Black Rings

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Black Rings" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by "Sharvaree Vadgama" at "Indian Institute of Science, Bangalore", under the supervision of "Prof. Chethan Krishnan" during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled "Black Rings" are the results of the investigations carried out by me at Center of High Energy Physics, Indian Institute of Science, Bangalore, under the supervision of Prof. Chethan Krishnan and the same has not been submitted elsewhere for any other degree.

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Abstract

Higher dimensional Gravity is important to understand Gravity in general. As we see that the Uniqueness of Black Holes is not followed in higher dimensions as it does in four dimensions. Apart from the five dimensionsal spherical black hole solution given by Myers and Perry there is a Black ring for Einstein's equations in five dimensions.

String theory, till date the most consistent theory of Gravity suggests higher dimensions. We try to understand the solutions given by Emparan and Reall termed "Black Rings" which are objects with horizons having $S^1 \times S^2$ topology.

We try to understand the general Black ring solution and also look at the thin and thick ring solutions. Using the phase diagrams we try to show that the two solutions for five dimensions are unique. We conclude by looking at the Stability of the solutions

Contents

Chapter 1

Introduction

1.1 History

In the year 1687[1], Newton proposed this theory of Gravity which gave us some inital idea which later became one of the four fundamental forces of Nature. His theory calculates the gravitational pull to be directly proportional to the masses of the objects and inversely to the square of distance between the two. This theory was quiet extra-ordinary in the way-it applied to all massive objects. As we now this theory didn't give a complete picture as it was relevant in only the non-relativistic regime.

In the year 1905, Einstein wrote the paper "On Electrodynamics of the moving bodies" which is now what we call Special Theory of Relativity. (as it is applicable to only special case where the curvature of the earth is considered almost negligible) In that thoery he postulated that the speed of light in vacuum is same for any observer (whether stationary or moving) and all the laws of mechanics are invariant in any non-accelerating (interial) frames of reference.

Later, Einstein published a revolutionary paper in 1915 where Gravity was described not as a simple force but as a property of the space-time. Any object with Energy and momentum can bend spacetime.This came to be known as the theory of General Relativity. So the concept of only massive objects having gravitatioanl force was changed and also gravity was considered as an intrinsic property of the spacetime manifold rather as a force.
In that same paper Einstein predicted objects which we now call as "Black

[1] Although, Newton's book *Principia* was submitted to the Royal society in 1686

holes" [2]

As Wald [12] defined it "A black hole is a region of spacetime exhibiting strong gravitational effects that nothing including particles and electromagnetic radiation such as light can escape from inside it." Thus they are considered as objects with massive gravitational attraction like no other and justified the name "black"-as nothing, not even light could escape it once it crossed the event horizon of the black hole.

In the beginning they were studied in four dimensions (three space and one time) and as Hawking spherical event horizons [3] meaning, having symmetries of that of S^2 [4]

They were majorly studied in four dimensions till Myers and Perry in the year 1986 gave a solution of a Black Hole in five dimensions.These solutions had properties similar to those of standard Kerr Black Holes[5] in four dimensions,are rotating and have charge. These solutions also gave insights about event horizons and extended horizons in higher dimensions.

1.2 Work in Higher Dimensions

Roberto Emparan and Harvey S. Reall published "A rotating ring in five dimensions" in the year 2001, which gave an idea of a different kind of event horizon for a black hole in five dimensions-that of $S^2 \times S^1$ topology, a non-spherical ring-like horizon. These new solutions were termed "Black Rings".

They described that such a topology of a event horizon can exist unqiuely and thus a new set of Uniqueness theorems have to formulated for higher dimensions.[6]. In the papers they showed that this solution is diferent from that of Meyers and Perry's Black hole solution and showed that at a fixed mass (M), and an appropriate choice of area a_H and angular momentum (J), a sperical and non sperical solution to Einstein's equations is found in d=5.

[2]This term was coined by John Wheeler in the year 1967

[3]Event horizon roughly means a surface past which particles can never escape to infinity

[4]For d dimensional object, spherical symmetries mean having symmetries of that of S^{d-2}

[5]Section 4: Kerr solution

[6]One that incorporates that a few conserved charges not necessarily fixes the Black Hole solution

1.3 Outline

In the following set of chapters I would like to go through the details of the Black Rings in five dimensions. In Motivation and Background we will first try to understand the necessity of studying higher dimenisonal gravity and in Basics we will look into Laws of Black Hole Physics, differential forms and E-M duality.

In Black Hole solutions we will study the general Black Hole formulation in four dimensions- Schwarschild solution,Tangherlini solution, Kerr solution and Myers perry solution in d=5. In the next section, Black Rings, we will look into the C-metric, the derivation of Black rings and their basic properties.

In the following section Shape and Stability of Black Ring, we will understand the topology of Black ring. We also look at the instability of the Black Rings along with other higher dimensional solutions. Lastly we will conclude with challenges and futher problems with the Black Rings solutions and Higher dimensional gravity in general along.

Chapter 2

Motivation

Our world is percieved in three dimensions. So intuitively forces in nature are taken in d=3. As time was added as a dimension, it became d=4. But as we extended field theories in n dimensions, gravity was also studied in higher dimensions. Theories in 4 dimensions cannot just be directly extended to higher dimensions, as there are a few factors which have to be taken care of.

A different rotation dynamics comes into play and this affects the appearance of the extended black objects to a great extent.
With higher dimensions, there is a possibility of more independent rotation planes. The rotation group $SO(d-1)$ has Cartan subgroup $U(1)^N$ with $N = \lfloor \frac{d-1}{2} \rfloor$ hence there is a possibility of N independent angular momenta.

As number of dimensions increases, the balance between the gravitational and centrifugal potentials change.The peculiar feature about higher dimensional rotations is that Newtonian potential is dependent on dimenions while the centrifugal potential is not. So when they compete to balance dimensionality plays a crucial role. The radial fall-off of the newtonian potential which is given by

$$-\frac{GM}{r^{d-3}} \tag{2.1}$$

has a dependence on dimension d, while the centrifugal potential is considered to be on a plane and thus is always,

$$\frac{J^2}{m^2 r^2} \tag{2.2}$$

where G is Gravitational constant, J is angular momentum, r distance and m is mass.

This is not what exactly happens while solving higher dimensional Einstein's equations but it gives an intutive idea.

2.1 Some important theorems

Israel in 1967, published a paper titled "Event horizons in Static Vacuum Space times" where he looked at spacetimes in four dimensions- which solved Einstein's equations. He came up with certain conditions which needs to be satisfied by the static space times using properties of four dimensional Geometry arguments mentioned in [23].

