Connes spectral distance on Noncommutative spaces

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Connes spectral distance on Non-commutative spaces" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by " Alpesh Avinash Patil" at "S N Bose National Center for Basic Sciences, Kolkata", under the supervision of "Prof. Biswajit Chakraborty" during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled "Connes' spectral distance on Noncommutative spaces " are the results of the investigations carried out by me at the Department of Theoritical Sciences, S N Bose National Centre for Basic Sciences, Kolkata, under the supervision of Prof. Biswajit Chakraborty and the same has not been submitted elsewhere for any other degree.

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Abstract

In [4], Hilbert-Schmidt operator formulation of Noncommutative Quantum Mechanics was put forward and in [3] exact formulation and interpretation of the formalism was given. Since due to noncommutativity of the position coordinates the notion of a space in a geometric sense(points,lines) is lost and consequently the distance cannot be defined. The framework of Noncommutative Geometry essentially deals with these type of spaces, where spectral triples are defined which encodes the topological and geometrical information of the space in algebraic terms. In particular we are interested in the Connes distance function defined on these spectral triples to give distance between states of the algebra - pure states of the algebra have one-to -one correspondence with the points of the noncommutative space. In [15], a algorithm was developed to compute Connes distance function for noncommutative spaces. We found that the algorithm works only for computing infinitesimal distances and therefore, modified the algorithm such that finite distances can also be calculated. But the modified algorithm becomes highly nontrivial for calculating the Connes distance and path forward is not clear till now. Therefore, we use a alternative approach developed in [11]. Distances between discrete orthogonal basis states and coherent states are calculated for two different types of noncommutative spaces: Moyal plane and Fuzzy sphere. Coherent states(minimal uncertainty states) are significant, through which a POVM(Positive-Operator Valued Measure) is defined for weak position measurement. It is shown that the metric on the set of coherent states of Moyal plane is flat, as expected due to infinitesimal distance calculation in [15]. For fuzzy sphere even though the infinitesimal distance between coherent states is the geodesic distance on Sphere S^2 (as calculated in [16]) up to a overall numerical constant, we show the finite distance between coherent states is not equal to corresponding geodesic distance on sphere S^2. We calculate the Connes distance between coherent states exactly only for the n=1/2 case(i.e for the n=1/2 representation of the su(2) lie algebra representing the fuzzy sphere).

Contents

Chapter 1

Introduction

The noncommutative nature of space-time has been widely established in the literature. In [5], Doplicher et al. argued that space-time loses any operational meaning below Planck's length $\lambda_p = \left(\frac{G\hbar}{c^3}\right)^{1/2} \approx 1.6\text{x}10^{-33}$ cm, when implications from quantum theory and Einstein's theory of gravity are considered together. Since from Heisenberg uncertainty principle in quantum theory, localization of a space-time event with a greater accuracy implies increase in the uncertainty of the energy in that region at some time due to the measurement. While according to the classical theory of gravitation, concentration of large amount of energy in small region (below Planck's volume) will lead to formation of black holes. Therefore, limitations on the localization of space-time event should be considered in any quantum theory incorporating gravity. A natural way to achieve this, is by introducing commutation relations on the space-time co-ordinates thereby making the space-time noncommutative.

$$[q_\mu, q_\nu] = iQ_{\mu\nu} \tag{1.1}$$

where $Q_{\mu\nu}$ is a antisymmetric tensor. In [5], above commutations relations were put forward and a quantum field theory was constructed on this noncommutative space-time. The idea that space-time can be noncommutative was also considered in early days of quantum field theories, in order to get rid of the divergences occurring in the field theory. In [1], Snyder showed that a natural unit of length can be introduced in a Lorentz invariant way, thereby removing the divergences in the field theory partially. It was also shown that to introduce a unit of length it is necessary to drop the commutativity of coordinates.

In [17], Seiberg and Witten showed that effective quantum field theories on noncommutative space-time called Noncommutative field theories represents some lower energy limits of string theory. Thus, noncommutative field theories, where $Q_{\mu\nu}$ in (1.1) is taken to be constant are widely studied. The noncommutativity in this field theories is usually incorporated by deforming the algebra of functions, by introducing a new product rule which make the algebra noncommutative. Noncommutativity also arises in condense matter physics. A simple example is the Landau problem - motion of electron in a 2 dimensional plane subject to a perpendicular magnetic field. When the system is projected in the lowest Landau level the two dimensional plane becomes noncommutative. Despite this advances the physical implications of noncommutative space-time are not well understood and we refer to [2] for the review of the noncommutative field theories. In order to better understand the consequences of noncommutative space-time and to provide a theoretical prediction for the noncommutative parameter in (1.1), generalization of quantum mechanics to noncommutative space-time are also studied. Noncommutative quantum mechanical models for harmonic oscillator [8],Coulomb problem [9], spherical well potential [4] have been investigated. The two types of noncommutative spaces considered in this investigations are the following:

$$\text{Moyal Plane:} \quad [x_i, x_j] = i\theta_{ij} \tag{1.2}$$

$$\text{Fuzzy Sphere:} \quad [x_i, x_j] = i\theta\epsilon_{ijk}x_k \tag{1.3}$$

where θ_{ij} is a constant anti-symmetric matrix and ϵ_{ijk} is the anti-symmetric tensor. In [3], a general formulation and interpretational framework of Noncommutative quantum mechanics in terms of Hilbert Schmidt operators was put forward. In this framework weak position measurements in terms of coherent states were given thereby providing a meaning to position measurements. The subject of our study is the geometric structure of the above mentioned two noncommutative spaces.

The best example of a noncommutative space in the context of physics is the phase space in quantum mechanics. The position and momenta coordinates are replaced by noncommutating operators $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$. John von Neumann studied the mathematical structure of such quantum phase space which he called "pointless geometry" - as due to Heisenberg uncertainty relations the notion of point is lost. His investigation in this direction led to the theory of Von Neumann algebras. This work was carried forward by Gelfand, Naimark and Segal by defining C^* algebras and establishing a link between commutative C^* algebras and algebra of continuous functions on a space(discussed in chap(3). In 1980's, Alain Connes [6] generalized these concepts to the setting of noncommutative C^* algebras by providing a differential structure on them thereby establishing the field of Noncommutative Geometry. In noncommutative geometry, the focus is shifted from the space itself to the algebra of continuous functions on them and the pure states on the algebra represents the points of the underlying noncommutative space. The framework of noncommutative geometry has many applications in quantum physics [24] such as standard model in elementary particle physics, renormalization in quantum field theory, quantum hall effect in solid state physics and many more since its main inspiration was from quantum mechanics itself. Also, Connes along with his collaborators [7] has developed a model to describe standard model of particle physics weakly coupled to gravity in the framework of noncommutative geometry. The Standard model is built on a manifold called "Almost Commutative" manifold M x F, where M is the space-time manifold $\mathbb{M}^4$ and F is a finite space representing the gauge content of the theory. Therefore, the framework of noncommutative geometry provides the mathematical setup to deal with the noncommutative space-times discussed above.

The geometric structure of the noncommutative spaces (1.2)(1.3) has been studied recently [10] - [16] by calculating the Connes spectral distance between pure states on the algebra of functions on this space. As mentioned earlier, pure states corresponds to the points in the underlying noncommutative space. In [15], a general algorithm was developed to calculate Connes distance between states of a noncommutative space in the setup of Hilbert-Schmidt operator formulation of noncommutative quantum mechanics [3]. Subsequently, the Connes distance between infinitesimally separated coherent states and discrete states was calculated for Moyal plane [15] and for Fuzzy sphere [16]. We started by investigating the above mentioned algorithm in order to calculate finite distances. To calculate Connes distance between finitely separated states we had to modify the algorithm, as it was found out that the algorithm worked well only for calculating infinitesimal distances. But in the modified algorithm, calculation of a particular factor becomes very difficult, as discussed in sec(4.3). We therefore, take a alternative approach to calculate the Connes distance between finitely separated coherent and discrete states in the case of Moyal plane and Fuzzy sphere. This alternative approach was adopted from [11], where Connes distance between coherent states of Moyal plane was calculated for the spectral triple (3.12).

The thesis is organized as follows. In chapter 2 , we review the Hilbert-Schmidt formulation of noncommutative quantum mechanics. We also discuss the Positive Operator Valued Measure (POVM) position measurement as constructed in the formulation using coherent states. We then go on to review the basics objects - classical Hilbert space ($\mathcal{H}_c$), quantum Hilbert space ($\mathcal{H}_q$), and coherent states on Moyal plane and Fuzzy sphere, which are required for our analysis. In chapter 3 we motivate the construction of spectral triples in noncommutative geometry, which are generalization of Riemannian spin manifolds. Subsequently we define spectral triple and Connes distance function on states of the algebra of the spectral triple. Thus, setting the general stage, we review the spectral triple constructed on Moyal plane and fuzzy sphere using which we calculated finite Connes distance between coherent and discrete states. In chapter 4, we present our analysis to calculate the Connes distance. First, we find the Connes distance on Moyal plane and Fuzzy sphere by an alternative approach as mentioned previously. Then in the last section we propose a general method to find the Connes distance by modifying the algorithm in [15] and discuss the issues with the previous algorithm. In chapter 5, we conclude the thesis.

Chapter 2

Noncommutative Quantum Mechanics

In Noncommutative(NC) Quantum Mechanics, we consider a non-commutative configuration space of the following type:

$$[X_i, X_j] = i\theta_{ij} \tag{2.1}$$

A framework of NC Quantum Mechanics deriving from the analogy with the phase space of commutative quantum mechanics was established in [3] [4], where a 2-D configuration space was considered. In standard quantum mechanics, physical states are represented by rays in a Hilbert space and observables as self-adjoint operators acting on this Hilbert space. The time evolution is then given by unitary transformations on the set of states. As known, this Hilbert space is the space of square integrable functions $f(\vec{x}), \vec{x} \in \mathbb{R}^d$ on the configuration space $\mathbb{R}^d$. But, in NC quantum mechanics, since the space is quantized due to the noncommutative relation (2.1) the configuration space loses the geometric structure (as the notion of a point cannot be defined). Therefore, we represent the configuration space by a Hilbert space called classical Hilbert space ($\mathcal{H}_c$) on which the representation of the noncommutative algebra (2.1) is constructed. Thus, the vectors in classical Hilbert space represents the configuration space of the physical system . We proceed further by defining the states of a physical system in NC quantum mechanics.

A state of a physical system in standard quantum mechanics is given by a square integrable function i.e $\psi(\vec{x})$ s.t $\int |\psi(\vec{x})|^2 < \infty$, similarly we define a state in NC Quantum mechanics to be a Hilbert-Schmidt operator acting on classical Hilbert space ($\mathcal{H}_c$) i.e $\psi : \mathcal{H}_c \to \mathcal{H}_c$ s.t $tr_c(\psi^\dagger \psi) < \infty$. The space of Hilbert-Schmidt operators is a Hilbert space [20], which we call quantum Hilbert space ($\mathcal{H}_q$) - whose elements(rays) are states of the physical system in

NC quantum mechanics. On quantum Hilbert space $\mathcal{H}_q$, we built a unitary representation of abstract non-commutative Heisenberg algebra,

$$[x_i, p_j] = i\hbar\delta_{ij} \tag{2.2}$$

$$[x_i, x_j] = i\theta_{ij} \tag{2.3}$$

$$[p_i, p_j] = 0 \tag{2.4}$$

in terms of position operator $\hat{X}_i$ and momentum operator $\hat{P}_i$ acting on $\mathcal{H}_q$. It is shown in [3] that the standard interpretation of quantum mechanics stills holds, albeit with a weak position measurement in the sense that instead of a projective measurement we have only a POVM(Positive Operator Valued Measure) for a position measurement. Since the position coordinates $\hat{x_i}, \hat{x_j}$ do not commute with each other, according to the Heisenberg uncertainty principle we cannot have a precise measurement of them simultaneously. Therefore, even though the notion of point is lost due to non-commutativity, in order to preserve the notion of a particle being localized at a certain point the position measurement are defined in terms of coherent states(minimum uncertainty states).

We considered here two different types of Noncommutative space

1. Moyal Plane: $[x_i, x_j] = i\theta_{ij}$: θ_{ij} is a constant anti-symmetric matrix
2. Fuzzy Sphere:$[x_i, x_j] = i\theta\epsilon_{ijk}x_k$: ϵ_{ijk} is the anti-symmetric tensor

In the following section we briefly construct the classical Hilbert space $\mathcal{H}_c$, quantum Hilbert space $\mathcal{H}_q$ and coherent states on both the noncommutative spaces.

2.1 Moyal Plane

The following construction was put forward in [3]. We restrict our analysis to two dimensional Moyal plane where θ_{ij} becomes a scalar.

$$[x_i, x_j] = i\theta\epsilon_{ij} \quad : i, j = 1, 2, \quad \epsilon_{12} = -\epsilon_{21} = 1 \tag{2.5}$$

The algebra(2.5) is actually same as the algebra on phase space $[x, p] = i\hbar$ of a 1-D harmonic oscillator and therefore $\mathcal{H}_c$ is the usual boson fock space:

$$\mathcal{H}_c = span\left\{ |n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n |0\rangle \right\} \tag{2.6}$$

where $b = \frac{\hat{x}_1 + i\hat{x}_2}{\sqrt{2\theta}}$ and $|n\rangle$ are eigenvectors of the radial operator $r = b^\dagger b$.

The quantum Hilbert space $\mathcal{H}_q$ as mentioned is the set of Hilbert-Schmidt operators acting on $\mathcal{H}_c$:

$$\mathcal{H}_q = \left\{ \psi \in \mathcal{B}(\mathcal{H}_c) : tr_c(\psi^\dagger \psi) < \infty \right\} = span \left\{ |m\rangle\langle n| \right\} \tag{2.7}$$

where subscript 'c' implies the trace is over $\mathcal{H}_c$ and $\mathcal{B}(\mathcal{H}_c)$ is the space of bounded operators. The inner product on $\mathcal{H}_q$ is defined as

$$(\psi|\phi) = tr_c(\psi^\dagger \phi) \tag{2.8}$$

On this general setup we now built a unitary representation of non-commutative Heisenberg algebra(2.2 − 2.4) analogous to the SchrÃűdinger representation, through the action,

$$X_i \psi(\hat{x}_1, \hat{x}_2) = \hat{x}_i \psi(\hat{x}_1, \hat{x}_2) \tag{2.9}$$

$$P_i \psi(\hat{x}_1, \hat{x}_2) = \frac{\hbar}{\theta} \epsilon_{ij} [\hat{x}_j, \psi(\hat{x}_1, \hat{x}_2)] \tag{2.10}$$

Notations: We denote the elements of $\mathcal{H}_c$ by $|.\rangle$ and $\mathcal{H}_q$ by $|.)$. Capital letters are reserved for operators acting on $\mathcal{H}_q$ and we use small letters with hat notation to denote operators acting on $\mathcal{H}_c$. In order to distinguish the hermitian conjugation on $\mathcal{H}_c$ and $\mathcal{H}_q$ corresponding to there respective inner products , '†' is used for $\mathcal{H}_c$ and '‡' for $\mathcal{H}_q$

We now introduce the following useful operators on $\mathcal{H}_q$:

$$B = \frac{X_1 + iX_2}{\sqrt{2\theta}} \quad \Rightarrow \quad B|\psi) = |b\psi) \tag{2.11}$$

$$B^\ddagger = \frac{X_1 - iX_2}{\sqrt{2\theta}} \quad \Rightarrow \quad B^\ddagger|\psi) = |b^\dagger\psi) \tag{2.12}$$

$$P = P_1 + iP_2 \quad \Rightarrow \quad P|\psi) = -i\frac{\hbar}{\theta}|[b, \psi]) \tag{2.13}$$

$$P^\ddagger = P_1 - iP_2 \quad \Rightarrow \quad P^\ddagger|\psi) = i\frac{\hbar}{\theta}|\left[b^\ddagger, \psi\right]) \tag{2.14}$$

Now the physical states of a system are represented by normalized vectors in $\mathcal{H}_q$. The above framework is interpreted in the same way as the standard quantum mechanics. But due to the non-commutativity (2.5), the precise measurement of position of a particle is lost. This is restored in a weak sense i.e a particle localized at a particular point by using minimal uncertainty coherent states as follows:

A coherent state of a harmonic oscillator is a eigenvector of the annihilation operator b s.t $\Delta x \Delta p = \frac{\hbar}{2}$. Similarly, $|z\rangle \in \mathcal{H}_c$ s.t $b\,|z\rangle = z\,|z\rangle$ is a coherent state in $\mathcal{H}_c$ $\quad (z = \frac{x_1 + ix_2}{\sqrt{2\theta}} \in \mathbb{C})$

$$|z\rangle = \exp\big(-\bar{z}b + zb^\dagger\big)\,|0\rangle = \exp\Big(-\frac{z\bar{z}}{2}\Big)\exp\big(zb^\dagger\big)\,|0\rangle \tag{2.15}$$

and

$$\frac{1}{\pi}\int d^2z\,|z\rangle\langle z| = \mathbb{1}_c \tag{2.16}$$

From this we define states $|z) = |z\rangle\langle z| \in \mathcal{H}_q$ which are eigenvectors of B: $B|z) = z|z)$. Therefore, $z : (x_1, x_2)$ can be interpreted as the position coordinates of a particle. This states are non-orthogonal and give a resolution of identity in $\mathcal{H}_q$

$$(z_1|z_2) = e^{-|z_1-z_2|} \tag{2.17}$$

$$\mathbb{1}_q = \int \frac{\mathrm{d}z\,\mathrm{d}\bar{z}}{\pi}|z)e^{\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}}(z| \tag{2.18}$$

Even though the set of coherent states form a basis for $\mathcal{H}_c$, it is a non-orthogonal and over complete basis. Hence, we cannot define projective measurement as in standard quantum mechanics and the set of complete,non-orthogonal, positive operators π_z provides a POVM for position measurement.

$$\pi_z = \frac{1}{2\pi\theta}|z)e^{\overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z}}(z| \tag{2.19}$$

This implies

$$\mathbf{1}_q = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\pi_z \tag{2.20}$$

$$(\psi|\pi_z|\psi) \;\geq\; 0 \;\;\forall\;|\psi) \in \mathcal{H}_q : \quad \pi_z\pi_w \neq \delta(z-w) : \quad \pi_z^2 \propto \pi_z \tag{2.21}$$

such that the probability of finding a particle in a state represented by the density matrix ρ at $z : (x_1, x_2)$ is given by

$$p(x_1, x_2) = tr_q(\pi_z\rho) \tag{2.22}$$

and if $\rho = |\psi)(\psi|$ is a pure state then

$$p(x_1, x_2) = tr_q(\pi_z\rho) = (\psi|\pi_z|\psi) \tag{2.23}$$

2.2 Fuzzy Sphere

In this section we review the construction of classical and quantum Hilbert space as carried out in [16]. We also discuss the generalized coherent states called Perelomov coherent sates, which will be essential in order to define position measurement via POVM as shown in the Moyal plane case.

