Functional Analysis And Operator Theory

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by

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Certificate

This is to certify that this dissertation entitled Functional Analysis And Operator Theorytowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Akash Gupta at Indian Institute of Science Education and Research under the supervision of Dr. Tirthankar Bhattacharyya, Professor, Department of Mathematics, Indian Institute Of Science, Bengaluru , during the academic year 2020-2021.

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This thesis is dedicated to my father Mr.Subodh Gupta
without whom none of this would have been possible

Declaration

I hereby declare that the matter embodied in the report entitled Functional Analysis And Operator Theory are the results of the work carried out by me at the Department of Mathematics, Indian Institute Of Science, Bengaluru, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Tirthankar Bhattacharyya and the same has not been submitted elsewhere for any other degree.

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Acknowledgments

I would like to thank my supervisor Dr.Tirthankar Bhattacharyya, for taking me as his student and constantly supporting me throughout my 5th year. I really liked the heuristic approach, which my guide took in teaching me the basics. He made me learn most of the concepts by finding answers to a set of questions related to it. Through this method I not only learned the concept but also got a glimpse of how Mathematics is done in practice. My 5th year journey wasn't so smooth, which should be hardly surprising, but the motivation, courage and guidance I got from my guide helped me a lot. The presentations that I gave were very helpful in getting crystal clear understanding of concepts. I am also grateful to IISC Bangalore for providing me an office in their Maths Department.

Abstract

This project is mainly about The Spectral Theorem for Normal Operators, Hyponormal Operators, Berger-Shaw Theorem and an important corollary of it, the Putnam's Inequality. We do have spectral theorem for Compact Normal Operators, the poof of which is not hard. But to relax the requirement of compactness and still get a similar result for a Normal Operator is quite a challenge. But it can be done and the countable sum in the compact case is replaced by an integral. Hyponormal operators share remarkable number of properties with normal operators. But what is even more remarkable is that for hyponormal operators which are multicyclic we have Berger-Shaw theorem. If an Operator is purely hyponomral, meaning it is hyponormal and the only reducing subspace of it where it is normal is the trivial space, then the real and the imaginary part of it is absolutely continuous w.r.t the Lebesgue Measure on its spectrum. The Putnam inequality tells us that the norm of the commutator of a Hyponormal Operator is bounded by the two-dimensional Lebesgue Measure of the spectrum of the operator.

Contents

Introduction

This thesis starts with the discussion on some of the fundamental topics in functional analysis. Hahn-Banach Theorem has been stated in the first chapter and using it separation results and norm preserving extension theorem are proved. Open Mapping and Closed Graph Theorem have been stated and some not so intuitive applications of them have been mentioned, the exacting task of proving that a linear map from a Banach space to a Banach space is continuous can be easily done easily using Closed Graph Theorem, which reduces the work to just establishing that the map is closed. This technique has been extensively used in the first chapter. Some results about a Schauder basis in a Banach space have also been proved, here also I have employed the Closed Graph theorem. All these results emphasizes the importance of Open Mapping and Closed Graph Theorem have in Functional Analysis.

Pure contractions are bounded linear maps with $\|T\| \leq 1$ and $(T^*)^n \to 0$ as $n \to \infty$ strongly. The famous left Shift operator on l^2 is an example of a pure contraction. Turns out all the pure contractions on a Hilbert space are unitary equivalent to $S \otimes Id$ on $l^2 \otimes E$ for some Hilbert space E. But to arrive at this result one needs to get accustomed to Tensor product of Hilbert Spaces, hence for the sake of completeness this thesis has a complete discussion on Tensor Product of Hilbert spaces.

One of the main theorems in this thesis is the spectral theorem of normal operators in a Hilbert space. Given a normal operator T on a Hilbert space H, this theorem gives for every $x, y \in H$ the existence of a complex measure $E_{x,y}$ on the spectrum of the T, such that $< T(x), y >= \int_{\sigma(T)} \lambda dE_{x,y}(\lambda)$. In a concise manner we write $T = \int \lambda dE$. This is a very important result in the theory of normal operators, as many results about normal operators can proved using this decomposition. But to understand the proof of this result one needs to know the fundamentals of

Weak topology, Banach algebra, complex measures and the Gelfand-Naimark Theorem. All of these have been discussed in sufficient details in this thesis.

Hyponoraml Operator T on a Hilbert space H is an operator for which $T^*T - TT^* \geq 0$, which is equivalent to $\|T^*x\| \leq \|Tx\| \, \forall x \in H$. One can see that a normal operator is obviously hyponormal, any Isometry of a Hilbert space is also hyponormal. Many results about normal operators are also true for hyponormal operators and the proofs are very similar. But what is very unique to a pure hyponormal operator(A hyponormal operator such that if a restriction of it on a reducing subspace is normal then that reducing subspace is trivial.) is that its real and the imaginary self-adjoint parts are absolutely continuous. This is a remarkable result and the proof of it is very fascinating despite being quite long and difficult. Then we move on to Berger-Shaw Theorem, the main of this thesis. $\mathbf{T} \in B(H)$, is said to be a multicyclic operator if $\exists$ a finite set of vectors $\{g_1, g_2, \ldots, g_m\}$, such that linear span of $\{R(T)g_i \mid R \in RAT(\sigma(T)), 1 \leq i \leq m)\}$ is dense in H. The set of above vectors is a generating set. **T** is said to be m-multicyclic if the cardinality of the smallest generating set is m.

Now the Berger-Shaw theorem says that for a m-multicyclic hyponomral operator $tr([T^*, T]) \leq \frac{m}{\pi} Area(\sigma(T))$. A most important corollary of the Berger-Shaw theorem is the Putnam$'$s inequality, which is that for a hyponormal operator T $\|[T^*, T]\| \leq \frac{Area(\sigma(T))}{\pi}$. Though originally Putnam$'$s inequality was not discovered as a corollary of the Berger-Shaw theorem, the proof of it becomes a lot easy once Berger-Shaw theorem is known.

My Contributions

This thesis is mostly a literary review. But some of the proofs, discussions and examples in this thesis are my own. In the first chapter I have proved a result regarding strict separation of two disjoint convex sets, in which one is compact and the other is closed. It is formally stated in theorem 3 of this chapter. I have also proved some non-intuitive results which require open mapping theorem and closed graph theorem. The proofs of theorem 3 and the results under the section of open mapping theorem are my own. The Discussion of Tensor product was found from the internet, but the original discussion at some places lacked rigour and some of the results were assumed, I made the discussion rigourous and logically coherent.

In the chapter Banach Algebra I added some necessary details in the proof of proposition 2. I also proved that the mapping $a \to r(a)$, where r(a) is the spectral radius of a, is an upper semi-continuous map. In the chapter about spectral Theorem of normal operators, the proofs in the section Alegbra of $L^{\infty}(E)$ are my own, statements were just mentioned in the original source. I have made the proof of theorem 5 in this chapter easy by filling in some gaps and explaining each part of the proof. Generalizing the definition of exponential of an element in a Banach Algebra, I proved that for two commuting elements a and b $e^{a+b} = e^{a}e^{b}$. I derived the resolution of identity for a compact self-adjoint operator. This chapter about the normal operators is quite difficult as compared to the other chapters , but I have tried to keep discussion as simple as I could.

In the chapter Riesz Functional Calculus, the proofs of the statements in the proposition 3 are my own. Most of this chapter is a literary review, as it's a standard topic in Functional Calculus.

In the chapter about Hyponormal operator, after the proof of a result regarding pure hyponormal operator, I have given an example of a hyponormal operator which is not pure and have shown that real and imaginary self-adjoint parts of it are not absolutely continuous, hence showing that we cannot relax the pure condition on an hyponormal operator for the theorem to hold. Following this example, is an example of a pure hyponormal operator at the end of this chapter.

In the chapter Berger-Shaw theorem, I have simplified the proofs of some lemma by providing every minor detail in them. With some help from my guide I was able to prove lemma 5, which is crucial in the proof of Berger-Shaw. The proof of lemma 5 was absent in the original source. I have briefly discussed the facts in notes, which were assumed and were used in the proof of Berger-Shaw theorem in the original source. In this thesis I have worked hard to make each discussion rigourous and to leave no logical gaps in the proofs.

0.1 Hahn-Banach Theorem

Introduction: In this chapter I have discussed Hahn Banach Theorem and proofs of separation results in a NLS. Here I have proved a result regarding strict separation of two disjoint convex sets, in which one is compact and the other is closed. It is formally stated in theorem 3. I have also proved some non-intuitive results which require open mapping theorem and closed graph theorem. These results emphasis the importance of these two fundamental theorems in Functional analysis. The proofs of theorem 3 and the results under the section of open mapping theorem are my own.

Let V be a linear space over $\mathbb{R}$.

Definition: $p : V \to \mathbb{R}$ is said to sublinear if $p(x + y) = p(x) + p(y)$ and $p(\alpha x) = \alpha p(x)$ $\forall x, y \in V, \alpha \geq 0$.

Hahn-Banach Theorem: Let $M \subset V$ be a linear subspace of V, $p : V \to \mathbb{R}$ be a sublinear map. If there is a linear map $g : M \to \mathbb{R}$ such that $g \leq p$ on M, then $\exists$ a linear extension f of g on V such that $f \leq p$ $\forall x \in V$.

We will not prove this result, but we will see some of its applications.

Separation Results:

Proposition 1: Let C be an open convex subset in a NLS V over $\mathbb{R}$ and assume $0 \in C$, we then define $p : V \to \mathbb{R}$, as $p(x) = \inf\{\alpha > 0 \mid \frac{1}{\alpha}x \in C\}$. This functional is know as the **Minkowski functional**. We will now prove the following results about p.

1. p is a sublinear map
2. $\exists \beta > 0$ such that $p(x) \leq \beta \|x\|$ $\forall x \in V$.
3. $C = \{x \mid p(x) < 1\}$

Proof. As $0 \in C$, $\exists \delta > 0$ such that $B(0, \delta) \subset C$, pick $x \neq 0 \in V$, then $\frac{\delta x}{2\|x\|} \in C$, hence for $\alpha = \frac{2\|x\|}{\delta} > 0$ $\frac{1}{\alpha}x \in C$, so $\{\alpha > 0 \mid \frac{1}{\alpha}x \in C\} \neq \emptyset$, hence inf is not ∞ and also from the definition $p(x) \geq 0$. Hence p is a non-negative function. $p(x) \leq \frac{2\|x\|}{\delta}$ as $\frac{\delta x}{2\|x\|} \in C$ $\forall x \neq 0$, taking $\beta = \frac{2}{\delta}$ we have $p(x) \leq \beta \|x\|$ $\forall x \in V$. This proves 2.

Now for $\alpha > p(x)$, by definition $\exists\alpha_0 > 0$ such that $p(x) \leq \alpha_0 < \alpha$ and $\frac{x}{\alpha_0} \in C$. Now $\frac{\alpha}{\alpha_0} < 1$, $\frac{x}{\alpha} = \frac{\alpha_0}{\alpha}\left(\frac{x}{\alpha_0}\right) + \left(1 - \frac{\alpha_0}{\alpha}\right) 0$, as $0 \in C$ and C is convex we have $\frac{x}{\alpha} \in C$.

Let x,y in V and $\epsilon > 0$, then $\frac{x}{p(x)+\frac{\epsilon}{2}}$ and $\frac{y}{p(y)+\frac{\epsilon}{2}} \in C$. Take $\theta = \frac{p(x)+\frac{\epsilon}{2}}{p(x)+p(y)+\epsilon} < 1$, then $\theta\left(\frac{x}{p(x)+\frac{\epsilon}{2}}\right) + (1-\theta)\left(\frac{y}{p(y)+\frac{\epsilon}{2}}\right) \in C$, so $\frac{x}{p(x)+p(y)+\epsilon} + \frac{y}{p(x)+p(y)+\epsilon} \in C \implies \frac{x+y}{p(x)+p(y)+\epsilon} \in C$. Hence $p(x+y) \leq p(x) + p(y) + \epsilon$. As ϵ was arbitrary, we have $p(x+y) \leq p(x) + p(y)$.

Let $\lambda > 0$, $p(\lambda x) = \inf\{\alpha > 0 \mid \frac{\lambda x}{\alpha} \in C\} = \inf\{\lambda\alpha > 0 \mid \frac{x}{\alpha} \in C\} = \lambda\{\alpha > 0 \mid \frac{x}{\alpha} \in C\} = \lambda p(x)$. This proves 1.

Let $x \in C$ if $x = 0$, then $p(x) = 0 < 1$, but if $x \neq 0$, then take a $\delta > 0$ such that $B(x,\delta) \subset C$, so by taking $\alpha = \frac{2\|x\|}{2\|x\|+\delta} < 1$, we have $\frac{x}{\alpha} \in C$, so $p(x) \leq \alpha < 1$. Now let $x \in V$ with $p(x) < 1$, choose a δ such that $p(x) < \delta < 1$ so $\frac{x}{\delta} \in C$ and $\delta\left(\frac{x}{\delta}\right) + (1-\delta).0 \in C \implies x \in C$. Hence $C = \{x \mid p(x) < 1\}$. With this all our claims are proved. □

Proposition 2: Let V be vector space over $\mathbb{C}$.

1. Let f be a $\mathbb{C}$ linear functional, then Re(f) is $\mathbb{R}$ linear functional on V.
2. Let f be a $\mathbb{R}$ linear functional, then $\widetilde{f}(x) = f(x) - if(ix)$ is a $\mathbb{C}$ linear functional and any other $\mathbb{C}$ linear functional g such that Re(g)=f will agree with $\widetilde{f}$ on V.
3. Let $p : V \to \mathbb{R}$ be any semi-norm, then $|f| \leq p \iff |\widetilde{f}| \leq p$.
4. Let f and $\widetilde{f}$ be defined as above, then $\|f\| = \left\|\widetilde{f}\right\|$.

Proof. 1. Let $f = f_1 + if_2$, where f_1 and f_2 are real valued functions. take $r \in \mathbb{R}$, $f(rx) = f_1(rx) + if_2(rx) = rf(x) = rf_1(x) + irf_2(x)$ hence by comparison $f_1(rx) = rf_1(x)$.

$f(x+y) = f_1(x+y) + if_2(x+y) = f_1(x) + if_2(x) + f_1(y) + if_2(y) = f_1(x) + f_1(y) + i(f_2(x) + f_2(y))$. Again by comparison we have $f_1(x+y) = f_1(x) + f_2(y)$. This proves 1.

2. Let f be a $\mathbb{R}$ linear functional.

$\widetilde{f}(x+y) = f(x+y) - if(ix+iy) = f(x) + f(y) - if(ix) - if(iy) = \widetilde{f}(x) + \widetilde{f}(y)$. It's enough to check that $\widetilde{f}(ix) = i\widetilde{f}(x)$. $\widetilde{f}(ix) = f(ix) - if(-x) = i(f(x) - if(ix)) =$

$i\widetilde{f}(x)$. Hence $\widetilde{f}$ is a $\mathbb{C}$ linear functional. Now if $g = f + ig'$, where g' is real valued, consider $(\widetilde{f} - g)(x) = -i(g'(x) + f(ix))$. As both $\widetilde{f}$ and g are $\mathbb{C}$ linear, $\widetilde{f} - g$ is also $\mathbb{C}$ linear. So $(\widetilde{f} - g)(ix) = -i(g'(ix) + f(-x)) = g'(x) + f(ix)$ as g' and f are real valued, we have $g'(x) = -f(ix)$. Hence $g = \widetilde{f}$.

3. If $|\widetilde{f}| \leq p(x)$, then $|f(x)| = |Re(\widetilde{f}(x))| \leq |\widetilde{f}(x)| \leq p(x)$.

   Now if $|f(x)| \leq p(x)$, let $\widetilde{f}(x) = |\widetilde{f}(x)|e^{i\theta}$ for some θ, then $|\widetilde{f}(x)| = e^{-i\theta}\widetilde{f}(x) = Re(e^{-i\theta}\widetilde{f}(x)) = f(e^{-i\theta}x) \leq p(e^{-i\theta}x) = p(x)$. Hence $|\widetilde{f}(x)| \leq p(x)$.

4. Take $p(x) = \|f\|\|x\|$, it is a semi norm and $|f(x)| \leq p(x)\ \forall x \in V$. So $|\widetilde{f}(x)| \leq p(x) = \|f\|\|x\|$. Hence $\left\|\widetilde{f}\right\| \leq \|f\|$ and it is easy to see that $\|f\| \leq \left\|\widetilde{f}\right\|$, so $\|f\| = \left\|\widetilde{f}\right\|$.

□

Theorem 1: Let V be NLS over $\mathbb{C}$. C be a non-empty open convex subset and let $x_0 \notin C$, then there exist a bounded linear functional f such that $Re(f)(x) < Re(f)(x_0)\ \forall x \in C$.

Proof. We first consider V as NLS over the sub-field $\mathbb{R}$ of $\mathbb{C}$. Assume that $0 \in C$, then let p be the Minkowski functional for C. Let M= span$\{x_0\}$, define $g : M \to \mathbb{R}$ as $g(tx_0) = t$. Now for $t > 0$, $\frac{1}{t}(tx_0) = x_0 \notin C$, so $g(tx_0) = t \leq p(tx_0)$(by definition), for $t \leq 0$. it holds obviously as p is a non-negative. Hence $g \leq p$ on M. Now by Hanh-Banach theorem we have linear functional $f : V \to \mathbb{R}$ such that $f_{|_M} = g$ and $f \leq p$ on V. Now for $x \in C$, $f(x) \leq p(x) < 1 = g(x_0) = f(x_0)$. Moreover $\exists \beta > 0$ such that $f(x) \leq p(x) < \beta\|x\|$. Hence $\|f\| \leq \beta$.

Now if $0 \notin C$, then take $y \in C$. Let $C' = C - y$ then C' is also an open convex set, which contains 0 and $x_0 - y \notin C'$. Now find a function f as above for C', then for x in C $f(x - y) < f(x_0 - y) \implies f(x) < f(x_0)$.

Now consider V as a NLS over $\mathbb{C}$(original V), the function f we have found is going to be $\mathbb{R}$ linear, then by proposition 2, we have $\widetilde{f}$ a $\mathbb{C}$ linear map with $Re(\widetilde{f}) = f$ and also $\left\|\widetilde{f}\right\| = \|f\|$. And also $Re(\widetilde{f})(x) = f(x) < f(x_0) = Re(\widetilde{f})(x_0)\ \forall x \in C$. □

Lemma 1: Let V be a NLS and f be a bounded linear functional, then f is an open map.

Proof. Let K=Ker(f), choose a $\widetilde{x} \in V - K$ such that $f(\widetilde{x}) = 1$. Take any open set U in V, for $x_0 \in U$, $f(x_0) = t_0 \in f(U)$. For some $\delta > 0$, we have $B(x_0, \delta) \subset U$. $x_0 = t_0\widetilde{x} + y$ for some y in K. Now for $t' \in B(t_0, \frac{\delta}{\|\widetilde{x}\|})$, take $x' = t'\widetilde{x} + y$. We have $f(x') = t'$ and $\|x_0 - x'\| = \|\widetilde{x}\| \, |t' - t_0| < \delta$, hence $x' \in B(x_0, \delta) \subset U$. So $t' \in f(U)$, hence f(U) is open. $\square$

Theorem 2: Let C_1 and C_2 be two non-empty disjoint convex subsets, such that C_1 is open, then there exists $f \in V^*$ and $\alpha \in \mathbb{R}$ such that $Re(f)(x_1) < \alpha \leq Re(f)(x_2)\ \forall x_1 \in C_1, x_2 \in C_2$.

Proof. Since C_1 is open and C_1 snd C_2 are both convex, $C_1 - C_2$ is open and convex. As C_1 and C_2 are disjoint $0 \notin C_1 - C_2$ by applying theorem 1, we get a bounded linear function $f : V \to \mathbb{C}$ such that $Re(f)(x_1 - x_2) < Re(f)0 = 0 \implies Re(f)x_1 < Re(f)x_2\ \forall x_1 \in C_1, x_2 \in C_2$. Since $f(C_1)$ is open by lemma 1 and is convex, $Re(f)(C_1)$ is also open and convex , as it just a projection of an open convex set on the real line. Hence $Re(f)C_1$ is an open interval, letting $\alpha = \sup_{x_1 \in C_1} Re(f)x_1$ we get $Re(f)(x_1) < \alpha \leq Re(f)(x_2)\ \forall x_1 \in C_1, x_2 \in C_2$. $\square$

Theorem 3: Let E_1 and E_2 be two non-empty, disjoint convex subsets in a NLS V, such that E_1 is closed and E_2 is compact, then $\exists f \in V^*$, $\alpha \in \mathbb{R}$ and $\epsilon > 0$ such that $Re(f)(x_2) \leq \alpha - \epsilon < \alpha < \alpha + \epsilon \leq Re(f(x_1))\ \forall x_1 \in E_1, x_2 \in E_2$.

Proof. If E is a non-empty set in V, define $d_E(x) = \inf_{y \in E} \|x - y\|$ for x $\in$ V and $d(E_1, E_2) = \inf_{x \in E_1, y \in E_2} \|x - y\|$, for any two non-empty sets. Moreover $d_E(x)$ is a continuous function on V. Since E_1 and E_2 are disjoint sets, where the former is closed and latter is compact, $d(E_1, E_2) > 0$. Choose a d_0 such that $0 < d_0 < d(E_1, E_2)$. Let $U = \{x \in V \mid d_{E_2}(x) < d_0\}$, then U is an open set which contains E_2 and is disjoint from E_1, as for x in E_1 $d_{E_2}(x) \geq d(E_1, E_2) > d_0$. Now we will show that U is convex. Now for any x in V there exists $y_x \in E_2$ such that $d_{E_2}(x) = \|x - y_x\|$, as follows consider $d_{\{x\}}$ it is a continuous function on V, as E_2 is compact $\exists y_x \in E_2$ such that $d_{\{x\}}(y_x) \leq d_{\{x\}}(y) \forall y \in E_2$, which is same as $d_{E_2}(x) = \|x - y_x\|$.

Let $x_1, x_2 \in U$, so $\exists y_1, y_2 \in E_2$ such that $d_{E_2}(x_1) = \|x_1 - y_1\|$ and $d_{E_2}(x_2) = \|x_2 - y_2\|$. Take $0 < \theta < 1$, $\theta y_1 + (1 - \theta) y_2 \in E_2$ as it is convex.

$$d_{E_2}(\theta x_1 + (1-\theta)x_2) \leq \|\theta x_1 + (1-\theta)x_2 - \theta y_1 - (1-\theta)y_2\| \leq \theta\|x_1 - y_1\| + (1-\theta)\|x_2 - y_2\|$$

$$= \theta d_{E_2}(x_1) + (1-\theta) d_{E_2}(x_2) < \theta d_0 + (1-\theta) d_0 = d_0$$

Hence U is convex. Now we apply theorem 2 to get $f \in V^*$, $\widetilde{\alpha} \in \mathbb{R}$ such that $Re(f)(x) < \widetilde{\alpha} \leq Re(f)(y)\ \forall x \in U, y \in E_1$. As E_2 is compact and convex $Re(f)(E_2)$ is a closed interval in

Re(f)(U), which is an open interval. So if $c = \sup Re(f)(E_2)$, take $\alpha = \frac{\widetilde{\alpha}+c}{2}$ and $\epsilon = \frac{\widetilde{\alpha}-c}{2}$, hence we have $Re(f)(x) \leq c = \alpha - \epsilon < \alpha < \alpha + \epsilon = \widetilde{\alpha} \leq Re(f)(y)\ \forall x \in E_2, y \in E_1$. With this our proof is complete. □

0.2 Norm-Preserving Extension Theorem

Lemma 2: Let V be a $\mathbb{F}$ linear space($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.) Let $p : V \to \mathbb{R}$ be a semi-norm, M be a subspace of V and $g : M \to \mathbb{F}$ be a linear functional such that $|g| \leq p(x)$, then there exists f linear extension of g such that $|f| \leq p(x)$ on V.

Proof. First consider the case in which $\mathbb{F} = \mathbb{R}$. $g \leq |g| \leq p(x)$ and p(x) is also a sub-linear map, so by Hahn-Banach theorem there exists $f : V \to \mathbb{R}$ such that $f_{|_M} = g$ and $f \leq p$ on V. Now $-f(x) = f(-x) \leq p(-x) = p(x)$, hence $|f| \leq p$.

Now come to the case in which $\mathbb{F} = \mathbb{C}$. First consider V as a space over sub-field $\mathbb{R}$ of $\mathbb{C}$. If M is a subspace of V when considered as a space over $\mathbb{C}$, then it will remain as a subspace of V when considered as a space over $\mathbb{R}$. Re(g) is now a $\mathbb{R}$ linear map on M and $|Re(g)| \leq |g| \leq p$, so there exists f a $\mathbb{R}$ linear map such that $|f| \leq p$ and $f_{|_M} = Re(g)$. Now take $\widetilde{f}(x) = f(x) - if(ix)$, as $Re(\widetilde{f}_{|_M}) = Re(g)$ hence we must have $\widetilde{f}_{|_M} = g$ and as $|f| \leq p$ we have $|\widetilde{f}| \leq p$. □

Theorem 4: Let V be a NLS, M be a subspace of V and $g : M \to \mathbb{F}$ be a bounded linear functional, then there exists a linear extension of g which is continuous and $\|f\| = \|g\|$.

Proof. As g is a bounded linear map on M, hence $|g(x)| \leq \|g\|\|x\|\ \forall x \in M$. Define $p(x) = \|g\|\|x\|$, for all x in V. It is clearly a semi-norm, and we have $|g| \leq p(x)$, so by lemma 2 there exists a linear extension f of g, such that $|f| \leq p(x) = \|g\|\|x\|\ \forall x \in V$. So f is continuous and $\|f\| \leq \|g\|$ and we always have $\|g\| \leq \|f\|$, so $\|f\| = \|g\|$. With this our proof is completed. □

0.3 Open Mapping And Closed Graph Theorem

Open mapping theorem: Let X and Y be two Banach spaces, $f : X \to Y$ be a bounded linear map which is surjective also, then f is an open map.

It is one of the most fundamental theorems in functional analysis, it gives rise to closed graph theorem. We will not do the proof of it, as it is a very standard theorem.

Closed graph theorem : Let X and Y be two Banach spaces, $f : X \to Y$ be a closed linear map, then f is a continuous map.

Lets see some applications of the above two theorem.

1. $(A)_{ij} := a_{ij}$ be a matrix. Let $(x) \in l^p$, define Ax by declaring $Ax(i) = \sum_{j=1}^{\infty} a_{ij}x_j$, where $1 \leq p \leq \infty$, if $A(x) \in l^p \forall x \in l^p$, then $A : l^p \to l^p$ is a bounded linear operator.

   Proof. **Case** 1: Let $p = \infty$. Define $sgn(z) = \frac{\bar{z}}{|z|}$, when $z \neq 0$ and 0 when $z = 0$ $\forall z \in \mathbb{C}$. So we have that $z(sgn(z)) = |z|$ $\forall z \in \mathbb{C}$. We show that for each "i" $\{a_{ij}\}_{j=1}^{\infty}$ is in l^1. Take x=$\{sgn(a_{ij})\}_{j=1}^{\infty}$, then x is in l^∞, hence A(x) is in l^∞. So $Ax(i) = \sum_{j=1}^{\infty} a_{ij}sgn(a_{ij}) = \sum_{j=1}^{\infty} |a_{ij}| < \infty$, so $\{a_{ij}\}_{j=1}^{\infty}$ is in l^1 for each i.

   Case 2: Let $p = 1$. Assume that for some i $\{a_{ij}\}_{j=1}^{\infty}$ is unbounded, then $\exists$ an increasing sequence $\{j_k\}_{k=1}^{\infty}$ of natural numbers such that such that $|a_{ij_k}| \geq k$ for each k. Define $x(j) = 0$ $j \neq j_k$ for any k and $\frac{sgn(a_{ij_k})}{k^{\frac{3}{2}}}$ when $j = j_k$ for some k. Now $\sum_{j=1}^{\infty} |x(j)| = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$, so x in in l^1, hence Ax is in l^1. But $Ax(i) = \sum_{j=1}^{\infty} a_{ij}x(j) = \sum_{k=1}^{\infty} \frac{|a_{ij_k}|}{k^{\frac{3}{2}}} \geq \sum_{k=1}^{\infty} \frac{k}{k^{\frac{3}{2}}} = \infty$. But this is a contradiction. Hence $\{a_{ij}\}_{j=1}^{\infty}$ is bounded for each i.

   Case 3: Let $1 < p < \infty$. Assume that for some "i" $\sum_{j=1}^{\infty} |a_{ij}|^q = \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. Define $1, n_1, n_2, n_3, \ldots$ such that

$$\sum_{j=1}^{n_1} |a_{ij}|^q \geq 1^{\frac{2}{p-1}}$$

$$\sum_{j=n_1+1}^{n_2} |a_{ij}|^q \geq 2^{\frac{2}{p-1}}$$

$$\vdots$$

$$\sum_{j=n_{k-1}+1}^{n_k} |a_{ij}|^q \geq k^{\frac{2}{p-1}}$$

$$\vdots$$

   We can see that $1 \leq n_1 < n_2 < n_3 < \ldots$.

Define x(j) =

$$\frac{sgn(a_{ij})|a_{ij}|^{\frac{q}{p}}}{\sum_{k=1}^{n_1}|a_{ik}|^q} \quad j = 1, \ldots, n_1$$

$$\frac{sgn(a_{ij})|a_{ij}|^{\frac{q}{p}}}{\sum_{k=n_1+1}^{n_2}|a_{ik}|^q} \quad j = n_1+1, \ldots, n_2$$

$$\vdots$$

$$\frac{sgn(a_{ij})|a_{ij}|^{\frac{q}{p}}}{\sum_{k=n_{l-1}+1}^{n_l}|a_{ik}|^q} \quad j = n_{l-1}+1, \ldots, n_l$$

$$\vdots$$

Then $\sum_{j=1}^{\infty}|x(j)|^p = \sum_{l=1}^{\infty}\frac{1}{\left(\sum_{k=n_{l-1}+1}^{n_l}|a_{ik}|^q\right)^{p-1}} \leq \sum_{l=1}^{\infty}\frac{1}{k^2} < \infty$. Hence x is in l^p.

But $Ax(i) = \sum_{j=1}^{\infty}a_{ij}x_j = \sum_{l=1}^{\infty}\left(\frac{\sum_{j=n_{l-1}+1}^{n_l}|a_{ij}|^q}{\sum_{j=n_{l-1}+1}^{n_l}|a_{ij}|^q}\right) = \sum_{l=1}^{\infty}1 = \infty$, Which is a contradiction and hence for each "i" $\{a_{ij}\}_{j=1}^{\infty}$ is in l^q.

Now we come to the end of the proof. We will show that $x \to Ax$ is a closed map in each case and as l^p is Banach for each p, we can conclude that the map is continuous by closed graph theorem.

Let $\{x^n\}_{n=1}^{\infty}$ be a sequence in l^{∞} such that $\|x^n\|_{\infty} \to 0$ as $n \to \infty$ and $Ax^n \to y$ in l^{∞}, then we have $Ax^n(i) \to y(i)$ for each "i" and then

$$|\sum_{j=1}^{\infty}a_{ij}x_j^n| \leq \sum_{j=1}^{\infty}|a_{ij}||x_j^n| \leq \|x^n\|_{\infty}\underbrace{\sum_{j=1}^{\infty}|a_{ij}|}_{<\infty} \tag{1}$$

So for a fixed i right hand side in $(1) \to 0$ as $n \to \infty$, hence $Ax^n(i) \to 0$ and $y(i) = 0$. So $y = 0$.

Now let $\{x^n\}_{n=1}^{\infty}$ be in l^1 , $\|x^n\|_1 \to 0$ and $Ax^n \to y$ in l^1. Then we have $Ax^n(i) \to y(i)$ for each i.

$$|\sum_{j=1}^{\infty}a_{ij}x_j^n| \leq \sum_{j=1}^{\infty}|a_{ij}||x_j^n| \leq \underbrace{\sup_j |a_{ij}|}_{<\infty}|\|x^n\|_1$$

as $\|x^n\|_1 \to 0$ we have that $y(i) = 0$ for each i.

Now for $1 < p < \infty$, let $(x^n) \to 0$ in l^p and $Ax^n \to y$ in l^p. Then again we have $Ax^n(i) \to y(i)$ for each i.

$$|\sum_{j=1}^{\infty} a_{ij}x_j^n| \leq \sum_{j=1}^{\infty} |a_{ij}||x_j^n| \leq \underbrace{\left(\sum_{j=1}^{\infty} |a_{ij}|^q\right)^{\frac{1}{q}}}_{<\infty} \underbrace{\left(\sum_{j=1}^{\infty} |x_j^n|^p\right)^{\frac{1}{p}}}_{\text{goes to zero as n goes to } \infty}$$

Hence $y(i) = 0$ for each i and we have $y = 0$. So A is a closed map and we are done.

□

2. (X, Ω, μ) be a σ finite measure space. $K : X \times X \to \mathbb{C}$ is an $\Omega \times \Omega$ measurable function such that for each f in $L^p(\mu)$ $a.e[\mu]$ x $K(x, .)f(.) \in L^1(\mu)$ and $Kf(x) = \int K(x,y)f(y)d\mu(y)$ is an element of $L^p(\mu)$, then $f \to Kf(x)$ is a bounded linear map on $L^p(\mu)$, where $1 \leq p < \infty$.

Proof. Linearity is not hard to prove. So we shall only prove that it is a continuous map.

Case 1: Let $p = 1$. Define $k_x(y) = K(x,y)$, let "x" be such that K_x is not essentially bounded. Let $E_k = \{y \mid |k_x(y)| \geq k\}$, then for each $k \in \mathbb{N}$, $\mu(E_k) > 0$, now as the space is $\sigma-$finite, we can get a subset $\widetilde{E_k}$ of E_k such that $0 < \mu(\widetilde{E_k}) < \infty$.

