathcal{O}_W(W) = B$ should be equal to the given ring homomorphism ψ (This particular construction will gives an anti equivalence between the category of Rings and the category of Affine schemes)

Let $s \in A$ then $f^{-1}(D(s)) = D(\psi(s))$, we define

$$f^{\flat}_{D(s)} : \mathcal{O}_Z(D(s)) = A_s \longrightarrow B_{\psi(s)} = (f_*\mathcal{O}_W)(D(s))$$

as the ring hommorphism induced by ψ. This ring homomorphism is compatible with the restriction map of principal open subsets. Let U be an open set of Z then for every principal open set $D(f)$ contained in U we have morphisms from $\mathcal{O}_Z(U)$ to $(f_*\mathcal{O}_W)(D(f))$ (by composing the restriction from $\mathcal{O}_Z(U)$ to $\mathcal{O}_Z(D(f))$ with $f^{b}_{D(f)}$). We can view $\mathcal{O}_Z(U)$ as the inverse limit of the inverse system containing $\mathcal{O}_Z(D(f))$ ($D(f) \subseteq U$), similarly $(f_*\mathcal{O}_W)(U)$ is the inverse limit of the inverse system containing $(f_*\mathcal{O}_W)(D(f))$ ($D(f) \subseteq U$). Hence from the universlity of inverse limit we have a unique morphism from $\mathcal{O}_Z(U)$ to $(f_*\mathcal{O}_W)(U)$ which is compatible with the restriction morphisms, define this as $f^{\flat}_U$. So we get a morphism of sheaves. For $w \in W$

$$f^{\sharp}_w : \mathcal{O}_{Z,f(w)} = A_{\psi^{-1}(p_w)} \longrightarrow B_{p_w} = \mathcal{O}_{W,w}$$

is the homomorphism induced by ψ and in particular it is a local ring homomorphism. Hence $(f, f^{\flat})$ is a morphism of affine schemes. for simplicity we denote this morphism by $Spec(\psi)$, if $\phi : B \longrightarrow C$ then $Spec(\phi \circ \psi) = Spec(\psi) \circ Spec(\phi)$. Hence we get a contravariant functor

$$Spec : (Ring) \longrightarrow (Aff)$$

Define the functor Γ as follows If $(f, f^{\flat}) : (W, \mathcal{O}_W) \longrightarrow (Z, \mathcal{O}_Z)$ is a morphism of ringed spaces, define $\Gamma(W) := \mathcal{O}_W(W)$ and $\Gamma(f, f^{\flat}) := f^{\flat}_Z$. Restricting Γ to the category of Affine schemes we get a contravarinat functor

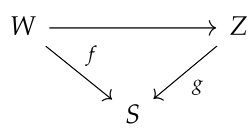
$$\Gamma : (Aff) \longrightarrow (Ring)$$

Proposition 1.2.4. *The functors $Spec$ and Γ define an anti-equivalence between the category of rings and the category of affine schemes.*

1.3 Schemes

Definition 1.3.1. *Consider a locally ringed space $(W, \mathcal{O}_W)$, If there exist some open cover $W = \bigcup_{i \in I} U_i$ such that the locally ringed spaces $(U_i, \mathcal{O}_X|_{U_i})$ obtained by restriction is an affine scheme for all i then we say that $(W, \mathcal{O}_W)$ is a scheme. We define morphism between schemes as same as the morphism between the underlying locally ringed spaces.*

By (Sch) we denote the category of schemes.
Given a scheme S consider the collection of tuples (W, f) where W is a scheme and $f : W \longrightarrow S$ is a morphism of schemes, if (Z, g) is another tuple then we define a morphism between (W, f) and (Z, g) as a scheme morphism $W \longrightarrow Z$ such that the following diagram commutes.



This collection of tuples with morphisms form a category called schemes over S (or of S-schemes) which we denote by Sch/S. f is called the structural morphism of the S–scheme W. We say S is the base scheme and it is evident that (S, id_S) is the final object of the category (Sch/S).

1.3.1 Open subschemes

Proposition 1.3.2. 1. *Consider a scheme W and an open subset U of W. The locally ringed space $(U, \mathcal{O}_W|_U)$ is a scheme which is called an open subscheme of W. If U is an "affine scheme", then U is called an "affine open subscheme".*

2. *The affine open subschemes of a scheme forms a basis of its topology.*

Proposition 1.3.3. *Consider two affine open subschemes U, V of a scheme W then for each $w \in U \cap V$ there exist a open subscheme W_w (depending on w) such that $w \in W_w$ and it is principally open in both U, V.*

1.3.2 Morphisms into affine schemes, gluing of morphisms .

Schemes are obtained by gluing affine schemes together so if we have proper "gluing lemmas"(If we know how to glue along open affine schemes) many facts about schemes can be verified by evaluating them at the level of open affine schemes

Proposition 1.3.4. *Consider locally ringed spaces W, Z and an open subset U of W. The map $U \mapsto Hom(U, Y)$ (the set of morphism between the locally ringed spaces $U, \mathcal{O}_W|_U$ and $Z\mathcal{O}_Z$) is a presheaf of sets on W.*

Proposition 1.3.5. *(Gluing of morphisma) Consider locally ringed spaces W, Z and let $Hom(U, Z)$ has the same meaning as in the previous proposition for any given open set U of W. Then the presheaf $U \mapsto Hom(U, Z)$ is a sheaf on W.*

Proof. Let $W = \bigcup_i U_i$ be an open covering. If $(\psi, \psi^\flat), (\phi, \phi^\flat) \in Hom(W, Z)$ such that $res^W_{U_i}(\psi, \psi^\flat) = res^W_{U_i}(\phi, \phi^\flat)$ for all i, then the continous maps $\psi = \phi$. Let $f \in \mathcal{O}_Z(Z)$ then $(j^W_{U_i})^\flat \circ \psi^\flat(f) = (j^W_{U_i})^\flat \circ \phi^\flat(f)$ that is $\psi^\flat(f)|_{U_i} = \phi^\flat(f)|_{U_i}$ for all i, since W is a locally ringed space $\psi(f) = \phi(f)$ (from the sheaf property). Since ψ, ϕ agrees on all elements of $\mathcal{O}_Z(Z)$, $\psi = \phi$

Let $(\psi_i, \psi_i^\flat) \in Hom(U_i, Z)$ be such that $res^{U_i}_{U_i \cap U_j}(\psi_i, \psi_i^\flat) = res^{U_j}_{U_i \cap U_j}(\psi_j, \psi_j^\flat)$. Then from pasting lemma of continous maps there exist unique $\psi : W \longrightarrow Z$. Let $f \in \mathcal{O}_Z(Z)$ then the compatibility condition of $\psi_i^\flat$ implies that $\psi_i^\flat(f)|_{U_i \cap U_j} = \psi_j^\flat(f)|_{U_i \cap U_j}$ then there exist $g \in \mathcal{O}_W(W)$ such that $g|_{U_i} = \psi_i^\flat(f)$, define $\psi^b(f) = g$. $\psi^\flat|_{U_i} = \psi_i$ Since the maps ψ_i are compatible with restriction morphism of the locally ringed space W, $\psi^\flat$ is also compatible with restrictions hence $(\psi, \psi^\flat) \in Hom(W, Z)$

■

From the above proposition it follows that.

Proposition 1.3.6. *Consider the schemes $(W, \mathcal{O}_W)$ and $Z = Spec(B)$ then the natuaral map*

$$Hom(W, Z) \longrightarrow Hom(B, \Gamma(W, \mathcal{O}_W)) \quad (f, f^\flat) \mapsto f^\flat_Z$$

is a bijection.

Corollary 1.3.7. *$Spec(\mathbb{Z})$ is the final object in the category of Schemes.*

We have $Hom(W, Spec(\mathbb{Z}[T])) \simeq \Gamma(W, \mathcal{O}_W)$ (this follows from the fact that $Hom(\mathbb{Z}[T], R) \simeq R$). More generally for an R scheme X we have $Hom_R(W, Spec(R[T]) = B^1_R) = Hom_{R-alg}(R[T], \Gamma(W, \mathcal{O}_W)) = \Gamma(W, \mathcal{O}_W)$ (by definition of an R−scheme morphism the image of R in $\Gamma(W, \mathcal{O}_W)$ is already determined hence we only have the freedom to choose where T goes) .

Remark 1.3.8. *We may apply Proposition 1.3.6 also to $B = \Gamma(W, \mathcal{O}_W)$. Thus for every scheme W there is a morphism $c_W : W \longrightarrow Spec(\Gamma(W, \mathcal{O}_W))$ which corresponds to $id_{\Gamma(W, \mathcal{O}_W)}$ we call it as the canonical morphism .*

1.4 Basic properties of schemes and morphisms of schemes

1.4.1 Topological Properties

Definition 1.4.1. *Consider a scheme $(W, \mathcal{O}_W)$*

- *$(W, \mathcal{O}_W)$ is called a connected scheme if W is connected as a topological space.*
- *$(W, \mathcal{O}_W)$ is said to be quasi-compact if W is quasi compact as a topological space.*
- *$(W, \mathcal{O}_W)$ is said to be irreducible if W is irreducible as a topological space.*

We have already seen that Affine schemes are quasi compact (for any ring B, $Spec(B)$ is quasis compact)

Definition 1.4.2. *Consider a morphism of schemes $f : W \longrightarrow Z$, it is said to be surjective, injective,bijective,open,closed, or a homeomorphism, respectively if the underlying map between the topological spaces has this property.*

1.4.2 Noetherian Schemes

Definition 1.4.3. *Consider a scheme $(W, \mathcal{O}_W)$, if there exist an affine open cover $W = \bigcup_i U_i$ such that $\gamma(U_i, \mathcal{O}_W)$ are noetherian rings then we say that W is a locally noetherian scheme. $(W, \mathcal{O}_W)$ is said to be a noetherian scheme if its locally noetherian and the underlying topological space W is quasi-compact.*

In the case of affine scheme the notion of locally noetherian coincides with the notion of noetherian (being isomorphic to $Spec(B)$ for some ring B affine schemes are quasi compact). Since localisation of a noetherian ring is again noetherian it follows that affine open subschemes of a locally noetherian scheme are noetherian. In particular locally noetherian schemes admits a basis consisting of noetherian affine open subschemes.
Given a locally noetherian scheme W, for every $w \in W$ the local rings $\mathcal{O}_{W,w}$ are noetherian. But even for affine schemes W it is not true that if $O_{W,w}$ is noetherian for all $w \in W$, then W is noetherian.

Proposition 1.4.4. *Consider an affine scheme $W = Spec(B)$ for some ring B then B is a noetherian ring if and only if W is a noetherian scheme .*

Remark 1.4.5. *The underlying topological space of an affine noetherian scheme is a noetherian topological space (Since the ring assosciated to the affine scheme is noetherian). If W is a noetherian scheme then it is quasi compact hence it can be covered by finitely many open affine noetherian subschemes. It follows that the underlying topological space of W is noetherian (and in particular has only finitely many irreducible components).*

Corollary 1.4.6. *Open subscheme of a (locally) noetherian scheme is again (locally) noetharian.*

1.4.3 Reduced and integral schemes, function fields.

Definition 1.4.7.
- *Consider a scheme W, if all local rings $\mathcal{O}_{W,w}, w \in W$ are reduced (rings without nilpotent elements) then W is said to be reduced.*
- *A reduced scheme which is irreducible is called an integral scheme.*

Proposition 1.4.8. *Consider a scheme W.*

1. *The ring $\Gamma(U, \mathcal{O}_X)$ is reduced for every open set U of W if and only if W is a reduced scheme.*
2. *The ring $\Gamma(U, \mathcal{O}_X)$ is an integral domain for every non empty open set U of W if and only if W is an integralscheme.*
3. *If W is integral then $\mathcal{O}_{W,w}$ is an integral domain for every $w \in W$.*

Let $W = Spec(B)$ be an affine scheme then W is an integral scheme if and only if B is an integral domain. If $\eta \in W$ is a generic point of W then η corresponds to the zero ideal (Since nilradical is the zero ideal). Then $O_{W,\eta} = B_0$ which is equal to the field of fractions of B (hence a field).

Proposition 1.4.9. *Let W be a scheme the mapping*

$$W \xrightarrow{w \mapsto \overline{\{w\}}} \{Y \subseteq W : Y \quad irreducible, closed\}$$

is a bijection. That is every closed irreducible subset of W contains a unique generic point (that is W is a sober space).

Definition 1.4.10. *Let η be the generic point of an integral scheme W. We set $K(W) = \mathcal{O}_{W,\eta}$, $K(W)$ is a field which will be referred as the function field of W.*

Since W is irreducible existence of η is assured, η will be contained in some affine scheme hence $O_{W,\eta}$ will be a field as desired.

Lemma 1.4.11. *Let $W = Spec(B)$ be an integral affine scheme and η be its generic point. If $U \subseteq W$ is an open set then*

$$\Gamma(U, \mathcal{O}_W) = \bigcap_{w \in U} O_{W,w}$$

Proposition 1.4.12. *Let η be the generic point of an integral scheme W whose function field is $K(X)$.*

1. *Let U be a non empty affine open subscheme of W, that is $U = Spec(B)$ for some ring B, then $K(W) = Frac(B) = Frac(\mathcal{O}_{W,w})$ for every $w \in W$.*
2. *Consider the non empty open sets $U \subseteq V \subseteq W$. Then the maps*

$$\Gamma(V, \mathcal{O}_W) \xrightarrow{res^V_U} \Gamma(U, \mathcal{O}_W) \xrightarrow{f \mapsto f_\eta} K(W) = O_{W,\eta}$$

are injective.

3. *Let U be an open subset of W which is not empty then given any open covering $U = \bigcup_i U_i$ we have*

$$\Gamma(U, \mathcal{O}_W) = \bigcap_i \Gamma(U_i, \mathcal{O}_W) = \bigcap_{w \in U} O_{W,w}$$

where the intersection takes place in $K(W)$.

1.5 Prevarieties as Schemes

In some sense schemes are generalisation of Prevearieties. Prevarieties themselves are not schemes. In this section we try to assosciate a scheme to any given prevarierty (we try to construct a functor from the category of prevarieties to the category of schemes). In the case of affine variety the assosciation is clear, given any affine variety $(W, \mathcal{O}_W)$ we assosciate it with the scheme $Spec(\Gamma(W, \mathcal{O}_W))$.

1.5.1 Schemes (locally) of finite type over a field.

Definition 1.5.1. *Given a field k let W be a scheme over $Spec(k)$ (a $k-$scheme). If W admits an open cover $W = \bigcup_{i \in I} U_i$ such that $U_i = Spec(B_i)$ where B_i are finitely generated $k-$algebras we say that W is "locally of finite type over k" in addition if W is quasi-compact then we say that W is of "finite type over k".*

If B is a finitely generated $k-$ algebra then B is a noetherian ring it follows that schemes (locally) of finite type over k are (locally) noetherian.

1.5.2 Equivalence of the category of integral schemes of finite type over k and prevarieties over k .

A significant difference in the underlying topological spaces of a prevariety and a scheme is that the underlying topological space of a prevariety need not contain a unique generic point correspond to every irreducible closed subset but this is true in the case of schemes (proposition 4.3.8) and in the case of prevariety every point is closed which is not true in the case of a scheme

Definition 1.5.2. *Consider a topological space W. If every closed irreducible subset V of W contains a unique generic point then we say that W is a sober space.*

As a first step to associate schemes to prevariety we will assosciate a sober space to the underlying topological space of a prevariety .

Let X be the underlying topological space of a prevariety then every point in X is closed. Consider the set of all irreducible closed subsets of X we denote this set by $t(X)$. Now we try to define a topology on $t(X)$. Let Z be a closed subset of X then every irreducible closed subset of Z is also irreducible an closed in X hence $t(Z)$ can be conidered as a subset of $t(X)$. We say a subset of $t(X)$ is closed if and only if it is of the form $t(Z)$ for some closed subset Z of X.

- Clearly ϕ and $t(X)$ are closed.
- For closed subsets Z_1, Z_2 of X $t(Z_1) \cup t(Z_2) = t(Z_1 \cup Z_2)$. If $U \in t(Z_1) \cup t(Z_2)$ then $U \in t(Z_1 \cup Z_2)$ (Since irreducible closed subset of Z_1, Z_2 are also irreducible closed subsets of $Z_1 \cup Z_2$) that is $t(Z_1) \cup t(Z_2) \subseteq t(Z_1 \cup Z_2)$. If $U \in t(Z_1 \cup Z_2)$ then either $U \subseteq Z_1$ or $U \subseteq Z_2$ (otherwise U can be written as the unoin of two proper closed subsets $U = (U \cap Z_1) \cup (U \cap Z_2)$) that is $t(Z_1 \cup Z_2) \subseteq t(Z_1) \cup t(Z_2)$. Hence $t(Z_1) \cup t(Z_2) = t(Z_1 \cup Z_2)$ as desired.
- $t(\bigcap_i Z_i) = \bigcap_i t(Z_i)$ for closed subsets $Z_i \subseteq X$

So our definition of closed sets define a topology on $t(X)$. Let $f : X \longrightarrow Y$ then we can define a map $t(f) : t(X) \longrightarrow t(Y)$ by mapping each point of $t(X)$ which corresponds to a irreducible closed subset of X to its closure in the image of f (Since irreducibility of Z implies irreducibility of $f(Z)$ the map is well defined). Let F be a closed subset of Y then

$t(f)^{-1}(t(F)) = \{U \subseteq X \quad irreducible \quad closed : \overline{f(U)} \subseteq F\} = \{U \subseteq X \quad irreducible \quad closed : f(U) \subseteq F\} = \{U \subseteq f^{-1}(F) \quad irreducible \quad closed\} = t(f^{-1}(F))$

That is $t(f)$ is continous. If $Z \subseteq X$ is irreucible and closed then $t(Z)$ is irreucible and closed (Let $t(Z) = t(Z_1) \cup t(Z_2)$ for two closed subsets, $Z \in t(Z)$ without loss of generality assume $Z \in t(Z_1) \Rightarrow Z \subseteq Z_1 \Rightarrow t(Z) \subseteq t(Z_1)$ so $t(Z)$ can not be written as the union of two proper closed subsets). Let $U \subseteq t(X)$ be closed and irreducible then $U = t(Z)$ for some closed set Z. If $Z = Z_1 \cup Z_2$ for some closed subsets Z_1, Z_2 of X then $t(Z) = t(Z_1 \cup Z_2) = t(Z_1) \cup t(Z_2)$ without loss of generality assume that $t(Z) = t(Z_1)$ (Since $t(Z)$ is irreducible). Every $x \in X$ is closed and irreducible so $t(Z) = t(Z_1) \Rightarrow Z = Z_1$. That is Zcan not be written as the union of two proper closed subsets hence it is irreducible. Irreducible closed subsets of $t(X)$ are in the form $t(Z)$ where $Z \subseteq X$ is closed and irreducible. Given any irreducible closed subset $t(Z) \subseteq t(X)$ there is a unique generic point $Z \in t(Z)$ (Z is not contained in any proper closed subset of $t(Z)$ hence $\overline{Z} = t(Z)$). t defines a functor from the category of topological spaces all of whose points are closed to the category of sober spaces. We say $t(X)$ is the sober space associated to X or the soberification of X

Given X we have a natural continous map $\alpha_X : X \longrightarrow t(X)$ which maps each $x \in X$ to the irreducible closed subset $\{x\} \in t(X)$. Let $Z \in t(X)$ then $\overline{Z} = t(Z)$ where the closure is taken inside $t(X)$ so the only closed points of $t(X)$ are

in the form $\{x\} = t(\{x\})$ for some $x \in X$. So α_X is a bijection between X and the closed points of $t(X)$. α_X defines a bijection between the closed subsets of X and the closed subsets $\alpha_X(X)$ (because $t(Z) \cap \alpha_X(X) = \{\{x\} : x \in Z\}$). Hence α_X is a homeomorphism from X onto the set of closed points in $t(X)$. If $t(Z_1) \neq t(Z_2)$ then $t(Z_1) \cap \alpha_X(X) \neq t(Z_2) \cap \alpha_X(X)$ (the map $t(Z) \mapsto t(Z) \cap \alpha_X(X)$ is a bijection) hence by proposition 5.4.5(2) the set of closed points of $t(X)$ is very dense in $t(X)$
This construction can be generalized to give a functor from the category of topological spaces to the full sub category of sober spaces.
Let k be an algebraically closed field then we already showed that the folowing categories are equivalent

1. the category of integral affine schemes of finite type over k
2. the opposed category of integral finitely generated k-algebras
3. the category of affine varieties

For a k-scheme W locally of finite type we will identify $W(k) = Hom_k(Spec(k), W)$ with the set of closed points of W. Define $\alpha : W(k) \longrightarrow W$ as the inclusion map. We define a sheaf of rings as follows

$$O_{W(k)} := \alpha^{-1}\mathcal{O}_W$$

From $(\alpha^{-1}\mathcal{O}_W)_w = O_{W,\alpha(w)}$ we get that $(W(k), O_{W(k)})$ is a locally ringed space.

Theorem 1.5.3. *The above construction $(W, \mathcal{O}_W) \mapsto (W(k), O_{W(k)})$ give rise to an equivalence of the following categories*

- *the category of integral schemes of finite type over k*
- *the category of prevariety over k*

Consider a integral scheme W which is of finite type over k where $k-$ is an algebraically closed field and $W(k)$ be the associated prevariety. Let $\eta \in W$ denotes the generic point of W then $\eta \in U$ for some open affine scheme $U \subseteq W$ (then $U \cap W(k)$ is an open affine variety of $W(k)$).

$$K(W) = Frac(O_{W,\eta}) = Frac(\mathcal{O}_W(U) = O_{W(k)(U)}) = K(W(k))$$

.
Via the equivalence of categories described earlier the k-scheme $\mathbb{A}^n(k)$ (resp. $\mathbb{P}^n(k)$) corresponds to the prevariety $\mathbb{A}^n(k)$ (resp. $\mathbb{P}^n(k)$).

1.6 Subschemes and Immersions

1.6.1 Open Immersions

Definition 1.6.1. *A morphism $j : Z \longrightarrow W$ of schemes is called an open immersion, if the underlying continuous map is a homeomorphism of Z with an open subset U of W, and the sheaf homomorphism $\mathcal{O}_W \longrightarrow j_*\mathcal{O}_Z$ induces an isomorphism $\mathcal{O}_W|_U \simeq= j_*\mathcal{O}_Z$ (of sheaves on U) .*

That is if we have an open immersion from Z to W then Z can be considered as an open subscheme of W.

1.6.2 Closed subschemes

Here we discuss the notion of a closed subscheme. As a starting point lets consider the case of an affine sheme $Spec(B)$. closed subsets of $Spec(B)$ are of the form $V(b)$ and it is homemorphic to $Spec(B/b)$, So we can define a scheme structure on $V(b)$ induced from $Spec(B/b)$ Via this homeomorphism. We want schemes of this form to be closed subschemes of $Spec(B)$ (and any closed subschme to have this form) . It is to be noted that a closed subset $Y \subseteq Spec(B)$ can be represented using different ideals and each ideal yeilds a different scheme structure on Y. That is there may be many closed subchemes with the same underlying closed subset.

Definition 1.6.2. *Let W be a topological space and $\mathcal{G}$ a sheaf of rings on W. An ideal sheaf $\mathcal{F}$ in $\mathcal{G}$ is a subobject of $\mathcal{G}$ in the category of sheaves of $\mathcal{G}$-modules, i.e., a subsheaf of $\mathcal{G}$ viewed as a sheaf of abelian groups such that* $\Gamma(U,\mathcal{G}).\Gamma(U,\mathcal{F}) \subseteq \Gamma(U,\mathcal{F})$ *for all open subsets U of W. In other words, $\mathcal{F}(U)$ is an ideal of $\mathcal{G}(U)$ for all open sets $U \subseteq W$*

Definition 1.6.3. *If I is an ideal sheaf in a sheaf $\mathcal{F}$ of rings, then the quotient sheaf $\mathcal{F}/I$ is the sheaf associated to the presheaf $U \mapsto F(U)/I(U)$.*

Lemma 1.6.4. *Let I be an ideal sheaf in $\mathcal{F}$ (sheaf of rings on the topological space W) then the canonical map $\mathcal{F} \longrightarrow \mathcal{F}/I$ is surjective. The canonical map $\mathcal{F}(U)/I(U) \longrightarrow (\mathcal{F}/I)(U)$ is injective for all open subsets $U \subseteq W$ (it is not surjective in general)*

Lemma 1.6.5. *Let $\mathcal{F}$ and $\mathcal{G}$ be ringed spaces defined on a topological space W. Let $f : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of ringed spaces then $U \mapsto Ker(f)(U) : Ker(f_U)$ for all open set $U \subseteq W$ is an ideal sheaf (It is denoted by $Ker(f)$) in $\mathcal{F}$*

Definition 1.6.6. *Consider a scheme W.*

1. *A closed subscheme of W is given by a closed subset $Y \subseteq W$ (let $i : Y \longrightarrow W$ be the inclusion) and a sheaf $\mathcal{O}_Y$ on Y, such that $(Y,\mathcal{O}_Y)$ is a scheme, and such that the sheaf $i_*\mathcal{O}_Y$ is isomorphic to $\mathcal{O}_W/\mathcal{I}$ for a sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_W$.*

2. *A morphism $i : Y \longrightarrow W$ of schemes is called a closed immersion, if the underlying continuous map is a homeomorphism between Y and a closed subset of W, and the sheaf homomorphism $i^\flat : \mathcal{O}_W \longrightarrow i_*\mathcal{O}_Y$ is surjective .*

Let $Y \subseteq W$ be a closed subscheme in the sense of (1) . Set $i^\flat : (W,\mathcal{O}_W) \longrightarrow (W,\mathcal{O}_W/\mathcal{I}) \longrightarrow (W, i_*\mathcal{O}_Y)$ then (1) of the above definition imply that $i^\flat$ is surjective. That is for every closed subscheme Y of W we have $(i,i^\flat) : Y \longrightarrow W$ such that $(i,i^\flat)$ is a closed immersion. Note that in part (1) of the above definition we explicitly demand that $(Y,\mathcal{O}_Y)$ is a scheme, this wont be true for an arbitary sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_W$. The investigation regarding which sheaf of ideals give rise to a closed subscheme leads to the notion of "quasi coherent" sheaves.
Let $(i,i^\flat) : Y \longrightarrow W$ be a closed immersion and let $Y' = i(Y)$, by definition Y' is closed subset of W. Let $\mathcal{I}$ be the ideal sheaf defined by $\mathcal{I} = ker(i^\flat)$. We denote the presheaf $U \mapsto \mathcal{O}_W(U)/\mathcal{I}(U)$ (for $U \subseteq W$ open) by $\mathcal{F}$. Then we have $j^\flat : \mathcal{F} \longrightarrow i_*\mathcal{O}_Y$ induced from $i^\flat$. Since $j^\flat_U$ is injective for all open sets U of W the induced maps on the stalks are injective. As $j^\flat$ is induced from $i^\flat$ then maps on stalks induced by $j^\flat$ is surjective. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{j^\flat} & i_*\mathcal{O}_Y \\ \uparrow & \nearrow_{\exists!\overline{j^\flat}} & \\ \mathcal{O}_W/\mathcal{I} & & \end{array}$$

and the maps on the stalks induced by the unique map $\overline{j^\flat}$ is bijective hence $\overline{j^\flat} : \mathcal{O}_W/\mathcal{I} \longrightarrow i_*\mathcal{O}_Y$ is an isomorphism. We can have a scheme structure on Y' induced from Y via the map i. If $k : Y' \longrightarrow W$ is the inclusion then $\mathcal{O}_W/\mathcal{I} \simeq k_*\mathcal{O}_Y$. That is every closed immersion to W give rise to a closed subscheme of W.