The non-commutative algebra of the fuzzy sphere is the su(2) lie-algebra with the parameter λ

$$[x_i, x_j] = i\lambda\epsilon_{ijk}x_k : \epsilon \text{ is the antisymmetric tensor with } \epsilon_{123} = 1 \tag{2.24}$$

The classical Hilbert Space $\mathcal{H}_c$ can be represented as span of eigenvectors of two uncoupled harmonic oscillators as follows:

Let $b_1, b_1^\dagger$ and $b_2, b_2^\dagger$ are the annihilation-creation operators of the two harmonic oscillator respectively, satisfying

$$\left[b_i, b_j^\dagger\right] = \frac{\lambda}{2}\delta_{ij} \quad : [b_1, b_2] = 0 : \quad \left[b_1^\dagger, b_2^\dagger\right] = 0 \tag{2.25}$$

Then the eigenvectors of this system are the eigenvectors of number operators $N_1 = b_1^\dagger b_1$ and $N_2 = b_2^\dagger b_2$

$$|n_1 n_2\rangle = \frac{1}{\sqrt{n_1!}\sqrt{n_2!}}(b_1^\dagger)^{n_1}(b_2^\dagger)^{n_2}|0\rangle \tag{2.26}$$

$N_1 |n_1 n_2\rangle = n_1 |n_1 n_2\rangle \quad N_2 |n_1 n_2\rangle = n_2 |n_1 n_2\rangle$

Now we get the $su(2)$ lie algebra (2.24) by the Jordan-Schwinger map:

$$\hat{x}_i = b_\alpha^\dagger \sigma_i^{\alpha\beta} b_\beta \tag{2.27}$$

and the eigenvectors $|n_1 n_2\rangle$ become eigenvectors of radial operator $\hat{\vec{x}}^2$ and $\hat{x}_3$:

$$\hat{\vec{x}}^2 |n, n_3\rangle = \lambda^2 n(n+1)|n, n_3\rangle \quad \hat{x}_3 |n, n_3\rangle = \lambda n_3 |n, n_3\rangle \tag{2.28}$$

where $n = \frac{n_1+n_2}{2}$ and $n_3 = \frac{n_1-n_2}{2}$, $n, n_3 \in \frac{\mathbb{Z}}{2} \quad : -n \leq n_3 \leq n$. The annihilation-creation operators $\hat{x}_\pm = \hat{x}_1 \pm \hat{x}_2$ which satisfy

$$[\hat{x}_3, \hat{x}_\pm] = \pm\lambda\hat{x}_\pm \quad [\hat{x}_+, \hat{x}_-] = 2\lambda\hat{x}_3 \tag{2.29}$$

$$\hat{x}_\pm |n, n_3\rangle = \lambda\sqrt{n(n+1) - n_3(n_3 \pm 1)}\,|n, n_3 \pm 1\rangle \tag{2.30}$$

Therefore, it can be easily seen that the Hilbert spaces $\mathcal{H}_c$ and $\mathcal{H}_q$ are

$$\mathcal{H}_c = span\left\{|n, n_3\rangle\right\} \tag{2.31}$$

$$\mathcal{H}_q = span\left\{\Big|n, n_3\Big\rangle\Big\langle n', n_3'\Big|\right\} \tag{2.32}$$

Since the radial operator $\hat{\vec{x}}^2$ is Casimir operator, $\mathcal{H}_c$ and $\mathcal{H}_q$ can be divided into subspaces characterized by n.

$$\mathcal{H}_c = \oplus\mathcal{H}_c^n \;:\; \mathcal{H}_c^n = span\left\{|n, n_3\rangle : -n \leq n_3 \leq n\right\} \tag{2.33}$$

$$\mathcal{H}_q = \oplus\mathcal{H}_q^n \quad:\quad \mathcal{H}_q^n = span\left\{\Big|n, n_3\Big\rangle\Big\langle n, n_3'\Big| = |n_3, n_3') : -n \leq n_3 \leq n\right\} \tag{2.34}$$

The fuzzy $\mathbb{R}^3_*$ space can be visualized as made of fuzzy spheres of different radii characterized by n and the classical Hilbert space $\mathcal{H}^n_c$ corresponding to the fuzzy sphere characterized by n, is the irreducible n^{th} representation of su(2) lie-algebra. The radius of this fuzzy sphere as seen from (2.28) is given by

$$r_n = \lambda\sqrt{n(n+1)}$$

Now generalized coherent states can be constructed for the fuzzy sphere case known as Perelomov coherent states. We briefly review here the construction of Perelomov coherent state as discussed in [[21]]. Let G be a general lie group with a unitary irreducible representation T(g) on some Hilbert space $\mathcal{H}$. Let $|x_0\rangle$ be a vector in $\mathcal{H}$ and $O(x_0)$ be the orbit of $|x_0\rangle$ w.r.t action of G on $\mathcal{H}$ i.e $\{T(g)\,|x_o\rangle : \forall\ g \in G\}$. We define a equivalence relation as two vectors are equivalent if they differ from each other up to a constant phase. Then, the generalized coherent states are defined as elements $\in [O(x_0)]$,where $[O(x_0)]$ is the set of equivalence classes of $O(x_0)$. Thus, this implies a generalized coherent state is $|g\rangle = T(g)\,|x_0\rangle \in [O(x_0)]$ where $g \in G/H$, H is the stability group of $|x_0\rangle$ i.e for $h \in H \ \ T(h)\,|x_0\rangle = e^{i\alpha}\,|x_0\rangle$, α is constant. If we choose $|x_0\rangle$ to be such that its isotropy subalgebra(as defined below) is maximal, we get coherent states with minimal uncertainty.

Let $\mathcal{G}$ be the lie algebra of the G and T_g be its representation. Let $\mathcal{G}^c$ be the complexification of $\mathcal{G}$ i.e all linear combinations of elements of $\mathcal{G}$ with complex coefficients. A subalgebra $\mathcal{B}$ of $\mathcal{G}_c$ is called isotropy subalgebra if for $b \in \mathcal{B}$ implies $T_b\,|x_0\rangle = \sigma_b\,|x_0\rangle \ : \sigma_b \in \mathbb{C}$. The subalgebra $\mathcal{B}$ is called maximal if $\mathcal{B} \oplus \bar{\mathcal{B}} = \mathcal{G}_c$ where sub-algebra $\bar{\mathcal{B}}$ is conjugate of $\mathcal{B}$. If we choose $|x_0\rangle$ to be such that its isotropy subalgebra is maximal, we get coherent states with minimal uncertainty.

For Fuzzy sphere we have the Lie group $G = SU(2)$ corresponding to the non-commutative algebra (2.24). The stability group $U(1)$ is generated by $J_3 = \frac{\hat{x_3}}{\lambda}$. This implies the coherent states corresponds to point in $S^2 = SU(2)/U(1)$. There exist two vectors $|n, \pm n\rangle \in \mathcal{H}^n_c$ for which the isotropy algebra is maximal. We consider the orbit of $|n, n\rangle \in \mathcal{H}^n_c$ to get coherent states with minimal uncertainty $\Delta\hat{\vec{x}}^2 = (\Delta\hat{\vec{x}}^2)_{min} = n\lambda^2$. We know $g \in X = SU(2)/U(1)$ can be written as

$$g = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \alpha \end{bmatrix} : \beta = \beta_1 + i\beta_2 \text{ and } \alpha^2 + |\beta|^2 = 1 \tag{2.35}$$

Now we parametrize $\alpha = \cos\frac{\theta}{2}$ and $\beta = -\sin\frac{\theta}{2}e^{-i\phi}$, where $0 \le \theta < \pi$ and $0 \le \phi < 2\pi$. Thus any element $g \in X$ can be represented as a point in

S^2, $p = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ such that

$$g_p = \exp\left(i\frac{\theta}{2}(m_1\sigma_1 + m_2\sigma_2)\right) \tag{2.36}$$

where $m_1 = \sin\phi, m_2 = -\cos\phi$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are Pauli matrices. Thus, in general for the n^{th} representation of $SU(2)$ group, $g_p \in X$ can be written as

$$g_p = \exp(i\theta(m_1 J_1 + m_2 J_2)) \quad : J_i = \hat{x}_i/\lambda \tag{2.37}$$

Therefore, the Perelomov coherent states $|z\rangle \in \mathcal{H}_c^n$ are elements of the set generated by the action of group $X = SU(2)/U(1)$ on the state $|n, n\rangle$ given by

$$|z\rangle = exp\left(-\tan^{-1}|z|(e^{-i\phi}J_+ - e^{i\phi}J_-)\right)|n, n\rangle \tag{2.38}$$

where $z \in \mathbb{C}, z = -\tan\left(\frac{\theta}{2}\right)e^{-i\phi}$ represents the stereographic projected coordinates of the points on S^2 from the south pole and $J_\pm = J_1 \pm iJ_2$.

In the next chapter we see, how in the framework of noncommutative geometry, the coherent states can represent the noncommutative space. There is one-to-one correspondence between the points in the noncommutative space and the coherent state labeled by $z : (x_1, x_2)$. As discussed in the Moyal plane case, this coherent states provide us with a weak position measurement. Therefore, by defining a distance on the set of coherent states we investigate the geometry of the underlying non-commutative space.

Chapter 3

Noncommutative Geometry

In the framework of Noncommutative geometry, the topological and geometrical data when generalized for noncommutative algebras can be written in compact form called spectral triples. We first sketch some important steps that leads us from usual notions in geometry to spectral triples. The main tool of our analysis is the Connes distance function, which gives distance between states of the algebra. Therefore, next we define Connes distance function and show how it is equivalent to a metric given on a Riemannian manifold. Finally, we construct the spectral triples for two non-commutative spaces of our study: Moyal plane and Fuzzy sphere.

3.1 Spectral Triple

In geometry, a space is basically a set of points with additional structure defined on it (such as manifolds). The topology on the space provide us with a distinction between points. The notion of how far or close the points are from each other is given by defining a distance function on the space. But in noncommutative geometry, the main emphasis is shifted from the space to the collection of functions on the space. A strong motivation of this can also be seen from the physics point of view. In physics, we always deal with the coordinates defined on the space rather than points itself on the space and we measure this coordinates in order to give a location of a event. The notion of points on a space becomes even more elusive in quantum physics, as due to Heisenberg uncertainty principle we cannot localize the coordinates of a event up to arbitrary small accuracy. The phase space in quantum mechanics provides a good example of noncommutative space where the coordinates are replace by operators. Therefore, the collection of functions on the space is a better notion of the space. We show how the collection of function provides us with the topological information of the space. First we define some preliminary objects, we follow mostly [24] and for some preliminary definition in Functional analysis [20]. A good review for some relevant concepts of Noncommutative Geometry is [22].

Algebra : An algebra $\mathcal{A}$ over a field $\mathbb{F}$ is a vector space over $\mathbb{F}$ with a multiplication operation defined which is associative and distributive.

Banach space: A normed space is a pair $(B, ||.||)$, where B is a vector space and $||.||$ is the norm defined on it. A Banach space B is a normed space which is complete(every Cauchy sequence converges in B) in the metric on B defined w.r.t its norm.

Banach algebra: A Banach algebra $\mathcal{A}$ is a algebra $\mathcal{A}$ over a field $\mathbb{F}$ which is also a Banach space relative to a norm $||.||$ such that $\forall a, b \in \mathcal{A}, ||a.b|| \leq ||a||\ ||b||$

Involution: For a Banach algebra $\mathcal{A}$, a involution is a map $a \to a^*$ from $\mathcal{A}$ to $\mathcal{A}$ such that for $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$: $(a^*)^* = a,\ (a.b)^* = b^*a^*, (\alpha a+b)^* = \bar{\alpha}a^* + b^*$

C*-algebra: A C*-algebra $\mathcal{A}$ is a Banach algebra with involution and a C*-identity

$$||a|| = ||a^*a||^{1/2} \text{ for } \forall\ a \in \mathcal{A}$$

Character: A character of a Banach algebra $\mathcal{A}$ is a nonzero homomorphism $\mu : \mathcal{A} \to \mathbb{C}$, which is surjective. M($\mathcal{A}$) denotes the set of characters on $\mathcal{A}$

Let X be locally compact Hausdorff (any two points can be separated by two disjoint sets) space, then the space of continuous functions $C(X)$ forms an commutative C^* algebra. The algebra also becomes unital (i.e a identity exist) if we consider only compact Hausdorff space. Now, from a commutative C^* algebra, we can recover the the topological space in the following manner. The set of characters $M(\mathcal{A})$ is actually a topological space with a well defined topology and there exist a correspondence between $\mathcal{A}$ and $M(\mathcal{A})$ via Gelfand transform (for proof refer to [24]) by which for each $a \in \mathcal{A}$ we can define a function $\hat{a} : M(\mathcal{A}) \to \mathcal{C}$ s.t $\hat{a}(\mu) = \mu(a)$. Hence, Gelfand transform is map from $\mathcal{A}$ to $C_0(M(\mathcal{A}))$ and thereby we recover the points of space X as the characters of the algebra. Therefore, the topological properties of a space can be recovered from the algebra of function on the space and from the following theorem it is established that we can associate a locally compact Hausdorff topological space to every C*-algebra.

Theorem(Gelfand-Naimark)([24] chap 1): For a commutative C*-algebra $\mathcal{A}$, the Gelfand transformation is an isometric *-isomorphism between $\mathcal{A}$ and $C_0(M(\mathcal{A}))$.

This paves the way for us to consider noncommutative C^* algebras, as locally compact Hausdorff space even though we cannot recover this whole space only some points may be corresponding to center of the algebra. Now, due to another theorem by Gelfand and Naimark any abstract C*-algebras (commutative or noncommutative) as defined above can be characterized, thereby giving a concrete meaning to them by the following theorem :

Theorem(Gelfand-Naimark)([[24] chap 1):Any C*-algebra has a isometric representation as a C*-algebra of closed subalgebra of algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on some Hilbert space

Thus we have a concrete realization of this abstract C^* algebras as some subalgebra of $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$. Hence, we don't have to work with this abstract C^* algebras instead we can confine our analysis only to algebra of bounded operators on a Hilbert space. We now have to introduce how to do calculus on this noncommutative algebras. For this we have to consider what does vector fields(smooth derivation on space of continuous function on a manifold M) means on noncommutative spaces. We here only briefly show how universal 1-forms are defined on the algebras and how this 1-forms are given through the action of a first-order differential operator called Dirac operator.

Module: Let $\mathcal{A}$ be a algebra and $\mathcal{N}$ be a linear space, then $\mathcal{N}$ is called a left module over algebra $\mathcal{A}$ if there exist a bilinear map $\mathcal{A}$ x $\mathcal{N} \to \mathcal{N}$: $(a, n) \to a.n$ s.t

$$a.(b.n) = (a.b)n \quad a, b \in \mathcal{A}, n \in \mathcal{N}$$

Similarly, a right module can be defined and $\mathcal{N}$ is called a bimodule over algebra $\mathcal{A}$ if it is a left and right module over $\mathcal{A}$ s.t $a.(n.b) = (a.n)b$. Let $E \xrightarrow{\pi} M$ be a vector bundle on manifold M and $\Gamma(M, E)$ be the linear space of sections on E. It is easy to see that $\Gamma(M, E)$ is a bimodule over the algebra of $C^\infty(M)$ - space of smooth functions on the manifold. Therefore, let E be a bimodule over complex unital algebra $\mathcal{A}$. A derivation on it is defined as follows[[24] chap 9].