Define f= $\sum_{k=1}^{\infty} \frac{\chi_{\widetilde{E_k}}}{k^{\frac{3}{2}}\mu(\widetilde{E_k})}$, then

$$\int |f| = \int \sum_{k=1}^{\infty} \frac{\chi_{\widetilde{E_k}}}{k^{\frac{3}{2}}\mu(\widetilde{E_k})} = \sum_{k=1}^{\infty} \int \frac{\chi_{\widetilde{E_k}}}{k^{\frac{3}{2}}\mu(\widetilde{E_k})} = \sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$$

So $f \in L^1(\mu)$, but

$$\begin{aligned} \int |K_x(y)||f(y)|d\mu(y) &= \int \sum_{k=1}^{\infty} \frac{|K_x(y)|\chi_{\widetilde{E_k}}}{k^{\frac{3}{2}}\mu(\widetilde{E_k})} \\ &= \sum_{k=1}^{\infty} \int_{\chi_{\widetilde{E_k}}} \frac{|K_x(y)|}{k^{\frac{3}{2}}\mu(\widetilde{E_k})} \\ &\geq \sum_{k=1}^{\infty} \frac{k\mu(\widetilde{E_k})}{k^{\frac{3}{2}}\mu(\widetilde{E_k})} \\ &\geq \sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{2}}} = \infty. \end{aligned} \tag{0.3.1}$$

Hence $K_x(.)f(.)$ is not in $L^1(\mu)$, so the measure of set of x such that $K_x(.)$ is not essentially bounded is zero.

Case 2: Let $1 < p < \infty$. Let x be such that $K_x(.)$ is not in $L^q(\mu)$. Consider $E^k = \{y \mid k \leq |K_x(y)| < k+1\}$ as the space is $\sigma-$finite, write $E^k = \sqcup_{n=1}^{\infty} E_n^k$, where $\mu(E_n^k) < \infty$, so $X = \sqcup_{k=0}^{\infty} \sqcup_{n=1}^{\infty} E_n^k$. As the collection $\{E_n^k \mid 0 \leq k < \infty, 1 \leq n < \infty\}$ is countable, reorder it in the form $\{E_v\}_{v=1}^{\infty}$. So $X = \sqcup_{v=1}^{\infty} E_v$ and $\mu(E_v) < \infty$. $\int_{E_v} |K_X(y)|^q d\mu(y) < \infty$ as $|k_x(y)|$ is bounded in each E_v.

Now $\int_X |K_x(y)|^q d\mu(y) = \sum_{v=1}^{\infty} \int_{E_v} |K_x(y)|^q d\mu(y) = \infty$

As before let $1 \leq v_1 < v_2 < \dots$ be an increasing sequence such that

$$\sum_{v=1}^{v_1} \int_{E_v} |K_x(y)|^q \geq 1^{\frac{2}{p-1}}$$

$$\vdots$$

$$\sum_{v=v_{l-1}+1}^{v_l} \int_{E_v} |K_x(y)|^q \geq l^{\frac{2}{p-1}}$$

$$\vdots$$

Define f(y)= $\sum_{l=1}^{\infty} \left(\frac{sgnK_x(y)|K_x(y)|^{\frac{q}{p}} \sum_{v=v_{l-1}+1}^{v_l} \chi_{E_v}(y)}{\sum_{v=v_{l-1}+1}^{v_l} \int_{E_v} |K_x(y)|^q d\mu} \right)$

Then

$$\begin{aligned}\int |f|^p d\mu &= \sum_{l=1}^{\infty} \left(\frac{\sum_{v=v_{l-1}+1}^{v_l} \int_{E_v} |K_x(y)|^q d\mu}{\left(\sum_{v=v_{l-1}+1}^{v_l} \int_{E_v} |K_x(y)|^q d\mu\right)^p} \right) \\ &= \sum_{l=1}^{\infty} \left(\frac{1}{\left(\sum_{v=v_{l-1}+1}^{v_l} \int_{E_v} |K_x(y)|^q d\mu\right)^{p-1}} \right) \qquad (0.3.2) \\ &\leq \sum_{l=1}^{\infty} \frac{1}{l^2} < \infty.\end{aligned}$$

So f is in $L^p(\mu)$. But $\int |k_x(y)f(y)| = \int_X \sum_{l=1}^{\infty} \frac{|K_x(y)|^q \sum_{v=v_{l-1}+1}^{v_l} \chi_{E_v}(y)}{\sum_{v=v_{l-1}+1}^{v_l} \int_{E_v} |K_x(y)|^q d\mu} = \sum_{l=1}^{\infty} 1 = \infty.$

Hence the measure of set of all x such that $K_x(y)$ is not in $L^q(\mu)$ is zero.

Now assume that, $\{f^n(y)\}$ is a sequence in $L^p(\mu)$ such that $\|f^n\|_p \to 0$ as $n \to \infty$ and $\{Kf^n\}$ converges to f in $L^p(\mu)$, then there exists a subsequence of $\{Kf^n\}$ which converges to f point wise $a.e[\mu]$, WLOG assume that the subsequence is the $\{Kf^n\}$, then $Kf^n(x) \to f(x)$ a.e[x]. Let x be such that $Kf^n(x) \to f(x)$ and $K_x(.)$ is in $L^q(\mu)$, then

$$|Kf^n(x)| = |\int K_x(y) f^n(y) d\mu(x)| \leq \int |K_x(y)||f^n(y)| d\mu \leq \left(\int |K_x(y)|^q\right)^{\frac{1}{q}} \|f^n\|_p$$

Now the right hand side of the above inequality goes to zero as n goes to infinity. Hence $Kf^n(x) \to 0$ as $n \to \infty$, So f(x)=0 $a.e[\mu]$. Similarly if $p = 1$, then again by taking x such that $K_x(.)$ is essentially bounded and $Kf^n(x)$ converges to f, by same method we can show that $f(x) = 0$.

So the mapping $f \to Kf(x)$ is a closed mapping in each case and as $L^p(\mu)$ is Banach for each p, so by closed graph theorem the mapping is continuous.

$\square$

3. Let X be a Banach space and assume there exists $\{x_n\}$ a sequence in X such that for each x in X there are unique scalars $\{\alpha_n\}$ such that $\lim_{n\to\infty} \|x - \sum_{k=1}^{n} \alpha_k x_k\| = 0$. Such a sequence is called a Schauder basis.

   Define Y=$\{(\alpha_k) \mid \sum_{k=1}^{\infty} \alpha_k x_k$ converges in X$\}$. Addition and scalar multiplication are defined point wise. It's easy to see that Y is a linear space. Define $\|.\|_Y$ on Y as follows $\|(\alpha_k)\|_Y =$

$\sup_l \left\| \sum_{k=1}^{l} \alpha_k x_k \right\|$. First we need to check that $\|(\alpha_k)\|$ is finite for elements in Y. Let $(\alpha_k) \in Y \implies \sum_{k=1}^{n} \alpha_k x_k \to x \in X$ and hence $\left\|\sum_{k=1}^{n} \alpha_k x_k\right\| \to \|x\|$. Hence the sequence $\left\|\sum_{k=1}^{n} \alpha_k x_k\right\|$ is bounded and $\|(\alpha)_k\|_Y < \infty$.

If $\|(\alpha)_k\|_Y = 0$, then let $x = \sum_{k=1}^{\infty} \alpha_k x_k$, now $\left\|\sum_{k=1}^{l} \alpha_k x_k\right\| \le \|(\alpha)_k\|_Y = 0\ \forall l$ and since $\|x\| = \lim_{l\to\infty} \left\|\sum_{k=1}^{l} \alpha_k x_k\right\|$, we have $\|x\| = 0 \implies x = 0$. Now by uniqueness we must have $\alpha_k = 0\ \forall k$, hence $(\alpha_k) = 0$. If $(\alpha_k) = 0$ then obviously $\|(\alpha_k)\|_Y = 0$. Triangle inequality and homogeneity can be easily verified, hence $\|.\|_Y$ is a norm on Y.

Now lets show that it is Banach with this norm. Let $(\alpha^n)_{n=1}^{\infty}$ be a Cauchy sequence in Y. First we show that $(\alpha_l^n)_{n=1}^{\infty}$ converges in $\mathbb{C}$, for each fixed "l", as n→ ∞. Let l be fixed, consider

$$\begin{aligned}\left\| \left(\sum_{k=1}^{l} \alpha_k^n x_k \right) - \left(\sum_{k=1}^{l} \alpha_k^m x_k \right) \right\| &\le \sup_p \left\| \left(\sum_{k=1}^{p} \alpha_k^n x_k \right) - \left(\sum_{k=1}^{p} \alpha_k^m x_k \right) \right\| \\ &= \sup_p \left\| \sum_{k=1}^{p} (\alpha_k^n - \alpha_k^m) x_k \right\| \\ &= \|(\alpha^n) - (\alpha^m)\|_Y \to 0 \text{ as } n, m \to \infty\end{aligned} \tag{0.3.3}$$

So $\{\sum_{k=1}^{l} \alpha_k^n x_k\}$ is Cauchy. As $\{x_1, x_2, \ldots, x_l\}$ is an independent set we have

$$\alpha_1^n \to \widetilde{\alpha_1} \text{in } \mathbb{C}$$

$$\vdots$$

$$\alpha_l^n \to \widetilde{\alpha_l}$$

and $\sum_{k=1}^{l} \alpha_k^n x_k \to \sum_{k=1}^{l} \widetilde{\alpha_k} x_k$. So we have proved that for each l $\alpha_l^n \to \widetilde{\alpha_l}$ in $\mathbb{C}$. Consider $(\widetilde{\alpha_k})$, we show that it is in Y. Given $\epsilon > 0$, $\exists N_\epsilon$ such that for $n, m \ge N_\epsilon\ \forall l$ $\left\|\sum_{k=1}^{l} (\alpha_k^n - \alpha_k^m) x_k\right\| < \epsilon$, as (α^n) is Cauchy in Y. Keeping l fixed and taking limit $m \to \infty$ we get

$$\left\| \sum_{k=1}^{l} (\alpha_k^n - \widetilde{\alpha_k}) x_k \right\| < \epsilon\ \forall n \ge N_\epsilon\ \forall l \tag{1}$$

$$\begin{aligned}\left\|\sum_{k=l_1}^{l_2}\widetilde{\alpha_k}x_k\right\| &\leq \left\|\left(\sum_{k=l_1}^{l_2}\widetilde{\alpha_k}x_k\right)-\left(\sum_{k=l_1}^{l_2}\alpha_k^{N_\epsilon}x_k\right)\right\|+\left\|\sum_{k=l_1}^{l_2}\alpha_k^{N_\epsilon}x_k\right\| \\ &\leq 2\epsilon+\underbrace{\left\|\sum_{k=l_1}^{l_2}\alpha_k^{N_\epsilon}x_k\right\|}_{\to 0 \text{ as } l_1,l_2\to\infty}\end{aligned} \tag{0.3.4}$$

So we have that $(\sum_{k=1}^{n}\widetilde{\alpha_k}x_k)_{n=1}^{\infty}$ is Cauchy and as X is Banach this converges, so $(\widetilde{\alpha_k})\in Y$.

Now in (1) by taking supremum over l we get $\|(\alpha^n)-(\widetilde{\alpha})\|_Y\leq\epsilon\ \forall n\geq N_\epsilon$. So $(\alpha^n)\to(\widetilde{\alpha})$ in Y, hence Y is Banach.

Define $T: X\to Y$ as $T(x)=(\alpha_k)$, where $x=\sum_{k=1}^{\infty}\alpha_kx_k$ as (α_k) is unique, so T(x) is well defined. It is easy to see that it is a linear bijection. Now $T^{-1}: Y\to X$ is $T^{-1}(\alpha_k)=\sum_{k=1}^{\infty}\alpha_kx_k=x\in X$.

So $\left\|T^{-1}(\alpha_k)\right\|=\|x\|=\lim_{l\to\infty}\left\|\sum_{k=1}^{l}\alpha_kx_k\right\|\leq\|(\alpha_k)\|_Y$, hence T^{-1} is a bounded linear map and by Open mapping theorem T will also be a bounded linear map.

Define $\pi_n: Y\to\mathbb{C}$ as follows $\pi_n(\alpha_k)=\alpha_n$. It is projection onto n^{th} coordinate.

Now $|\alpha_n|\underbrace{\text{dist}(x_n, span\{x_1,\ldots,x_{n-1}\})}_{>0 \text{ as } x_1,\ldots,x_n \text{ are independent}}\leq\|\sum_{k=1}^{n}\alpha_kx_k\|\leq\|(\alpha_n)\|_Y$. Hence π_n is a bounded linear map on Y.

Now $f_n=\pi_n\circ T$ is also a bounded linear map on X. And $f_n(\sum\alpha_kx_k)=\alpha_n$. Let $Z_n=$ kernel(f_n), obviously $x_n\notin Z_n$, but $\text{span}\{x_i\mid i\neq n\}\subset Z_n\implies\overline{\text{span}\{x_i\mid i\neq n\}}\subset Z_n$, as Z_n is closed. This tells us that $x_n\notin\overline{\text{span}\{x_i\mid i\neq n\}}\implies\text{dist}(x_n,\overline{\text{span}\{x_i\mid i\neq n\}})>0$.

Now we will finally show that X is separable. Let $D=\{a_k\}_{k=1}^{\infty}$ be a countable dense subset in $\mathbb{C}$. Let $E_n=\{\sum_{i=1}^{n}a_ix_i\mid a_i\in D\}$. E_n is a countable set and hence $E=\cup_{n=1}^{\infty}E_n$ is also a countable set. Given $x=\sum_{i=1}^{\infty}\alpha_ix_i$, there exists n_0, such that $\left\|x-\sum_{i=1}^{n_0}\alpha_ix_i\right\|<\frac{\epsilon}{2}$. Let $M=max_{k=1,2,\ldots,n_0}\|x_k\|$, choose $a_i\in D$ such that $|a_i-\alpha_i|<\frac{\epsilon}{2Mn_0}\ \forall i$, then we have that

$$\left\|x-\underbrace{\sum_{i=1}^{n_0}a_ix_i}_{\in E_{n_0}\subset E}\right\|\leq\left\|x-\sum_{i=1}^{n_0}\alpha_ix_i\right\|+\left\|\sum_{i=1}^{n_0}(a_i-\alpha_i)x_i\right\|\leq\frac{\epsilon}{2}+M\sum_{i=1}^{n_0}|a_i-\alpha_i|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.$$

Hence E is a countable dense set in X.

4. First let X be a NLS and $E \in BL(X)$, such that $E^2 = E$, then M=range E and N=kernel E are closed and complementary subspaces of X.

   Assume $E(x) = y \in N$, then $Ex = E^2x = Ey = 0$, hence $M \cap N = \{0\}$. For x in X, $E(x - Ex) = Ex - E^2x = 0$ so $x - Ex \in N$, hence $x = \underbrace{Ex}_{\in M} + \underbrace{(x - Ex)}_{\in N}$. So M and N are algebraically complemented. But they are more than that, N is closed as E is a bounded linear map. Assume $E(x_n) \to y$ in X. Now $y = E(x) + y'$ where $y' \in N$ and $E(x) \in M$, as E is bounded linear we have $E^2x_n \to E(y) = E^2x + E(y') \implies Ex_n \to Ex$, hence $y = Ex \in M$. So M is closed and if $x \in M$ then $x - Ex \in M \cap N = \{0\}$ so $E(m) = m$ $\forall m \in M$. So E is the projection map onto M.

   Now take X to be a Banach space, let M and N be two closed subspaces in X such that $M \cap N = \{0\}$ and $X = M + N$, so every element x in X can be uniquely written as a sum of an element in M and an element in N. Let E be the projection map onto M, then E is a bounded linear map, with $E^2 = E$. It's easy to show to see that $E^2 = E$. Define $\|.\|'$ on X as $\|x = m + n\|' = \|m\| + \|n\|$, one can easily verify that it is norm on X. Moreover $(X, \|\|')$ is a Banach space since $(X, \|.\|)$ is Banach. Now consider $id : (X, \|.\|') \to (X, \|.\|)$ it is a linear bijection and $\|m + n\| \leq \|m\| + \|n\| = \|m + n\|'$, hence id is a bounded linear bijection, so by Open mapping theorem id is a homeomorphism. So $\exists c > 0$ such that $c(\|m\| + \|n\|) \leq \|m + n\| \; \forall m \in M, n \in N$. Now we are done as, $\|E(m + n)\| = \|m\| \leq \frac{1}{c}\|m + n\|$, hence E is a bounded linear map.

0.4 Tensor Product of Hilbert Spaces

Introduction: Here I have discussed Tensor Product of two Hilbert spaces. I found the original discussion in internet, but at some places rigour was missing and some results were assumed, so I filled in the gaps and made the discussion rigourous and logically coherent. After Tensor product I have discussed an application of it, in which one classifies the Pure contractions on a Hilbert space.

Let H and K be two Hilbert spaces. $(H \otimes K, \phi)$ is said to be the tensor product of H and K if

1. $\phi : H \times K \to H \otimes K$ is a bilinear map. $H \otimes K$ is a Hilbert space.
2. Span of $\phi(H \times K)$ is dense in $H \otimes K$.

3. $< \phi(x,y), \phi(x',y') >=< x,x' >< y,y' > \forall x,x' \in H \; y,y' \in Y.$

We will first prove the existence of $H \otimes K$ and then the uniqueness.

Let $B_c(K,H)$ denote the set of Bounded conjugate linear maps from K into H. So if $T \in B_c(K,H)$, then $T(y+y') = T(y) + T(y')$, $T(\lambda y) = \overline{\lambda} T(y)$ and $\sup\{\|Ty\| \mid \|y\| \leq 1\} < \infty$.

Proposition 1: Given T in $B_c(K,H)$, there exists a unique $T^\# \in B_c(H,K)$ such that

$< Ty, x >=< T^\# x, y > \forall x \in H, y \in K.$

Proof. Let $x \in H$ be fixed. Define $f_x(y) =< x, Ty >$, f_x is a bounded linear functional on K, hence there exists unique $y_x \in K$ such that $f_x(y) =< y, y_x > \forall y \in K$. Define $T^\#(x) = y_x$. Now $\left\|T^\# x\right\| = \|y_x\| = \|f_x\| \leq \|x\| \|T\| \implies \left\|T^\#\right\| \leq \|T\|$.

$$< \lambda x, Ty >= \lambda < x, Ty >= \lambda < y, T^\# x >=< y, \overline{\lambda} T^\# x > \forall y \in K \implies T^\#(\lambda x) = \overline{\lambda} T^\#(x)$$

by uniqueness.

$$< x + x', Ty >=< x, Ty > + < x', Ty >=< y, T^\# x > + < y, T^\# x' >=< y, T^\#(x) + T^\#(x') >$$

$$\implies T^\#(x+x') = T^\#(x) + T^\#(x').$$

Hence $T^\# \in B_c(H,K)$. □

Let I be any set and $f : I \to [0, \infty)$ be a non-negative function on I. Define $\sum_{i \in I} f(i) = \sup_{I' \subset I} \sum_{i \in I'} f(i)$, where I' is a finite subset of I.

Proposition 2: Let $a : I \times J \to [0, \infty)$ be a non-negative function. Denote $a(i,j) = a_{ij}$. Let $\sum_{j \in J} a_{ij} = b_i$ and $\sum_{i \in I} a_{ij} = c_j$, then $\sum_{i \in I} b_i = \sum_{j \in J} c_j$, that is $\sum_{i \in I} \sum_{j \in J} a_{ij} = \sum_{j \in J} \sum_{i \in I} a_{ij}$.

Proof. Let J' and I' be any finite subset of J and I, then

$$\sum_{j\in J'}\sum_{i\in I'} a_{ij} = \sum_{i\in I'}\sum_{j\in J'} a_{ij} \le \sum_{i\in I'} b_i \le \sum_{i\in I} b_i \tag{1}$$

Inequality (1) holds for any $J' \subset J$ and $I' \subset I$, where both J' and I' are finite.

Now my claim is

$$\sup_{I'\subset I}\sum_{j\in J'}\sum_{i\in I'} a_{ij} = \sum_{j\in j'}\left(\sup_{I'\subset I}\sum_{i\in I'} a_{ij}\right) = \sum_{j\in J'} c_j$$

lets prove it. $J' = \{j_1, j_2, \dots, j_m\}$ be fixed, then

$$\sum_{j\in J'}\sum_{i\in I'} a_{ij} = \sum_{i\in I'} a_{i,j_1} + \cdots + \sum_{i\in I'} a_{ij_m}$$

$$\implies \sup_{I'\subset I}\sum_{j\in J'}\sum_{i\in I'} a_{ij} \le c_{j_1} + \cdots + c_{j_m}$$

Now if for any $j_k \in J'$, $c_{j_k} = \infty$, then we are done as $\sum_{i\in I'} a_{ij_k} \le \sum_{j\in J'}\sum_{i\in I'} a_{ij}\ \forall I' \subset I$. So $\infty = c_{j_k} = \sup_{I'\subset I}\sum_{i\in I'} a_{ij_k} \le \sup_{I'\subset I}\sum_{j\in J'}\sum_{i\in I'} a_{ij}$. But if $\forall j_k\ c_{j_k} < \infty$, let $I_1, I_2, \dots, I_m$ be finite subsets of I such that $\sum_{i\in I_k} a_{ij_k} > c_{j_k} - \frac{\epsilon}{m}\ \forall k = 1, \dots, m$. Now let $\widetilde{I} = \cup_{k=1}^m I_k$, $\widetilde{I}$ is again a finite subset of I and

$$\left(\sum_{k=1}^m c_{j_k}\right) - \epsilon \le \sum_{k=1}^m\sum_{i\in I_k} a_{ij_k} \le \sum_{k=1}^m\sum_{i\in\widetilde{I}} a_{ij_k} = \sum_{j\in J'}\sum_{i\in\widetilde{I}} a_{ij} \le \sup_{I'\subset I}\sum_{j\in J'}\sum_{i\in I'} a_{ij}$$

Hence we have $\sup_{I'\subset I}\sum_{j\in J'}\sum_{i\in I'} a_{ij} = \sum_{j\in J'} c_j$ for every finite subset J' of J. So from (1) we have $\sum_{j\in J'} c_j \le \sum_{i\in I} b_i$ for every finite subset J' of J. Hence we have

$$\sum_{j\in J} c_j \le \sum_{i\in I} b_i \text{ and by symmetry } \sum_{i\in I} b_i \le \sum_{j\in J} c_j.$$

So $\sum_{j\in J} c_j = \sum_{i\in I} b_i$. □

Now let $\{f_k\}_{k\in K}$ be a orthonormal basis for K and $\{e_j\}_{j\in J}$ be orthonormal basis for H. Then $Tf_k = \sum_j < Tf_k, ej > e_j$. So, $\left\|Tf_k\right\|^2 = \sum_j | < Tf_k, e_j > |^2$.

$$\sum_k \left\|Tf_k\right\|^2 = \sum_k\sum_j | < Tf_k, e_j > |^2 = \sum_j\sum_k | < T^\# e_j, f_k > |^2 = \sum_j \left\|T^\# e_j\right\|^2$$

So $\sum_k \left\|Tf_k\right\|^2$ is independent of the orthonormal basis chosen.

Define $B(K,H)$ to be the set of all $T \in B_c(K,H)$ such that $\sum_{k\in K}\left\|Tf_k\right\|^2 < \infty$, where $\{f_k\}_{k\in K}$ is an orthonormal basis for K. It's easy to see that it is a linear space. Since for T in B(K,H) $\sum_{k\in K}\left\|Tf_k\right\|^2 < \infty$, Tf_k can be non-zero only for countably many f_k's. Let S,T be in B(K,H), then $\sum_k < Sf_k, Tf_k >$ is a countable sum, therefore it is well defined. Moreover the series converges absolutely and is independent of the basis chosen for K.

By polarization identity we have,

$$< Sf_k, Tf_k >= \frac{1}{4}\left(\left\|Sf_k + Tf_k\right\|^2 - \left\|Sf_k - Tf_k\right\|^2 + i\left\|Sf_k + iTf_k\right\|^2 - i\left\|Sf_k - iTf_k\right\|^2\right) \quad (2)$$

From (2) we can see that the series converges absolutely.

$$\begin{aligned}
\sum_k < Sf_k, Tf_k > &= \frac{1}{4}\left(\sum_k \left\|S + Tf_k\right\|^2 - \sum_k \left\|S - Tf_k\right\|^2 + \sum_k i\left\|S + iTf_k\right\|^2 - \sum_k i\left\|S - iTf_k\right\|^2\right) \\
&= \frac{1}{4}\left(\sum_j \left\|(S+T)^{\#}e_j\right\|^2 - \sum_j \left\|(S-T)^{\#}e_j\right\|^2 + \sum_j i\left\|(S+iT)^{\#}e_j\right\|^2 - \sum_j i\left\|(S-iT)^{\#}e_j\right\|^2\right) \\
&= \frac{1}{4}\left(\sum_j \left\|S^{\#} + T^{\#}e_j\right\|^2 - \sum_j \left\|S^{\#} - T^{\#}e_j\right\|^2 + \sum_j i\left\|S^{\#} + iT^{\#}e_j\right\|^2 - \sum_j i\left\|S^{\#} - iT^{\#}e_j\right\|^2\right) \\
&= \sum_j < S^{\#}e_j, T^{\#}e_j >
\end{aligned} \quad (0.4.1)$$

Since $T \to T^{\#}$ is a linear mapping, we have

So $\sum_k < Sf_k, Tf_k >$ is independent of the orthonormal basis chosen for K.

Now we are ready to define inner product on B(K,H), $< S, T >:= \sum_k < Sf_k, Tf_k > \forall S, T \in B(K,H)$.

$< T, T >\geq 0$ and $< T, T >= 0 \implies \sum_k \left\|Tf_k\right\|^2 = 0 \implies Tf_k = 0 \; \forall f_k$, hence $T = 0$. Linearity w.r.t first argument and conjugate linearity for second argument can be easily verified.

Now we show that B(K,H) is a Hilbert space with the defined inner product. Fix an orthonormal basis $\{f_k\}_{k\in K}$ for K and $\{e_j\}_{j\in J}$ for H. We will create an surjective isometry from B(K,H) to

$l^2(K \times J)$. Now as $l^2(K \times J)$ is a Hilbert space, B(K,H) will also be Hilbert space.

For T in B(K,H) define $g : B(K,H) \to l^2(K \times J)$ as follows $g(T)(k,j) =< Tf_k, e_j >$. Since $\sum_{j,k} |g(T)(k,j)|^2 = \sum_k \|Tf_k\|^2 < \infty$, the map is well defined, moreover it is a linear map and an isometry. It's easy to see the inverse map. Given $\mu = (\mu_{k,j}) \in l^2(K \times J)$, $g^{-1}(\mu)(y) = \sum_j \left(\sum_k \overline{< y, f_k >} \mu_{kj} \right) e_j$ for y in K. Hence B(K,H) is a Hilbert space.

Define $H \otimes K := B(K,H)$ and $\phi(x,y) :=< y, . > x$.

$\sum_k \|\phi(x,y)(f_k)\|^2 = \sum_k \|< y, f_k > x\|^2 = \|x\|^2 \sum_k | < y, f_k > |^2 = \|x\|^2 \|y\|^2 < \infty$. So $\phi(x,y) \in H \otimes K$. ϕ is clearly a bilinear map. We denote $\phi(x,y)$ as $x \otimes y$.

$< x \otimes y(y'), x' >=<< y, y' > x, x' >=< y, y' >< x, x' >=<< x, x' > y, y' > \ \forall x' \in H, y' \in K$

So $(x \otimes y)^\# =< x, . > y$. For T in $H \otimes K$,

$$\begin{aligned} < x \otimes y, T > &= \sum_k < x \otimes y f_k, Tf_k > \\ &= \sum_k << y, f_k > x, Tf_k >= \sum_k < y, f_k >< x, Tf_k > \\ &= \sum_k < y, f_k >< f_k, T^\# x >=< y, T^\# x > \end{aligned} \tag{0.4.2}$$

Now $< x \otimes y, x' \otimes y' >=< y, < x', x > y' >=< x, x' >< y, y' >$. Now we show that $\{f_k \otimes e_j\}$ is an orthonormal basis for $H \otimes K$. $< f_k \otimes e_j, f'_k \otimes e'_j >=< f_k, f'_k >< e_j, e'_j >= \delta_{k,k'} \delta_{j,j'}$, so these vectors clearly form an orthonormal set. Now if $< f_k \otimes e_j, T >= 0 \ \forall k, j$, then $< e_j, Tf_k >= 0 \ \forall j, k$, so by fixing k ,we see that $Tf_k = 0$ for all k and hence $T = 0$. So $\{f_k \otimes e_j\}$ is an orthonormal set in $H \otimes K$, and the span of $\phi(H \times K)$ is dense in it. So we have completed the proof of existence. Now we shall prove the uniqueness.

Lemma 1: If $\sum_{k=1}^n x_k \otimes y_k = 0$ and $\{y_k\}_{k=1}^n$ is an independent set, then $x_k = 0$ $k = 1, 2, \ldots, n$.

Proof. Let $\{z_1, \ldots, z_m\}$ be an orthonormal basis for the space spanned by $\{x_1, \ldots, x_n\}$. Write $x_k = \sum_{r=1}^m \lambda_r^k z_r$,

$$\sum_k (\sum_r \lambda_r^k z_r) \otimes y_k = \sum_k \sum_r \left(\lambda_r^k z_r \otimes y_k \right) = \sum_r \sum_k \left(z_r \otimes (\lambda_r^k y_k) \right) = \sum_r z_r \otimes (\sum_k \lambda_r^k y_k) = 0$$

Let $\sum_k \lambda_r^k y_k = y'_r$, then we have $\sum_r z_r \otimes y'_r = 0$. Now as $\{z_r \otimes y'_r\}_{r=1}^m$ is an orthonormal set, hence $\sum_r \|z_r \otimes y'_r\|^2 = \sum_r \|y'_r\|^2 = 0$. So $y'_r = 0 = \sum_k \lambda_r^k y_k$, now as $\{y_k\}$ is an independent set, we have $\lambda_r^k = 0 \ \forall r, k$. Hence $x_k = 0 \ \forall k$. □

Lemma 2: Let $\psi : H \times K \to E$ be a bilinear map. If $\sum_{k=1}^n x_k \otimes y_k = 0$ then $\sum_{k=1}^n \psi(x_k, y_k) = 0$.

Proof. Let $\{y'_1, \ldots, y'_m\}$ be an independent set for the space spanned by $\{y_1, \ldots, y_n\}$. Write $y_k = \sum_r \lambda_r^K y'_r$.

$$\sum_k x_k \otimes y_k = \sum_k x_k \otimes (\sum_r \lambda_r^k y'_r) = \sum_k \sum_r \left(\lambda_r^K x_k \otimes y_r\right) = \sum_r \left((\sum_k \lambda_r^k x_k) \otimes y'_r \right) = 0$$

Now by lemma 1 we have $\sum_k \lambda_r^k x_k = 0$ for all r. Since ψ is bilinear

$$0 = \sum_r \psi(0, y'_r) = \sum_r \psi(\sum_k \lambda_r^k x_k, y'_r) = \sum_r \sum_k \psi(x_k, \lambda_r^k y'_r) = \sum_k \psi(x_k, \sum_r \lambda_r^k y'_r) = \sum_k \psi(x_k, y_k)$$

So we are done. □

Let $H \odot K =$ span$\{x \otimes y \mid x \in H, y \in K\}$. We know that $H \odot K$ is a dense subspace in $H \otimes K$. Let (E, ψ) be a tensor product of H and K. Define $U : H \odot K \to E$ as follows $U(\sum_{k=1}^n x_k \otimes y_k) = \sum_{k=1}^n \psi(x_k, y_k)$. The map is well defined by Lemma 2. It's easy to see that U is a linear map.

$$\begin{aligned} < U(\sum_{k=1}^n x_k \otimes y_k), U(\sum_{k=1}^n x_k \otimes y_k) > &= < \sum_{k=1}^n \psi(x_k, y_k), \sum_{k=1}^n \psi(x_k, y_k) > \\ &= \sum_{i=1}^n \sum_{j=1}^n < \psi(x_i, y_i), \psi(x_j, y_j) > \\ &= \sum_{i=1}^n \sum_{j=1}^n < x_i, x_j >< y_i, y_j > \qquad (0.4.3) \\ &= \sum_{i=1}^n \sum_{j=1}^n < x_i \otimes y_i, x_j \otimes y_j > \\ &= \left\| \sum_{i=1} x_i \otimes y_i \right\|^2 \end{aligned}$$

Hence U is an isometry and range of U is the span$\{\psi(x,y) \mid x \in H, y \in K\}$. As U is an isometry on $H \odot K$, it extends uniquely to an isometry V on $H \otimes K$, and range of V will be closed in E and as it contains the span$\{\psi(x,y) \mid x \in H, y \in K\}$ which is dense in E, range of V is E. Hence V is a surjective isometry from $H \otimes K$ to E. Moreover $\psi = V \circ \phi$.

Now we shall discuss tensor product of two operators. Let $A \in B(H)$ and $B \in B(K)$, then $A \otimes B$ is a map from $H \otimes K$ into itself, defined as follows $A \otimes B(T) = ATB^*$ for $T \in H \otimes K$. First we need to check that $ATB^* \in H \otimes K$. Let $\{e_j\}$ be a orthonormal basis for H.

$$\begin{aligned}
\sum_k \left(\|ATB^*(f_k)\|^2\right) &\le \|A\|^2 \sum_k \left(\|TB^*(f_k)\|^2\right) \\
&= \|A\|^2 \sum_k \sum_j \left(| < TB^*(f_k), e_j > |^2\right) \\
&= \|A\|^2 \sum_k \sum_j \left(| < T^\#(e_j), B^*(f_k) > |^2\right) \\
&= \|A\|^2 \sum_j \sum_k \left(| < BT^\#(e_j), f_k > |^2\right) \\
&= \|A\|^2 \sum_j \left(\left\|BT^\#(e_j)\right\|^2\right) \le \|A\|^2 \|B\|^2 \sum_j \left(\left\|T^\#(e_j)\right\|^2\right) \\
&= \|A\|^2 \|B\|^2 \|T\|^2
\end{aligned} \tag{0.4.4}$$

So we can conclude that $A \otimes B$ is a bounded linear operator on $H \otimes K$. Lets calculate $A \otimes B(x \otimes y)$, it is a map on K, so lets try to evaluate it on some y' in K.

$$A(x \otimes y)B^*(y') = A(x \otimes y)B^*y' = A < y, B^*y' > x =< By, y' > Ax = Ax \otimes By(y')$$

Hence $A \otimes B(x \otimes y) = Ax \otimes By$.

Let $T \in B_c(K,H),\ A \in B(H),\ B \in B(K)$, then

$$< ATBy, x >=< TBy, A^*x >=< T^\# A^*x, By >=< B^*T^\# A^*x, y >$$

Hence $(ATB)^\# = B^*T^\# A^*$

Proposition 3: The following are true.

1. $(A \otimes B)(A' \otimes B') = AA' \otimes BB'$

2. $(A \otimes B)^* = A^* \otimes B^*$
3. $Id \otimes Id = Id.$
4. $A \otimes B$ is invertible $\iff$ A and B are invertible.