He stated the following theorem "Israel theorem states that the only static and asymptotically flat vacuum space time posessing a regular horizon is the Schwarzchild solution This thoerem can be generalized to obtain the result which states the uniqueness of Reisnner Nordstrom black holes as the only solution for the charges black holes." [23]

Similarly he showed that Kerr solution is a unque solution for a rotating black hole in d= 4.

"Uniqueness in black holes means, the choice of all of these asymptotic black hole parameters select a unique black hole rather than a continuous set."[8]

John Wheeler conjectured that "Black Holes have No hair" . This was backed by a uniqueness theorems for Schwarzschild and Kerr Newmann solutions. Black Hole solutions are defined by a small set of parameters.

"Whether this no hair property continues to hold for higher dimensional black holes could depend on the way one chooses to generalize it. If one generalises no hair to mean that the solutions are determined in term of a small number of (not necessairly conserved)asymptotic data than it continues to hold in higher dimensions as far as we know. However, if one choose to more restrictive definition which requires conserved charges then this property fails in higher dimensions as there are objects with have non-conserved charges." [8] [For example, Rotating black rings have parameters which include non-conserved dipole charges]

2.2 Applications of studing Higher Dimensional Gravity

There are also a few applications of higher dimensions which give us more reason to study it.

1. String theory, one of the most consistent theory for quantum gravity till date requires extra dimensions. This theory has successfully calculated the microscopic counting of entropy of a black hole.
2. The production of higher dimensional black holes in future colliders becomes a concievable possibility in scenarios involving extra dimensions.[6]

Chapter 3

Important results and defintions

3.1 Black Hole Physics Laws

Barden, Cartan and Hawking gave the laws of Black Holes Mechanics [22] in the form similar to Laws of Thermodynamics They can be summarised as the following:

1. The surface gravity κ is a constant over the event horizon of a stationary Black Hole. Like the zeroth law of thermodynamics.

2. Any two neighbouring stationery axisymmetric solutions containng a perfect fluid with circular flow and a central black hole are related by

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega_H \delta J_H + \int \Omega \delta dJ + \int \mu \delta dN + \int \theta \delta dS \tag{3.1}$$

where A is area of event horizon, Ω_H is angular velocity , $J =$ angular momentum, N is S is . This is called the differential mass formula.

3. $\delta A \geq 0$
with A as the area of the horizon. This bears similarity with the second law of thermodynamics with anology between area and entropy.

4. It is not possible to reduce κ to zero by a finite sequence of operations.

Laws of Thermodynamics

1. If two systems are both in thermal equilibrium with the third system then in thermal equilibrium with each other

2. Energy cannot be created or destroyed : it can only be changed from one form to another.

3. For an isolated system, process with $\delta S \geq 0$ are possible, where S is Entropy of the system.

4. For a crystaline solid the entropy of a the state is zero at absolute zero temperature. For non-crystaline solids the entropy doesn't reach zero at absolute zero.

Here, we can see that κ plays role of Temperature and A plays role of entropy.

3.2 Differential forms

A differential p-form is a p rank tensor that is antisymmetric under exchange of any pair of indices. tensor. Thus, scalars are automatically 0-forms and liner functions are differential 1-form.

Hodge star operator

In an n-dimensional manifold, Hodge star is an operator is a map from *p* form to *(n-p)* form given by

$$_*\omega_{\mu_1\mu_2\mu_3\mu_1..\mu_{n-p}} = \frac{\sqrt{|g|}}{p!}\epsilon_{\mu_1\mu_2\mu_3\mu_1..\mu_p} g^{\mu_{n-(p+1)}\nu_1}...g^{\mu_n\nu_p} w_{\nu_1\nu_2...\nu_p} \tag{3.2}$$

It is denoted by star $*$. In the above equation, $_*\omega$ is the Hodge dual of ω.

3.3 E- M duality

Maxwell's equations in vacuum are given by [29]

$$\begin{aligned} \nabla.\vec{E} &= 0 \\ \nabla.\vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{3.3}$$

These equations are invariant when we switch Electric field and Magnetic field as

$$(\vec{E}, \vec{B}) \longrightarrow (-\vec{B}, \vec{E}) \tag{3.4}$$

We denote the field strength by $F_{\mu\nu}$

$$F^{0i} = -F^{i0} = -E^i \qquad\qquad F^{ij} = \epsilon_{ijk} B^k \tag{3.5}$$

So the Maxwell's equations can be written as

$$\partial_\nu F^{\mu\nu} = 0 \qquad\qquad \partial^*_\nu F^{\mu\nu} = 0 \tag{3.6}$$

with ${}^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$ So the E- M Duality take the F to the ${}^*F$ and ${}^*F$ to negative of F.

For Maxwell's equations with a source are as follows

$$\begin{aligned} \nabla.\vec{E} &= \frac{\rho}{\epsilon_0} \\ \nabla.\vec{B} &= 0 \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \times \vec{B} &= \mu_0 J + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{3.7}$$

As we got the relation for F and ${}^*F$ the sourceless Maxwell's equations, we get a relation with a source term for Maxwell's equations in matter.

Chapter 4

Some Black Hole solutions

4.1 Schwarzschild solution

Schwarzschild solution is spherically symmetric [1] Einstein's equation in vaccum

$$R_{\mu\nu} = 0 \tag{4.1}$$

Schwarzschild solution is as given below [13]

$$(ds)^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left((d\theta)^2 + \sin^2\theta (d\phi)^2\right) \tag{4.2}$$

with Gravitational constant G, Mass M, speed of light c, θ and ϕ being azimuthal and polar angles respectively.

We can see that the metric breaks down at $r = 0, 2M$ which are called singularities. The singularty at $r = 2M$ is a coordinate singularity, which can be removed with another coordinate system called Kruskal Szkeres coordinates[14], while the one at $r = 0$ is a curvature singularity where the metric scalars like $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ diverges.
This solution is characterised by Mass, Charge and Spin.

4.2 Tangherlini Solution

Tangherlini found a solution to d- dimensional static spinning spherical black holes which solves Einstein's vacuum equations in $d > 4$.This solution [32] is

[1] with symmteries of that of a sphere and in d dimensions that of $S^{(d-2)}$

a simple generalisaton to the Schwarzschild solution (for d=4) given above.

$$ds^2 = -\left(1 - \frac{\mu}{r^{d-3}}\right) dt^2 + \left(1 - \frac{\mu}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \tag{4.3}$$

where $\mu = \frac{16\pi GM}{(d-2)\Omega_{(d-2)}}$ and Ω is line element of unit $(d-2)$ sphere.

4.3 Kerr solution

After almost 50 years after the Schwarzschild's solution to get the solution of a rotating black hole. It is non static, stationery solution to Einstein's vacuum equations which is also axisymmetric.