Derivation: A derivation d is a linear map from $\mathcal{A}$ to E which satisfies Leibniz rule, for $a, b \in \mathcal{A}$

$$d(ab) = a.db + da.b$$

A derivation is called inner derivation $(ad(m))$ if it is defined by a element $m \in E$ s.t for $a \in \mathcal{A}$: $ad(m)a = m.a - a.m$. The derivation which are not inner are called outer derivation.

Let $d : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ s.t for $a \in \mathcal{A}$ $da = 1 \otimes a - a \otimes 1$. This is a derivation as it is a linear map by construction and satisfies Leibniz rule:

$$\begin{aligned} d(ab) &= 1 \otimes ab - ab \otimes 1 = a \otimes b - ab \otimes 1 + 1 \otimes ab - a \otimes b \\ &= a(1 \otimes b - b \otimes 1) + (1 \otimes a - a \otimes 1)b = a(db) - (da)b \end{aligned}$$

We define $\Omega^1\mathcal{A}$ to be bimodule over $\mathcal{A}$ s.t

$$\Omega^1\mathcal{A} = ker(m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A})$$

where $m(a\otimes b) = a.b$ is the multiplication map. Now $\sum_j a_j \otimes b_j \in \Omega^1\mathcal{A}$ implies $\sum_j a_j b_j = 0$, thus

$$\sum_j a_j \otimes b_j = \sum_j a_j \otimes b_j - \sum_j a_j b_j \otimes 1 = \sum_j a_j db_j$$

Therefore , $\Omega^1\mathcal{A}$ is a subbimodule of $\mathcal{A}\otimes\mathcal{A}$ generated by elements $a\ db$. $\Omega^1\mathcal{A}$ is called the bimodule of universal 1-forms over $\mathcal{A}$ and $(\Omega^1\mathcal{A}, d)$ is called universal first-order differential calculus. As stated earlier there is an equivalence between the bimodules over C^∞(M) and the space of sections of a vector bundle due to Serre-Swan theorem as follows [[24] chap 2]:

Definition(Projective Module):The module $\mathcal{M}$ over a algebra $\mathcal{A}$ is projective if there exist a module $\mathcal{M}^\perp$ such that $\mathcal{M}\oplus\mathcal{M}^\perp \equiv \mathcal{A}^n, n > 0$. It is called finitely generated if there exist a finite no of elements $m_1, m_2, ...m_k$ such that

$$\mathcal{M} = \left\{\sum_{i=1}^{k} m_i a_i\right\}_{a_i\in\mathcal{A}}$$

Then by Serre-Swan theorem, it can be said that $C^\infty(M, E)$ forms a finitely generated projective module over C(M) and every finitely generated projective module over a algebra $\mathcal{A}$ is of that form.

Theorem(Serre-Swan): The Γ functor from the category of vector bundles on a manifold M to a category of finitely generated projective modules over C(M) is an equivalence of categories

The significance of the projective module can be seen from the following property of them.

Connection: Let $\mathcal{M}$ be a right module over $\mathcal{A}$. A connection is a linear mapping from $\Delta : \mathcal{M} \to \mathcal{M}\otimes_{\mathcal{A}} \Omega^1\mathcal{A}$ which satisfy the Leibniz rule, for $a \in \mathcal{A}, s \in \mathcal{M}$

$$\Delta(as) = a\Delta s + s\otimes da$$

Theorem: A right module admits a universal connection if and only if it is projective

We can now introduce a first-order self-adjoint operator D acting on a Hilbert space which is the space of sections of the vector bundle on a manifold M such that the algebra of continuous function C^∞(M) is represented on it. Now from this operator which falls into the category of generalized Dirac operator, we can recover the 1-forms by letting for $a \in \mathcal{A}$

$$da := [D\ a]$$

as operators acting on the Hilbert space $\mathcal{H}$, for $\psi \in \mathcal{H}$

$$(da)\psi = D(a\psi) - aD\psi = [D\ a]\psi$$

This Dirac operator also stores the information of the metric of the manifold as shown below. There is a lot more additional structure associated with Dirac operator through which we can get the dimension of the manifold and do integration by using the spectral properties of it, as shown in [24]. Now the whole structure discussed above can be written in a compact form known as spectral triple. Spectral triples are generalization of Riemannian spin manifolds to the non-commutative algebras.

Definition:(Spectral Triple)[23], A Spectral Triple $(\mathcal{A}, \mathcal{H}, D)$ comprises of the following: a involutive algebra $\mathcal{A}$(a dense subalgebra of a C*-algebra) , a Hilbert space $\mathcal{H}$ where $\mathcal{A}$ acts through a representation π and a self-adjoint, densely defined operator D(Dirac operator) on $\mathcal{H}$ which satisfies:

1. D can be unbounded operator in general but $[D, \pi(a)]$ is bounded

2.D has compact resolvent i.e for $\lambda \in \mathbb{C}/\mathbb{R}$, $(D - \lambda)^{-1}$ is compact when the algebra $\mathcal{A}$ is unital(there exist a identity element) or $\pi(a)(D - \lambda)^{-1}$ be compact if it is non-unital

The conditions imposed on the Dirac operator ensure that spectrum of Dirac operators is real and discrete i.e the collection of eigenvalues $\{\mu_n\}$ is a discrete set in $\mathbb{R}$. Also, the eigenspace corresponding to each eigenvalue is finite dimensional. The second condition implies that the eigenvalues follows a growth property such that there is no accumulation point for the set of eigenvalues other than at infinity i.e as $n \to \infty, \lambda_n \to \infty$ [23].

Operator norm[20]: Let T be a bounded operator acting on a Hilbert space $\mathcal{H}$. Let $||.||$ be the norm defined on the Hilbert space $\mathcal{H}$. Then the operator norm of T is

$$||T||_{op} = sup\frac{||Tv||}{||v||} \quad v \in \mathcal{H}$$

State on algebra $\mathcal{A}$ [20]: A state ω on *-algebra $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \to \mathbb{C}$ which is positive i.e $\omega(a^*a) \geq 0\ \forall a \in \mathcal{A}$ and has a norm 1.

Any convex linear combination of states is again a state. A state is called pure state if it cannot be written as convex combination of some other states. If the algebra is commutative then the space of pure states is same as the space of characters as defined above. Therefore, the pure states have a one-to-one correspondence with the points in the space. Now the distance on noncommutative space is defined as follows:

For a general spectral triple $(\mathcal{A}, \mathcal{H}, D)$, **Connes Distance Function** defines the distance between two states ω, ω' of algebra($\mathcal{A}$) as

$$d_D(\omega, \omega') = sup_{a \in \mathcal{A}} \left\{ \left| \omega(a) - \omega'(a) \right| \ \ | : \ \ ||[D, \pi(a)]||_{op} \leq 1 \right\} \tag{3.1}$$

We first give the spectral triple corresponding to the Riemannian manifold and then recover the usual definition of distance on the Riemannian manifold from the Connes distance function. Connes distance function has the advantage of providing us with a distance on discrete spaces, we therefore calculate Connes distance on a two point space as an example.

Canonical spectral triple

Let M be a compact Riemannian spin manifold.

- $\mathcal{A} = C^{\infty}(M)$ be the algebra of complex-valued smooth function under point-wise multiplication:$(f.g)(x) = f(x)g(x), f, g \in \mathcal{A}, x \in M$

- Let S be the spinor bundle on M, $\mathcal{H} = L^2(MS)$ be the Hilbert space of square integrable spinorial section on M. The algebra elements act by multiplication: $(f\psi)(x) = f(x)\psi(x), \psi \in \mathcal{H}$

- D be the Dirac operator associated with the Levi-Civita connection, $D = -i\gamma^{\mu}\nabla^{s}_{\mu}$

then it can be proved that $(\mathcal{A} = C^{\infty}(M), \mathcal{H} = L^2(MS), D)$ is a spectral triple called Canonical spectral triple[[[23]]]. Thus, the spectral triples are algebraic descriptions of Riemannian manifolds which can be generalized to the case of noncommutative space considering the corresponding noncommutative algebra.

On a Riemannian manifold M the distance between two points $x, y \in M$ is define by a metric $g_{\mu\nu}$ as:

$$d_g(x, y) = \inf \int_{\sigma} ds \quad : ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

where σ represent paths from x to y and infimum is attained along a geodesic from x to y. This same distance can also be given for the manifold M by Connes Distance function :

$$d_D(x, y) = sup_{a \in \mathcal{A}} \left\{ |\phi_x(a) - \phi_y(a)| \ \ | : \ \ ||[D, a]||_{\infty} \leq 1 \right\}$$

where ϕ are pure states of algebra $\mathcal{A}$ s.t $\phi_x(a) = a(x)$.

For $f \in \mathcal{A}$ *and* $\psi \in \mathcal{H}$, $[D\ f]\psi = -i\gamma^\mu(\partial_\mu f)\psi$. Therefore $[D\ f]$ acts by multiplication on $\mathcal{H}$ and $[D\ f] = -i\gamma(df) \in \mathcal{A}$

$$\begin{aligned}
||[D\ f]||_\infty &= sup|(\gamma^\mu\partial_\mu f)(\gamma^\nu\partial_\nu f)^*|^{1/2} \quad \text{where } |.| \text{ is the modulus on complex numbers} \\
&= sup|\gamma^\mu\gamma^\nu\partial_\mu f\partial_\nu f^*|^{1/2} \\
&= sup|\frac{[\gamma^\mu\gamma^\nu]}{2}\partial_\mu f\partial_\nu f^* + \frac{\{\gamma^\mu\gamma^\nu\}}{2}\partial_\mu f\partial_\nu f^*|^{1/2} \\
&= sup|g^{\mu\nu}\partial_\mu f\partial_\nu f^*|^{1/2} \quad since\{\gamma^\mu\gamma^\nu\} = 2g^{\mu\nu} \\
&= ||grad(f)||_\infty \quad : grad(f) = gradient
\end{aligned}$$

Let $\sigma(t) : [0\ 1] \to M$ be a smooth path in M s.t $\sigma(1) = y$ *and* $\sigma(0) = x$

$$\begin{aligned}
\phi_y(f) - \phi_x(f) &= f(\sigma(1)) - f(\sigma(0)) = \int_0^1 \frac{\mathrm{d}f(\sigma(t))}{\mathrm{d}t}dt \\
&= \int_0^1 grad(f).\dot\sigma(t)dt
\end{aligned}$$

$$\begin{aligned}
|\phi_y(f) - \phi_x(f)| &\le \int_0^1 |grad(f)||\dot\sigma(t)|dt \\
&\le ||grad(f)||_\infty \int_0^1 |\dot\sigma(t)|dt = ||grad(f)||_\infty length(\sigma) \\
&\le ||[D\ f]||_\infty length(\sigma)
\end{aligned}$$

Thus, we get
$sup_{a\in\mathcal{A}}\{|\phi_x(f) - \phi_y(f)|\ \ | : \ \ ||[D, f]||_\infty \le 1\} \le \inf_\sigma length(\sigma) = d_\sigma(x, y)$

Define $f_{\sigma,z}(x) = d_\sigma(z, x)$ and $f_{\sigma,z} \in \mathcal{A}$
Now $|f_{\sigma,z}(y) - f_{\sigma,z}(x)| \le d_\sigma(y, x)$ by triangle inequality and for σ being such that it is a geodesic $||[D\ f_{\sigma,z}]||_\infty = ||grad(f_{\sigma,z})|| = 1$, thus $f = f_{\sigma,z}$ saturates the above inequalities . Hence we get that both the distance function defined on a manifold M are equal

$$d_D(x, y) = d_\sigma(x, y)$$

A simple example is: when $M = \mathbb{R}$ and $D = \frac{\mathrm{d}}{\mathrm{d}x}$, then the condition $||[D\ f]||_\infty \le 1$ becomes $\frac{\mathrm{d}f}{\mathrm{d}x} \le 1$ and the supremum is saturated by functions $f(x) = \pm x + constant$ which gives the distance.

Two-point space

Lets consider the case of a discrete space. Let $X = \{1, 2\}$ be the space of two points. The algebra of continuous complex-valued function is taken as $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, since for any $f \in \mathcal{A}$ it will be a pair of complex numbers (c_1, c_2) s.t $f(i) = c_i,\ i = 1, 2$. We take the Hilbert space to be $\mathcal{H} = \mathbb{C}^2$, which can also be thought in a loose way, to be the spinor bundle on the two point space. The algebra $\mathcal{A}$ acts on $\mathcal{H}$ via the representation π ,which are diagonal 2 x 2 complex-valued matrices. For $a = (a_1, a_2) \in \mathcal{A}$ *and* $\psi = (\psi_1, \psi_2) \in \mathcal{H}$

$$\psi \to \pi(a)\psi = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{3.2}$$

We take the Dirac operator to be 2x2 off-diagonal hermitian matrix

$$D = \begin{bmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{bmatrix} \quad \Lambda \in \mathbb{C} \tag{3.3}$$

We let the diagonal terms to be zero as in the commutator $[D\ \pi(a)]$ the diagonal terms will always vanish. Therefore, the spectral triple for the two-point space will be

$$\mathcal{A} = \mathbb{C}^2 : H = \mathbb{C}^2 : D = \begin{bmatrix} 0 & \Lambda \\ \bar{\Lambda} & 0 \end{bmatrix} \tag{3.4}$$

Let ω_1 *and* ω_2 be the two states of the algebra $\mathcal{A}$ s.t for $a = (a_1, a_2) \in \mathcal{A},\ \omega_i(a) = a_i \quad i = 1, 2$. It can be easily seen that any other state can be written as convex combination of the above two states. Therefore, this are only two pure states of the algebra and they corresponds to the two points of the space. We can also define the action of ω_1 *and* ω_2 on $\mathcal{A}$ as $\omega_1(a) = tr(\rho_1\pi(a)) = a_1$ and $\omega_2(a) = tr(\rho_2\pi(a)) = a_2$ where ρ_1 *and* ρ_2 are the basis of representation $\pi(\mathcal{A})$

$$\rho_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \rho_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

For any $a = (a_1, a_2)$ we have $[D\ \pi(a)] = (a_1 - a_2) \begin{bmatrix} 0 & -\Lambda \\ \bar{\Lambda} & 0 \end{bmatrix}$. Therefore, for a s.t $||[D\pi(a)]||_{op} \leq 1 \implies |a_1 - a_2| \leq \frac{1}{|\Lambda|}$, since

$$\begin{aligned} ||[D\ \pi(a)]||^2_{op} &= ||[D\ \pi(a)]^\dagger [D\ \pi(a)]||_{op} \quad (\mathcal{A} \text{ is } C^* \text{ algebra}) \\ &= |a_1 - a_2|^2 \left\| \begin{bmatrix} |\Lambda|^2 & 0 \\ 0 & |\Lambda|^2 \end{bmatrix} \right\|_{op} \\ &= |a_1 - a_2|^2 |\Lambda|^2 \end{aligned}$$

Hence, the Connes distance between two states ω_1 and ω_2 which corresponds to the two points of the space is

$$
\begin{aligned}
d(\omega_1, \omega_2) &= sup\,\{|\omega_1(a) - \omega_2(a) : ||[D\ \pi(a)]||_{op} \leq 1|\} \\
&= sup\left\{|a_1 - a_2| : |a_1 - a_2| \leq \frac{1}{|\Lambda|}\right\} \quad where\ a = (a_1, a_2) \\
&= \frac{1}{|\Lambda|}
\end{aligned}
$$

Thus, the above two examples show how geometric information can be stored in the Dirac operator. By changing the Dirac operator we can also change the geometry of the space. Therefore, Connes distance function provides a general formulation of the distance function through which we can find distance between noncommutative spaces as well as discrete spaces.

3.2 Spectral Triple for Moyal plane and Fuzzy Sphere

In this section we construct the spectral triple for Moyal plane and Fuzzy sphere. In sec (2), the definition of quantum Hilbert space was motivated such that its elements represents states of a physical system in comparison with square integrable functions on $\mathbb{R}^d$ in standard quantum mechanics. As mentioned in the previous section, the space of bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is a C^* algebra. It can be shown that space of Hilbert-Schmidt operators $\mathcal{B}_2(\mathcal{H})$ is a two-sided *-ideal in this C^* algebra $\mathcal{B}(\mathcal{H})$. Therefore, as per our construction the quantum Hilbert space $\mathcal{H}_q$ naturally provides a C^* algebra acting on the classical Hilbert space $\mathcal{H}_c$. In the following, we construct spectral triples in order to provide a geometric structure(calculating Connes distance between states acting on $\mathcal{H}_q$) to the configuration space i.e classical Hilbert space. Similarly, to provide distance function on quantum Hilbert space the corresponding spectral triples can be constructed as done in [15],[16], but we restrict our analysis to finding Connes distance on classical Hilbert space.