Proof. 1. $(A \otimes B)(A' \otimes B')(T) = A \otimes B(A'TB'^*) = A(A'TB'^*)B' = AA'T(BB')^* = AA' \otimes BB'(T)$.

Hence (1) is true.

2. Let $\{f_k\}$ be orthonormal basis for K and $\{e_j\}$ be orthonormal basis for H. Take $T, \widetilde{T} \in H \otimes K$.

$$\begin{aligned}< A \otimes B(T), \widetilde{T} > &= \sum_k < ATB^*(f_k), \widetilde{T}(f_k) >= \sum_k < TB^*(f_k), A^*\widetilde{T}(f_k) > \\ &= \sum_k \sum_j \Big(< TB^*(f_k), e_j >< e_j, A^*\widetilde{T}(f_k) > \Big) \\ &= \sum_k \sum_j \Big(< T^{\#}(e_j), B^*(f_k) >< A(e_j), \widetilde{T}(f_k) > \Big) \\ &= \sum_j \sum_k \Big(< BT^{\#}(e_j), f_k >< f_k, \widetilde{T}^{\#}A(e_j) > \Big) = \sum_j < BT^{\#}(e_j), \widetilde{T}^{\#}A(e_j) > \\ &= \sum_j < T^{\#}(e_j), B^*\widetilde{T}^{\#}A(e_j) >= \sum_j < T^{\#}(e_j), (A^*\widetilde{T}B)^{\#}(e_j) > \\ &= \sum_k < T(f_k), A^*\widetilde{T}B(f_k) >=< T, A^* \otimes B^*(\widetilde{T}) >\end{aligned} \tag{0.4.5}$$

Hence $(A \otimes B)^* = A^* \otimes B^*$.

3. $Id \otimes Id(T) = Id(T)Id^* = T$

4. If A and B are invertible then $(A \otimes B)(A^{-1} \otimes B^{-1}) = AA^{-1} \otimes BB^{-1} = Id \otimes Id = Id$. Similarly we have $(A^{-1} \otimes B^{-1})(A \otimes B) = Id$.

Now assume that $A \otimes B$ is invertible, then it is lower bounded also, so there exists $\beta > 0$ such that $\beta \|x\| \|y\| = \beta \|x \otimes y\| \leq \|A \otimes B(x \otimes y)\| = \|Ax \otimes By\| = \|Ax\| \|By\|$. So we have that $\frac{\beta}{\|B\|} \|x\| \leq \|Ax\|$, hence A is lower bounded, similarly we can show that A^* is also lower bounded, and hence A is invertible. Similarly one can show B is invertible.

$\square$

0.5 Pure Contractions

Let H be a Hilbert space, $T \in B(H)$ if $\|T\| \leq 1$ and $(T^*)^n \to 0$ as $n \to \infty$ strongly, then we say T is a pure contraction.

Let $l^2 = l^2(\mathbb{N})$ and $S : l^2 \to l^2$, $S(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$, i.e S is a unilateral shift on l^2.

Consider $S \otimes Id : l^2 \otimes H \to l^2 \otimes H$, where H is some Hilbert space. Let L be a invariant subspace of $S \otimes Id$, then define $T = P_{|_{L^\perp}} \circ S \otimes Id_{|_{L^\perp}} : L^\perp \to L^\perp$.

For x,y$\in L^\perp$, $< Tx, y > = \underbrace{< (S \otimes Idx)_{L^\perp}, y > = < S \otimes Idx, y >}_{\text{as } y\in L^\perp} = < x, \underbrace{S^* \otimes Idy}_{\in L^\perp} >$

So $T^* = (S^* \otimes Id)_{L^\perp}$.

$$(S^* \otimes Id)^n (x \otimes y) = (S^*)^n \otimes Id(x \otimes y) = (S^*)^n x \otimes y \quad x \in l^2 \text{ and } y \in H$$

Hence $\|(S^* \otimes Id)^n (x \otimes y)\| = \|(S^*)^n x\| \|y\| \to 0$ as $n \to \infty$. Hence $(S^*)^n \otimes Id \to 0$ in $l^2 \odot H$ and finally because of the denseness of $l^2 \odot H$ it goes to zero in $l^2 \otimes H$. One easily see's that $\|S \otimes Id\| \leq 1$, hence it is a pure contraction. Moreover T is also a pure contraction.

Now let H be a Hilbert space and $T \in B(H)$ be a pure contraction. We will construct a Hilbert space $l^2 \otimes E$, for some Hilbert space E and find an invariant subspace L in $l^2 \otimes E$ for $S \otimes Id$. Moreover $L^\perp$ will be unitary equivalent to H and T will be unitary equivalent to $P_{|_{L^\perp}} \circ S \otimes Id_{|_{L^\perp}}$.

As $\|T^*\| = \|T\| \leq 1$ we have $< (Id - TT^*)(x), x > = < x, x > - < (TT^*)(x), x > = \|x\|^2 - \|T^* x\|^2 \geq 0$. So $Id - TT^* \geq 0$, let $\Delta = (Id - TT^*)^{\frac{1}{2}}$ and $E = \overline{R(\Delta)}$.

Define W: $H \to l^2 \otimes E$ as follows $W(h) = \sum_{k=0}^{\infty} \left(e_k \otimes \Delta (T^*)^k h \right)$, where $e_k(i) = \delta_{i,k+1}$. We need to check if W(h) converges or not. $\{e_k \otimes \Delta (T^*)^k h\}_{k=1}^{\infty}$ is an orthogonal sequence, so it's enough to check that $\sum_{k=0}^{\infty}\left\|e_k \otimes (T^*)^k h\right\|^2 < \infty$.

$$\begin{aligned}
\sum_{k=0}^{\infty}\left\|e_k \otimes (T^*)^k h\right\|^2 &= \sum_{k=0}^{\infty}\|e_k\|^2\left\|\Delta(T^*)^k h\right\|^2 \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} < \Delta(T^*)^k h, \Delta(T^*)^k h > \\
&= \lim_{n\to\infty}\sum_{k=1}^{n} < (T^*)^k h, (Id - TT^*)(T^*)^k h > \qquad (0.5.1)\\
&= \lim_{n\to\infty}\sum_{k=1}^{n}\left(\left\|(T^*)^k h\right\|^2 - \left\|(T^*)^{k+1} h\right\|^2\right) \\
&= \lim_{n\to\infty}(\|h\|^2 - \left\|(T^*)^{n+1} h\right\|^2) = \|h\|^2
\end{aligned}$$

As T is a pure contraction. So we have that $\left\|W(h)\right\|^2 = \|h\|^2$, hence W is an isometry. We will need W^*, so lets calculate it on $e_k \otimes y$ where $y \in E$.

$$\begin{aligned}
< W^*(e_k \otimes y), h > &=< e_k \otimes y, W(h) > \\
&=< e_k \otimes y, \sum_{n=1}^{\infty} e_n \otimes \Delta(T^*)^n h > \qquad (0.5.2)\\
&=< y, \Delta(T^*)^k h > \\
&=< T^k \Delta y, h > \forall h \in H
\end{aligned}$$

Hence $W^*(e_k \otimes y) = T^k\Delta y$. Now $T \circ W^*(e_k \otimes y) = T^{k+1}\Delta y = W^*(e_{k+1} \otimes y) = W^* \circ (S \otimes Id)(e_k \otimes y)$. Since the span of $\{e_k \otimes y \mid k \geq 0, y \in E\}$ is dense in $l^2 \otimes E$, we have $T \circ W^* = W^* \circ (S \otimes Id)$ on $l^2 \otimes E$.

As W is as isometry, R(W) is closed. And $S^* \otimes Id(W(h)) = \sum_{k=0}^{\infty} e_k \otimes \Delta(T^*)^{k+1} h = \sum_{k=0}^{\infty} e_k \otimes \Delta(T^*)^k (T^* h) = W(T^* y)$. So R(W) is an invariant subspace for $S^* \otimes Id$ and as result of it L= $R(W)^{\perp}$ is an invariant subspace for $S \otimes Id$. We have that $L^{\perp}$ is unitary equivalent to H.

$$W \circ T \circ W^* = WW^* \circ (S \otimes Id) = \underbrace{P_{L^{\perp}}}_{\text{projection onto } L^{\perp}} \circ (S \otimes Id)$$

As W is an isometry. So we have $W \circ T \circ W^*_{|_{L^{\perp}}} = P_{L^{\perp}}(S \otimes Id)_{|L^{\perp}}$ on $L^{\perp} = R(W)$.

So we finally conclude that $L^{\perp} = R(W)$ is unitary equivalent to H and T is unitary equivalent to $P_{L^{\perp}}(S \otimes Id)_{|L^{\perp}} \in B(L^{\perp})$. With this we have classified all the pure Contractions on a Hilbert space.

0.6 Weak Topology

Introduction: This chapter is a Literary review. This chapter is important for this Thesis as the concepts in it are heavily used in the Gelfand-Naimark theorem, and besides the topic is itself very beautiful, so it certainly deserved an entire chapter devoted to it. The main result in this chapter is Banach-Alaoglu Theorem. There were some serious gaps in the proof of theorem 2 in the book, so I have filled in the gaps and made the proof simple by providing every detail.

Topological Vector Space: Let V be a vector space over either $\mathbb{R}$ or $\mathbb{C}$. Let τ be a topology on V such that every singleton set is closed and vector addition and scalar multiplication becomes continuous map. More explicitly, given $x, y \in V$ and a nbd W of $x+y$, there exist a nbd U_1 of x and a nbd U_2 of y such that $x' + y' \in W$ for every $x' \in U_1$ and $y' \in U_2$. Similarly given a nbd W of αx, where α is a scalar, then there exist a ball $B(\alpha, r)$ and a nbd U_2 of x such that $ax' \in W$ $\forall a \in B(\alpha, r)$ and $x' \in U_2$.

A local base around a point is a collection β of nbd's of x, such that for any open set U containing x, there exist an element in β contained in it. Now as we shall see because of the fact that addition is continuous, knowing local base around one point gives local base at any other point just by translations.

Theorem 1 : Let V be a topological vector space. T_a defined as $T_a(x) = a + x$ and for $\alpha \neq 0$, $M_\alpha(x) = \alpha x$ are homeomorphism from V to V.

Proof. It's easy to see that that both T_a and M_α are bijective, and their inverses are given by T_{-a} and $M_{\frac{1}{\alpha}}$. Now as scalar multiplication and vector addition are continuous, its basic topology to see that T_a and M_α are continuous for any a$\in$ V and for any scalar. $\square$

Now a very important conclusion of the above theorem is that translation of a open is open and closed is closed. So given a local bases β of 0 in V, local basis for $x \in V$ is collection of translation of elements by x in β and any open set is union of translation of elements by in β.

Lemma 1 : Let W be any nbd of 0 in V, given any $x \in V$ $\exists \alpha \neq 0$ such that $\alpha x \in W$. This type of nbd is known as an absorbing nbd.

Proof. From simple algebra we know $0.x = 0 \in V$. Now scalar multiplication is continuous, so given a nbd W of 0 in V there exist a ball around 0 in field and a nbd around x such that multiplication of a scalar from the ball and an element in the nbd of x lies in W. So pick a non-zero element α from the ball, and then $\alpha x \in W$. □

Lemma 2 : Let V be a topological vector space. Λ be a linear functional on V, if $\Lambda \neq 0$ then $ker(\Lambda)$ has non- empty interior.

Proof. Assume that there exist a nbd $W \subset ker(\Lambda)$. Pick any $x \in W$, take $W - x$ it is again a open set containing zero and is contained in $ker(\Lambda)$ as it is subspace. Now for any $y \in V$ there exists $\alpha \neq 0$ such that $\alpha y \in W - x$. Hence $\Lambda(\alpha y) = \alpha\Lambda(y) = 0$, which tells us that $\Lambda(y) = 0$ but y was any element in V, so $\Lambda = 0$. So our lemma is proved. □

Now let's discuss some Weak topology. Given a set X and "F" a non empty collection of maps of X into some topological space. We would like to give some topology to X such that all maps $f : X \to Y_f$ for $f \in F$ are continuous. One way is to form basis with finite intersection of sets like $f^{-}(U)$, where U is an open set in Y_f and $f \in F$. Then give topology on X by declaring arbitrary union of elements in the basis to be open sets. It is easy to see that this topology is the weakest such that all $f \in F$ are continuous. This topology is called the weak topology induced by F. One very important and well known example of a set given weak topology is product of topological spaces. Let $\{X_\alpha\}$ be a collection of topological spaces and X be product of these spaces. Then the product topology on X is precisely the weak topology induced by all the projection maps from X onto X_α for each α.

Given a topological vector space X, we define X' as collection of all continuous linear functional on X. X' is known as dual of X. A collection F of maps of a set X is said to separate points in X, if given two distinct points x and y in X there exist $f \in F$ such that $f(x) \neq f(y)$.

Lemma 3 : Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ and Λ be linear functional on a vector space V. $N = \{x \mid \Lambda_1(x) = \cdots = \Lambda_n(x) = 0\}$.Then following are equivalent.

1. There exist scalars $\alpha_1, \ldots, \alpha_n$ such that $\Lambda = \alpha_1\Lambda_1 + \cdots + \alpha_n\Lambda_n$.

2. There exist a γ, such that $|\Lambda(x)| \leq \gamma \max_i |\Lambda_i(x)| \, \forall x \in V$.

3. $\Lambda(x) = 0 \, \forall x \in N$.

Proof. If $\Lambda = \alpha_1\Lambda_1 + \cdots + \alpha_n\Lambda_n$ then $\forall x \in V$ we have

$$|\Lambda x| = |\alpha_1\Lambda_1 x + \cdots + \alpha_n\Lambda_n x| \leq |\alpha_1\Lambda_1 x| + \cdots + |\alpha_n\Lambda_n x| \leq (|\alpha_1| + \cdots + |\alpha_2|) \max_i |\Lambda_i(x)|$$

So by taking $\gamma = |\alpha_1| + \cdots + |\alpha_2|$ we have 1 $\implies$ 2. 2 $\implies$ 3 is easy to see. Lets prove 3 $\implies$ 1. Let F denote the underlying field. Define $\phi : V \to F^n$ as follows, $\phi(x) = (\Lambda_1 x, \ldots, \Lambda_n x)$ $\forall x \in V$. It's easy to see that ϕ is linear map, now define a linear functional on $\phi(V)$ as follow $f(\phi(x)) = \Lambda(x)$. It's well defined as, if $\phi(x) = \phi(x') \implies \Lambda_i(x) = \Lambda_i(x') \, \forall i$ and hence $x - x' \in N$ so $\Lambda(x - x') = 0 \implies \Lambda x = \Lambda x'$. Checking f is linear is easy. Now we extend f from $\phi(V)$ to all of F^n linearly and hence there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $f(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n \, \forall (x_1, \ldots, x_n) \in F^n$. Hence $\Lambda x = \alpha_1\Lambda_1 x + \cdots + \alpha_n\Lambda_n x$, $x \in V$. □

In continuation to the above lemma, I would like to prove a fact which we shall use in our next theorem. If V is a topological vector space and $\Lambda, \Lambda_1, \ldots, \Lambda_n$ are continuous linear functionals. If $|\Lambda(x)| \leq \gamma \max_i |\Lambda_i(x)| \, \forall x \in N^c$, then this inequality hold for all points in V. It's enough to show that that $\Lambda(x) = 0 \, \forall x \in N$. Fix a point x in N and given $\epsilon > 0$ there exist a nbd W of x, such that for all $x' \in W$ we have $|\Lambda x - \Lambda x'| < \frac{\epsilon}{2}$. And pick a nbd U of x such that $|\Lambda_i x'| < \frac{\epsilon}{2\gamma}$ $\forall i, \forall x' \in U$. Now $W \cap U$ is a nbd of x and since N has empty interior by lemma 2(assuming all Λ_i are non trivial), there must exist a point x' in $W \cap U \cap N^c$, and hence we have

$$|\Lambda(x)| \leq |\Lambda(x) - \Lambda(x')| + |\Lambda(x')| < \frac{\epsilon}{2} + \gamma \max_i |\Lambda_i x'| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence $\Lambda(x) = 0$. So the inequality holds for all points in V.

Theorem 2 : Let X be a vector space and X' be a vector space of linear functional on X which separates X, then the weak topology τ' induced by X' makes X into a locally convex topological vector space and whose dual is again X'. We will denote the underlying field by F.

Proof. Lets first understand open sets in (X, τ'). Let $W \in \tau'$ and $x' \in W$, so by definition there exist a basis element containing x', contained in W, meaning we have $\Lambda_1, \ldots, \Lambda_n \in X'$

and $U_1,\dots,U_n$ open sets in F, such that $x' \in \{x \in X \mid x \in \Lambda_i^{-1}(U_i) \forall i = 1,\dots,n\} \subset W$. Now $\Lambda_i(x') \in U_i$ so $\exists r_i > 0$ such that $B(\Lambda_i(x'), r_i) \subset U_i$, so $x' \in \{x \in X \mid \Lambda_i(x) \in B(\Lambda_i(x'), r_i), \forall i = 1,\dots,n\} \subset \{x \in X \mid x \in \Lambda_i^{-1}(U_i)\} \subset W$.

So sets of the form $\{x \mid |\Lambda_i x - \Lambda_i x'| < r_i, i = 1,\dots,n, r_i > 0, \Lambda_i \in X'\}$ forms a local base at x'. Now we show that τ' is translational invariant. It is enough to show that translation of a basis element is again a basis element.

$$\beta = \{x \mid |\Lambda_i x - \Lambda_i x'| < r_i\}$$

$$\beta + a = \{x \mid |\Lambda_i(x-a) - \Lambda_i(x')| < r_i\}$$

$$= \{x \mid |\Lambda_i(x) - \Lambda_i(x'+a)| < r_i\}$$

Which is again a basis element at $x' + a$. Hence τ' is translational invariant.

Let β_0 be a local basis at 0 consisting the sets of the form $\{x \mid |\Lambda_i x| < r_i, i = 1,\dots,n, r_i > 0, \Lambda_i \in X'\}$. Now we show elements in β_0 are convex, balancing and absorbent. Let $V = \{x \mid |\Lambda_i x < r_i\}$. Pick x,y in V and $0 < \theta < 1$, then

$$|\Lambda_i(\theta x + (1-\theta)y)| = |\theta \Lambda x + (1-\theta)\Lambda_i y| \leq \theta |\Lambda_i x| + (1-\theta)|\Lambda_i y| < \theta r_i + (1-\theta) r_i = r_i$$

Hence V is convex. Now let $s \in F$ with $|s| \leq 1$ then if $x \in V$ $|\Lambda_i(sx)| = |s||\Lambda_i(x)| \leq |\Lambda_i(x)| < r_i$, hence $sV \subset V$, Hence it is balanced. Let $x \in X$ if $\Lambda_i x = 0$ for all i then it is automatically in V, but if not then take $s \in F$ with $|s| < \min_i \frac{r_i}{|\Lambda_i x|}$ for those i for which $|\Lambda_i x| \neq 0$, then $sx \in V$.

As X' separates X and and F is Hausdorff, (X, τ') is also Hausdorff. Now we shall show addition and scalar multiplication are continuous. Its enough to show "+" is continuous at (0,0). Take V an element in β_0, then $\frac{V}{2} + \frac{V}{2} = V$, as we have just shown that elements in β_0 are convex. So $\frac{V}{2} X \frac{V}{2}$ is our required nbd around $(0,0)$.

Let $\alpha \in F$ and $x \in X$. Let V be an element in β_0, then $\alpha x + V$ is a general basis element at αx. Now $\beta y = \alpha x + (\beta y - \alpha x)$, so if $\beta y - \alpha x \in V$ then $\beta y \in \alpha x + V$.

$$\beta y \in \alpha x + V \iff |\Lambda_i(\beta y - \alpha x)| < r_i$$

$$\iff |\alpha \Lambda_i x - \beta \Lambda_i x + \beta \Lambda_i x - \beta \Lambda_i y| < r_i$$

$$\iff |(\alpha - \beta)\Lambda_i x + \beta(\Lambda_i(x-y))| < r_i$$

So let $r_0 = \min\{r_i\}$, $M = \max_i |\Lambda_i x|$ (assume that $M > 0$). Take β such that $|\beta - \alpha| < \frac{r_0}{2M}$, then $|\beta - \alpha||\Lambda_i x| \leq |\beta - \alpha| M < \frac{r_0}{2} \leq \frac{r_i}{2}$. Moreover $|\beta| \leq \frac{r_0}{2M} + |\alpha|$.

Take W= $\{y \mid |\Lambda_i y - \Lambda_i x| < \frac{r_0 M}{r_0 + 2M|\alpha|}\}$, it's a nbd of x, and we have that if $y \in W$ and $\beta \in F$ with $|\beta - \alpha| < \frac{r_0}{2M}$ then

$$|\beta||\Lambda_i x - \Lambda_i y| < (\frac{r_0}{2M} + |\alpha|)|\Lambda_i x - \Lambda_i y| < \frac{r_0}{2} \leq \frac{r_i}{2}.$$

Hence by taking $(\beta, y) \in (B(\alpha, \frac{r_0}{2M}) X W)$, we have

$$|(\alpha - \beta)\Lambda_i x + \beta(\Lambda_i(x - y))| \leq |\beta - \alpha||\Lambda_i x| + |\beta||\Lambda_i x - \Lambda_i y| < \frac{r_i}{2} + \frac{r_i}{2} = r_i$$

Hence $\beta y \in \alpha x + V$. If $M = 0$, then simply take β such that $|\alpha - \beta| < 1$, so we would have $|\beta| < 1 + |\alpha|$ and then take $W = x + \{y \mid |\Lambda_i y| < \frac{r_i}{|\alpha|+1}\}$. Now if $(\beta, y) \in (B(\alpha, 1) X W)$, then

$$|\Lambda_i(\beta y - \alpha x)| = |\beta||\Lambda_i y| < r_i.$$

Hence scalar multiplication is continuous. All $\Lambda \in X'$ are obviously in the dual of (X, τ'). Now let Λ be continuous linear functional on X, then there exist an element V= $\{x \mid |\Lambda_i(x)| < r_i, \Lambda_i \in X'\}$in β_0 such that for all $x \in V$ $|\Lambda x| < 1$. Let $N = \{x \mid \Lambda_1 x = \cdots = \Lambda_n x = 0\}$. Now take x in N^c. Let $r_0 = \min r_i$. Then $\frac{x r_0}{2 \max |\Lambda_i x|} \in V$, hence $|\Lambda(\frac{x r_0}{2 \max |\Lambda_i x|})| < 1 \implies |\Lambda(x)| < \frac{2}{r_0} \max |\Lambda_i x|$. So taking $\gamma = \frac{2}{r_0}$ we have $|\Lambda(x)| < \gamma \max |\Lambda_i x|$ $\forall x \in N^c$. Now from our previous discussion, we know that if this inequality holds for points in N^c then it must hold for all points in X and hence by lemma 3, there exist scalars $\alpha_1, \ldots, \alpha_n$ such that $\Lambda = \alpha_1 \Lambda_1 + \cdots + \alpha_n \Lambda_n$. Now as X' is linear space $\Lambda \in X'$. So we have shown that the dual of (X, τ') is X' itself. And this completes our proof.

□

Weak topology induced by dual of X

Let (X, τ) be a topological vector space and X^* be the dual of X. Assume that X^* separates X, then let X_w denote the X topologized with weak topology τ_w induced by X^*. By X, we would mean (X, τ) and by X_w, we would mean (X, τ_w). By theorem 2 we know that X_w is a locally convex topological vector space, with dual as X^* itself. As all linear functional in X^* are by definition continuous in X, and τ_w is the weakest topology such that all linear functions in X^* are continuous, we have that $\tau_w \subset \tau$.

To say that $x_n \to x$ in X we would mean every nbd of x in X contains all x_n after some large n. But when we say $x_n \to x$ weakly, we mean that every nbd in X_w contains the sequence after some n. Now an element in local base at x in X_w looks like $\{x' \mid |\Lambda_i x' - \Lambda_i x| < r_i, \Lambda_i \in X^*, i = 1, \ldots, n\}$. So if we have $x_n \to x$ weakly, then $\exists N$ such that $\forall k \geq N$ we must have $|\Lambda_i x_k - \Lambda_i x| < r_i$ for $i = 1, \ldots, n$. So we can see that $\Lambda x_n \to \Lambda x \ \forall \Lambda \in X^*$, converse is easily seen to be true. Hence $x_n \to x$ in $X_w \iff \Lambda x_n$ converges to $\Lambda x \ \forall \Lambda \in X^*$. So if a sequence is strongly convergent then automatically it converges weakly. But it is well know that the converse is false. Let $I = [0, 1]$ with usual topology. It is a compact Hausdorff space. Let X= $C(I)$ be space of continuous complex valued function, with sup norm it is a Banach space. And by Riesz Representation theorem we know the dual of X. Every bounded linear functional on it is integration w.r.t some regular complex borel measure. Now for a sequence in X to converge strongly means uniform convergence on I. Consider $x_n(t) = t^n$ where $t \in I$, now $x_n \in X$, but it is well known that it does not converge in X. Now let ϕ be any element in X^*, then there exist μ a complex borel measure such that $\phi(x) = \int_I x(t) d\mu(t) \ x \in X$. So $\phi(x_n) = \int_I x_n(t) d\mu(t) \to$ zero by DCT. Hence x_n in X_w converges to zero, but it does not converge in X.

Another well known example is of $X = l^2$. Consider $\{e_n\}$ the standard basis elements in X. It cannot converge in X as it is not Cauchy. But if ϕ is an element in X^* then by classification we know there exist $\{x(n)\} \in X$ such that $\phi(y) = \sum_{n=1}^{\infty} x(n) y(n) \ \forall y \in X$. Now it is easy to see that $\phi(e_n) = x(n) \to 0$, hence $\{e_n\}$ converges to 0 in X_w.

These two examples tells us that weak topology can be strictly smaller than the original topology, as it was in the above two cases.

Weak * Topology On The Dual: Let X be a topological vector space and X^* be its dual. Given $x \in X$ define $f_x : X^* \to F$, (F here denote the field) as $f_x(\Lambda) = \Lambda x \ \forall \Lambda \in X^*$. It's a linear functional on X^*, as $f_x(\Lambda + \Lambda') = \Lambda + \Lambda'(x) = \Lambda x + \Lambda' x = f_x(\Lambda) + f_x(\Lambda')$ and similarly $f_x(c\Lambda) = c f_x(\Lambda)$. Let Y= $\{f_x \mid x \in X\}$. Y is a linear space that separates X^*.

$$f_x + f_y(\Lambda) = \Lambda(x) + \Lambda(y) = \Lambda(x + y) = f_{x+y}(\Lambda)$$

So $f_x + f_y = f_{x+y} \in Y$. And similarly

$$\lambda f_x(\Lambda) = \lambda \Lambda(x) = \Lambda(\lambda x) = f_{\lambda x}(\Lambda)$$

Now if $\Lambda \neq \Lambda'$ then there $\exists x \in X$ such that $\Lambda x \neq \Lambda' x$, hence $f_x(\Lambda) \neq f_x(\Lambda')$. So we have proved that Y is a separating linear space of linear functional on x^*. Now topologize X^* with the weak topology induced by Y. This topology on X^* is known as weak * topology. By theorem 2 we know that X^* with this topology is a locally convex topological vector space, and its dual is Y. Now we come to a very important theorem about X^* with weak * topology.

0.7 Banach - Alaoglu Theorem

Theorem 3: Let X be a topological vector space. X^* denotes (X^*, τ^*), where τ^* is the weak * topology. Let V be a nbd of zero in X then $K = \{\Lambda \in X^* \mid |\Lambda x| \leq 1 \forall x \in V\}$ is compact in X^*.

Proof. Let $x \in X$, as every nbd of zero in X is absorbing, for V and x there exist $\gamma(x) > 0$ such that $x \in \gamma(x)V$. Now for $\Lambda \in K$ we have $\frac{|\Lambda(x)|}{\gamma(x)} \leq 1 \implies |\Lambda(x)| \leq \gamma(x)\ \forall \Lambda \in K$.

Define $D_x = \{\alpha \in \mathbb{C} \mid |\alpha| \leq \gamma(x)\}$ then D_x are compact sets. Let $P = \prod_{x \in X} D_x$ with product topology. Now elements in P are $f : X \to \cup_{x \in X} D_x$ such that $f(x) \in D_x$ for each $x \in X$. As every D_x is compact, so P is compact by Tychonoff's Theorem. Now $K \subset P \cap X^*$, we shall show that the topology induced by P and X^* on K are one and the same. We do this by showing every element in K has same local base in X^* and in P.

Let $\Lambda_0 \in K$, $W_1 = \{\Lambda \in X^* \mid |f_{x_i}(\Lambda) - f_{x_i}(\Lambda_0)| < r_i i = 1, \ldots, n\}$ collection of all such sets after intersecting with K gives a local base for Λ_0 in K when considered as a subspace of X^*. But when considered as a subspace of P, sets like $W_2 = \{\Lambda \in P \mid |\Lambda(y_i) - \Lambda_0(y_i)| < a_i, y_i \in X, i = 1, \ldots, m\}$ forms local bases after intersecting with K. But then it is obvious that the collection of set like W_1 and W_2 are same after intersecting with K. Hence local base for Λ_0 in each topology is same and hence topologies are also same.

Now if we can show that K is a closed set in P then as P is compact, K will also be compact and our proof will be complete. Let $\Lambda_0 \in cl(K)$, we need to show that it is linear and $|\Lambda_0| \leq 1$ $\forall x \in V$. Take x,y in X and $\alpha \in \mathbb{C}$. Let $U = \{\Lambda \in P \mid |\Lambda(x) - \Lambda_0(x)| < \epsilon, |\Lambda(y) - \Lambda_0(y)| <$

$\epsilon, |\Lambda(\alpha x + y) - \Lambda_0(\alpha x + y)| < \epsilon\}$ it's a nbd of Λ_0 in P. Hence there exist $\Lambda \in K \cap U$, so

$$\begin{aligned} |\Lambda_0(\alpha x + y) - \alpha\Lambda_0(x) - \Lambda_0(y)| &= |\Lambda_0 - \Lambda(\alpha x + y) + \alpha\Lambda - \Lambda_0(x) + \Lambda - \Lambda_0(y)| \\ &\leq (2 + |\alpha|)\epsilon \end{aligned} \tag{0.7.1}$$

as ϵ was arbitrary, we have that $\Lambda_0(\alpha x + y) = \alpha\Lambda_0(x) + \Lambda_0(y)$, hence Λ_0 is linear. Now let $x \in V$ and $\epsilon > 0$ so $\{\Lambda \in P \mid |\Lambda(x) - \Lambda_0(x)| < \epsilon\}$ is a nbd around Λ_0, so $\exists \Lambda \in K \cap P$, so

$$|\Lambda(x) - \Lambda_0(x)| < \epsilon \implies |\Lambda_0(x)| < \epsilon + 1$$

as $\Lambda(x) \leq 1$. Since $\epsilon > 0$ was arbitrary we have $|\Lambda_0(x)| \leq 1$, hence $\Lambda_0 \in K$. So K is closed and hence is compact under weak * topology. This completes our proof. $\square$

0.8 Banach Algebra

Introduction: This chapter is a Literary review. Banach Algebra is the generalization of the space of all bounded linear maps on a Hilbert spaces. The structure of a Banach algebra is similar to that of space of bounded linear maps, so we have invertible elements, spectral radius and many other similar concepts in it. I have added some necessary details in the proof of proposition 2. After this proposition, I have proved that the mapping $a \to$ r(a), where r(a) is the spectral radius of a, is an upper semi-continuous map.

Definition: Let A be a complex vector space, with a multiplication defined on it such that

1. x.(y.z)= (x.y).z $\forall x, y, z \in A$. That is multiplication is associative.
2. $x.(y+z) = x.y + x.z$ and $(y+z).x = y.x + z.x$ $\forall x, y, z \in A$. It distributes over addition.
3. $\alpha(x.y) = (\alpha x).y = x.(\alpha y)$ $\forall x, y \in A$ and $\forall \alpha \in \mathbb{C}$.

A with above properties is known as a complex algebra. Now a complex algebra A is called a Banach algebra if it has a norm, with which it is a Banach space, $\|xy\| \leq \|x\|\|y\|$ $\forall x, y \in A$ and if there exist an unit element e in A, with $e.x = x.e = x$ $\forall x \in A$ then $\|e\| = 1$.

If A is a Banach algebra without an unit element, then we can easily attach an unit element to it as follows. Define $A_1 = \{(x, \alpha) \mid x \in A, \alpha \in \mathbb{C}\}$. Define addition and scalar multiplication component wise and let $(x, \alpha)(y, \beta) = (xy + \beta x + \alpha y, \alpha\beta)$. With these operation it becomes a complex algebra. Now define $\|(x, \alpha)\| = \|x\| + |\alpha|$. It is obvious with this norm A_1 is a Banach space and a simple calculation shows

$$\begin{aligned} \|(x,\alpha)(y,\beta)\| &= \|xy + \beta x + \alpha y\| + |\alpha\beta| \\ &\leq \|x\|\|y\| + |\beta|\|x\| + |\alpha|\|y\| + |\alpha||\beta| \\ &= (\|x\| + |\alpha|)(\|y\| + |\beta|) = \|(x,\alpha)\|\|(x,\beta)\| \end{aligned} \tag{0.8.1}$$

Moreover $(x, \alpha)(0, 1) = (x, \alpha)$ $\forall (x, \alpha) \in A_1$. So A_1 is a Banach algebra with an unit element. Now $x \in A$ can be identified with $(x, 0)$, and this mapping is even an isomorphic isometry from

A onto $A' = \{(x,0) \mid x \in A\}$. One can see that A_1 mod A' is nothing but $\mathbb{C}$ itself. So A sits inside A_1 as a Banach sub algebra.

In a Banach algebra A, multiplication is a continuous map, as let $x_n \to x$ and $y_n \to y$ then

$$\left\|x_n y_n - xy\right\| = \left\|x_n(y_n - y) + (x_n - x)y\right\| \leq \|x_n\| \left\|y_n - y\right\| + \|x_n - x\| \left\|y\right\|$$

Here we have used a property of Banach algebra that $\left\|xy\right\| \leq \|x\| \left\|y\right\|$. Hence $x_n y_n \to xy$.

Let X be a Banach space, B(X) is the set of all bounded linear operator on X. Then B(X) with usual operator norm and operations becomes a Banach algebra with an unit element. Turns out every Banach algebra A with an unit element can be identified with a Banach sub algebra of B(A).