The metric is given as [13]

$$\begin{aligned} ds^2 = -\left(1 - \frac{2GMr}{\Sigma}\right)(dt)^2 - \left(\frac{2GMar\sin^2\theta}{\Sigma}\right)(d\phi dt + dt d\phi)\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ + \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\sin^2\theta d\phi^2 \end{aligned} \tag{4.4}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ $\Delta = r^2 - 2Mr + a^2$

The killing vectors of the above metric is $K = \partial_t$ and $R = \partial_\phi$. The vector K^μ is not orthogonal to t= constant hypersurfaces and thus the metric is stationary, as it is rotating it is not static. The norm of K^μ is given by

$$K^\mu K_\mu = -\frac{\Delta - a^2\sin^2\theta}{\Sigma} \tag{4.5}$$

At $r = r_+$ (root of $\Delta = 0$). As

$$K^\mu K_\mu = \frac{a^2}{\Sigma}\sin^2\theta \geq 0 \tag{4.6}$$

The region between the two surfaces-the outer horizon and stationary limit surface(locus of $K^\mu K_\mu = 0$). This region is called the ergosphere.

4.4 Myers Perry solution

In the year 1986 Myers and Perry found an exact solution to Black Hole in any dimension $d > 4$ rotating in all possible independent rotation planes. The solutions belong to the class of solutions called the Kerr-Schild class

$$g_{\mu\nu} = \eta_{\mu\nu} + 2H(x^\rho)k_\mu k_\nu \tag{4.7}$$

where k_ν is the null vector with respect to both $g_{\mu\nu}$ and the Minkowski space $\eta_{\mu\nu}$.This approach takes a form of the general metric $g_{mu\nu}$ like the one of linearised gravity. (It is not exactly linearised gravity as the $H(x^\rho)$ is a general function). This approach gives the solution in a relatively simple manner.

We first look at the solutions with rotation in one plane. The metric take the following form [3]

$$\begin{aligned} ds^2 = -dt^2 + \frac{\mu}{r^{d-5}\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \\ \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + r^2\cos^2\theta d\Omega^2_{d-4} \end{aligned} \tag{4.8}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - \frac{\mu}{r^{d-5}}$

We can calculate the mass and angular momentum by comparison the asymptotic field to the equation mentioned above [Check Appendix A] and we get

$$M = \frac{(d-2)\Omega_{d-2}}{16\pi G}\mu \tag{4.9}$$

$$J = \frac{2}{d-2}Ma \tag{4.10}$$

The above equation has its similarity with the Kerr solution. The $\frac{1}{r}$ is replaced by $\frac{1}{r^{d-3}}$ and it gives an idea that the higher dimensional black holes could just normally extended from four dimensions and don't differ much from them.

When we take $a = 0$ in the above metric we get the Tangherlini solution mentioned above.

4.4.1 General Solution

The general solution [3] with arbitary rotation in odd d dimension is given as

$$ds^2 = -dt^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu^2 d\phi_i^2) + \frac{\mu r^2}{\Pi F}(dt - a_i\mu_i^2 d\phi_i)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \tag{4.11}$$

and for even d

$$ds^2 = -dt^2 + r^2 d\alpha^2 (r^2 + a_i^2)(d\mu_i^2 + \mu^2 d\phi_i^2) + \frac{\mu r}{\Pi F}(dt - a_i\mu_i^2 d\phi_i)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \tag{4.12}$$

Where mass paramter is μ, i runs from 1 to N and $\mu_i^2 + \alpha^2 = 1$.

$$F(r, \mu_i) = 1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \qquad \Pi(r) = \prod_{i=1}^{N} (r^2 + a_i^2) \tag{4.13}$$

Chapter 5

Black Rings

Roberto Emparan and Harvey S. Reall gave the solution to Einstein's vacuum equations in $d = 5$. These were called Black rings as they are black holes with the horizon topology $S^1 \times S^2$ in d=5.

5.1 Solution using the C metric

As it is almost impossible to directly solve Einstein's equations to get the solution, a Wick rotated version of special metric is taken which is given below. This metric belongs to the bigger class of metric called the C- metric. It was first discovered by Levi-Civita [31] in 1918 as the class of metrics having timelike killing vector orthogonal to three space whose Ricci curvature tensor is of the given form as

$$R^a_b = \alpha \eta^a \eta_b + \beta \delta^a_b \tag{5.1}$$

It belongs to the family with three parameters which are solution to the vacuum Einstein's equations. It provides new examples of items like Killing horizons, trapped surfaces, incomplete geodesics, etc [25] C- metric has a clear and unambiguous physical interpretation as the combined gravitational and electromagnetic field of the uniformly accelerating charged mass. The following metric is the Wick rotated version of the metric in [25]

$$\begin{gathered} ds^2 = -\frac{F(x)}{F(y)}\left(dt + \sqrt{\frac{\nu}{\xi_1}}\frac{\xi_2 - y}{A} d\psi\right)^2 \\ + \frac{1}{A^2(x-y)^2}\left[-F(x)\left(G(y)d\psi^2 + \frac{F(y)}{G(y)}dy^2\right) + F(y)^2\left(\frac{dx^2}{G(x)} + \frac{G(x)}{F(x)}d\phi^2\right)\right] \end{gathered} \tag{5.2}$$

where

$$F(\xi) = 1 - \frac{\xi}{\xi_1} \qquad\qquad G(\xi) = 1 - \xi^2 + \nu\xi^3 \tag{5.3}$$

We follow the notations as given in the [6] where ξ_1 is used to define $F(\xi)$ and ξ_2, ξ_3 and ξ_4 are the roots of $G(\xi)$.

A condition is imposed on the value of ν to obtain only real and distinct roots of $G(\xi)$ which is

$$0 < \nu < \nu_* \equiv \frac{2}{3\sqrt{3}} \tag{5.4}$$

And also the roots are arranged such that the condition can be imposed

$$-1 < \xi_2 < 0 < 1 < \xi_3 < \xi_4 < \frac{1}{\nu} \tag{5.5}$$

A double root appears only when ν is ν^*.
Consider x to be in the region $[\xi_2, \xi_3]$

Case 1
where $\xi_1$is greater then ξ_3 and $g_{\phi\phi}$ vanishes at $x = \xi_3$ and to avoid conical singularity we have the following condition:

$$\Delta\phi' = \frac{4\pi\sqrt{F(\xi_3)}}{G'(\xi_3)} = \frac{4\pi\sqrt{\xi_1 - \xi_3}}{\nu\sqrt{\xi_1}(\xi_3 - \xi_2)(\xi_4 - \xi_3)} \tag{5.6}$$

This is the has to be in equal to the periodicty imposed on ϕ [check Appendix 1] and so the value of ξ_1 is fixed using ξ_2 and ξ_3.
This gives the realtion betweeen the roots as given below:

$$\xi_1 = \frac{\xi_4^2 - \xi_2\xi_3}{2\xi_4 - \xi_2 - \xi_3} \tag{5.7}$$

This value of ξ_1 gives the black ring solution.