3.2.1 Moyal plane

The following spectral triple was constructed in [15].

$$\mathcal{A} = \mathcal{H}_q = span\,\{|m\rangle\langle n|\}\,(2.7) \tag{3.5}$$

$$\mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2\ ; \quad \mathcal{H}_c = span\left\{|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n\,|0\rangle\right\}\ \text{from}(2.6) \tag{3.6}$$

$$D_M = \sqrt{\frac{2}{\theta}}\begin{bmatrix} 0 & b^\dagger \\ b & 0 \end{bmatrix} \tag{3.7}$$

The algebra $\mathcal{A}$ acts on $\mathcal{H}$ through the representation π as:

$$\pi(a)\begin{bmatrix} |\psi\rangle \\ |\phi\rangle \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} |\psi\rangle \\ |\phi\rangle \end{bmatrix} = \begin{bmatrix} a\,|\psi\rangle \\ a\,|\phi\rangle \end{bmatrix} \quad a \in \mathcal{A} \tag{3.8}$$

The Dirac operator is first constructed on $\mathcal{H}_q$ where it is defined as a hermitian operator $D_M = \rho_\alpha P_\alpha = \rho_1 P_1 + \rho_2 P_2$, ρ_α are Pauli matrices and P_α as defined in(2.10). Therefore, D_M acts on $\psi = \begin{bmatrix} |\psi)_1 \\ |\psi)_2 \end{bmatrix} \in \mathcal{H}_q \otimes \mathbb{C}^2$ as

$$D_M\psi = \sqrt{\frac{2}{\theta}}\left[\begin{bmatrix} 0 & ib^\dagger \\ -ib & 0 \end{bmatrix}, \psi\right] \tag{3.9}$$

This Dirac operator which acts adjointly on $\mathcal{H}_q \otimes \mathbb{C}^2$ also naturally provides a left action on $\mathcal{H}_c \otimes \mathbb{C}^2$. Thus now by transforming $\hat{b} \to i\hat{b}$ and $\hat{b}^\dagger \to -i\hat{b}^\dagger$ which means a $SO(2)$ rotation in the $\hat{x}_1, \hat{x}_2$ space by $\frac{\pi}{2}$, we define the Dirac operator (3.7) on $\mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2$. Since $\mathcal{H}_q$ is a infinite dimensional vector space, identity is not a Hilbert-Schist operator. Therefore the algebra $\mathcal{A} = \mathcal{H}_q$ is a non-unital algebra. Thus, the spectral triple above will be a legitimate spectral triple if $\forall a \in \mathcal{A}, [D, \pi(a)]$ is bounded operator and $\pi(a)\,(D - \lambda)$ is compact, where λ is in resolvent set of D.

The boundedness of operators$[D_M, \pi(a)]$ follows easily from the fact that $[b, a]$ and $\left[b^\dagger, a\right]$ are bounded operators, since

$$[D_M, a] = \sqrt{\frac{2}{\theta}}\begin{bmatrix} 0 & \left[b^\dagger, a\right] \\ [b, a] & 0 \end{bmatrix} \tag{3.10}$$

To prove that $\pi(a)\,(D_M - \lambda)$ is compact is much involved. For this we refer to [[28]], where it is proved for a spectral triple on Moyal plane which is intimately connected to the spectral triple discussed above. This spectral triple is the isospectral deformation of the canonical spectral triple of Euclidean space $\mathbb{R}^2$ in which, while retaining the same Hilbert space and

Dirac operator of the canonical spectral triple, the commutative algebra of Schwartz function is deformed into a noncommutative algebra by defining a new product rule called Moyal $\star$ product given as:

$$(f \star g)(x) := \frac{1}{(\pi\theta)^2} \int d^2 s d^2 t f(x+s) g(x+t) e^{-2is.\Theta^{-1}t} \tag{3.11}$$

where $\Theta = \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a 2 x 2 real skew symmetric matrix

Therefore, the spectral triple corresponding to above deformation is the following:

$$\mathcal{A} = (\mathcal{S}, \star) \quad \mathcal{H} = L^2(\mathbb{R}^2) \quad D = -i\sigma^\mu \partial_\mu \quad : \mu = 1, 2 \tag{3.12}$$

where $\mathcal{A} = (\mathcal{S}, \star)$ algebra of complex Schwartz(smooth,rapidly decreasing) function on $\mathbb{R}^2$ equipped with Moyal $\star$ product and σ^μ are Pauli matrices.

3.2.2 Fuzzy sphere

In case of fuzzy sphere space, we consider the following spectral triple corresponding a particular fuzzy sphere indexed by n. The construction of the spectral triple can be found in [16] and from (2.33),(2.34)

$$\mathcal{A} = H_q^n : \quad \mathcal{H}_q^n = span\left\{ \Big|n, n_3\Big\rangle\Big\langle n, n_3^{'}\Big| = |n_3, n_3^{'}) : -n \leq n_3 \leq n \right\} \tag{3.13}$$

$$\mathcal{H} = \mathcal{H}_c^n \otimes \mathbb{C}^2 \; ; \quad \mathcal{H}_c^n = span\left\{ |n, n_3\rangle : -n \leq n_3 \leq n \right\} \tag{3.14}$$

$$D_F = \frac{1}{r} \begin{pmatrix} J_3 & J_- \\ J_+ & -J_3 \end{pmatrix} \tag{3.15}$$

where $J_\pm = \frac{\hat{x}_\pm}{\lambda}$,$J_i = \frac{\hat{x}_i}{\lambda}$ as defined for Fuzzy sphere space in sec(2.2). As in the Moyal plane case the algebra $\mathcal{A}$ acts on $\mathcal{H}$ through the representation π as:

$$\pi(a) \begin{bmatrix} |\psi\rangle \\ |\phi\rangle \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} |\psi\rangle \\ |\phi\rangle \end{bmatrix} = \begin{bmatrix} a\,|\psi\rangle \\ a\,|\phi\rangle \end{bmatrix} \quad a \in \mathcal{A} \tag{3.16}$$

The Dirac operator D_F was constructed in [[25]] and also reviewed in [[16]], where the Dirac operator was constructed first on Sphere S^2 and then deformed for the fuzzy sphere. Since the algebra is a finite dimensional vector space, it is unital. Therefore, the conditions, the above spectral triple should satisfy are, $\forall a \in \mathcal{A}, [D, \pi(a)]$ is bounded operator and $(D - \lambda)^{-1}$ is compact, where $\lambda \in \mathbb{C}/\mathbb{R}$ is in resolvent set of D.

We proceed to show that the above spectral triple satisfies the above condition as proved in [16]. For a finite dimensional Hilbert space $\mathcal{H}$, the trace norm and the operator norm is equivalent (i.e they give the same topology on $\mathcal{B}(\mathcal{H})$) due to the relation $||T||_{op} \leq ||T||_{tr} \leq \sqrt{d}||T||_{op}$. Therefore, we show that the trace norm $||[D,\pi(a)]||_{tr} < \infty$. The $||[D,\pi(a)]||_{tr}$ can be written as

$$||[D,\pi(a)]||_{tr}^2 = 2||[J_3,a]||_{tr}^2 + ||[J_+,a]||_{tr}^2 + ||[J_-,a]||_{tr}^2 \tag{3.17}$$

using the C^* algebra property $||A||^2 = ||A^*A||$. Now since $||A+B||_{tr} = ||A||_{tr}+||B||_{tr}$ and $||AB||_{tr} \leq ||A||_{tr}||B||_{tr}$, this implies $||[B,a]||_{tr} \leq 2||B||_{tr}||a||_{tr}$ and by using the following results

$$||J_3||_{tr}^2 = \frac{1}{2}||J_+||_{tr}^2 = \frac{1}{2}||J_-||_{tr}^2 = \frac{1}{3}n(n+1)(2n+1) \tag{3.18}$$

It follows

$$||[D,\pi(a)]||_{tr} \leq \frac{2\sqrt{2(2n+1)}}{\lambda}||a||_{tr} \leq \frac{2\sqrt{2}(2n+1)}{\lambda}||a||_{op} < \infty \tag{3.19}$$

Now for the second condition ,the resolvent operator $(D-\mu)^{-1}, \quad \mu \notin \mathbb{R}$ can be written as

$$(D-\mu)^{-1} = \frac{r^2}{n(n+1) - r\mu(r\mu+1)}\left(D + \frac{1}{r} + \mu\right) \tag{3.20}$$

since $\left(D+\frac{1}{r}+\mu\right)(D-\mu) = \frac{1}{r^2}\begin{pmatrix} J^2 & 0 \\ 0 & J^2 \end{pmatrix} - \frac{r\mu(r\mu+1)}{r^2}\mathbb{I}_2 = \frac{n(n+1)-r\mu(r\mu+1)}{r^2}$ as J^2 is the Casimir operator $J^2\,|n,n_3\rangle = n(n+1)\,|n.n_3\rangle$. Again by calculating trace norm it can be shown that the operator $\left(D+\frac{1}{r}+\mu\right)$ is bounded. Since the Hilbert space is finite dimensional this operator is a finite rank operator(i.e image of the operator is finite dimensional). Therefore the compactness of the resolvent operator follows from the fact that a bounded thereby continuous and finite rank operator is a compact operator [20].

Chapter 4

Connes distance on Moyal plane and Fuzzy sphere

In sec(2.1), we introduced two different set of basis for classical Hilbert space $\mathcal{H}_c$ of Moyal plane and Fuzzy sphere: one is the discrete basis (2.6) , (2.31) also called harmonic oscillator basis and the other one is coherent state basis(2.15),(2.38) characterized by a continuous parameter $z \in \mathbb{C}$. We now calculate Connes distance between pure states on algebra $\mathcal{A} = \mathcal{H}_q$ which correspond to the two basis mentioned above. As discussed in previous chapter, this pure states have one-to-one correspondence with the points in the space.

In sec(2), we constructed the quantum Hilbert space $\mathcal{H}_q$ such that it is the tensor product of the $\mathcal{H}_c$ and its dual $\mathcal{H}_c^* : \mathcal{H}_q = \mathcal{H}_c \otimes \mathcal{H}_c^*$. From this fact it is evident the $\mathcal{H}_q$ is self-dual $\mathcal{H}_q = \mathcal{H}_q^*$. Therefore, all pure states on $\mathcal{H}_q$ are normal states. The following are equivalent statements [26]:

- A state ω is a normal state

- the state ω can be given by a trace class operator ρ_ω acting on $\mathcal{H}_c$, which is a hermitian semi-positive operator with trace $\rho_\omega = 1$ s.t for $a \in \mathcal{A} = \mathcal{H}_q$.

$$\omega(a) = tr_c(\rho_\omega a) \quad \text{where c implies trace over } \mathcal{H}_c \tag{4.1}$$

The normal state ω is pure state if $\rho_\omega^2 = \rho_\omega$. Since $tr_c(\rho_\omega^\dagger \rho_\omega) = tr_c(\rho_\omega^2) \leq tr_c(\rho_\omega) = 1$, $\rho_\omega \in \mathcal{H}_q$ is a Hilbert-Schmidt operator. Therefore, the pure states corresponds to $\rho_{|n\rangle} = |n\rangle\langle n| \in \mathcal{H}_q$ ($i.e$ $\omega_{|n\rangle}$) in the harmonic oscillator basis and $\rho_z = |z\rangle\langle z| \in \mathcal{H}_q$ ($i.e$ $\omega_{|z\rangle}$) in the coherent state basis.

4.1 Moyal plane

In this section, we calculate the Connes distance between coherent states ρ_z and the harmonic oscillator states $\rho_{|n\rangle}$ of Moyal plane. In [15], the Connes distance for infinitesimally separated coherent states was calculated by using a algorithm developed there. It turns out that the infinitesimal distance can be calculated correctly up to a numerical constant only. We discuss this issue in the last section (4.3). We extend this results by giving a alternative approach to calculate the Connes distance between finitely separated coherent states. In [11] [10], the Connes distance between coherent states were calculated for the spectral triple(3.12) which is the isospectral deformation of canonical spectral triple for $\mathbb{R}^2$. We follow similar approach here and calculate Connes distance for the spectral triple discussed in sec(3.2.1).

4.1.1 Connes distance between coherent states

The spectral triple on Moyal plane as shown in sec (3.2.1) is:

$$\mathcal{A} = \mathcal{H}_q = span\left\{|m\rangle\langle n|\right\} \quad \mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2 \quad D_M = \sqrt{\frac{2}{\theta}} \begin{bmatrix} 0 & b^\dagger \\ b & 0 \end{bmatrix}$$

where $\mathcal{H}_c = span\left\{|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle\right\}$

Therefore, the Connes distance (3.1) between the states ω_z and $\omega_{z'}$ is

$$d(\omega_z, \omega_{z'}) = sup_{a\in B}\left\{|\omega_z(a) - \omega_{z'}(a)|\right\} \tag{4.2}$$

$$B = \{a \in \mathcal{A} = \mathcal{H}_q : ||[D, \pi(a)]||_{op} \leq 1\} \quad \text{(Lipschitz ball)} \tag{4.3}$$

From (4.1), the action of the state ω_z on $\mathcal{H}_q$ can be written as

$$\begin{aligned} \omega_z(a) = tr_c(\rho_z a) &= tr_c\left(U(z,\bar{z})\,|0\rangle\langle 0|\,U^\dagger(z,\bar{z})a\right) \\ &= \langle 0|\left(U^\dagger(z,\bar{z})aU(z,\bar{z})\right)|0\rangle \end{aligned} \tag{4.4}$$

where $|z\rangle = U(z,\bar{z})\,|0\rangle$, $U(z,\bar{z}) = \exp\left(-\bar{z}b + zb^\dagger\right)$. It implies the algebra element $a \in \mathcal{A} = \mathcal{H}_q$ gets translated by the adjoint action of $U(z,\bar{z})$ thereby furnishing a proper representation of the translational group. We now first calculate the distance between ω_z and ω_0 i.e between origin and a point $z = (x_1, x_2)$ and afterwards prove that the Connes distance is translationally invariant. By eq(4.4) , eq(4.2) becomes

$$\begin{aligned} d(\omega_z,\ \omega_0) &= \sup_{a\varepsilon B}\left|\langle 0|\left(U^\dagger(z,\bar{z})aU(z,\bar{z})\right)|0\rangle - \langle 0|\,a\,|0\rangle\right| \\ &= \sup_{a\varepsilon B}\left|\langle 0|\left(U^\dagger(z,\bar{z})aU(z,\bar{z})) - a\right)|0\rangle\right| \end{aligned} \tag{4.5}$$

Intuitively, $d(\omega_z, \omega_0)$ is the maximum change in the expectation values of the $a \in B$ and the translated algebra element $U^\dagger(z,\bar{z})aU(z,\bar{z})$ in the same state $|0\rangle \in \mathcal{H}_c$.

The strategy we adopt from here on is, first we find a upper bound(such that it is the least upper bound) on Connes distance. Once we find the upper bound we prove that it is the Connes distance by: (i) finding a algebra element $a_s \in B$ called supremum element s.t it attains the upper bound or by (ii) finding a sequence $\{a_n\}$ $a_n \in B$ s.t $\lim\limits_{n\to\infty} |\omega_z(a_n) - \omega_{z'}(a_n)|$ accumulates on the upper bound

Now to find the upper bound, we make use of the additional structure of state ω_z as shown in (4.4). Each state is labeled by two real parameters $z : (x_1, x_2),\ x_1, x_2 \in \mathbb{R}$ and on this set $\{(x_1,x_2) : x_1, x_2 \in \mathbb{R}\}$ there is a natural metric defined through $z, z^{'} \in \mathbb{C}$ $\left|z - z^{'}\right| = \frac{1}{\sqrt{2\theta}}\left|(x_1 - x_1^{'}) - i(x_2 - x_2^{'})\right|\sqrt{2\theta} = \frac{1}{\sqrt{2\theta}}\sqrt{(x_1 - x_1^{'})^2 + (x_2 - x_2^{'})^2}$ (since $z = \frac{x_1 + ix_2}{\sqrt{2\theta}}$ (2.15)). Hence we can regard $(x_1, x_2) \in \mathbb{R}^2$ and therefore for fixed $a = a^\dagger \in \mathcal{A}$, $\omega_z(a)$ is a function on $\mathbb{R}^2$, $\ \omega_{z:(x_1,x_2)} : \mathbb{R}^2 \to \mathbb{R}$. We now find the upper bound on $|\omega_z(a) - \omega_0(a)|$ for a fixed $a \in B$ which turns out to be independent of a. From here onwards we only consider hermitian algebra elements $a = a^\dagger \in \mathcal{A}$, since it was proved in [19], that the supremum element $a_s \in \mathcal{A}$ which attains the supremum in Connes distance function belongs to the subset of hermitian elements of algebra $\mathcal{A}$.