Let A be a Banach algebra with an unit element. Take $x \in A$, define $M_x : A \to A$ as follows $M_x(y) = xy \ \forall y \in A$. M_x is a linear operator, and $\left\|M_x(y)\right\| = \left\|xy\right\| \leq \|x\| \left\|y\right\|$, hence it is a bounded linear operator on A. Let Y= $\{M_x \mid x \in A\} \subset B(A)$. Moreover, $M_{x+z}(y) = (x+z)y = xy + zy = M_x(y) + M_z(y)$, So $M_{x+z} = M_x + M_z$ and $M_{xz}(y) = (xz)y = x(zy) = M_x(zy) = M_x M_z(y)$, so $M_{xz} = M_x M_z$. Now the mapping $\phi : A \to Y$ defined as $\phi(x) = M_x$, is an isomorphic isometry, as we have $\|M_x\| \leq \|x\|$ and $\left\|M_x(e)\right\| = \|xe\| = \|x\|$ and hence $\|M_x\| = \|x\|$. So we see that A sits inside B(A) as a Banach sub algebra.

Let's see some examples of Banach algebra.

1. Let K be a non-empty compact set and C(K) denotes class of all continuous complex valued function on K. Then with sup norm it is a Banach space and define multiplication point wise, then it becomes a Banach Algebra.

2. Let K be non-empty compact set, B be the collection of functions in C(K) which are holomorphic in the interior of K, then with sup norm it is a complete space. One can use Morera's theorem to prove that B is complete. Hence B is a sub algebra of A.

0.9 Complex Homomorphism

Let A be complex algebra, a complex homomorphism $\phi : A \to \mathbb{C}$ is a non-zero linear map and $\phi(xy) = \phi(x)\phi(y)\ \forall x, y \in A$.

An element $x \in A$ is said to be invertible if $\exists x^{-1} \in A$ such that $x^{-1}x = e = xx^{-1}$. Some easy conclusions that we can make right away from the definitions are:

1. If ϕ is any complex homomorphism on A, then $\phi(e) = 1$, as ϕ is a non zero map so there exist $y \in A$ such that $\phi(y) \neq 0$, then $\phi(y) = \phi(ye) = \phi(y)\phi(e)$, hence $\phi(e) = 1$.
2. If $x \in A$ is invertible , ϕ is any complex homomorphism, then $\phi(x) \neq 0$, as $1 = \phi(e) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1})$.

Now I would like to give a few sufficient conditions for the existence of inverse of an element in a Banach algebra A.

Proposition 1 :

1. Let $x \in A$, if either $\sum_{n=0}^{\infty} x^n$ exist or $\|x\| < 1$ then $e - x$ is invertible.
2. If x is invertible and we have $\left\|(x - y)x^{-1}\right\| < 1$ then y is also invertible. Set of all invertible elements G in A is open and the mapping $x \to x^{-1}$ from G to G, is a continuous map.
3. If ϕ is a complex homomorphism and x is such that $\|x\| < 1$ then $\phi(x) < 1$.

Proof. 1. We have that $\|x^n\| \leq \|x\|^n\ \forall n$, now if $\|x\| < 1$ then $\sum_{n=0}^{\infty} \|x\|^n$ exists as it is nothing but a GP, hence $\sum_{n=0}^{\infty} \|x^n\|$ exist. As A is Banach $\sum_{k=0}^{n} x^k$ converges as $n \to \infty$.

$$\begin{aligned}
\left\| \left(\sum_{k=0}^{\infty} x^k \right) (e - x) - e \right\| &= \lim_{n\to\infty} \left\| \left(\sum_{k=0}^{n} x^k \right) (e - x) - e \right\| \\
&= \lim_{n\to\infty} \left\| e + \left(\sum_{k=1}^{n} x^k \right) - \left(\sum_{k=1}^{n+1} x^k \right) - e \right\| \qquad (0.9.1) \\
&= \lim_{n\to\infty} \left\| x^{n+1} \right\| = 0
\end{aligned}$$

So $\left(\sum_{k=0}^{\infty} x^k\right)(e-x) = e$ and similarly one can show $(e-x)\left(\sum_{k=0}^{\infty} x^k\right) = e$. And hence $e-x$ is invertible.

2. $\left\|(x-y)x^{-1}\right\| < 1 \implies \left\|e - yx^{-1}\right\| < 1$, now by what we have just proved, we have yx^{-1} to be invertible and $\left(\sum_{k=0}^{\infty}(e-yx^{-1})^k\right) yx^{-1} = e$. Now consider

$$\begin{aligned}\left\|x^{-1}\left(\sum_{k=0}^{\infty}(e-yx^{-1})^k\right)y - e\right\| &= \lim_{n\to\infty}\left\|x^{-1}\left(\sum_{k=0}^{n}(e-yx^{-1})^k\right)y - e\right\| \\ &= \lim_{n\to\infty}\left\|x^{-1}\left[\left(\sum_{k=0}^{n}(e-yx^{-1})^k\right)yx^{-1} - e\right]x\right\| \\ &\leq \left\|x^{-1}\right\|\|x\|\lim_{n\to\infty}\left\|\left(\sum_{k=0}^{n}(e-yx^{-1})^k\right)yx^{-1} - e\right\| = 0\end{aligned} \tag{0.9.2}$$

Hence $x^{-1}\left(\sum_{k=0}^{\infty}(e-yx^{-1})^k\right)y = e$ and similarly one can show $yx^{-1}\left(\sum_{k=0}^{\infty}(e-yx^{-1})^k\right) = e$. So $y^{-1} = x^{-1}\left(\sum_{k=0}^{\infty}(e-yx^{-1})^k\right)$.

Now if y is such that $\|x-y\| < \frac{1}{\|x^{-1}\|}$, then we have y to be invertible and $y^{-1} = x^{-1}\left(\sum_{k=0}^{\infty}(e-yx^{-1})^k\right)$.

$$\begin{aligned}\left\|y^{-1} - x^{-1}\right\| &= \left\|x^{-1}\left(\sum_{n=1}^{\infty}(e-yx^{-1})^n\right)\right\| \\ &\leq \left(\left\|x^{-1}\right\|\right)\left(\sum_{n=1}^{\infty}\left\|e-yx^{-1}\right\|^n\right) \\ &\leq \left\|x^{-1}\right\|\left(\frac{\|x-y\|\left\|x^{-1}\right\|}{1-\|x-y\|\|x^{-1}\|}\right).\end{aligned} \tag{0.9.3}$$

Now by keeping x fixed and making $\|x-y\| \to 0$ will make right hand side of the above inequality goes to zero, and hence $\left\|y^{-1} - x^{-1}\right\| \to 0$. So $x \to x^{-1}$ is continuous.

3. Let $\|x\| < 1$, take any $\lambda \in \mathbb{C}$, with $|\lambda| \geq 1$. Then $\left\|\frac{x}{\lambda}\right\| < 1$ and hence $e - \frac{x}{\lambda}$ is invertible.

Now if ϕ is a complex homomorphism then $\phi(e-\frac{x}{\lambda})$ is not zero, so $1-\frac{\phi(x)}{\lambda}\neq 0$, hence $\phi(x)\neq\lambda$ and $|\phi(x)|<1$.

$\square$

Part 3 of the above proposition tell us that a complex homomorphism ϕ is always a bounded linear map with $\|\phi\|\leq 1$.

0.10 Note 1

Let p$\in\mathbb{N}$, Now consider

$$\sum_{k=1}^{p(n+1)}\left\|x^k\right\|\leq\left(\sum_{k=0}^{n}\left\|x^{kp}\right\|\right)\left(\|x\|+\left\|x^2\right\|+\cdots+\|x^p\|\right)$$

Taking limit $n\to\infty$ we get $\sum_{k=1}^{\infty}\left\|x^k\right\|\leq\left(\sum_{k=0}^{\infty}\left\|x^{kp}\right\|\right)\left(\|x\|+\left\|x^2\right\|+\cdots+\|x^p\|\right)$. So if we carefully observe that if $\|x^p\|<1$ then $\sum_{k=0}^{\infty}\left\|x^{kp}\right\|\leq\frac{1}{1-\|x^p\|}<\infty$. Hence again we have that $\sum_{k=0}^{n}x^k$ converges as $n\to\infty$, so by the above proposition $e-x$ is invertible.

Now if $\lambda\in\mathbb{C}$ is such that $|\lambda|^p>\|x^p\|$ then $\left\|\left(\frac{x}{\lambda}\right)^p\right\|<1$ where $p\in\mathbb{N}$, then by our above discussion we have that $e-\frac{x}{\lambda}$ is invertible, hence $x-\lambda$ is invertible and $(x-\lambda)^{-1}=\frac{-1}{\lambda}\left(\sum_{k=0}^{\infty}(\frac{x}{\lambda})^k\right)$.

If $a\in A$ is a nilpotent element, meaning for some $n_0\in\mathbb{N}$ $a^{n_0}=0$, then for any $\lambda\neq 0$, we have $|\lambda|^{n_0}>\|a^{n_0}\|=0$ and hence $a-\lambda$ is invertible for all $\lambda\neq 0$. Isn't it magical that, Just because for some n, $a^n=0$, $(a-\lambda)^{-1}$ exist $\forall\lambda\neq 0$.

0.11 Basic Properties Of Spectra

Let A be Banach algebra. Spectrum of $x \in A$ is defined as the collection of $\lambda \in \mathbb{C}$ such that $x - \lambda$ is not invertible, it is denoted by $\sigma(x)$. Spectral radius of x denoted by r(x) is defined as the $\sup\{|\lambda| \mid \lambda \in \sigma(x)\}$.

Proposition 2

1. $\sigma(x)$ is a compact subset of $\mathbb{C}$.
2. $\sigma(x)$ is always non empty.
3. $r(x) = \lim\|x^n\|^{\frac{1}{n}}$

Proof. 1. We know that if $|\lambda| > \|x\|$ then $x - \lambda$ is invertible, hence if $\lambda \in \sigma(x)$ then we must have $|\lambda| \leq \|x\|$. So $\sigma(x)$ is bounded. Now let $\lambda \in \sigma(x)^c$, so $x - \lambda$ is invertible, now by proposition 1 we have that if $\left\|(x-\lambda') - (x-\lambda)\right\| < \frac{1}{\left\|(x-\lambda)^{-1}\right\|}$ then $x - \lambda'$ is also invertible. Hence $B\left(\lambda, \frac{1}{\left\|(x-\lambda)^{-1}\right\|}\right) \subset \sigma(x)^c$. So $\sigma(x)$ is closed and hence compact.

2. Let $G = \sigma(x)^c$, it is an open set. Define $\phi : G \to A$ as follows $\phi(\lambda) = (x-\lambda)^{-1}$ $\forall \lambda \in G$.

$$\begin{aligned}
\lim_{\lambda' \to \lambda} \frac{\phi(\lambda') - \phi(\lambda)}{\lambda' - \lambda} &= \lim_{\lambda' \to \lambda} \frac{(x-\lambda')^{-1} - (x-\lambda)^{-1}}{\lambda' - \lambda} \\
&= \lim_{\lambda' \to \lambda} \frac{(x-\lambda')^{-1}\left((x-\lambda) - (x-\lambda')\right)(x-\lambda)^{-1}}{\lambda' - \lambda} \\
&= \lim_{\lambda' \to \lambda} = (x-\lambda')^{-1}(x-\lambda)^{-1} \\
&= (x-\lambda)^{-2} \\
&= \phi'(\lambda)
\end{aligned} \tag{0.11.1}$$

It is easy to see that $\phi'(\lambda)$ is a continuous map, hence ϕ is holomorphic map. Many results of complex analysis are true in this setup also, and the proofs are identical.

Now if $G = \mathbb{C}$ then ϕ is an entire function. Moreover $\left\|(x-\lambda)^{-1}\right\| = \frac{1}{|\lambda|}\left\|(\frac{x}{\lambda} - e)^{-1}\right\|$, so if $|\lambda| \to \infty$ then $\left\|(\frac{x}{\lambda} - e)^{-1}\right\| \to 1$. Hence $\left\|(x-\lambda)^{-1}\right\| \to 0$ as $|\lambda| \to \infty$. So ϕ is an entire bounded function and hence it has to be constant, but this is contradiction as ϕ is certainly not constant. So $G \neq \mathbb{C}$ and $\sigma(x) \neq \emptyset$.

3. In note 1 we saw that if $|\lambda|^p > \|x^p\|$ for any $p \in \mathbb{N}$, then $x - \lambda$ is invertible. So if $\lambda \in \sigma(x)$ we must have $|\lambda| \leq \|x^n\|^{\frac{1}{n}}$ $\forall n$, hence $|\lambda| \leq \liminf \|x^n\|^{\frac{1}{n}}$ $\forall \lambda \in \sigma(x)$. So we conclude that $r(x) \leq \liminf \|x^n\|^{\frac{1}{n}}$.

Define G=$\{k \in \mathbb{C} \mid \frac{1}{k} \in \sigma(x)^c\} \cup \{0\}$, it is an open set. To see this consider $0 < |k| < \frac{1}{\|x\|} \iff \frac{1}{|k|} > \|x\|$, then $\frac{1}{|k|} \in \sigma(x)^c$ so $\{k \mid |k| < \frac{1}{\|x\|}\} \subset G$. So 0 is in interior of G. Now take $k \neq 0 \in G$ so $\frac{1}{k} \in \sigma(x)^c$ which is an open set. Let $r > 0$ be such that $B(\frac{1}{k}, r) \subset \sigma(x)^c$, as $x \to x^{-1}$ is a continuous map there exist a $\delta > 0$ such that $B(k, \delta) \subset \mathbb{C} - 0$ and if $\alpha \in B(k, \delta)$ then $\frac{1}{\alpha} \in B(\frac{1}{k}, r) \subset \sigma(x)^c$. Hence $B(k, \delta) \subset G$. So G is open.

Define $\phi : G \to A$ as follows $\phi(k) = (x - \frac{e}{k})^{-1}$ for $k \neq 0$ and 0 when $k = 0$. Its easy to see that ϕ is continuous and is holomorphic everywhere in G except may be at 0. But then from complex analysis we can conclude that ϕ will be holomorphic on all of G. Let $|k| < \frac{1}{\|x\|}$ then

$$\phi(k) = \sum_{n=0}^{\infty} -k^{n+1} x^n$$

Now from complex analysis R-radius of convergence of the above series equals $\frac{1}{\limsup \|x^n\|^{\frac{1}{n}}}$. Which is also the $dist(0, \partial G)$. Now my claim is $dist(0, \partial G) = \frac{1}{r(x)}$. If $r(x) = 0$ then $G = \mathbb{C}$ and $dist(0, \partial G) = \infty$. If $r(x) \neq 0$, then $\exists \lambda_0 \in \sigma(x)$ such that $|\lambda_0| = r(x)$ so $\frac{1}{\lambda_0} \in G^c$ and there will exist a sequence $\{\lambda_n\} \to \lambda_0$ in $\sigma(x)^c$, hence by continuity $\frac{1}{\lambda_n} \to \frac{1}{\lambda_0}$ but $\{\frac{1}{\lambda_n}\} \in G$ so $\frac{1}{\lambda_0} \in cl(G)$ and hence $\frac{1}{\lambda_0} \in \partial G$. This gives us $dist(0, \partial G) \leq \frac{1}{|\lambda_0|} = \frac{1}{r(x)}$. But if $dist(0, \partial G) < \frac{1}{|\lambda_0|}$ then $\exists \lambda \in \partial G$ such that $|\lambda| < \frac{1}{|\lambda_0|}$, now as $\partial G = \overline{G} - G$ so λ does not belong to G, meaning $\frac{1}{\lambda} \in \sigma(x)$ but we also have $\frac{1}{|\lambda|} > |\lambda_0|$ and this contradicts the definition of λ_0. Hence we must have $dist(0, \partial G) = \frac{1}{r(x)}$. Now finally we have

$$r(x) \leq \liminf \|x^n\|^{\frac{1}{n}} \leq \limsup \|x^n\|^{\frac{1}{n}} = \frac{1}{R} = \frac{1}{dist(0, \partial G)} = r(x)$$

So we have that $r(x) = \lim \|x^n\|^{\frac{1}{n}}$. This completes our proof.

$\square$

0.12 Notes 2

The formula $r(x) = \lim \|x^n\|^{\frac{1}{n}}$ is quite interesting, it not only tells us that $\lim \|x^n\|^{\frac{1}{n}}$ exists but the limit is even equal to the spectral radius of x $\forall x \in A$.

One can prove that the function $r : A \to [0, \infty)$, where r(a) is the spectral radius of $a \in A$, is an upper-semi continuous function and is continuous at a if $r(a) = 0$.

First observe that if $a_n \to a$ in A, $\alpha_n \in \sigma(a_n)$ and $\alpha_n \to \alpha$ in $\mathbb{C}$, then $\alpha \in \sigma(a)$. As we have that $a_n - \alpha_n \to a - \alpha$ if $a - \alpha$ was invertible then there would exist a nbd U of $a - \alpha$ in G(set of all invertible elements in A), as G is open. So after some n_0, $a_n - \alpha_n \in U \implies a_n - \alpha_n$ is invertible after n_0 which is a contradiction. Hence $\alpha \in \sigma(a)$.

Let "a" be fixed in A. Take α such that $0 \leq r(a) < \alpha$. Now assume that $\exists \{a_n\} \to a$ and $r(a_n) \geq \alpha \ \forall n$. Now for each n take $\lambda_n \in \sigma(a_n)$ such that $|\lambda_n| = r(a_n)$. So we have $|\lambda_n| \geq \alpha > r(a) \ \forall n$. If $|\lambda_n| \to \infty$ then $\frac{1}{\left\|(a-\lambda_n)^{-1}\right\|} = \frac{|\lambda_n|}{\left\|\left(e-\frac{a}{\lambda_n}\right)^{-1}\right\|} \to \infty$, hence for 1, $\exists N_1$ such that $\forall n \geq N_1$ we have $1 \leq \frac{1}{\left\|(a-\lambda_n)^{-1}\right\|}$. Take a n' such that $n' > N_1$ and $\|a - a_{n'}\| < 1$, Hence $\left\|(a - \lambda_{n'}) - (a_{n'} - \lambda_{n'})\right\| < 1 \leq \frac{1}{\left\|(a-\lambda_{n'})^{-1}\right\|}$ which implies $a_{n'} - \lambda_{n'}$ is invertible and this is a contradiction, hence $|\lambda_n| \nrightarrow \infty$. So there must exist a bounded subsequence of it and a convergent subsequence in that bounded subsequence. So let $\lambda_{n_k} \to \lambda$ and as $|\lambda_{n_k}| \geq \alpha$ we have $|\lambda| \geq \alpha$, moreover $a_{n_k} \to a$, so we must have $\lambda \in \sigma(a)$, which is a contradiction as $|\lambda| \geq \alpha > r(a)$. Hence $\exists$ a nbd U of "a" such that $r(a') < \alpha \ \forall a' \in U$. This completes our proof.

Gelfand - Mazur Theorem

Theorem 1: If A is a Banach algebra, such that every non zero element is invertible in A, then A is isomorphically isometric to $\mathbb{C}$.

Proof. Consider $\phi : \mathbb{C} \to A$, where $\phi(\lambda) = \lambda e$. It is an isometry which is an injective homomorphism also. If we can show that ϕ is surjective then ϕ is an isomorphic isometry. Let $a \in A$, then as we have seen that the spectrum of any element in a Banach algebra is non empty, $\exists\lambda$ such that $a-\lambda$ is not invertible, but then $a-\lambda=0$ and $a=\lambda e$, as every non zero element is invertible. Thus ϕ is surjective also.

□

Lemma 1 : Let V and U be two open sets, such that $V \subset U$ and U contains no boundary points of V. Then V is a union of some components of U.

Proof. Let C be a component of U such that $C \cap V \neq \emptyset$. Now as $C = (C \cap V) \cup (C \cap (\overline{V})^c)$, because if $x \in C$ is not in V then it cannot be in $\overline{V}$ as if it were then x$\in \partial V = \overline{V} - V$, but we have $\partial V \cap U = \emptyset$. Thus $C \cap (\overline{V})^c$ must be empty as C is a connected set. This gives us $C = C \cap V \subset V$. Hence V is the union of all those components of U with which it intersects.

□

Lemma 2 : Let A be a Banach algebra. Let $\{x_n\}$ be a sequence in G(A)(the set of all invertible elements in A) which converges to x$\in \partial G(A)$, then$\left\|(x_n)^{-1}\right\| \to \infty$.

Proof. $x \in \partial G = \overline{G} - G$, which means x is not invertible. Now assume that $\exists M > 0$ such that $\left\|(x_n)^{-1}\right\| \leq M \implies 0 < \frac{1}{M} \leq \frac{1}{\|(x_n)^{-1}\|}$ $\forall n$. Take n' such that $\|x - x_{n'}\| < \frac{1}{M}$. Thus we have $\|x - x_{n'}\| < \frac{1}{M} \leq \frac{1}{\|(x_{n'})^{-1}\|}$ which implies that x is also invertible, which is a contradiction as $x \in \partial G$.

□

Theorem 2:

1. If A is a closed sub-algebra of a Banach algebra B and A contains unit element of B, then G(A) is union of some components of $A \cap G(B)$.
2. Let $x \in A$, then $\sigma_A(x)$ is equal to $\sigma_B(x)$ union some components of $\sigma_B(x)^c$.

Proof. 1. We have that $G(A) \subset G(B) \cap A$. Now both of these sets are open in A, if we can show $G(B) \cap A$ contains no boundary points of $G(A)$ then by Lemma 1 we are done. Now as A is closed sub-algebra of a Banach algebra, hence it is also a Banach algebra. Take $x \in \partial G(A)$ so we can find a sequence $\{x_n\}$ in G(A) such that $x_n \to x$. Now if $x \in G(B)$ then by the continuity of the inverse map, we will have $x_n^{-1} \to x^{-1}$ and hence $\left\| x_n^{-1} \right\|$ will be bounded, which is a contradiction to the lemma 2.

2. Let $x \in A$. First note that $\sigma_B(x) \subset \sigma_A(x)$, as if $x - \lambda$ is not invertible in B then it cannot be invertible in A also. let Ω_A denote $(\sigma_A(x))^c$ and Ω_B denote $(\sigma_B(x))^c$. Then $\Omega_A \subset \Omega_B$ and both of them are open sets in $\mathbb{C}$. Take any $\lambda \in \partial\Omega_A$, then $x - \lambda$ will be a boundary point in G(A) and hence by the previous result $x - \lambda$ cannot belong to G(B). As a result λ does not belong to Ω_B. Now by lemma 1 Ω_A is union of some components of Ω_B. Now since, $\sigma_A(x) = \sigma_B(x) \cup (\sigma_A(x) \cap \Omega_B)$ and $\sigma_A(x) \cap \Omega_B$ is union of components of Ω_B which do not intersect with Ω_A and are bounded. And hence $\sigma_A(x)$ equals $\sigma_B(x)$ union some bounded components of Ω_B.

□

Corollaries

1. If Ω_B is connected then $\sigma_A(x) = \sigma_B(x)$
2. If $\sigma_A(x)$ has empty interior then $\sigma_A(x) = \sigma_B(x)$.

Proof. 1. As $\sigma_B(x)$ is a bounded set in $\mathbb{C}$, Ω_B is connected unbounded set. Now spectrum's are compact sets and as result of it they are bounded. Hence $\sigma_A(x) = \sigma_B(x)$.

2. $\sigma_A(x) \cap \Omega_B$ is an open set, as it is union of some components of Ω_B and components of an open set are open. So we must have $\sigma_A(x) \cap \Omega_B = \emptyset$, since $\sigma_A(x)$ has empty interior.

□

0.13 Commutative Banach Algebra

Introduction: This is a very important chapter as Gelfand Transform is discussed in it. I have tried to simplify the original discussion by providing every details wherever they were missing.

We start by giving some definitions. An ideal I in a commutative Complex algebra A is a linear subspace of A such that if $x \in I$ and $a \in A$ then $ax = xa \in I$. So one can observe that an Ideal is a sub ring, that has one extra property. A proper ideal is an ideal which is proper subset of A. A maximal ideal is a proper ideal which is not contained in any other proper ideal, meaning if I is a maximal ideal, and I' is an ideal in A such that $I \subset I'$, then either $I = I'$ or $I' = A$.

0.14 Proposition 1

In this proposition A stands for a commutative Banach algebra and I is an ideal in A.

1. If I is an ideal then $\overline{I}$ is also an ideal.
2. If I is a proper ideal then $\overline{I}$ is also a proper ideal.
3. A maximal ideal I is always closed.

Proof. 1. $\overline{I}$ is a linear space is basic. If $x \in \overline{I}$ then there exist $\{x_n\} \in I$ such that $x_n \to x$. Let $a \in A$, since multiplication is continuous $ax_n \to ax$ and $ax_n \in I \forall n$ we have $ax \in \overline{I}$.

2. If I is a proper ideal then $I \cap G(A) = \emptyset$, as if any $x \in G(A) \cap I$, then $e = xx^{-1} \in I$ and if $e \in A$ then $I = A$. So $I \subset G(A)^c$, which is a closed set in A, so $\overline{I} \subset G(A)^c$. Hence $\overline{I}$ is also a proper ideal.

3. Let I be a maximal ideal. This means I and $\overline{I}$ both are proper ideals and also we have $I \subset \overline{I}$, so we must have $I = \overline{I}$. Hence I is closed.

□

Theorem 1: Let A be commutative complex algebra, I be a proper ideal in A, then there exist a maximal ideal I' containing I in A.

Proof. Let $\sum$ denote the collection of proper ideals in A which contains I. $\sum$ is a non empty collection, as $I \in \sum$. Define the partial order in $\sum$ as the usual subset relation. Given any ordered chain $\{I_\alpha\}$ in $\sum$, then $\cup_\alpha I_\alpha$ is an ideal containing I and it is proper as e does not belong to any I_α hence e does not belong to the union also. So this union is in $\sum$ and is an upper bound for $\{I_\alpha\}$. Now by Zorn's lemma there exist a maximal element in $\sum$ and this I' is our maximal ideal containing I. □

0.15 Quotient Banach algebra

Given a homomorphism ϕ from a commutative Banach algebra A into a Banach algebra B, then the $ker(\phi)$ is an ideal in A and is closed if ϕ is continuous. Now if we start with a closed ideal I, we can construct the Quotient space A/I, which is a typical Banach space with Quotient norm, but will become a commutative Banach algebra if we define multiplication as follows $(x+I)(y+I) = (xy+I)$. Well definedness can be easily checked. We need to check that $|||xy+I||| \leq (|||x+I|||)(|||y+I|||)\ \forall x,y \in A$ and $|||e+I||| = 1$. let a and a' in I be such that $||x+a|| < |||x+I||| + \epsilon$ and $||y+a'|| < |||y+I||| + \epsilon$, then

$$|||xy+I||| \leq ||(x+a)(y+a')|| = ||xy+xa'+ay+aa'|| \leq (||x+a||)(||y+a'||)$$

$$< (|||x+I|||)(|||y+I|||) + \epsilon(|||x+I||| + |||y+I||| + \epsilon)$$

As ϵ is arbitrary, we have that $|||xy+I||| \leq (|||x+I|||)(|||y+I|||)$

Now $0 < |||e+I||| \leq ||e|| = 1$ and also $|||e+I||| = |||e.e+I||| \leq (|||e+I|||)(|||e+I|||)$, by what we have just proved, hence $1 \leq |||e+I|||$. So we have $|||e+I||| = 1$, and with this we have proved that A/I is a Commutative Banach algebra. Let $\pi : A \to A/I$ be the usual project map on a quotient space, then it is a continuous homomorphism with kernel equal to I.

Given a complex algebra A, Δ is defined as the collection of all complex homomorphism on A. In the next theorem A denotes a commutative Banach algebra.

Theorem 2 :

1. Every maximal ideal M of A is the kernel of some complex homomorphism on A.
2. Kernel of a complex homomorphism is always a maximal ideal.
3. $x \in A$ is invertible iff $h(x) \neq 0 \; \forall h \in \Delta$.
4. $\lambda \in \sigma(x)$ iff $\lambda = h(x)$ for some $h \in \Delta$.

Proof.

1. Let M be a maximal ideal in A, then M is closed and A/M is a commutative Banach algebra. Now take $a \notin M$, let I_a be the ideal generated by a, then $M + I_a$ is also an ideal which is larger than M and contains M, hence $M + I_a = A$. For some $x \in A$ and $z \in M$ we have $xa + z = e$. Now $\pi(xa + z) = e + M \implies \pi(x)\pi(a) = e + M$, what we have proved is that every non zero element in A/M is invertible, hence by Gelfand-Mazur theorem A/M is isomorphically isometric to $\mathbb{C}$. Let $j : A/M \to \mathbb{C}$ be the isomorphism, then $j \circ \pi = h : A \to \mathbb{C}$ is a complex homomorphism on A with kernel= M.

2. As the co-dimension of the kernel of a complex homomorphism is one, that's why it is a maximal ideal.

3. Let $h \in \Delta$ and $x \in G(A)$, then $xx^{-1} = e \implies h(xx^{-1}) = 1 \implies h(x)h(x^{-1}) = 1$, hence $h(x) \neq 0 \; \forall h \in \Delta$. Now if $x \notin G(A)$ then ideal generated by x I_x is a proper ideal, then by theorem 1 we have a maximal ideal I' containing I. Now by what we have just proved $\exists h \in \Delta$, with kernel(h)= I', and hence we have $h(x) = 0$. So if $h(x) \neq 0 \; \forall h \in \Delta$ then x must be in G(A).

4. If $\lambda \in \sigma(x)$, then $x - \lambda \notin G(A)$ and by the previous result there exist $h \in \Delta$ such that $h(x - \lambda) = 0 \implies h(x) = \lambda$. Moreover if $\lambda = h(x)$ for some $h \in \Delta$, then we have that $h(x - \lambda e) = h(x) - \lambda = 0$ and hence $\lambda \in \sigma(x)$.

□

From above theorem, we can see that maximal ideals in A are in one-one correspondence with complex homomorphism on A. That's why we also call Δ as the maximal ideal space of A.

0.16 Gelfand Transform

Let $x \in A$, define $\widehat{x} : \Delta \to \mathbb{C}$ as follows $\widehat{x}(h) = h(x)\ \forall h \in \Delta$. Let $\widehat{A} = \{\widehat{x} \mid x \in A\}$, then $\widehat{A}$ is a complex algebra. One can easily verify that $\widehat{x} + \widehat{y} = \widehat{x+y}$, $\lambda\widehat{x} = \widehat{\lambda x}$ and $\widehat{x}\widehat{y} = \widehat{xy}$ $\forall x, y \in A$. If $h \neq h'$ then by the definition of not being equal, there exists $x \in A$ such that $h(x) \neq h'(x) \implies \widehat{x}(h) \neq \widehat{x}(h')$. So $\widehat{A}$ separates point in Δ. We call $\widehat{x}$ as the Gelfand transform of x. Now topologize Δ with the weak topology induced by $\widehat{A}$. This topology is known as the Gelfand topology on Δ. Radical of a commutative complex algebra A is intersection of all maximal ideals in A, it is denoted by rad(A). If rad(A)=$\{0\}$, then A is said to be a semi-simple algebra.

Theorem 3

1. Let A be Banach algebra, then Δ with Gelfand topology is a compact and Hausdorff space.
2. Let A be commutative Banach algebra, then $x \to \widehat{x}$ is a homomorphism into $C(\Delta)$ with range $\widehat{A}$ and its an isomorphism iff A is semi-simple.
3. Again A is a commutative Banach algebra. $x \in A$ the spectrum of x is the range of $\widehat{x}$ and hence $||\widehat{x}||_\infty = r(x) \leq ||x||$.

Proof. 1. As $\widehat{A}$ is separating and $\mathbb{C}$ is Hausdorff, Δ is also Hausdorff. A is a Banach algebra, let A' denote the dual of A, By taking V to be $\{x \in A \mid ||x|| < 1\}$ in Banach-Alaoglu Theorem we get $K = \{\Lambda \in A' \mid ||\Lambda|| \leq 1\}$ is compact in weak * topology on A'. Now Δ is subset of K and Gelfand topology is nothing but restriction of weak * topology on Δ. Hence if we can show that Δ is a closed subset of K then we are done.

Let $\Lambda_0 \in \overline{\Delta}$, we need to show $\Lambda_0 \in \Delta \iff \Lambda_0(e) = 1$ and $\Lambda_0(xy) = \Lambda_0(x)\Lambda_0(y)$ $\forall x, y \in A$. Take U=$\{\Lambda \in A' \mid |\Lambda(e) - \Lambda_0(e)| < \epsilon\}$, this is a nbd of Λ_0 in A'. Then $\exists h \in U \cap \Delta$, so $|1 - \Lambda_0(e)| = |h(e) - \Lambda_0(e)| < \epsilon$, hence $\Lambda_0(e) = 1$.

Let U $= \{\Lambda \in A' \mid |\Lambda_0(xy) - \Lambda(xy)| < \epsilon, |\Lambda(x) - \Lambda_0(x)| < \epsilon, |\Lambda(y) - \Lambda_0(y)| < \epsilon\}$, its again a nbd of Λ_0 in A'. So $\exists h \in \Delta \cap U$,

$$\begin{aligned} |\Lambda_0(xy) - \Lambda_0(x)\Lambda_0(y)| &= |-h(xy) + h(x)h(y) + \Lambda_0(xy) - \Lambda_0(x)\Lambda_0(y)| \\ &\leq |\Lambda_0 - h(xy)| + |\Lambda_0(x)(h(y) - \Lambda_0(y))| + |h(y)(h(x) - \Lambda_0(x))| \\ &\leq (1 + |\Lambda_0(x)| + ||y||)\epsilon. \end{aligned} \tag{0.16.1}$$

So, $\Lambda_0(xy) = \Lambda_0(x)\Lambda_0(y)$. Hence Δ is a compact in Gelfand topology.

2. We have already seen that $\widehat{x} + \widehat{y} = \widehat{x+y}$, $\lambda\widehat{x} = \widehat{\lambda x}$ and $\widehat{x}\widehat{y} = \widehat{xy}$ $\forall x, y \in A$. So $x \to \widehat{x}$ is a homomorphism. Now $\widehat{x} = 0$ means $h(x) = 0$ $\forall h \in \Delta$. This implies x belongs to intersection of kernel of h for all h in Δ. Now this is where we use the one-one correspondence between maximal ideals and complex homomorphism. Intersection of kernel of all complex homomorphism is equal to the rad(A), which is trivial if A is semi-simple. Hence if A is semi-simple then the map is injective. But if it is injective then it would mean intersection of kernel of all h is trivial and hence rad(A) is trivial.