This can be explained as mentioned in [2] that with given relation and constriants in the values of the roots, factors of F(x) in the metric are non zero. When t and y have constant values, the cross section has a topology of a ring, while x and ϕ show the regular surface of the sphere.

Case 2
$\xi_1 = \xi_3$ Unlike the previous case, $g_{\phi\phi}$ does not vanish, so the periodicity condition of ϕ' is not imposed. The sections of constant y and t have S^3 topology withh ψ and ϕ as independent angles of rotation. The metric on the spatial cross section is written as

$$ds_H^2 = R^2 \left(\frac{\nu^2(1+\lambda x)\lambda}{(1+\nu x)^3} \frac{dx^3}{(1-x^2)} + \frac{\nu^2(1-x^2)}{1+\nu x} d\phi^2 + \frac{\lambda(1+\lambda)(1-\nu)^2}{\nu(1-\lambda)(1+\lambda x)} d\psi^2 \right) \tag{5.8}$$

But this does not give any clear geometric picture.
To get some idea the topology we take the ring coordinates and a flat four dimension metric.

5.2 Visualising S2 * S1 using Ring coordinates

The flat metric is written in spherical form as two spheres of radius r_1 and r_2. And then with a suitable set of coordinates which we get but taking the ones which give equipotential surface of the 2 form potential $B_{\mu\nu}$ and its Hodge Dual A_ϕ.

The ring coordinates are defined as the following
$x^1 = r_1 \cos\phi$ $\quad$ $x^2 = r_1 \sin\phi$
$x^3 = r_2 \cos\psi$ $\quad$ $x^4 = r_2 \sin\psi$

In a four dimensional spacetime two independent roatations planes ϕ and ψ are possible. They have independent angular momenta J_ϕ and J_ψ

The flat metric of four dimensional ring co-ordinates given above is of the form as given in [2]

$$dx_4^2 = dr_1^2 + r_1^2 d\phi^2 + dr^2 + r_2^2 d\psi^2 \tag{5.9}$$

We take the rings which extend along (x^3, x^4) plane and rotate along ψ and this gives a non-vanishing angular momentum term J_ψ. We take the ring as the circular string which is like the electric source of the 3-form field strength H which gives the two form potential B as $H = dB$

We have the field strength H obeying the following equation

$$\partial_\mu(\sqrt{(-g)}H^{\mu\nu\rho}) = 0 \tag{5.10}$$

We construct the solution of the field equation with a special condition where the electric source is circular and is at $r_1 = 0$ and $r_2 = R$ and

$$0 \leq \psi \leq 2\pi \tag{5.11}$$

This gives us a special case where a point source is located at the circumference (or circular boundary) of the circular source.

In order to look at the equipotential surfaces of 2-form B, we find a solution with a fixed guage [1] and we get

$$B_{t\psi} = \frac{R}{2\pi}\int_0^{2\pi} d\psi \frac{r_2 \cos\psi}{r_1^2 + r_2^2 - 2Rr_2\cos\psi} \tag{5.12}$$

$$= -\frac{1}{2}\left(1 - \frac{R^2 + r_1^2 + r_2^2}{\Sigma}\right) \tag{5.13}$$

where

$$\Sigma = \sqrt{(r_1^2 + r_2^2 + R^2)^2 - 4R^2r_2^2} \tag{5.14}$$

The above solution is given in [1].

We can note that it is very similar to the solution of the potential of that of a circular ring for a point on the ring-except that in them denominator term we find another r_1 term which is present due to the point source present at that point.

The Hodge dual of the field F is ${}_*H = dA$ where A is the one form potential so the dual is

$$A_\phi = -\frac{1}{2}\left(1 + \frac{R^2 - r_1^2 + r_2^2}{\Sigma}\right) \tag{5.15}$$

Now, we define our coordinates x and y, that correspond to the values of constant $B_{t\psi}$ and its Hodge dual A_ϕ as

$$y = -\frac{R^2 + r_1^2 + r_2^2}{\Sigma} \qquad\qquad x = \frac{R^2 - r_1^2 - r_2^2}{\Sigma} \tag{5.16}$$

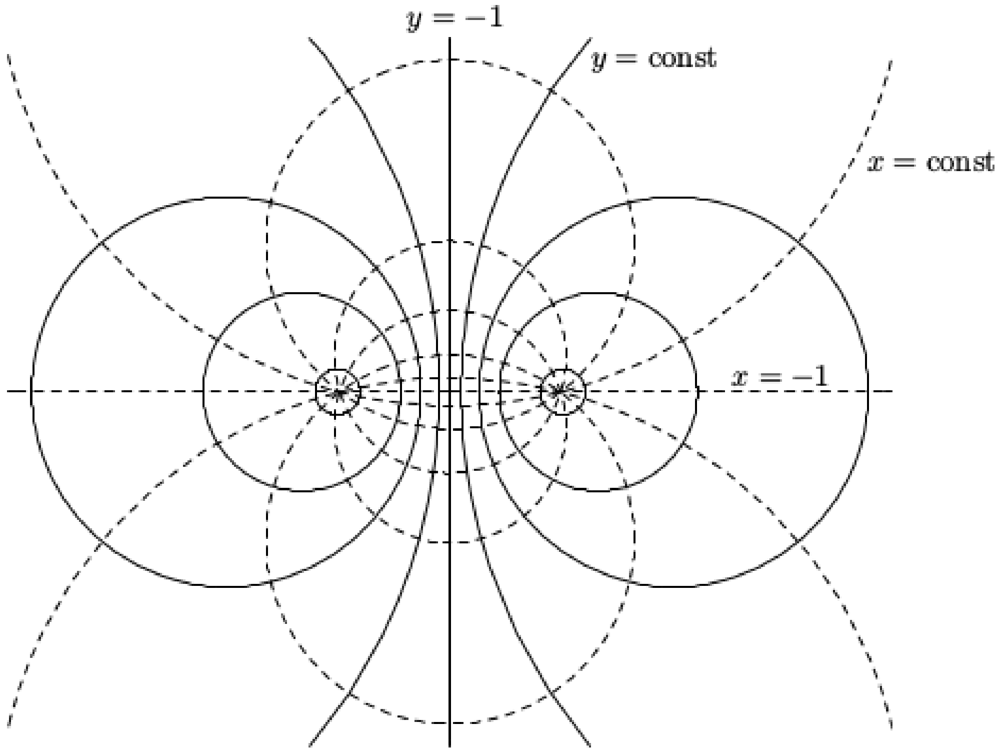


Figure 5.1: Ring coordinates in flat d=4 [1]

Taking the values of r_1 and r_2 from the above equation as

$$r_1 = R\frac{\sqrt{1-x^2}}{x-y} \qquad\qquad x = R\frac{\sqrt{y^2-1}}{x-y} \tag{5.17}$$

and the coordinates ranges are

$$-\infty \leq y \leq -1 \qquad\qquad -1 \leq x \leq 1 \tag{5.18}$$

where $y = -\infty$ refers to the ring source position and asymptotic infinty is recoverd as $x \longrightarrow y \longrightarrow -1$.