Let us define a map $W : \mathbb{R} \to \mathbb{R}$ by

$$W(t) = \omega_{zt}(a) = tr(\rho_{zt}a) \tag{4.6}$$

with $t \in [0,1]$ i.e the function $\omega_{z'}(a)$ restricted on the straight line connecting $z^{'} = 0$ to $z^{'} = z$. We can then write

$$|\omega_z(a) - \omega_0(a)| = \left|\int_0^1 \frac{\mathrm{d}W(t)}{\mathrm{d}t}dt\right| \leq \int_0^1 \left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|dt \tag{4.7}$$

Now since,

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \frac{\mathrm{d}\left(\omega_{zt}(a)\right)}{\mathrm{d}t} = \frac{\mathrm{d}\left\langle 0\right|\left(U^\dagger(zt,\bar{z}t)aU(zt,\bar{z}t)\right)\left|0\right\rangle}{\mathrm{d}t}$$

We can make use of Hadamard identity

$$\begin{aligned}\left(U^\dagger(z,\bar{z})aU(z,\bar{z})\right) &= \exp(G)a\exp(-G)\\ &= a + [G,a] + \frac{1}{2!}[G,[G,a]] + \frac{1}{3!}[G,[G,[G,a]]] + \cdots\end{aligned} \tag{4.8}$$

where $G = \bar{z}b - zb^\dagger$, to get

$$\begin{aligned}\frac{\mathrm{d}W(t)}{\mathrm{d}t} &= \langle 0|\,[G,a]\,|0\rangle + t\,\langle 0|\,[G,[G,a]]\,|0\rangle + \frac{t^2}{2!}\,\langle 0|\,[G,[G,[G,a]]]\,|0\rangle + \cdots \\ &= \langle 0|\,(\exp(tG)\;[G,a]\;\exp(-tG))\,|0\rangle\end{aligned}$$

On further simplification, this can be recast as

$$\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \bar{z}\omega_{zt}([b,a]) + z\omega_{zt}([b,a]^\dagger) \tag{4.9}$$

Now we get the following upper bound for $\left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|$, by making use of Cauchy-Schwartz inequality :

$$\begin{aligned}\left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| &= \left|\bar{z}\omega_{zt}([b,a]) + z\omega_{zt}([b,a]^\dagger)\right| &(4.10)\\ &\leq \sqrt{2}|z|\sqrt{|\omega_{zt}([b,a])|^2 + |\omega_{zt}([b^\dagger,a])|^2} &(4.11)\\ &\leq \sqrt{2}|z|\sqrt{||[b,a]||_{op}^2 + ||[b^\dagger,a]||_{op}^2} &(4.12)\end{aligned}$$

Note that in the last step, we have made use of the fact that states ω's are linear functionals of unit norm.

Now with Dirac operator D (3.7), one can prove (see Appendix(A)) the following identity

$$\|[\mathcal{D},\pi(a)]\|_{op} = \sqrt{\frac{2}{\theta}}\|[b,a]\|_{op} = \sqrt{\frac{2}{\theta}}\|[b^\dagger,a]\|_{op} \tag{4.13}$$

Using this, the “ball" condition (4.3) reduces for $a \in B$ as(A.3)

$$\|[b,a]\|_{op} = \|[b^\dagger,a]\|_{op} \leq \sqrt{\frac{\theta}{2}} \tag{4.14}$$

Therefore from (4.12) and (4.14), one can write

$$\left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| \leq \sqrt{2\theta}|z| \tag{4.15}$$

Hence from eq (4.5, 4.7, 4.15) we have the following upper bound for Connes distance :

$$d(\omega_z,\omega_0) \leq \sqrt{2\theta}|z| \tag{4.16}$$

Now, the upper bound can be identified as Connes distance, provided there exists the so-called supremum element $a_s \in B$ for which this inequality in (4.16) will be saturated. We therefore from (4.5) look for an optimal element $a = a_s$, satisfying $U^\dagger a_s U = \left(a_s + \sqrt{2\theta}|z|\right)$ s.t.

$$d(\omega_z, \omega_0) = \sqrt{2\theta}|z| \tag{4.17}$$

A simple inspection into (4.8) shows that a_s should satisfy

$$[G, a_s] = \sqrt{2\theta}|z| \quad \text{and} \quad [G, [G, a_s]] = 0 \tag{4.18}$$

where $G = \bar{z}b - zb^\dagger$, ensuring that all higher order nested commutators vanish. Observe that since $b, b^\dagger$ act irreducibly on $\mathcal{H}_c$, we must have, using Schur's lemma, $[G, a_s]$ to be proportional to the identity operator, as happens here. This yields,

$$a_s = \sqrt{\frac{\theta}{2}}\left(be^{-i\alpha} + b^\dagger e^{i\alpha}\right) \tag{4.19}$$

where $z = |z|e^{i\alpha}$.

It can be seen that even though the above element is inside Lipschitz ball B but is an unbounded operator, $a_s \in B$, as

$$\|[\mathcal{D}, \pi(a_s)]\|_{op} = 1, \tag{4.20}$$

but

$$\|a_s\|_{tr} = \sqrt{\frac{\theta}{2}}\sum_{n=1}(2n+1) = \infty \tag{4.21}$$

Consequently $a_s \notin \mathcal{H}_q = \mathcal{A}$, but can be thought of belonging to the multiplier algebra [1]. We therefore look for a sequence $\{a_n\}$, $a_n \in B$, s.t. $\lim_{n\to\infty} a_n = a_s$ and

$$d(\omega_z, \omega_0) = \lim_{n\to\infty} |\omega_z(a_n) - \omega_0(a_n)| = \sqrt{2\theta}|z|$$

This can be achieved with

$$a_n = \sqrt{\frac{\theta}{2}}\left(be^{-i\alpha}(e^{-\lambda_n b^\dagger b}) + (e^{-\lambda_n b^\dagger b})b^\dagger e^{i\alpha}\right) \tag{4.22}$$

by proving the following proposition as in [11] (proposition 3.5).

[1]Multiplier algebra $M = M_L \cap M_R$ where $M_L = \{T \in \beta(\mathcal{H}_q)|\ \psi T \in \mathcal{H}_q\ \forall\ \psi \in \mathcal{H}_q\}$ and $M_R = \{T \in \beta(\mathcal{H}_q)|T\psi \in \mathcal{H}_q\ \forall\ \psi \in \mathcal{H}_q\}$

Proposition: Let $z = |z|e^{\,i\alpha}$ be a fixed translation and $\lambda > 0$. Define $a = \sqrt{\frac{\theta}{2}}\left(b' + b'^{\dagger}\right)$, where $b' = be^{-i\alpha}\left(e^{-\lambda b^{\dagger}b}\right)$. Then there exists a $\gamma > 0$ s.t. $a \in B$ (Lipschitz ball) for any $\lambda \le \gamma$.

The proposition was proved in [11]. Here we provide a rough sketch of the proof highlighting the essential points in Appendix (B.
Hence, from the above proposition we can construct a sequence $\{\lambda_n\}$, $\lambda_n < \gamma$ s.t. $\lim_{n\to\infty}\lambda_n = 0$. To prove (4.17) , it only remains to show that

$$\lim_{\lambda\to 0}|\omega_z(a) - \omega_0(a)| = \sqrt{2\theta}|z| \tag{4.23}$$

where $a = \sqrt{\frac{\theta}{2}}(b' + (b')^{\dagger})$; $b' = be^{-i\alpha}(e^{\,-\lambda b^{\dagger}b})$ and $|z| = ze^{i\alpha}$.

Now,

$$\omega_0(a) = 0 \quad \text{and} \quad \omega_z(a) = \sqrt{2\theta}\ |z|\exp\!\big(-|z|^2(1-e^{-\lambda})\big)$$

Therefore,

$$\lim_{\lambda\to 0}|\omega_z(a) - \omega_0(a)| = \lim_{\lambda\to 0}\ \sqrt{2\theta}\ |z|\exp\!\big(-|z|^2(1-e^{-\lambda})\big) = \sqrt{2\theta}\ |z| \tag{4.24}$$

Now it follows from the translational invariance of the Connes distance as shown below:

$$d(\omega_z, \omega_{z'}) = \sqrt{2\theta}\Big|z - z^{'}\Big| \tag{4.25}$$

let U be a unitary transformation acting on $\mathcal{H}_c$, the map $\alpha_u : \mathcal{A} \to \mathcal{A}$ s.t $\alpha_u(a) = UaU^{\dagger}$ is an automorphism. Therefore the Connes distance(4.2)

$$\begin{aligned}
d(\omega_{U|z\rangle}, \omega_{U|z'\rangle}) &= sup_{a\in\mathcal{A}}\left\{\Big|\omega_{U|z\rangle}(a) - \omega_{U|z'\rangle}(a)\Big| \ :\ ||[D,\pi(a)]||_{op} \le 1\right\} \\
from(4.4) &= sup_{a\in\mathcal{A}}\left\{\Big|\omega_{|z\rangle}(UaU^{\dagger}) - \omega_{|z'\rangle}(UaU^{\dagger})\Big| \ :\ ||\big[D,\pi(UaU^{\dagger})\big]||_{op} \le 1\right\} \\
&= sup_{a'\in\mathcal{A}}\left\{\Big|\omega_{|z\rangle}(a^{'}) - \omega_{|z'\rangle}(a^{'})\Big| \ :\ ||\Big[D,\pi(a^{'})\Big]||_{op} \le 1\right\} \\
&= d(\omega_{|z\rangle}, \omega_{|z'\rangle})
\end{aligned}$$

Hence (4.25) follows when we take $U = U^{\dagger}(z^{'}, \bar{z}^{'}) = \exp\!\big(\bar{z}^{'}b - z^{'}b^{\dagger}\big) = \exp\!\big(-(-\bar{z}^{'})b + (-z^{'})b^{\dagger}\big) = U(-z^{'}, -\bar{z}^{'})$, s.t $d(\omega_{|z\rangle}, \omega_{|z'\rangle}) = d(\omega_{U|z\rangle}, \omega_{U|z'\rangle}) = d(\omega_{|z-z'\rangle}, \omega_{|0\rangle})$
We found the Connes distance between set of states $\{\omega_z\}$. This states corresponds to the coherent basis $|z\rangle \in \mathcal{H}_c$ and $|z\rangle\langle z| \in \mathcal{H}_q$ through which we

defined weak position measurements in sec(2.1), eq(2.22) by providing the interpretation that $z : (x_1, x_2), z = \frac{x_1+ix_2}{\sqrt{2\theta}}$ are position co-ordinates. Therefore from (4.25) we conclude that the metric on the Moyal plane is Euclidean(flat). From (4.25) is also follows that the metric(infinitesimal distance) as calculated in [15] is correct upto a numerical constant.

4.1.2 Connes Distance between Discrete states: Harmonic oscillator basis

For the discrete basis case, first we compute the distance between the states $\rho_{n+1} \equiv |n+1\rangle\langle n+1|$ and $\rho_n \equiv |n\rangle\langle n|$ (can also be termed as "infinitesimal distance") and then for states $\rho_n = |n\rangle\langle n|$ and $\rho_n = |m\rangle\langle m|, m-n \geq 2$. We proceed through the same approach adopted above. Accordingly, we first re-write the Connes distance as the difference in the expectation value of the transformed algebra element and that of itself in the same state $|n\rangle$ starting with

$$d(\omega_{n+1}, \omega_n) = \sup_{a\in B} |tr(\rho_{n+1}a) - tr(\rho_n a)| \tag{4.26}$$

we get,

$$\begin{aligned} d(\omega_{n+1}, \omega_n) &= \sup_{a\in B} \left| \langle n| \frac{b}{\sqrt{n+1}} a \frac{b^\dagger}{\sqrt{n+1}} |n\rangle - \langle n|a|n\rangle \right| \\ &= \sup_{a\in B} \frac{1}{n+1} \left| \langle n|([b,a] + ab)b^\dagger - (n+1)a|n\rangle \right| \end{aligned}$$

Which on simplification yields

$$\begin{aligned} d(\omega_{n+1}, \omega_n) &= \sup_{a\in B} \frac{1}{\sqrt{n+1}} |\langle n|[b,a]|n+1\rangle| \\ &= \sup_{a\in B} \frac{1}{\sqrt{n+1}} \left| \langle n+1|[b^\dagger, a]|n\rangle \right| \end{aligned} \tag{4.27}$$

We can now invoke Bessel's inequality

$$\|A\|_{op}^2 \geq \sum_i |A_{ij}|^2 \geq |A_{ij}|^2 \tag{4.28}$$

(written in terms of the matrix elements A_{ij} of an operator $\hat{A}$ in some orthonormal bases), to write (using 4.14)

$$\begin{aligned} d(\omega_{n+1},\omega_n) &= \sup_{a\in B}\frac{1}{\sqrt{n+1}}|\langle n|[b,a]|n+1\rangle| \\ &= \sup_{a\in B}\frac{1}{\sqrt{n+1}}\left|\langle n+1|[b^\dagger,a]|n\rangle\right| \\ &\leq \frac{1}{\sqrt{n+1}}\|[b,a]\|_{op} = \frac{1}{\sqrt{n+1}}\|[b^\dagger,a]\|_{op} \end{aligned} \tag{4.29}$$

This finally yields

$$d(\omega_{n+1},\omega_n) \leq \sqrt{\frac{\theta}{2(n+1)}} \tag{4.30}$$

Again the RHS will correspond to the required distance, provided that we can find at least one optimal element a_s s.t. the above inequality is saturated. For this, let us write, using (4.27), (4.30)

$$d(\omega_n,\omega_{n+1}) = \sup_{a\in B}\frac{1}{\sqrt{n+1}}|\langle n|\,[b,a]\,|n+1\rangle| = \sqrt{\frac{\theta}{2}}\frac{1}{\sqrt{n+1}} \tag{4.31}$$

Now expressing $a\in B$ as $a = \|a\|_{tr}\hat{a}$ in terms of the unit vector $\hat{a}\in\mathcal{A} = \mathcal{H}_q$, satisfying $\|\hat{a}\|_{tr} = 1$, we get an upper bound of $\|a\|_{tr}$, by making use of the ball condition (4.3) and the identity (4.14) as

$$\|a\|_{tr} \leq \frac{1}{\|[\mathcal{D},\pi(\hat{a})]\|_{op}} \leq \sqrt{\frac{\theta}{2}}.\frac{1}{\|[b,\hat{a}]\|_{op}} \tag{4.32}$$

Also the Connes distance (4.31) can be re-written as

$$d(\omega_n,\omega_{n+1}) = \sup_{a\in B}\frac{\|a\|_{tr}}{\sqrt{n+1}}|\langle n|\,[b,\hat{a}]\,|n+1\rangle| \tag{4.33}$$

Using the bound (4.32) of $\|a\|_{tr}$ we get, by multiplying both sides of inequality by $|\langle n|\,[b,\hat{a}]\,|n+1\rangle|$,

$$\|a\|_{tr}\,\langle n|\,[b,\hat{a}]\,|n+1\rangle \leq \sqrt{\frac{\theta}{2}}\frac{|\langle n|\,[b,\hat{a}]\,|n+1\rangle|}{\|[b,\hat{a}]\|_{op}} \leq \sqrt{\frac{\theta}{2}} \tag{4.34}$$

The second inequality follows from Bessel inequality (4.28), $\frac{|\langle n|[b,\hat{a}]|n+1\rangle|}{\|[b,\hat{a}]\|_{op}} \leq 1$. We therefore look for an optimal element $a_s = \frac{\hat{a_s}}{\||D\langle\pi(\hat{a_s})\||\|}$, so that the supremum value of LHS is $\sqrt{\frac{\theta}{2}}$ and (4.33) saturates the inequality in (4.30):by

$$\sup_{\hat{a}\in\mathcal{A}}\frac{|\langle n|\,[b,\hat{a}]\,|n+1\rangle|}{\|[b,\hat{a}]\|_{op}} = \frac{|\langle n|\,[b,\hat{a}_s]\,|n+1\rangle|}{\|[b,\hat{a}_s]\|_{op}} = 1 \tag{4.35}$$

We now decompose the unit vector $\hat{a}_s$ as,

$$\hat{a}_s = \cos\theta \hat{d\rho} + \sin\theta \hat{d\rho_\perp} \tag{4.36}$$

where $\|\hat{d\rho}\|_{tr} = \|\hat{d\rho_\perp}\|_{tr} = 1$ and $\hat{d\rho_\perp} \in W = \{a \in \mathcal{A} : (d\rho, a) = 0\}$ is taken to be a unit vector orthogonal to $\hat{d\rho}$ and belong to the plane formed by $\hat{a}$ and $\hat{d\rho}$. Substituting in (4.35) yields,

$$\left| \langle n| \left[b, \cos\theta \hat{d\rho} + \sin\theta \hat{d\rho_\perp}\right] |n+1\rangle \right| = \|[b, \cos\theta \hat{d\rho} + \sin\theta \hat{d\rho_\perp}]\|_{op}$$

Now using $\hat{d\rho} = \frac{1}{\sqrt{2}} d\rho = \frac{1}{\sqrt{2}}(|n+1\rangle\langle n+1| - |n\rangle\langle n|)$ and $\langle n| [b, \hat{d\rho}] |n+1\rangle = \sqrt{2(n+1)}$, one gets

$$\left| \cos\theta \sqrt{2(n+1)} + \sin\theta \frac{\sqrt{(n+1)}}{\|d\rho_\perp\|_{tr}} \left((d\rho_\perp)_{n+1,n+1} - (d\rho_\perp)_{n,n}\right) \right|$$
$$= \|\cos\theta[b, \hat{d\rho}] + \sin\theta[b, \hat{d\rho_\perp}]\|_{op}$$

But since $tr(d\rho, d\rho_\perp) = 0 \implies (d\rho_\perp)_{n+1,n+1} - (d\rho_\perp)_{n,n} = 0$ this further simplifies on dividing by $\cos\theta \neq 0$ to get

$$\|[b, d\rho] + \tan\theta \frac{\|d\rho\|_{tr}}{\|d\rho_\perp\|_{tr}} [b, d\rho_\perp]\|_{op} = 2\sqrt{n+1} \tag{4.37}$$

Therefore the supremum in (4.31) is attained by $a_s \in B$ which belong to the following set as follows from (4.35)

$$a_s = a_s(\theta, d\rho_\perp) \in A_s$$
$$A_s = \left\{ a_s = \frac{a}{||[D, \pi(a)]||_{op}} :: \hat{a} \in \mathcal{A} \;\; s.t \; ||[b, \;\; d\rho] + \tan\theta \; \frac{||d\rho||_{tr}}{||d\rho_\perp||_{tr}} \; [b, \; d\rho_\perp]||_{op} = 2\sqrt{n+1} \right\} \tag{4.38}$$

where $\hat{a} = \cos\theta \; \hat{d\rho} + \sin\theta \; \hat{d\rho_\perp} \in \mathcal{A}$

Now using triangle inequality in eq(4.37) we get,

$$||[b, \; d\rho] + \tan\theta \, \tfrac{||d\rho||_{tr}}{||d\rho_\perp||_{tr}} \, [b, \; d\rho_\perp]||_{op} \leq ||[b, \; d\rho]||_{op} + |\tan\theta| \, \tfrac{||d\rho||_{tr}}{||d\rho_\perp||_{tr}} || \, [b, \; d\rho_\perp]||_{op}$$

Since $\sqrt{\frac{\theta}{2}}||[D,\ \pi(d\rho)]||_{op} = ||[b,\ d\rho]||_{op} = \ 2\sqrt{n+1}$

$\Rightarrow \ ||[b,\ d\rho]{+}\tan\theta\ \frac{||d\rho||_{tr}}{||d\rho_{\perp}||_{tr}}\ [b,\ d\rho_{\perp}]||_{op} \leq 2\sqrt{n+1}{+}|\tan\theta|\ \frac{||d\rho||_{tr}}{||d\rho_{\perp}||_{tr}}||\ [b,\ d\rho_{\perp}]||_{op}$

It can be easliy seen that one way to attain equality in the above relation is for $\theta = 0$ for which the corresponding optimal element is identified as $a_s = \frac{d\rho}{||[D,\pi(d\rho)]||_{op}} \in A_s$, in agreement with the result of [15] [10], indicating that $a_s \propto d\rho$ for this infinitesimal case. Therefore, for any element $a_s \in A_s$ saturates the inequality in (4.30) one of which is $a_s = \frac{d\rho}{||[D,\pi(d\rho)]||_{op}}$ and we get the "infinitesimal distance " for harmonic oscillator basis to be

$$d(\omega_{n+1},\omega_n) = \sqrt{\frac{\theta}{2(n+1)}} \tag{4.39}$$

Same result as (4.39) was obtained by using the algorithm in [15], therefore the algorithm gives exact infinitesimal distance for discrete states.