3. We have already seen in theorem 2 that $\lambda \in \sigma(x)$ iff $\lambda = h(x)$ for some $h \in \Delta$. So it is clear that $\sigma(x)$ is the range of $\widehat{x}$. And then we have that $||\widehat{x}||_\infty = r(x) \leq ||x||$.

□

We know that a complex homomorphism on a Banach algebra is always continuous, this can be generalized where we can replace the range space from $\mathbb{C}$ to any semi-simple commutative Banach algebra.

Theorem 4: Let A be a Banach algebra and B be a semi simple commutative Banach algebra. $\phi : A \to B$ be a homomorphism, then ϕ is continuous.

Proof. As B is semi-simple commutative Banach algebra, $h(a) = h(a')$ $\forall h \in \Delta$ for some $a, a' \in B$ implies that $a = a'$.

We will use closed graph theorem to prove ϕ is continuous. let $x_n \to x$ in A and $\phi(x_n) \to y$ in B, we need to show $y = \phi(x)$ to show ϕ is a closed a map. Let h be a complex homomorphism on B, then $h \circ \phi$ is a complex homomorphism on A, which is a continuous map.

$$h(y) = \lim h \circ \phi(x_n) = h \circ \phi(x)$$

h was any complex homomorphism on B, hence $y = \phi(x)$ and ϕ is continuous. $\square$

Examples

1. $\mathbb{C}$ is the simplest commutative Banach algebra. And any ideal in $\mathbb{C}$ is either trivial or is all of $\mathbb{C}$, as an ideal is subspace also and can have dimension either 0 or 1. So there is only one maximal ideal in $\mathbb{C}$ that is $\{0\}$, and complex homomorphism corresponding to it is identity function. So from the one-one correspondence between maximal ideal and complex homomorphism, identity is the only non-zero complex homomorphism on $\mathbb{C}$.

2. Let X be a compact Hausdorff space. C(X) the collection of all complex-valued continuous function on X, with sup norm it is a commutative Banach algebra. Now for any $x \in X$ define $h_x : C(X) \to \mathbb{C}$ as $h_x(f) = f(x)\ \forall f \in C(X)$, h_x is a complex homomorphism on C(X), with kernel=$\{f \in C(X) \mid f(x) = 0\}$. So $h_x \in \Delta$. Now our claim is that every element in Δ is h_x for some x. Assume that if it was not true then there would exist a Maximal ideal M which is not the kernel for any h_x. So if we take a x in X, there must exist a function f in M such that $f(x) \neq 0$, otherwise $M \subset$ kernel(h_x) and then M = kernel(h_x). But then we can a find a nbd around each point in X where there exist a function in M not vanishing in that nbd. So by compactness of X there exists finite number of functions in M $f_1, \ldots, f_n$ such that for each point in X at least one of them is non-zero. Let $g = f_1\overline{f_1} + \cdots + f_n\overline{f_n}$, then $g \in M$, as M is an ideal. But now $g > 0$ and hence it is invertible in C(X) and this is a contradiction, as a proper ideal can never contain an invertible element. C(X) separates points in X by Urysohn's lemma. So the mapping $x \to h_x$ is a nice bijection between X and Δ, because if $x \neq x'$ then we can find a function in C(X) separating these two points and hence $h_x \neq h'_x$. Not just bijection X and Δ are topologically equivalent also. As the Gelfand topology on Δ is nothing but weak topology induced by C(X), now let γ be the weak topology induced on X by C(X) then we must have $\gamma \subset \tau$(original topology of X). But as τ is compact and γ is Hausdorff, we must have $\gamma = \tau$.

0.17 Involution

Let A be a complex algebra, an involution on A is a map " * " from A to A, such that $\forall x, y \in A$ we have

1. $(x^*)^* = x$
2. $(x+y)^* = x^* + y^*$.
3. $(\lambda x)^* = \overline{\lambda} x^*$
4. $(xy)^* = x^* y^*$

We already know some examples of involution on a Banach algebra. In $\mathbb{C}$ the map $z \to \overline{z}$ satisfies all the properties of an involution. If H is a Hilbert space, then the map $T \to T^*$, where T^* is the adjoint operator, satisfies all the properties of an involution. An element "x" for which $x = x^*$, is called self-adjoint.

Proposition 2: Let A be complex algebra, with an involution *.

1. $x + x^*$, $i(x - x^*)$ and xx^* are self adjoint for any $x \in A$.
2. Every element x in A, has unique decomposition $x = u + iv$, where both u and v are self adjoint.
3. Unit e is self adjoint.
4. x is invertible iff x^* is invertible and in that case $(x^*)^{-1} = (x^{-1})^*$.
5. $\lambda \in \sigma(x) \iff \overline{\lambda} \in \sigma(x^*)$.

Proof.

1. $(x + x^*)^* = x^* + (x^*)^* = x^* + x$, $(i(x - x^*))^* = -i(x^* - x) = i(x - x^*)$, $(xx^*)^* = (x^*)^* x^* = xx^*$
2. if $x = u + iv = u' + iv'$, where u,v,u' and v' are all self adjoint, then $u - u' = i(v' - v)$, by taking adjoint both side we can see $u - u' = 0$ and $v' - v = 0$. Write $x = \frac{x+x^*}{2} + i(\frac{x-x^*}{2i})$.

3. Consider $ea^* = a^*e = a^*$ for any $a \in A$, now by taking adjoint both side we get $e^*a = ae^* = a \, \forall a \in A$, hence by uniqueness of identity element, we have $e = e^*$.
4. if x^{-1} exists then $xx^{-1} = x^{-1}x = e$, now by taking adjoint on both side we get $(x^{-1})^*x^* = x^*(x^{-1})^* = e^* = e$. Hence $(x^*)^{-1} = (x^{-1})^*$.
5. Just observe that $x - \lambda$ is invertible $\iff (x-\lambda)^* = x^* - \overline{\lambda}$ is invertible.

□

A B^* algebra B is a Banach algebra with an involution, such that $||xx^*|| = ||x||^2 \, \forall x \in B$. $||x||^2 = ||xx^*|| \leq (||x||)(||x^*||) \implies ||x|| \leq ||x^*||$, w by reversing role of x and x^* we get that $||x|| = ||x^*||$.

Now we come to the principal theorem of this chapter, The Gelfand-Naimark Theorem

Theorem 5: Let B be a commutative B * algebra, then $x \to \widehat{x}$ from B to $C(\Delta)$ is *-isometric isomorphism.

Proof. Denote the above map by ϕ. We have already seen that $\phi : A \to C(\Delta)$ is homomorphism.

Now we show that h(x) is real for every self adjoint element x and for $h \in \Delta$. For $t \in \mathbb{R}$, let $z = x + ite$.

$$h(x) = \alpha + i\beta$$

$$zz^* = (x + ite)(x - ite) = x^2 + t^2e$$

$$|h(z)|^2 = |\alpha + i(\beta + t)|^2 = \alpha^2 + (\beta + t)^2 \leq ||z||^2 = ||zz^*|| \leq ||x||^2 + t^2$$

$$\implies \alpha^2 + 2\beta t + \beta^2 \leq ||x||^2 < \infty$$

Now the above inequality can hold $\forall t \in \mathbb{R}$ only if $\beta = 0$. Hence h(x) is real valued whenever $x = x^*$.

Now for $h \in \Delta$ and $X \in B$, $\overline{h(x)} = h(u) - ih(v) = h(x^*)$, where u and v are self adjoint and $x = u + iv$. This tells us that $\widehat{x^*} = \overline{\widehat{x}} \, \forall x \in B$.

Now we show that $||\widehat{x}||_\infty = ||x||\ \forall x \in B$, thus proving that ϕ is an isometry. First assume that $x = x^*$, then $||x||^2 = ||xx^*|| = ||x^2||$, now assume that $||x^{2^k}|| = ||x||^{2^k}$ for $k \leq n-1$. Now $||x^{2^n}|| = ||x^{2^{n-1}}x^{2^{n-1}}|| = ||x^{2^{n-1}}||^2 = ||x||^{2^{n-1}2} = ||x||^{2^n}$. Hence by principle of mathematical induction we have $||x^{2^m}|| = ||x||^{2^m}\ \forall m \in \mathbb{N}$. By using the formula for r(x), we have $r(x) = \lim ||x^n||^{\frac{1}{n}} = ||x|| = ||\widehat{x}||_\infty$, as $\sigma(x) = \widehat{x}(\Delta)$. Now let $x \in B$, consider $y = xx^*$. $\widehat{y} = \widehat{x}\widehat{x^*} = \widehat{x}\overline{\widehat{x}} = |\widehat{x}|^2$, hence $||\widehat{y}||_\infty = ||\widehat{x}||^2_\infty = ||y|| = ||xx^*|| = ||x||^2$. So we have $||\widehat{x}||_\infty = ||x||\ \forall x \in B$.

Now $\widehat{A} = \{\widehat{x} \mid x \in B\}$ is an algebra in $C(\Delta)$, which separates points in Δ and if $h \in \Delta$, means $h \neq 0$ by definition of Δ and hence $\exists x \in B$ such that $h(x) \neq 0 \implies \widehat{x}(h) \neq 0$. So $\widehat{A}$ does not vanish on Δ, moreover it is self adjoint also, since $\overline{\widehat{x}} = \widehat{x^*} \in \widehat{A}$. So by Stone-Weierstrass Theorem $\widehat{A}$ is dense in $C(\Delta)$ and we have that $\widehat{A} = C(\Delta)$. With this completes our proof.

□

Theorem 6: Let B be a commutative B * algebra and there is an element $x \in B$ such that the collection of all polynomials in x and x^* is dense in B, then the map $\widehat{x} : \Delta \to \sigma(x)$ is a homeomorphism.

Proof. We know that $\widehat{x}$ is continuous by definition of Gelfand topology, it is surjective by theorem 2. If we can prove that this map is injective then it will be open also, as Δ is compact and $\sigma(x)$ is Hausdorff. So all we have to show to complete the proof is that $\widehat{x}$ is injective.

Let h and $h' \in \Delta$, such that $h(x) = h'(x)$ and also $h(x^*) = \overline{h(x)} = \overline{h'(x)} = h'(x^*)$. Since h and h' are homomorphism, they agree on every polynomial in x and x^*, since h and h' are continuous and polynomials in x and x^* are dense in B, we have $h(x) = h'(x)\ \forall x \in B$. So we have proved that if $h \neq h'$, then we must have $h(x) \neq h'(x)$. This completes our proof. □

0.18 Note

In continuation to the above theorem, by the Gelfand Theorem we know that there exist a * isomorphic isometry $\psi : B \to C(\Delta)$, where $\psi(y) = \widehat{y}\ \forall y \in B$. And in the previous theorem, we have seen that $\widehat{x} : \Delta \to \sigma(x)$ is a homeomorphism and hence $\phi : C(\sigma(x)) \to C(\Delta)$, defined as $\phi(f) = f \circ \widehat{x}, \forall f \in C(\sigma(x))$ is a * isomorphic isometry. Now consider $\Psi = \psi^{-1} \circ \phi : C(\sigma(x)) \to B$, it is also a * isomorphic isometry. This map means that given a $f \in C(\sigma(x))$, $\Psi(f) = y$, where $\widehat{y} = f \circ \widehat{x}$. Now if f is identity function on $\sigma(x)$ then $\Psi(f) = x$.

0.19 The Spectral Theorem

Introduction: In this chapter we discuss the spectral theorem for normal operator. The proofs in the section Alegbra of $L^{\infty}(E)$ are my own, statements were just mentioned in the original source. I have made the proof of theorem 5 easy by filling in some gaps and explaining each part of the proof. Then I have defined the exponential of an element in a Banach Algebra, and proved that for two commuting elements a and b $e^{a+b} = e^a e^b$. Then I have derived the resolution of identity for a compact self-adjoint operator. This chapter is quite difficult as compared to the other chapters , but I have tried to keep discussion as simple as I could.

In this chapter we prove the spectral theorem for a normal operator on a Hilbert space H. We have already proved almost all the results required to prove the spectral theorem. Only two more preliminary results are left before we see the spectral theorem.

Theorem 1: Let H be a Hilbert space and $\{x_n\}$ be a sequence of orthogonal vectors in H. Then the following are equivalent.

1. $\sum_{n=1}^{\infty} x_n$ converges in H.
2. $\sum_{n=1}^{\infty} ||x_n||^2 < \infty$.
3. $\sum_{n=1}^{\infty} < x_n, y >$ converges for each $y \in H$.

Proof. $|| \sum_{n=1}^{\infty} x_n||^2 = \lim_{n\to\infty} || \sum_{k=1}^{n} x_k||^2 = \lim_{n\to\infty} \sum_{k=1}^{n} ||x_k||^2 = \sum_{k=1}^{\infty} ||x_k||^2$, hence $1 \implies 2$ is true.

For $m \geq n$, $|\sum_{k=n}^{m} < x_k, y > | = | < \sum_{k=n}^{m} x_k, y > | \leq (||\sum_{k=n}^{m} x_k||)(||y||) = (\sum_{k=n}^{m} ||x_k||^2)^{\frac{1}{2}} ||y||$, so if 2 is true then the left hand side of the inequality can be made arbitrarily small by taking n large enough. Hence $s_n(y) = \sum_{k=1}^{n} < x_k, y >$ is a Cauchy sequence for each $y \in H$ in $\mathbb{C}$, and hence is convergent. So $2 \implies 3$ is true.

Now assume 3, $s_n(y) = \sum_{k=1}^{n} \overline{< x_k, y >}$ is a bounded linear functional on H $\forall n$. Moreover by hypothesis, $s_n(y)$ converges for each $y \in H$. Hence by Banach-Steinhaus theorem $||s_n||^2 \leq M <$

$\infty\ \forall n$ for some $M > 0$. Now it is easy to see that $||s_n||^2 = \sum_{k=1}^{n} ||x_k||^2$ and hence we are done. So 3 $\implies$ 1.

$\square$

Theorem 2: Let $f : H \times H \to \mathbb{C}$ be a sesquilinear map, which is bounded in the sense that M = $\sup\{|f(x,y)| \mid ||x|| = ||y|| = 1\} < \infty$. Then there exist a bounded linear operator S such that $f(x,y) =< x, Sy > \forall x, y \in H$. Moreover $||S|| = M$.

Proof. For a fixed y in H, $g_y(x)$ = f(x,y) is bounded linear functional on H as $|f(x,y)| \leq M(||x||)(||y||)\ \forall x, y \in H$, hence $||g_y|| \leq M||y||$. Now by Riesz Representation theorem there exist a unique element y' in H, such that $f(x,y) =< x, y' >$ for all x in H and $||y'|| = ||g_y||$. So we define $S(y) = y'$ for all y in H. Linearity of S can be easily verified. Now $||S(y)|| = ||g_y|| \leq M||y||$ for all y and hence $||S|| \leq M$. Now for $\epsilon > 0$ take $\tilde{x}$ and $\tilde{y}$ with norm equal to one, such that $M - \epsilon < |f(\tilde{x}, \tilde{y})| = | < \tilde{x}, S\tilde{y} > | \leq ||S||$, hence $M - \epsilon < ||S||$, as ϵ was arbitrary we have that $||S|| = M$.

$\square$

0.20 Resolution Of The Identity

Let (X,M) be measurable space and H be a Hilbert space, a resolution of identity E on (X,M) is a map from M to B(H) such that

1. $E(\emptyset) = 0$ and $E(X) = I$.
2. $E(\omega)$ is a projection operator on H for each $\omega \in M$.
3. $E(\omega \cap \omega') = E(\omega)E(\omega')$.
4. If $\omega \cap \omega' = \emptyset$, then $E(\omega \cup \omega') = E(\omega) + E(\omega')$.
5. For x,y in H, $E_{x,y}(\omega) =< E(\omega)x, y >$ is complex measure on M.

If M is a borel sigma algebra of a locally compact Hausdorff space X, then we want the measure $E_{x,y}$ to be regular also. Now we will discuss some important facts about a resolution of identity.

Property 3 tells us that $E(\omega)$ and $E(\omega')$ commutes for all ω, $\omega' \in M$. If $\omega \cap \omega' = \emptyset$, then we have $E(\omega)E(\omega') = 0$, hence the range of $E(\omega)$ is orthogonal to that of $E(\omega')$. E is finitely additive, but it is not countably additive, as if $\{\omega_n\}$ is a sequence of disjoint sets in M, then $\sum_{k=m}^{n} E(\omega_k)$ is a projection operator onto the subspace $R(E(\omega_m)) + \cdots + R(E(\omega_n))$, and hence norm of it is either zero or one. Hence we cannot always expect $\sum_{n=1}^{\infty} E(\omega_n)$ to converge in norm. But given x in H, $E(\omega_n)x$ is an orthogonal sequence in H. Also $\sum_{n=1}^{\infty} < E(\omega_n)x, y >=< E(\omega)x, y > \forall y \in h$, where $\omega = \cup\omega_n$, as $E_{x,y}$ is a complex measure. Now by theorem 1, we can say that $\sum_{n=1}^{\infty} E(\omega_n)x$ converges in H, moreover we have $< E(\omega)x, y >= \sum_{n=1}^{\infty} < E(\omega_n)x, y >= \lim \sum_{k=1}^{n} < E(\omega_k)x, y >= \lim < \sum_{k=1}^{n} E(\omega_k)x, y >=< \sum_{k=1}^{\infty} E(\omega_k)x, y > \forall y \in H$, Hence we have that $\sum_{n=1}^{\infty} E(\omega_n)x = E(\omega)x$. So what we can finally conclude is that for a given x, $\omega \to E(\omega)x$ is H valued measure, in the sense that it is same as a normal measure, except that its range is neither $[0, \infty]$ nor $\mathbb{C}$ but H. As $E_{x,x}(\omega) =< E(\omega)x, x >\geq 0$, $E_{x,x}$ is positive measure on M, hence if $\omega \subset \omega'$ then $< E(\omega)x, x >\leq< E(\omega')x, x > \implies < E(\omega' - \omega)x, x >\geq 0 \; \forall x \in H$. So we have that if $\omega \subset \omega'$ then $E(\omega) \leq E(\omega')$.

Theorem 3: If $\{\omega_n\}$ is a sequence in M such that $E(\omega_n) = 0 \; \forall n$, then for $\omega = \cup\omega_n$ we have $E(\omega) = 0$.

Proof. As $E(\omega_n) = 0$, we have $E_{x,x}(\omega_n) = 0 \; \forall x$, so by sub-additivity of a positive measure we have $E_{x,x}(\omega) = 0 \; \forall x \in H$, and hence $E(\omega) = 0$. □

0.21 Algebra of L^{∞}(E)

Let E be a resolution of identity on (X,M). f be complex measurable function on X. Let $\{D_i\}$ be a countable basis in $\mathbb{C}$. Define V to be union of those D_i for which $E(f^{-1}D_i) = 0$. Then V is the largest open set for which $E(f^{-1}(V)) = 0$. If W is open. then it can be written as a union of a sub-collection $\{D_{i_k}\}$ of $\{D_i\}$, now if $E(f^{-1}(W)) = 0$, then $E(f^{-1}(D_{i_k})) = 0 \; \forall k$, so $D_{i_k} \subset V$

and hence $W \subset V$.

V^c is defined as the essential range of f. Now let B be the commutative B * algebra of all bounded complex-valued measurable functions. Just for the sake of clarity $||f|| = \sup_{x \in X} |f(x)|$ for f in B. Now define N as $\{f \in B \mid ||f||_\infty\} = 0$, here $||f||_\infty$ is the essential norm. N is a closed ideal, as let $\{f_n\}$ be a sequence in N and $f_n \to f$ in B then the convergence is uniform and hence point-wise. Let F_n be the set where $f_n \neq 0$ then we have $E(F_n) = 0$ and hence $E(F) = 0$, where $F = \cup F_n$. Now on F^c we will have $f_n = 0\ \forall n$ and hence $f = 0$. So $f \in N$.

Now the Quotient space B/N is also a commutative B * algebra, which is denoted by $L^\infty(E)$. We shall show that $|||g+N||| = ||g||_\infty$ for g in B. Pick a f in N and let F be a set such that $E(F) = 0$ and $f = 0$ on F^c.

$$||g+f|| = \sup_{x \in X} |f + g(x)| \geq \sup_{x \in F^c} |f + g(x)| = \sup_{x \in F^c} |g(x)| \geq ||g||_\infty$$

The above inequality is true for any f in N, and hence $|||g+N||| \geq ||g||_\infty$. Now $||g||_\infty$ is also an essential bound for g, so there exists F such that $E(F) = 0$ and $|g(x)| \leq ||g||_\infty$ in F^c. Define $f = (-\chi_F)g$, it is in N.

$$|||g+N||| \leq ||f+g|| = \sup_{x \in X} |f + g(x)| = \sup_{x \in F^c} |g(x)| = ||g||_\infty$$

So we can conclude that $|||g+N||| = ||g||_\infty$.

We will also show that $\sigma(g+N) = R(g)$, the essential range of g. Let $\lambda \in (R(g))^c$, define f=$\frac{1}{g-\lambda}$ on $g^{-1}(R(g))$ and zero on $(g^{-1}R(g))^c = g^{-1}(R(g)^c)$. $E(g^{-1}(R(g)^c) = E(g^{-1}(V)) = 0$. As R(g) is closed dist($\lambda, R(g)) = \delta > 0$. So $|g(x) - \lambda| > \delta\ \forall x \in g^{-1}(R(g))$ and hence $|f| < \frac{1}{\delta}\ \forall x \in X$. So f is in B and $f.(g-\lambda) = 1$ on $g^{-1}(R(g))$, hence $(f+N)(g-\lambda+N) = f(g-\lambda)+N = 1+N$, so $g-\lambda+N$ is invertible in L^∞. So we have $\sigma(g+N) \subset R(g)$.

Now assume that $(g-\lambda)+N$ is invertible in $L^\infty(E)$, then there exist a f in B and a set F in M, such that $f = \frac{1}{g-\lambda}$ on F and $E(F^c) = 0$. Now $|f(x)| \leq ||f||$ on X, hence $|g(x) - \lambda| \geq \frac{1}{||f||} > 0$ on F, so $\lambda \notin \overline{g(F)}$, hence $\lambda \in (\overline{g(F)})^c$, which is an open set and $g^{-1}(\overline{g(F)}) \supset g^{-1}(g(F)) \supset F$, so $(g^{-1}\overline{g(F)})^c \subset F^c$, hence $E(g^{-1}(\overline{g(F)})^c) = 0$ as $E(F^c) = 0$. So, $(\overline{g(F)})^c \subset V \implies \overline{g(F)} \supset V^c = R(g)$, hence $\lambda \notin R(g)$. Finally we have $\sigma(g+N) \supset R(g)$ and $\sigma(g+N) = R(g)$.

As we know that $L^\infty(E)$ is a commutative B * algebra, lets employ the Gelfand-Naimark theorem on it. Pick an element $g+N$, now according to the theorem $|||g+N||| = ||\widehat{g+N}||_\infty = r(g+N)$. We have shown that $\sigma(g+N) = R(g)$, the essential range of g. Hence essential norm of g equals $|||g+N|||$ which is equal to $r(g+N) = \sup_{\lambda \in R(g)} |\lambda|$.

If f is in B, then there exist a sequence of simple function $\{s_n\}$ in B, such that $s_n \to f$ uniformly and hence in B. As $|||f - s_n + N||| \leq ||f - s_n||$, so $s_n + N \to f + N$ in $L^\infty(E)$. We will denote g+N as g itself in the remaining chapter.

Theorem 4: Given a resolution of identity E as above, there exist a * isomorphic isometry $\Psi : L^\infty(E) \to A$, where A is a normal Banach sub-algebra in B(H), such that for $\forall x, y \in H$ we have $< \Psi(f)x, y >= \int_X f dE_{x,y}$. This is denoted by $\int_x f dE = \Psi(f)$. We also have $||\Psi(f)x||^2 = \int_X |f|^2 dE_{x,x}$ $\forall x \in H$ and $\forall f \in L^\infty(E)$.

Proof. We start by defining Ψ for simple functions in B. Let $s = \sum_{i=1}^n c_i \chi_{\omega_i}$, where $\{\omega_i\}$ is a partition of X. Define $\Psi(s) = \sum_{i=1}^n c_i E(\omega_i)$. Let $s = \sum_{i=1}^n c_i \chi_{\omega_i}$ and s'=$\sum_{j=1}^m c'_j \chi_{\omega'_j}$ be two simple functions in B, then $s + s' = \sum_{i,j} (c_i + c'_j) \chi_{\omega_i \cap \omega'_j}$, hence

$$\Psi(s+s') = \sum_{i,j} (c_i + c'_j) E(\omega_i \cap \omega'_j) = \sum_{i,j} c_i E(\omega_i) E(\omega'_j) + \sum_{i,j} c'_j E(\omega_i) E(\omega'_j)$$

$$= \sum_i c_i E(\omega_i) E(X) + \sum_j c'_j E(\omega'_j) E(X) = \Psi(s) + \Psi(s')$$

As $E(X) = I$. Now $ss' = \sum_{i,j} c_i c'_j \chi_{\omega_i \cap \omega'_j}$, so

$$\Psi(ss') = \sum_{i,j} c_i c'_j E(\omega_i) E(\omega'_j) = (\sum_i c_i E(\omega_i))(\sum_j c'_j E(\omega'_j)) = \Psi(s)\psi(s')$$

It's easy to verify that $\Psi(\lambda s) = \lambda \Psi(s)$, where s is simple function. Now consider $\Psi(s)^* =$

$\sum_i \overline{c_i} E(\omega_i) = \Psi(\bar{s})$, as $E(\omega_i)$'s are projection.

Let x,y be in H,

$$< \psi(s)x, y >=< \sum_i c_i E(\omega_i)x, y >= \sum_i c_i < E(\omega_i)x, y >= \sum_i c_i E_{x,y}(\omega_i) = \int_X s dE_{x,y}$$

$||\Psi(s)x||^2 = \sum_i |c_i|^2 ||E(\omega_i)x||^2 = \sum_i |c_i|^2 < E(\omega_i)x, x >= \int_X |s|^2 dE_{x,x}$. As $\{\omega_i\}$ are disjoint $\{E(\omega_i)x\}$ are orthogonal vectors.

$$||\Psi(s)x||^2 = \sum_i |c_i|^2 ||E(\omega_i)||^2 \leq ||s||_\infty^2 ||E(\omega)x||^2 \leq ||s||_\infty^2 ||x||^2.$$

Hence $||\Psi(s)|| \leq ||s||_\infty$. Here $||.||_\infty$ is the norm in $L^\infty(E)$. Now pick j such that $|c_j| = ||s||_\infty$ and a vector x in the range of $E(\omega_j)$ with norm one, then $||\Psi(s)x|| = |c_j| = ||s||_\infty$, as the ranges of $E(\omega_i)$ are orthogonal. So we have that $||\Psi(s)|| = ||s||_\infty$ for simple functions.

Now take a f in $L^\infty(E)$, let $\{s_n\}$ be a sequence converging to f in $L^\infty(E)$. Define $\Psi(f) = \lim \Psi(s_n)$. This limit exists as $||\Psi(s_n) - \Psi(s_m)|| = ||\Psi(s_n - s_m)|| = ||s_n - s_m||_\infty \to 0$ as n,m$\to \infty$, hence $\Psi(s_n)$ is Cauchy and convergent. It's easy to see that the definition is independent of the sequence chosen. It's easy to check that Ψ is a homomorphism. And $||\Psi(f)|| = \lim ||\Psi(s_n)|| = \lim ||s_n||_\infty = ||f||_\infty$. So Ψ is an isometry also.

For x,y in H $< \Psi(f)x, y >= \lim < \Psi(s_n)x, y >= \lim \int_X s_n dE_{x,y} = \int_X f dE_{x,y}$. DCT can be used to pass the limit inside the integral as the functions involved are bounded and measure is also bounded. Similarly we have that $||\Psi(f)x||^2 = \int_X |f|^2 dE_{x,x}$. If Q in B(H) commutes with with each $E(\omega)$ then it will commute with $\Psi(s)$ for every simple function s, and then by continuity we will have that Q commutes with every $\Psi(f)$ for f in $L^\infty(E)$. This completes our proof.

□

0.22 Spectral Theorem

Let B be a B-* algebra and x be a self adjoint element in B. Let A be the closure of the algebra generated by e and x. Then A is commutative B-* algebra in B. Let Δ be the maximal ideal space of A. Now $\sigma_A(x)$ is nothing but range of $\widehat{x}$. But we also have $\widehat{x} = \widehat{x^*} = \overline{\widehat{x}}$, this tells us that $\widehat{x}$ is a real-valued function, hence $\sigma_A(x)$ is a compact subset of $\mathbb{R}$. Now any subset of $\mathbb{R}$ will have empty interior in $\mathbb{C}$ and hence $\sigma_A(x) = \sigma_B(x)$. So spectrum of any self-adjoint element in a B-* algebra is always a compact subset of $\mathbb{R}$.

Let H be a Hilbert space, then we know that B(H) is a B-* algebra. Let A be any closed * algebra in B(H) containing I. Now take T in A such that T^{-1} exist in B(H), then TT^* is a self adjoint operator, hence $\sigma_{B(H)}(TT^*)$ is a compact subset of $\mathbb{R}$ and hence $(\sigma_{B(H)}(TT^*))^c$ is a connected subset of $\mathbb{C}$. Hence $\sigma_A(TT^*) = \sigma_{B(H)}(TT^*)$. So $(TT^*)^{-1} \in A$ and $T^{-1} = T^*(TT^*)^{-1}$, hence $T^{-1} \in A$. Thus T has the same spectrum relative to all closed * algebra in B(H) containing T.

Theorem 5: Let H be a Hilbert space and A be a closed normal algebra in B(H). Δ be the maximal ideal space of A, then the following statements are true.

1. There exists a unique resolution of identity E on borel sets of Δ such that $T = \int_\Delta \widehat{T} dE$ $\forall T \in A$.
2. There exist a *-isomorphic isometric extension of the inverse Gelfand transform map from $L^\infty(E)$ onto B a closed normal algebra in B(H)
3. B is the closure of the collection of finite linear combinations of the projections $E(\omega)$.
4. if ω is an open and non-empty set in Δ, then $E(\omega) \neq 0$.
5. $S \in B(H)$ commutes with each element in A iff it commutes with each projection $E(\omega)$.

Proof. Assume that there exist a resolution of identity E such that $T = \int_\Delta \widehat{T} dE$ $\forall T \in A$. This means for given $x, y \in H$ we have

$$< Tx, y >= \int_\Delta \widehat{T} dE_{x,y} \forall \widehat{T} \in C(\Delta) \tag{0.22.1}$$

Now by Gelfand-Naimark theorem each function in C(Δ) is $\widehat{T}$ for some T in A. So we know the map $\int_\Delta (.) dE_{x,y}$ on each function in C(Δ) and by definition $E_{x,y}$ is regular measure, so by Riesz Representation theorem for each x,y in H, $E_{x,y}$ is a unique regular measure such that 0.22.1 holds $\forall T \in A$. Now $< E(\omega)x, y >= E_{x,y}(\omega)$, so we know $< E(\omega)x, y >$ uniquely for each x,y in H and hence $E(\omega)$ is determined uniquely. So the uniqueness is done.

Lets come to existence. For $x, y \in H$ consider the map $\widehat{T} \to < Tx, y >$. It is easy to see that it is a linear functional. Moreover $| < Tx, y > | \leq (||T||)(||x||)(||y||) = ||\widehat{T}||_\infty(||x||)(||y||)$, as $||T|| = ||\widehat{T}||$ by Gelfand-Naimark theorem. Hence $\widehat{T} \to < Tx, y >$ is a bounded linear functional on $C(\Delta)$. Now by Riez Representation theorem there exist a unique regular borel measure $\mu_{x,y}$ such that

$$< Tx, y >= \int_\Delta \widehat{T} d\mu_{x,y} \tag{2}$$

$\forall \widehat{T} \in C(\Delta)$ and $\forall x, y \in H$, moreover we have that $|\mu_{xy}|(\Delta) \leq (||x||)(||y||)$. Now observe that

$$< T(x + x'), y >= \int_\Delta \widehat{T} d\mu_{x+x',y} =< Tx, y > + < Tx', y >= \int_\Delta \widehat{T} d\mu_{x,y} + \int_\Delta \widehat{T} d\mu_{x',y}$$

so we have $\int_\Delta \widehat{T} d\mu_{x+x',y} = \int_\Delta \widehat{T} d(\mu_{x,y} + \mu_{x',y})$ $\forall \widehat{T} \in C(\Delta)$, hence by uniqueness we have $\mu_{x+x',y} = \mu_{x,y} + \mu_{x',y}$. By the same method one can verify that $\mu_{\lambda x,y} = \lambda \mu_{x,y}$, $\mu_{x,\lambda y} = \overline{\lambda} \mu_{x,y}$ and $\mu_{x,y+y'} = \mu_{x,y} + \mu_{x,y'}$. Now take any bounded borel function f then by what we have just proved $(x, y) \to \int_\Delta f d\mu_{x,y}$ is bounded sesquilinear map and hence by theorem 2 there exists $\Phi(f) \in B(H)$ such that $< \Phi(f)x, y >= \int f d\mu_{x,y}$ $\forall x, y \in H$ and for any bounded borel function f. Linearity of Φ can be easily verified. Now from Gelfand-theory we know that a normal operator is self adjoint iff $\widehat{T}$ is real valued. So let $T \in A$ be a self adjoint operator, so

$$\int \widehat{T} d\mu_{x,y} =< Tx, y >=< x, Ty >= \overline{< Ty, x >} = \overline{\int \widehat{T} d\mu_{y,x}} = \int \widehat{T} d\overline{\mu_{y,x}}$$

Now let $T \in A$ write T as $T = T_1 + iT_2$, then $\widehat{T} = \widehat{T_1} + i\widehat{T_2}$.

$$\int \widehat{T} d\mu_{x,y} = \int \widehat{T_1} d\mu_{x,y} + i \int \widehat{T_2} d\mu_{x,y} = \int \widehat{T_1} d\overline{\mu_{y,x}} + i \int \widehat{T_2} d\overline{\mu_{y,x}}$$

$$= \int \left(\widehat{T_1} + i\widehat{T_2}\right) d\overline{\mu_{y,x}} = \int \widehat{T} d\overline{\mu_{y,x}} \tag{3}$$

So from 3 we conclude that $\mu_{x,y} = \overline{\mu_{y,x}}\ \forall x, y \in H$. Now

$$\begin{aligned} < (\Phi\overline{f})(x), y > &= \int \overline{f} d\mu_{x,y} = \overline{\int f d\mu_{y,x}} = \overline{< (\Phi f)(y), x >} =< x, (\Phi f)(y) > \\ &=< (\Phi f)^* x, y > \forall x, y \in H \end{aligned} \tag{0.22.2}$$

Hence we have $\Phi(\overline{f}) = (\Phi f)^*$ Now we prove that Φ is multiplicative also. Let $S, T \in A$, we have $\widehat{ST} = \widehat{S}\widehat{T}$.

$$\int \widehat{S}\widehat{T} d\mu_{x,y} =< STx, y >= \int \widehat{S} d\mu_{Tx,y}$$

Hence we can replace S by any bounded borel function f. So we have,

$$\int f\widehat{T} d\mu_{x,y} = \int f d\mu_{Tx,y} =< (\Phi f)T(x), y >=< Tx, z >= \int \widehat{T} d\mu_{x,z}$$

where $z = (\Phi f)^* y$. Replacing T by a bounded borel function g, we get

$$< \Phi(fg)x, y >= \int fg d\mu_{x,y} = \int g d\mu_{x,z} =< (\Phi g)x, z >=< (\Phi f)(\Phi g)x, y >$$

Hence we have $\Phi(fg) = (\phi f)(\Phi g)$.