In the newly defined coordinates the flat metric takes the form

$$dx_4^2 = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1+x^2)d\phi^2\right] \tag{5.19}$$

To get some clarity we can rewrite the same metric in a different way with the sphercial components r and ϕ which are defined in [1] as a

$$r = -\frac{R}{y} \qquad \cos\theta = x \tag{5.20}$$

with the coordiants ranging in

$$0 \leq r \leq R, \qquad 0 \leq \theta \leq \pi \tag{5.21}$$

The flat metric then is transformed as

$$dx_4^2 = \frac{1}{1 + \left(\frac{r\cos\theta}{R}\right)^2} \left[\left(1 - \frac{r^2}{R^2}\right) R^2 d\psi^2 + \frac{dr^2}{1 - \frac{r^2}{R^2}} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \right] \tag{5.22}$$

There is an apparent singularity at $r = R$ which corresponds to ψ axis of rotation.

The surfaces of constant r which is actually constant y have a ring-like topology $S^1 \times S^1$ where S^2 is parameterised by (θ, ϕ) coordinates and S^1 by ψ. This metric is also Reimann flat which is just the trivial solution to Einstein's equations, whereas the actual solution is only Ricci flat. But this solution gives a very clear idea of the topology of the $S^2 \times S^1$ event horizon of the Black ring solution.

5.3 Neutral Ring

Another way to write black ring solution is to write it [1] as follows:

$$ds^2 = -\frac{F(y)}{F(x)} \left(dt - CR\frac{1+y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} F(x) \left[-\frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right] \tag{5.23}$$

where

$$F(\zeta) = 1 + \lambda\zeta \qquad G(\zeta) = (1 - \zeta^2)(1 + \nu\zeta) \tag{5.24}$$

and

$$C = \sqrt{\lambda(\lambda - \nu)\frac{1+\lambda}{1-\lambda}} \tag{5.25}$$

The dimensionless parameters λ and ν are in the range $(0,1)$ with the condition $\nu \leq \lambda$ The coordinates vary in the ranges

$$-\infty \leq y \leq -1 \qquad\qquad -1 \leq x \leq 1 \tag{5.26}$$

When both the parameter λ and ν vanishes we recover the flat form of the metric. Asympotic infinity occurs at $x \longrightarrow y \longrightarrow -1$. In order to avoid conical singularity[1] angular variables are identified with periodicity

$$\Delta\psi = \Delta\phi = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu} \tag{5.27}$$

The two parameters must satisfy the following condition

$$\lambda = \frac{2\nu}{1+\nu^2} \tag{5.28}$$

which comes from the cubic equation in [1]

5.4 Phase diagram

The figure below [2] is obtained when horizon area and spin squared is plotted for a fixed mass for the neutral black ring and Myers Perry Black Hole.

The two figures below are plotted using the range of values of ν for the following equations: For black ring,

$$a_H = 2\sqrt{\nu(1-\nu)} \tag{5.29}$$
$$j^2 = \frac{(1+\nu)^3}{8\nu}$$

where parameter ν can vary as $0 < \nu \leq 1$

For MP Black Hole

$$a_H = 2\sqrt{2(1-j^2)} \tag{5.30}$$

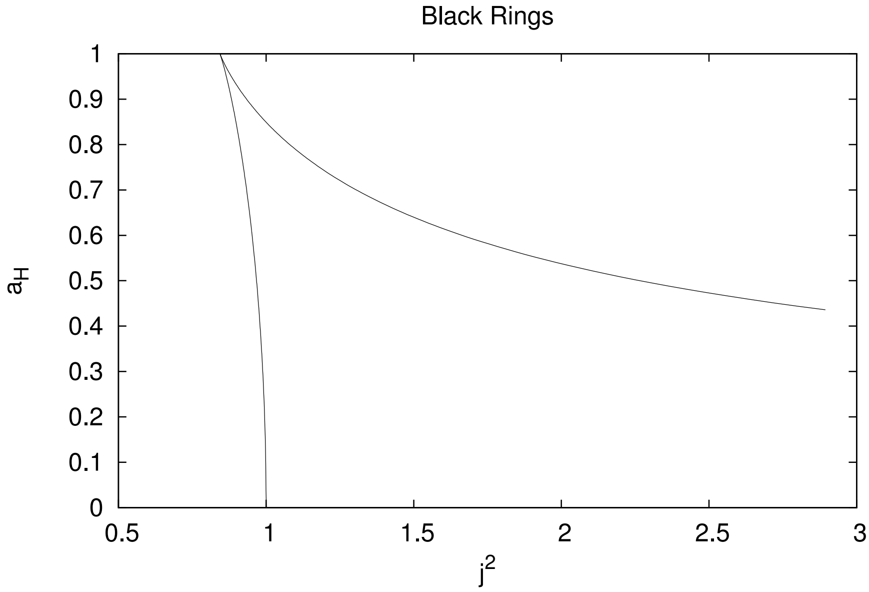


Figure 5.2: Black rings

In the last figure, the dotted branch shows the MP Black Solution, while the thick branches represent thin and fat black ring. Out of the two thick lines, the lower one is the Fat ring solution while the one above is the thin ring solution.

In the region,

$$\frac{27}{32} < j^2 < 1 \tag{5.31}$$

three different solutions - MP black hole and two rings is found.