We now find distance between finitely separated discrete “harmonic oscillator" states $|n\rangle$ and $|m\rangle$ in the Moyal plane

For the finite case, to compute the distance between $\rho_n \equiv |n\rangle\langle n|$ and $\rho_m \equiv |m\rangle\langle m|$ with the difference between the two integers m and n being $|m-n| \geq 2$. We start by writing,

$$\begin{aligned}
d(\omega_m,\omega_n) &= \sup_{a\in B} |tr(\rho_{n+k}a) - tr(\rho_n a)| \quad ; \quad \text{where } k = m-n \\
&= \sup_{a\in B} \left|tr(\rho_{n+k} - \rho_{n+(k-1)} + \rho_{n+(k-1)} - \rho_{n+(k-2)} \cdots + \rho_{n+1} - \rho_n, a)\right| \\
&= \sup_{a\in B} \left|tr\left(\sum_{i=1}^{k}(\rho_{n+i} - \rho_{n+(i-1)}), a\right)\right| \\
&= \sup_{a\in B} \left|\sum_{i=1}^{k} tr\left((\rho_{n+i} - \rho_{n+(i-1)}), a\right)\right|
\end{aligned}$$

As shown in the infinitesimal case (4.27),

$$tr\left(\left(\rho_{n+i} - \rho_{n+(i-1)}\right)a\right) = \frac{1}{\sqrt{n+i}}\langle n+(i-1)|[b,a]|n+i\rangle$$

Therefore, proceeding forward as in the infinitesimal case,

$$\begin{aligned} d(\omega_m,\omega_n) &= \sup_{a\in B}\left|\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}}\langle n+(i-1)|[b,a]|n+i\rangle\right| \\ &\le \sup_{a\in B}\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}}|\langle n+(i-1)|[b,a]|n+i\rangle| \\ &\le \sqrt{\frac{\theta}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}} \end{aligned} \tag{4.40}$$

by using eq (4.28) and (4.14).

Now to find an supremum element $a_s \in B$ for which the above inequality is saturated, we demand

$$\left|\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}}|\langle n+(i-1)|[b,a_s]|n+i\rangle|\right| = \sqrt{\frac{\theta}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}}$$

Equivalently,

$$\begin{aligned} \left|\sum_{i=1}^{k}(a_s)_{n+i,n+i}-(a_s)_{n+(i-1),n+(i-1)}\right| &= |(a_s)_{n+k,n+k}-(a_s)_{n,n}| \\ &= \sqrt{\frac{\theta}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}} \end{aligned}$$

Now if we let $(a_s)_{n+k,n+k} = 0$ it implies $|(a_s)_{n,n}| = \sqrt{\frac{\theta}{2}}\sum_{i=1}^{k}\frac{1}{\sqrt{n+i}}$ constructing such a_s so that $a_s \in B$ we get,

$$a_s = \sum_{p=n}^{m-1}\left(\sqrt{\frac{\theta}{2}}\sum_{i=1}^{m-p}\frac{1}{\sqrt{p+i}}\,|p\rangle\langle p|\right) \tag{4.41}$$

where $m = n+k$.

Which gives us,

$$d(\omega_m,\omega_n) = \sqrt{\frac{\theta}{2}}\sum_{i=1}^{m-n}\frac{1}{\sqrt{n+i}} \tag{4.42}$$

By above equation it is also seen that for “harmonic oscillator" basis,

$$d(\omega_m,\omega_n) = d(\omega_m,\omega_l)+d(\omega_l,\omega_n) \quad \text{for} n \le l \le m \tag{4.43}$$

Finally note that a_s (4.41) is no longer proportional to $\Delta\rho = \rho_m - \rho_n = |m\rangle\langle m| - |n\rangle\langle n|$.

4.2 Fuzzy sphere

In this section, we calculate the Connes distance between the discrete states $\rho_{|n_3\rangle}$ and coherent states ρ_z for fuzzy sphere indexed by n. For Discrete state, the analysis is similar to the case of Moyal plane with some complexity arising from the fact that the commutator of the raising and lowering operator is not identity but J_3. Therefore, we first calculate the Connes distance for discrete basis providing only important steps in the analysis.

4.2.1 Connes distance on discrete state basis

The spectral triple corresponding to particular fuzzy sphere indexed by n as shown in sec(3.2.2) is.

$$\mathcal{A} = H_q^n = span\left\{\Big|n, n_3\Big\rangle\Big\langle n, n_3^{'}\Big|\right\} : \ \mathcal{H} = \mathcal{H}_c^n \otimes \mathbb{C}^2 : \ D = \frac{1}{r}\begin{pmatrix} J_3 & J_- \\ J_+ & -J_3 \end{pmatrix} \tag{4.44}$$

where $\mathcal{H}_c^n = span\left\{|n, n_3\rangle : -n \leq n_3 \leq n\right\}$. From here on since the index n is fixed we omit its reference while writing the states $|n, n_3\rangle = |n_3\rangle$

We first compute the distance between the states $\rho_{n_3+1} \equiv |n_3+1\rangle\langle n_3+1|$ and $\rho_{n_3} \equiv |n_3\rangle\langle n_3|$ ("infinitesimal distance") again starting with

$$d(\omega_{n+1}, \omega_n) = \sup_{a\in B} |tr(\rho_{n+1}a) - tr(\rho_n a)| \tag{4.45}$$

we get,

$$\begin{aligned} d(\omega_{n+1}, \omega_n) &= \sup_{a\in B}\left|\langle n_3|\frac{J_- a J_+}{\sqrt{n(n+1) - n_3(n_3+1)}}|n_3\rangle - \langle n_3|a|n_3\rangle\right| \\ &= \sup_{a\in B}\frac{1}{\sqrt{n(n+1) - n_3(n_3+1)}}|\langle n_3|[J_-, a]|n_3+1\rangle| \end{aligned}$$

We now invoke Bessel's inequality(4.28)

$$d(\omega_{n+1}, \omega_n) \leq \frac{\|[J_-, a]\|_{op}}{\sqrt{n(n+1) - n_3(n_3+1)}} = \frac{\|[J_+, a]\|_{op}}{\sqrt{n(n+1) - n_3(n_3+1)}} \tag{4.46}$$

Using the following condition as proved in appendix (A), for $a \in B$

$$\|[J_-, a]\|_{op} = \|[J_+, a]\|_{op} \leq r : \quad \text{where } r = \lambda\sqrt{n(n+1)} \tag{4.47}$$

This finally yields

$$d(\omega_{n_3+1},\omega_{n_3}) \leq \frac{r}{\sqrt{n(n+1)-n_3(n_3+1)}} \tag{4.48}$$

Here we don't construct the set of supremum element which can be done similar to the Moyal plane case but provide two supremum elements which saturates the above inequality $a_s = \frac{r}{\sqrt{n(n+1)-n_3(n_3+1)}} |n_3+1\rangle\langle n_3+1|$ and the other one as also shown in [16] $a_s = \frac{d\rho}{||[D\ d\rho]||_{op}}$ where $d\rho = |n_3+1\rangle\langle n_3+1| - |n_3\rangle\langle n_3|$. Therefore the "infinitesimal distance " is

$$d(\omega_{n_3+1},\omega_{n_3}) = \frac{\lambda\sqrt{(n(n+1))}}{\sqrt{n(n+1)-n_3(n_3+1)}} \tag{4.49}$$

In the case of fuzzy sphere also the algorithm in [15], gives the exact infinitesimal distance for discrete which was calculated in [16].

For the finite case, to compute the distance between $\rho_{n_3} \equiv |n_3\rangle\langle n_3|$ and $\rho_{m_3} \equiv |m_3\rangle\langle m_3|$ with $|m_3-n_3| \geq 2$ and $-n \leq m_3, n_3 \leq n$. As in the Moyal plane case we get

$$\begin{aligned}
d(\omega_{m_3},\omega_{n_3}) &= \sup_{a\in B} |tr(\rho_{n_3+k}a) - tr(\rho_{n_3}a)| \quad ; \quad \text{where } k = m_3 - n_3 \\
&= \sup_{a\in B} \left| \sum_{i=1}^{k} tr\left((\rho_{n_3+i} - \rho_{n_3+(i-1)}), a\right) \right| \\
&\leq \sup_{a\in B} \sum_{i=1}^{k} \frac{|\langle n_3+(i-1)|[J_-,a]|n_3+i\rangle|}{\sqrt{n(n+1)-(n_3+i)(n_3+i-1)}} \\
&\leq \sum_{i=1}^{k} \frac{r}{\sqrt{n(n+1)-(n_3+i)(n_3+i-1)}}
\end{aligned}$$

Now to find an supremum element $a_s \in B$ for which the above inequality is saturated, similar to Moyal plane we get

$$a_s = \sum_{p=n_3}^{m_3-1} \left(\sum_{i=1}^{m_3-p} \frac{r}{\sqrt{n(n+1)-(p+i)(p+i-1)}} |p\rangle\langle p| \right) \tag{4.50}$$

Which gives us,

$$d(\omega_{m_3},\omega_{n_3}) = \sum_{i=1}^{k} \frac{r}{\sqrt{n(n+1)-(n_3+i)(n_3+i-1)}} \tag{4.51}$$

Again by above equation it is also seen that for discrete basis,

$$d(\omega_{m_3},\omega_{n_3}) = d(\omega_{m_3},\omega_{l_3}) + d(\omega_{l_3},\omega_{n_3}) \quad \text{for } n_3 \leq l_3 \leq m_3 \tag{4.52}$$

4.2.2 Connes distance on coherent states

In [16], the Connes distance for infinitesimally separated coherent states ρ_z and ρ_{z+dz} was calculated using the algorithm in [[15]]. This turns out to be the geodesic distance up to a numerical constant between the two point on the sphere S^2 whose stereographic projected coordinates are z and $z + dz$. From this motivation we hypothesize that the Connes distance between finitely separated coherent states will be the geodesic distance on sphere S^2. But in [[27]], Connes distance between coherent states was calculated for a similar spectral triple on fuzzy sphere corresponding to the spectral triple as discussed here. There it was proved that the distance on the set of coherent states is same as the geodesic distance on S^2 only in the limit $n \to \infty$. We therefore first show that for any fuzzy sphere indexed by n, the Connes distance is bounded above by the geodesic distance on the sphere. Then we go on to show that for any finite n the the Connes distance is not equal to the geodesic distance on S^2 and calculate the distance exactly for $n = 1/2$ case.

The Perelomov coherent state as constructed in sec(2.2) is (2.38)

$$|z\rangle = \exp(-\bar{\alpha}J_- + \alpha J_+)\,|n\rangle = U_F(z,\bar{z})\,|n\rangle \tag{4.53}$$

where $\alpha = -\frac{\theta}{2}e^{-i\phi} = -(\tan^{-1}|z|)\,e^{-i\phi}$ and for simplicity we write $|n, n_3\rangle = |n_3\rangle$

Now the state $\omega_z(a)$ can be simplified as

$$\begin{aligned}\omega_z(a) &= tr(\rho_z a) = tr\left(U_F(z,\bar{z})\,|n\rangle\langle n|\,U_F^\dagger(z,\bar{z})a\right)\\ &= \langle n|\,(U_F^\dagger(z,\bar{z})aU_F(z,\bar{z}))\,|n\rangle\end{aligned} \tag{4.54}$$

First, we show that the Connes spectral distance is bounded above by geodesic distance on the sphere. From above the Connes distance becomes

$$d(\omega_z,\,\omega_0) = \sup_{a\varepsilon B}\left|\langle n|\,(U_F^\dagger(z,\bar{z})aU_F(z,\bar{z}))\,|n\rangle - \langle n|\,a\,|n\rangle\right| \tag{4.55}$$

Let $W(t) = \omega_{zt}(a) = tr(\rho_{zt}a)$, with $t \in [0,1]$ being a real parameter and $a = a^\dagger$. Therefore we get a upper bound

$$|\omega(a) - \omega'(a)| = \left|\int_0^1 \frac{\mathrm{d}W(t)}{\mathrm{d}t}dt\right| \leq \int_0^1 \left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right|dt \tag{4.56}$$

Now $\frac{\mathrm{d}W(t)}{\mathrm{d}t} = \frac{\mathrm{d}\omega_{zt}}{\mathrm{d}t} = \frac{\mathrm{d}\langle n|(U_F^\dagger(zt,\bar{z}t)aU_F(zt,\bar{z}t))|n\rangle}{\mathrm{d}t}$

By Hadamard identity :

$$
\begin{aligned}
(U_F^\dagger(z,\bar{z})aU_F(z,\bar{z})) &= \exp(G)a\exp(-G) \\
&= a + [G,a] + \frac{1}{2!}[G,[G,a]] + \frac{1}{3!}[G,[G,[G,a]]] + \dots\dots\dots
\end{aligned}
\tag{4.57}
$$

where $G = |\alpha|\left(e^{-i\phi}J_+ - e^{-i\phi}J_-\right) \quad |\alpha| = \tan^{-1}|z|$

$$
\left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| = \left(\frac{|z|dt}{1+|z|^2t^2}\right)\left|\omega_{zt}\left(\frac{[G,a]}{|\alpha|}\right)\right| \tag{4.58}
$$

Now, since(see appendix (A))(A.7)

$$
\frac{1}{r}||[J_+,a]||_{op} \leq ||[D,\pi(a)]||_{op} \; and \; \frac{1}{r}||[J_-,a]||_{op} \leq ||[D_F,\pi(a)]||_{op}
$$

for $a \in B$ this implies $||[J_+,a]||_{op} \leq r$ and $||[J_-,a]||_{op} \leq r$

where $r = \lambda\sqrt{n(n+1)}$ and $D = \frac{1}{r}\begin{pmatrix} J_3 & J_- \\ J_+ & -J_3 \end{pmatrix}$. Therefore by using Cauchy-Schwartz inequality we get

$$
\begin{aligned}
\left|\omega_{zt}\left(\frac{[G,a]}{|\alpha|}\right)\right| &= \left|e^{-i\phi}\omega_{zt}([J_+,a]) - e^{i\phi}\omega_{zt}([J_-,a])\right| \\
&\leq \sqrt{2}\sqrt{|\omega_{zt}([J_+,a])|^2 + |\omega_{zt}([J_-,a])|^2} \\
&\leq \sqrt{2}\sqrt{||[J_+,a]||_{op}^2 + ||[J_-,a]||_{op}^2} \\
&\leq 2r
\end{aligned}
$$

Thus we get from (4.56) and (4.58) ,

$$
\begin{aligned}
|\omega_z(a) - \omega_0(a)| \leq \int_0^1 \left|\frac{\mathrm{d}W(t)}{\mathrm{d}t}\right| dt &= \int_0^1 \left(\frac{|z|dt}{1+|z|^2t^2}\right)\left|\omega_{zt}\left(\frac{[G,a]}{|\alpha|}\right)\right| \\
\leq (2r)\int_0^1 \left(\frac{|z|dt}{1+|z|^2t^2}\right) &= 2r\tan^{-1}|z| = 2r|\alpha|
\end{aligned}
\tag{4.59}
$$

Therefore from (4.59) we get that the Connes distance is bounded above by the geodesic distance on the sphere

$$
d(\omega_z,\ \omega_0) = \sup_{a\varepsilon B}|\omega_z(a) - \omega_0(a)| \leq 2r\tan^{-1}|z| = r\theta \tag{4.60}
$$

Now to prove that the Connes distance on coherent states is equal to corresponding geodesic distance on the sphere we have to find a supremum element $a_s \in B$ s.t $|\omega_z(a_s) - \omega_n(a_s)| = r\theta$. But by the following simple analysis it can be seen that Connes distance depends on n and therefore for each n we get different metric on the coherent states not equal to the geodesic distance on sphere.