Theorem 1.6.7. *Consider an affine scheme $W = Spec(B)$. Consider closed subsets of W of the form $V(b)$ where b is an ideal of B, we give $V(b)$ the structure of a scheme via the homeomorphism $Spec(B/b) \simeq V(b)$. $V(b)$ with this scheme structure is a closed subscheme of W. Every closed subscheme of W looks like this, that is there is a bijective correspondence between the set of closed subschemes of W and the set of ideals of B.*

1.6.3 Subschemes and immersions.

Definition 1.6.8.
- *Consider a scheme W, we say that $(Z, \mathcal{O}_Z)$ is a subscheme of W if Z is locally closed in W and W contains an open subscheme U such that Z is a closed subscheme of U. (i. e. U is the complement of $\overline{Z}\backslash Z$. since Z is locally closed Z is open in $\overline{Z}$ and $\overline{Z}\backslash Z$ is closed in $\overline{Z}$). We then have a natural morphism of schemes $Z \longrightarrow W$ ($Z \longrightarrow U \longrightarrow W$).*
- *Consider a morphism of schemes $i : Z \longrightarrow W$. If $i(Z)$ is locally closed, i is an homeomorphism between Z and $i(Z)$ and for every $z \in Z$ the ring homomorphism $i_z : O_{W,i(z)} \longrightarrow O_{Z,z}$ is surjective.*

Let Z be a subscheme of W and U be the largest open set containing W. Then we know that Z is closed subscheme of U. Let $j : Z \longrightarrow U$ be the inclusion then the sheaf morphism $j^b : \mathcal{O}_W|_U \longrightarrow j_*\mathcal{O}_Z$ is surjective by definition. Let $i : U \longrightarrow W$ be the inclusion then the map at the level of stalks $i^\sharp_w : O_{W,w} \longrightarrow (i_*\mathcal{O}_W|_U)_w$ is surjective for all $w \in U$. That is the map on the stalk induced by $i \circ j : Z \longrightarrow W$ is surjective for all $z \in Z$. Hence the natuaral map $Z \hookrightarrow W$ is an immersion.
Let $k : Z' \longrightarrow W$ be an immersion and Let $Z \subseteq W$ be locally closed set which is homeomorphic to Z' via the underlying continous map (and Z get a scheme structure from Z' via this identification). Define U as the complement of $\overline{Z}\backslash Z$ in W. Then $Z \subseteq U$ is closed, let $j : Z \longrightarrow U$ be a morphism of schemes induced by k. For any $z \in Z$ we have $i^\sharp_z : (\mathcal{O}_W|_U)_z \longrightarrow (j_*\mathcal{O}_Z)_z$ is surjective (Since j is defined using k) if $z \notin Z$ then the surjectivity follows trivially hence Z is a closed subscheme of U, and it is a subscheme of W. That is every immersion give rise to a subscheme. (note the similarity in the definition of quasi projective variety and subscheme). If Z is a subscheme of W, whose underlying subset is closed in W, then Z is a closed subscheme of W. (The corresponding statement for open subschemes is false) .
Let W be a k–scheme of finite type. That is $W = \bigcup_{i=1}^n U_i$ such that $U_i = Spec(B_i)$ and B_i are f.g k–algebras. It follows immediately that principal open sets of U_i are k–schemes of finite type. U_i is a noetherian space (Since B_i is noetherian) hence every open set $U \subseteq U_i$ is quasi compact and can be covered using finitely many principal open sets. Hence every open set of of U_i is a k-scheme of finite type. If $Y \subseteq W$ is a closed scheme then it is a k–scheme of finite type because the affine coordinate rings are just quotients of the corresponding rings of W (W is noetherian hence it is quasi-compact which implies Y is quasi compact). It follows immediately that any subscheme of W is a k-scheme of finite type.

1.6.4 Projective and quasi-projective schemes over a field.

Sometimes it is helpful if we know how a particular scheme is embedded as a subscheme of projective space.

Definition 1.6.9. *Consider a field k.*

- *A k- scheme W is called projective, if there exist $n \geq 0$ and a closed immersion $W \hookrightarrow \mathbb{P}^n(k)$.*
- *A k-scheme W is called quasi-projective, if there exist $n \geq 0$ and an immersion $W \hookrightarrow \mathbb{P}^n(k)$.*

1.6.5 The underlying reduced subscheme of a scheme

Let W be a scheme then it is possible that there exist many closed subschemes of W with the underlying topological space as W. Let nil_W denotes the map $U \mapsto nil(\mathcal{O}_W(U))$ for some open set $U \subseteq W$. Let $U \subseteq V$ be open sets and $f \in nil_W(V)$ then $res^V_U(f) \in nil_W(U)$ hence nil_W is a presheaf (which takes values in the category of abelian groups). Let $f, g \in nil(U)$ and $U = \bigcup_i U_i$ be an open cover, if $f|_{U_i} = g|_{U_i}$ then $f = g$. But (W, nil_W) is not a sheaf in general because the axiom of glueing is not true in general. Let $\mathcal{N} := \mathcal{N}_W$ be the sheafification of nil_W ($\mathcal{N}_W$ is a sheaf of abelian groups). Let $f \in \mathcal{N}_W(U)$ and $g \in \mathcal{O}_W(U)$, if $f \in nil_W(U)$ we have $gf \in \mathcal{N}_W(U)$, if $f \notin nil_W(U)$ then there exist an open cover $u = \bigcup_i U_i$ such that $f|_{U_i} \in nil_W(U_i)$ then $g|_{U_i} f_{U_i} = (gf)|_{U_i} \in nil_W(U_i)$ since $\mathcal{N}_W$ is a sheaf this would imply that $gf \in \mathcal{N}_W(U)$ hence $N_W \subseteq \mathcal{O}_W$ is a sheaf of ideals.

As usual $\mathcal{O}_W/\mathcal{N}_W$ denotes the sheaf associated to the presheaf $U \mapsto \mathcal{O}_W(U)/\mathcal{N}_W(U)$.Let $W = \bigcup_i U_i$ be an open cover by affine subschemes and $A_i = \Gamma(U_i, \mathcal{O}_W)$. Let $\mathcal{B}_i$ denotes the basis of U_i consisting of principal open subsets. we claim that $\mathcal{N}_W|_{\mathcal{B}_i} = nil_W|_{\mathcal{B}_i}$. Let $g \in A_i$ and $f \in \mathcal{N}_W(D(g))$ then either $f \in nil_W(D(g))$ or there exist an open cover $D(g) = \bigcup_j V_j$ such that $f|_{V_j} \in nil_W(V_j)$. if $f \in nil_W(D(g))$ we are done (we already know that $nil_W(D(g)) \subseteq \mathcal{N}_W(D(g))$). Principal open sets are quasi-compact so there exist a finite indexing set J such that $D(g) = \bigcup_{j \in J} V_j$. There exist $n_j \in \mathbb{N}$ such that $res^{D(g)}_{V_j}(f^{n_j}) = 0$ set $N := max_{j \in J}\{n_j\}$ then we have $f^N|_{V_j} = 0 \quad \forall j \in J$ since $\mathcal{N}_W$ is a sheaf it follows that $f^N = 0$ that is $f \in nil_W(D(g))$ (since $\mathcal{N}_W \subseteq \mathcal{O}_W$) and $\mathcal{N}_W|_{\mathcal{B}_i} = nil_W|_{\mathcal{B}_i}$ as desired.
Let $\mathcal{F}$ denotes the presheaf $U \mapsto \mathcal{O}_W(U)/\mathcal{N}_W(U)$ then $\mathcal{F}(D(g)) = \mathcal{O}_W(D(g))/\mathcal{N}_W(D(g)) = (A_i)_g/nil((A_i)_g) = (A_i)_g/(A_i)_g nil(A_i) = (A_i/nil(A_i))_g = O_{Spec(A_i/nil(A_i))}(D(g))$. We have $Spec(A_i) = Spec(A_i/nil(A_i))$ and $\mathcal{F}|_{\mathcal{B}_i} = O_{Spec(A_i/nil(A_i))}|_{\mathcal{B}_i}$. Let $f \in (\mathcal{O}_W/\mathcal{N}_W)(D((g))$ then there exist an open cover $D(g) = \bigcup_j(D(g_j))$ such that $f|_{D(g_j)} \in \mathcal{F}(D(g_j))$ (this happens because $\mathcal{B}_i$ is a basis) it is easy to see that $f \in \mathcal{F}(D(g))$ (Since $\mathcal{F}$ is a sheaf on $\mathcal{B}_i$) that is $\mathcal{F}|_{\mathcal{B}_i} = (\mathcal{O}_W/\mathcal{N}_W)|_{\mathcal{B}_i}$. Hence $\mathcal{O}_W/\mathcal{N}_W|_{U_i} = O_{Spec(A_i/nil(A_i))}$ (Since the evaluation on a basis completely determines the sheaf). This shows that $\mathcal{O}_W/\mathcal{N}_W$ is a scheme. By definition this is a closed subscheme of $(W, \mathcal{O}_W)$
Let $U \subseteq U_i$ and $g \in A_i$ such that $D(g) \subseteq U$. $(\mathcal{O}_W/\mathcal{N}_W)(D(g)) = (A_i/nil(A_i))_g = (A_i)_g/(A_i)_g nil(A_i) = (A_i)_g/nil((A_i)_g)$ hence it is a reduced ring, that is intersection of all prime ideals of $\mathcal{O}_W/\mathcal{N}_W(D(g))$ is the zero ideal. We have a ring homomorphism $res^U_{D(g)} : (\mathcal{O}_W/\mathcal{N}_W)(U) \longrightarrow (\mathcal{O}_W/\mathcal{N}_W)(D(g))$. Pullbacks of prime idelas of $(\mathcal{O}_W/\mathcal{N}_W)(D(g))$ are prime in $(\mathcal{O}_W/\mathcal{N}_W)(U)$ so $nil((\mathcal{O}_{XW}/\mathcal{N}_W)(U)) \subseteq Ker(res^U_{D(g)})$. Let $U = \bigcup_j D(g_j)$ then $nil((\mathcal{O}_W/\mathcal{N}_W)(U)) \subseteq \bigcap_i Ker(res^U_{D(g_i)})$. If $f \in \bigcap_i Ker(res^U_{D(g_i)})$ that is $f|_{D(g_i)} = 0$ for all i hence $f = 0$ which implies that $nil((\mathcal{O}_W/\mathcal{N}_W)(U)) = 0$. Similar arguments for an arbitary ring shows that $(W, \mathcal{O}_W/\mathcal{N}_W)$ is a reduced scheme we denote this scheme by W_{red}
The following proposition shows that in some sense W_{red} is the smallest closed scheme of $(W, \mathcal{O}_W)$ with the underlying topological space same as W.

Proposition 1.6.10. *Let W' denotes a closed subscheme of $(W, \mathcal{O}_W)$ whose underlying topological space is W then the inclusion morphism $W_{red} \longrightarrow X$ factors through a closed immersion $W_{red} \longrightarrow W'$.*

We get a functor from the "category of schemes" to the "category of reduced schemes" by sending a scheme W to its underlying reduced subscheme W_{red}.

Proposition 1.6.11. *Given a morphism $f : W \longrightarrow Z$ between schemes its possible to define a morphism $f_{red} : W_{red} \longrightarrow Z_{red}$ such that the diagram*

$$\begin{array}{ccc} W_{red} & \xrightarrow{i_W} & W \\ \Big\downarrow{\scriptstyle f_{red}} & & \Big\downarrow{\scriptstyle f} \\ Z_{red} & \xrightarrow{i_Z} & Z \end{array}$$

commutes. The map from the category of schemes which takes any scheme W to the reduced scheme W_{red} and any morphism of schemes f to f_{red} defines a functor from the "category of schemes" to the "category of reduced schemes".

Proposition 1.6.12. *Consider a locally closed subset Y of a scheme W, then there exist a unique reduced subscheme Y_{red} of W whose undrlying topological space is Y.*

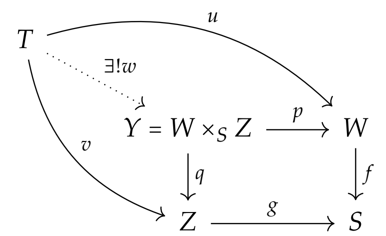
Chapter 2

More on Schemes

2.0.1 Fiber products in arbitrary categories.

Consider a fixed object S in the category $\mathcal{C}$.

Definition 2.0.1. *For two morphisms $f : W \longrightarrow S$ and $g : Z \longrightarrow S$ in $\mathcal{C}$ we call a triple (Y, p, q) consisting of an object Y in $\mathcal{C}$ and morphisms $p : Y \longrightarrow W$ and $q : Y \longrightarrow Z$ such that $f \circ p = g \circ q$, a fiber product of f and g or a fiber product of W and Z over S (with respect to f and g), if for every object T in (C) and for all pairs (u, v) of morphisms $u : T \longrightarrow W$ and $v : T \longrightarrow Z$ such that $f \circ u = g \circ v$ there exists a unique morphism $w : T \longrightarrow Y$ such that $p \circ w = u$ and $q \circ w = v$.*



From the universal property it is clear that the fibre product is uniquely determined upto unique isomorphism if it exists. We denote the fibre product Y using $W \times_S Z$ or $W \times_{f,g,S} Z$. The map $p : W \times_S Z \longrightarrow W$ is called the "first projection" and the map $q : W \times_S Z \longrightarrow Z$ is called the "second projection".
From the definition it follows that there is a bijective correspondence between $Hom(T, W \times_S Z)$ and the collection of pairs $(u,v) \in Hom(T, W) \times Hom(T, Z)$ such that $f \circ u = g \circ v$. This can be reformulated as follows. Define a category $\mathcal{C}/S$ whose objects are pairs (W, f) where $W \in Obj(\mathcal{C})$ and $f : W \longrightarrow S$ is a morphism in $\mathcal{C}$. (W, f) is called a S–object and f is called the structure morrphism of W. Some times we will simply write W instead of (W, f). For two S–Objects (W, f) and (Z, g) we define $Hom_S(W, Z)$ as the collection of morphism $l : W \longrightarrow Z$ such that $g \circ l = f$, these morphisms are called S–morphisms. Its easy to see that S–objects and S–morphisms constitute a category which is denote by $\mathcal{C}/S$. Usually we write $Z_S(W)$ instead of $Hom_S(W, Z)$ and call $Z_S(W)$ the set of W-valued points of Z (over S).
Given S–objects (W, f) and (Z, g), $((W \times_S Z, l), p, q)$ is the unique triple (up to unique isomorphism) such that for any S–object (T, h) the map

$$Hom_S(T, W \times_S Z) \longrightarrow Hom_S(T, W) \times Hom_S(T, Z) \quad w \mapsto (p \circ w, q \circ w)$$

is bijective.
Fiber product in $\mathcal{C}$ can be viewd as product in $\mathcal{C}/S$ with p, q as the respective projection morphisms.
Example:

- In the "category of sets (Sets)" arbitrary fiber products exist: Consider a fixed set S and maps $f : W \longrightarrow S$ and $g : Z \longrightarrow S$. Then we get that

$$\begin{array}{ccccc} W & \xleftarrow{p} & W \times_S Z = \{(w,z) \in W \times Z : f(w) = g(z)\} & \xrightarrow{q} & Z \\ \\ w & \longleftarrow & (w,z) & \longrightarrow & z \end{array}$$

  is a fiber product in the category of sets.

- Consider the category of topological spaces and S be a fixed topological space. Consider continuous maps $f : W \longrightarrow S$ and $g : Z \longrightarrow S$ then the fiber product $W \times_S Z$ is the fiber product of the underlying sets $\{(w,z) \in W \times Z : f(w) = g(z)\}$ with the topology induced by the product topology on $W \times Z$.

In the further discussions we assume that all fiber product exist in $\mathcal{C}$. Let $(W,f),(Y,g),(W',f'),(Z',g') \in \mathcal{C}/S$ and $u : W \longrightarrow W'$, $v : Z \longrightarrow Z'$ be S−morphisms then we get the following commutative diagram.

$$\begin{array}{ccccccc} W \times_S Z & \xrightarrow{p} & W & & & & \\ \downarrow q & \searrow^{u \times_S v} & & \searrow^{u} & & & \\ Z & & W' \times_S Z' & \xrightarrow{p'} & W' & & \\ & \searrow^{v} & \downarrow q' & & \downarrow f' & & \\ & & Z' & \xrightarrow{g'} & S & & \end{array}$$

Since u, v are S−morphisms we get $f = f' \circ u$ and $g = g' \circ v$. $u \times_S v$ is the unique map whose existance follows from universality of the fiber product $W' \times_S Z'$. The commutativity of the diagram is a consequence of all these facts. We say fiber product is functorial in the above sense.

2.0.2 Fiber products of schemes.

Here we will prove that fiber product always exist in the category of schemes. As a first step we prove that fiber product of affine schemes always exist and then use this to prove the general case.

Proposition 2.0.2. *Consider the rings A, B and R and the morphisms $R \longrightarrow A$ and $R \longrightarrow B$. Let S, W and Z be the affine schemes corresponding to the rings R, A and B and set $Y = Spec(A \otimes_R B)$. Similarly we get morphisms of schemes $p : Y \longrightarrow W$ and $q : Y \longrightarrow Z$ corresponding to the ring homomorphism*

$$\alpha : A \longrightarrow A \bigotimes_R B, \quad a \longrightarrow a \bigotimes 1,$$

$$\beta : B \longrightarrow A \bigotimes_R B, \quad b \longrightarrow 1 \bigotimes b.$$

Then the schem Y together with the morphism p and q is the fiber product of the schemes X and Z over S.

Proof. We know that for any Scheme T and for any affine scheme $Spec(C)$ we have the following bijection $Hom_{Sch}(T, Spec(C)$ $Hom_{Ring}(C, \Gamma(T, O_T))$. Extending this we get that If T is an S−scheme there is a bijection functorial in T as follows $Hom_{Sch/S}(T, Y) \simeq Hom_{R-alg}(A \otimes_R B, \Gamma(T, O_T)) \simeq Hom_{R-alg}(A, \Gamma(T, O_T)) \times Hom_{R-alg}(B, \Gamma(T, O_T)) \simeq Hom_{Sch/S}(T, W) \times Hom_{Sch/S}(T, Z)$ where the second bijection is induced by composition with α and β. It follows that Y is the fiber product.

■

If $f : W \longrightarrow Z$ is a morphism of schemes then by f_t we denote the continous map between the underlying topological space and by f_s we denote the sheaf morphism involved.

Theorem 2.0.3. *Fiber product exist in the category of schemes.*

Consider a fixed scheme S and the $S-$schemes W and Z then we have to show that $W \times_S Z$ exist in the category of schemes. The proof is done in several steps. The idea is to cover S, W and Z using affine schemes, and we already know that fibre product of affine schemes exist, we try to glue these together to get the desired fiber product. We denote by $w : W \longrightarrow S$ and $z : Z \longrightarrow S$ the structure morphisms.
Step 1: Assume that $(W \times_S Z, p, q)$ exist. Then prove that for any open subscheme $U \subseteq W$, $U \times_S Z = p^{-1}(U)$ and the first and second projections are given by the restriction of p and q respectively (here $p^{-1}(U)$ is viewed as an open subscheme of $W \times_S Z$).
Step 2: Assume $W = \bigcup_i U_i$ is an open covering of W and set $Y_i := U_i \times_S Z$ exists for all i then show that $W \times_S Z$ exists.
Step 3: Assume that $(W \times_S Z, p, q)$ exist. Then prove that for any open subscheme $X \subseteq S$, $w^{-1}(X) \times_S z^{-1}(X) = (w \circ p)^{-1}(X) = (z \circ q)^{-1}(X)$ and the first and second projections are given by the restriction of p and q respectively (here $(w \circ p)^{-1}(X) = (z \circ q)^{-1}(X)$ is viewed as an open subscheme of $W \times_S Z$).
Step 4: Let $S = \bigcup_i X_i$ be an open cover of S and set $W_i := w^{-1}(X_i)$ and $Z_i := z^{-1}(X_i)$. If $W^i \times_{X_i} Z^i$ exist for all i then show that $W \times_S Z$ exists and $W^i \times_{X_i} Z^i$ is an open cover of $W \times_S Z$.
Step 5: Proposition 2.0.2 together with all the previous four steps implies the theorem.

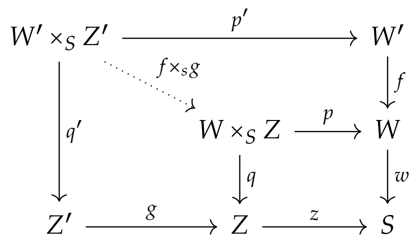
If $S = Spec(R)$ is affine, we will often write $W \times_R Z$ instead of $W \times_S Z$. If $Z = Spec(B)$ is affine, we also write $W \otimes_S B$ or, for $S = Spec(R)$ affine, $W \otimes_R B$ instead of $W \times_S Z$.

Corollary 2.0.4. *Consider $S-$schemes W and Z and an open cover $S = \bigcup_i S_i$ of S. Let W_i and Z_i be inverse image of SS_i in W and Z. Assume that for each i we have $W_i = \bigcup_j W_{ij}$ and $Z_i = \bigcup_j Z_{ij}$ Then*

$$W \times_S Z = \bigcup_i \bigcup_{j \in J_i} \bigcup_{k \in K_i} W_{ij} \times_{S_i} Z_{ik}$$

and $W_{ij} \times_{S_i} Z_{ik}$ are open in $W \times_S Z$.

Let $(W, w), (Z, z), (Z', z')$ and (W', w') be $S-$schemes. As mentioned earlier fiber product of schemes over S can be viewed as product in the category of S-schemes. Let $f : W' \longrightarrow W$, $g : Z' \longrightarrow Z$ be morphism of $S-$schemes (that is $w \circ f = w'$ and $z \circ g = z'$). Consider the following commutative diagram



We have $f \circ p' = p \circ (f \times_S g)$ and $g \circ q' = q \circ (f \times_S g)$ which justifies the notation. If $Z' = Z$ and $g = id_Z$ then as seen in the above propositoin $W' \times_S Z' = (W \times_S Z) \times_W W'$ (if T is any scheme and $u : T \longrightarrow W'$ and $v : T \longrightarrow W \times_S Z$ such that $f \circ u = p \circ v$ then $z \circ (q \circ v) = w \circ p \circ v = w \circ (f \circ u)$ then from the universality of $W' \times_S Z'$ there exist a map $h : T \longrightarrow W' \times_S Z'$ such that $q \circ v = q' \circ h = q \circ f \times_S g \circ h$ that is $v = f \times_S g \circ h$ and $f \circ u = p' \circ h$. Since $W' \times_S Z'$ satisfies the defining property of fiber product $(W \times_S Z) \times_W W' = W' \times_S Z'$).

Proposition 2.0.5. *Let (W, w) , (Z, z) and (W', w') be $S-$schemes and $f : W' \longrightarrow W$ be a morphism of $S-$schemes. Set*

$g := f \times_S Id_Z$. We obtain the following commutative diagram where all squares are cartesian.

$$\begin{array}{ccccc} W' \times_S Z & \xrightarrow{g} & W \times_S Z & \xrightarrow{q} & Z \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle z} \\ W' & \xrightarrow{f} & W & \xrightarrow{w} & S \end{array}$$

Suppose that f can be written as the composition of scheme morphisms where each morphism is a homeomorphism onto its image and also admits one of the following assumptions:

1. *Given any $w' \in W'$ there exist an affine open set U (depending on w') which contains $f(w')$ and $f^{-1}(U)$ is quasicompact moreover the morphism $f^{\sharp}_{w'} : O_{W,f(w')} \longrightarrow O_{W',w'}$ is surjective.*
2. *Given any point $w' \in W'$, the homomorphism $f^{\sharp}_{w'} : O_{W,f(w')} \longrightarrow O_{W',w'}$ is bijective.*

We set $Y' = W' \times_S Z$ and $Y = W \times_S Z$. then

1. *g is a homeomorphism of Y' onto $g(Y') = p^{-1}(f(W'))$.*
2. *For all points $y' \in Y'$ consider the commutative diagram induced on local rings by the left square of the proposition*

$$\begin{array}{ccc} \mathcal{O}_{Y',y'} & \xleftarrow{g^{\sharp}_{y'}} & \mathcal{O}_{Y,g(y')} \\ \uparrow & & \uparrow{\scriptstyle p^{\sharp}_{g(y')}} \\ \mathcal{O}_{W',p'(y')} & \xleftarrow[f^{\sharp}_{p'(y')}]{} & \mathcal{O}_{W,p(g(y'))} \end{array}$$

then the homomorphism $g^{\sharp}_{y'}$ is surjective and its kernel is generated by the image of the kernel of $f^{\sharp}_{p'(y')}$ under $p^{\sharp}_{g(y')}$

2.0.3 Examples

Products of affine spaces

Let R be a ring and $\mathbb{B}^n_R = Spec(R[T_1, \ldots, T_n])$ be the affine space over then we have $\mathbb{B}^n_R \times_R \mathbb{B}^m_R = \mathbb{B}^{n+m}_R$ (because from proposition 2.0.2 we have $Spec(R[T_1, \ldots, T_n]) \times_R Spec(R[T_1, \ldots, T_m]) = Spec(R[T_1, \ldots, T_n] \otimes_R R[T_1, \ldots, T_m]) = Spec(R[T_1, \ldots, T_{n+m}]))$

Products of prevarieties

Lemma 2.0.6. *Consider $k-$schemes (locally) of finite type W and Z over some field k then the fiber product $W \times_k Z$ is (locally of) finite type over k.*

Proof. Let $W = \bigcup_i W_i$ and $Z = \bigcup_j Z_j$ be an (finite) affine open cover such that $W_i = Spec(A_i)$ and $Z_j = Spec(B_j)$ where A_i, B_j are finitely generated $k-$algebras. From corollary 2.0.4 we get that $W \times_k Z = \bigcup_{i,j} W_i \times_k Z_j$ is an (finite) affine open cover. We have $W_i \times_k Z_j = Spec(A_i \otimes)_k B_j$ hence $W \times_k Z$ is locally of finite type (finite type). ■

Lemma 2.0.7. *Consider integral $k-$schemes W and Z if k is algebraically closed then the fiber product of W and Z, $W \times_k Z$ is an integral $k-$scheme.*

2.1 Base change, Fibers of a morphism

2.1.1 Base change in categories with fiber products

Let $\mathcal{C}$ be a catgory where arbitary fiber product exist (like the category of schemes). $u : S' \longrightarrow S$ then u induces a functor from $(\mathcal{C}/S)$ to $(\mathcal{C}/S')$. Let $l : W \longrightarrow S$ be an $S-$object then set $u^*(W) := W \times_S S'$ this is an $S'-$object whose structure morphism is given by the second projection. Let Z be an $S-$object and $f : W \longrightarrow Z$ be morphism of $S-$objects then set $u^*(f) = f \times_S Id_{S'} : u^*(W) = W \times_S S' \longrightarrow u^*(Z) = Z \times_S S'$ which is a morphism of $S'-$objects. Sometimes we denote $u^*(W)$ by $W_{(S')}$ it is called the inverse image or the base change of W by u. We denote $u^*(f)$ by $f_{(S')}$ and it is called as the inverse image or the base change of f by u.
$u^* : (\mathcal{C}/S) \longrightarrow (\mathcal{C}/S')$ defines a covariant functor. Let $u' : S'' \longrightarrow S'$, we have $(u \circ u')^* = u'^* \circ u^*$
Let $h : T \longrightarrow S'$ then we can consider $S-$object $(u \circ h : T \longrightarrow S)$. Let p be the first projection and q be the second projection of $W_{(S')}$. Let $k \in Hom_{S'}(T, W_{(S')})$ that is $h = q \circ k$. Set $k' = p \circ k : T \longrightarrow W$ then $l \circ p \circ k = u \circ h$ that is $k' \in Hom_S(T, W)$. Similarly given $k' \in Hom_S(T, W)$ we have $(k', h)_S : T \longrightarrow W_{(S,)}$ (from the universlaity of fiber product) such that $(k', h)_S \in Hom_{S'}(T, W_{(S')})$. We obtain mutually inverse bijections, functorial in T and W.

$$Hom_S(T, W) \underset{p\circ k \leftarrow k}{\overset{k' \mapsto (k'h)_S}{\rightleftarrows}} Hom_{S'}(T, W_{(S')})$$

2.1.2 Fibers of morphisms

Let $f : W \longrightarrow S$ be a morphism of schemes. Consider (W, f) and (S, Id_S) as $S-$schemes then $(W, Id_W, f) = W \times_S S$. Let $U \subseteq S$ be an open set then we have $f^{-1}(U) = W \times_S U$ (where f^{-1} denotes the pullback of the continous map). We wish to define the fiber $f^{-1}(s)$ for some $s \in S$ in similar manner. Points of the topological space S can be viewed as morphism from the residue field $\kappa(s)$ to S. consider the following cartesian diagram diagram

$$\begin{array}{ccccc} W \otimes_S \kappa(s) & \longrightarrow & W \times_S S \simeq W & \xrightarrow{id_W} & W \\ \downarrow & & \downarrow f & & \downarrow f \\ spec(\kappa(s)) & \longrightarrow & S & \xrightarrow{Id_S} & S \end{array}$$

First hypothesis of proposition 2.0.5 is valid here it follows that $f^{-1}(s)$ can be identified with the underlying topological space of $W \otimes_S \kappa(s)$

Definition 2.1.1. *Let $Spec\kappa(s) \longrightarrow S$ be the canonical morphism. Then we call*

$$W_S := W \bigotimes_S \kappa(s)$$

the fiber of f in s. The notation $f^{-1}(s)$, when understood as a scheme, will always refer to the $\kappa(s)$-scheme W_s.