Now we are in the position to define E on borel subsets of Δ. Define $E(\omega) = \Phi(\chi_\omega)$. Lets verify some properties of E.

1. $E(\emptyset) = \Phi(\chi_\emptyset)$, now $< \Phi(\chi_\emptyset)x, y >= \int \chi_\emptyset d\mu_{x,y} = 0$. So we have $\Phi(\chi_\emptyset) = 0$.
2. $< E(\Delta)x, y >=< \Phi(\chi_\Delta)x, y >= \int \widehat{I} d\mu_{x,y} =< x, y >$ so we have $E(\Delta) = I$.
3. $E(\omega)^* = (\Phi\chi_\omega)^* = \Phi\overline{\chi_\omega} = \Phi\chi_\omega = E(\omega)$. So $E(\omega)$ is self adjoint. $E(\omega)E(\omega) = \Phi(\chi_\omega)(\Phi\chi_\omega) = \Phi(\chi_\omega)^2 = \Phi(\chi_\omega) = E(\omega)$. Hence $E(\omega)$ is an idempotent and a self adjoint operator, hence it is a projection operator.
4. $E(\omega \cap \omega') = \Phi(\chi_{\omega\cap\omega'}) = \Phi(\chi_\omega \chi_{\omega'}) = \Phi(\chi_\omega)\Phi(\chi_{\omega'}) = E(\omega)E(\omega')$.
5. $E(\omega \cup \omega') = \Phi(\chi_{\omega\cup\omega'}) = \Phi(\chi_\omega + \chi_{\omega'}) = E(\omega) + E(\omega')$ provided $\omega \cap \omega' = \emptyset$.
6. $< E(\omega)x, y >=< \Phi\chi_\omega x, y >= \int \chi_\omega d\mu_{x,y} = \mu_{x,y}(\omega)$. So $E_{x,y}$ is a regular measure.

So E is a resolution of identity. Now from theorem 4 we have that Φ is *-isomorphic isometry from $L^\infty(E)$ onto a closed normal sub algebra B of B(H). Also $< \Phi(\widehat{T})x, y >= \int \widehat{T} d\mu_{x,y} =< Tx, y >$

from definition and for all x,y in H, hence $\Phi(\widehat{T}) = T$, so this mapping is an extension of the inverse Gelfand transform $C(\Delta)$. Now f in $L^\infty(E)$ can be approximated by simple functions in $L^\infty(E)$, hence B is closure of the set of all finite linear combination of projections $E(\omega)$.

Now let $\widehat{T}$ be in $C(\Delta)$ with support in ω a non-empty open set in Δ. If $E(\omega) = 0$, consider $\widehat{T} = \widehat{T}.\chi_\omega$, then $\Phi\widehat{T} = \Phi\widehat{T}\Phi\chi_\omega = \Phi\widehat{T}.E(\omega) = 0$. So $T = 0$ and hence $\widehat{T} = 0$ but this contradicts the Urysohn's lemma.

Now for the last part, let $S \in B(H)$, $T \in A$, x,y in H and $z = S^*y$, then

$$< STx, y > = < Tx, S^*y > = \int \widehat{T} d\mu_{x,z}$$

$$< TSx, y > = \int \widehat{T} d\mu_{Sx,y}$$

$$< SE(\omega)x, y > = < E(\omega)x, z > = E_{x,z}(\omega)$$

$$< E(\omega)Sx, y > = E_{Sx,y}(\omega)$$

Now if S commutes with every T in A then $\mu_{x,z} = \mu_{Sx,y}$ and hence S commutes with every projection $E(\omega)$. Similarly if S commutes with every projection $E(\omega)$, then S commutes with every element in A.

□

Now I would like to present a result on commutation of two operators. Let B be a Banach algebra, for x in B, define exp(x)= $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. The series converges absolutely as follows

$$\sum_{n=0}^{\infty} \frac{||x^n||}{n!} \leq \sum_{n=0}^{\infty} \frac{||x||^n}{n!} = e^{||x||} < \infty$$

hence the series converges in B for all x in B. Now for any x and y in B such that x commutes with y we have $(x+y)^n = \sum_{k=0}^{n} n! \frac{x^k y^{n-k}}{k!(n-k)!}$.

We will show that for any two commuting element x any y in B we have $e^{(x+y)} = e^x e^y$.

$$\left(\sum_{p=0}^{2n} \frac{x^p}{p!}\right)\left(\sum_{l=0}^{2n} \frac{y^l}{l!}\right) = \sum_{k=0}^{2n}\sum_{p=0}^{k} \frac{1}{k!}\left(\frac{k!x^p y^{k-p}}{p!(k-p)!}\right) + \delta_n$$

$$= \sum_{k=0}^{2n} \frac{(x+y)^k}{k!} + \delta_n$$

$$\implies \left\|\left(\sum_{p=0}^{2n} \frac{x^p}{p!}\right)\left(\sum_{l=0}^{2n} \frac{y^l}{l!}\right) - \sum_{k=0}^{2n} \frac{(x+y)^k}{k!}\right\| = ||\delta_n||$$

All we have to show now is that $\|\delta_n\| \to 0$ as $n \to \infty$.

$$\delta_n = \frac{x^{2n}}{(2n)!}\left(\sum_{k=1}^{2n} \frac{y^k}{k!}\right) + \frac{x^{2n-1}}{(2n-1)!}\left(\sum_{k=2}^{2n} \frac{y^k}{k!}\right) + \cdots + x\left(\frac{y^{2n}}{(2n)!}\right)$$

$$\|\delta_n\| \le \frac{\|x\|^{2n}}{(2n)!}\left(\sum_{k=1}^{2n} \frac{\|y\|^k}{k!}\right) + \frac{\|x\|^{2n-1}}{(2n-1)!}\left(\sum_{k=2}^{2n} \frac{\|y\|^k}{k!}\right) + \cdots + \|x\|\left(\frac{\|y\|^{2n}}{(2n)!}\right)$$

$$\le \left(\frac{\|x\|^{n+1}}{(n+1)!} + \cdots + \frac{\|x\|^{2n}}{(2n)!}\right)\left(\frac{\|y\|}{1!} + \cdots + \frac{\|y\|^{2n}}{(2n)!}\right) + \left(\frac{\|y\|^{n+1}}{(n+1)!} + \cdots + \frac{\|y\|^{2n}}{(2n)!}\right)\left(\frac{\|x\|}{1!} + \cdots + \frac{\|x\|^{n}}{(n)!}\right)$$

$$\le (e^{\|y\|})\left(\frac{\|x\|^{n+1}}{(n+1)!} + \cdots + \frac{\|x\|^{2n}}{(2n)!}\right) + (e^{\|x\|})\left(\frac{\|y\|^{n+1}}{(n+1)!} + \cdots + \frac{\|y\|^{2n}}{(2n)!}\right) \qquad (1)$$

Now we are done as the right hand side in (1) can be made arbitrarily small by taking n large enough. Hence $\|\delta_n\| \to 0$ as $n \to \infty$ and we have that $e^{x+y} = e^x e^y$ for commuting x and y in B. Now for x and $-x$ in B we have that $e^{x-x} = e^x e^{-x} = e^0 = I$, where I is the identity element of B. So we see that e^x is invertible for every x in B.

Theorem 6 Let M and N be two normal operators in B(H), and $T \in B(H)$ such that $MT = TN$ then we have $M^*T = TN^*$.

Proof. First take $S \in B(H)$, let $V = S^* - S$ and $Q = exp(V)$. Now $V^* = -V$, so $Q^* =$

$exp(V^*) = exp(-V) = Q^{-1}$, hence Q is a unitary operator and $\|Q\| = 1$.

By induction one can prove that $M^kT = TN^k \ \forall k$, hence exp(M) T= T exp(N) $\implies$ T= exp(-M)T exp(N). Let $U_1 = M^* - M$ and $U_2 = N - N^*$. Now because M and N are normal, we have that

$$\begin{aligned} exp(M^*)Texp(-N^*) &= exp(U_1)Texp(U_2) \\ \implies \left\| exp(M^*)Texp(-N^*) \right\| &\leq \|T\| \end{aligned} \tag{1}$$

Define $f : \mathbb{C} \to B(H)$ as $f(\lambda) = exp(\lambda M^*)Texp(-\lambda N^*)$. Now inequality 1 is true for any M and N which are normal and $MT = TN$, so it should also be true for $\overline{\lambda}M$ and $\overline{\lambda}N$, hence we get $\left\| f(\lambda) \right\| \leq \|T\|$. So f is a bounded entire function and hence has to be constant. So

$$f(\lambda) = f(0) = T \implies exp(\lambda M^*)T = Texp(\lambda N^*) \tag{2}$$

So by comparing coefficient of λ in (2) we get $M^*T = TN^*$.

□

Theorem 7: Let T be a normal operator in B(H), then there exists a unique resolution of identity E on spectrum of T, such that

$$T = \int \lambda dE \tag{1}$$

Moreover, if $S \in B(H)$ commutes with T then S commutes with every projection $E(\omega)$.

Proof. First assume that there exists a resolution of identity E such that (1) holds. Then by theorem 4, we have $T^* = \int \overline{\lambda} dE$ and if $P(\lambda, \overline{\lambda})$ is any polynomial in λ and $\overline{\lambda}$, then $P(T, T^*) = \int P(\lambda, \overline{\lambda})dE$. Now by the Stone-Weierstrass theorem the set of polynomials in λ and $\overline{\lambda}$ is dense in $C(\sigma(T))$. Hence we know the value of $\int f dE_{x,y}$ for every $f \in C(\sigma(T))$, So by Riesz representation theorem, $E_{x,y}$ is uniquely determined. So E is determined uniquely.

Consider the closure of the algebra generated by A, A^* and identity, denote it by B. B is a commutative B-* algebra and polynomials in A and A^* are dense in B, so we have that Δ the maximal ideal space of B and $\sigma(A)$ are homeomorphic, and hence we identify Δ by $\sigma(T)$. Now by theorem 5 there exists a resolution of identity E on $\sigma(T)$ such that $T = \int \widehat{T} dE$, where $\widehat{T}(\lambda) = \lambda$.

Now if there is an operator S in B(H) which commutes with T, then in theorem 6 by taking M=N=T, we get $T^*S = ST^*$. So S will commute with every element in B, and hence by theorem 5, we have that it commutes with every projections $E(\omega)$. $\square$

0.23 Note

We can easily compute the resolution of identity for a compact self adjoint operator. From spectral theorem for compact self adjoint T we know that the spectrum of it is discrete, and if $\{\lambda_1, \lambda_2 \dots\}$ are non-zero distinct points in its spectrum, then $T = \sum_{i=1}^{\infty} \lambda_i P_{\lambda_i}$, where P_{λ_i} is projection onto the eigen-space of λ_i. From basic theory we know that $R(P_{\lambda_i})$ is orthogonal to $R(P_{\lambda_j})$ for $i \neq j$. Now one can easily check that mapping $\{\lambda_i\} \to P_{\lambda_i}$, $\{0\} \to P_0$, P_0 is projection onto kernel of T, is a resolution of identity on $\sigma(T)$ (obviously by extending it naturally on the power set of $\sigma(T)$ like a counting measure.) The uniqueness of such a resolution of identity was proved in the theorem 7, hence this is our spectral decomposition of T on $\sigma(T)$.

0.24 Symbolic Calculus For Normal Operators

If T is a normal operator on H, let E be the spectral decomposition of it as in theorem 7. Now given a bounded borel function on $\sigma(T)$ from theorem 5 we know there exists the operator $\Phi(f)$ such that $\Phi(f) = \int f dE$. The mapping $f \to \Phi(f)$ is a homomorphism from set of all bounded borel function on $\sigma(T)$ into B(H). And also $\Phi(f)^* = \phi(\bar{f})$. But we can only claim that $\left\|\Phi(f)\right\| \leq \sup_{\lambda \in \sigma(T)} |f(\lambda)|$. Equality happens when f is a continuous function as one recalls that Φ is nothing but an extension of inverse Gelfand transform on $C(\sigma(T))$. So ϕ when restricted to $C(\sigma(T))$ it is an isomorphism onto the closure of algebra generated by A, A^* and I. We also have the formula $\left\|\Phi(f)x\right\|^2 = \int |f|^2 dE_{x,x}$ for all bounded borel functions.

After doing a lot of hard work, lets enjoy the fruits of it. Now we can easily prove the existence of square root of a positive operator.

Theorem 8**:** Let T be an operator in B(H), then the following are equivalent.

1. $< Tx, x > \geq 0 \; \forall x \in H$.

2. $T = T^*$ and $\sigma(T) \subset [0, \infty)$.

Proof. Assume 1 to be true. Write $T = T_1 + iT_2$, where T_1 and T_2 are self adjoint. Now this gives $< Tx, x > = < T_1x, x > + i < T_2x, x >$, now as $< Tx, x > \geq 0$, we have $< T_2x, x > = 0$ $\forall x \in H$ and hence $T_2 = 0$. So $T = T^*$. We have proved that spectrum of a self adjoint operator is a compact subset of $\mathbb{R}$. Now let $\lambda > 0 < T - \lambda x, T - \lambda x > = \|Tx\|^2 + 2\lambda < Tx, x > + \lambda^2 < x, x > \geq \lambda^2 < x, x >$, hence $T + \lambda$ is lower bounded and hence is invertible as it is self adjoint. So $\lambda < 0$ cannot belong to $\sigma(T)$. So it must be a subset of $[0, \infty)$.

Now assume 2 is true, Let E be the spectral decomposition of T. Now $< Tx, x > = \int_\Delta \widehat{T} dE_{x,x}$ for all x in H. From the basic theory we have $\widehat{T}(\Delta) = \sigma_T$, so $\widehat{T}$ is a positive map on Δ. Hence $< Tx, x > = \int \widehat{T} dE_{x,x} \geq 0$. This completes our proof.

□

Theorem 9: Let T be a positive operator then there exists a unique positive square root S in B(H) of T.

Proof. Let A be any commutative B-* algebra containing T and I in B(H). Let Δ be its maximal ideal space. As T is a positive operator, by theorem 8 we have that $\widehat{T}$ is positive function, hence it has unique continuous square root function on Δ. The operator corresponding to the square root function in A, will be our square root of T. And it is easy to see that the square root of T is a positive operator.

Now let A_0 denote the closure of the algebra generated by T and I. It is the smallest commutative B-* algebra containing T and I. Now by the preceding arguments there exists a positive square root S_0 of T in A_0. Now let S be a positive square root of T in B(H). Let A be the algebra generated by S and I. Then $T = S^2$ belongs to A, hence $A_0 \subset A$ and $S_0 \in A$, now let Δ be the maximal ideal space of A. Then $\widehat{S_0}^2 = \widehat{T} = \widehat{S}^2$ and hence $\widehat{S_0} = \widehat{S}$, so we must have $S_0 = S$. This completes our proof. □

0.25 Hyponormal operators

Introduction: In this chapter I have defined hyponormal operator and proved some properties of it. Then I have discussed an application of Stone-Weierstrass Theorem in the setting of bounded of linear operators, it really amazes me every time I see this application. Then we have the main the theorem of this chapter regarding a pure hyponormal operator, the proof of which is quite long and exacting, so I have tried to put every simple detail and give justifications wherever they were absent in the original source. Then I have given an example, which shows that pure part in the hypothesis is crucial and cannot be relaxed. Then I have given an example of a pure hyponormal operator. These examples were my own.

$T \in B(H)$, is a hyponormal operator if $T^*T - TT^* \geq 0$. Which is equivalent to $\|T^*x\| \leq \|Tx\| \ \forall x \in H$. We shall prove some facts about them. In the following proposition, T denotes a hyponormal operator.

0.26 Propositions

1. let $a, r \in \mathbb{C}$, then $a + rT$ is a hyponormal operator.
2. If T is invertible, then T^{-1} is also a hyponormal operator.
3. let r(T) denote spectral radius of T. Then we have $r(T) = \|T\|$.
4. If $Tx = zx$, then $T^*x = \bar{z}x$.
5. If $Tx_1 = z_1x_1$, $Tx_2 = z_2x_2$ and $z_1 \neq z_2$, then $< x_1, x_2 >= 0$.
6. Let M be an invariant subspace of T and $T_{|_M}$ is normal, then M reduces T.
7. $M = \{x \in H \mid Tx = zx\}$, then M reduces T and $T_{|_M}$ is normal.

Proof. 1.

$$\begin{aligned}(a+rT)^*(a+rT)-(a+rT)(a+rT)^* &= (\overline{a}+\overline{r}T^*)(a+rT)-(a+rT)(\overline{a}+\overline{r}T^*)\\ &= |a|^2+\overline{a}rT+\overline{r}aT^*+|r|^2T^*T-|a|^2-a\overline{r}T^*-r\overline{a}T\\ &- |r|^2TT^*\\ &= |r|^2(T^*T-TT^*)\geq 0\end{aligned} \tag{0.26.1}$$

Hence a hyponormal operator remain hyponormal even after a translation and scaling.

2. In order to prove 2, we first need to prove a general fact about positive operators. Let $0 \leq A \leq I$, then $0 \leq A^2 \leq A \leq I$. We use generalized Cauchy-Schwarz inequality for A. For any $x, y \in H$, we have $| < Ax, y > | \leq < Ax, x >< Ay, y >$.

   Take $y = Ax$ and assume $\|x\| \leq 1$ in the above inequality.

$$< A^2x, x >=< Ax, Ax >\leq< Ax, x >< A^2x, Ax >$$

   As both $\|A\|$ and $\|x\|$ both are less than 1. We obtain

$$< A^2x, x >\leq< Ax, x >$$

   for $\|x\| \leq 1$. Using this special case the general case easily follows . And hence $0 \leq A^2 \leq A \leq I$.

   Now additionally, if A is invertible also. Then $A^{-1} \geq I$. We need to show

$$< A^{-1}x, x >\geq< x, x > \forall x \in H$$

   WLOG take $x = Ay$ for some $y \in H$.

$$\begin{aligned}< y, Ay > - < Ay, Ay > &=< y, Ay > - < y, A^2y >\\ &=< y, A - A^2(y) >\geq 0.\end{aligned} \tag{0.26.2}$$

   By using what we have just proved. And hence $A^{-1} \geq I$.

   Now we come to our main result.

$$T^*T - TT^* \geq 0$$

$$\begin{aligned}
&\implies T^*(I-(T^*)^{-1}TT^*T^{-1})T \geq 0 \\
&\implies I \geq (T^*)^{-1}TT^*T^{-1} \\
&\implies T(T^*)^{-1}T^{-1}T^* \geq I \\
&\implies T(T^*)^{-1}T^{-1}T^* - I \geq 0 \\
&\implies T((T^{-1})^*T^{-1} - T^{-1}(T^{-1})^*)T^* \geq 0 \\
&\implies (T^{-1})^*T^{-1} - T^{-1}(T^{-1})^* \geq 0
\end{aligned} \tag{0.26.3}$$

And hence T^{-1} is also hyponormal.

3. We will show that $\|T\|^n = \|T^n\| \, \forall n$.

   For n= 2.

   let $\|x\| \leq 1$, $< Tx, Tx >=< T^*Tx, x > \leq \|T^*Tx\| \leq \left\|T^2x\right\|$. Taking supremum over x, we get $\|T\|^2 \leq \left\|T^2\right\|$. Assume it to be true for less than or equal to $n-1$.

   $$< T^{n-1}x, T^{n-1}x >=< T^*T^{n-1}x, T^{n-2}x > \leq \left\|T^*T^{n-1}x\right\|\left\|T^{n-2}x\right\| \leq \|T^nx\|\left\|T^{n-2}x\right\|$$

   Taking supremum over x, we get

   $$\left\|T^{n-1}\right\|^2 \leq \|T^n\|\left\|T^{n-2}\right\|$$

   Now by induction,

   $$\|T\|^{2n-2} \leq \|T^n\|\|T\|^{n-2}$$
   $$\implies \|T\|^n \leq \|T^n\|$$

   And hence $\|T\|^n = \|T^n\|$. So by the principle of induction it holds for all n. And we have that $\|T\| = \|T^n\|^{\frac{1}{n}} \, \forall n$. Now by using the formula for spectral radius of an operator, that is $r(T) = \lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$ we have our result.

4. Given that, $Tx = zx \implies \left\|(T-zI)x\right\| = 0$. As we have just seen that operator remains hyponoraml after translation, so $T-zI$ is also hyponormal. So, $\left\|(T^*-\bar{z}I)x\right\| \leq \left\|(T-zI)x\right\| = 0$. Hence $T^*x = \bar{z}x$.

5.

   $$z_1 < x_1, x_2 >=< z_1x_1, x_2 >=< Tx_1, x_2 >=< x_1, T^*x_2 >=< z_1, \overline{z_2}x_2 >= z_2 < x_1, x_2 >$$

So, $(z_1 - z_2) < x_1, x_2 >= 0$, as $z_1 \neq z_2$ we have $< x_1, x_2 >= 0$.

6. Let $x \in M$, since $B = T_{|_M}$ is normal, we have $\|Bx\| = \|Tx\| = \|B^*x\|$. Now if P is the projection on M, then $B^* = PT^*_{|_M}$. So, $\|T^*x\| \leq \|Tx\| = \|PT^*x\|$. So this implies $\|T^*x\| = \|PT^*x\|$, and hence $T^*x \in M$. So M reduces T.

7. To prove 7, we first show M reduces T.

   Let x$\in M$, $T(Tx) = T(zx) = z(Tx)$, hence $Tx \in M$. Now as T is hyponormal, we have just seen $T^*x = \bar{z}x$.

   $$T(T^*x) = T(\bar{z}x) = \bar{z}zx = z(\bar{z}x) = z(T^*x)$$

   so, $T^*x \in M$. we have shown that M reduces T. Now again let $x \in M$

   $$\|Tx\| = |z|\|x\| = |\bar{z}|\|x\| = \left\|T^*x\right\|.$$

   $$\implies \|Tx\| = \left\|T^*x\right\|.$$

   Hence $T_{|_M}$ is normal.

□

In our next theorem about hyponormal operator, we will need an application of Stone-Weierstrass theorem. So lets discuss the theorem.

Stone-Weierstrass theorem:Let X be a compact space, C(X)(collection of real-valued continuous function on X) is a Banach algebra with sup norm. Let "A" be an algebra in C(X), such that it separates points in X and vanishes at no point, then it is dense in C(X).

When you take collection of complex valued continuous function, in addition to the above conditions, we need an algebra to be self adjoint also, for it to be dense in C(X).

Now we shall discuss a surprising consequence of the above theorem. Let K be a compact set in $\mathbb{R}$. C(K) denotes the Banach algebra of continuous complex-valued functions on K. Take any $\lambda \in \mathbb{R} - K$. "A" be the algebra generated by $(x - \lambda)^{-1}$ and Identity map. This algebra in C(K), certainly separates points in K and does not vanishes on K, take $f(x) = \frac{1}{(x-\lambda)}$, it's a non-vanishing and one-one function. Further, this algebra is self-adjoint also, since $\lambda \in \mathbb{R}$ and x is a real number in K. Then by Stone-Weierstrass theorem ”A” is dense in C(K), meaning any continuous function on K can approximated uniformly by polynomials in $\frac{1}{x-\lambda}$. Isn’t it amazing.

Now be ready to get amazed even more. Let B be a self-adjoint operator. Take any $\lambda \in \mathbb{R} - \sigma(B)$. Let L denote the closure of the algebra generated by B and the Identity operator. L is a commutative B * algebra. Now by Gelfand theory L is isomorphically isometric to the $C(\sigma(B))$(algebra of continuous complex-valued function on $\sigma(B) \subset \mathbb{R}$.) Let this isomorphism be $\phi : C(\sigma(B)) \rightarrow L$. let ”I” and ”1” denote the identity map and constant map mapping everything to 1 on $\sigma(B)$.

Then $\phi(I) = B$ and $\phi(1) =$ identity operator. Hence $\phi((x-\lambda)^{-1}) = (B-\lambda)^{-1}$. So the algebra generated $(B-\lambda)^{-1}$ and identity operator will be isomorphic to that generated by $(x-\lambda)^{-1}$ and I. Isomorphism will be just restriction of ϕ on the generated algebra. But we have just discussed that the algebra generated by $(x-\lambda)^{-1}$ is dense in $C(\sigma(B))$, and hence the algebra generated by $(B-\lambda)^{-1}$ will be dense in L. This means that B can be approximated in norm by polynomials in $(B-\lambda)^{-1}$. We will be using this result in our next proof.

0.27 Definitions

”A” a self-adjoint operator is said to be absolutely continuous if its spectral measure is absolutely continuous w.r.t Lebesgue measure on $\mathbb{R}$. An operator S is said to be a purely hyponormal operator if it is hyponormal and only reducing space on which it is normal is the trivial space.

Remark: If S is a purely hyponormal operator, then it cannot have any eigen value, as we have proved that eigen spaces of a hyponormal operator reduces it and the operator when restricted to it is normal. This also tell us that a non-injective operator can never be a purely hyponormal operator.

0.28 Theorem

Let S be a pure hyponormal operator, $S = A + iB$ where A and B are self adjoint. Then A and B are absolutely continuous.

Proof. We will prove that A is absolutely continuous, for B we can consider $-iS = B - iA$, one can easily check that this is also a pure hyponormal operator. So what we do for S can be applied to $-iS$ and we can conclude that is B also absolutely continuous.

Define C:= $\frac{(S^*S - SS^*)}{2} = i(AB - BA)$. C is a positive operator as S is hyponormal. Let $''\Delta''$ be an open interval with mid point $''\alpha''$.

$$(A - \alpha)B - B(A - \alpha) = -iC$$

let $A = \int tdE(t)$ be spectral decomposition of "A". Multiplying $E(\Delta)$ from both right and left side of the above eqn, we get

$$E(\Delta)(A - \alpha)BE(\Delta) - E(\Delta)B(A - \alpha)E(\Delta) = -iE(\Delta)CE(\Delta)$$

$$(\int_\Delta (t - \alpha)dE(t))E(\Delta)BE(\Delta) - E(\Delta)BE(\Delta)(\int_\Delta (t - \alpha)dE(t)) = -iE(\Delta)CE(\Delta) \quad (0.28.1)$$

As $c \geq 0$, $E(\Delta)CE(\Delta) = E(\Delta)C^{\frac{1}{2}}C^{\frac{1}{2}}E(\Delta) = E(\Delta)C^{\frac{1}{2}}(E(\Delta)C^{\frac{1}{2}})^*$

$$\left\|E(\Delta)CE(\Delta)\right\| = \left\|E(\Delta)C^{\frac{1}{2}}\right\|^2 = \left\|C^{\frac{1}{2}}E(\Delta)E(\Delta)C^{\frac{1}{2}}\right\| = \left\|C^{\frac{1}{2}}E(\Delta)C^{\frac{1}{2}}\right\| \quad (0.28.2)$$

From (2) and (3) we have

$$\left\|C^{\frac{1}{2}}E(\Delta)C^{\frac{1}{2}}\right\| \leq 2\|B\| \frac{|\Delta|}{2} = \|B\| \, |\Delta|$$

Where $|\Delta|$ is Lebesgue measure of Δ. We have used the following fact that $\left\|(\int_\Delta (t - \alpha)dE(t))\right\| \leq$ supremum of function $t - \alpha$ over the interval Δ and as α is the mid point, the supremum is half the length of Δ.

Now let Δ be an open set. Let $\Delta = \cup_n \Delta_n$, where Δ_n are disjoint open intervals.

$E(\Delta) = \sum E(\Delta_n)$ in (SOT).

$$\left\| C^{\frac{1}{2}} E(\Delta) C^{\frac{1}{2}} \right\| = \left\| \sum C^{\frac{1}{2}} E(\Delta_n) C^{\frac{1}{2}} \right\| \leq \sum \|B\| \, |\Delta_n| = \|B\| \, |\Delta|$$

Now let Δ be a borel set with $|\Delta| = 0$. Then we can find a nested sequence of open sets $\{U_n\}$ such that $\Delta \subset U_{n+1} \subset U_n \forall n$ and $|\Delta| < \frac{1}{2^n}$. We use the fact that if $\chi_\Delta \leq \chi_{\cap U_n}$ then $E(\Delta) \leq E(\cap U_n)$.

$$0 \leq C^{\frac{1}{2}} E(\Delta) C^{\frac{1}{2}} \leq C^{\frac{1}{2}} E(\cap U_n) C^{\frac{1}{2}} = \lim_{n \to \infty} C^{\frac{1}{2}} E(U_n) C^{\frac{1}{2}} = 0$$

As $\left\| C^{\frac{1}{2}} E(U_n) C^{\frac{1}{2}} \right\| \leq \|B\| \, 2^{-n} \to 0$ as $n \to \infty$.

So $C^{\frac{1}{2}} E(\Delta) C^{\frac{1}{2}} = 0$. Since, $\left\| C^{\frac{1}{2}} E(\Delta) C^{\frac{1}{2}} \right\| = \left\| E(\Delta) C^{\frac{1}{2}} \right\|^2 = 0$, we have $E(\Delta) C^{\frac{1}{2}} = 0$. And as a result $E(\Delta) C = 0$.

So what we have concluded is very important, lets state it in words. If S is a hyponormal operator, let A be its real part and E denote the spectral measure corresponding to A. C be defined as $\frac{[S^*,S]}{2}$ and Δ be any borel with Lebesgue measure zero, then we have $E(\Delta) C = 0$.

Let H_a be the set of all $x \in H$ such that $< E(.)x, x >$ is absolutely continuous w.r.t Lebesgue measure. From the previous statement it is clear that $R(C) \subset H_a$. Lets check if this is a closed subspace or not.

Let x,y $\in H_a$, ω be a borel set with Lebesgue measure zero. Then $< E(\omega)x + y, x + y >= \left\| E(\omega)x + y \right\|^2 = \left\| E(\omega)x + E(\omega)y \right\|^2 \leq (\left\| E(\omega)x \right\| + \left\| E(\omega)x \right\|)^2 = 0$. As x,y$\in H_a$ both $< E(\omega)x, x >= E(\omega)y, y >= 0$. So $x + y \in H_a$.

$< E(\omega)cx, cx >= |c|^2 < E(\omega)x, x >$ so it can be seen easily that if $x \in H_a$ then $cx \in H_a$.

Now let $\{x_n\}$ be a sequence in H_a, such that $x_n \to x$. Then $< E(\omega)x, x >= \lim_{n \to \infty} < E(\omega)x_n, x_n >$. So if $|\omega| = 0$, then $< E(\omega)x, x >$ is the limit of zero sequence, and hence is zero. So $x \in H_a$. Hence H_a is a closed subspace. If we can show that $H_a = H$, then the proof is complete, as if this was true then for any $|\omega| = 0$, we would have $< E(\omega)x, x >= 0 \forall x \in H$, and hence $E(\omega) = 0$. So we finally show that $H_a = H$.

Till now we haven't used the fact S is purely hyponormal, we will use it now. Let L be any subspace which reduces S and contains R(C), then $R(C) \subset L \implies L^{\perp} \subset R(C)^{\perp} = ker(C) = ker([S^*, S])$. As L reduces S and hence $L^{\perp}$ will also reduce S and $S_{|_{L^{\perp}}}$ is normal. so $L^{\perp} = 0$ and $L = H$.

Now the idea is to construct L the smallest reducing subspace of S, which contains range of C, and show that $L \subset H_a$. Which from the previous argument will show that $H = L \subset H_a$, and hence $H_a = H$. Completing our proof.

A subspace L reduces S iff it is invariant under both A and B. So our L is going to be the closure of span$\{A^{m_k}B^{n_k} \ldots A^{m_1}B^{n_1}Cx \mid x \in H, k \geq 0, m_1, n_1, \ldots, m_k, n_k \in \mathbb{N}\}$. It's clear that it reduces S and contains range of C. To show this L is contained in H_a, it's enough to show that $E(\Delta)A^{m_k}B^{n_k} \ldots A^{m_1}B^{n_1}C = 0$, whenever $|\Delta| = 0$.

We proceed by induction on k. Let $s \in \mathbb{R} - \sigma(B)$.

$$A(B - s) - (B - s)A = -iC$$

Multiplying $(B - s)^{-1}$ to both left and right side of the above equation.

$$(B - s)^{-1}A - A(B - s)^{-1} = -i(B - s)^{-1}C(B - s)^{-1}$$

As $(B - s)^{-1}$ is self adjoint, $(B - s)^{-1}C(B - s)^{-1} \geq 0$. Taking $T = A - i(B - s)^{-1}$, we see that it is a hyponormal operator. Now fix a $|\Delta| = 0$. We now apply what we proved in the earlier part of the proof for T.

$$E(\Delta)(B - s)^{-1}C(B - s)^{-1} = 0$$
$$\implies E(\Delta)(B - s)^{-1}C = 0.$$

Differentiating above equation w.r.t s, n-times we get.

$$E(\Delta)(B - s)^{-n}C = 0$$

So we can see that for any polynomial P in $((B - s)^{-1})$, we have

$$E(\Delta)P((B - s)^{-1})C = 0.$$

Now we use our previous discussion that we did before this proof under Stone-Weierstrass theorem.