We know that the Connes distance between the discrete basis states $|i\rangle$ and $|j\rangle$: $-n \leq i, j \leq n$ is (4.51) and(4.52)

$$\begin{aligned} d(\omega_{|i\rangle}, \omega_{|j\rangle}) &= \sum_{k=i+1}^{j1} d(\omega_{|k-1\rangle}, \omega_{|k\rangle}) \\ \text{where } \; d(\omega_{|k-1\rangle}, \omega_{|k\rangle}) &= \frac{r}{\sqrt{n(n+1) - (k)(k-1)}} \end{aligned} \tag{4.61}$$

Thus the Connes distance between north pole $|n\rangle$ and south pole$|-n\rangle$ becomes

$$d(\omega_{|n\rangle}, \omega_{|-n\rangle}) = \sum_{k=-n+1}^{n} d(\omega_{|k-1\rangle}, \omega_{|k\rangle}) = \sum_{k=1}^{2n} \frac{r}{\sqrt{k(2n+1-k)}} \tag{4.62}$$

Now north pole $|n\rangle$ and south pole$|-n\rangle$ are also coherent states i.e $z = 0$ and $z \to \infty$ respectively. Hence from eq(4.62) it is evident that Connes distance between $|n\rangle$ and $|-n\rangle$ is not equal to $r\pi$ as expected.
Consider the following

$$\begin{aligned} \sum_{k=1}^{2n} \frac{r}{\sqrt{k(2n+1-k)}} &= \sum_{k=1}^{2n} \frac{r(1/n)}{\sqrt{\frac{k}{n}(2 + \frac{1}{n} - \frac{k}{n})}} \\ for \xrightarrow{n \to \infty} : &= \int_0^2 \frac{r\ dx}{\sqrt{x(2-x)}} \\ &= 2\int_0^1 \frac{r\ dt}{\sqrt{(1-t^2)}} \quad \text{(substituting } x = 1-t\text{)} \\ &= r\pi \end{aligned} \tag{4.63}$$

Therefore, for $n \to \infty$ the distance between north pole and south pole is the geodesic distance.

We now explicitly calculate the Connes distance between coherent states for the $n = 1/2$ case:

For $n = 1/2$ any general hermitian $a = a^\dagger$ can be written as

$$a = \begin{pmatrix} a_{11} & \gamma \\ \bar{\gamma} & a_{22} \end{pmatrix} \tag{4.64}$$

The Connes distance $eq(4.55)$ becomes

$$\begin{aligned} d(\omega_z,\ \omega_0) &= \sup_{a\varepsilon B} \left| \langle n| \left(U_F^\dagger(z,\bar{z}) a U_F(z,\bar{z})\right) |n\rangle - \langle n| a |n\rangle \right| \\ &= \sup_{|\gamma|,\beta} \ \sin|\alpha| \left(2|\gamma| \cos|\alpha| + \beta \sin|\alpha|\right) \end{aligned} \tag{4.65}$$

where $\beta = a_{22} - a_{11}$ and since it can be calculated

$$\begin{aligned} ||[D, \pi(a)]||_{op} &= \frac{1}{r}\sqrt{4|\gamma|^2 + \beta^2}, \\ \text{for } \ a \in B \text{ it impies that } &\sqrt{4|\gamma|^2 + \beta^2} \leq r \end{aligned} \tag{4.66}$$

Hence, maximizing (4.65) w.r.t $|\gamma|$ by using (4.66) we get for $n = 1/2$

$$d(\omega_z,\ \omega_0) = r\sin|\alpha| = r\sin\frac{\theta}{2} \tag{4.67}$$

The corresponding values of $|\gamma|$ and β which gives the supremum are $|\gamma| = \frac{r}{2}\cos|\alpha|$ and $\beta = r\sin|\alpha|$ and hence the supremum element is

$$a_s = \frac{r\cos|\alpha|}{2}\left(e^{-i\phi}J_+ + e^{i\phi}J_-\right) - \frac{r\sin|\alpha|}{2}J_3 = \frac{d\rho}{||[D, \pi(d\rho)]||_{op}} \tag{4.68}$$

where $d\rho = |z\rangle\langle z| - |n\rangle\langle n|$.

In conclusion, we find that for each n the Connes distance will give a different metric for the coherent states. For $n = 1/2$ case, the $\mathcal{H}_c$ is the span of two basis vectors $\left|n = \pm\frac{1}{2}\right\rangle$, therefore the corresponding fuzzy sphere space represent highest degree of noncommutativity(fuzziness). As we go to higher representations n, the no of basis vectors($\mathcal{H}_c$ is $2n+1$ dimensional) increases. For the limit $n \to \infty$ s.t $r_n = \lambda\sqrt{n(n+1)} \to 1$ we recover the commutative sphere S^2 [27] and the Connes distance between coherent states is the geodesic distance on S^2, where z represents the stereographic projected variable. From (4.67) it can be seen that for $n = 1/2$ the infinitesimal Connes distance is $r\frac{d\theta}{2}$ which implies that locally the metric on coherent states is the metric of sphere. Next we investigate whether this feature is retained for each n by calculating the Connes distance between two infinitesimally separated coherent states.

To find the Connes distance between two infinitesimally separated coherent states $|z + dz\rangle$ and $|z\rangle$

$$d(\omega_{z+dz},\ \omega_z) = \sup_{a\varepsilon B} |\omega_{z+dz}(a) - \omega_z(a)| \tag{4.69}$$

Now since $\omega_z(a) : \mathbb{R}^2 \rightarrow \mathbb{R}$ for $a = a^\dagger$, we have

$$\omega_{z+dz}(a) = \omega_z(a) + dz \left(\frac{d\omega_z}{dz}\right) \tag{4.70}$$

By using Hadamard identity, we get

$$\frac{d\omega_z}{dz} = \frac{1}{1+|z|^2} \;\; \omega_z \left(\frac{[G,a]}{|\alpha|}\right) \tag{4.71}$$

Therefore, the Connes distance $eq(4.69)$ becomes

$$d(\omega_{z+dz},\ \omega_z) = \frac{|dz|}{1+|z|^2} \;\; \left(\sup_{a\varepsilon B} \left|\omega_z \left(\frac{[G,a]}{|\alpha|}\right)\right|\right) \tag{4.72}$$

Since the distance is SU(2)-invariant (can be proved similarly as done in Moyal plane case) thus $d(\omega_{z+dz},\ \omega_z) = d(\omega_{dz},\ \omega_0)$ i.e distance between any two coherent states is equal to distance between north pole $|n\rangle$ and the corresponding state. This shows as following that for each the Connes distance between coherent states locally has the metric of sphere up to a numerical constant. (4.72) becomes

$$d(\omega_{z+dz},\ \omega_z) = d(\omega_{dz},\ \omega_0) = \frac{|dz|}{1+|z|^2} \;\; \left(\sup_{a\varepsilon B} \left|\omega_0 \left(\frac{[G,a]}{|\alpha|}\right)\right|\right) \tag{4.73}$$

Now, $\omega_0 \left(\frac{[G,a]}{|\alpha|}\right) = \langle n| \left[e^{-i\phi}J_+ - e^{-i\phi}J_-, a\right] |n\rangle = \sqrt{2n}\left(e^{-i\phi}a_{n-1,n} - e^{-i\phi}a_{n,n-1}\right)$ for $a = \sum_{i,j} a_{i,j}\, |i\rangle\langle j|$

Since $a = a^\dagger$, let $a_{n-1,n} = a^\dagger_{n,n-1} = \sigma = |\sigma|e^{i\psi}$ and substituting this in $eq(4.73)$ we get

$$d(\omega_{z+dz},\ \omega_z) = d(\omega_{dz},\ \omega_0) = \frac{|dz|}{1+|z|^2} \;\; \left(\sup_{a\varepsilon B}\ \sqrt{2n}\ |\sigma|\ 2|\cos(\phi-\psi)|\right) \tag{4.74}$$

It remains to show that $|\sigma|$ is bounded. From Bessel's inequality it can be proved that, $||A||^2_{op} \geq \sum_i |A_{ij}|^2 \geq |A_{ij}|^2$ and we know $||A||_{op} \leq ||A||_{tr} \leq \sqrt{d}||A||_{op}$, d is the dimension of the matrix A. Therefore since $a \in \mathcal{A}$ is trace class operator it implies $|\sigma| \leq \infty$.The value of σ will be fixed by the Lipschitz Ball condition and will be dependent on n and r. By letting $\psi = \phi$ and $|\sigma| = rf(n)$ in $eq(4.74)$, we get

$$d(\omega_{z+dz},\ \omega_z) = d(\omega_{dz},\ \omega_0) = \frac{|dz|}{1+|z|^2} \;\; \left(\sqrt{2n}\ 2r\ f(n)\ \right) = r\ d\theta \left(\sqrt{2n}f(n)\right) \tag{4.75}$$

Hence for each n representation , the Connes distance distance for coherent states locally retains the metric of sphere upto a n dependent constant. This same result as mentioned earlier was also obtained in [16] by applying the algorithm in [15]

4.3 Connes Distance Function

In this section we take the issue with the algorithm developed in [15]. As mentioned earlier in previous sections, in [15], a algorithm was developed to compute Connes distance function for noncommutative spaces and Connes distance between infinitesimally separated states for coherent state basis and discrete state basis was calculated for Moyal plane. It turns out that the algorithm works only for computing infinitesimal distances for discrete states and upto a overall constant for the case of coherent states, which can be seen by comparing the results in the previous section with the results in [15] for Moyal plane and [16] for fuzzy sphere. Thus, we modify the algorithm such that finite distance can also be calculated.

Let $(\mathcal{A} = \mathcal{H}_q : \mathcal{H} = \mathcal{H}_c \otimes \mathbb{C}^2 : \mathcal{D})$ be a general spectral triple. The Connes spectral distance between two states (3.1) is defined by

$$d(\omega, \omega') = \sup_{a \in B} |\omega(a) - \omega'(a)| \tag{4.76}$$

$$B = \{a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\|_{op} \leq 1\} \tag{4.77}$$

now as in [[15]], we consider the states which satisfy the following conditions :

- The states ω, ω' are normal states(4.1),which implies for each state ω there exist a density operator ρ_w s.t

$$\omega(a) = tr(\rho_\omega a) \tag{4.78}$$

- The states ω and ω' are separately bounded on B, i.e. $\omega(a) < \infty$ and $\omega'(a) < \infty, \forall\ a \in B$.
- Let $V_0 = \{a \in \mathcal{A} : \|[\mathcal{D}, \pi(a)]\|_{op} = 0\}$, then the states ω, ω' are such that $\omega(a) - \omega'(a) = 0$, $\forall\ a \in V_0$.

and writing $a = \|a\|_{tr}\hat{a}$ in terms of the "unit vector" $\hat{a}$ satisfying $\|\hat{a}\|_{tr} = 1$,the Connes distance eq(4.76) becomes

$$d(\omega, \omega') = \sup_{a \varepsilon B'} |tr_c(d\rho, a)| = \sup_{a \varepsilon B'} |(d\rho, a)| = \sup_{a \varepsilon B'} \|a\|_{tr} |(d\rho, \hat{a})| \tag{4.79}$$

where $d\rho = \rho_\omega - \rho_{\omega'}$, $(A, B) = tr_c(A^\dagger, B)$ is the inner product defined on $\mathcal{H}_q$ and since $d\rho \in \mathcal{H}_q$(as discussed at start of the chapter)

$$(d\rho, a) = tr_c(d\rho^\dagger a) = tr_c(d\rho \ a) \tag{4.80}$$

$B^{'}$ is defined as follows

$$V_o^\perp = \{a \in \mathcal{A} : \| \, [\mathcal{D}, \pi(a)] \, \|_{op} \neq 0\} \tag{4.81}$$

$$B^{'} = \{a \in V_o^\perp : \| \, [\mathcal{D}, \pi(a)] \, \|_{op} \leq 1\} \tag{4.82}$$

$$W = \{a \in \mathcal{A} : (d\rho, a) = 0\} \tag{4.83}$$

Now an element $a \in B'$ is such that $\| \, [\mathcal{D}, \pi(a)] \, \|_{op} \leq 1$ i.e $||a||_{tr} \| \, [\mathcal{D}, \pi(\hat{a})] \, \|_{op} \leq 1$ where, $\hat{a} \in \mathcal{A}$. Therefore $||a||_{tr}$ is bounded above by

$$\|a\|_{tr} \leq \frac{1}{\| \, [\mathcal{D}, \pi(\hat{a})] \, \|_{op}} \tag{4.84}$$

Thus, for a particular $\hat{a} \in \mathcal{A}$ we take in eq(4.79) $||a||_{tr} = \frac{1}{\|[\mathcal{D}, \pi(\hat{a})]\|_{op}}$.Therefore we get,

$$d(\omega, \omega') = \sup_{\hat{a} \in \mathcal{A}} \frac{|(d\rho, \hat{a})|}{\| \, [\mathcal{D}, \pi(\hat{a})] \, \|_{op}} = \sup_{a \in \mathcal{A}} \frac{|(d\rho, a)|}{\| \, [\mathcal{D}, \pi(a)] \, \|_{op}} \tag{4.85}$$

The eq(4.85) means that, for two states which satisfies the above given three condition, the Connes distance can be obtained by finding the supemum of the set $\left\{ \frac{|(d\rho, \hat{a})|}{\|[\mathcal{D}, \pi(\hat{a})]\|_{op}} : \hat{a} \in \mathcal{A} \text{ and } ||\hat{a}||_{tr} = 1 \right\}$ and if the supremum is attained by an $\hat{a}_s \in \mathcal{A}$, then the supremum element $a_s \in B^{'}$ is given by

$$a_s^{'} = \frac{\hat{a}_s}{\| \, [\mathcal{D}, \pi(\hat{a}_s)] \, \|_{op}} = \frac{a_s}{\| \, [\mathcal{D}, \pi(a_s)] \, \|_{op}} \quad \text{s.t} \ \ d(\omega, \omega') = \left(d\rho, a_s^{'} \right) \tag{4.86}$$

Now by making use of the inner product structure in eq(4.85), we can further simplify the Connes distance function. For this we decompose $\hat{a}$ as $\hat{a} = \cos\theta \hat{d\rho} + \sin\theta \hat{d\rho_\perp}$, where $\|\hat{d\rho}\|_{tr} = \|\hat{d\rho_\perp}\|_{tr} = 1$ and $\hat{d\rho_\perp} \in W$, is taken to be orthogonal to $\hat{d\rho}$ and corresponds to be a unit vector in the plane formed by $\hat{a}$ and $\hat{d\rho}$. We therefore write $||[D, \pi(\hat{a})]||_{op}$ as

$$\| \, [\mathcal{D}, \pi(\hat{a})] \, \|_{op} = \| \left[\mathcal{D}, \cos\theta \pi(\hat{d\rho}) + \sin\theta \pi(\hat{d\rho_\perp}) \right] \|_{op} \tag{4.87}$$

substituting this in eq(4.85), the Connes distance becomes

$$\begin{aligned} d(\omega, \omega') &= \sup_{\hat{a} \in \mathcal{A}} \left| \frac{(d\rho, \hat{a})}{\| \, [\mathcal{D}, \pi(\hat{a})] \, \|_{op}} \right| \\ &= \sup_{\substack{d\rho_\perp \in W \\ \theta \in [0, \pi/2]}} \left(\frac{|||d\rho||_{tr} \cos\theta|}{\| \left[\mathcal{D}, \cos\theta \pi(\hat{d\rho}) + \sin\theta \pi(\hat{d\rho_\perp}) \right] \|_{op}} \right) \end{aligned}$$

Since $\cos\theta \neq 0$, we get

$$d(\omega,\omega') = N||d\rho||_{tr} \tag{4.88}$$

where N is given by

$$N = \frac{1}{\inf\limits_{\substack{d\rho_\perp \in W \\ \theta\in[0,\pi/2]}} \|[\mathcal{D},\pi(\hat{d\rho})] + \tan\theta[\mathcal{D},\pi(\hat{d\rho_\perp})]\|_{op}} \tag{4.89}$$

whereas in [15] , it was given to be

$$d(\omega,\omega') = \frac{||d\rho||_{tr}}{||\left[D,\pi(\hat{d\rho})\right]||_{op}} \tag{4.90}$$

which can be clearly seen to be a lower bound on the Connes distance (by using the triangle inequality for the denominator in(4.88) and comparing with (4.90)). We did not find the Connes distance by using (4.88) because the calculation of the factor N is nontrivial. But as shown in previous section (4.90) gives exact result for infinitesimal distance between discrete case for Moyal plane and Fuzzy sphere, while up to a overall numerical constant for infinitesimal distance between coherent states. In the following we provide a explanation for this.