Let $f : W \longrightarrow S$ be a morphism of schemes then given any $s\ inS$ we get a $\kappa(s)-$scheme W_s. In other words the morphism f can be viewed as a family of schemes over fields parameterized by the points of S .
Examples:

- Consider a field k which is algebraically closed and define

$$W(k) := \{(u, t, s) \in \mathbb{A}^3(k) : ut = s\}$$

Since $UT - S$ is an irreducible polynomial of $k[U,T,S]$ we may consider $W(k)$ as an affine variety. The associated integral k-scheme of finite type is

$$W := Spec(k[U,T,S]/(UT - S))$$

Let $f : W \longrightarrow \mathbb{A}^1_k$ denotes that map $(u,t,s) \mapsto s$ then for $s \in \mathbb{A}^1(k)$, $f^{-1}(s) = W_s = Spec(A_s)$ where

$$A_s := k[U,T,S]/(UT - S) \bigotimes_{k[S]} k[S]/(S - s) = k[U,T]/(UT - s).$$

$UT - s \in k[U,T]$ is irreducible for $s \neq 0$ and reducible for $s = 0$. f defines a family of k–schemes W_s parameterized by $s \in \mathbb{A}^1(k)$ such that W_0 is reducible and W_s is irreducible for all $s \neq 0$.

- Let k be a field and $a \in k^\times$ and set $W := V(Z^2 - W^2(W+1) - aY) \subset \mathbb{A}^3_k$. Let $f : W \longrightarrow \mathbb{A}^1_k = Spec(k[Y])$ be the morphism induced by the canonical ring homomorphism $k[Y] \longrightarrow k[W,Z,Y]/(Z^2 - W^2(W+1) - aY)$. Let $y \in \mathbb{A}^1(k) = k$, considered as a closed point of A^1_k . We have by definition $W_y = Spec(A_y$, where

$$A_y := k[W,Z]/(Z^2 - W^2(W+1) - ay).$$

2.1.3 Inverse images and schematic intersections of subschemes

Consider an immersion $i : Y \longrightarrow Z$ (this allow us to consider Y as a subscheme of Z) and a morphism of schemes $f : W \longrightarrow Z$. Consider the base change $i_{(W)} : Y \times_Z W \longrightarrow W$, proposition 2.0.5 implies that the induced at the level stalks is surjective and $i_{(W)}$ is a homeomorphism onto $f^{-1}(Y)$ that is $i_{(W)}$ is an immersion. This allow us to view $Y \times_Z W$ as a subscheme of W we call it the *inverse image of Y under f*. $f^{-1}(Y)$ will always denote this subscheme
Example: Consider affine schemes $W = Spec(A)$ and $Z = Spec(B)$ and a morphism of schemes $f : W \longrightarrow Z$ ($f^\flat_Z : B \longrightarrow A$). Consider a closed subscheme Y of Z then we know that Y looks like $V(\mathfrak{b}) = Spec(B/\mathfrak{b})$ for some ideal $\mathfrak{b}$ of B. Then we get $f^{-1}(Y) = V(f^\flat_Z(\mathfrak{b})A)$.
Consider $i : Y \longrightarrow W$ and $j : Z \longrightarrow W$ be two subschemes. Intuitively we want $Z \cap Y = i^{-1}(Z)$ or $Z \cap Y = j^{-1}(Y)$ this demand leads us to the following

$$Z \cap Y := Z \times_W Y = i^{-1}(Z) = j^{-1}(Y)$$

We call this the "*schematic intersection of Z and Y in W*".
From now on, $Z \cap Y$, when seen as a scheme, will always mean this subscheme. From the universal property of the fiber product we get a universal property for $Z \cap Y$: Any morphism $T \longrightarrow W$ factors through $Z \cap Y$ if and only if it factors through Z and through Y (this is an intutively desired universal property).
Example:

- Let $W := Spec(A)$, $Z = V(\mathfrak{a})$ and $Y = \mathfrak{b}$ then we get that

$$V(\mathfrak{a}) \cap V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$$

- Let R be a ring, if $f_1,\dots,f_r,g_1,\dots,g_s \in R[X_0,\dots,X_n]$, $V_+(f_1,\dots,f_r)$ and $V_+(g_1,\dots,g_s)$ be closed subschemes of $\mathbb{P}^n_R$ then

$$V_+(f_1,\dots,f_r) \cap V_+(g_1,\dots,g_s) = V_+(f_1,\dots,f_r,g_1,\dots,g_s) \subseteq \mathbb{P}^n_R$$

2.2 Dimension of schemes over a field

Definition 2.2.1. *Consider the topological space W. By $dim(W)$ we denote the* ***Dimension*** *of the space W which is defined to be the supremum of all lengths of chains*

$$W_0 \supset W_1 \supset \cdots \supset W_l \quad \text{l=length of the chain}$$

of irreducible closed (proper) subsets of W. If W posses the additional structure of a scheme, then we define its dimension to be the dimension of the topological space W. The space W is said to be equidimensional (with dimension d), if every irreducible components of W has the same dimension (d).

If $W = \phi$ we set $dim(W) = -\infty$, otherwise dimension is a non-negative integer or ∞

Definition 2.2.2. *Consider the ring B. We define the dimension $dim(B)$ of B as the supremum of all lengths of chains*

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \ldots \mathfrak{p}_l$$

prime ideals of B. The length of the above chain is l. This is called the Krull dimension of the ring B.

Let $W = Spec(B)$ then we get an inclusion reversing bijective correspondece between prime ideals of B and the irreducible subsets of W it follows that $dim(B) = dim(W)$. For any field k we have $dim(Spec(k)) = 0$. If B is a PID which is not a field then $dim(B) = dim(Spec(B)) = 1$. If we apply this to the case of the ring $k[T]$ we get that $dim(Spec(k[T])) = 1$. Let B be any ring then and $\mathfrak{p}_0 \subset \mathfrak{p}_1 \ldots \mathfrak{p}_l$ be a chain of prime ideals then we have $\mathfrak{p}_0 \subset \mathfrak{p}_1 \ldots \mathfrak{p}_l \subset (\mathfrak{p}_l, T)$ a chain of prime ideals of $B[T]$, hence $dim(B[T]) \geq dim(B) + 1$.

Lemma 2.2.3. *Consider the topological space W.*

1. *If $Z \subseteq W$ then $dim(Z) \leq dim(W)$. If W is irreducible, $dim(W) < \infty$, moreover if Z is a proper subset of W which is closed, then $dim(Z) < dim(W)$.*
2. *Given a covering $W = \bigcup_\alpha U_\alpha$ by open subsets U_α we get that*

$$dim(W) = sup_\alpha dim(U_\alpha)$$

3. *Consider the collection of irreducible components of W which we denote by I then*

$$dim(W) = sup_{Z \in I} dim(Z)$$

4. *If W is a scheme then*

$$dim(W) = sup_{w \in W} dim(O_{W,w})$$

Corollary 2.2.4. *Consider the closed immersion $j : V \longrightarrow W$ of schemes . Assume W is integral and $dim(W) = dim(V) < \infty$, then j is an isomorphism.*

Lemma 2.2.5. *Consider the open morphism $g : W \longrightarrow V$ of schemes. Given any $w \in W$ and any generization v' of $v := g(w)$ we can find $w' \in W$ where w' is the generization of w such that $g(w') = v'$.*

Proposition 2.2.6. *Consider the open morphism $g : W \longrightarrow V$ of schemes, then $dim(W) \geq dim(g(W))$.*

Proposition 2.2.7. *Consider the locally noetherian scheme W. The following assertions are equivalent:*

1. *$dim(W) = 0$.*
2. *Every subset of W is open in W.*

3. *Every local ring of W is a local Artin ring.*
4. *The canonical morphism*

$$\coprod_{w \in W} Spec(O_{W,w}) \longrightarrow W$$

is an isomorphism.

2.2.1 Integral morphisms of affine schemes

A homomorphism $f : A \longrightarrow B$ of rings is said to be intgral if B is integral over $f(A)$. Given below is a "geometric version" of the "Going Up" theorem .

Proposition 2.2.8. *Consider the affine schemes $W = Spec(B)$ and $V = Spec(A)$, consider an integral homomorphism $\psi : A \longrightarrow B$ which give rise to a morphism $g : W \longrightarrow V$ of scheme. Let $\mathfrak{b} \subseteq B$ be an ideal of B and $Y = V(\mathfrak{b}) \subseteq W$ be the corresponding closed subspace, then $g(Y) = V(\psi^{-1}(\mathfrak{b}))$. That is g is closed. Moreover*

1. $dim(g(Y)) = dim(Y)$.
2. *injectivity of ψ implies surjectivity of g.*

Lets recall the definition of a norm. Consider a finite field extension L of K then for any element $\alpha \in L$

$$N_{L/K}(\alpha) = (\prod_{i=1}^{n} \alpha_i)^{[L:K(\alpha)]}$$

where α_i are the roots listed with multiplicity of the minimal polynomial of α over K which is lying in some extension field of L.

Theorem 2.2.9. *Consider the integral injective homomorphism $\psi : A \longrightarrow B$ of integral domains. Set $M = Frac(A)$ and $S = Frac(B)$. Set $g := Spec(\psi) : Spec(B) \longrightarrow Spec(A)$. If A is integrally closed and S is a finite field extension of M then $N_{S/M}(B) \subseteq A$ where $N_{S/M} : S \longrightarrow M$ is the norm. We get $f(V(b)) = V(N_{S/M}(b))$ (equality of sets) for every $b \in B$, and $dim(V(b)) = dim(V(N_{L/K}(b)))$.*

Lemma 2.2.10. *Consider the rings A, B. The cardinality of the underlying set of fibers of the morphism $Spec(B) \longrightarrow Spec(A)$ are finite if it is induced from a finite homomorphism $A \longrightarrow B$ of rings.*

2.2.2 Dimensions of schemes of finite type over a field

Fix a field k. One of the important things we do here is the geometric interpretation of the result given below, which is a refined form of the Noether Normalization theorem

Theorem 2.2.11. *Consider a non-empty ring A which is a finitely generated algebra over k.*

1. *We can find $t_1, \dots, t_d \in A$ with the property that the k-algebra homomorphism $\psi : k[T_1, \dots, T_d] \longrightarrow A$, induced by mapping $T_i \mapsto t_i$, is injective and finite.*
2. *If $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_r \neq A$ is a chain of ideals in A ($r \geq 0$), then the t_i in (1) can be chosen such that $\psi^{-1}(\mathfrak{a}_i) = (T_1, \dots, T_{h(i)})$ for all $i = 0, \dots, r$ and suitable $0 \leq h(0) \leq h(1) \leq \cdots \leq h(r) \leq d$.*

We can have the following geometric intepretation; Let W is an affine scheme of finite type over k then $\Gamma(W, O_W)$ is a finitely generated k−algebra that is we have a morphism $f : W \longrightarrow \mathbb{A}^d_k$ of k−schemes.

- Let ψ be the corresponding k-algebra of f homomorphism then ψ is finite and injective. Hence by Proposition 2.2.8 and Lemma 2.2.10, f is closed, surjective, and has finite fibers.

- If $Z_r \subseteq \cdots \subseteq Z_0$ is a chain of closed subschemes of W, then we can construct f in such a way that each Z_i is mapped on to $V(T_1, \ldots, T_{h(i)})$ in $\mathbb{A}^d_k$.
- If $Z_r \subset \cdots \subset Z_0$ is a chain of integral closed subschemes (i.e. $Z_i = V(p_i)$ for some prime ideal $p_i \subset A$), From corollary 5.9 of **[AT18]** we get that $h(0) < h(1) < \cdots < h(r)$.

Corollary 2.2.12. *Consider a non-empty ring A which is a finitely generated algebra over k. Then $dim(A) = d$ for some integer $d \geq 0$ if and only if there exists a finite injective homomorphism $k[T_1, \ldots, T_d] \hookrightarrow A$ of k-algebras.*

Corollary 2.2.13. *For any non negative integer n we get that $dim(\mathbb{A}^n_k) = dim(\mathbb{P}^n_k) = n$.*

Proof. From the above corolloary it follows that $dim(\mathbb{A}^n_k) = n$ and we know that $\mathbb{P}^n_k$ can be obtained by glueing n copies of $\mathbb{A}^n_k$ hence by proposition 2.2.3 we have $dim(\mathbb{P}^n_k) = n$.

∎

Definition 2.2.14. *Consider a ring B, a chain of prime ideals of B is called maximal if it is maximal with respect to refinement. Similarly, we call a chain of closed irreducible subsets of a topological space maximal, if it does not admit a refinement.*

Theorem 2.2.15. *Consider an integral domain B which is a finitely generated k–algebra. Let $d = dim(B)$. Let $q_{h(1)} \subset \cdots \subset q_{h(r)}$ be a chain of prime ideals of B such that $dim(V(q_{h(i)}) = d - h(i)$.*

- *We can find a finite injective homomorphism $\psi : k[T_1, \ldots, T_d] \longrightarrow B$ of k-algebras with $\psi^{-1}(q_{h(i)}) = (T_1, \ldots, T_{h(i)})$ for all $i = 1, \ldots, r$.*
- *Given any ψ as in (a) the chain $(q_{h(i)})_i$ of prime ideals can be refined to a $q_0 \subset \cdots \subset q_d$ of A such that $\psi^{-1}(q_j) = (T_1, \ldots, T_j)$ for all $j = 1, \ldots, d$.*

That is, every chain of prime ideals in B can be extended to a chain which is maximal and all maximal chains have equal length.

Proposition 2.2.16. *Consider non-empty k-scheme W of finite type. all the assertions given below are equivalent:*

1. *$dim(W) = 0$*
2. *W is an affine scheme, and $\Gamma(W, O_W)$ is a finite dimensional k-vector space, moreover $\Gamma(W, O_W) = \prod_{w \in W} O_{W,w}$.*
3. *Every subset of W is open in W.*
4. *The cardinality of the underlying set of W is finite.*

Corollary 2.2.17. *Consider a zero dimensional integral k-scheme W of finite type. Then $W \cong Spec(k')$, for some finite field extension k' of k .*

Let L be a field extension of a field K and n be the largest integer such that there exist an injective hoommorphism $K[T_1, \ldots, T_n] \longrightarrow L$ of K–algebras. We call n the transendnce degree of L over K. Let S be a subset of L such that L is algebraic over the field $K(S)$ and S is algebraically independent over K then S is said to be a transcendence basis of L/K. The size of the transcendence basis is equal to the transcendence degree of the extension and is denoted $trdeg_K L$ or $trdeg(L/K)$ (for any field extension there exist a transcendence basis and any two such basis has the same size).

Theorem 2.2.18. *Let α be a generic point of the scheme W (that is W is irreducible). If W is a k-scheme of locally finite type .*

1. *$dim(W) = trdeg_k(\kappa(\alpha))$.*
2. *For any closed point $w \in W$ $dim(O_{W,w}) = dim(W)$.*

3. *Consider the morphism $g : V \longrightarrow W$ of k-schemes of locally finite type. If there exist $v \in V$ such that $g(v) = \alpha$. Then $dim(V) \geq dim(W)$. Since every open subset U of X contains the generic point α, we obtain $dim(U) = dim(W)$.*

4. *Consider the morphism $g : V \longrightarrow W$ of k-schemes of locally finite type. If g has finite fibers then $dim(V) \leq dim(W)$.*

Corollary 2.2.19. *Consider a closed point $w \in W$ for some k-scheme locally of finite type W. Then $dim(O_{W,w}) = sup_Y dim(Y)$, where the supremum is taken over all (finitely many) irreducible components of W containing w.*

2.2.3 Local dimension in a point

Definition 2.2.20. *Consider a topological space W and a point $w \in W$. The dimension of W in w is*

$$dim_w W = inf_U dim(U)$$

where the infimum is taken over all open sets U containing w.

Lemma 2.2.21. *Consider a quasi-compact topological space W. If $\{Z_\alpha\}_{\alpha \in I}$ is a collection of closed subsets such that the intersection of each finite subcollection is nonempty, then $\bigcap_{\alpha \in I} Z_\alpha$ is nonempty.*

Lemma 2.2.22. *Let W be a non empty quasi-compact topological space which is kolmogrov then W contains a closed point.*

Lemma 2.2.23. *Consider the topological space W.*

1. *For any open neighborhood U of $w \in W$, we get $dim_w U = dim_w W$.*

2. *One has $dim(W) = sup_{w \in W} dim_w W$. Let F denotes the collection of closed points of W. If W is a quasi-compact scheme, then $dim(W) = sup_{w \in F} dim_w W$.*

3. *For any integer n the $V_n := \{w \in W; dim_w W \leq n\}$ is open in W.*

Proposition 2.2.24. *Let W be a topological space and $w \in W$ be a point, let I denote the collection of irreducible components of W containing w. If W is a k–scheme of locally finite type then $dim_w W = sup_{Z \in I} dim Z$. For any closed point $w \in W$ is we get $dim_w W = dim(O_{W,w})$.*

2.2.4 Codimension of closed subschemes

Definition 2.2.25. *Consider the topological space W.*

- *For any closed irreducible subset Z of W we define the codimension $codim_W Z$ of Z in W as the supremum of the lengths of chains of irreducible closed subsets $Z_0 \supset Z_1 \supset \cdots \supset Z_l$. such that $Z_l = Z$.*
- *A closed subset Z of W is said to be equi-codimensial (of codimension d), if every irreducible components of Z have the same codimension in W (equal to d).*

Consider the affine scheme $W = Spec(B)$. Let $Z = V(\mathfrak{p})$ be a closed irreducible subset of W then $codim_W Z = dim(B_{\mathfrak{p}})$ (it is the supremum of lengths of of chains of prime ideals of B that have $\mathfrak{p}$ as its maximal element) it is also known as the height of $\mathfrak{p}$. Suppose Z is a closed irreducible subset of an arbitary scheme W and let η be the generic point of Z. Let U be an affine open set of W containing η then $codim_U(Z \cap U) = dim(O_{W,\eta})$. Since Z makes non empty intersection with U every chain of closed irreducible sets that end in Z produces a chain in U which ends in $Z \cap U$ of the same length (because for any closed irreducible set Z if $Z \cap U \neq \phi$ then $Z = \overline{(Z \cap U)}$), similarly any chain of closed ireeducible sets of U that ends in $Z \cap U$ give rise to a chain in W which ends in Z (by taking the closure). Hence $dim(O_{W,\eta}) = codim_U Z \cap U = codim_W Z$. For any $z \in Z$ there exist an affine open set U such that $z \in Z \cap U$ so $\eta \in Z \cap U$ hence $dim(O_{W,\eta}) \leq dim(O_{W,z})$ (because $\overline{\eta}$ in U is $Z \cap U$). So we have

$$codim_W Z = dim(O_{W,\eta}) = inf_{z \in Z} dim(O_{W,z})$$

Definition 2.2.26. *Consider a subset Z of a scheme W. Then*

$$codim_W(Z) := inf_{z \in Z} dim(O_{W,z})$$

is called the codimension of Z in W.

The previous discussion shows that this definition coincide with the definition 2.2.25 when Z is a closed irreducible subset of W.

- For any closed subset Y of W, we find

$$codim_W Y = inf_Z codim_W Z$$

  , where Z runs through the set of irreducible components of Y .

- A closed subset Y of W is of codimension 0 if and only if Y contains an irreducible component of W. If Y contains an irreducible component this is clear. If $codim_W(Y) = 0$ then there exist an ireeducible component Z of Y such that $codim_W(Z) = 0$ that is Z is not contained in any irreducible set but we know that Z is irreducible in W hence Z must be one of the irreducible components of W.

Proposition 2.2.27. *Let W be k–scheme of finite type. If W is irreducible and have dimension d then.*

1. *Every maximal chains of closed irreducible subsets of W have equal length.*
2. *If Y of W we have $dim(Y) + codim_W(Y) = dim(W)$.*

2.2.5 Dimension of projective varieties

From previously described results we obtain analogues statements in the case of projective varieties.

Lemma 2.2.28. *Consider the cone $C(W) \subseteq \mathbb{A}_k^{n+1}$ for some integral closed subscheme W of $\mathbb{P}_k^n$. Then $dim(C(W)) = dim(W) +$* 1.

Proposition 2.2.29. *Consider an integral closed subscheme W of $\mathbb{P}_k^n$ which non zero dimension. If $g \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial such that $V_+(g)$ is non empty and does not contains W then $W \cap V_+(g) \neq \emptyset$, and $W \cap V_+(g)$ is equi-codimensional of codimension 1 in W.*

Applying induction to this we can get a generaliztion.

Proposition 2.2.30. *Consider an integral closed subscheme W of $\mathbb{P}_k^n$, let $f_1, \ldots, f_r \in k[X_0, \ldots X_n]$ be non-constant homogeneous polynomials. Then every irreducible components of $W \cap V_+(f_1, \ldots, f_r)$ has codimension $\leq r$ in W. If $dim(W) \geq r$ then $W \cap V_+(f_1, \ldots, f_r \neq \emptyset$.*

Corollary 2.2.31. *Consider an integral closed subscheme W of $\mathbb{P}_k^n$. Then $W = V_+(g)$ for some homogeneous polynomial g which is irreducible if and only if W has codimension 1.*

Corollary 2.2.32. *Consider an integer $n \geq 2$, let g be a non-constant homogeneous polynomial set $W := V_+(g) \subset \mathbb{P}_k^n$. Then W is connected.*

2.3 Intersections of plane curves

In this section we discuss about hypersurfaces in $\mathbb{P}_k^2$ and will give outline of an elementary proof of the Bézout's theorem which says (in a crude way) that given two curves described by polynomials of degree d and e in $\mathbb{P}_k^2$, the number of "intersection points" of these curves "counted with multiplicity" is equal to $d * e$.

2.3.1 Intersection numbers of plane curves

Assume k is a field.

Definition 2.3.1. *Consider a non zero non-constant homogeneous polynomial $g \in k[X,Y,T]$ with $n = \deg(g)$. The closed subscheme $V_+(g)$ of $\mathbb{P}^2_k$ is known as a plane curve. We denote it by C and we set n as the degree of C.*

Note: Consider a plane curve $C \subset \mathbb{P}^2_k$ the we see that its degree may depend on the embedding (rather than only on the isomorphism class of the k-scheme C). But if the degree of C is greater than or equal to 3, it depends only on the isomorphism class of the k−scheme C (more precisely on the arithmetic genus of C).
Let $C = V_+(f)$ be a plane curve then proposition 2.2.29 implies that C is equi-codimensional of codimension 1. Let $f = f_1^{e_1} \dots f_r^{e_r}$ be the distinct irreducible factors of f. Then $V_+(f_i^{e_1}) \quad 1 \leq i \leq r$ are the irreducible components of C. The scheme $V_+(f)$ is reduced if and only if (f) is a radical ideal that is, if and only if the power of every factor in the decomposition of f (given above) is one.

Lemma 2.3.2. *Let $f, g \in k[X,Y,T]$ be non zero non-constant homogeneous polynomials. Then $dim(V_+(f,g)) = 0$ if and only if g.c.d of f and g is one (that is they dont have any common facotrs).*

Considre two plane curves C and D described by polynomials f and g respectively then the schematic intersection

$$C \cap D = V_+(f) \cap V_+(g) = V_+(f,g)$$

Definition 2.3.3. *Let $C, D \subset \mathbb{P}^2_k$ be two plane curves such that $Z := C \cap D$ is a k-scheme of dimension 0. Then we call $i(C,D) := dim_k(\Gamma(Z, \mathcal{O}_Z))$ the intersection number of C and D. For $z \in Z$ we call $i_z(C,D) := dim_k(\mathcal{O}_Z, z)$ the intersection number of C and D at z.*

We have $i(C,D) = dim_k(\Gamma(Z, \mathcal{O}_Z))$. Since $dim(Z) = 0$ proposition 2.2.16 implies that $\Gamma(Z, \mathcal{O}_Z) = \prod_{z \in Z} \mathcal{O}_{Z,z}$. Hence $i(C,D) = dim_k(= \prod_{z\in Z} \mathcal{O}_{Z,z}) = \sum_{z \in Z} dim_k(\mathcal{O}_{Z,z})) = \sum_{z \in C \cap D} i_z(C,D)$ (proposition 2.2.16 also implies that the underlying topological space of Z has only finite number of points).

2.3.2 Bézout's theorem

Lemma 2.3.4. *Consider a field extension K of k and set $C_K := C \otimes_k K$ and $D_K = D \otimes_k K$. Then $C_K = V_+(f_K) \subset \mathbb{P}^2_K$, where f_K is the polynomial f considered as a homogeneous polynomial with coefficients in K. Similarly, $D_K = V_+(g_K) \subset \mathbb{P}^2_K$. We have*

$$i(C,D) = i(c_K, D_K)$$

This shows that if needed we can replace k with another field extension of k which suits our need when talking about the intersection number of two plane curves. From now onwards we take k to be algebraically closed. Let $Z := C \cap D$ (where $C = V_+(f)$, $D = V_+(g)$ and $\deg(f) = n$, $\deg(g) = m$) be the intersection of two plane curves such that $dim(Z) = 0$ hence the underlying topological space of Z contains only finitely many points. So we can construct an hypersurface $L \subset \mathbb{P}^2_k$ such that $L \cap Z = \varnothing$. Its possible to choose coordinates X, Y and T of $\mathbb{P}^2_k$ such that $V_+(T) = L$ that is $V_+(T) \cap Z = \varnothing$. Define $S := k[X,Y,T]$ and let $\mathfrak{a} = (f,g) \subset S$. Then $S = \bigoplus_d S_d$ is a graded k−algebra where S_d is the subring of S constitute of homogeneous polynomials of degree d. Since $\mathfrak{a}$ is an ideal generated by homogeneous polynomial $B := S/\mathfrak{a}$ is also a graded k−algebra, that is $B \bigoplus_d B_d$ where B_d is the image of S_d under the canonical homomorphism. We know that $dim_k(S_d)$ is finite hence B_d is a finite dimensional k−vector space.