[1] See appendix:Conical singularity

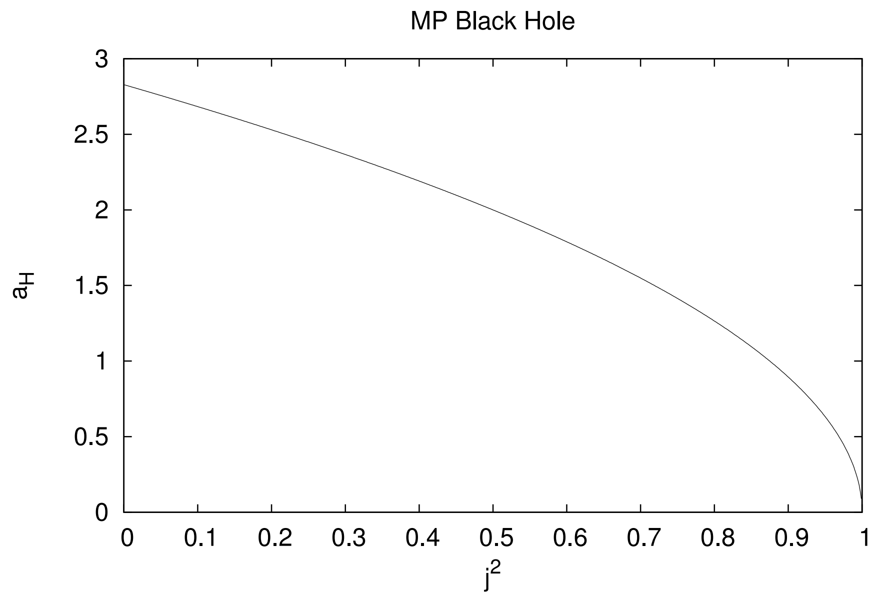


Figure 5.3: Angular momentum

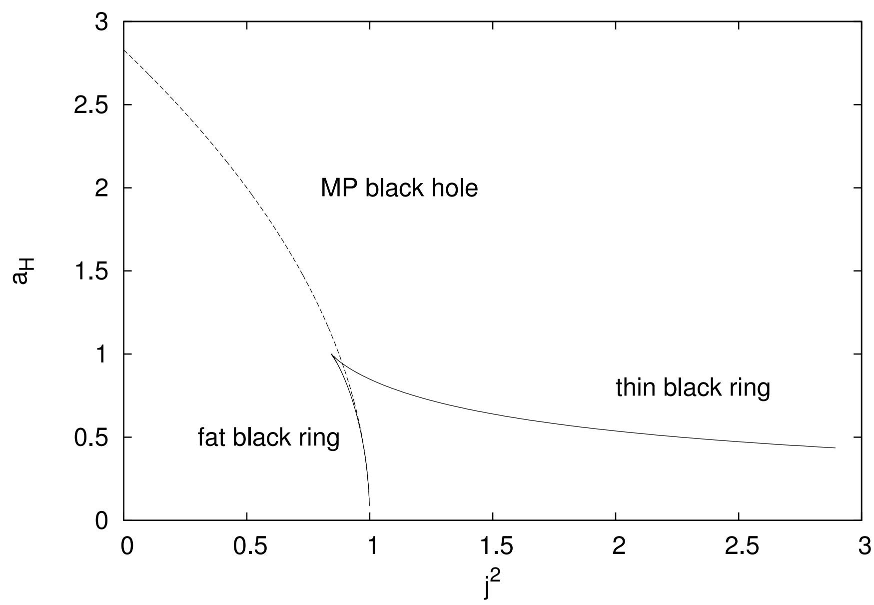


Figure 5.4: Phase-diagram

Chapter 6

Shape and Stability of the Ring

Shape of the Black ring

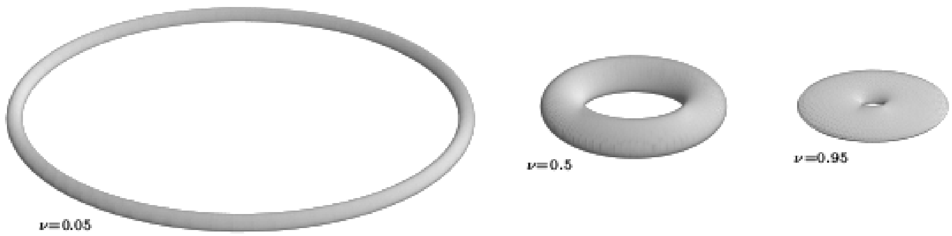


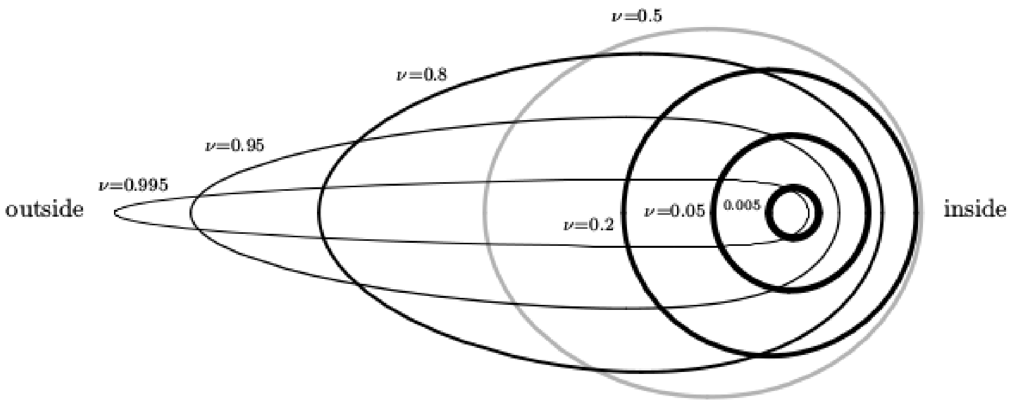
Figure 6.1: Black rings [7]

The above figure taken from [7] show that a ring with same mass can vary a lot in shape just by varying the parameter ν.

The topology $S^2 \times S^1$ is exactly S^2 but it is a distorted S^2 in an isometric embedding [1] of the shape of the ring.

Figure shows the isometric emedding of cross section of the black ring 2-sphere (with azimuthal angle suppressed) with varied value of ν and j. The size of the S^1 is estimated as the inner radius of the horizon.

[1]It is an idea to visualise a curved geometry in flat Euclidean space in a way it preserves the distance



ν	0.005	0.05	0.2	0.5	0.8	0.95	0.995
j	5.04	1.70	1.04	0.92	0.95	0.988	0.999

Figure 6.2: Isometric-Embedding [7]

The distortion $j \longrightarrow 1$) in the fat black ring branch when $\nu \longrightarrow 1$, the S^2 flattens out (for a fixed mass) and when $j = 1$ the horizon disappears and a singular ring remains. Similarly if you look at the MP Black Hole at $j = 1$ the S^3 horizon flattens out. And thus at $j = 1$ MP black Hole and fat black ring give the same solution.

6.1 Stability

Supersymmetric black rings are stable as Supersymmetry ensures the stability to quadratic perturbations. As we take the linearized gravity to study the vacuum solutions, we need to check stability of the solutions in those perturbations.

Looking at it qualitatively, we know that thin rings undergo Gregory-Laflamme instability. The black ring formed in between the thin and the fat ring must be highly unstable as adding extra matter that gives mass but no angular momentum, there is no black ring the system can evolve too which will make it react backwards forcefully. [7]

In the radial stability section we notice the the fat black ring is radially unstable while thin one is. It can be backed by the argument derived from the study of the black ring and MP black hole phase diagram which shows that one unstable mode is added from thin to at black ring making the latter more unstable. The dipole charge solution can remove the unstability (G-L type) as the charges can balance it.