Let us first consider the case of coherent states on Moyal plane. From (4.88), the Connes distance between states ω_z and ω_{z+dz} is $d(\omega_z,\omega_{z+dz}) = N||d\rho||_{tr}$ where $d\rho = dz\,|1\rangle\langle 0| + d\bar{z}\,|0\rangle\langle 1|$ and $||d\rho||_{tr} = \sqrt{2}|dz|$. Comparing this with distance calculated in sec(4.1.1), (4.25),

$$d(\omega,\omega') = N\sqrt{2}|dz| = \sqrt{2\theta}|dz|$$

we find $N = \sqrt{\theta}$ i.e N should be a constant and the infinitesimal Connes distance is given by $||d\rho||_{tr}$ up to $\sqrt{\theta}$. Therefore due to $||d\rho||_{tr}$ in (4.90) we get the correct infinitesimal distance between coherent states upto a numerical constant by (4.90), since $||\left[D,\pi(\hat{d\rho})\right]||_{op} = \sqrt{\frac{3}{\theta}}$. We now show that for infinitesimal Connes distance between coherent states N is indeed a constant. In sec(4.1.1), it was proven that the Connes distance between coherent states on Moyal plane is translationally invariant for the action of unitary operators called displacement operators $U(z,\bar{z})$ (4.4) on coherent states $|z\rangle$. Also notice that the set of coherent states is the orbit of action of this displacement operators(translational group) on $|0\rangle$. Now from (4.4)

$$d(\rho_z,\rho_{z'}) = d(U\rho_z U^\dagger, U\rho_{z'}U^\dagger) \tag{4.91}$$

From (4.91) and (4.88) we get for $N = N(\rho_z, \rho_{z'})$ that

$$\begin{aligned}
N(\rho_z, \rho_{z'}) &= N(U\rho_z U^\dagger, U\rho_{z'} U^\dagger) \quad \text{since} ||d\rho||_{tr} = ||Ud\rho U^\dagger||_{tr} \\
&= N(\rho_0,\ U(-z, -\bar{z})\rho_{z'} U(z, \bar{z})) \\
\text{(by taking } U = U^\dagger(z, \bar{z}) &= U(-z, -\bar{z}) \text{ and since } \rho_z = U(z, \bar{z})\rho_0 U(-z, -\bar{z})) \\
&= N(\rho_0, U(z^{'} - z, \bar{z}^{'} - \bar{z})\rho_0 U(z - z^{'}, \bar{z} - \bar{z}^{'}))
\end{aligned} \tag{4.92}$$

This implies for infinitesimal case

$$\lim_{z \to z^{'}} N(\rho_z, \rho_{z'}) = N(\rho_0, \rho_0) = \text{constant}$$

Notice that this analysis holds in general for any set of states on the so called homogeneous spaces, which are a orbit of action of a unitary representation of the unitary operator acting on some vector $|\psi\rangle \in \mathcal{H}_c$. Therefore the results holds for fuzzy sphere also. Hence, we find that the N is constant in (4.88) for infinitesimal Connes distance between coherent states. A natural line of further investigation will be to get finite distance by integrating the infinitesimal distance. For this we have to find what are the geodesics along which to integrate to get the finite distance. Investigation along this direction has not carried out by us yet.

For the case of discrete states. we re-write the (4.90) from (4.79)

$$d(\omega, \omega^{'}) = (d\rho, a_s) \quad \text{where } a_s = \frac{d\rho}{||[D, \pi(d\rho)]||_{op}}$$

For the "infinitesimal distance" between discrete states the above a_s was shown to be in the set of supremum elements A_s (4.38) for Moyal plane and fuzzy sphere. From (4.79) ,the Connes distance becomes the inner product of $d\rho$ with the supremum element a_s. Since the above a_s was shown to be the supremum element, (4.88) which is the inner product of this a_s with the $d\rho$ it gives the exact infinitesimal distance on discrete states for Moyal plane and fuzzy sphere.

Chapter 5

Conclusion / Results

Connes distance for finitely separated coherent states and discrete states was calculated for Moyal plane and Fuzzy sphere.

Moyal plane

We get the Connes distance between coherent states ρ_z and $\rho_{z'}$ for spectral triple in sec(3.2.1) to be (4.25)

$$d(\rho_z, \rho_{z'}) = \sqrt{2\theta}\left|z - z^{'}\right|$$

Therefore, the metric on Moyal plane is flat (via the one-to-one correspondence between coherent states and points of space).

The Connes distance for finitely separated discrete states was obtained as :(4.42) (4.43)

$$d(\omega_m, \omega_n) = \sqrt{\frac{\theta}{2}} \sum_{i=1}^{m-n} \frac{1}{\sqrt{n+i}}$$

and the triangle inequality is saturated for

$$d(\omega_m, \omega_n) = d(\omega_m, \omega_l) + d(\omega_l, \omega_n) \quad \text{for} n \leq l \leq m$$

Therefore

$$d(\omega_m, \omega_n) = \sum_{i=n+1}^{m} d(\omega_i, \omega_{i-1}) \quad where\ d(\omega_{i+1}, \omega_i) = \sqrt{\frac{\theta}{2(i+1)}}$$

Fuzzy sphere

We get that the Connes distance between coherent states for a fuzzy sphere indexed by n will be bounded above by the corresponding geodesic distance on sphere S^2 and will be equal to the geodesic distance only in the limit $n \to \infty$. Therefore the metric on the set of coherent states on each fuzzy sphere n will be different. For n=1/2 case, the Connes distance is obtained as (4.67) for $z = -\tan\frac{\theta}{2}e^{-i\phi}$

$$d(\omega_z,\ \omega_0) = r\sin\frac{\theta}{2}$$

This is half the distance of the chord connecting the two corresponding points on the sphere. The n=1/2 case has the highest degree of fuzziness and as n increases we approach towards the commutative limit i.e sphere S^2.

The Connes distance on finitely separated discrete states was obtained as:(4.51)(4.52)

$$d(\omega_{m_3}, \omega_{n_3}) = \sum_{i=1}^{k} \frac{r}{\sqrt{n(n+1) - (n_3+i)(n_3+i-1)}} \tag{5.1}$$

and the triangle inequality is saturated for

$$d(\omega_{m_3}, \omega_{n_3}) = d(\omega_{m_3}, \omega_{l_3}) + d(\omega_{l_3}, \omega_{n_3}) \quad \text{for } n_3 \leq l_3 \leq m_3 \tag{5.2}$$

Therefore,

$$d(\omega_{|i\rangle}, \omega_{|j\rangle}) = \sum_{k=i+1}^{j1} d(\omega_{|k-1\rangle}, \omega_{|k\rangle}) \text{ where } d(\omega_{|k-1\rangle}, \omega_{|k\rangle}) = \frac{r}{\sqrt{n(n+1) - (k)(k-1)}}$$

In the two noncommutative spaces above, the Connes distance for discrete states follows a nice structure - the distance between two states can be obtained by adding the "infinitesimal distances" along the discrete states in between the two discrete states between which the distance is to be calculated.

In sec(4.3), we put forward a general formula (4.88) subject to some condition , to calculate Connes distance. This was obtained by modifying the algorithm developed in [[15]].The general formula is :

$$d(\omega, \omega') = N||d\rho||_{tr} \ \ where \ \ N = \frac{1}{\inf\limits_{\substack{d\rho_\perp \in W \\ \theta\in[0,\pi/2]}} ||[\mathcal{D}, \pi(\hat{d\rho})] + \tan\theta[\mathcal{D}, \pi(\hat{d\rho_\perp})]||_{op}}$$

Here the calculation of the factor N is highly non-trivial and thats why we proceed through a alternative approach as in sec(4.1)(4.2). From this we found that the formula(4.90) for Connes distance as given in [15] is actually a lower bound on the Connes distance function. According to our analysis in chapter 4, the formula (4.90) obtained in [15] gives Connes distance (up to a overall constant numerical factor) for infinitesimally separated coherent states and exact Connes distance for infinitesimally separated discrete states. This is because in the infinitesimal calculation the factor N above becomes a overall numerical constant in the case of coherent states. Therefore, further investigation can be done to find out whether, for Connes distance between coherent states we can get the finite distances from integrating the infinitesimal distances along a geodesic. In fuzzy sphere, for the n=1/2 case we get 4.67 : $d(\omega_z,\ \omega_0) = r \sin\frac{\theta}{2}$, which corresponds to half of the chordal distance, where the two pure states ω_z and ω_0 are joined along the chord passing through the interior of the sphere where each point corresponds to a mixed state. It is evident from this that the conventional geodesic on a sphere S^2 will not give the correct finite distance and we have to consider mixed states along with pure states to find the geodesics for fuzzy sphere.

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Appendix A

Identities on $||[[D, \pi(a)]]||_{op}$

A.1 Moyal plane

The Dirac operator for Moyal plane is (3.7)

$$D_M = \sqrt{\frac{2}{\theta}} \begin{bmatrix} 0 & b^\dagger \\ b & 0 \end{bmatrix}$$

and by using the C^* algebra property of $\mathcal{B}(\mathcal{H})$: $||A||_{op}^2 = ||A^\dagger A||$, for $a = a^\dagger \in \mathcal{A}$

$$\begin{aligned}
||[D_M, \pi(a)]||_{op}^2 &= ||[D_M, \pi(a)]^\dagger [D_M, \pi(a)]||_{op} \\
&= \frac{2}{\theta} \left\| \begin{bmatrix} [b, a]^\dagger [b, a] & 0 \\ 0 & \left[b^\dagger, a\right]^\dagger \left[b^\dagger, a\right] \end{bmatrix} \right\|_{op} \\
&= \frac{2}{\theta} max \left\{ ||[b, a]^\dagger [b, a]||_{op},\ ||\left[b^\dagger, a\right]^\dagger \left[b^\dagger, a\right]||_{op} \right\} \\
&= \frac{2}{\theta} max \left\{ ||\left[b^\dagger, a\right][b, a]||_{op},\ ||[b, a]\left[b^\dagger, a\right]||_{op} \right\} \quad \text{since } a = a^\dagger \\
&= \frac{2}{\theta} ||[b, a]^\dagger [b, a]||_{op} = \frac{2}{\theta} ||\left[b^\dagger, a\right]^\dagger \left[b^\dagger, a\right]|| \\
&= \frac{2}{\theta} ||[b, a]||_{op}^2 = \frac{2}{\theta} ||\left[b^\dagger, a\right]||_{op}^2
\end{aligned} \tag{A.1}$$

We get the second last line from the property of C^* algebra $||A||_{op} = ||A^\dagger||_{op}$ which can be seen as follows:

$$||A||_{op}^2 = ||A^\dagger A||_{op} \leq ||A^\dagger||_{op} ||A||_{op} \implies ||A||_{op} \leq ||A^\dagger||_{op}$$

$$||A^\dagger||_{op}^2 = ||A A^\dagger||_{op} \leq ||A||_{op} ||A^\dagger||_{op} \implies ||A^\dagger||_{op} \leq ||A||_{op}$$

and we get $||A||_{op} = ||A^\dagger||_{op}$

Therefore

$$||[D_M, \pi(a)]||_{op} = \sqrt{\frac{2}{\theta}}||[b,a]||_{op} = \sqrt{\frac{2}{\theta}}||\left[b^\dagger, a\right]||_{op} \tag{A.2}$$

now for $a = a^\dagger \in B$ i.e $||[D_M, \pi(a)]||_{op} \leq 1$, we get

$$||[b,a]||_{op} = ||\left[b^\dagger, a\right]||_{op} \leq \sqrt{\frac{\theta}{2}} \tag{A.3}$$

A.2 Fuzzy sphere

The Dirac operator for fuzzy sphere is (3.15)

$$D_F = \frac{1}{r}\begin{pmatrix} J_3 & J_- \\ J_+ & -J_3 \end{pmatrix}$$

For $a = a^\dagger \in \mathcal{A}$

$$\begin{aligned}
||[D_F, \pi(a)]||^2_{op} &= ||[D_F, \pi(a)]_{op}[D_F, \pi(a)]||_{op} \\
&= \frac{1}{r^2}\left\|\begin{bmatrix} [J_+,a]^\dagger[J_+,a] + [J_3,a]^\dagger[J_3,a] & [[J_-,a],[J_3,a]] \\ -[[J_+,a],[J_3,a]] & [J_-,a]^\dagger[J_-,a] + [J_3,a]^\dagger[J_3,a] \end{bmatrix}\right\| \\
&\geq \frac{1}{r^2}\sup_{\substack{\psi\in\mathcal{H} \\ ||\psi||=1}} \langle\psi_1|\, [J_+,a]^\dagger[J_+,a] + [J_3,a]^\dagger[J_3,a]\,|\psi_1\rangle \quad (\psi = \begin{pmatrix} |\psi_1\rangle \\ 0 \end{pmatrix}) \\
&\geq \frac{1}{r^2}\sup_{\substack{|\psi_1\rangle\in\mathcal{H}_c \\ \langle\psi_1|\psi_1\rangle=1}} \langle\psi_1|\, [J_+,a]^\dagger[J_+,a]\,|\psi_1\rangle + \frac{1}{r^2}\sup_{\substack{|\psi_1\rangle\in\mathcal{H}_c \\ \langle\psi_1|\psi_1\rangle=1}} \langle\psi_1|\, [J_3,a]^\dagger[J_3,a]\,|\psi_1\rangle \\
&\geq ||[J_+,a]||^2_{op} + ||[J_3,a]||^2_{op}
\end{aligned} \tag{A.4}$$

Therefore we get

$$\frac{1}{r}||[J_+,a]||_{op} \leq ||[D_F, \pi(a)]||_{op} \quad \frac{1}{r}||[J_3,a]||_{op} \leq ||[D_F, \pi(a)]||_{op} \tag{A.5}$$

now for $a = a^\dagger \in \mathcal{A}, [J_+,a]^\dagger = -[J_-,a]$ and using $||A||_{op} = ||A^\dagger||_{op}$ as shown above, we get

$$\frac{1}{r}||[J_-,a]||_{op} \leq ||[D_F, \pi(a)]||_{op} \tag{A.6}$$

now for $a = a^\dagger \in B$ i.e $||[D_F, \pi(a)]||_{op} \leq 1$, we get

$$||[J_+,a]||_{op} \leq r \quad ||\left[J^-, a\right]||_{op} \leq r \tag{A.7}$$

Appendix B

Proof of proposition(3.5) in [11]

Proposition: Let $z = |z|e^{\,i\alpha}$ be a fixed translation and $\lambda > 0$. Define $a = \sqrt{\frac{\theta}{2}}(b' + b'^{\dagger})$ where $b' = be^{-i\alpha}\ (e^{\,-\lambda b^{\dagger}b})$. Then there exist a $\gamma > 0$ $s.t$ $a \in$ B(Lipschitz ball) for any $\lambda \leq \gamma$.

The proposition was proved in [[11]], we only give a rough sketch of the proof highlighting the essential points.

Proof: For $a \in \mathcal{A}$ s.t $a' \in B$ (4.3) , it has to satisty the Lipschitz ball condition,

$$||[D, \pi(a)]||_{op} = \sqrt{\frac{2}{\theta}}||[b, a]||_{op} = ||[b, (b' + b'^{\dagger})]||_{op} \leq 1 \tag{B.1}$$

Let $C = [b, (b' + b'^{\dagger})]$, we first prove the propositon for $\alpha = 0$. For C, we get as in [[11]](prop. 3.5, eq(3.21))

$$C = [b, (b' + b'^{\dagger})] = \left(e^{-\lambda} - e^{\lambda}(1 - e^{-\lambda})b^2 - (1 - e^{-\lambda})N\right) e^{-\lambda N} \tag{B.2}$$

where $N = b^{\dagger}b$ s.t $N\,|m\rangle = m\,|m\rangle$ for $|m\rangle \in \mathcal{H}_c$

In the "harmonic oscillator" basis (2.6)

$$C_{n,n} = (e^{-\lambda} - (1 - e^{-\lambda})n)e^{-\lambda n} \tag{B.3}$$

$$C_{n-2,n} = -e^{\lambda}(1 - e^{-\lambda})\sqrt{n(n-1)}e^{-\lambda n} \tag{B.4}$$

The Lipschitz ball conditon (B.1) can be proven by using Schur's test,

$$||C||_{op} \leq \left(sup_n \sum_m |C_{m,n}|\right)^{1/2} \left(sup_m \sum_n |C_{m,n}|\right)^{1/2} \tag{B.5}$$

It is shown that for λ sufficiently small

$$|C_{n-2,n}| + |C_{n,n}| \leq e^{-\lambda} \tag{B.6}$$

$$|C_{n+2,n}| + |C_{n,n}| \leq e^{\lambda} \tag{B.7}$$

now it implies by Schur's test,

$$||C||_{op} \leq (e^{-\lambda}e^{\lambda})^{1/2} = 1 \;\text{ if } \lambda \leq \gamma = \ln\left(\frac{1}{2}\sqrt{1+4e} - 1\right)$$

Now , for the case $\alpha \neq 0$, the elements $C_{n,n}$ get multiplied by $e^{-i\alpha}$ which disappears after taking the mod $|C_{n,n}|$.Hence the proof remains unchanged

Therefore, $a = \sqrt{\frac{\theta}{2}}(b' + b'^{\dagger}) \in B$ if $\lambda \leq \gamma$.