Lemma 2.3.5. *For $d \geq n + m$ we have $dim_k(B_d) = nm$.*

Theorem 2.3.6. ***Theorem of Bézout:*** *Consider two plane curves $C = V_+(f)$ and $D = V_+(g)$ in $\mathbb{P}^2_k$ whose schematic intersection has dimension zero. Then*

$$i(C,D) = (\deg(f))(\deg(g))$$

Outline of proof: Let

$$\phi : k[X,Y,T] \longrightarrow k[X,Y], \quad h \mapsto \overline{h} = h(X,Y,1)$$

be the dehomegenization with respect to T. As $Z = D_+(T)$ we have $Z = Spec(A)$ with $A = k[X,Y]/(\overline{f},\overline{g})$. To prove Bézout theorem we have to show that $dim_k(A) = nm$. The map ϕ induces a surjective $k-$algebra homomorphism $B = S/(f,g) \longrightarrow A$ from this we obtain a $k-$linear map $v_d : B_d \longrightarrow A$. To show $dim_k(A) = nm$ it is enough to show that v_d is an isomorphism for some $d \geq n+m$.

2.4 Local properties of schemes

2.4.1 Formal derivatives

Definition 2.4.1. *Consider a ring R and let $f = \sum_{i=0}^{d} a_i T^i$ be a polynomial in $R[T]$, we define the formal derivative*

$$\frac{\partial f}{\partial T} := \sum_{i=1}^{d} i a_i T^{i-1}$$

If $f \in R[\hat{T}] = R[T_1, \dots, T_n]$ we define the "partial derivative" $\frac{\partial f}{\partial T_i}$ by viewing the ring $R[\hat{T}]$ as the ring $R[T_1, \dots, \hat{T}_i, \dots, T_n][T_i]$ and applying definition 2.4.1.
$\frac{\partial f}{\partial T_i} : R[\hat{T}] \longrightarrow R[\hat{T}]$ is an $R-$linear map and it obeys the "Leibniz rule" that is

$$\frac{\partial fg}{\partial T_i} = f\frac{\partial g}{\partial T_i} + g\frac{\partial f}{\partial T_i}, \quad \forall f,g \in R[\hat{T}]$$

Lemma 2.4.2. *Consider a ring R, and let $f \in R[T_0, \dots, T_n]$ be homogeneous of degree d. Then the partial derivatives satisfy the Euler relation*

$$\sum_{j=0}^{n} \frac{\partial f}{\partial T_j}.T_j = d.f$$

2.4.2 Zariski's definition of the tangent space

Definition 2.4.3. *Consider a scheme W, and let $w \in W$. We see that $\mathfrak{m}_w/\mathfrak{m}_w^2$ ($\mathfrak{m}_w$ is the maximal ideal in the local ring $O_{W,w}$) is a vector space over $O_{W,w}/\mathfrak{m}_w = \kappa(w)$, Set*

$$T_w W = (\mathfrak{m}_w/\mathfrak{m}_w^2)^*$$

where (*) refers to the dual vector space. $T_w W$ is said to be the (Zariski, or absolute) tangent space of W in w.

Remark 2.4.4. *Consider a point w of the k−scheme W.*

- *Let $\{\overline{w_1}, \dots, \overline{w_n}\}$ be the image of elements of $\mathfrak{m}_w$ in $(\mathfrak{m}_w/\mathfrak{m}_w^2)$ such that they form a basis of the vector space $(\mathfrak{m}_w/\mathfrak{m}_w^2)$ (over $\kappa(w)$). From Nakayama's lemma we know that $\{w_1, \dots, w_n\}$ generates $\mathfrak{m}_w$ (as an $O_{W,w}$ module). Similarly any set of elements $\{w_1, \dots, w_n\}$ which generates $\mathfrak{m}_w$ will generate $\mathfrak{m}_w/\mathfrak{m}_w^2$. Hence if $\mathfrak{m}_w$ is finitely generated then $dim_{\kappa(w)}(\mathfrak{m}_w/\mathfrak{m}_w^2)$ is the cardinallity of a minimal generating set of $\mathfrak{m}_w$. A finitely generated vector space and its dual has the same dimension hence $dim_{\kappa(w)} T_w W$ is the cardinallity of a minimal generating set of $\mathfrak{m}_w$. In particular if W is locally noetherian then $dim_{\kappa(w)} T_w W$ is finite.*

- *If $U \subseteq W$ is an open set containing w, then $T_w W = T_w U$ (because $O_{W|_U,w} = O_{W,w}$).*

- *Consider a morphism $f : W \longrightarrow V$ of schemes. Let $w \in W$ then we have an induced morphism $f_w^\sharp : O_{V,f(w)} \longrightarrow O_{W,w}$ since this is a local ring homomorphism it takes $\mathfrak{m}_{f(w)}$ inside $\mathfrak{m}_w$. Let $\psi : \mathfrak{m}_{f(w)}/\mathfrak{m}_{f(w)}^2 \longrightarrow \mathfrak{m}_w/\mathfrak{m}_w^2$ be the induced morphism. Consider $\mathfrak{m}_{f(w)}/\mathfrak{m}_{f(w)}^2 \otimes_{\kappa(f(w))} \kappa(w)$ we can view this as a $\kappa(w)$ module (vector space) (Let*

$v \otimes k1 \in \mathfrak{m}_{f(w)}/\mathfrak{m}^2_{f(w)} \otimes_{\kappa(f(w))} \kappa(w)$ and $k_2 \in \kappa(w)$ then define $k_2(v \otimes k_1) := v \otimes k_1 k_2$). ψ induces a $\kappa(w)$-linear map $\mathfrak{m}_{f(w)}/\mathfrak{m}^2_{f(w)} \otimes_{\kappa(f(w))} \kappa(w) \longrightarrow \mathfrak{m}_w/\mathfrak{m}^2_w$ (such that $v \otimes k \mapsto k\psi(v)$). Let $\phi : \mathfrak{m}_w/\mathfrak{m}^2_w \longrightarrow \kappa(w)$ be a $\kappa(w)$ linear map then by composing with the linear map described before we get a $\kappa(w)$ linear map $\mathfrak{m}_{f(w)}/\mathfrak{m}^2_{f(w)} \otimes_{\kappa(f(w))} \kappa(w) \longrightarrow \kappa(w)$ such that $v \otimes k \mapsto \phi(k\psi(v)) = k\phi \circ \psi(v)$. Let $\kappa(w)/\kappa(f(w))$ is finite (that is $\kappa(w)$ is a finite dimensional $\kappa(f(w))$ vector space) and $k^w_1, \dots, k^w_n$ (with $k_1 = 1$) be a basis of $\kappa(w)$ over $\kappa(f(w))$. Then $\phi \circ \psi(v) = \sum_{i=1}^n k_i k^w_i$ using this we define a map $\mathfrak{m}_{f(w)}/\mathfrak{m}^2_{f(w)} \otimes_{\kappa(f(w))} \kappa(w) \longrightarrow \kappa(f(w)) \times \kappa(w)$ such that $v \otimes k \mapsto k\phi \circ \psi(v) \mapsto (\sum_{i=1}^n k_i, k\sum_{i=1}^n k^w_1)$. That is given a $\kappa(w)$–linear map $\mathfrak{m}_w/\mathfrak{m}^2_w \longrightarrow \kappa(w)$ we produced a $\kappa(f(w))$–linear map $\mathfrak{m}_{f(w)}/\mathfrak{m}^2_{f(w)} \longrightarrow \kappa(f(w))$ and an element of $\kappa(w)$. We get a $\kappa(w)$–linear map

$$df_w : T_w W \longrightarrow T_{f(w)} V \times \kappa(w) \longrightarrow T_{f(w)} V \bigotimes_{\kappa(f(w))} \kappa(w)$$

.

If $\mathfrak{m}_{f(w)}/\mathfrak{m}^2_{f(w)}$ is finite dimensional then $Im(\phi \circ \psi)$ will be finite dimensional, then by considering this finite dimensional subspace of $\kappa(w)$ and proceeding as in the above discussion we get a $\kappa(w)$–linear map

$$df_w : T_w W \longrightarrow T_{f(w)} V \bigotimes_{\kappa(f(w))} \kappa(w)$$

.

2.4.3 Tangent spaces of affine schemes over a field

The notion of absolute tangent space is well behaved when W is a scheme over a filed k and $w \in W$ have residue field equal to k. Consider the morphism $k \longrightarrow \kappa(w)$ induced from $W \longrightarrow Spec(k)$ that is $\kappa(w) = k$ if and only if w is a k–valued point (Since all morphism are k–schemes). We start with situation of an affine space.

Tangent spaces of k-valued points of $\mathbb{A}^n_k$: Let $x \in \mathbb{A}^n_k$ be a k valued point that is $x \in A^n_k(k) = Hom_k(Spec(k), \mathbb{A}^n_k) \cong Hom_k(k[T_1, \dots, T_n], k) \cong k^n$. All morphism from $k[T_1 \dots, T_n]$ to k are evaluation homomorphisms whose kernel looks like $(T_1 - x_1, \dots, T_n - x_n)$ for some point $x = (x_i, \dots x_n) \in \mathbb{A}^n_k$ then the unique point of $Spec(k)$ is mapped to the maximal ideal $(T_1 - x_1, \dots, T_n - x_n)$ (which is the prime ideal corresponds to x) of $k[\mathbb{T}]$. From the elements $T_i - x_i$ we obtain a basis for the k–vector space $\mathfrak{m}_x/\mathfrak{m}^2_x$ Let $(m_i)_{i=1}^n$ be this basis then any k– linear map $\mathfrak{m}_x/\mathfrak{m}^2_x \longrightarrow k$ is completely charecterized by where m_i goes hence $k^n \cong T_x X$. The resulting isomorphism can be explicitly described as follows

$$(v_1, \dots, v_n) \mapsto (\mathfrak{m}_x/\mathfrak{m}^2_x \longrightarrow k, \quad \overline{g} \mapsto \sum v_i \frac{\partial g}{\partial T_i}(x))$$

Let $f : \mathbb{A}^n_k \longrightarrow \mathbb{A}^r_k$ be the map given by $(f_1, \dots, f_r)$ such that $f_i \in k[T_1, \dots, T_n]$ for all $1 \le 1 \le r$. Let $x = (x_1, \dots, x_n) \in \mathbb{A}^n(k)$, then the induced map $df_x : T_x\mathbb{A}^n_k \longrightarrow T_{f(x)}\mathbb{A}^r_k$ is given, using the identifications of the tangent spaces with k^n and k^r , resp., as above, by the matrix

$$\left(\frac{\partial f_i}{\partial T_j}(x)\right)^{1 \le i \le r}_{1 \le j \le n}$$

Given polynomials $f_1, \dots, f_r \in R[T_1, \dots, T_n]$ for some ring R, we denote by

$$J_{f_1,\dots,f_r} := \left(\frac{\partial f_i}{\partial T_j}(x)\right)_{i,j} \in M_{r \times n}(R[T_1, \dots, T_n])$$

the *jacobian matrix* of the f_i.

Definition 2.4.5. *Consider a morphism $f : W \longrightarrow W'$ of schemes and an integer $d \ge 0$.*

- *f is said to be smooth of relative dimension d at $w \in W$, if we can find an affine set U of W containing w and affine open* *set $V = Spec(R)$ of W' containing $f(w)$ such that $f(U) \subset V$, and an open immersion*

$$j : U \hookrightarrow Spec(R[T_1, \dots, T_n]/(f_1, \dots, f_{n-d}))$$

of $R-$schemes of an appropriate n and f_i, such that the jacobian matrix $J_{f_1,\dots,f_{n-d}}(x)$ has rank $n-d$.

- *f is called a smooth morphism if f is smooth (of relative dimension d) at every $w \in W$. If f is smooth we say W is smooth over W'*

Recall for this definition that we denote for $g \in R[T_1, \dots, T_n]$ (e.g.,$g = \dfrac{\partial f_i}{\partial T_j}$ and for $x \in \mathbb{A}^n_R$ (or x in a subscheme U of $\mathbb{A}^n_R$ by $g(x) \in \kappa(x)$ the image of g in $\mathcal{O}_{\mathbb{A}^n_R,x}/\mathfrak{m}_x$.

Definition 2.4.6. *A noetherian local ring B is said to be a regular local ring if the cardinality of the set of minimal number of generators of its maximal ideal is equal to $dim(B)$ (Krull dimension)*

Let B be a noetherian local ring with maximal ideal $\mathfrak{m}$, let $k = B/\mathfrak{m}$ be the residue field of B. B is regular if and only if $dim_k(\mathfrak{m}/\mathfrak{m}^2) = dim(B)$

Definition 2.4.7. *Consider a noetherian ring B, if the localization of B at every prime ideal is a regular local ring then we say that B is a regular ring.*

Definition 2.4.8. *Consider a locally noetherian scheme W. W is said to be a regular scheme if for every $w \in W$ the local ring $\mathcal{O}_{W,w}$ is regular.*

Remark 2.4.9.

- *Consider the noetherian ring B, it is regular if and only if $Spec(B)$ is regular.*
- *A point $w \in W$ is regular if and only if $dim(T_w W)$ (as $\kappa(w)$ vector space) is equal to $dim(\mathcal{O}_{W,w})$*

2.5 Normal schemes

Definition 2.5.1. *A domain R is called normal if it is integrally closed in its field of fractions.*

Definition 2.5.2. *Consider a scheme W, we say that W is a normal scheme if for every $w \in W$ the local ring $\mathcal{O}_{W,w}$ is a normal domain.*

Lemma 2.5.3.

- *Consider a locally noetherian scheme W then for every connected open subset U of W $\Gamma(U, \mathcal{O}_W)$ is a normal domain.*
- *Consider a scheme W which is quasi compact, we say that W is normal if for every closed point $w \in W$ the local ring $\mathcal{O}_{W,w}$ is normal.*
- *Consider a scheme W if there exist an open cover $W = \bigcup_i U_i$ with $\Gamma(U_i, \mathcal{O}_X)$ are normal for all i then W is normal.*

Corollary 2.5.4. *In a locally noetherian scheme every regular point is a normal point. In particular every regular locally noetherian scheme is also a normal locally noetherian scheme.*

Proposition 2.5.5. *Consider a locally noetherian scheme W and a normal point $w \in W$ with $dim(\mathcal{O}_{W,w}) \leq 1$ then w is a regular point.*

Consider a locally noetherian scheme W, if $dim(\mathcal{O}_{W,w}) = 1$ implies that $\mathcal{O}_{W,w}$ is regular for any $w \in W$ then we say that W is "regular in codimension 1".

Proposition 2.5.6. *Assume that the affine scheme $W = Spec(B)$ is regular. Consider a closed integral scheme $Z = V(f) = Spec(A/(f))$ of W for some $f \in B$ then Z is normal if and only if it is "regular in codimension 1".*

Let W be a locally noetherian scheme. We call the set

$$W_{norm} := \{w \in W : \mathcal{O}_{W,w} \text{ normal } \}$$

the normal locus of W.

2.5.1 Geometric concept of normality, Hartogs' theorem

Recall the Hartog's theorem: Let $U \subseteq \mathbb{C}^n, \quad n > 1$ be an open set and $x \in U$. If $f : U \backslash \{x\}$ is a holomorphic function then we can extend f to all of U. The following is an analogues statement in our settings.

Theorem 2.5.7. *Consider an open subset U of the locally noetherian normal scheme W with $codim_W(W \; U) \geq 2$ then the restriction map $\Gamma(W, \mathcal{O}_W) \longrightarrow \Gamma(U, \mathcal{O}_W)$ is an isomorphism.*

Chapter 3

Cohomological Algebra

3.1 Abelian categories

Definition 3.1.1. *A category $\mathcal{C}$ is called additive if it satisfies the following condition:*

1. *Given any pair (X, Y) of $Ob(\mathcal{C})$, $Hom_{\mathcal{C}}(X, Y)$ has a structure of additive (i.e. abelian) group, and the composition law is bilinear,*
2. *there exists an object 0 such that $Hom_{\mathcal{C}}(0, 0) = 0$ (0 in the RHS is the additive identity of the homomorphism group while 0 in the LHS is an object of $\mathcal{C}$),*
3. *Given any pair (X, Y) of $Ob(\mathcal{C})$ the functor*

   $$W \longrightarrow Hom_{\mathcal{C}}(X, W) \times Hom_{\mathcal{C}}(Y, W)$$

   is representable,
4. *Given any pair (X, Y) of $Ob(\mathcal{C})$ the functor*

   $$W \longrightarrow Hom_{\mathcal{C}}(W, X) \times Hom_{\mathcal{C}}(W, Y)$$

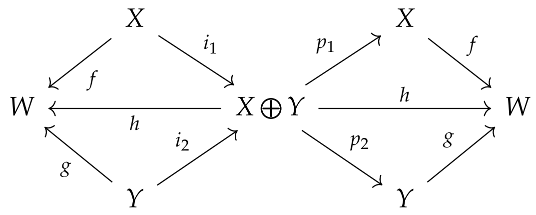
   is representable,

Proposition 3.1.2. *Under the conditions (i), (ii), of Definition 3.1.1, Z is a representative of the functor $W \mapsto Hom_{\mathcal{C}}(X, W) \oplus Hom_{\mathcal{C}}(Y$ if and only if there are morphisms $i_1 : X \longrightarrow Z, i_2 : Y \longrightarrow Z, p_1 : Z \longrightarrow X, p_2 : Z \longrightarrow Y$, such that $p_2 \circ i_1 = 0, p_1 \circ i_2 = 0, p_1 \circ i_1 = id_X, p_2 \circ i_2 = id_Y$ and $i_1 \circ p_1 + i_2 \circ p_2 = id_Z$.*

Similarly we can show that Z' is a representative of the functor $W \mapsto Hom_{\mathcal{C}}(W, X) \oplus Hom_{\mathcal{C}}(W, Y)$ if and only if there are morphisms $i_1 : X \longrightarrow Z', i_2 : Y \longrightarrow Z', p_1 : Z' \longrightarrow X, p_2 : Z' \longrightarrow Y$, such that $p_2 \circ i_1 = 0, p_1 \circ i_2 = 0, p_1 \circ i_1 = id_X, p_2 \circ i_2 = id_Y$ and $i_1 \circ p_1 + i_2 \circ p_2 = id_{Z'}$.

Corollary 3.1.3. *The representative of the functor $W \mapsto Hom_{\mathcal{C}}(W, X) \oplus Hom_{\mathcal{C}}(W, Y)$ and the representative of the functor $W \mapsto Hom_{\mathcal{C}}(X, W) \oplus Hom_{\mathcal{C}}(Y, W)$ are isomorphic. We denote this unique object by $X \oplus Y$.*

Remark 3.1.4.



(Warning: The diagram is not commutative). All the maps are as defined in proposition 3.1.2. Then we have $h \circ i_1 = f$, $h \circ i_2 = g$ and $f \circ p_1 + g \circ p_2 = h$

Definition 3.1.5. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two additive categories a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ is called additive if for any pair of objects (X, Y) of $\mathcal{C}$, F defines a group homomorphism $Hom_{\mathcal{C}}(X, Y) \longrightarrow Hom_{\mathcal{C}'}(F(X), F(Y))$*

The functor $W \mapsto Hom_{\mathcal{C}}(Z, W)$ is an additive functor. If $f : W \longrightarrow W'$ then we have $Hom_{\mathcal{C}}(Z, f) : Hom_{\mathcal{C}}(Z, W) \longrightarrow Hom_{\mathcal{C}}(Z, W')$; for any $g \in Hom_{\mathcal{C}}(Z, W)$ we have $Hom_{\mathcal{C}}(Z, f)(g) = f \circ g$. Since composition is bilinear this map is a group homomorphism.

Lemma 3.1.6. *Let $\mathcal{A}, \mathcal{B}$ be additive categories. Let $F : \mathcal{A} \longrightarrow \mathcal{C}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

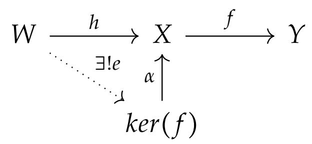
Definition 3.1.7. *Let $f \in Hom_{\mathcal{C}}(X, Y)$*

- *If the functor*
$$Z \mapsto ker(Hom_{\mathcal{C}}(Z, f)) = \{u \in Hom_{\mathcal{C}}(Z, X) : f \circ u = 0\}$$
is representeble, its representative is called the kernel of f and it is denoted $ker(f)$.
- *If the fucntor*
$$Z \mapsto ker(Hom_{\mathcal{C}}(f, Z)) = \{v \in Hom_{\mathcal{C}}(Y, Z) : v \circ f = 0\}$$
is representeble, its representative is called the cokernel of f and it is denoted by $coker(f)$.

Let $f \in Hom_{\mathcal{C}}(X, Y)$ and assume $ker(f)$ exists. Let

$$\psi : Hom_{\mathcal{C}}(-, ker(f)) \longrightarrow ker(Hom_{\mathcal{C}}(-, f))$$

be a natuaral transformation and ϕ be its inverse. Set $\alpha := \psi_{ker(f)}(id_{ker(f)})$ then $f \circ \alpha = 0$ (by construction). If $g \in Hom_{\mathcal{C}}(W, ker(f))$ then $\psi_W(g) = \psi_W(id_{ker(f)} \circ g) = \psi_{ker(f)}(id_{ker(f)}) \circ g = \alpha \circ g$. If $h \in ker(Hom_{\mathcal{C}}(-, f))$ then $h = \alpha \circ \phi(h)$ for a unique map $\phi(h) : W \longrightarrow ker(f)$ (if $h = \alpha \circ g$ hence $\alpha \circ (g - \phi(h))$. As ψ is bijective, composition with α is also bijective and composition is also compatible with the group structure of Hom hence $g - \phi(h) = 0$ which implies $g = \phi(h)$). Conversely if (Z, α) is such that $f \circ \alpha = 0$ and for any $h \in ker(Hom_{\mathcal{C}}(-, f))$ there exist unique $e : W \longrightarrow ker(f)$ such that $h = \alpha \circ e$ then $Z \cong ker(f)$. $ker(f)$ can be charecterized by this properties.



The natuaral transformation ψ is completely determined by the morphism α, sometimes we refer this morphism using the notation $ker(f)$, most of the time $ker(f)$ denotes the representative object. When it is necessary we use $ker(f)$ to denote both the morphism and the representative object.

Assume that $coker(f)$ exist. Let $\Lambda : Hom_{\mathcal{C}}(coker(f), -) \longrightarrow ker(Hom(f, -))$ be a natuaraltrasformation and Ω be its inverse. Set $\beta := \Lambda_{coker(f)}(id_{coker(f)})$ then $\beta \circ f = 0$(by construction). Let $g \in Hom_{\mathcal{C}}(Coker(f), W)$ then $\Lambda_W(g) = \Lambda_W(g \circ id_{coker(f)}) = g \circ \Lambda_{coker(f)}(id_{coker(f)}) = g \circ \beta$. Let $h \in ker(Hom(f, -))$ then there exist unique map $\Omega(h) \in Hom_{\mathcal{C}}(coker(f), W)$ such that $h = \Omega(h) \circ \beta$. Coversely if (Z, β) is such that $\beta \circ f = 0$ and for any $h \in ker(Hom(f, -))$ there exist a unique morphism $e \in Hom_{\mathcal{C}}(coker(f), W)$ such that $h = e \circ \beta$ then $Z \cong coker(f)$. $coker(f)$ is charecterized by this property.

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{h} & W \\ & & \downarrow{\scriptstyle\beta} & \overset{e}{\nearrow} & \\ & & coker(f) & & \end{array}$$

The natuaral transformation Λ is completely determined by the morphism β, sometimes we refer this morphism using the notation $coker(f)$, most of the time $coker(f)$ denotes the representative object. When it is necessary we use $coker(f)$ to denote both the morphism and the representative object

Remark 3.1.8. *Let $\alpha : ker(f) \longrightarrow X$ and $\beta : Y \longrightarrow coker(f)$ be as defined above.*

- *Let $g, h : W \longrightarrow ker(f)$ be such that $\alpha \circ g = \alpha \circ h$. Then $\alpha \circ (g - h) = 0_{(W,X)}$. But $0_{(W,X)} \in ker(Hom_{\mathcal{C}}(-, f))$ hence there exist unique map from $W \longrightarrow ker(f)$ whose composition with α gives $0_{(W,X)}$; we have $\alpha \circ (g - h) = \alpha \circ 0_{(W,ker(f))} = 0_{(W,X)}$ hence $g - h = 0_{(W,ker(f))}$ and α is monomorphism*
- *Similarly we obtain that β is an epimorphism.*
- *Let $X, X', Y, Y' \in \mathcal{C}$ such that $X \cong X'$ and $Y \cong Y'$. Let $f : X \longrightarrow Y$ and $g : X' \longrightarrow Y'$ be morphisms compatible with the isomorphisms, that is the following diagram commutes*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \updownarrow & & \updownarrow \\ X' & \xrightarrow{g} & Y' \end{array}$$

where the vertical arrows are isomorphisms. Then we have the following commutative diagram

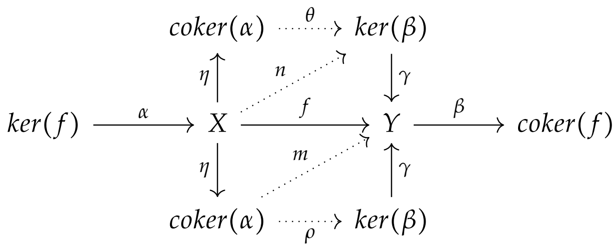
$$\begin{array}{ccc} ker(f) & \xrightarrow{\alpha_f} & X \\ \updownarrow & & \updownarrow \\ ker(g) & \xrightarrow{\alpha_g} & X' \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\beta_f} & coker(f) \\ \updownarrow & & \updownarrow \\ Y' & \xrightarrow{\beta_g} & coker(g) \end{array}$$

Where the isomorphism between X and X', Y and Y' are the same as before. Isomorphism between $ker(f)$ and $ker(g)$, $coker(f)$ and $coker(g)$ are obtained from their universal properties. If such a diagram exist we say that $ker(f) \cong ker(g)$ and $coker(f) \cong coker(g)$.

Definition 3.1.9. *Let $\alpha : ker(f) \longrightarrow X$ and $\beta : Y \longrightarrow coker(f)$ be as defined above.*

- *$coker(\alpha)$ (if it exists) is called the coimage of f and it is denoted by **Coim(f)**.*
- *$ker(\beta)$ (if it exists) is called the image of f and it is denoted by **Im(f)***

Consider the following commutative diagram

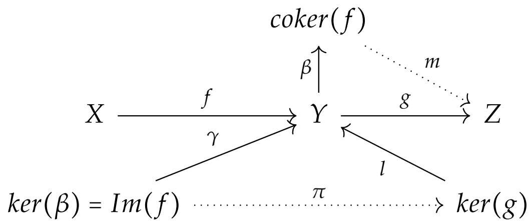


We have $f \circ \alpha = 0$ hence there exist a unique morphism $m : coker(\alpha) \longrightarrow Y$ such that $f = m \circ \eta$. We have $0 = \beta \circ f = \beta \circ (m \circ \eta)$. Since η is an epimorphism it follows that $\beta \circ m = 0$, hence there exist a unique morphism $\rho : coker(f) \longrightarrow ker(f)$ such that $m = \gamma \circ \rho$ (Since both the triangles in the lower rectangle are commutative the lower rectangle is commutative).
Similarly $\beta \circ f = 0$ hence there exist a unique morphism $n : X \longrightarrow ker(\beta)$ such that $f = \gamma \circ n$. We have $0 = f \circ \alpha = (\gamma \circ n) \circ \alpha$. Since γ is a monomorphism it follows that $n \circ \alpha = 0$ hence there exist a unique morphism $\theta : coker(f) \longrightarrow ker(f)$ such that $\theta \circ \eta = n$ (Since both the triangles in the upper rectangle are commutative the upper rectangle is commutative).
Since η is an epimorphism we get that $\rho = \theta$. That is there exist a natural morphism $Coim(f) \longrightarrow Im(f)$

Definition 3.1.10. *An additive category $\mathcal{C}$ is called an abelian category if its satisfies the two following conditions.*

- *For any morphism $f : X \longrightarrow Y$, $ker(f)$ and $coker(f)$ exist.*
- *The canonical morphism $Coim(f) \longrightarrow Im(f)$ is an isomorphism.*

In some sense the second condition in the above definition is eqivalent to the first isomorphism theorem (in the category of groups, rings,modules etc).
From now onwards we take $\mathcal{C}$ to be abelian. Let f be a morphism in an abelian category then f can be factored uniquely as $x \longrightarrow Coim(f) \longrightarrow Im(f) \longrightarrow y$ where the first arrow is an epimorphism the second arrow is an isomorphism and the third arrow is a monomorphism. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ such that $g \circ f = 0$. Then we have the following commutative diagram.