We can approximate B^n by polynomials in $(B-s)^{-1}$ in norm. And hence we have,

$$E(\Delta)B^nC = 0.$$

$$E(\Delta)A^mB^nC = A^mE(\Delta)B^nC = 0$$

As A commutes with $E(\Delta)$. Hence we have proved the statement for $k=1$. Assume it to be true for less than or equal to $k-1$.

$$BA = AB + iC$$

$$B^2A = BAB + iBC = AB^2 + iCB + iBC$$

$$B^3A = AB^3 + iCB^2 + iBCB + iB^2C$$

We can generalize and say that

$$B^nA = AB^n + iCB^{n-1} + i\sum B^lCB^k$$

If we multiply $E(\Delta)$ both side of the above equation we get

$$E(\Delta)B^nA = E(\Delta)AB^n + 0$$

As $E(\Delta)C = 0$ and $E(\Delta)B^lC = 0$, for any l.

$$E(\Delta)B^nA = E(\Delta)AB^n = AE(\Delta)B^n$$

So we can iterate this process and conclude that

$$E(\Delta)B^nA^m = A^mE(\Delta)B^n$$

$\forall m, n.$

$$\begin{aligned} E(\Delta)A^{m_k}B^{n_k}\dots A^{m_1}B^{n_1}C &= A^{m_k}E(\Delta)B^{n_k}A^{m_{k-1}}\dots A^{m_1}B^{n_1}C \\ &= A^{m_k}A^{m_{k-1}}E(\Delta)B^{n_k}B^{n_{k-1}}\dots A^{m_1}B^{n_1}C = 0 \end{aligned} \tag{0.28.3}$$

By induction hypothesis. So by principle of induction the statement is true for all k. And our proof is complete.

□

We cannot relax the pure hyponormal condition on an operator, for the above result to be true. For example, take any compact self-adjoint operator ”T”, we know that its spectrum is countable and contains atleast one non-zero eigenvalue. So if λ is a non-zero eigen vaule of T, We can see that Lebesgue measure of $\{\lambda\}$ is zero, as it is just a singleton set, But spectral measure of it is a projection operator on the eigenspace of λ, which is certainly a non-zero operator. Hence the spectral measure of any compact self adjoint operator is not absolutely continuous w.r.t Lebesgue measure on $\mathbb{R}$.

Lets discuss an example of pure hyponormal operator. Let ”S” be an unilateral shift on a Hilbert space H. If there is any reducing subspace ”M” of S, where it is normal, then it is unitary on M. Since any isometry which is normal, is unitary also. But from the Von Neumann-Wold Decomposition theorem, we have that the only reducing subspace where a unilateral shift can be unitary is the trivial space . Hence any reducing subspace of S where it is normal is the trivial subspace. And it is certainly hyponormal, because $[S^*, S] =$ projection onto $R(S)^\perp$. So it is a pure hyponormal operator. And hence $\frac{S^*+S}{2}$ and $\frac{S-S^*}{2i}$ have spectral measure which are absolutely continuous w.r.t Lebesgue measure on $\mathbb{R}$.

0.29 Riesz Functional Calculus

Introduction: In this chapter I have discussed the Riesz Functional Theorem. It is quite a standard topic in functional analysis. So most of it is a literary review. At the end of this chapter I have proved some important facts in the proposition 3, these proofs are my own.

Let G be an open set in $\mathbb{C}$, X be a Banach space. $f : G \to X$ is said to be a holomorphic function on G, if $\lim_{z' \to z} \frac{f(z')-f(z)}{z'-z}$ exists in X for every $z \in G$ and and limit is a continuous function. Let X^* be the dual of X, then for $\phi \in X^*$ we have $\phi \circ f : G \to \mathbb{C}$ holomorphic whenever f is holomorphic, since

$$|\frac{\phi(f(z')) - \phi(f(z))}{z'-z} - \phi(f'(z))| = |\phi\left(\frac{f(z')-f(z)}{z'-z} - f'(z)\right)| \leq \|\phi\| \left\| \frac{f(z')-f(z)}{z'-z} - f'(z) \right\|.$$

As the right hand side can be made arbitrarily small, we have $(\phi \circ f)'(z) = \phi(f'(z))$ and as f' is continuous so is $\phi \circ f'$.

Let $f : [a,b] \to X$ be a continuous map and $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path, then we define $\int_\gamma f dz$ as an element I in X, such that given $\epsilon > 0$ there exist a $\delta > 0$ such that $\left\| I - \sum_{k=1}^n \gamma(t_{k-1} - \gamma_{t_k}) f(\tau_k) \right\| < \epsilon$, where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ is a partition of [a,b] such that $\|P\| < \delta$ and $\tau_k \in [t_{k-1}, t_k]$. Its a standard result that such a limit always exists when the curve is rectifiable and f is continuous. Now if f is a continuous function defined near $\{\gamma\}$ into X then $\int_\gamma f dz = \int_\gamma f \circ \gamma dz$. From definition it is easy to see that $\int_\gamma \phi \circ f dz = \phi(\int_\gamma f dz)$, where $\phi \in X^*$.

Now by Hahn-Banach extension theorem, we know that if $\phi(x) = 0 \ \forall \phi \in X^*$, then $x = 0$. We will use this fact in our next result.

Cauchy Theorem: Let $f : G \to X$ be a holomorphic function. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be closed rectifiable curves in G, such that $\sum_{k=1}^n n(\gamma_k; a) = 0 \ \forall a \in \mathbb{C} - G$ then $\sum_{k=1}^n \int_{\gamma_k} f(z) dz = 0$.

Proof. Let $\phi \in X^*$, then $\phi\left(\sum_{k=1}^n \int_{\gamma_k} f(z) dz\right) = \sum_{k=1}^n \int_{\gamma_k} \phi \circ f(z) dz = 0$, by scalar valued version of Cauchy theorem. As it is true for any linear functional in X^*, we have $\sum_{k=1}^n \int_{\gamma_k} f(z) dz =$

0.

$\square$

$\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a set of closed rectifiable curves. We say that Γ is positively oriented if

1. $\{\gamma_i\} \cap \{\gamma_j\} = \emptyset$ if $i \neq j$.
2. $n(\Gamma : z) := \sum_{k=1}^{n} n(\gamma_k, z) = 1$ or 0 $\forall z \in \mathbb{C} - \cup_k \{\gamma_k\}$.
3. Each γ_k is simple, meaning non-intersecting.

Inside of Γ denoted by ins(Γ)=$\{z \in \mathbb{C} - \cup_k \{\gamma_k\} \mid n(\Gamma; z) = 1\}$ and similarly out(Γ)=$\{z \in \mathbb{C} - \cup_k \{\gamma_k\} \mid n(\Gamma; z) = 0\}$.

let $\underbrace{K}_{comapct} \subset \underbrace{U}_{open}$ as $\mathbb{C}$ is locally compact and Hausdorff, there exists ”V” an open set whose closure is compact such that $K \subset V \subset \overline{V} \subset U$. So we can always enlarge a compact set K in U, such that the enlarged K contains an open subset which is dense inside it. Now if D is any dense set in $\mathbb{C}$ then $V \cap D$ is dense in V and hence $\overline{V} \cap D$ is dense in $\overline{V}$.

Proposition 1: Let G be an open set in $\mathbb{C}$, $K \subset G$ be a compact set. Then there exists $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ a positively oriented set in $G - K$, such that $K \subset ins(\Gamma)$ and $G - \mathbb{C} \subset out(\Gamma)$.

Proof. Enlarge K if needed so as to have an open subset dense inside it. Choose a $\delta > 0$ such that $0 < \delta < \frac{d}{2}$, where $d = dist(K, \partial G) > 0$. Form a grid in the plane by vertical and horizontal lines, such that distance between any two consecutive lines is less than δ. Now collect all rectangles which intersect with K, let them be $R_1, R_2, \ldots, R_m$. They will be only finitely many as K is bounded. Now let $z \in K \cap R_j$, then $R_j \subset B(z, \sqrt{2}\delta) \subset B(z, \frac{d}{\sqrt{2}}) \subset G$, hence each R_j is contained in G.

Let ∂R_j denote boundary of R_j with anti-clockwise orientation. Now if $\sigma(j)$ and $\sigma(i)$ be two directed segments shared by R_j and R_i then $\int_{\sigma(j)} f dz = -\int_{\sigma(i)} f dz$, where f is any continuous function on the common side, because according to the given orientation $\sigma(j) = -\sigma(i)$.

Now let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the sides such that each belongs to only one of the ∂R_i in $\{\partial R_j\}_{j=1}^m$. Then we can say that

$$\sum_{k=1}^{n} \int_{\gamma_k} f dz = \sum_{j=1}^{m} \int_{\partial R_j} f dz \text{ where f is any continuous function on } \cup_j \partial R_j$$

Since on any side which is shared by two rectangles, the integrals will cancel in pairs and the integral will survive only on those sides which are not shared.

Now take $z \in K - \cup_j \partial R_j$, then z will be in the interior of a unique R_j. Let f be a holomorphic function on G, then $g(w) = \frac{f(w)}{w-z}$ is continuous on $\cup_j \partial R_j$. Now for R_i such that $z \notin R_i$, we can get a slightly bigger rectangle $\widetilde{R_i}$ such that $R_i \subset \widetilde{R_i} \subset G$ and $z \notin \widetilde{R_i}$. Now $\widetilde{R_i}$ is simply connected domain and g(w) is holomorphic in it, hence by Cauchy's theorem $\int_{\partial R_i} g(w)dw = 0$. And $\frac{1}{2\pi i} \int_{\partial R_j} g(w)dw = n(\partial R_j; z)f(z) = f(z)$ by Cauchy's theorem as $\mathbb{C} - G$ is contained in the unbounded component of $(\partial R_j)^c$ and hence $n(\partial R_j; z) = 0 \; \forall z \in \mathbb{C} - G$. Finally we have

$$\sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw = f(z) \; \forall z \in K - \cup_j \partial R_j. \tag{1}$$

Any γ_k will not intersect with K, because if γ_k intersects with K, then it will be shared among two rectangles in $\{R_1, \ldots R_m\}$, which will contradict the definition of γ_k, hence $\gamma_k \cap K = \emptyset$ for each k. Hence both sides are continuous w.r.t z in K in (1). As $K - \cup_j \partial R_j$ is dense in K, we have

$$\sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw = f(z) \; \forall z \in K \tag{2}$$

Now the segments $\gamma_1, \ldots, \gamma_n$ will form finitely many closed polygons, which if intersects can only intersect on corners. So we can round up the corners of the polygons so that they become disjoint and still remain disjoint with K. Formula (2) will still hold after rounding up the corners. Moreover these are simple curves.

Now by taking $f = 1$ in (2) we get that $\sum_{k=1}^{n} n(\gamma_k; z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int_{\gamma_k} \frac{1}{w-z} dw = 1 \; \forall z \in K$. And also $n(\gamma_k; z) = 0 \; \forall z \in \mathbb{C} - G$ as $\mathbb{C} - G$ is contained in the unbounded component of each R_i. So we are done.

□

Definition: Let X be a Banach algebra and $a \in X$. G be an open set in $\mathbb{C}$ such that $\sigma(a) \subset G$. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be set of positively oriented curves in $G - \sigma(a)$ such that $\sigma(a) \subset ins(\Gamma)$ and $\mathbb{C} - G \subset out(\Gamma)$. Let f holomorphic function in G, then we define f(a)= $\frac{1}{2\pi i} \int_\Gamma f(w)(w - a)^{-1} = \frac{1}{2\pi i} \sum_{k=1}^n \int_{\gamma_k} f(w)(w-a)^{-1} dw$.

We need to check that the definition does not depend on the choice of Γ.

Proposition 2: Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ be sets of positively oriented curves in $G - \sigma(a)$, such that $\sigma(a) \subset ins(\Gamma)$ and $\mathbb{C} - G \subset out(\Gamma)$ and similar properties are satisfied by Λ. Then $\frac{1}{2\pi i} \int_\Gamma f(w)(w-a)^{-1} = \frac{1}{2\pi i} \int_\Lambda f(w)(w-a)^{-1}$.

Proof. Consider $\widehat{\Gamma} = \{\gamma_1, \ldots, \gamma_n, -\lambda_1, \ldots, -\lambda_m\}$, it is a set of closed rectifiable curves in $G - \sigma(a)$. Now if $z \notin U = G - \sigma(a)$, then either $z \notin G$ or $z \in \sigma(a)$. If $z \notin G$ then $n(\widetilde{\Gamma}; z) = \sum_{k=1}^n n(\gamma_k; z) - \sum_{k=1}^m n(\lambda_k; z) = 0 - 0 = 0$, and if $z \in \sigma(a)$ then $n(\widetilde{\Gamma}; z) = \sum_{k=1}^n n(\gamma; z) - \sum_{k=1}^m n(\lambda_k; z) = 1 - 1 = 0$.

The function $w \to f(w)(w-a)^{-1}$ is holomorphic in U, so by Cauchy theorem we have $\frac{1}{2\pi i} \int_{\widetilde{\Gamma}} f(w)(w-a)^{-1} dw = \frac{1}{2\pi i} \int_\Gamma f(w)(w-a)^{-1} - \frac{1}{2\pi i} \int_\Lambda f(w)(w-a)^{-1} = 0$. So we are done.

□

Let Hol(a) denotes the set of all holomorphic functions in a nbd of $\sigma(a)$. If $f, g \in$ Hol(a) then f+g, fg are also holomorphic with their domains taken as intersection of the domains of f and g. So it is an algebra.

Riesz Functional Theorem: Let X be a Banach algebra with identity and a be in X then

1. $f \to f(a)$ is a homomorphism from Hol(a) into X.
2. If $f(z) = \sum_{k=1}^\infty \alpha_k z^k$ is power series with $R > r(a)$, then $f(a) = \sum_{k=1}^\infty \alpha_k a^k$.
3. for f=1, f(a)= I and when f(z)=z then f(a)=a.
4. Let $f, f_1, f_2 \ldots$ be a sequence of holomorphic function defined on G, with $\sigma(a) \subset G$. If $f_n \to f$ uniformly on compact subsets of G then $f_n(a) \to f(a)$ in norm.

Proof. Let f,g$\in$ $Hol(a)$ defined in a common domain G. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a set of positively oriented closed curves in $G - \sigma(a)$ such that $\sigma(a) \subset ins(\Gamma)$. Now $ins(\Gamma) \cup \gamma = cl(ins(\Gamma))$ is a compact set in G, then again find $\Lambda = \{\lambda_1, \dots, \lambda_m\}$ set of positively oriented closed curves in $G - cl(ins(\Gamma))$ such that $cl(ins(\Gamma)) \subset ins(\Lambda)$. And obviously we would have $\cup_k \lambda_k \subset out(\Gamma)$.

$$\begin{aligned}
f(a)g(a) &= \left(\frac{1}{2\pi i}\int_\Gamma f(z)(z-a)^{-1}dz\right)\left(\frac{1}{2\pi i}\int_\Lambda g(w)(w-a)^{-1}dw\right) \\
&= \frac{-1}{4\pi^2}\int_\Gamma\int_\Lambda f(z)g(w)(z-a)^{-1}(w-a)^{-1}dwdz \\
&= \frac{-1}{4\pi^2}\int_\Gamma\int_\Lambda f(z)g(w)(z-a)^{-1}(\frac{(w-a)-(z-a)}{w-z})(w-a)^{-1}dwdz \\
&= \frac{-1}{4\pi^2}\int_\Gamma\int_\Lambda f(z)g(w)\frac{(z-a)^{-1}-(w-a)^{-1}}{w-z}dwdz \\
&= \frac{-1}{4\pi^2}\int_\Gamma f(z)\left(\int_\Lambda \frac{g(w)}{w-z}dw\right)(z-a)^{-1}dz + \frac{-1}{4\pi^2}\int_\Lambda g(w)\left(\int_\Gamma \frac{f(z)}{z-w}dz\right)(w-a)^{-1}dw \\
&= \frac{-1}{4\pi^2}\int_\Gamma f(z)g(z)(z-a)^{-1}dz = fg(a)
\end{aligned} \tag{0.29.1}$$

As $\mathbb{C} - G$ is contained in both $out(\Gamma)$ and $out(\Lambda)$ we have $\int_\Lambda \frac{g(w)}{w-z}dw = n(\Lambda; z)g(z)$ and $\int_\Gamma \frac{f(z)}{z-w}dz = n(\Gamma; w)f(w)$ by Cauchy's theorem. And as $z \in \cup_k\{\gamma_k\} \subset ins(\Lambda),\ w \subset \cup_k\{\lambda_k\} \subset out(\Gamma)$ we have $n(\Lambda; z) = 1$ and $n(\Lambda; w) = 0$. Linearity is easily verified, hence it is a homomorphism.

Let $f(z) = z^k$,$k \geq 0$ and $\gamma(t) = Re^{i\pi t}$ for some $R > \|a\| \geq r(a)$, then certainly $\sigma(a) \subset ins(\gamma)$. Then by definition we have

$$\begin{aligned}
f(a) &= \frac{1}{2\pi i}\int_\gamma z^k(z-a)^{-1}dz = \frac{1}{2\pi i}\int_\gamma z^{k-1}\left(I - \frac{a}{z}\right)^{-1}dz \\
&= \frac{1}{2\pi i}\int_\gamma z^{k-1} \underbrace{\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n}_{\text{as } \|\frac{a}{z}\| = \frac{\|a\|}{|z|} = \frac{\|a\|}{R} < 1} dz \\
&= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty}\frac{a^n}{z^{n-k+1}}dz = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\int_\gamma \frac{a^n}{z^{n-k+1}}dz = a^k
\end{aligned} \tag{0.29.2}$$

As the series $\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n$ converges uniformly on $\{\gamma\}$ we can take integral inside the sum. And $\frac{1}{z^n}$ has anti-derivative for every n except $n=1$, but we know that $\int_\gamma \frac{1}{z}dz = 2\pi i$. So $f(a) = a^k$.

Now if $f = \sum_{k=1}^{n} \alpha_k z^k$ then by linearity $f(a) = \sum_{k=1}^{1} \alpha_k a^k$. Assume that $\{f_n\}$ is a sequence of holomorphic defined on G, with $\sigma(a) \subset G$ and $f_n \to f$ uniformly on compact subsets of G. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a set of positively oriented closed curves such that $\sigma(a) \subset ins(\Gamma)$.

As $t \to \left\|(\gamma_k(t) - a)^{-1}\right\|$ is continuous function on $[0,1]$ for each k, we must have a $M > 0$ such that $\left\|(\gamma_k(t) - a)^{-1}\right\| \leq M$ for each k and $t \in [0,1]$. As we uniform convergence on $\cup_k\{\gamma_k\}$ given an $\epsilon > 0$ we can find an N such that $\forall n \geq N$ we have $|f_n(\gamma_k(t)) - f(\gamma_k(t))| < \epsilon$ for each $t \in [0,1]$ and for each k.

$$\begin{aligned}
\|f_n(a) - f(a)\| &= \left\|\int_\Gamma (f_n(z) - f(z))(z-a)^{-1}dz\right\| \\
&\leq \sum_{k=1}^{m}\left\|\int_{\gamma_k} (f_n(z) - f(z))(z-a)^{-1}dz\right\| \\
&\leq \sum_{k=1}^{n}\int_0^1 |f_n(\gamma_k(t)) - f(\gamma_k(t))| \left\|(\gamma_k(t) - a)^{-1}\right\| d|\gamma_k|(t) \\
&\leq \epsilon M \sum_{k=1}^{n} V(\gamma_k)
\end{aligned} \tag{0.29.3}$$

Where $V(\gamma_k)$ is the total variation of γ_k. Hence we have $f_n(a) \to f(a)$ in norm.

Now let $f(z) = \sum_{n=1}^{\infty} \alpha_n z^n$ be a power series with $R > r(a)$, the we have $s_n(z) = \sum_{k=1}^{n} \alpha_k z^k$ converges uniformly to f(z) in compact subsets of $B(0,R)$. So by the previous result we must have $s_n(a) = \sum_{k=1}^{n} \alpha_k a^k \to f(a)$ as $n \to \infty$, hence $f(a) = \sum_{n=1}^{\infty} \alpha_n a^n$. This completes our proof.

□

Theorem 2: If $\tau : Hol(a) \to X$ be any homomorphism, such that

1. $\tau(1) = I$ and $\tau(z) = a$.

2. Let f_n defined in an open set G, such that $\sigma(a) \subset G$, if $f_n \to f$ uniformly on compact subsets of G then $\tau(f_n) \to \tau(f)$ in norm.

Then $\tau(f) = f(a)\ \forall f \in Hol(a)$.

Proof. As τ is homomorphism, $\tau(z^k) = (\tau(z))^k = a^k\ \forall k \geq 0$ and if $p(z) = \sum_{k=1}^{n} \alpha_k z^k$ then $\tau(p) = \sum_{k=1}^{n} \alpha_k a^k = p(a)$. Let $\lambda \notin \sigma(a)$, then $I = \tau(1) = \tau((z-\lambda)^{-1}(z-\lambda)) = \tau((z-\lambda)^{-1})\tau(z-\lambda) = \tau((z-\lambda)^{-1})(a-\lambda)$, similarly $I = (a-\lambda)\tau((z-\lambda)^{-1})$ so $\tau((z-\lambda)^{-1}) = (a-\lambda)^{-1}$. So for any rational function R with poles outside $\sigma(a)$, we have $\tau(R) = R(a)$. Now let $f \in Hol(a)$, so $f : G \to \mathbb{C}$ with $\sigma(a) \subset G$. Let V be an open set such that $\overline{V}$ is compact and $\sigma(a) \subset V \subset \overline{V} \subset G$. Now by Runge's Theorem there exists a sequence of rational functions $\{R_n\}$ with poles outside $\overline{V}$ and $R_n \to f$ uniformly on $\overline{V}$ and hence on V. Now we have $\tau(f) = \lim \tau(R_n) = \lim R_n(a) = f(a)$, Hence $\tau(f) = f(a)$ for every f in Hol(a). □

Theorem 3: Let $f \in Hol(a)$ then $\sigma(f(a)) = f(\sigma(a))$.

Proof. Let $\lambda \notin f(\sigma(a))$, then as $f(\sigma(a))$ is a compact set we can find an open set U such that $f(\sigma(a)) \subset U$ and $\lambda \notin U$. Hence $f(z) - \lambda$ is invertible in $f^{-1}(U)$. So $f(z) - \lambda \in Hol(a)$ is invertible and hence $f - \lambda(a) = f(a) - \lambda$ will also be invertible. So $\lambda \notin \sigma(f(a))$.

Now pick $\widetilde{\lambda} \notin \sigma(f(a))$, if $\widetilde{\lambda} = f(\lambda')$ for some $\lambda' \in \sigma(a)$, then $h(\lambda) = f(\lambda) - \widetilde{\lambda} = (\lambda - \lambda')g(\lambda)$ where g is a holomorphic function, defined where f is defined. Now $f(a) - \widetilde{\lambda} = (a - \lambda')g(a) = g(a)(a - \lambda')$ is invertible hence $a - \lambda'$ is also invertible, which is a contradiction as $\lambda' \in \sigma(a)$. Hence $\widetilde{\lambda} \notin f(\sigma(a))$. □

Proposition 3:

1. Let X be Banach algebra, $a \in X$ let $f \in Hol(a)$ and g be a holomorphic in a nbd of $f(\sigma(a))$, then g(f(a))=$g \circ f(a)$.
2. Let H be Hilbert space, $A \in B(H)$ for $f \in Hol(A)$ define $\widetilde{f}(z) = \overline{f(\overline{z})}$, then $\widetilde{f}$ is holomorphic and $f(A)^* = \widetilde{f}(A^*)$. Now If A is normal so is f(A).
3. If X is a Banach space, $A \in B(X)$ and M is closed subspace of X, such that $(\alpha - A)^{-1}M \subset M$ for every $\alpha \in \sigma(A)^c$. Then f(A)M$\subset M$ for every $f \in Hol(A)$.

4. If X is a Banach algebra, I is an ideal in X. Let $a \in I$ then for every $f \in Hol(a)$ with $f(0) = 0$ we have $f(a) \in I$.

Proof. 1. Let $g : U \to \mathbb{C}$ such that $f(\sigma(a)) \subset U$ and $f : G \to \mathbb{C}$ such that $\sigma(a) \subset G$. Take $G' = G \cap f^{-1}(U)$, then $g \circ f : G' \to \mathbb{C}$ is a holomorphic function in a nbd of $\sigma(a)$ and hence $g \circ f \in Hol(a)$.

So we take $f : G' \to \mathbb{C}$ and $g : U \to \mathbb{C}$. Since $f(\sigma(a))$ is a compact subset in U, there exist an open set V such that $f(\sigma(a)) \subset V \subset \overline{V} \subset U$ and $\overline{V}$ is compact. Get $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ a set of positively oriented closed curve in $U - \overline{V}$ such that $\overline{V} \subset ins(\Gamma)$ and $\mathbb{C} - U \subset out(\Gamma)$.

Now $f^{-1}(V)$ is open in G' and $\sigma(a) \subset f^{-1}(V)$, get $\Lambda = \{\lambda_1, \ldots, \lambda_m\}$ a set of positively oriented closed curves in $f^{-1}(V) - \sigma(a)$ such that $\sigma(a) \subset ins(\Lambda)$ and $\mathbb{C} - f^{-1}(V) \subset out(\Lambda)$. As $\sigma(f(a)) = f(\sigma(a))$, then by definition

$$g(f(a)) = \frac{1}{2\pi i} \int_\Gamma g(w)(w - f(a))^{-1} dw$$

Now for $w \in \cup_k \{\gamma_k\} \subset (\overline{V})^c$, we have $w - f$ invertible in $f^{-1}(V)$. So $(w - f(a))^{-1} = (w - f(a))^{-1} = (w - f)^{-1}(a) = \frac{1}{2\pi i} \int_\Lambda \frac{1}{w - f(z)} (z - a)^{-1} dz$

$$\begin{aligned} g(f(a)) &= \frac{-1}{4\pi^2} \int_\Gamma \int_\Lambda \frac{g(w)}{w - f(z)} (z - a)^{-1} dz dw \\ &= \frac{1}{2\pi i} \int_\Lambda \left(\frac{1}{2\pi i} \int_\Gamma \frac{g(w)}{w - f(z)} dw \right) (z - a)^{-1} dz \\ &= \frac{1}{2\pi i} \int_\Lambda g \circ f(z) (z - a)^{-1} dz = g \circ f(a) \end{aligned} \tag{0.29.4}$$

As $f(z) \in V$ and $\overline{V} \subset ins(\Gamma)$ so $n(\Gamma; f(z)) = 1$ and hence $\frac{1}{2\pi i} \int_\Gamma \frac{g(w)}{w - f(z} dw = g \circ f(z)$. So we are done.

2. If $f : G \to \mathbb{C}$ then $\widetilde{f} : \overline{G} \to \mathbb{C}$. For $z, z' \in \overline{G}$,

$$\frac{\overline{f(\overline{z'})} - \overline{f(\overline{z})}}{z' - z} = \overline{\left(\frac{f(\overline{z'}) - f(\overline{z})}{\overline{z'} - \overline{z}} \right)} \to \overline{f'(\overline{z})} \text{ as } z' \to z$$

Hence $\widetilde{f}$ is holomorphic, whenever f is holomorphic.

Let $\gamma : [0,1] \to \mathbb{C}$ be a closed rectifiable curve. Define $\widetilde{\gamma} : [0,1] \to \mathbb{C}$ as $\widetilde{\gamma}(t) = \overline{\gamma(1-t)}$. Then one can easily see that $\widetilde{\gamma}$ is also a closed rectifiable curve. If f is continuous function defined near γ, then $\widetilde{f}$ will be a continuous function defined near $\widetilde{\gamma}$.

Let P= $\{0 = t_0 < t_1 < t_2 < \cdots < t_n = 1\}$ be partition of $[0,1]$, then define $\widetilde{P} = \{0 = 1 - t_n < \cdots < 1 - t_0 = 1\}$ is also a partition of $[0,1]$ and one can see that the mesh size of P and $\widetilde{P}$ are same.

Let $R(P, f, \gamma) = \frac{1}{2\pi i} \sum_{k=1}^{n} (\gamma(t_k) - \gamma(t_{k-1}) f(\gamma(t_k)))$. Now

$$
\begin{aligned}
R(P, \widetilde{f}, \widetilde{\gamma}) &= \frac{1}{2\pi i} \sum_{k=1}^{n} (\widetilde{\gamma}(t_k) - \widetilde{\gamma}(t_{k-1}))\widetilde{f}(\widetilde{\gamma}(t_k)) \\
&= \frac{1}{2\pi i} \sum_{k=1}^{n} (\overline{\gamma}(1-t_k) - \overline{\gamma}(1-t_{k-1}))\overline{f(\overline{\bar{\gamma}(1-t_k)})} \\
&= \overline{\frac{1}{2\pi i} \sum_{k=1}^{n} (\gamma(1-t_{k-1}) - \gamma(1-t_k))f(\gamma(1-t_k)} = \overline{R(\widetilde{P}, f, \gamma)}
\end{aligned}
\tag{0.29.5}
$$

So if we take mesh size of P goes to zero we have $\frac{1}{2\pi i} \int_{\widetilde{\gamma}} \widetilde{f}(z)dz = \overline{\frac{1}{2\pi i} \int_{\gamma} f dz}$.

So when we take $f(w) = \frac{1}{w-z}$ for $z \notin \{\gamma\}$, we will have $n(\gamma; z) = \overline{n(\widetilde{\gamma}; \overline{z})} = n(\widetilde{\gamma}; \overline{z})$, as index is always an integer. Now Let $K \subset G$, where K is compact and G is open. Let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ be a set of positively oriented closed curves in $G - K$ such that $K \subset ins(\gamma)$ and $\mathbb{C} - G \subset out(\gamma)$.

Then $\widetilde{\Gamma} = \{\widetilde{\gamma_1}, \ldots, \widetilde{\gamma_n}\}$ will be a set of positively oriented closed curves in $\overline{G} - \overline{K}$ and $\overline{K} \subset ins(\widetilde{\Gamma})$ and $\mathbb{C} - \overline{G} \subset out(\widetilde{\Gamma})$, by our previous discussion. Here $\overline{G} = \{\overline{z} \mid z \in G\}$ and similarly $\overline{K}$ is defined.

Let $A \in B(H)$. $f \in Hol(A)$ so $f : G \to \mathbb{C}$ such that $\sigma(A) \subset G$. Let γ be a closed rectifiable curve in $G - \sigma(A)$.

Let $R(P,A,f,\gamma) = \frac{1}{2\pi i}\sum_{k=1}^{n}(\gamma(t_k)-\gamma(t_{k-1}))f(\gamma(t_k))(\gamma(t_k)-A)^{-1}$.

$$\begin{aligned}
R(P,A,f,\gamma)^* &= \frac{-1}{2\pi i}\sum_{k=1}^{n}(\overline{\gamma}(t_k)-\overline{\gamma}(t_{k-1}))\overline{f(\gamma(t_k))}(\overline{\gamma(t_k)}-A^*)^{-1}\\
&= \frac{-1}{2\pi i}\sum_{k=1}^{n}(\widetilde{\gamma}(1-t_k)-\widetilde{\gamma}(1-t_{k-1}))\widetilde{f}(\widetilde{\gamma}(1-t_k))(\widetilde{\gamma}(1-t_k)-A^*)^{-1}\\
&= \frac{1}{2\pi i}\sum_{k=1}^{n}(\widetilde{\gamma}(1-t_{k-1})-\widetilde{\gamma}(1-t_k))\widetilde{f}(\widetilde{\gamma}(1-t_k))(\widetilde{\gamma}(1-t_k)-A^*)^{-1}\\
&= R(\widetilde{P},A^*,\widetilde{f},\widetilde{\gamma})
\end{aligned} \tag{0.29.6}$$

So by taking mesh size of P goes to zero we have

$$\left(\frac{1}{2\pi i}\int_{\gamma} f(z)(z-A)^{-1}dz\right)^* = \frac{1}{2\pi i}\int_{\widetilde{\gamma}}\widetilde{f}(z)(z-A^*)^{-1}dz$$

Now lets come to the proof of our main result. Let $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ be a set of positively oriented closed curves in $G-\sigma(A)$ such that $\sigma(A)\subset ins(\Gamma)$ and $\mathbb{C}-G\subset out(\Gamma)$.

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(z-A)^{-1}dz$$

$$f(A)^* = \frac{1}{2\pi i}\int_{\widetilde{\Gamma}}\widetilde{f}(z)(z-A^*)^{-1}dz = \widetilde{f}(A^*)$$

Now if A is normal then

$$\begin{aligned}
f(A)f(A)^* &= f(A)\widetilde{f}(A^*)\\
&= \frac{-1}{4\pi^2}\int_{\Gamma}\int_{\widetilde{\Gamma}} f(w)\widetilde{f}(z)(w-A)^{-1}(z-A^*)^{-1}dwdz\\
&= \frac{-1}{4\pi^2}\int_{\Gamma}\int_{\widetilde{\Gamma}} f(w)\widetilde{f}(z)(z-A^*)^{-1}(w-A)^{-1}dwdz\\
&= f(A)^*f(A)
\end{aligned} \tag{0.29.7}$$

Hence f(A) is normal.

3. Let $f\in Hol(A)$ so $f: G\to\mathbb{C}$ such that $\sigma(A)\subset G$. . Let $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ be a set of positively oriented closed curves in $G-\sigma(A)$ such that $\sigma(A)\subset ins(\Gamma)$ and $\mathbb{C}-G\subset out(\Gamma)$.

For $\gamma \in \Gamma$ let $R(P, A, f, \gamma) = \frac{1}{2\pi i} \sum_{k=1}^{n} (\gamma(t_k) - \gamma(t_{k-1})) f(\gamma(t_k))(\gamma(t_k) - A)^{-1}$. As mesh goes to zero we have R(P,A,f,γ) converges to $\frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1} dz$. Hence R(P,A,f,$\gamma$)x converges to $\left(\frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1} dz\right) x$ where "x" is in M, but by the hypothesis R(P,A,f,γ)x$\in M$ for every partition P, and hence $\left(\frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1} dz\right) x \in M$. But γ was any curve in Γ so we are done.

4. Let $f \in Hol(a)$ such that $f(0) = 0$. Then $f(z) = zg(z)$ where g is also in Hol(a). Hence $f(a) = ag(a)$, now as $a \in I$, we have $f(a) \in I$ also.

□

0.30 Berger-Shaw Theorem

Introduction: This is the main chapter in this thesis. Here we use all the fundamentals that we have developed in the previous chapters. The proof of this theorem is not easy, it requires several lemmas to be proved first. I have simplified the proofs of some lemmas and provided every minor detail in each lemma. I have proved some simple results regarding the direct sum of two hyponormal operators. With some help from my guide I was able to prove lemma 5, which is used in the proof of Berger-Shaw theorem. The proof of this lemma is not in the original source and is assumed there. Finally I have explained the proof of Putnam Inequality.