6.2 Off shell perturbation

Introducing an external forces to understand the equilibrium is very useful. So we can take the system a little away from the equilibrium and we can understand the potential that the equilibrium extremizes. [7]

We take a radial force[2] for the black ring.

Create a conical defect δ in the disk inside the ring. It creates a tension τ acting per unit length of the black ring circle

$$\tau = \frac{3}{16\pi G}\delta = \frac{3}{8G}\left(1 - \frac{1+\nu}{1-\nu}\sqrt{\frac{1-\lambda}{1+\lambda}}\right) \tag{6.1}$$

When τ is zero it is in equilibrium.

We take the radial potential as

$$V(R_1) = -\int^{R_1} \tau(R_1^{'})d(R_1^{'}) \tag{6.2}$$

although we are interested only at the points around the equilibrium values where $V^{'} = \tau = 0$ where τ is tension per unit length. Perturbating away from equilibrium we find

$$V^{''} = -\left.\frac{d\tau}{dR_1}\right|_{equil} > 0 \tag{6.3}$$

which is stable equilibrium.

[2]which keeps the Killing symmtries but deforms the radius and takes it away from it value at equilibrium

With some outword pressure the ring can be made static at

$$R_1^{equil} + dR_1 \tag{6.4}$$

If

$$V'' = -\frac{d\tau}{dR_1}\bigg|_{equil} < 0 \tag{6.5}$$

is interpreted as the inward pulling tension required to prevent the runaway increase of the ring radius from equilibrium.

6.3 Radial stability

The above mentioned radial off-shell purturbations are applied to black rings. In order to do that we choose the radius of ring as the radius of the inner[3] ring $R_1^{inner} = R_1^1$ This gives us

$$R_1 = R\sqrt{\frac{\lambda}{\nu}} \tag{6.6}$$

We keep J and M to be fixed while we perturb the ring radius.

$$\left(\frac{d\lambda}{d\nu}\right) = -\frac{\frac{\partial j}{\partial \nu}}{\frac{\partial j}{\partial \lambda}} \tag{6.7}$$

We look at reduced area, $\hat{a}_H = \frac{A_H}{J}$ and reduced radius $r = \frac{R_1}{J^{\frac{1}{3}}}$.
So that

$$\left(\frac{d\lambda}{d\nu}\right)_{A_H,J} = -\frac{\partial \hat{a_H}/\partial \nu}{\partial \hat{a_H}/\partial \lambda} = \frac{2(2-\nu)(1-\nu)}{(1+\nu^2)^2} \tag{6.8}$$

We use the above results to commute

$$\left(\frac{d\tau}{dr}\right)_{*,J} = \frac{(d\tau/d\nu)_{*,J}}{(dr/d\nu)_{*,J}} = \frac{\partial_\nu \tau + \left(\frac{d\lambda}{d\nu}\right)_{*,J}\partial_\lambda \tau}{\partial_\nu r + \left(\frac{d\lambda}{d\nu}\right)_{*,J}\partial_\lambda r} \tag{6.9}$$

where $*$ is used for M or A_H

From the sign of the above equation; sign of $\left(\frac{d\tau}{dr}\right)_{*,J}$, we get that the thin rings are radially stable while the fat ones are unstable, as we get a

[3]as we need to study the inner hole

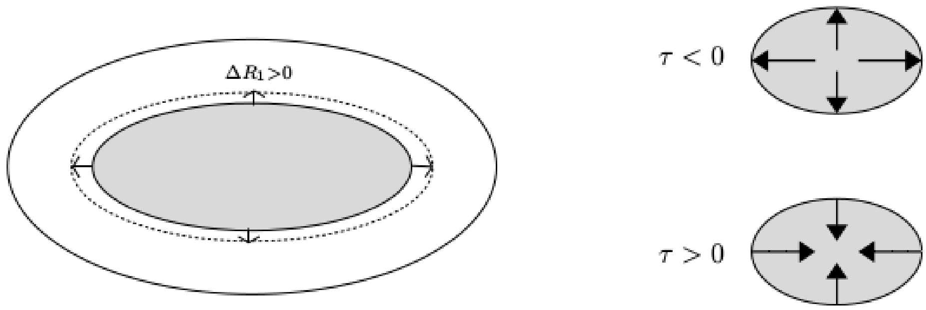


Figure 6.3: Radial-perturbations [7]

positive sign for thin black rings ($0 \leq \nu < 1/2$) and negative for fat ring ($1/2 \leq \nu < 1$)[7]

The diagram below shows Radial Potential V(r) for fixed values of mass and spin. Fat black ring have unstable equilibrium at local maxima while thin black ring have stable equilibrium at local minima.

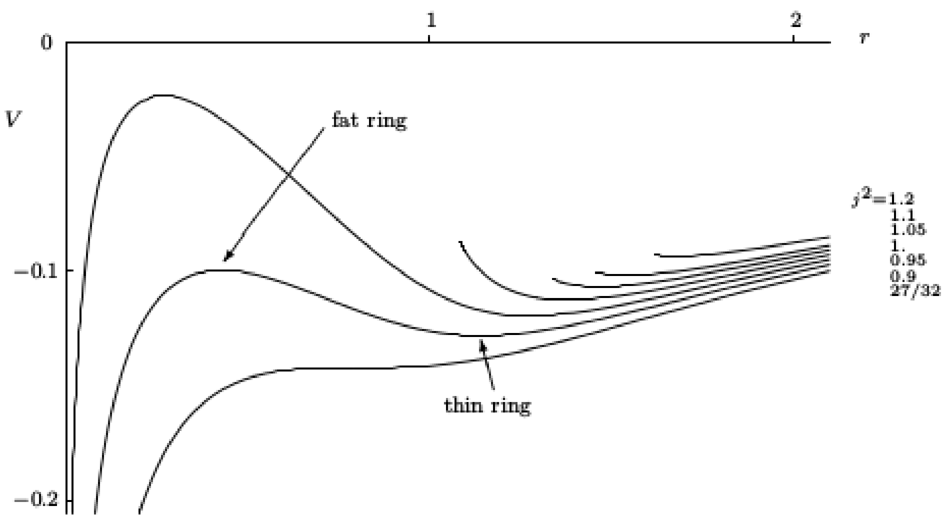


Figure 6.4: Radial-Potential [7]

In the recent paper by Jorge Santos and Benson Way[30], it was shown using numerical methods that not just fat black rings but also thin black rings are unstable by studying the non-axisymmetric linearised gravitational perturbations on the Black rings.