We know that $g \circ f = 0$ hence there exist a unique morphism m such that $m \circ \beta = g$. So $g \circ \gamma = (m \circ \beta) \circ \gamma = m \circ (\beta \circ \gamma) = 0$. Hence there exist a unique morphism $\pi : Im(f) \longrightarrow ker(g)$ such that $l \circ \pi = \gamma$.

Definition 3.1.11. *A sequence of morphisms:*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called an exact sequence if (or exact at Y):

- $g \circ f = 0$
- *the natural morphism* $Im(f) \longrightarrow ker(g)$ *is an isomorphism.*

More generally a sequence of morphisms is called exact if any successive pair of arrows is exact.

Proposition 3.1.12. *The sequence*

$$0 \longrightarrow X \xrightarrow{f} Y$$

is exact if and only if f *is a monomorphism. Similarly the sequence*

$$X \xrightarrow{f} Y \longrightarrow 0$$

is exact if and only if f *is an epimorphism.*

Proposition 3.1.13. *Let* $f : X \longrightarrow Y$ *be a morphism in* $\mathcal{C}$ *then the following sequences are exact.*

- $0 \longrightarrow ker(f) \longrightarrow X \longrightarrow Im(f) \longrightarrow 0$
- $0 \longrightarrow Im(f) \longrightarrow Y \longrightarrow coker(f) \longrightarrow 0$

Definition 3.1.14. *Let* $\mathcal{C}$ *and* $\mathcal{C}'$ *be two abelian categories. An additive functor* F *from* $\mathcal{C}$ *to* $\mathcal{C}'$ *is called left (resp. right) exact if for any exact sequence in* $\mathcal{C}$*:*

$$0 \longrightarrow X \longrightarrow X' \longrightarrow X''$$

(resp.:$X \longrightarrow X' \longrightarrow X'' \longrightarrow 0$*) the sequence:*

$$0 \longrightarrow F(X) \longrightarrow F(X') \longrightarrow F(X'')$$

(resp.:$F(X) \longrightarrow F(X') \longrightarrow F(X'') \longrightarrow 0$*) is exact. If* F *is both left and right exact,* F *is called exact. A contravariant functor* F *from* $\mathcal{C}$ *to* $\mathcal{C}'$ *is called left exact (resp. right exact, resp. exact), if so is* F *regarded as a functor from* $\mathcal{C}^o$ *to* $\mathcal{C}'$*.*

Proposition 3.1.15. *Let* $Z \in \mathcal{C}$ *then* $Hom_{\mathcal{C}}(-, Z)$ *and* $Hom_{\mathcal{C}}(Z, -)$ *are left exact functors.*

Definition 3.1.16. *Let* $Z \in \mathcal{C}$*. One says that* Z *is injective (resp. projective) if the functor* $Hom_{\mathcal{C}}(-, Z)$ *(resp.* $Hom_{\mathcal{C}}(Z, -)$*) is exact.*

Proposition 3.1.17. *Let* $Z \in \mathcal{C}$*.* Z *is injective if and only if for every monomorphism* $f : X \longrightarrow Y$ *in* $\mathcal{C}$*,* $Hom_{\mathcal{C}}(f, Z)$ *is surjective.*

Proposition 3.1.18. *Let* $\mathcal{C}$ *be an abelian category and Let*

$$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

be such that $g \circ f = 0$*.*

- $$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow Hom_{\mathcal{C}}(M_3, N) \xrightarrow{Hom_{\mathcal{C}}(gN)} Hom_{\mathcal{C}}(M_2, N) \xrightarrow{Hom_{\mathcal{C}}(fN)} Hom_{\mathcal{C}}(M_1, N)$$

is an exact sequence of abelian groups for all objects N of $\mathcal{C}$.

- $$0 \longrightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

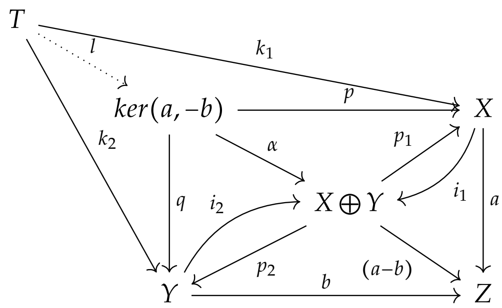
  is exact if and only if

$$0 \longrightarrow Hom_{\mathcal{C}}(N, M_1) \xrightarrow{Hom_{\mathcal{C}}(Nf)} Hom_{\mathcal{C}}(N, M_2) \xrightarrow{Hom_{\mathcal{C}}(Ng)} Hom_{\mathcal{C}}(N, M_3)$$

is an exact sequence of abelian groups for all objects N of $\mathcal{C}$.

Lemma 3.1.19. *Fiber product (pull back) exist in an abelian category.*

Proof. Let $\mathcal{C}$ be an abelian category and $X, Y, Z \in \mathcal{C}$. If $a : X \longrightarrow Z$ and $b : Y \longrightarrow Z$ are morphisms then we have a morphism $(a, -b) : X \bigoplus Y \longrightarrow Z$ such that $(a, -b) \circ i_1 = a$ and $(a, -b) \circ i_2 = -b$ where $i_1 : X \longrightarrow X \bigoplus Y$, $i_2 : Y \longrightarrow X \bigoplus Y$ are the canonical morphism (defined in proposition 3.1.2 or refer remark 3.1.4). Consider the following diagram (warning:it's not a commutative diagram)



where $p = p_1 \circ \alpha$ and $q = p_2 \circ \alpha$. We have $0 = (a, -b) \circ \alpha = (a \circ p_1 + -b \circ p_2) \circ \alpha = a \circ p - b \circ q \Rightarrow a \circ p = b \circ q$. Now asumme that $a \circ k_1 = b \circ k_2 \Rightarrow (a, -b) \circ i_1 \circ k_1 = -(a, -b) \circ i_2 \circ k_2 \Rightarrow (a, -b)(i_1 \circ k_1 + i_2 \circ k_2) = 0$. Hence there exist a unique morphism $l : T \longrightarrow ker(a, -b)$ such that $\alpha \circ l = i_1 \circ k_1 + i_2 \circ k_2$. We have $p \circ l = p_1 \circ \alpha \circ l = p_1 \circ (i_1 \circ k_1 + i_2 \circ k_2) = k_1$ similarly $q \circ l = p_2 \circ \alpha \circ l = p_2 \circ (i_1 \circ k_1 + i_2 \circ k_2) = k_2$. That is $(ker(a, -b), p, q) = X \times_Z Y$.

■

Remark 3.1.20. *Arguments dual to those in lemma 3.1.19 will show that push out exist in an abelian category.*

3.2 Diagram chasing

Definition 3.2.1. *Let A be an abelian category. Let $i : A \longrightarrow B$ and $q : B \longrightarrow C$ be morphisms of A such that $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact (such exact sequence are called short exact) sequence. We say the short exact sequence is split if there exist morphisms $j : C \longrightarrow B$ and $p : B \longrightarrow A$ such that (B, i, j, p, q) is the direct sum of A and C.*

Lemma 3.2.2. *Let $\mathcal{C}$ be an abelian category. Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be a short exact sequence.*

1. *Given a morphism $s : C \longrightarrow B$ left inverse to $B \longrightarrow C$, there exists a unique $\pi : B \longrightarrow A$ such that (s, π) splits the short exact sequence as in Definition 3.2.1.*

2. *Given a morphism $\pi : B \longrightarrow A$ right inverse ($\pi \circ f = Id_A$) to $A \longrightarrow B$, there exists a unique $s : C \longrightarrow B$ such that (s, π) splits the short exact sequence as in Definition 3.2.1.*

Lemma 3.2.3. *(short five lemma:)*
consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & . & \xrightarrow{m} & . & \xrightarrow{e} & . & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & . & \xrightarrow{m'} & . & \xrightarrow{e'} & . & \longrightarrow & 0
\end{array}$$

with short exact rows,f and h monic imply g monic, and f and h epi imply g epi.

Lemma 3.2.4. *(five lemma) Consider the following commutative diagram*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{g_1} & A_2 & \xrightarrow{g_2} & A_3 & \xrightarrow{g_3} & A_4 & \xrightarrow{g_4} & A_5 \\
\Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle f_3} & & \Big\downarrow{\scriptstyle f_4} & & \Big\downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow{h_1} & B_2 & \xrightarrow{h_2} & B_3 & \xrightarrow{h_3} & B_4 & \xrightarrow{h_4} & B_5
\end{array}$$

with exact rows then

- *If f_1 is epi, and f_2 and f_4 are monic, then f_3 is monic.*
- *If f_2 and f_4 are epi while f_5 is monic then f_3 is epic.*
- *If f_1 is epi, f_2 and f_4 are isomorphisms, and f_5 is monic, then f_3 is an isomorphism.*

Consider the commutative diagram give in lemma 3.2.3 add the kernels and cokernels of $f, g,$ and h to form a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(f) & \xrightarrow{m_0} & ker(g) & \xrightarrow{e_0} & ker(h) & & \\
 & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle k} & & \\
0 & \longrightarrow & a & \xrightarrow{m} & b & \xrightarrow{e} & c & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & a' & \xrightarrow{m'} & b' & \xrightarrow{e'} & c' & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle i'} & & \Big\downarrow{\scriptstyle j'} & & \Big\downarrow{\scriptstyle k'} & & \\
 & & coker(f) & \xrightarrow{m_1} & coker(g) & \xrightarrow{e_1} & coker(h) & \longrightarrow & 0
\end{array}$$

Each coloumn is exact (consider the first coloumn then $Im(i) = ker(coker(ker(f))) \cong ker(f)$ hence it is exact at a, we know $Im(f) \cong ker(coker(f)) = ker(j)$ hence it is exact at b) and both middle rows are given to be exact. We have $g \circ m \circ i = m' \circ f \circ i = 0$ hence there exist unique map $m_0 : ker(f) \longrightarrow ker(g)$ such that $m \circ i = j \circ m_0$ using similar arguments define e_0, m_1 and e_1.
Consider the first raw, it is exact at $ker(f)$ (we know $j \circ m_0$ is monic hence m_0 must be monic). We have $k \circ e_0 \circ m_0 = e \circ m \circ i = 0$, since k is a monomorphism we get that $e_0 \circ m_0 = 0$. Let $x \in_m ker(g)$ such that $e_0 \circ x \equiv 0$. Hence $e \circ j \circ x \equiv 0$.

Since the second raw is exact there exist $y \in_m a$ such that $m \circ y \equiv j \circ x$. $m' \circ f \circ y = g \circ m \circ y \equiv g \circ j \circ x = 0$ since m' is monic this would imply that $f \circ y \equiv 0$. We know that the coloumns are exact hence there exist $z \in_m ker(f)$ such that $y \equiv i \circ z$ that is $j \circ x \equiv m \circ i \circ z = j \circ m_0 \circ z$ since j is a monomorphism this would imply that $x \equiv m_0 \circ z$. Hence the sequence is exact at $ker(g)$. From duality it follows that the fourth sequence is also exact.

Remark 3.2.5. *We can refine the result in the above discussion as follows. Consider the following commutative diagram*

$$\begin{array}{ccccc} a & \xrightarrow{m} & b & \xrightarrow{e} & c \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ a' & \xrightarrow{m'} & b' & \xrightarrow{e'} & c' \end{array}$$

- *If the first raw is exact and m' is monic then $ker(f) \longrightarrow ker(g) \longrightarrow ker(h)$ is exact (since we used only this much information to prove the analogues statement in the above discussion the same proof works here).*
- *If the second raw is exact and e is an epi then $coker(f) \longrightarrow coker(g) \longrightarrow coker(h)$ is exact.*

Lemma 3.2.6. *(Ker-coker sequence = Snake lemma). Given a morphism $< f, g, h >$ of short exact sequences, as in the above diagram, there is an arrow $\delta : Ker(h) \longrightarrow Coker(f)$ such that the following sequence is exact*

$$0 \longrightarrow ker(f) \xrightarrow{m_0} ker(g) \xrightarrow{e_0} ker(h) \xrightarrow{\delta} coker(f) \xrightarrow{m_1} coker(g) \xrightarrow{e_1} coker(h) \longrightarrow 0$$

Remark 3.2.7. *Consider the following commutative diagram with exact raws*

$$\begin{array}{cccccccc} & & a & \xrightarrow{m} & b & \xrightarrow{e} & c & \longrightarrow 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & \\ 0 & \longrightarrow & a' & \xrightarrow{m'} & b' & \xrightarrow{e'} & c' & \end{array}$$

then the sequence $ker(f) \longrightarrow ker(g) \longrightarrow ker(h) \longrightarrow coker(f) \longrightarrow coker(g) \longrightarrow coker(h)$ is exact.

3.3 Category of Complexes

Let $\mathcal{C}$ be an additive category.

Definition 3.3.1. *A complex X in $\mathcal{C}$ consists of the data $\{X^n, d^n_X\}_{n \in \mathbb{Z}}$ such that for every $n \in \mathbb{Z}$:*

$$X^n \in \mathcal{C} \quad d^n_X \in Hom_{\mathcal{C}}(X^n, X^{n+1}) \quad and \quad d^{n+1} \circ d^n = 0$$

A morphism f from a complex X to a complex Y is a sequence $\{f^n\}_{n \in \mathbb{Z}}$ of morphisms $f^n : X^n \longrightarrow Y^n$, such that for any $n \in \mathbb{Z}$:

$$d^n_Y \circ f^n = f^{n+1} \circ d^{n+1}_X$$

We denote by $\mathbf{C}(\mathcal{C})$ the category of complexes in $\mathcal{C}$ (composition of morphism is defined in the obvious way). This is an additive category, if $\mathcal{C}$ is abelian $\mathbf{C}(\mathcal{C})$ is abelian.
The family $d_X = \{d^n_{xX}\}_n$ is called the differential of the complex X. A complex X is said to be bounded (resp. bounded below, resp. bounded above) if $X^n = 0$ for $|n| \gg 0$ (resp. $n \ll 0$, resp. $n \gg 0$). The full subcategory of $\mathbf{C}(\mathcal{C})$ consisting of bounded complexes (resp. complexes bounded below, resp. complexes bounded above), is denoted

$\mathbf{C}^b(\mathcal{C})$ (resp. $\mathbf{C}^+(\mathcal{C})$, resp. $\mathbf{C}^-(\mathcal{C})$). We identify $\mathcal{C}$ with the full subcategory of $\mathbf{C}(\mathcal{C})$ consisting of complexes X such that $X^n = 0$ for $n \neq 0$.

Definition 3.3.2. *Let k be an integer, and let $X \in \mathbf{C}(\mathcal{C})$. One defines a new complex $X[k]$ by setting:*

$$X[k]^n := X^{n+k} \quad d^n_{X[k]} := (-1)^n d^{n+k}_X$$

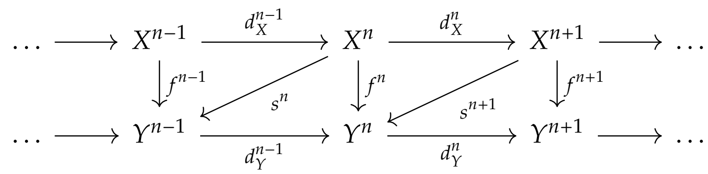
For a morphism $f : X \longrightarrow Y$ in $\mathbf{C}(\mathcal{C})$, one defines $f[k] : X[k] \longrightarrow Y[k]$ by setting:

$$f[k]^n := f^{n+k}$$

The functor $[k]$ from $\mathbf{C}(\mathcal{C})$ to $\mathbf{C}(\mathcal{C})$ is called the shift functor of degree k.

Definition 3.3.3. *A morphism $f : X \longrightarrow Y$ in $\mathbf{C}(\mathcal{C})$ is called homotopic to zero if there exist morphisms $s^n : X^n \longrightarrow Y^{n-1}$ in $\mathcal{C}$ such that for any n:*

$$f^n = d^{n-1}_Y \circ s^n + s^{n+1} \circ d^n_X$$

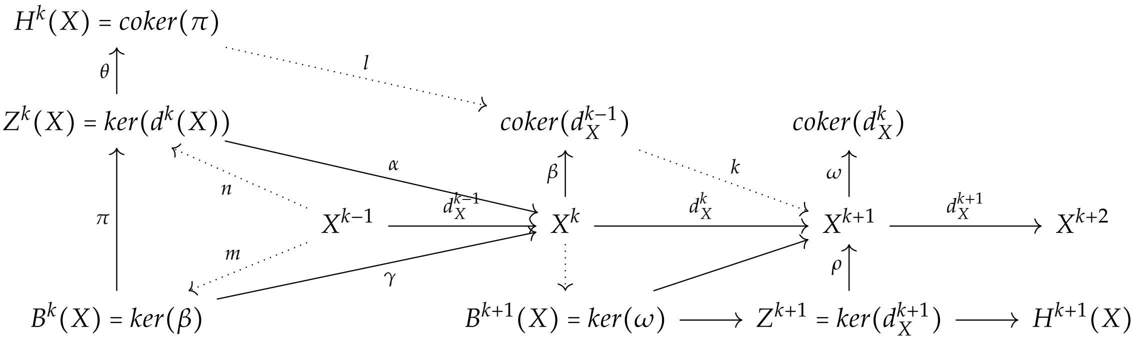


One says f is homotopic to g if $f - g$ is homotopic to zero. Let X, Y and Z be in $\mathbf{C}(\mathcal{C})$ and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be morphisms of complexes. Assume f is homotopic to zero that is there exist a map $s^n : X^n \longrightarrow Y^{n-1}$ such that $f^n = d^{n-1}_Y \circ s^n + s^{n+1} \circ d^n_X$. Then $g^n \circ f^n = g^n \circ d^{n-1}_Y \circ s^n + g^n \circ s^{n+1} \circ d^n_X$. But we know that $g^n \circ d^{n-1}_Y = d^{n-1}_Z \circ g^{n-1}$ define $h^n := g^{n-1} \circ s^n : X^n \longrightarrow Z^{n-1}$. That is $(g \circ f)^n = d^{n-1}_Z \circ h^n + h^{n+1} \circ d^n_X$, hence $g \circ f$ is homotopic to zero.
Now assume g is homotopic to zero that is there exist $s^n : Y^n \longrightarrow Z^{n-1}$ such that $g^n = d^{n-1}_Z \circ s^n + s^{n+1} \circ d^n_Y$. Then $g^n \circ f^n = d^{n-1}_Z \circ s^n \circ f^n + s^{n+1} \circ d^n_Y \circ f^n$. We know that $d^n_Y \circ f^n = f^{n+1} \circ d^n_X$. Define $h^n := s^n \circ f^n$ then $(g \circ f)^n = d^{n-1}_Z \circ h^n + h^{n+1} \circ d^n_X$. Hence $g \circ f$ is homotopic to zero.
We denote by $Ht(X, Y)$ the subgroup of $Hom_{\mathbf{C}(\mathcal{C})}(X, Y)$ consisting of morphisms homotopic to zero.

Definition 3.3.4. *Define a category $\mathbf{K}(\mathcal{C})$ by setting $ob(\mathbf{K}(\mathcal{C})) := ob(\mathbf{C}(\mathcal{C}))$ and $Hom_{\mathbf{K}(\mathcal{C})}(X, Y) := Hom_{\mathbf{C}(\mathcal{C})}(X, Y)/Ht(X, Y)$ (From the above discussion it follows that the composition is well defined).*

From now onwards throughout this section we assume that $\mathcal{C}$ is abelian.

Definition 3.3.5. *Let $X \in \mathcal{C}$ set $Z^k(X) := ker(d^k_X)$, $\quad B^k(X) := Im(d^{k-1}_X)$ and $H^k(X) = coker(B^k(X) \longrightarrow Z^k(X))$. One calls $H^k(X)$ the k^{th} cohomology of the complex X (sometimes we write $Z^k(X)/B^k(X)$ instead of $coker(B^k(X) \longrightarrow Z^k(X))$).*



We obtain the following exact sequences:

- $B^k(X) \longrightarrow Z^k(X) \longrightarrow H^k(X) \longrightarrow 0$.
- $0 \longrightarrow B^k(X) \longrightarrow X^k \longrightarrow coker(d_X^{k-1}) \longrightarrow 0.$
- $X^{k-1} \longrightarrow Z^k(X) \longrightarrow H^k(X) \longrightarrow 0.$
- $0 \longrightarrow Z^k \longrightarrow X^k \longrightarrow B^{k+1} \longrightarrow 0.$
- $0 \longrightarrow H^k(X) \longrightarrow coker(d_X^{k-1}) \longrightarrow X^{k+1}.$
- $0 \longrightarrow H^k(X) \longrightarrow coker(d_X^{k-1}) \longrightarrow Z^{k+1}(X) \longrightarrow H^{k+1}(X) \longrightarrow 0.$

Lemma 3.3.6. *$H^n(-)$ is an additive functor from $\mathbf{C}(\mathcal{C})$ to $\mathcal{C}$ and $H^{n+k}(X) \cong H^n(X[k])$.*

Definition 3.3.7. *A morphism of complexes $f : X \longrightarrow Y$ is called a quasi isomorphism if the induced map $H^n(f) : H^n(X) \longrightarrow H^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Definition 3.3.8. *Let $X, Y \in \mathbf{C}(\mathcal{C})$ a morphism $f : X \longrightarrow Y$ is called a homotopy equivalance if there exist $g : Y \longrightarrow X$ so that $f \circ g$ is homotopic to Id_Y and $g \circ f$ is homotopic to Id_X. If such an homotopy equivalance exist we say that X and Y are homotopically equivalant. In other words a homotopy equivalance is an isomorphism in $\mathbf{K}(\mathcal{C})$.*

Lemma 3.3.9.
- *Let $f, g : X \longrightarrow Y$ be morphism of complexes, if they are homotopic then the induced morphisms $H^n(f)$ and $H^n(g)$ are equal for all $n \in \mathbb{Z}$*
- *If f is a homotopy equivalance then f is a qusi isomorphism.*

Corollary 3.3.10. *$H^n(-)$ is an additive functor from $\mathbf{K}(\mathcal{C})$ to $\mathcal{C}$.*

Remark 3.3.11. *Let $X, Y, Z \in \mathbf{C}(\mathcal{C})$ and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be morphisms of complexes*

- *$f : X \longrightarrow Y$ is monic if and only if $f^n : X^n \longrightarrow Y_n$ is monic for all n.*
- *$f : X \longrightarrow Y$ is epi if and only of $f^n : X^n \longrightarrow Yn$ is epi for all n.*
- *The sequence $X \longrightarrow Y \longrightarrow Z$ is exact if and onlly if the sequence $X^n \longrightarrow Y^n \longrightarrow Z^n$ is exact for all n.*

Proposition 3.3.12. *Let $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ be an exact sequence in $\mathbf{C}(\mathcal{C})$ then there exist a canonical long exact sequence in $\mathcal{C}$*

$$\ldots \longrightarrow H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \longrightarrow H^{n+1}(X) \longrightarrow \ldots$$

Proof. We already know that $ker(coker(d_X^{n-1}) \longrightarrow Z^{n+1}(X)) = H^n(X)$ and $coker(coker(d_X^{n-1}) \longrightarrow Z^{n+1}(X)) = H^{n+1}(X)$. Consider the following commutative diagram

$$\begin{array}{ccccccc} & & coker(d_X^{n-1}) & \longrightarrow & coker(d_Y^{n-1}) & \longrightarrow & coker(d_Z^{n-1}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z^{n+1}(X) & \longrightarrow & Z^{n+1}(Y) & \longrightarrow & Z^{n+1}(Z) & & \end{array}$$

With exact raws (Since $X^n \longrightarrow Y^n \longrightarrow Z^n$ is exact the corresponding *ker* and *coker* sesquences are exact). Applying snake lemma (remark 9.2.8) to the above diagram we obtain an exact sequence

$$H^n(X) \longrightarrow H^n(Y) \longrightarrow H^n(Z) \longrightarrow H^{n+1}(X) \longrightarrow H^{n+1}(Y) \longrightarrow H^{n+1}(Z)$$

. The proposition follows immediately.

■

Definition 3.3.13. *Let $X \in \mathbf{C}(\mathcal{C})$ then we define the truncated complexes $\tau^{\geq n}(X)$ and $\tau^{\leq n}(X)$ by:*

$$\tau^{\leq n}(X) : \ldots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow ker(d_X^n) \longrightarrow 0 \longrightarrow \ldots$$

$$\tau^{\geq n}(X) : \ldots \longrightarrow 0 \longrightarrow coker(d_X^{n-1}) \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \ldots$$

Its easy to see that

$$H^i(\tau^{\leq n}(X)) = \begin{cases} 0 & \text{if } i > n \\ H^i(X) & \text{if } i \leq n \end{cases}$$

and

$$H^i(\tau^{\geq n}(X)) = \begin{cases} H^i(X) & \text{if } i \geq n \\ 0 & \text{if } i < n \end{cases}$$

Notation:

- We set $\tau^{<n}(X) := \tau^{\leq n-1}(X)$ and $\tau^{>n}(X) := \tau^{\geq n+1}(X)$.

  Proposition 3.3.14. *Let $X \in \mathbf{C}(\mathcal{C})$*

- *The natural morphism $H^k(tr^{\leq n})(X) \longrightarrow H^k(X)$ (induced from $tr^{\leq n}(X) \longrightarrow X$) is an isomorphism for $k \leq n$ and $H^k(tr^{\leq n})(X) = 0$ for $k > n$.*

Proposition 3.3.15. *$\tau^{\leq n}$ and $\tau^{\geq n}$ defines functors from $\mathbf{C}(\mathcal{C})$ to $\mathbf{C}(\mathcal{C})$ which transforms morphisms homotopic to zero into morphisms homotopic to zero (hence defines a functor from $\mathbf{K}(\mathcal{C})$ to $\mathbf{K}(\mathcal{C})$).*

3.4 Mapping Cones

Let $\mathcal{C}$ be an additive category

Definition 3.4.1. *Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}(\mathcal{C})$, the mapping cone of f, denoted by $M(f)$, is the object of $\mathbf{C}(\mathcal{C})$ defined as follows:*

$$\begin{cases} M(f)^n := X^{n+1} \bigoplus Y^n \\ d^n_{M(f)} := \begin{pmatrix} d^n_{X[1]} & 0 \\ f^{n+1} & d^n_Y \end{pmatrix} \end{cases}$$

(recall that $d^n_{X[1]} = -d^{n+1}_X$)

Remark 3.4.2. *We have morphisms $d^n_{X[1]} : X^{n+1} \longrightarrow X^{n+2}$ and $0 : Y^n \longrightarrow X^{n+2}$ this induces a unique morphism $l : X^{n+1} \bigoplus Y^n \longrightarrow X^{n+2}$. Similarly we have morphisms $f^{n+1} : X^{n+1} \longrightarrow Y^{n+1}$ and $d^n_Y : Y^n \longrightarrow Y^{n+1}$ this induces a unique morphism $m : X^{n+1} \bigoplus Y^n \longrightarrow Y^{n+1}$. We denote by $d^n_{M(f)} : X^{n+1} \bigoplus Y^n \longrightarrow X^{n+2} \bigoplus Y^{n+1}$ the unique morphism induced by l and m.*

We define morphisms $\alpha(f) : Y \longrightarrow M(f)$ and $\beta(f) : M(f) \longrightarrow X[1]$ by:

$$\alpha(f)^n := \begin{pmatrix} 0 \\ Id_{Y^n} \end{pmatrix}$$

$$\beta(f)^n \begin{pmatrix} Id_{X^{n+1}}, & 0 \end{pmatrix}$$

Lemma 3.4.3. *For any $f : X \longrightarrow Y$ in $\mathbf{C}(\mathcal{C})$, there exists $\phi : X[1] \longrightarrow M(\alpha(f)^n)$ where α is an isomorphism in $\mathbf{K}(\mathcal{C})$ such that the following diagram commutes in $\mathbf{K}(\mathcal{C})$.*

$$\begin{array}{ccccccc} Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & X[1] & \xrightarrow{-f[1]} & Y[1] \\ \downarrow{Id_Y} & & \downarrow{Id_{M(f)}} & & \downarrow{\phi} & & \downarrow{Id_{Y[1]}} \\ Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\alpha(\alpha(f))} & M(\alpha(f)) & \xrightarrow{\beta(\alpha(f))} & Y[1] \end{array}$$

Note that such a result would not hold in $\mathbf{C}(\mathcal{C})$. Note further that ϕ is not unique even in $\mathbf{K}(\mathcal{C})$.