In order to prove Berger-Shaw theorem, we will need a few lemmas.

Lemma 1 Let H be a Hilbert space, T∈ B(H) and P be a finite rank projection. Then **tr**$(P[T^*,T]P) \leq \left\| P^{\perp}TP \right\|_2^2$.

Proof. Let $(e_1, e_2, \ldots, e_n)$ be an orthonormal basis for P(H), then $P(x) = \sum_{i=1}^{n} < x, e_i > e_i$.

$$\begin{aligned} \mathrm{tr}(P[T^*T]P) &= \sum_{i=1}^{n} < P(T^*T - TT^*)(e_i), e_i > \\ &= \sum_{i=1}^{n} (< PT^*Te_i, e_i > - < PTT^*e_i, e_i >) \\ &= \left(\sum_{i=1}^{n} \|T(e_i)\|^2 \right) - \left(\sum_{i=1}^{n} \|T^*(e_i)\|^2 \right) \end{aligned} \tag{0.30.1}$$

And also,

$$\left\| P^{\perp}TP \right\|_2^2 = \sum_{i=1}^{n} \left\| P^{\perp}TPe_i \right\|^2 = \left(\sum_{i=1}^{n} \|Te_i\|^2 \right) - \left(\sum_{i=1}^{n} \|PTe_i\|^2 \right)$$

It's enough to show that $\sum_{i=1}^{n} \|T^*e_i\|^2 \geq \sum_{i=1}^{n} \|PTe_i\|^2$. By Pythagoras Theorem we have

$\|PTe_i\|^2 = \sum_{j=1}^{n} | < Te_i, e_j > |^2$.

$$\begin{aligned}\sum_{i=1}^{n}\|PTe_i\|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(| < Te_i, e_j > |^2\right)\\ &= \sum_{j=1}^{n}\sum_{i=1}^{n}\left(| < T^*e_j, e_i > |^2\right)\\ &= \sum_{j=1}^{n}\left\|PT^*e_j\right\|^2 \leq \sum_{j=1}^{n}\left\|T^*e_j\right\|^2\end{aligned} \tag{0.30.2}$$

Hence, we are done. □

Looking at the proof, we see that we have used the finiteness of rank of **P** in interchanging sums. More importantly we need **P** to be in trace class. But in our case the operator $P[T^*, T]P$ is a finite rank operator and hence it's automatically in trace class. But when the the projection operator is of infinite rank it's illegitimate to interchange sums and more seriously, it might not be in trace class also. Moreover if we take **P** to be identity map on H and **T** to be any non-surjective isometry on H , provided H is infinite dimensional, we can see that left hand side of the inequality will be strictly positive and right hand side will be zero as $I^{\perp} = 0$. So the finiteness of **P** cannot be relaxed.

Definition: **T** $\in B(H)$, is said to be a multicyclic operator if $\exists$ a finite set of vectors $\{g_1, g_2, \ldots, g_m\}$, such that linear span of $\{R(T)g_i \mid R \in RAT(\sigma(T)), 1 \leq i \leq m)\}$ is dense in H. The set of above vectors is a generating set. **T** is said to be m-multicyclic if the cardinality of the smallest generating set is m.

0.31 Notes

1. Let P and P' be two projection maps, then $P \leq P' \iff P(H) \subseteq P'(H)$.

   Let $R(P) \subseteq R(P')$, then $R(P') = R(P) + R(P)^{\perp} \cap R(P')$ and hence $P' = P + P''$, where P" is projection onto $R(P)^{\perp} \cap R(P')$.

   $$< P'x, x >= \left\|P'x\right\|^2 = \|Px\|^2 + \|P''x\|^2 \geq \|Px\|^2 =< Px, x > \forall x \in H$$

And if $P \leq P'$ and $x \in R(P)$, then

$$\|x\|^2 =< Px, x > \leq < P'x, x >= \left\|P'x\right\|^2 \leq \|x\|^2$$

$$So, \left\|P'x\right\| = \|x\| \implies P'x = x \implies x \in P'(H)$$

2. If A and B are two commuting operators, and A is invertible then $A^{-1}B = BA^{-1}$.

   Let $x \in H$ and $A^{-1}x = y$.

   $$BA^{-1}x = By \tag{1}$$

   $$Ay = x \implies BAy = Bx = ABy \implies A^{-1}Bx = By \tag{2}$$

   So, from (1) and (2), we see that $BA^{-1} = A^{-1}B$. And if both are invertible then it's easy to see that their inverses also commute.

Lemma 2: Let $\{A_n\}$ be a sequence of self-adjoint operators, such that $0 \leq A_n \leq A_{n+1} \leq I$ $\forall n$. Then there exist an operator $A \in B(H)$, such that $A_n \to A$, in SOT.

Proof. Let $x \in H$, Consider $\{< A_n x, x >\}$ it's a monotonically increasing, positive sequence, and is bounded by $\|x\|^2$. So it is convergent and hence cauchy.

Now we would use generalized Cauchy-Schwarz inequality, which says that if A or -A is a positive operator and $x, y \in H$, then $| < Ax, y > | \leq < Ax, x >< Ay, y >$.

$$\begin{aligned} \left\|(A_n - A_m)(x)\right\|^2 &=< (A_n - A_m)x, (A_n - A_m)x > \\ &\leq \left(< (A_n - A_m)x, x >\right) \left(< (A_n - A_m)^2 x, (A_n - A_m)x >\right) \end{aligned} \tag{0.31.1}$$

As $< A_n x, x > \leq \|x\|^2$, $\forall x \in H$ we have $\|A_n\| \leq 1$ $\forall n$. And hence $< (A_n - A_m)^2 x, A_n - A_m x >$ is bounded $\forall n, m$ for a fixed x . Now as $< A_n - A_m x, x > \to 0$ as $n, m \to \infty$ the left hand side also goes to zero. So $\{A_n x\}$ is Cauchy and hence must converge to some element in H. Define Ax = $\lim_{n\to\infty} A_n x$. Now all we have to show is that A is a bounded operator and that easily follows from uniform boundedness principle. So we are done.

□

Lemma 3: If **T** is a m-multicyclic operator, then there is a sequence of finite rank projection $\{P_k\}$, such that $P_k \to I$ (SOT), and $rank(P_k^{\perp}TP_k) \leq m \; \forall k \geq 1$.

Proof. Let $g_1, g_2, \dots, g_m$ be any set of generating vectors for **T**. $\{\lambda_j\}$ be a countable dense subset of $\mathbb{C} - \sigma(T)$. Arrange it so that each λ_j occurs infinitely often. Let $P_k :=$ projection onto $\vee$ $\{T^j(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i \mid 0 \leq j \leq 2k, 1 \leq i \leq m\}$

P_k is of finite rank and $P_k \leq P_{k+1}$. To see this, it's enough to show that spanning set of $R(P_k) \subset R(P_{k+1})$.

For $0 \leq j \leq 2k, 1 \leq i \leq m$

$$\begin{aligned} T^j(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i &= T^j(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}(T-\lambda_{k+1})(T-\lambda_{k+1})^{-1})g_i \\ &= T^j(T-\lambda_{k+1})(T-\lambda_1)^{-1}\dots(T-\lambda_{k+1})^{-1}g_i \\ &= T^{j+1}(T-\lambda_1)^{-1}\dots(T-\lambda_{k+1})^{-1}g_i \\ &- \lambda_{k+1}T^j(T-\lambda_1)^{-1}\dots(T-\lambda_{k+1})^{-1}g_i \end{aligned} \tag{0.31.2}$$

Since for λ outside $\sigma(T)$, $(T-\lambda)^{-1}$ and $(T-\lambda')$ will commute as $(T-\lambda)$ and $(T-\lambda')$ commutes.

As $0 \leq j+1 \leq 2k+1$, we can see that both terms in the last expression are in $R(P_{k+1})$, and hence it is in $R(P_{k+1})$. So we are done.

Now only vectors in the spanning set of $R(P_k)$ that can be mapped by T outside of $R(P_k)$ are $\{T^{2k}(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i \mid 0 \leq i \leq m\}$. Hence $rank(P_k^{\perp}TP_k) \leq m \; \forall k$, as all those vectors which stays in $R(P_k)$ after applying T will be mapped to zero by $P_k^{\perp}$.

Now all we have to show is $P_k \to I$. Let L be the closure of $\cup_{k=1}^{\infty}R(P_k)$. As we know $0 \leq P_k \leq P_{k+1} \leq I$, by lemma 2 we have a bounded operator P as the strong limit of $\{P_k\}$. We show that P agrees with Identity in L, and $L = H$ to complete the proof. Let $x \in \cup R(P_k)$, then for some k, $x \in R(P_k)$ and hence $\forall n \geq k$ we have $P_n x = x$. So $Px = x$. So P agrees with Identity on L, by continuity.

$T^j(T-\lambda_1)^{-1})g_i$ for $j = 0, 1 \in R(P_1)$. So $(T-\lambda_1)(T-\lambda_1)^{-}1g_i \in R(P_1)$, that is $g_i \in$

$R(P_1) \forall i$. So we just need to show $F(T)L \subset L \; \forall F \in RAT(\sigma(T))$. For $(T-\lambda)^{-1}$, where $\lambda \in \sigma(T)^c$, we can find a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_n\}$ such that $\{\lambda_{n_k}\} \to \lambda$. And by continuity $(T-\lambda_{n_k})^{-1} \to (T-\lambda)^{-1}$. So it's enough to show that for all n, $(T-\lambda_n)^{-1}L \subset L$ and also that $TL \subset L$.

We start with $T^j(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i$ in $R(P_k) \subset L$.

$$T\left(T^j(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i\right) = T^{j+1}(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i$$

if $0 \le j \le 2k-1$ then we are done, as the above expression will again stay in $R(P_k)$ and hence in L. But when $j = 2k$,

$$\begin{aligned} T^{2k+1}(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i &= T^{2k+1}(T-\lambda_{k+1})(T-\lambda_1)^{-1}\dots(T-\lambda_{k+1})^{-1}g_i \\ &= T^{2(k+1)}(T-\lambda_1)^{-1}\dots(T-\lambda_{k+1})^{-1}g_i \\ &- \lambda_{k+1}T^{2k+1}(T-\lambda_1)^{-1}\dots(T-\lambda_{k+1})^{-1}g_i \end{aligned} \tag{0.31.3}$$

Both the terms are in $R(P_{k+1})$ and hence the whole expression is in $R(P_{k+1})$. So by linearity $T(L) \subset L$.

we can assume $m \ge k+1$, as each λ_m occurs infinitely often in the sequence.

$$\begin{aligned} &(T-\lambda_m)^{-1}T^j(T-\lambda_1)^{-1}\dots(T-\lambda_k)^{-1}g_i = \\ &\qquad T^j(T-\lambda_{k+1})\dots(T-\lambda_{m-1})(T-\lambda_1)^{-1}\dots(T-\lambda_m)^{-1}g_i \end{aligned}$$

Now the above expression is a polynomial in T multiplied with $(T-\lambda_1)^{-1}\dots(T-\lambda_m)^{-1}$. The degree of polynomial is nothing but $j+m-1-k$, which is going to be strictly less than $2m$. Hence the above expression is in $R(P_m)$, so in L also. Again by linearity $(T-\lambda_m)^{-1}L \subset L \; \forall m$. And our proof is completed.

$\square$

0.32 Notes

Let H be a separable Hilbert space, $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis. Then we define the trace of an operator "A" as $\sum_{i=1}^{\infty} < Ae_i, e_i >$. But the definition will make sense only if this sum does not depends on the choice of basis. Now one sufficient condition for this is that A is in trace class, meaning A can be written as product of two Hilbert-Schmidt operators. But we can see that if A is any positive operator, it might not be in trace class but still the sum will be invariant under the choice of basis.

$$\sum_{i=1}^{\infty} < Ae_i, e_i >= \sum_{i=1}^{\infty} < A^{\frac{1}{2}}e_i, A^{\frac{1}{2}}e_i >= \sum_{i=1}^{\infty} \left\| A^{\frac{1}{2}}e_i \right\|^2$$

So we see that trace of A is exactly the Hilbert-Schmidt norm of $A^{\frac{1}{2}}$, which does not depends on the basis, so the trace of a positive operator is well defined. Now if T is any hyponormal operator, then $[T^*, T]$ is a positive operator and hence its trace is well defined.

Lemma 2 : Let T be a m-multicyclic hyponormal operator, then $tr[T^*, T] \leq m\|T\|^2$.

Proof. By lemma 3 there exist a sequence $\{P_k\} \to I$ and $rank(P_k^{\perp} T P_k) \leq m \; \forall k$. Let E_1 be a set of orthonormal basis for $R(P_1)$. And E_k be a orthonormal basis for $R(P_k) \cap R(P_{k-1}^{\perp})$ $k \geq 2$, now $\cup_{k=1}^{n} E_k$ is an orthonormal basis for $R(P_n)$, and $E = \cup_{k=1}^{\infty} E_k$ is an orthonormal basis for H, as $\cup_{k=1}^{\infty} R(P_k)$ is dense in H.

$$tr[T^*, T] = \sum_{e_i \in E} < [T^*, T]e_i, e_i >= \lim_{n \to \infty} \sum_{e_i \in \cup_{k=1}^{n} E_k} < [T^*, T]e_i, e_i >= \lim_{n \to \infty} tr(P_n[T^*, T]P_n)$$

Now by lemma 1, we have $tr(P_n[T^*, T]P_n) \leq \left\| P_n^{\perp} T P_n \right\|_2^2 \leq m\|T\|^2$, as rank $(P_n^{\perp} T P_n) \leq m$ $\forall n$. So we have $\lim_{n \to \infty} tr(P_n[T^*, T]P_n) \leq m\|T\|^2$. And our proof is complete. □

Now we will prove some facts.

1. $[(T_1 \oplus T_2)^*, (T_1 \oplus T_2)] = [T_1^*, T_1] \oplus [T_2^*, T_2]$.
2. If T_1 and T_2 are positive operators, then $T_1 \oplus T_2$ is also a positive operator.
3. If T_1 and T_2 are hyponormal, then $T_1 \oplus T_2$ is also hyponormal.

4. $tr(T_1 \oplus T_2) = tr(T_1) + tr(T_2)$.

5. $tr([(a+rS)^*, (a+rS)]) = r^2$, where $a \in \mathbb{C}$, $r \geq 0$. and S is a unilateral shift.

Proof. 1.

$$\begin{aligned}[(T_1 \oplus T_2)^*, (T_1 \oplus T_2)] &= (T_1^* \oplus T_2^*)(T_1 \oplus T_2) - (T_1 \oplus T_2)(T_1^* \oplus T_2^*) \\ &= (T_1^*T_1 \oplus T_2^*T_2) - (T_1T_1^* \oplus T_2T_2^*) \\ &= (T_1^*T_1 - T_1T_1^*) \oplus (T_2^*T_2 - T_2T_2^*) \\ &= [T_1^*, T_1] \oplus [T_2^*, T_2]\end{aligned} \tag{0.32.1}$$

2.

$$\begin{aligned}< T_1 \oplus T_2(x \oplus y), x \oplus y > &=< T_1x \oplus T_2y, x \oplus y > \\ &=< T_1x, x > + < T_2x, x >\end{aligned} \tag{0.32.2}$$

As both T_1 and T_2 are positive, the above expression is positive, hence $T_1 \oplus T_2$ is a positive operator.

3. This easily follows from part 1 and part 2.

4. Let $\{e_i\}$ be an orthonormal basis for H. Then $(\{e_i \oplus 0\}, \{0 \oplus e_j\})$ forms orthonormal basis for $H \oplus H$.

$$\begin{aligned}tr(T_1 \oplus T_2) &= \sum < T_1 \oplus T_2(e_i \oplus 0), e_i \oplus 0 > + \sum < T_1 \oplus T_2(0 \oplus e_j), 0 \oplus e_j > \\ &= \sum < T_1e_i \oplus 0, e_i \oplus 0 > + \sum < 0 \oplus T_2e_j, 0 \oplus e_j > \\ &= \sum < T_1e_i, e_i > + \sum < T_2e_j, e_j > \\ &= tr(T_1) + tr(T_2).\end{aligned} \tag{0.32.3}$$

5.

$$\begin{aligned}[(a+rS)^*, (a+rS)] &= (a+rS)^*(a+rS) - (a+rS)(a+rS)^* \\ &= (\bar{a}+rS^*)(a+rS) - (a+rS)(\bar{a}+rS^*) \\ &= |a|^2 + \bar{a}rS + raS^* + r^2S^*S - |a|^2 - arS^* - r\bar{a}S - r^2SS^* \\ &= r^2(S^*S - SS^*)\end{aligned} \tag{0.32.4}$$

Since S is an isometry we have $S^*S = I$ and $SS^* =$ Projection onto $R(S)$. So $I - SS^* =$ projection onto $R(S)^\perp$. And hence has its trace well defined. S is an unilateral shift, so $R(S)^\perp$ is a one-dimensional space. Hence trace of $I - SS^*$ is 1. And as trace is a linear functional, we are done.

$\square$

0.33 Vitali's Covering theorem

Theorem: Let F be any collection of non-degenerate closed balls in $\mathbb{R}^n$, such that $\sup\{diamB \mid B \in F\} < \infty$, then there exist a disjoint sub-collection G of F, such that $\cup_{B\in F} B \subset \cup_{B\in G} \hat{B}$, where if B is a closed disc of radius r then $\hat{B}$ is the concentric closed disc of radius five times to that of B.

Proof. Let D $= \sup\{diamB \mid B \in F\}$

$$F_j := \{B \in F \mid \frac{D}{(2)^j} < diamB \leq \frac{D}{(2)^{j-1}}\}$$

where $j = 1, 2, \ldots$

Then $F = \sqcup_{j=1}^{\infty} F_j$. Now we would define G_k as follows.

1. Let G_1 be any maximaly disjoint sub-collection of F_1. Such a collection is guaranteed by Zorn' lemma and this collection will be countable, as we cannot have uncountably many disjoint non-empty open sets in $\mathbb{R}^n$, since $\mathbb{R}^n$ is second countable.
2. Now after defining $G_1, G_2, \ldots, G_{k-1}$ we define $G_k$to be any maximaly disjoint sub-collection of the set $\{B \in F_k \mid B \cap B' = \emptyset \forall B' \in \cup_{i=1}^{k-1} G_i\}$.
3. Define $G = \cup_{k=1}^{\infty}$. G is a countable disjoint sub-collection of F.

Claim: $\cup_{B\in F} B \subset \cup_{B\in G} \hat{B}$

Let $B \in F$, then B belongs to a unique F_j. If $B \in \{B \in F_j \mid B \cap B' = \emptyset \forall B' \in \cup_{i=1}^{j-1} G_i\}$ then there must exit a $B' \in G_j$ such that $B \cap B' \neq \emptyset$ by the definition of G_j, otherwise there must exist a B' in $\cup_{i=1}^{j-1} G_i$ such that B' intersects with B. So, altogether we have a $B' \in \cup_{i=1}^{j} G_i$, such that $B \cap B' \neq \emptyset$.

$diamB' > \frac{D}{(2)^j}$ and $diamB \leq \frac{D}{(2)^{j-1}} \implies diamB < 2diamB'$, hence $B \subset \hat{B}'$. So our proof is complete. □

0.34 A corollary to Vitali's covering theorem

Corollary: If U is an open set in $\mathbb{R}^n$, then $\exists \{B_k\}$, a disjoint collection of closed balls, which are contained in U, such that $\mu(U - \cup_{k=}^{\infty} B_k) = 0$. Given a $\delta > 0$ we can also make sure that the diameter of our closed balls is less than δ.

Proof. Given a $\delta > 0$, let $F := \{B \subset U \mid 0 < diamB < \delta\}$, where B is a closed ball. It's easy to see that $U = \cup_{B \in F} B$, moreover the collection F is non-degenerate and supremum of diam of balls is bounded by δ. Applying Vitali's thm to this collection, we get a disjoint sub-collection $\{B_K\}$, such that $U \subset \cup_{k=1}^{\infty} \hat{B}_k$. By countable sub-additivity of Lebesgue measure, and the fact that measure of a ball of radius r in $\mathbb{R}^n$ is proportional to r^n, we get

$$\mu(U) \leq \sum_{k=1}^{\infty} \mu(\hat{B}_k) = (5^n) \sum_{k=1}^{\infty} \mu(B_k)$$

Let's assume that $\mu(U) < \infty$.

$$(1 - \frac{1}{(5^n)})\mu(U) \geq \mu(U) - \sum_{k=1}^{\infty} \mu(B_k) = \mu(U - \cup_{k=1}^{\infty} B_k)$$

Since, B_k's are disjoint and are contained in U. Choose a θ such that, $1 - \frac{1}{5^n} < \theta < 1$.

$$\mu(U)\theta > \mu(U - \cup_{k=1}^{\infty} B_k) = \inf_p \mu(U - \cup_{k=1}^{p} B_k)$$

So there exists M_1, such that $\mu(U)\theta > \mu(U - \cup_{k=1}^{M_1} B_k)$

Now by letting $U_2 = (U - \cup_{k=1}^{M_1} B_k)$, which is again an open set, we do the same argument for it as above and obtain $B_{M_1+1}, \dots, B_{M_2}$ in U_2, such that

$$\mu(U - \cup_{k=1}^{M_2} B_k) = \mu(U_2 - \cup_{k=M_1+1}^{M_2} B_k) < \theta\mu(U_2) < (\theta)^2\mu(U)$$

So we keep on doing it, and obtain a disjoint collection $\{B_k\}$, in U such that at the $p^{th} - stage$ we have

$$\mu(U - \cup_{k=1}^{M_p} B_k) < (\theta)^p \mu(U)$$

As $0 < \theta < 1$, $(\theta)^p \to 0$, as $p \to \infty$. And we have

$$\mu(U) = \sum_{k=1}^{\infty} \mu(B_k)$$

Now if $\mu(U) = \infty$, we then apply the above procedure to sets $\{x \in U \mid m - 1 < |x| < m\}$, $m \geq 1$.

And hence we are done.

$\square$

0.35 Note

Now we prove an interesting result about the direct sum of two operators. But before coming to that result, lets revise some basics about direct sum of two operator. It's easy to see that $\sigma(T_1 \oplus T_2) = \sigma(T_1) \cup \sigma(T_2)$. As existence of inverse of $T_1 \oplus T_2$ is equivalent to the existence of inverses of T_1 and T_2, so $T_1 \oplus T_2$ is not invertible $\iff$ either T_1 or T_2 is not invertible.

Now if P(x) is a polynomial in x and $T \in B(H)$, we can naturally define P(T) as an operator in B(H). Moreover if, $\lambda \in \sigma(T)^c$ and $Q(x) = (x - \lambda)^{-1}$, then we Q(T) is defined as $(T - \lambda)^{-1} \in B(H)$. Generalizing it, if R(x) is any rational function with poles outside the spectrum of T, then we can define R(T) as an operator in B(H), simply by replacing x by T in the expression.

$$(T_1 \oplus T_2)^2 = (T_1 \oplus T_2)(T_1 \oplus T_2) = T_1^2 \oplus {T_2}^2$$

follows from definition of direct sum. And hence by induction we have

$$(T_1 \oplus T_2)^n = T_1^n \oplus {T_2}^n$$

And obviously we have $c(T_1 \oplus T_2) + (A \oplus B) = (cT_1 + A) \oplus (T_2 + B)$. So we have the result that $P(T_1 \oplus T_2) = P(T_1) \oplus P(T_2)$, where P is any polynomial.

Now with the fact that $(T_1 \oplus T_2)^{-1} = T_1^{-1} \oplus T_2^{-1}$, we can claim that $R(T_1 \oplus T_2) = R(T_1) \oplus R(T_2)$, where R(x) is any rational function with poles outside $\sigma(T_1) \cup \sigma(T_2)$. Even more, let f(x) be any holomorphic function in nbd of $\sigma(T_1) \cup \sigma(T_2)$. Now by Runge's theorem we have $\{R_n\}$ a sequence of rational function with poles outside $\sigma(T_1) \cup \sigma(T_2)$, and $R_n \to f$ uniformly in a nbd of $\sigma(T_1) \cup \sigma(T_2)$. By Riesz functional calculus we have $R_n(T_1 \oplus T_2) \to f(T_1 \oplus T_2)$. But we have just seen that $R_n(T_1 \oplus T_2) = R_n(T_1) \oplus R_n(T_2)$, and again by Riesz calculus $\lim_{n\to\infty} R_n(T_1 \oplus T_2) = \lim_{n\to\infty} R_n(T_1) \oplus R_n(T_2) = f(T_1) \oplus f(T_2)$. So finally we have,

$$f(T_1 \oplus T_2) = f(T_1) \oplus f(T_2).$$

Where f is any holomorphic function in a nbd of $\sigma(T_1) \cup \sigma(T_2)$.

Now we are in the position to state our result.

Lemma 5: let T_1 be m-multicyclic and T_2 be m'-multicyclic, such that $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, then $T_1 \oplus T_2$ is also a multicyclic operator with multiplicity $= \max\{m, m'\}$.

Proof. Let $\{g_1, \ldots, g_m\}$ be a generating set for T_1 and $\{v_1, \ldots, v_{m'}\}$ be a generating set for T_2. WLOG let $m > m'$, consider $\{g_i \oplus v_i \mid 1 \leq i \leq m, v_i = 0 \forall i \geq m' + 1\}$. We will show that this is a generating set for $T_1 \oplus T_2$.

First thing to observe is that closure of span$\{R(T_1 \oplus T_2) g_i \oplus v_i \mid R \in RAT(\sigma(T_1) \cup \sigma(T_2)), 1 \leq i \leq m\}$ = closure of span$\{f(T_1 \oplus T_2) g_i \oplus v_i \mid f \in Hol(\sigma(T_1) \cup \sigma(T_2)), 1 \leq i \leq m\}$, by Runge's theorem.

So it's enough to show that the set span$\{f(T_1 \oplus T_2)g_i \oplus v_i \mid f \in Hol(\sigma(T_1) \cup \sigma(T_2)), 1 \leq i \leq m\}$ is dense in H.

Now as $\sigma(T_1) \cap \sigma(T_2) = \emptyset$, we can find two disjoint open sets U_1 and U_2 such that $\sigma(T_1) \subset U_1$ and $\sigma(T_2) \subset U_2$. Let $f \in Hol(\sigma(T_1))$ and $g \in Hol(\sigma(T_2))$, we can assume that their domains U and V are contained in U_1 and U_2. Define F on $U \cup V$ as f on U and zero on V, similarly define G as g on V and zero on U. We can see that both F and G belong to $Hol(\sigma(T_1) \cup \sigma(T_2))$. Now let $H = F + G$, and consider

$$H(T_1 \oplus T_2) = H(T_1) \oplus H(T_2) = (F+G)(T_1) \oplus (F+G)(T_2) = f(T_1) \oplus g(T_2).$$

As G is zero in a nbd of $\sigma(T_1)$ and F is zero in a nbd of $\sigma(T_2)$.

Now let $x \oplus y$ be a element in $H \oplus H$. We can $f_1, \ldots, f_m$ in $Hol(\sigma(T_1))$ and $l_1, \ldots, l_{m'}$ in $Hol(\sigma(T_2)$ such that $\left\|\sum_{i=1}^{m} f_i(T_1)g_i - x\right\| < \epsilon$ and $\left\|\sum_{i=1}^{m'} l_i(T_2)v_i - y\right\| < \epsilon$. Let $H_i \in Hol(\sigma(T_1) \cup \sigma(T_2))$ such that $H_i(T_1 \oplus T_2) = f_i(T_1) \oplus l_i(T_2)$ for $1 \leq i \leq m$, let $l_i = 0$ for $m' + 1 \leq i \leq m$.

$$\sum_{i=1}^{m} H_i(T_1 \oplus T_2)(g_i \oplus v_i) = (\sum_{i=1}^{m} f_i(T_1)g_i) \oplus (\sum_{i=1}^{m} l_i(T_2)v_i)$$

Hence,

$$\left\|\left(\sum_{i=1}^{m} H_i(T_1 \oplus T_2)(g_i \oplus v_i)\right) - (x \oplus y)\right\| = \left\|\left(\sum_{i=1}^{m} f_i(T_1)g_i - x\right) \oplus \left(\sum_{i=1}^{m} l_i(T_2)v_i - y\right)\right\| < \sqrt{2}\epsilon.$$

So we see that the set span$\{f(T_1 \oplus T_2)(g_i \oplus v_i) \mid f \in Hol(\sigma(T_1) \cup \sigma(T_2)), 1 \leq i \leq m\}$ is dense in $H \oplus H$. And hence our proof is complete. □

There are two more points I would like to mention (proofs of which are easy) before going to our main Berger-Shaw theorem.

1. If T is a m multi-cyclic operator, then $a + rT$, where a is any complex number and r is positive, is also a m multi-cyclic operator with same generating sets.
2. $\oplus(S)^n := S \oplus \cdots \oplus S$ n times. If S is m- multicyclic operator then $\oplus(S)^n$ is nm multi-cyclic.

0.36 The Berger-Shaw Theorem

Theorem: If T is any m-multicyclic hyponormal operator, then $tr([T^*,T]) \leq \frac{m}{\pi}Area(\sigma(T))$.

Proof. Let $R = \|T\|$. D(a,r) is the closed disc in complex plane, centered at "a" and of radius r. And B(a,r) is an open ball with center "a" and radius r. Consider $B(0,R) - \sigma(T)$, it is an open set. We apply the corollary of the Vitali's theorem to this open set and obtain $\{D(a_i,r_i) \mid 1 \leq i \leq n\}$, a disjoint collection of closed balls in our open set such that

$$\pi R^2 - Area(\sigma(T) \cap B(0,R)) - \pi(r_1^2 + \cdots + r_n^2) < \epsilon$$

now $Area(\sigma(T) \cap B(0,R)) = Area(\sigma(T))$, as $\sigma(T) \subset D(0,R)$ and unit circle has measure zero.

So we have $\{D(a_i,r_i) \mid 1 \leq i \leq n\}$ contained in $D(0,R) - \sigma(T)$, and

$$\pi R^2 - Area(\sigma(T)) - \pi(r_1^2 + \cdots + r_n^2) < \epsilon$$

Let $S_i := \oplus(a_i + r_i S)^m$, where S is a unilateral shift. Now define $A = T \oplus S_1 \oplus \cdots \oplus S_n$. Observe a few points about A.

1. $\|A\| = max\{\|T\|,\|a_i + rS_i\|\}$. As $\|a_i + rS_i\| \leq |a_i| + r < R$ for all i. Since $D(a_i,r_i) \subset D(0,R)$. Hence $\|A\| = R$
2. A is a m-multicyclic operator as the spectrum of S_i is $D(a_i,r_i)$, and all $D(a_i,r_i)$ and $\sigma(T)$ are disjoint and also T and $S_i's$ are m -multicyclic. So by our lemma 5 A is a m-multicyclic operator.
3. A is a hyponormal opertor as T and S_i are, and direct sum of hyponormal operators is hyponormal.

Applying lemma 4 to A, we get

$$tr([A^*,A]) = tr([T^*,T]) + \sum_{i=1}^{n} mr_i^2 \leq mR^2$$

$$\implies tr([T^*,T]) \leq \frac{m}{\pi}(\pi R^2 - \sum_{i=1}^{n} \pi r_i^2) < \frac{m}{\pi}(Area(\sigma(T)) + \epsilon)$$

As ϵ is arbitrary, we have our theorem.

$$tr([T^*, T]) \leq \frac{m}{\pi} Area(\sigma(T)).$$

We are done. □

0.37 Putnam's Inequality

Putnam's inequality is a deep result in functional analysis. But the fact that it's an easy corollary of Berger-Shaw theorem is remarkable. The power of Berger-Shaw theorem is quite impressive. If you look at the original paper of Putnam in which he proves the inequality, he uses a lot of intricate techniques, which makes the paper abstruse. The route that I have taken is quite long but is not as exacting as the original proof.

We will be using the following simple lemma in the proof of Putnam inequality.

Lemma 6: Let T be a hyponormal operator, M be an invariant subspace of T. Then $S := T_{|_M}$ is also hyponormal.

Proof. let P be projection onto M, then $S^* = PT^*_{|_M}$. Let $x \in M$,

$$\|Sx\| - \|S^*x\| = \|Tx\| - \left\|PT^*_{|_M}x\right\| \geq \|Tx\| - \|T^*x\| \geq 0.$$

As T is hyponormal. Hence S is also hyponormal. □

Thoerem: If T is any hyponormal operator, then

$$\|[T^*, T]\| \leq \frac{Area(\sigma(T))}{\pi}.$$

Proof. Let f be a vector in H, such that $\|f\| \leq 1$. L:= closure of span$\{R(T)f \mid R \in RAT(\sigma(T))\}$. We see that L is invariant under T. Define $S = T_{|_L}$, We show that $\sigma(S) \subset \sigma(T)$. Let $\lambda \in (\sigma(T))^c$, so S-$\lambda$ is one-one as it's just a restriction of an invertible operator. We need to show that it is surjective also. We know that $T - \lambda(L)$ is closed as it's an image of a lower bounded operator. Now see this

$$R(T)f = (T - \lambda)(T - \lambda)^{-1}R(T)f$$

. where R$\in$ $RAT(\sigma(T))$. $(T - \lambda)^{-1}R(T)f$ is again an element of L. And hence $R(T)f \in T - \lambda(L)$. So we have span$\{R(T)f \mid R \in RAT(\sigma(T))\} \subset T - \lambda(L)$. Hence $L = T - \lambda(L)$ and $S - \lambda$ is surjective also. So $\lambda \in \sigma(S)^c$.

Now S is 1-multicyclic hyponormal operator, apply Berger-Shaw Theorem to S to obtain

$$\begin{aligned} < [T^*, T]f, f > &\leq < [S^*, S]f, f > \\ &\leq tr([S^*, S]) \\ &\leq \frac{Area(\sigma(S))}{\pi} \\ &\leq \frac{Area(\sigma(T))}{\pi} \end{aligned} \tag{0.37.1}$$

But this is true for any $f \in H$ with $\|f\| \leq 1$. Hence $\|[T^*, T]\| \leq \frac{Area(\sigma(T))}{\pi}$. This completes our proof. $\square$

One immediate corollary of Putnam inequality is that if T is a hyponormal operator, such that Lebesgue measure of its spectrum is zero, then it is a normal operator. So any hyponormal operator on a finite dimensional Hilbert space is always normal.

0.38 References

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