Chapter 7

Conclusions

After looking at the phase diagram of the Myers Perry Black Hole and Black Rings solution both solutions to Einstein's equations in vacuum in five dimensions, we see that the Uniqueness theorems for four dimensions are violated by this solution in five dimensions. Thus the earlier claims of only Spherical topology for event horizons were disproved in higher dimensions. With more number of dimensions, more degrees of freedoms and so we can expect new things as we study higher dimensions.

It is obvious to ask the next question about the possibility of Black rings with topology $S^1 \times S^{d-3}$ in d > 5. As suggested by the Higher dimensional topology theorem [26] these possibilities could be true.

IR and UV theory have been developed to describe Black Rings with conserved charges and dipole charges repectively. As in String theory, the microscopic description of the black holes is based on dynamics of a configurationof branes that has the same set of charges as Black Hole.

After the Black ring with one angular momentum and two angular momenta were found, a Black saturn solution was discovered. It is a black hole surrounded by concentric rotating black ring. These were constructed in [27]. Similarly two concentric Black rings called Di-rings solutions were constructed in [28]

Thus, we see that there is a lot of scope for different kind of solutions in higher dimensions.

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Appendix A

Linearised Gravity

Einstein-Hilbert action can be genralised in d $>$ 4 as given below as just a straightforward generalisation.

$$I = \frac{1}{16\pi G} \int d^d \sqrt{-g} R + I_{matter} \tag{A.1}$$

Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G 2(-g)^{-\frac{1}{2}} \left(\frac{\delta I_{matter}}{\delta g_{\mu\nu}} \right) \tag{A.2}$$

where $T_{\mu\nu} = 2(-g)^{-\frac{1}{2}} \left(\frac{\delta I_{matter}}{\delta g_{\mu\nu}} \right)$ This above form of gives a dimensionless definiton of g.
We can write the general Einstein metric as a small perturbation over Minkowski metric. It can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{A.3}$$

Thus, we get

$$\bar{h}_{\mu\nu} = 16\pi G T_{\mu\nu} \tag{A.4}$$

where $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}$ Solving the above equation for $T_{\mu\nu}$ while keeping in mind that we have localised sources and the fields in the asymptotic region are same as created by the pointlike sources of mass M and angular momentum with antisymmetric matrix J_{ij} at origin $X^k =$ [6]

$$
\begin{aligned}
T_{tt} &= M\delta^{d-1}(x^k) \\
T_{ti} &= -\frac{1}{2}J_{ij}\delta^{d-1}(x^k) \\
\bar{h}_{tt} &= \frac{16\pi G}{(d-3)\Omega_{d-2}}\frac{M}{r^{d-3}} \\
\bar{h}_{ti} &= -\frac{8\pi G}{\Omega_{d-2}}\frac{x^k J_{ki}}{r^{d-1}}
\end{aligned}
\tag{A.5}
$$

where $r = \sqrt{x^i x^i}$ and $\Omega_{d-2} = 2\pi^{\left(\frac{d-1}{2}\right)}\Gamma\left(\frac{d-1}{2}\right)$

We recover metric perturbation $h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{d-2}\bar{h}\eta_{\mu\nu}$ as

$$
\begin{aligned}
h_{tt} &= \frac{16\pi G}{(d-2)\Omega_{d-2}}\frac{M}{r^{d-3}} \\
h_{ij} &= \frac{16\pi G}{(d-2)(d-3)\Omega_{d-2}}\frac{M}{r^{d-3}}\delta_{ij} \\
h_{ti} &= -\frac{8\pi G}{\Omega_{d-2}}\frac{x^k J_{ki}}{r^{d-1}}
\end{aligned}
\tag{A.6}
$$

Appendix B

Conical singularity and Periodicity

Conical singularity is a non-curavature singularity. It can be visualised by taking a taking a flat sheet of paper and shaping it in a cone. The curvature of the flat paper is 0, but when shaped as a cone, it can a singularity at the top point.

We see that the coordinates in Section have conical singularity. To remove them, special periodicity conditions are imposed on the spherical coordinates in order to remove conical singularity.

Given below is the condition for ϕ to avoid conical singularity at $x = \xi_2$,

$$\Delta\phi = \frac{4\pi\sqrt{F(\xi_2)}}{G'(\xi_2)} = \frac{4\pi\sqrt{\xi_1 - \xi_2}}{\nu\sqrt{\xi_1}(\xi_3 - \xi_2)(\xi_4 - \xi_2)} \tag{B.1}$$

In the case where $\xi_1 > \xi_2$, there is another conical defect at $x = \xi_3$.To remove that we identify ϕ as

$$\Delta\phi' = \frac{4\pi\sqrt{F(\xi_3)}}{G'(\xi_3)} = \frac{4\pi\sqrt{\xi_1 - \xi_3}}{\nu\sqrt{\xi_1}(\xi_3 - \xi_2)(\xi_4 - \xi_3)} \tag{B.2}$$

We demand $\Delta\phi = \Delta\phi'$ for consistency and get the following result.

$$\xi_1 = \frac{\xi_4^2 - \xi_2\xi_3}{2\xi_4 - \xi_2 - \xi_3} \tag{B.3}$$

Appendix C

Einstein's field equations

Einstein's field equation show how $R_{\mu\nu}$ with $T_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2}T_{\mu\nu} \tag{C.1}$$

where $R_{\mu\nu}$ is Ricci tensor in four dimensions,

R Ricci scalar,
$g_{\mu\nu}$ space time metric,
Λ cosmological constant,
G Newton's gravitational constant,
$T_{\mu\nu}$ in stress energy tensor
in compact form taking

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} \tag{C.2}$$

gives the final equation as

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2}T_{\mu\nu} \tag{C.3}$$

Appendix D

Weyl Metric

In order to get the exact solutions of Einstein's equations a lot of efforts were made and techniques were developed. Weyl was the first to have found the general static axisymmetric solution of the vacuum Einstein equations which are given in [25] as:

$$ds^2 = -e^{2U}dt^2 + e^{-2U}(e^{2\gamma}(dr^2 + dz^2) + r^2 d\phi^2) \tag{D.1}$$

where U (r, z) is an axisymmetric harmonic solution of the Laplace's equations in a three dimensional flat space with metric

$$ds^2 = dr^2 + r^2 d\phi^2 + dz^2 \tag{D.2}$$

where γ satisfies

$$\begin{aligned} \frac{\partial \gamma}{\partial r} &= r\left[\left(\frac{\partial U}{\partial r}\right)^2 - \left(\frac{\partial U}{\partial z}\right)^2\right] \\ \frac{\partial U}{\partial z} &= 2r\frac{\partial U}{\partial r}\frac{\partial U}{\partial z} \end{aligned} \tag{D.3}$$

The C metric used in the solution of black rings is a subset of this large class of Weyl metrics.