Definition 3.4.4. *One defines a triangle in $\mathbf{K}(\mathcal{C})$ as being a sequence of morphisms $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ and a morphism of triangles as being a commutative diagram in $\mathbf{K}(\mathcal{C})$:*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\phi} & & \downarrow & & \downarrow & & \downarrow{\phi[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X[1]' \end{array}$$

Definition 3.4.5. *A triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $\mathbf{K}(\mathcal{C})$ is called a distinguished triangle, if it is isomorphic to a triangle*

$$X' \xrightarrow{f} Y' \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X'[1]$$

for some f in $\mathbf{C}(\mathcal{C})$.

Proposition 3.4.6. *The collection of distiguished triangles in $\mathbf{K}(\mathcal{C})$ satisfies the following properties*

- *(TR 0) A triangle isomorphic to a distinguished triangle is distinguished.*
- *(TR 1) For any $X \in \mathbf{K}(\mathcal{C})$*

  $$X \xrightarrow{Id_X} X \longrightarrow 0 \longrightarrow X[1]$$

  is a distinguished triangle.
- *(TR 2) Any $f : X \longrightarrow Y$ in $\mathbf{K}(\mathcal{C})$ can be embedded in a distinguished triangle*

  $$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

- *(TR 3)*

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

  is a distinguished triangle if and only if

  $$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

- *(TR 4) Given two distinguished triangles*

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1] \quad \textit{and} \quad X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow X'[1]$$

a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

can be embedded in a morphism of triangles (not necessarily unique).

- *(TR 5) (octahedral axiom). Suppose given distinguished triangles:*

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow X[1]$$

$$Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow Y[1]$$

$$X \xrightarrow{g \circ f} Z \longrightarrow Y' \longrightarrow X[1]$$

Then there exist a distinguished triangle

$$Z' \longrightarrow Y' \longrightarrow X' \longrightarrow Z'[1]$$

such that the following diagram is commutative

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & X[1] \\ \downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle Id_{X[1]}} \\ X & \xrightarrow{g \circ f} & Z & \longrightarrow & Y' & \longrightarrow & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle Id_Z} & & \downarrow & & \downarrow{\scriptstyle f[1]} \\ Y & \xrightarrow{g} & Z & \longrightarrow & X' & \longrightarrow & Y[1] \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle Id_{X'}} & & \downarrow \\ Z' & \longrightarrow & Y' & \longrightarrow & X' & \longrightarrow & Z'[1] \end{array}$$

3.5 Triangulated categories

Let $\mathcal{C}$ be an additive category, together with an automorphism (functor) $T : \mathcal{C} \longrightarrow \mathcal{C}$. We write sometimes $[1]$ for T and $[k]$ for T^k, (i.e. $X[1]$ for $T(X)$, or $f[1]$ for $T(f)$). A triangle in $\mathcal{C}$ is a sequence of morphisms $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$

Definition 3.5.1. *An additive category $\mathcal{C}$ consists of the following data*

- *An automorphism $T : \mathcal{C} \longrightarrow \mathcal{C}$.*
- *A family of triangles, called distinguished triangles.*

Which satisfies the axioms (TR 0)-(TR 5) of proposition 3.4.6 when setting $X[1] = T(X)$ is called a triangulated category.

Definition 3.5.2. *Let $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$ be two triangulated categories. We say that an additive functor F from $\mathcal{C}$ to $\mathcal{C}'$ is a functor of triangulated categories if $F \circ T = T' \circ F'$ and F sends distinguished triangles of $\mathcal{C}$ into distinguished triangles of $\mathcal{C}'$.*

Definition 3.5.3. *Let $\mathcal{C}$ be a triangulated category and $\mathcal{A}$ be an abelian category. An additive functor $F : \mathcal{C} \longrightarrow \mathcal{A}$ is called a cohomological functor if for any distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$, the sequence $F(X) \longrightarrow F(Y) \longrightarrow F(Z)$ is exact.*

For a cohomological functor F we write F^k for $F \circ T^k$. Let $X \longrightarrow Y \longrightarrow Z \longrightarrow T(X)$ be a distinguished triangle then by (TR 3) $Y \longrightarrow Z \longrightarrow T(X) \longrightarrow T(Y)$ is a distinguished triangle from this property we obtain a long exact sequence

$$\ldots \longrightarrow F^{k-1}(Z) \longrightarrow F^k(X) \longrightarrow F^k(Y) \longrightarrow F^k(Z) \longrightarrow F^{k+1}(Z) \longrightarrow \ldots$$

Proposition 3.5.4. *Let $\mathcal{C}$ be a triangulated category*

- *If*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow T(X)$$

  is a distinguished triangle then $g \circ f = 0$.

- *For any object $W \in \mathcal{C}$, $Hom_{\mathcal{C}}(W, -)$ and $Hom_{\mathcal{C}}(-, W)$ are cohomological functors.*

Note: Let $\mathcal{C}$ be an additive category and $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}(\mathcal{C})$ then $\alpha(f) \circ f : X \longrightarrow Y \longrightarrow M(f)$ is zero in $\mathbf{K}(\mathcal{C})$ but it need not be zero in $\mathbf{C}(\mathcal{C})$.

Corollary 3.5.5. *Let*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & T(X) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle T(\phi)} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & T(X') \end{array}$$

be a morphism of distinguished triangle (in $\mathcal{C}$). If ϕ and ψ are isomorphism then so is θ.

Proposition 3.5.6. *Let $\mathcal{C}$ be an abelian category then the functor $H^n(-) : \boldsymbol{k}(\mathcal{C}) \longrightarrow \mathcal{C}$ is a cohomological functor.*

Lemma 3.5.7. *Let $f : X \longrightarrow Y$ be a morphism in $\boldsymbol{K}(\mathcal{C})$ then f is a quasi isomorphism if and only if $H^n(M(f)) = 0$ for all $n \in \mathbb{Z}$.*

Definition 3.5.8. *Let $\mathcal{C}$ be a triangulated category. A triangle*

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} T(X)$$

in $\mathcal{C}$ is called antidistinguished if the triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{-\gamma} T(X)$$

is a distinguishede triangle.

3.6 Localisation of Categories

Definition 3.6.1. *Let $\mathcal{C}$ be a category and S be a family of morphisms in $\mathcal{C}$. S is a called a multiplicative system if it satisfies (S1) to (S4) below:*

- *(S1) For any $X \in \mathcal{C}$, $Id_X \in S$*
- *(S2) For any pair (f, g) of S such that the composition $g \circ f$ exists, $g \circ f \in S$.*
- *(S3) Any diagram*

$$\begin{array}{ccc} & & Z \\ & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with $g \in S$ may be completed to a commutative diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

with $h \in S$. Ditto with all the arrows reversed.

- *(S4) If $f, g \in Hom_{\mathcal{C}}(X, Y)$ the following conditions are equivalent*
  - *there exists $t : Y \longrightarrow Y'$, $t \in S$ such that $t \circ f = t \circ g$,*
  - *there exists $s : X' \longrightarrow X$, $s \in S$, such that $f \circ s = g \circ s$.*

Definition 3.6.2. *Let $\mathcal{C}$ be a category and S a multiplicative system.We can define a category $\mathcal{C}_S$ called the localization of $\mathcal{C}$ at S as follows:*

$$Ob(\mathcal{C}_S) := Ob(\mathcal{C})$$

For any $X, Y \in \mathcal{C}$ define

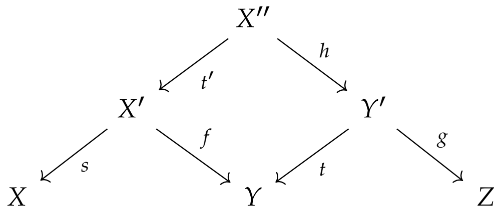
$$Hom_{\mathcal{C}_S}(X, Y) := \{(X', s, f) : X' \in \mathcal{C} \quad s : X' \longrightarrow X, \quad f : X' \longrightarrow Y \quad s \in S\} / \sim$$

where $(X', s, f) \sim (X'', t, g)$ if and only if there exist a commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \nearrow^{s} & \uparrow{\scriptstyle u} & \nwarrow^{t} & \\ X' & \longleftarrow & X''' & \longrightarrow & X'' \\ & \searrow_{f} & & \swarrow_{g} & \\ & & Y & & \end{array}$$

with $u \in S$.

The composition of $(X', s, f) \in Hom_{\mathcal{C}_S}(X, Y)$ *and* $(Y', t, g) \in Hom_{\mathcal{C}_S}(Y, Z)$ *is defined as follows. We use (S3) to find a commutative diagram:*



with $t' \in S$*. We set* $(Y', t, g) \circ (X', s, f) = (X'', s \circ t', g \circ h)$

The relation ~ given in the definition is an equivalence relation.Its easy to see that composition is well defined, assosciative and $Id_{X \in \mathcal{C}_S} = (X, Id_X, Id_X)$, that is $\mathcal{C}_S$ is indeed a category. We define a functor $Q : \mathcal{C} \longrightarrow \mathcal{C}_S$ by setting $Q(X) := X \quad \forall X \in \mathcal{C}$ and $Q(f) := (X, Id_X, f) \quad \forall f \in Hom_{\mathcal{C}}(X, Y)$.

Proposition 3.6.3. *Let* $\mathcal{C}$ *be a category and* S *be a multiplicative system.*

- *For any* $s \in S$ $Q(s)$ *is an isomorphism in* $\mathcal{C}_S$
- *Let* $\mathcal{C}'$ *bee another category,* $F : \mathcal{C} \longrightarrow \mathcal{C}'$ *be a functor such that* $F(s)$ *is an isomorphism for all* $s \in S$*. Then* F *factors uniquely through* Q*.*

Remark 3.6.4.
- *Let* $(X', s, f) \in Homm_{\mathcal{C}_S}(X, Y)$ *then* $(X', s, f) = (X', Id_{X'}, f) \circ (X', s, Id_X)$*. Let* α *be a functor from* $\mathcal{C}_S$ *to some other category. The action of* α *on morphisms is completely determined by its action on morphisms of the form* $(X', s, Id_{X'}), (X', Id_{X'}, f)$*. Let* β *be another functor from* $\mathcal{C}_S$ *with the same target as* α*, if* $\alpha \circ Q = \beta \circ Q$ *then* $\alpha = \beta$*.*
- *From the above remark and proposition 3.6.3 we get that* $(\mathcal{C}_S)^o \cong (\mathcal{C}^o)_S$*.*

Proposition 3.6.5. *Let* $\mathcal{C}$ *be a category and* $\mathcal{C}'$ *be a full subcategory. Let* S *be a multiplicative system in* $\mathcal{C}$ *and* S' *be the family of morphisms of* $\mathcal{C}'$ *which belongs to* S*. Assume* S' *is a multiplictive system in* $\mathcal{C}'$ *and one of the following condition holds.*

1. *For every morphism* $f : X \longrightarrow Y$ *in* S *with* $Y \in Ob(\mathcal{C}')$ *there exist* $g : W \longrightarrow X$ *with* $W \in Ob(\mathcal{C}')$ *and* $f \circ g \in S$*.*
2. *The same as (1) with arrows reversed.*

Then $\mathcal{C}'_{S'}$ *is a full subcategory of* $\mathcal{C}_S$

Definition 3.6.6. *Let* $\mathcal{C}$ *be a triangulated category, and let* $\mathcal{N}$ *be a subfamily of* $Ob(\mathcal{C})$*. One says* $\mathcal{N}$ *is a null system if it statisfies (N1)-(N3) below*

- *(N1)* $0 \in \mathcal{N}$*.*
- *(N2)* $X \in \mathcal{N}$ *if and only if* $X[1] \in \mathcal{N}$*.*
- *(N3) if* $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ *is a distinguished triangle. If* $X, Y \in \mathcal{N}$ *then* $Z \in \mathcal{N}$*.*

Set $S(\mathcal{N}) := \{f : X \longrightarrow Y :$ f is embedded into a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ with $Z \in \mathcal{N}\}$

Proposition 3.6.7. *Let* $\mathcal{C}$ *be a triangulated category. Assume* $\mathcal{N}$ *is a null system then* $S(\mathcal{N})$ *is a multiplicative system in* $\mathcal{C}$*.*

Notation: Let $\mathcal{C}$ be a triangulated category and $\mathcal{N}$ be a null system. We write $\mathcal{C}/\mathcal{N}$ instead of $\mathcal{C}_{S(\mathcal{N})}$.

Proposition 3.6.8. *Let* $\mathcal{C}$ *be a triangulated category and* $\mathcal{N}$ *a null system. Let* $Q : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{N}$ *be the canonical functor.*

- $\mathcal{C}/\mathcal{N}$ *becomes a triangulated category by taking for distinguished triangles those isomorphic to the image of a distinguished triangle in* $\mathcal{C}$*.*

- *We have* $Q(X) \cong 0$ *for all* $X \in \mathcal{N}$.
- *Any functor* $F : \mathcal{C} \longrightarrow \mathcal{C}'$ *of triangulated categories with* $F(X) \cong 0$ *for all* $X \in \mathcal{N}$ *factors uniquely through* Q.

Proposition 3.6.9. *Let* $\mathcal{C}$ *be a triangulated category and* $\mathcal{N}$ *be a null system in* $\mathcal{C}$, $\mathcal{C}'$ *be a full triangulated subcategory of* $\mathcal{C}$ *such that any distinguished triangle* $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ *in* $\mathcal{C}$ *with* $X, Y \in \mathcal{C}'$, *is a distinguished triangle in* $\mathcal{C}'$. *Set* $\mathcal{N}' := \mathcal{N} \cap Ob(\mathcal{C}')$

- $\mathcal{N}'$ *is a null system in* $\mathcal{C}'$.
- *Assume moreover that any morphism* $Y \longrightarrow Z$ *in* $\mathcal{C}$ *with* $Y \in \mathcal{C}', \quad Z \in \mathcal{N}$ *factorizes through an object of* $\mathcal{N}'$ *Then* $\mathcal{C}'/\mathcal{N}'$ *is a full subcategory of* $\mathcal{C}/\mathcal{N}$.

3.7 Derived categories

In this section we localise the category $\mathbf{K}(\mathcal{C})$ (we assume $\mathcal{C}$ is an abelian category through out this section) with respect to a multiplicative system which will be defined soon. Define

$$\mathcal{N} := \{X \in \mathbf{K}(\mathcal{C}) : H^n(X) = 0 \quad \forall n \in \mathbb{Z}\}$$

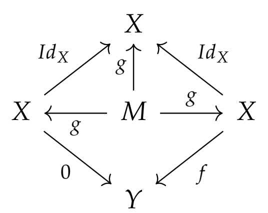
$0 \in \mathcal{N}$ and $X \in \mathcal{N} \Leftrightarrow X[1] \in \mathcal{N}$. Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{K}(\mathcal{C})$ if $X, Y \in \mathcal{N}$ then $H^n(f)$ (Since $H^X, H^n(Y) = 0$) is an isomorphism for all $n \in \mathbb{Z}$ hence by lemma 3.5.7 we get $H^n(M(f)) = 0$. Hence $\mathcal{N}$ is a null system
$f \in S(\mathcal{N}) \Leftrightarrow M(f) \in \mathcal{N} \Leftrightarrow H^n(M(f)) = 0 \quad \forall n \in \mathbb{Z} \Leftrightarrow f$ is a quasi isomorphim. The last implication follows from lemma 3.5.7. That is $S(\mathcal{N})$ is the collection of quasi isomorphisms of $\mathbf{K}(\mathcal{C})$.

Definition 3.7.1. *We set* $D(\mathcal{C}) := \mathbf{K}(\mathcal{C})/\mathcal{N}$ *and call* $D(\mathcal{C})$ *the derived category of* $\mathcal{C}$.

Let $s \in S(\mathcal{N})$ then $H^n(s)$ is ans isomorphism. From proposition 3.6.3 it follows that the functor $H^n(-)$ from $\mathbf{K}(\mathcal{C})$ to $\mathcal{C}$ factors through $D(\mathcal{C})$. By abusing the notaion we use H^n to denote the functor from $D(\mathcal{C})$ to $\mathcal{C}$ so obtained.
Using $\mathbf{K}^b(\mathcal{C})$ (resp. $\mathbf{K}^+(\mathcal{C})$,resp. $\mathbf{K}^-(\mathcal{C})$) instead of $\mathbf{K}(\mathcal{C})$ in the construction of $D(\mathcal{C})$ we construct derived category $D^b(\mathcal{C})$ (resp. $D^+(\mathcal{C})$, resp. $D^-(\mathcal{C})$).

Proposition 3.7.2.
- $D^b(\mathcal{C})$ *(resp.* $D^+(\mathcal{C})$*, resp.* $D^-(\mathcal{C})$*) is equivalent to the full subcategory of* $D(\mathcal{C})$ *consisting of objects* X *such that* $H^n(X) = 0$ *for* $|n| \gg 0$ *(resp.* $n \ll 0$*, resp.* $n \gg 0$*).*
- *By the composition of the functor* $\mathcal{C} \longrightarrow \mathbf{K}(\mathcal{C}) \longrightarrow D(\mathcal{C})$, $\mathcal{C}$ *is equivalent to the full subcategory of* $D(\mathcal{C})$ *consisting of objects* X *such that* $H^n(X) = 0$ *for* $n \neq 0$.

Let $X \in \mathbf{K}(\mathcal{C})$ then $Q(X) \cong 0$ ($Q : \mathbf{K}(\mathcal{C}) \longrightarrow D(\mathcal{C})$) if and only if there exist $Y \in \mathbf{K}(\mathcal{C})$ such that $X \oplus Y \in \mathcal{N}$. Then $0 = H^n(X \oplus Y) = H^n(X) \oplus H^n(Y)$ (Since $H^n(-)$ is a additive functor) this would imply that $H^n(X) = 0$ for all n, hence $X \in \mathcal{N}$ in other words $Q(X) \cong 0$ if and only if X is quasi isomorphic to 0 in $\mathbf{K}(\mathcal{C})$.
Let $f : X \longrightarrow Y$ be a morphism in $\mathbf{C}(\mathcal{C})$. f is zero in $D(\mathcal{C})$ if and only if there exist a quasi isomorphism $g : X' \longrightarrow X$ such that $f \circ g = 0$ in $\mathbf{K}(\mathcal{C})$ (or $f \circ g$ is homotopic to 0)



according to (S4) existence of such a g is equivalent to the existance of $h : Y \longrightarrow Y'$ such that $h \circ f = 0$ in $\mathbf{K}(\mathcal{C})$ (or $h \circ f$ is homotopic to 0) . That is f is zero in $D(\mathcal{C})$ if and only if there exist a quasi isomorphism $h : Y \longrightarrow Y'$ such that $h \circ f = 0$ in $\mathbf{K}(\mathcal{C})$ (or $h \circ f$ is homotopic to 0).

Proposition 3.7.3. *Let $\mathcal{C}$ be an abelian category and*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be an exact sequence in $\mathbf{C}(\mathcal{C})$. Let $M(f)$ be the mapping cone of f and $\phi^n : M(f)^n = X^{n+1} \bigoplus Y^n \longrightarrow Z^n$ be the morphism $(0, g^n)$. Then $\{\phi^n\}_n : M(f) \longrightarrow Z$ is a morphism of complexes, $\phi \circ \alpha(f) = g$ and ϕ is a quasi isomorphism.

That is an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\mathbf{C}(\mathcal{C})$ give rise to a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$ in $D(\mathcal{C})$ where $Z \longrightarrow X[1] = \beta(f) \circ \phi^{-1}$.

Remark 3.7.4.

1. *Isomorphisms satisfy "2 out of 3" property that is If any two of the morphisms $f : X \longrightarrow Y, g : Y \longrightarrow Z, g \circ f : X \longrightarrow Z$ is an isomorphism then the third one is also an isomorphism. If f, g are isomorphism then $g \circ f$ is an isomorphism. Now assume $f, g \circ f$ is an isomorphism, let h be the inverse of f and l be the inverse of $g \circ f$. We claim that $f \circ l$ is the inverse of g. $g \circ (f \circ l) = Id_Z$, we have $h = l \circ g \circ f \circ h = l \circ g$ then $(f \circ l \circ g) = f \circ h = Id_Y$ hence isomorphism satisfy "2 out of 3" property.*

2. *Let X, Y be complexes and $f : X \longrightarrow Y$ be a quasi isomorphism. From proposition 3.3.15 we know that $\tau^{\geq n}()$ and $\tau^{leqn}(-)$ defines functor from $\mathbf{K}(\mathcal{C})$ to $\mathbf{K}(\mathcal{C})$. consider the following diagram*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & Coker(d_X^{n-1}) & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow{\scriptstyle \alpha_X} & & \uparrow & & \\
\cdots & \longrightarrow & X^{n-1} & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & Y^{n-1} & \longrightarrow & Y^n & \longrightarrow & Y^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \alpha_Y} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & Coker(d_Y^{n-1}) & \longrightarrow & Y^{n+1} & \longrightarrow & \cdots
\end{array}$$

If $k > n$ then $\tau^{\geq n}(f)^k = f^k$ and if $k < n$ then $\tau^{\geq n}(f)^k = 0$ hence if $k \neq 0$, $H^k(\tau^{\geq n}(f))$ is an isomorphism. We have $\alpha_Y \circ f^n \circ d_X^{n-1} = \alpha_Y \circ d_Y^{n-1} \circ f^{n-1} = 0$ hence there exist $l : Coker(d_X^{n-1}) \longrightarrow Coker(d_Y^{n-1})$ such that $l \circ \alpha_X = \alpha_Y \circ f^n$. we know that $l = \tau^{\geq n}(f)^n$, hence $H^n(\tau^{\geq n}(f)) \circ H^n(X \longrightarrow \tau^{\geq n}(X)) = H^n(Y \longrightarrow \tau^{\geq n}(Y)) \circ H^n(f)$. From proposition 3.3.14 we know that $H^n(X \longrightarrow \tau^{\geq n}(X))$ and $H^n(Y \longrightarrow \tau^{\geq n}(Y))$ are isomorphisms. Hence (1) implies that $H^n(\tau^{\geq n}(f))$ is an isomorphism. So $\tau^{\geq n}(-)$ takes quasi isommorphisms to quasi isomorphisms hence by proposition 3.6.3 $\tau^{\geq n}(-)$ induces a functor $D(\mathcal{C}) \longrightarrow \mathbf{K}(\mathcal{C})$ composing this functor with the $\mathbf{K}(\mathcal{C}) \longrightarrow D(\mathcal{C})$ we get a functor $D(\mathcal{C}) \longrightarrow D(\mathcal{C})$ which we denote by $\tau^{\geq n}(-)$ (by abusing the notation). Strictly speaking $\tau^{\geq n} : D(\mathcal{C}) \longrightarrow D^+(\mathcal{C})$.

3. *Similarly $\tau^{\leq n}$ give rise to a functor $D(\mathcal{C}) \longrightarrow D^-(\mathcal{C})$ which we again denote by $\tau^{\leq n}$.*

Lemma 3.7.5. *Let*

$$\begin{array}{ccc}
W & \xrightarrow{a} & X \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{f} & Z
\end{array}$$

be a commutative diagram

- *If the diagram is cartesian then* $X \bigsqcup_W Y \longrightarrow Z$ *is a monomorphism.*
- *If the diagram is cocartesian then* $W \longrightarrow X \times_Z Y$ *is an epimorphism.*

Corollary 3.7.6. *Let*

$$\begin{array}{ccc} W & \xrightarrow{a} & X \\ \downarrow b & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

be a commutative diagram

- *If there exist a monomorphism* $X \bigsqcup_W Y \longrightarrow Z$ *then there exist an epimorphism* $W \longrightarrow X \times_Z Y$.
- *If there exist an epimorphism* $W \longrightarrow X \times_Z Y$ *then there exist a monomorphism* $X \bigsqcup_W Y \longrightarrow Z$.

Lemma 3.7.7. *Let*

$$\begin{array}{ccc} W & \xrightarrow{a} & X \\ \downarrow b & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

be a commutative diagram

- *If the diagram is cocartesian then the morphism* $l : Ker(a) \longrightarrow Ker(g)$ *induced by* b *is an epimorphism.*
- *If the diagram is cartesian then the morphism* $l : Coker(a) \longrightarrow Coker(g)$ *induced by* f *is a monomorphism.*

Corollary 3.7.8. *Let*

$$\begin{array}{ccc} W & \xrightarrow{a} & X \\ \downarrow b & & \downarrow f \\ Y & \xrightarrow{g} & Z \end{array}$$

be a commutative diagram.

- *Let* $n : X \bigsqcup_W Y \longrightarrow Z$ *be a monomorphism then* $c : Ker(a) \longrightarrow Ker(g)$ *induced by* b *is an epimorphism and* $d : Coker(a) \longrightarrow Coker(g)$ *induced by* f *is a monomorphism.*
- *Let* $m : W \longrightarrow X \times_Z Y$ *be a epimorphism then* $c : Ker(a) \longrightarrow Ker(g)$ *induced by* b *is an epimorphism and* $d : Coker(a) \longrightarrow Coker(g)$ *induced by* f *is a monomorphism.*

Proposition 3.7.9. *Let* $\mathcal{F}$ *be a full additive subcategory of* $\mathcal{C}$ *such that for any* $X \in \mathcal{C}$ *there exist* $X' \in \mathcal{F}$ *and an exact sequence* $0 \longrightarrow X \longrightarrow X'$ *then:*

- *For any* $X \in \boldsymbol{K}^+\mathcal{C}$ *there exist* $X' \in \boldsymbol{K}^+(\mathcal{F})$ *and a quasi isomorphism* $f : X \longrightarrow X'$.
- *Let* $\mathcal{N}$ *be as in definition and let* $\mathcal{N}' = \mathcal{N} \cap \boldsymbol{K}^+(\mathcal{F})$ *then the canonical functor* $\boldsymbol{K}^+(\mathcal{F})/\mathcal{N}' \longrightarrow D^+(\mathcal{C})$ *is an equivalance of categories.*

Lemma 3.7.10. *Let* $X \in \mathbf{C}(\mathcal{C})$*, if* $H^n(X) = 0$ *then the complex* X *is exact at* X^n.

Proof. We have $Im(d_X^{n-1}) \longrightarrow X^n = Ker(d_X^n) \longrightarrow X^n \circ Im(d_X^{n-1}) \longrightarrow Ker(d_X^n)$. $Im(d_X^{n-1}) \longrightarrow X^n$ is a monomorphism hence $Im(d_X^{n-1}) \longrightarrow Ker(d_X^n)$ is also a monomorphism. $H^n(X) = 0$ implies that $Im(d_X^{n-1}) \longrightarrow Ker(d_X^n)$ is an epimorphism hence $Ker(d_X^n) \cong Im(d_X^{n-1})$. That is the complex is exact at X^n.

■

Corollary 3.7.11. *Let $\mathcal{F}$ be a full additive subcategory of $\mathcal{C}$ such that for any $X \in \mathcal{C}$ there exist $X' \in \mathcal{F}$ and an exact sequence $0 \longrightarrow X \longrightarrow X'$ (hypothesis of proposition 3.7.9) and there exist an integer $d \geq 0$ such that for any exact sequence $X^0 \longrightarrow X^1 \longrightarrow \ldots \longrightarrow X^d \longrightarrow 0$ in $\mathcal{C}$ with $X^j \in \mathcal{F}$ whenever $j < d$ then $X^d \in \mathcal{F}$.*
Then for any $X \in \mathbf{K}^b(\mathcal{C})$ there exist $X'\mathbf{K}^b(\mathcal{F})$ and a quasi isomorphism $X \longrightarrow X'$.

Proof. Proposition 3.7.9 imples that there exist $X' \in \mathbf{K}^+(\mathcal{C})$ and a quasi isomorphism $X \longrightarrow X'$. Let n be such that $H^j(X) = 0$ for all $j > n$. Then we get that $H^j(X') = 0$ for all $j > n$. Therefore $\tau^{\leq n+d}(X') \longrightarrow X'$ is a quasi isomorphism. We know that if $k < n+d$ then $\tau^{\leq n+d}(X')^k \in \mathcal{C}$ and for $k > n+d$, $\tau^{\leq n+d}(X')^k = 0$. Since $H^k(\tau^{\leq n+d}(X')) = 0$ for all $k > n$ from lemma 3.7.10 we obtain an exact sequence $\tau^{\leq n+d}(X')^n \longrightarrow \tau^{\leq n+d}(X')^{n+1} \longrightarrow \ldots \longrightarrow \tau^{\leq n+d}(X')^{n+d} \longrightarrow 0$, from the hypothesis it follows that $\tau^{\leq n+d}(X')^{n+d} \in \mathcal{F}$. Hence $\tau^{\leq n+d}(X') \in \mathbf{K}^b(\mathcal{F})$. Its easy to see that the quasi isomorphism $X \longrightarrow X'$ induces a quasi isomorphism $X \longrightarrow \tau^{\leq n+d}(X')$.

■

Definition 3.7.12. *One says that $\mathcal{C}$ has enough injectives if for any $X \in \mathcal{C}$ there exist an injective object $X' \in \mathcal{C}$ and a monomorphism $X \longrightarrow X'$*

Proposition 3.7.13. *Assume $\mathcal{C}$ has enough injectives, and let $\mathcal{F}$ be the full subcategory of injective objects. Then the natuaral functor $\mathbf{K}^+(\mathcal{F}) \longrightarrow D^+(\mathcal{C})$ is an equivalance of categories.*

Definition 3.7.14. *$\mathcal{C}'$ is a thick subcategory of $\mathcal{C}$ if for any exact sequence $Y \longrightarrow Y' \longrightarrow X \longrightarrow Z \longrightarrow Z'$ in $\mathcal{C}$ with Y, Y', Z, Z' in $\mathcal{C}'$, X belongs to $\mathcal{C}'$*

Let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$, by $D^+_{\mathcal{C}'}(\mathcal{C})$ we denote the full subcategory of $D^+(\mathcal{C})$ consisting of complexes whose cohomology objects belongs to $\mathcal{C}'$, then there is a natural functor

$$\delta : D^+(\mathcal{C}') \longrightarrow D^+_{\mathcal{C}'}(\mathcal{C})$$

Proposition 3.7.15. *Let $\mathcal{C}$ be an abelian category, $\mathcal{C}'$ a thick full abelian subcategory. Assume that for any monomorphism $f : X' \longrightarrow X$ with $X' \in \mathcal{C}'$, there exists a morphism $g : X \longrightarrow Y$, with $Y \in \mathcal{C}'$ such that $g \circ f$ is a monomorphism. Then the functor δ (described above) is an equivalence of categories.*

3.8 Derived Functor

In this section $\mathcal{C}$ and $\mathcal{C}'$ denotes abelian categories and $F : \mathcal{C} \longrightarrow \mathcal{C}'$ denotes an additive functor. Q denotes the natuaral functor $\mathbf{K}^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C})$ or the natuaral functor $\mathbf{K}^+(\mathcal{C}') \longrightarrow D^+(\mathcal{C}')$

Definition 3.8.1. *Let $T : D^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C}')$ be a functor of triangulated categories, and let $s : Q \circ \mathbf{K}^+(F) \longrightarrow T \circ Q$ be a morphism of functors (where $\mathbf{K}^+(F) : \mathbf{K}^+(\mathcal{C}) \longrightarrow \mathbf{K}^+(\mathcal{C}')$ is the functor natuarally associated to F). Assume that for any functor of triangulated categories $G : D^+(\mathcal{C}) \longrightarrow D^+(\mathcal{C}')$, the morphism $Hom(T, G) \longrightarrow Hom(Q \circ \mathbf{K}^+(F), G \circ Q)$ induced by s is an isomorphism. Then (T, S), which is unique up to isomorphism, is called the right derived functor of F, and denoted RF. The functor $H^n \circ RF$, also denoted R^nF, is called the n-th derived functor of F.*

From nowonwards until proposition 3.8.4 we assume that F is left exact.

Definition 3.8.2. *A full additive subcategory $\mathcal{F}$ of $\mathcal{C}$ is called injective with respect to F (or F-injective, for short), if:*

1. *for any $X \in \mathcal{C}$ there exist $X' \in \mathcal{F}$ and an exact sequence $0 \longrightarrow X \longrightarrow X'$.*

2. *Let* $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ *be an exact sequence in* $\mathcal{C}$. *If* $X', X \in \mathcal{F}$ *then* $X'' \in \mathcal{F}$.

3. *Let* $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ *be an exact sequence in* $\mathcal{C}$. *If* $X', X, X'' \in \mathcal{F}$ *then* $0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0$ *is exact.*

Let $\mathcal{F}$ be an F injective full subcategory of $\mathcal{C}$ then F takes objects of $\mathbf{K}^+(\mathcal{F})$ quasi isomorphic to zero to objects of $\mathbf{K}^+(\mathcal{C}')$ quasi isomorphi to zero. Hence from proposition 3.6.8 it follows that the functor

$$\mathbf{K}^+(\mathcal{F}) \longrightarrow \mathbf{K}^+(\mathcal{C}') \longrightarrow D^+(\mathcal{C}')$$

factors through $\mathbf{K}^+(\mathcal{F})/\mathcal{N}'$ where $\mathcal{N}' = \mathcal{N} \cap \mathcal{F}$ (where $\mathcal{N}$ is as mentioned in definition 3.7.1). From proposition 3.7.9 we obtain that $\mathbf{K}^+(\mathcal{F})/\mathcal{N}'$ is equivalent to $D^+(\mathcal{C})$.

Proposition 3.8.3. *Assume that there exists an F-injective subcategory* $\mathcal{F}$ *of* $\mathcal{C}$. *Then the functor from* $\mathbf{K}^+(\mathcal{F})/\mathcal{N}' \longrightarrow D^+(\mathcal{C}')$ *constructed above is the right derived functor of F.*

It follows from the universal property of RF that the above construction is independent of the choice of $\mathcal{F}$

Proposition 3.8.4. *Let* $\mathcal{C}, \mathcal{C}'\mathcal{C}''$ *be three abelian categories and let* $F : \mathcal{C} \longrightarrow \mathcal{C}'$, $F' : \mathcal{C}' \longrightarrow \mathcal{C}''$ *be two left exact functors. Assume there exists a full additive sub- subcategory* $\mathcal{F}$ *of* $\mathcal{C}$ *(resp.* $\mathcal{F}'$ *of* $\mathcal{C}'$*) which is F-injective (resp.* F'*-injective), and such that* $F(Ob(\mathcal{F})) \subset Ob(\mathcal{F}')$. *Then* $\mathcal{F}$ *is* $(F' \circ F)$*-injective, and we have:*

$$R(F' \circ F) = RF \circ RF'$$

Chapter 4

Quasi-coherent modules

4.1 $\mathcal{O}_X$-modules

Let W be a topological space and $\mathcal{G}$ and $\mathcal{G}'$ be presheaves on W then we can define a new presheaf $\mathcal{G} \times \mathcal{G}'$ on W by setting

$$\mathcal{G} \times \mathcal{G}'(V) = \mathcal{G}(V) \times \mathcal{G}'(V) \quad \forall \text{ open sets } V \subseteq W$$

and by defining the restriction morphisms in the obvious way. Its straightforward to see that if $\mathcal{G}, \mathcal{G}'$ are sheaves then $\mathcal{G} \times \mathcal{G}'$ is a sheaf.
From now onwards we assume that $(W, \mathcal{O}_W)$ is a ringed space.

Definition 4.1.1. *Consider a ringed space $(W, \mathcal{O}_W)$. A sheaf $\mathcal{G}$ on W with the following (addition and scalar multiplication) sheaf morphisms is called an $\mathcal{O}_W$–module.*

- $\mathcal{G} \times \mathcal{G}, \longrightarrow \mathcal{G}, \quad (t, t') \mapsto t + t'$ *for* $t, t' \in \mathcal{G}(V)$, $V \subseteq W$ *open.*
- $\mathcal{O}_W \times \mathcal{G} \longrightarrow \mathcal{G}, \quad (a, t) \mapsto at$ *for* $a \in \mathcal{O}_W(V)$, $t \in \mathcal{G}(V)$, $V \subseteq W$ *open.*

such that these maps gives $\mathcal{G}(V)$ the structure of an $\mathcal{O}_W(V)$-module for every open set V of W.
Given $\mathcal{O}_W$–modules $\mathcal{G}_1$ and $\mathcal{G}_2$ we define a morphism of $\mathcal{O}_W$ modules as a sheaf morphism $x : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ such that $x_V : \mathcal{G}_1(V) \longrightarrow \mathcal{G}_2(V)$ is an $\mathcal{O}_W(V)$–module homomorphism for every open set V of W, That is

$$x_V(t + t') = x_V(t) + w_V(t'),$$

$$w_V(at) = a x_V(t)$$

for all $t, t' \in \mathcal{G}_1(V)$ and for all $a \in \mathcal{O}_W(V)$.

The collection of $\mathcal{O}_W$-modules forms a category which is denoted by $\mathcal{O}_W - Mod$. Using 0 we denote the trivial or zero $\mathcal{O}_W$–module which is defined by setting $0(V) = 0$ for all open set V of W
Examples:

- Let W be a topological space and let $\underline{\mathbb{Z}}$ be the constant sheaf of rings on W with value $\underline{\mathbb{Z}}$. Then a $\underline{\mathbb{Z}}$-module is simply a sheaf of abelian groups on W .
- Consider a topological space W consisting of a single point and a ring B. Set $\mathcal{O}_W(W) = B$ then $\mathcal{O}_W$ module is just an B–module M.

Let $\mathcal{G}$ be an $\mathcal{O}_W$ module and let $w \in W$. Let $V \subseteq W$ be an open set containing w then the $\mathcal{O}_W(V)$ module structure on $\mathcal{G}(V)$ induces an $\mathcal{O}_{W,w}$ module structure on $\mathcal{G}_w$. If $x : \mathcal{G} \longrightarrow \mathcal{G}'$ is a morphism of $\mathcal{O}_W$ modules then the morphism $x_w : \mathcal{G}_w \longrightarrow \mathcal{G}'_w$ induced on stalks is morphism of $\mathcal{O}_{W,w}$ modules.

Consider a locally ringed space $(W, \mathcal{O}_W)$. Set

$$\mathcal{G}(w) := \mathcal{G}_w/\mathfrak{m}_w\mathcal{G}_w = \mathcal{G}_w \bigotimes_{\mathcal{O}_{W,w}} \kappa(w)$$

(here $\mathfrak{m}_w$ denotes the maximal ideal of $\mathcal{O}_{W,w}$ and $\kappa(w) := \mathcal{O}_{W,w}/\mathfrak{m}_w$). This is a $\kappa(w)$ vector space called the fiber of $\mathcal{G}$ in w. Let $t \in \mathcal{G}(V)$ then by $t(w)$ we denote the image of $t_w \in \mathcal{G}_w$ in $\mathcal{G}(w)$

Definition 4.1.2. *Consider a $\mathcal{O}_W$−module $\mathcal{G}$. If $\mathcal{F}$ is another $\mathcal{O}_W$−module such that for every open set V of W $\mathcal{F}(V) \subseteq \mathcal{G}(V)$ and the inclusion $i_V : \mathcal{F}(V) \longrightarrow \mathcal{G}(V)$ give rise to a $\mathcal{O}_W$−module morphism $i : \mathcal{F} \longrightarrow \mathcal{G}$ we say that $\mathcal{F}$ is an $\mathcal{O}_W$−submodule of $\mathcal{G}$. The $\mathcal{O}_W$-submodules of $\mathcal{O}_W$ are called ideals of $\mathcal{O}_W$.*

Definition 4.1.3. *Let $\mathcal{F}$ be an $\mathcal{O}_W$-submodule of the $\mathcal{O}_W$ module $\mathcal{G}$. Define the quotient of $\mathcal{G}$ by $\mathcal{F}$ as the sheaf assosciated to the presheaf (sheafification)*

$$V \mapsto \mathcal{G}(V)/\mathcal{F}(V) \quad V \subseteq_{open} W$$

it denoted by $\mathcal{G}/\mathcal{F}$. The $\mathcal{O}_W$ module structure of $\mathcal{G}$ induces an $\mathcal{O}_W$-module structure on $\mathcal{G}/\mathcal{F}$.

The canonical homomorphism $\mathcal{G}(V) \longrightarrow \mathcal{G}(V)/\mathcal{F}(V)$ induces an $\mathcal{O}_W$-module homomorphism $\mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F}$.
Let $w \in W$ then we have

$$(\mathcal{G}/\mathcal{F})_w = \varinjlim_{w \in V} (\mathcal{G}/\mathcal{F})(V) = \varinjlim_{w \in V} \mathcal{G}(U)/\mathcal{F}(V) = \mathcal{G}_w/\mathcal{F}_w$$

where V runs through the open neighborhoods of w.
Let $x : \mathcal{G} \longrightarrow \mathcal{G}'$ be a morphism of $\mathcal{O}_W$-modules then we define kernel,image and cokernel of x as follows.

Definition 4.1.4.

- *The presheaf $V \mapsto Ker(x_V : \mathcal{G}(V) \longrightarrow \mathcal{G}'(V))$ is a sheaf hence an $\mathcal{O}_W$-submodule of $\mathcal{G}$. It is called the kernel of x and is denoted by $Ker(x)$. We get that $Ker(x)_w = Ker(x_w)$ for all $w \in W$ hence w is injective if and only if $Ker(x) = 0$*
- *The sheaf assosciated to the presheaf $V \mapsto Im(x_V : \mathcal{G}(V) \longrightarrow \mathcal{G}'(V))$ is an $\mathcal{O}_W$-submodule of $\mathcal{G}'$. It is called the image of x and is denoted by $Im(x)$. We get that $Im(x)_w = Im(x_w)$ for all $w \in W$ hence w is surjective if and only if $Im(x) = \mathcal{G}'$*
- *The sheaf associated to the presheaf $V \mapsto Coker(x_V : \mathcal{G}(V) \longrightarrow \mathcal{G}'(V))$ is an $\mathcal{O}_W$-submodule of $\mathcal{G}'$. It is called the cokernel of x and is denoted by $Coker(x)$. We get that $Coker(x)_w = Coker(x_w)$ for all $w \in W$*

All these definitions are compatible with the definitions of kernel, image and cokernel made in chapter 3. Its straightforward to see that $Coker(x) \cong \mathcal{G}'/Im(x)$

Proposition 4.1.5. *Every $\mathcal{O}_W$-module morphism $x : \mathcal{G} \longrightarrow \mathcal{G}'$ induces an isomorphism*

$$\mathcal{G}/Ker(x) \cong Im(x)$$

Definition 4.1.6. *A sequence of morphisms of $\mathcal{O}_W$-modules*

$$\mathcal{G} \xrightarrow{x} \mathcal{G}' \xrightarrow{x'} \mathcal{G}''$$

is said to be exact if it satisfies the consitions given below.

- $Im(x) \cong Ker(x')$
- *For every $w \in W$ the sequence at the level of stalks of $\mathcal{O}_{W,w}$−modules $\mathcal{G}_w \longrightarrow \mathcal{G}'_w \longrightarrow \mathcal{G}''_w$ is exact*

A sequence $\ldots \longrightarrow \mathcal{G}_{i-1} \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{G}_{i+1} \longrightarrow \mathcal{G}_{i+2} \longrightarrow \ldots$ is exact, if $\mathcal{G}_{i-1} \longrightarrow \mathcal{G}_i \longrightarrow \mathcal{G}_{i+1}$ is exact for all i.
Let $x, x' : \mathcal{G} \longrightarrow \mathcal{G}'$ be morphisms of $\mathcal{O}_W$-modules. By setting $(x + x')_V := x_V + x'_V$ for all open sets V of W we define $x + x'$ in the obvious way. Let $a \in \Gamma(W, \mathcal{O}_W)$ then we can define $ax : \mathcal{G} \longrightarrow \mathcal{G}'$ by setting $(ax)_V := a|_V x_V$. That is $Hom_{\mathcal{O}_W}(\mathcal{G}, \mathcal{G}')$ has a structure of $\Gamma(W, \mathcal{O}_W)$ module.

Proposition 4.1.7.
- *A sequence $0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}''$ of $\mathcal{O}_W$- modules is exact if and only if for all open subsets $V \subseteq W$ and for all $\mathcal{O}_V$-modules $\mathcal{F}$ the sequence*

$$0 \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G}'|_V) \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G}|_V) \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{F}, \mathcal{G}''|_V)$$

  of $\Gamma(V, \mathcal{O}_W)$-modules is exact.

- *A sequence $\mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$ of $\mathcal{O}_W$-modules is exact if and only if for all open subsets $V \subseteq W$ and for all $\mathcal{O}_V$-modules $\mathcal{F}$ the sequence*

$$0 \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{G}'|_V, \mathcal{F}) \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{G}|_V, \mathcal{F}) \longrightarrow Hom_{\mathcal{O}_V}(\mathcal{G}''|_V, \mathcal{F})$$

  of $\Gamma(V, \mathcal{O}_W)$-modules is exact.

4.1.1 Basic constructions of $\mathcal{O}_X$-modules

In this subsection $(X, \mathcal{O}_X)$ will always denote a ringed space
Direct Sum and Direct Product:
Consider a family of $\mathcal{O}_W$−modules $(\mathcal{G}_i)_{i\in I}$ then the sheafification of the presheaf

$$V \mapsto \bigoplus_{i\in I} \mathcal{G}_i(V)$$

is an $\mathcal{O}_W$-module (defining the addition and scalar multiplication component wise), it is called the direct sum of the family $(\mathcal{G}_i)_{i\in I}$ and is denoted by $\oplus_{i\in I} \mathcal{G}_i$.
Since Direct sum and inductive limit commute with each other we obtain the following $\mathcal{O}_{W,w}$-module isomorphism

$$(\bigoplus_{i\in I} \mathcal{G}_i)_w \cong \bigoplus_{i\in I} \mathcal{G}_{i,w}$$

The presheaf

$$V \mapsto \prod_{\in I} \mathcal{G}_i(V)$$

is a sheaf and an $\mathcal{O}_W$-module, it is said to be the direct product of the family $(\mathcal{G}_i)_{i\in I}$ and we use $\prod_{i\in I} \mathcal{G}_i$ to denote it.
We get the following $\mathcal{O}_{W,w}$-module homomorphism

$$(\prod_{i\in I} \mathcal{G}_i)_w \longrightarrow \prod_{i\in I} \mathcal{G}_{i,w}.$$

If $\mathcal{G}_i = \mathcal{G}$ for some $\mathcal{O}_W$-module $\mathcal{G}$, we use $\mathcal{G}^{(I)}$ (resp. $\mathcal{G}^I$) instead of $\oplus_{i\in I} \mathcal{G}_i$ (resp. $\prod_{i\in I} \mathcal{G}_i$).

Proposition 4.1.8. *$\mathcal{O}_W$−modules with their morphisms form an additive category this is indeed an abelian category*

Sums and intersections of submodules
Consider an $\mathcal{O}_W$−module $\mathcal{G}$ and a family $(\mathcal{G}_i)_{i\in I}$ of $\mathcal{O}_W$−submodules of $\mathcal{G}$ then we define $\sum_i \mathcal{G}_i$ to be the image of the canonical homomorphism

$$\bigoplus_i \mathcal{G}_i \longrightarrow \mathcal{G}$$

which is a $\mathcal{O}_W$–submodule of $\mathcal{G}$. The intersection $\bigcap_i \mathcal{G}_i$ of the family $(\mathcal{G}_i)$ is the $\mathcal{O}_W$-submodule of G defined as the kernel of the canonical homomorphism

$$\mathcal{G} \longrightarrow \prod_{i \in I} \mathcal{G}/\mathcal{G}_i$$

Tensor product

Let $\mathcal{G}$ and $\mathcal{F}$ be two $\mathcal{O}_W$-modules. The sheaf assosciated to the presheaf

$$V \mapsto \mathcal{G}(V) \bigotimes_{\mathcal{O}_W} \mathcal{F}(V)$$

is an $\mathcal{O}_W$-module. It is said to be the tensor product of $\mathcal{G}$ and $\mathcal{F}$ and we use $\mathcal{G} \otimes_{\mathcal{O}_W} \mathcal{F}$ to denote it.

Support

Consider an $\mathcal{O}_X$-module $\mathcal{G}$ then

$$Suppe(\mathcal{G}) := \{w \in W : \mathcal{G}_w \neq 0\}$$

Algebra

Consider a ringed space $(W, \mathcal{O}_W)$ then an $\mathcal{O}_W$-algebra is an $\mathcal{O}_W$-module $\mathcal{B}$ together with an $\mathcal{O}_W$ -bilinear multiplication

$$\mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}, \quad (b, b') \mapsto bb' \quad \forall b, b' \in \mathcal{B}(V), \quad V \subseteq W$$

this map has to be defined in such a way that for all open sets V of W it gives $\mathcal{B}(V)$ the structure of an $\mathcal{O}_W(V)$–algebra.

4.1.2 Direct and inverse image of $\mathcal{O}_X$-modules

We would like to define Direct and inverse image of $\mathcal{O}_W$-modules similar to the direct and inverse image of a sheaf

Consider a morphism of ringed spaces $f : (W, \mathcal{O}_W) \longrightarrow (Z, \mathcal{O}_Z)$.

Direct image:

Let $\mathcal{G}$ and $\mathcal{G}'$ be sheaves on W then we have

$$f_*(\mathcal{G} \times \mathcal{G}') = f_*(\mathcal{G}) \times f_*(\mathcal{G}')$$

Suppose $\mathcal{G}$ is an $\mathcal{O}_W$–module then from the functoriality of f_* we get morphisms

$$f_*(\mathcal{G}) \times f_*(\mathcal{G}) \longrightarrow f_*(\mathcal{G}), \quad f_*(\mathcal{O}_W) \times f_*(\mathcal{G}) \longrightarrow f_*(\mathcal{G})$$

and these morphisms give $f_*(\mathcal{G})$ a structure of an $f_*(\mathcal{O}_W)$-module. We can give $f_*(\mathcal{G})$ the structure of an $\mathcal{O}_Z$–module via the map $f^\flat$. $f_*(\mathcal{G})$ with this structure of an $\mathcal{O}_Z$–module is called the direct image of $\mathcal{G}$ under f. That is f_* defines a functor from the category of $\mathcal{O}_W$-modules to the category of $\mathcal{O}_Z$-modules.

Inverse image:

Consider the sheaves $\mathcal{F}$ and $\mathcal{F}'$ defined on Z, then

$$f^{-1}(\mathcal{F} \times \mathcal{F}') = f^{-1}(\mathcal{F}) \times f^{-1}(\mathcal{F}')$$

Let $\mathcal{F}$ is an $\mathcal{O}_Z$-module then similar to the case of direct image we can endow $f^{-1}(\mathcal{F})$ the structure of an $f^{-1}(\mathcal{O}_Z)$-module (addition and scalar multiplication is defined via the functoriality of f^{-1}). Via $f^\sharp : \mathcal{O}_Z \longrightarrow \mathcal{O}_W$, $\mathcal{O}_W$ is an $f^{-1}\mathcal{O}_Z$ algebra. Therefore

$$f^*\mathcal{F} := \mathcal{O}_W \bigotimes_{f^{-1}\mathcal{O}_Z} f^{-1}\mathcal{F}$$

is endowed with the structure of an $\mathcal{O}_W$-module which we call the inverse image of $\mathcal{F}$ under f .

f^* defines a functor from the catgeory of $\mathcal{O}_Z$-modules to the catgeory of $\mathcal{O}_W$-modules.

Proposition 4.1.9. *For every $\mathcal{O}_W$-module $\mathcal{G}$ and every $\mathcal{O}_Z$-module $\mathcal{F}$ there is an isomorphism of $\Gamma(Z, \mathcal{O}_Z)$-modules*

$$Hom_{\mathcal{O}_W}(f^*\mathcal{F}, \mathcal{G}) \longrightarrow Hom_{\mathcal{O}_Z}(\mathcal{F}, f_*\mathcal{G})$$

that is functorial in $\mathcal{G}$ and $\mathcal{F}$.

4.2 Quasi-coherent modules on a scheme

4.2.1 The $\mathcal{O}_{Spec(B)}$-module $\tilde{N}$ attached to an B-module N

Consider a ring B and the assosciated affine scheme $W = Spec(B)$, let N be an B-module. We define a presheaf $\tilde{N}$ on $\{D(f) : f \in B\}$ by setting

$$\Gamma(D(f), \tilde{N}) := N_f$$

Theorem 4.2.1. *Letf $\tilde{N}$ be defined as above then $\tilde{N}$ is a sheaf on $\{D(f) : f \in B\}$.*

We can extend this sheaf to W, again we denote this sheaf by $\tilde{N}$. For every $f \in B$, N_f is a B_f-module this induces an $\mathcal{O}_W$-module structure on $\tilde{N}$. If we view B as a module over itself we obtain that $\tilde{B} = \mathcal{O}_W$.
Let M be another B−module and $v : N \longrightarrow M$ be an B-module homomorphism then v induces homomorphisms $v_f : N_f \longrightarrow M_f$ of B_f modules for every $f \in B$. It is starightforward to see that v_fs give rise to an $\mathcal{O}_W$-module homomorphism $\tilde{v} : \tilde{N} \longrightarrow \tilde{M}$. That is $N \mapsto \tilde{N}$ is a functor from the category of B modules to the category of $\mathcal{O}_W$-modules.
a morphism $x : \mathcal{G} \longrightarrow \mathcal{F}$ of $\mathcal{O}_W$ modules give rise to an B-module homomorphism $x_W : \Gamma(W, \mathcal{G}) \longrightarrow \Gamma(W, \mathcal{F})$. Then Γ that takes $\mathcal{G}$ to $\Gamma(W, \mathcal{G})$ is a functor from the category of $\mathcal{O}_W$ modules to the category of B−modules.

Proposition 4.2.2. *Consider an affine scheme $W = Spec(B)$ then given any B−modules N and M*

$$Hom_B(N, M) \underset{\Gamma}{\overset{v \mapsto \tilde{v}}{\rightleftarrows}} Hom_{\mathcal{O}_W}(\tilde{N}, \tilde{M})$$

are mutually inverse. that is, the functor $N \mapsto \tilde{N}$ is fully faithful .

Proposition 4.2.3. *Consider a ring B and the assosciated affine scheme $W = Spec(B)$.*

1. *A sequence of B-modules $N \longrightarrow M \longrightarrow P$ is exact if and only if the corresponding sequence $\tilde{N} \longrightarrow \tilde{M} \longrightarrow \tilde{P}$ is an exact sequence of $\mathcal{O}_W$-modules.*
2. *Consider the B−module homomorphism $v : N \longrightarrow M$. Then*

$$Ker(\tilde{v}) = \widetilde{Ker}(v), \quad Im(\tilde{v}) = \widetilde{Im}(v), \quad Coker(\tilde{v}) = \widetilde{Coker}(v)$$

In particular, u is injective (resp. surjective, resp. bijective) if and only if $\tilde{u}$ is .

3. *Consider a family of B-modules $(N_i)_{i \in I}$. Then*

$$\bigoplus_{i \in I} \tilde{N}_i = (\widetilde{\bigoplus_{i \in I}} N_i)$$

4. *Let N be the filtered inductive limit of an inductive system of B-modules N_λ. Then $\tilde{N}$ is the inductive limit of the inductive system $\tilde{N}_\lambda$ of $\mathcal{O}_W$-modules .*

4.2.2 Quasi-coherent modules

Definition 4.2.4. *Consider a ringed space $W, \mathcal{O}_W$ and an $\mathcal{O}_W-$module $\mathcal{G}$. Suppose given any $w \in W$ we can find an open set V of W containing w such that there exist an exact sequence of $\mathcal{O}_{W|_U}-$modules as given below*

$$\mathcal{O}^J_{W|_V} \longrightarrow \mathcal{O}^I_{W|_V} \longrightarrow \mathcal{G}|_V \longrightarrow 0$$

(I and J are indexing sets that depends up on w) Then $\mathcal{G}$ is said to be quasi-coherent.

Consider a ringed space W. We call an $\mathcal{O}_W$ -algebra quasi-coherent if the corresponding O_W-module is quasi-coherent. Set

$$W_f := \{w \in W : f_w \text{ is invertible in } \mathcal{O}_{W,w}\}$$

Its easy to see that W_f is open in W. The image of f under the restriction homomorphism $\Gamma(W, \mathcal{O}_W) \longrightarrow \Gamma(W_f, \mathcal{O}_W)$ is invertible. That is given any $\mathcal{O}_W$-module $\mathcal{G}$ the restriction homomorphism $\Gamma(W, \mathcal{G}) \longrightarrow \Gamma(W_f, \mathcal{G})$ give rise to a homomorphism of $\Gamma(W, \mathcal{O}_W)$-modules

$$\Gamma(W, \mathcal{G})_f \longrightarrow \Gamma(W_f, \mathcal{G}) \quad (*)$$

If $W = Spec(B)$ then W_f is simply the principal open set $D(f)$.

Theorem 4.2.5. *Consider a scheme W and an $\mathcal{O}_W-$module $\mathcal{G}$. Then the assertions given below are equivalent.*

1. *The existance of a $B-$module N such that $\mathcal{G}|_V = \tilde{N}$ is guaranteed for any given affine subset $V = Spec(B)$ of W.*
2. *W admits an affine open covering $(V_i = Spec(B_i))_{i \in I}$ such that there exist B_i-module N_i and $\mathcal{G}|_{V_i} \cong \tilde{N}_i$ for all i.*
3. *$\mathcal{G}$ is quasi-coherent.*
4. *Given open subset of W of the form $V = Spec(B)$ and $f \in B$ the homomorphism $(*)$*

$$\Gamma(W, \mathcal{G})_f \longrightarrow \Gamma(D(f), \mathcal{G})$$

   is an isomorphism.

Corollary 4.2.6. *Consider a scheme $W = Spec(B)$ then we have an equivalence of the "category of $B-$modules" with the "category of $\mathcal{O}_W-$modules" via the functor $N \mapsto \tilde{N}$.*

Corollary 4.2.7. *Consider a scheme W.*

1. *Consider a morphism of quasi-coherent $\mathcal{O}_W-$modules $v : \mathcal{G} \longrightarrow \mathcal{F}$. Then $Ker(v)$, $Coker(v)$, and $Im(v)$ are quasi-coherent $\mathcal{O}_W$ -modules .*
2. *The direct sum of quasi-coherent $\mathcal{O}_W$-modules is again quasi-coherent .*
3. *Consider a quasi-coherent $\mathcal{O}_W-$module $\mathcal{G}$ and a family $(\mathcal{G}'_i)_{i \in I}$ of quasi-coherent submodules of $\mathcal{G}$. Then $\sum_i \mathcal{G}'_i$ and for finite I $\bigcap_i \mathcal{G}_i$ are quasi-coherent.*
4. *The tensor product $\mathcal{G} \otimes_{\mathcal{O}_W} \mathcal{F}$ is quasi-coherent, and for every open affine subset $V \subseteq W$ we have*

$$\Gamma(V, \mathcal{G} \bigotimes_{\mathcal{O}_W} \mathcal{F}) = \Gamma(V, \mathcal{G}) \bigotimes_{\Gamma(V, \mathcal{O}_W)} \Gamma(V, \mathcal{F}).$$

The corollary show that given a scheme W the "category of quasi-coherent $\mathcal{O}_X$-modules" is abelian.

4.2.3 Extending sections of quasi-coherent modules

Let $(W, \mathcal{O}_W)$ be a locally ringed space and let $\mathcal{T}$ be an invertible $\mathcal{O}_W$-module. For every $w \in W$ there exist an open neighborhood V of w and an isomorphism $x : \mathcal{T}|_V \cong \mathcal{O}_W|_V$. Let $t \in \Gamma(W, \mathcal{T})$, t is called invertible in w if $x_w(t_w)$ is a unit in $\mathcal{O}_{W,w}$ (that is if and only if $t(w) \neq 0 \in \mathcal{T}(w)$). We set

$$W_t(\mathcal{T}) := \{w \in W : t \text{ is invertible in } w\}$$

For $\mathcal{T} = \mathcal{O}_W$ we have $W_s(\mathcal{O}_W) = W_s$

Definition 4.2.8. *Consider a scheme W. Suppose for any affine open sets X, Y of W the intersection $X \cap Y$ is quasi-compact then we say that W is quasi seperated.*

In particular, if a scheme is locally noetherian then it is quasi seperated.

Theorem 4.2.9. *Consider a "quasi compact " and "quasi seperated" scheme W and a quasi-coherent $\mathcal{O}_W$−module $\mathcal{G}$. Let $t \in \Gamma(W, \mathcal{T})$ be a global section of an invertible $\mathcal{O}_W$−module $\mathcal{T}$.*

- *consider a global section $s \in \Gamma(W, \mathcal{G})$ such that $s|_{W_t} = 0$. Then existance of an integer $m > 0$ with $s \otimes t^{\otimes m} = 0 \in \Gamma(W, \mathcal{G} \otimes \mathcal{T}^{\otimes m})$ is assured.*
- *Given any $s' \in \Gamma(W, \mathcal{G})$ the existance of an integer $m > 0$ and and $s \in \Gamma(W, \mathcal{G} \otimes \mathcal{T}^{\otimes m})$ with $s|_{W_t} = s' \otimes t^{\otimes m}$ is assured.*

4.2.4 Direct and inverse image of quasi-coherent modules

Proposition 4.2.10. *Let $f : W = Spec(B) \longrightarrow Z = Spec(A)$ be a morphism of affine schemes and let $\psi : A \longrightarrow B$ be the corresponding ring homomorphism .*

- *Let N be a B-module and let $\psi_*(N)$ be the restriction of scalars to A, i.e., $\psi_*(N) = N$ considered as an A-module via ψ. Then there is a functorial isomorphism of $\mathcal{O}_Z$-modules*

$$f_*(\tilde{N}) \cong \psi_*\tilde{(}N)$$

- *Let N be an A-module. Then there is a functorial isomorphism of $\mathcal{O}_W$-modules*

$$f_*(\tilde{N}) \cong B \tilde{\bigotimes_A} N.$$

4.3 Properties of quasi-coherent modules

4.3.1 Modules of finite type and of finite presentation

Definition 4.3.1. *Let $(W, \mathcal{O}_W)$ be a ringed space. An $\mathcal{O}_W$-module $\mathcal{G}$ is called of finite type (resp. of finite presentation) if for all $w \in W$ there exists an open neighborhood V of w and an exact sequence of $\mathcal{O}_{W|_V}$-modules of the form*

$$\mathcal{O}^m_{W|_V} \longrightarrow \mathcal{G}|_V \longrightarrow 0$$

(resp. of the form $\mathcal{O}^n_{W|_V} \longrightarrow \mathcal{O}^m_{W|_V} \longrightarrow \mathcal{G}|_V \longrightarrow 0$) where $n, m \geq 0$ are integers (dependent on w) .

Proposition 4.3.2. *Let $W = Spec(B)$ be an affine scheme. An B-module N is of finite type (resp. of finite presentation) if and only if $\tilde{N}$ is an $\mathcal{O}_W$-module of finite type (resp. of finite presentation).*

Proposition 4.3.3. *Let $(W, \mathcal{O}_W)$ be a ringed space and let $\mathcal{G}$ be an $\mathcal{O}_W$-module of finite presentation .*

1. *For all $w \in W$ and for each $\mathcal{O}_W$-module $\mathcal{F}$, the canonical homomorphism of $\mathcal{O}_{W,w}$-modules*

$$Hom_{\mathcal{O}_W}(\mathcal{G},\mathcal{F})_w \longrightarrow Hom_{\mathcal{O}_{W,w}}(\mathcal{G}_w,\mathcal{F}_w)$$

is bijective.

2. *Let $\mathcal{G}$ and $\mathcal{F}$ be two $\mathcal{O}_W$-modules of finite presentation. Let $w \in W$ be a point and let $\theta : \mathcal{G}_w \cong \mathcal{F}_w$ be an isomorphism of $\mathcal{O}_{W,w}$-modules. Then there exists an open neighborhood V of w and an isomorphism $v : \mathcal{G}|_V \longrightarrow \mathcal{F}|_V$ of $\mathcal{O}_V$-modules such that $v_w = \theta$.*

Proposition 4.3.4. *Let $(W,\mathcal{O}_W)$ be a ringed space and $\mathcal{G}$ be an $\mathcal{O}_W$-module of finite type. Then $\mathcal{G}$ is of finite presentation if and only if for each open set $V \subseteq W$ and for each exact sequence of $\mathcal{O}_V$-modules*

$$0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}|_V \longrightarrow 0$$

where $\mathcal{F}$ is of finite type, $\mathcal{G}'$ is an $\mathcal{O}_V$-module of finite type .

4.3.2 Support of a module of finite type

Proposition 4.3.5. *Let $(W,\mathcal{O}_W)$ be a ringed space and let G be an $\mathcal{O}_W$-module of finite type. Let $w \in W$ be a point and let $t_i \in \Gamma(V,\mathcal{G})$ for $i = 1,\dots,m$ be sections over some open neighborhood of w such that the germs $(t_i)_w$ generate the stalk $\mathcal{G}_w$. Then there exists an open neighborhood $U \subseteq V$, such that the $t_i|_U$ generate $\mathcal{G}|_U$.*

Corollary 4.3.6. *Let $(W,\mathcal{O}_W)$ be a ringed space. For every $\mathcal{O}_W$-module $\mathcal{G}$ of finite type and any integer $r \geq 0$ the subset*

$$W_r := \{w \in W : \mathcal{G}_w \text{ can be generated by } r \text{ elements as } \mathcal{O}_{W,w} - module\}$$

is open in W.

If $(W,\mathcal{O}_W)$ is a locally ringed space the from Nakayama lemma it follows that

$$W_r = \{w \in W : dim_{\kappa(w)}\mathcal{F}(w) \leq r\}$$

Corollary 4.3.7. *Let $(W,\mathcal{O}_W)$ be a ringed space and let $\mathcal{G}$ be an $\mathcal{O}_W$-module of finite type. Then $Supp(\mathcal{G})$ is closed in W .*

Proposition 4.3.8. *Let W be a scheme, let $\mathcal{J}$ be an ideal of $\mathcal{O}_W$, and set*

$$Y := Supp(\mathcal{O}_W/\mathcal{J}) \quad \mathcal{O}_Y = i_Y^{-1}(\mathcal{O}_W/\mathcal{J})$$

Then Y is a closed subset of W, and $(Y,\mathcal{O}_Y)$ is a closed subscheme of W if and only if $\mathcal{J}$ is a quasi-coherent $\mathcal{O}_W$-module.

Corollary 4.3.9. *Let W be a scheme. Attaching to a quasi-coherent ideal $\mathcal{J}$ the closed subscheme ($Y := Supp(\mathcal{O}_W/\mathcal{J})$, $i_Y^{-1}(\mathcal{O}_W/\mathcal{J})$) defines a bijection between the set of quasi-coherent ideals of $\mathcal{O}_W$ and the set of closed subschemes of W. An inverse bijection is given by attaching to a closed subscheme $(Y,\mathcal{O}Y)$ the kernel of $\mathcal{O}_W \longrightarrow (i_Y)_*\mathcal{O}_Y$.*

Proposition 4.3.10. *Let W be a scheme and let $\mathcal{G}$ be a quasi-coherent $\mathcal{O}_W$-module of finite type . Then $Ann(\mathcal{G})$ is a quasi-coherent ideal of $\mathcal{O}_W$, for every open affine subset $V \subseteq W$ we have $\Gamma(V,Ann(\mathcal{G})) = Ann\Gamma(V,\mathcal{G})$, and the underlying topological space of $V(Ann(\mathcal{G}))$ is $Supp(\mathcal{G})$.*

4.3.3 Flat and finite locally free modules

Given a morphism of ringed spaces $f : W \longrightarrow Z$ and an $\mathcal{O}_W$–module $\mathcal{G}$, for each $w \in W$, we can give a $\mathcal{O}_Z,f(w)$-module structure to the $\mathcal{O}_{W,w}$-module $\mathcal{G}_w$ via the homomorphism $f_w^\sharp : \mathcal{O}_{Z,f(w)} \longrightarrow \mathcal{O}_{W,w}$.

Definition 4.3.11. *1. The O_W-module $\mathcal{G}$ is called flat over Z in w or f-flat in w if $\mathcal{G}_w$ is a flat $\mathcal{O}_{Z,f(w)}$-module. It is called flat over Z or f-flat if $\mathcal{G}$ is flat over Z in all points $w \in W$.*

2. *If $W = Z$ and $f = id_W$, we simply say that $\mathcal{G}$ is flat in w if it is id_W-flat in w, i.e. if $\mathcal{G}_w$ is a flat $\mathcal{O}_{W,w}$-module. Similarly, $\mathcal{G}$ is called flat, if $\mathcal{G}_w$ is a flat $\mathcal{O}_{W,w}$-module for all $w \in W$.*

3. *We say that f is flat, or that W is flat over Z , if $\mathcal{O}_W$ is flat over Z.*

Proposition 4.3.12. *Let $(W, \mathcal{O}_W)$ be a locally ringed space and let $\mathcal{G}$ be an $\mathcal{O}_W$-module. Then the following assertions are equivalent .*

1. *$\mathcal{G}$ is locally free of finite type .*

2. *$\mathcal{G}$ is of finite presentation and $\mathcal{G}_w$ is a free $\mathcal{O}_{W,w}$-module for all $w \in W$.*

3. *$\mathcal{G}$ is flat and of finite presentation .*

Corollary 4.3.13. *Let $W = Spec(B)$ be an affine scheme and let N be a B-module. Then the following assertions are equivalent .*

1. *$\tilde{N}$ is a locally free $\mathcal{O}_W$-module of finite type .*

2. *N is a finitely generated projective B-module.*

3. *N is a flat B-module of finite presentation.*

Lemma 4.3.14. *Let W be a scheme, let $Y \subseteq W$ be a finite set of points and let $V = Spec(B)$ be an open affine neighborhood of Y. Let E be a finite locally free $\mathcal{O}_W$-module of constant rank r. Then there exists an $s \in B$ such that $Y \subset D(s)$ and $E|_{D(s)} \cong \mathcal{O}^r_{D(s)}$*

4.3.4 Coherent modules

Definition 4.3.15. *Let $(W, \mathcal{O}_W)$ be a ringed space. An $\mathcal{O}_W$-module $\mathcal{G}$ is called coherent if $\mathcal{G}$ is of finite type and if for every open subset $V \subseteq W$, every integer $m \geq 0$, and for every homomorphism $x : \mathcal{O}^m_W|_V \longrightarrow \mathcal{G}|_V$ the kernel of x is of finite type .*

Proposition 4.3.16. *Let W be a locally noetherian scheme and let $\mathcal{G}$ be an $\mathcal{O}_W$-module. Then the following assertions are equivalent :*

1. *$\mathcal{G}$ is coherent.*

2. *$\mathcal{G}$ is of finite presentation.*

3. *$\mathcal{G}$ is of finite type and quasi-coherent .*

Corollary 4.3.17. *Let W be a locally noetherian scheme. Let $0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$ be an exact sequence of quasi-coherent $\mathcal{O}_W$-modules. Then $\mathcal{G}$ is coherent if and only if $\mathcal{G}'$ and $\mathcal{G}''$ are coherent .*

Chapter 5

Cohomology of Sheaves

Notation:

- *Ab* denotes the category of abelian groups
- $Mod(A)$ denotes the category of modules over a ring A.
- $Ab(X)$ denotes the category of sheaves of abelian groups on a topological space X.
- $Mod(X)$ denotes the category of sheaves of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$.
- $Qco(X)$ denotes the category of quasi-coherent sheaves of $\mathcal{O}_X$-modules on a scheme X.
- $Chs(X)$ the category of coherent sheaves of$\mathcal{O}_X$-modules on a noetherian scheme X.

5.1 Cohomology of Sheaves

In this section we define the cohomology of sheaves by taking the derived functor of the functor $(X, \mathcal{O}_X) \mapsto \Gamma(X, \mathcal{O}_X)$. As a first step we verify that all the categories we use has enough injectives

Proposition 5.1.1. *If A is a ring, then every A-module is isomorphic to a submodule of an injective A-module.*

Proposition 5.1.2. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $Mod(X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

Corollary 5.1.3. *If X is any topological space, then the category $Ab(X)$ of sheaves of abelian groups on X has enough injectives.*

Definition 5.1.4. *Let X be a topological space. Let $\Gamma(X, -)$ be the global section functor from $Ab(X)$ to Ab. We define the cohomology functors $H^i(X, -)$ to be the right derived functors of $\Gamma(X, -)$. For any sheaf $\mathcal{F}$, the groups $H^i(X, \mathcal{F})$ are the cohomology groups of $\mathcal{F}$.*

whenever we speak about the cohomology functor we only consider the underlying abelian sheaf structure of the given sheaf (in this chapter we only deal with sheaves which takes values in a cetgory whose objects has an abelian group structure)

Definition 5.1.5. *A sheaf $\mathcal{F}$ on a topological space X is flasque if for every inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is surjective.*

Lemma 5.1.6. *If $(X, \mathcal{O}_X)$ is a ringed space, any injective $\mathcal{O}_X$-module is fiasque.*

Proposition 5.1.7. *If $\mathcal{F}$ is a flasque sheaf on a topological space X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.*

The category of sheaves has enough injectives and finding cohomology groups $H^i(X, \mathcal{F})$ requires finding injective resolutions of sheaves. This is inconvenient but fortunately every sheaf has a flasque resolution and the cohomology can be computed using the flasque resolution.

Proposition 5.1.8. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X, -)$ from $Mod(X)$ to Ab coincide with the cohomology functors $H^i(X, -)$.*

Lemma 5.1.9. *On a noetherian topological space, a direct limit of flasque sheaves is flasque.*

Proposition 5.1.10. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)_\alpha$ be a direct system of abelian sheaves. Then there are natural isomorphisms, for each $i \geq 0$*

$$\varinjlim H^i(X, \mathcal{F}_\alpha) \longrightarrow H^i(X, \varinjlim \mathcal{F}_\alpha)$$

Lemma 5.1.11. *Let Y be a closed subset of X, let $\mathcal{F}$ be a sheaf of abelian groups on Y, and let $j : Y \longrightarrow X$ be the inclusion. Then $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$, where $j_*\mathcal{F}$ is the extension of $\mathcal{F}$ by zero outside Y*

Theorem 5.1.12. ***A Vanishing Theorem of Grothendieck:*** *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

outline of proof: The proof is based on induction on $n = dim(X)$
Step 1: Show that it is enough to prove the theorem for the case X is irreducible.
Step 2: Prove the theorem for X irreducible and $dim(X) = 0$
Step 3: Assume X is irreducible and execute the induction.

5.2 Cohomology of a Noetherian Affine Scheme

Lemma 5.2.1. *Let A be a noetherian ring, let $\mathfrak{a}$ be an ideal of A, and let I be an injective A-module. Then the submodule $J = \Gamma_{\mathfrak{a}}(I)$ is also an injective A-module.*

Lemma 5.2.2. *Let I be an injective module over a noetherian ring A. Then for any $f \in A$, the natural map of I to its localization I_f is surjective.*

Proposition 5.2.3. *Let I be an injective module over a noetherian ring A. Then the sheaf $\tilde{I}$ on $X = Spec(A)$ is fiasque.*

Theorem 5.2.4. *Let $X = Spec(A)$ be the spectrum of a noetherian ring A. Then for all quasi-coherent sheaves $\mathcal{F}$ on X, and for all $i > 0$, we have $H^i(X, \mathcal{F}) = 0$.*

Corollary 5.2.5. *Let X be a noetherian scheme, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then $\mathcal{F}$ can be embedded in a flasque, quasi-coherent sheaf $\mathcal{G}$.*

Theorem 5.2.6. *Let X be a noetherian scheme. Then the following conditions are equivalent:*

1. *X is affine.*
2. *$H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasi-coherent and all $i > 0$.*
3. *$H^1(X, \mathcal{J}) = 0$ for all coherent sheaves of ideals $\mathcal{J}$.*

5.3 Čech Cohomology

Let $\mathcal{F}$ be a sheaf of abelian groups on X and let $X = \bigcup_{i \in I} U_i$ be an open cover of X we denote this open cover by $\mathcal{U}$. Fix once and for all, a well ordering of the indexing set I. For any finite $J \subseteq I$ we set $U_J : \bigcap_{j \in J} U_j$. that is if $J = \{i_0, \ldots, i_p\}$ we define $U_{J=\{i_0,\ldots,i_p\}} = U_{i_0} \cap \cdots \cap U_{i_p}$.
Define a complex $C(\mathcal{U}, \mathcal{F})$ by setting

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} F(U_{i_0,\ldots,i_p})$$

an element $\alpha \in C^p(\mathcal{U}, \mathcal{F})$ is determined by giving $\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p})$ for each $p+1$ tuple $i_0 < \ldots i_p$ of elements of I. Hence we can define the cobundary map $d^p : C^p(\mathcal{U}, \mathcal{F}) \longrightarrow C^{p+1}(\mathcal{U}, \mathcal{F})$ by defining $(d^p(\alpha))_{i_0,\ldots,i_{p+1}}$ for all $p+2$ tuples $i_0 < \cdots < i_{p+1}$ of I. Define

$$(d^p(\alpha))_{i_0,\ldots,i_p} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0,\ldots,\hat{i_j},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_p}}$$

where $\hat{i_j}$ means we are omitting i_j

Definition 5.3.1. *Let X be a topological space and let $\mathcal{U}$ be an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ on X, we define the p^{th} Čech cohomology group of $\mathcal{F}$, with respect to the covering $\mathcal{U}$, to be*

$$H^p(\hat{\mathcal{U}}, \mathcal{F}) = h^p(C(\mathcal{U}, \mathcal{F}))$$

Note: here $h^p(C(\mathcal{U}, \mathcal{F}))$ is the p^{th} cohomology object of the chain $C(\mathcal{U}, \mathcal{F})$ in the sense of chapter 8.

Lemma 5.3.2. *Let X be a topological space and let $\mathcal{U}$ be an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ on X*

$$\hat{H}^0(\mathcal{U}, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$$

Let $V \subseteq X$ be an open set and $f : V \longrightarrow X$ denotes the inclusion map. Given $(X, \mathcal{U}, \mathcal{F})$ (where the symbols have their usual meaning) we construct a new complex $\mathcal{C}(\mathcal{U}, \mathcal{F})$ by setting

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{i_0 < \cdots < i_p} f_*(F|_{U_{i_0,\ldots,i_p}})$$

and we define the coboundary map in the obvious way.

Theorem 5.3.3. *For any sheaf of abelian groups on X, the complex $\mathcal{C}(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, i.e., there is a natural map $\epsilon : \mathcal{F} \longrightarrow \mathcal{C}^0$ such that the sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{C}^0 \longrightarrow \mathcal{C}^1 \longrightarrow \ldots$$

is exact.

Proposition 5.3.4. *Let X be a topological space, let U be an open covering, and let $\mathcal{F}$ be a fiasque sheaf of abelian groups on X. Then for all $p > 0$ we have $\hat{H}^P(\mathcal{U}, \mathcal{F}) = 0$.*

Proposition 5.3.5. *Let X be a topological space, and U an open covering. Then for each $p \geq 0$ there is a natural map, functorial in $\mathcal{F}$,*

$$\hat{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

Theorem 5.3.6. *Let X be a noetherian separated scheme, let $\mathcal{U}$ be an open affine cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, the natural maps of proposition 10.2.11 give isomorphisms*

$$\hat{H}^p(\mathcal{U}, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F})$$

5.4 Ext Group and Sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{F}$ and $\mathcal{F}'$ are $\mathcal{O}_X$–modules we denote by $Hom(\mathcal{F}, \mathcal{F}')$ the group of $\mathcal{O}_X$-module homomorphisms and by $Hom_{SH}(\mathcal{F}, \mathcal{F}')$ the morphisms of sheaves.

Definition 5.4.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define the functors $Ext^i(\mathcal{F}, -)$ as the right derived functors of $Hom(\mathcal{F}, -)$, and $\mathcal{E}xt^i(\mathcal{F}, -)$ as the right derived functors of $Hom_{SH}(\mathcal{F}, -)$.*

Lemma 5.4.2. *If $\mathcal{J}$ is an injective object of $Mod(X)$, then for any open subset $U \subseteq X$, $\mathcal{J}|_U$ is an injective object of $Mod(U)$.*

Proposition 5.4.3. *For any open subset $U \subseteq X$ and $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ we have*

$$\mathcal{E}xt^i_X(\mathcal{F}, \mathcal{G})|_U \cong \mathcal{E}xt^i(\mathcal{F}|_U, \mathcal{G}|_U)$$

Proposition 5.4.4. *For any $\mathcal{G}Mod(X)$, we have:*

- $\mathcal{E}xt^0(\mathcal{O}_X, \mathcal{G}) = \mathcal{G}$.
- $\mathcal{E}xt^i(\mathcal{O}_X, \mathcal{G}) = 0 \quad \forall i > 0$.
- $Ext^i(\mathcal{O}_X, \mathcal{G}) = H^i(\mathcal{O}_X, \mathcal{G}), \quad \forall i \geq 0$.

Proposition 5.4.5. *If $0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$ is a short exact sequence in $Mod(X)$, then for any $\mathcal{G}$ we have a long exact sequence*

$$0 \longrightarrow Hom(\mathcal{F}'', \mathcal{G}) \longrightarrow Hom(\mathcal{F}, \mathcal{G}) \longrightarrow Hom(\mathcal{F}', \mathcal{G})$$
$$\longrightarrow Ext^1(\mathcal{F}'', \mathcal{G}) \longrightarrow Ext^1(\mathcal{F}, \mathcal{G}) \longrightarrow \dots$$

and similarly for the $\mathcal{E}xt$ sheaves.

Proposition 5.4.6. *Let X be a noetherian scheme, let $\mathcal{F}$ be a coherent sheaf on X, let $\mathcal{G}$ be any $\mathcal{O}_X$-module, and let $x \in X$ be a point. Then we have*

$$\mathcal{E}xt(\mathcal{F}, \mathcal{G})_x \cong Ext^i_{\mathcal{O}_X}(\mathcal{F}_x, \mathcal{G}_x)$$

for any $i \geq 0$, where the right-hand side is Ext over the local ring $\mathcal{O}_{X,x}$.

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