Do We Really Require a Curved Description of Spacetime to Explain Gravity?

A Thesis

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by

Anmol Kumar Sahu

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Prof. Tejinder Pal Singh

Certificate

This is to certify that this dissertation entitled *Do We Really Require a Curved Description of Spacetime to Explain Gravity?* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune, represents study carried out by Anmol Kumar Sahu under the supervision of Prof. Tejinder Pal Singh, Department of Astronomy and Astrophysics, Tata Institute of Fundamental Research, Mumbai, during the academic year 2020-2021.

Prof. Tejinder Pal Singh

Committee:

Prof. Tejinder Pal Singh

Prof. Suneeta Vardarajan

This thesis is dedicated to Einstein's quest to understand the mind of God.

Declaration

I hereby declare that the matter embodied in the report entitled *Do We Really Require a Curved Description of Spacetime to Explain Gravity?* are the results of the work carried out by me at the Department of Astronomy and Astrophysics, Tata Institute of Fundamental Research, Mumbai, under the supervision of Prof. Tejinder Pal Singh and the same has not been submitted elsewhere for any other degree.

Anmol Kumar Sahu

Anmol Sahu

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I thank my supervisor Prof. Tejinder Pal Singh for introducing me to a whole realm of lesser known interesting ideas in fundamental physics. I thank him for allowing me to discover and work on my own research problem. I thank him for his guidance, kind suggestions, fruitful discussions, and unconditional support.

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Abstract

The most radical feature of General Relativity was its identification of gravity with spacetime curvature. However, in attempts to find a unified field theory, Einstein himself discovered an alternate way to look at gravity in which spacetime was globally flat, but gravity was mediated by torsion. In this thesis, we will discuss two such theories of gravity which are equivalent to general relativity but build upon globally flat spacetime. The theories which will be discussed are Teleparallel Equivalent of General Relativity (TEGR) and Symmetric Teleparallel Equivalent of General Relativity (STEGR). We will show that the identification of gravity with spacetime curvature is not unique but a mere convention. These theories give two other equivalent but conceptually very different ways to look at gravity. We will show that classically all three theories are indistinguishable and which one represents the reality we do not know.

Preface

My first introduction to the special theory of relativity came in class IX from a science magazine. I was amazed to learn that the speed of light is not relative, and time is relative. My entire summer vacation was spent chasing the speed of light. Taking hints from the article, I did numerous calculations to comprehend what I learned. I discovered the time dilation and the length contraction formulas with the help of thought experiments given in the article. After some time, I bought Einstein's biography, which among other things, introduced me to his last dream. At that moment it became absolutely clear to me what I want to do with my life. But I didn't know how. I had seen Prof. Jayant Narlikar in a documentary. I wrote to him, and he suggested me to join IISER after class XII. I did. The complex details didn't shift my aim but motivated me even more to pursue my goal.

When I was looking for my MS thesis project, I learned about a new approach to Quantum Gravity developed by by Prof. T.P. Singh from TIFR, Mumbai. The overview of the theory was convincing enough to make me join the project. As the project started, there were a lot of pre-requisites to understand the theory: Adler's trace dynamics [1], non-commutative geometry [2], GRW theory of Spontaneous localization [3], along with half dozen earlier papers on the theory. Since there were really a lot of pre-requisites which could easily take several months, what I did was, I first explored all of it without getting into too much details to get a big picture. Then I read some of the details which were required for calculations. As soon as I started the calculations, I figured that the calculations were easy but too messy and lengthy, taking several pages just for simple things. I struggled with it for 2-3 days and then I got an idea to introduce new variables. These variables simplified everything and calculations became easy and elegant. I re-did some of the calculations from the paper in these new variables and showed them to my supervisor. He appreciated me and asked me to do all of his work in these new variables. He said I can either work alone or take any student from his group for assistance. I choose the second option and picked Abhinash Kumar Roy from IISER Kolkata who was also doing his MS thesis under my supervisor. That's how the project actually began. We worked all day, and sometimes all night, without getting tired. I was experiencing the pleasure of findings things out for the very first time.

Since these variables made things simpler, they also opened new possibilities which were previously unknown. Apart from the representation of the theory in these new variables, we also solved the eigenvalue equation and found the ground state of the theory. According to my supervisor, the equation can in principle determine the values of the standard model parameters and the ground state could characterise non-singular initial epoch in quantum cosmology. These results are summarized in our paper which appeared in Modern Physics Letters A [4].

Around the same time a question occurred to me, Do we really require a curved description of spacetime to explain gravity? I did the literature survey and figured that there were some gravity theories which were equivalent to General Relativity but based on fixed Minkowskian background spacetime. Two examples of such theories were Teleparallel Gravity and Symmetric Teleparallel Gravity. In Teleparallel Gravity, curvature was zero and gravity was mediated by torsion. In symmetric teleparallel gravity both curvature and torsion were zero and gravity was mediated by non-metricity. Existence of such theories made me question the uniqueness of curved spacetime interpretation of classical gravity, which may have serious consequences for quantum gravity if taken seriously. For example, the sacred requirement of background independence in final theories may need revision in the light of these alternatives. I found these questions to be more interesting so I changed my thesis topic to explore the consequences of these alternative in detail.

I was in the dilemma on whether to include the previous work in the thesis, but finally I decided not to do that because it will require one more dedicated thesis to do justice with it.

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Chapter 1

Introduction

"One geometry cannot be more true than another; it can only be more convenient. Now, Euclidean geometry is and will remain, the most convenient ... What we call a straight line in astronomy is simply the path of a ray of light. If, therefore, we were to discover negative parallaxes, or to prove that all parallaxes are higher than a certain limit, we should have a choice between two conclusions: we could give up Euclidean geometry, or modify the laws of optics, and suppose that light is not rigorously propagated in a straight line. It is needless to add that every one would look upon this solution as the more advantageous."

— Henri Poincaré, Science and Hypothesis, 1902

Henri Poincaré wrote these lines after his thorough analysis of the relationship between physics and geometry. If we apply Poincaré's conclusion to the physics of gravity, it implies that the curved description of spacetime portrayed by General Relativity is mere a mathematical convention, and in principle, it can be replaced by an equivalent flat spacetime description. There have been numerous attempts at formulating a theory of gravity in flat spacetime, both before and after the inception of General Relativity. Albert Einstein himself made many failed attempts at formulating a theory in flat spacetime before he succeeded in 1915 with General Relativity [5]. However, with the birth of General Relativity, our understanding of spacetime completely changed because General Relativity was not only the theory of gravity but the theory of spacetime itself, and concluded that the spacetime was not flat, but it had curvature which changes depending on the presence of matter.

General Relativity has passed almost every empirical test that it has encountered since its birth. The empirical pieces of evidence in support of General Relativity piled up at such a scale that when we observed astronomical phenomena that were inconsistent with General Relativity, such as the mysterious rotation curves of the galaxies, we did not doubt that our fundamental theory of gravity could be wrong, rather we linked it to some mysterious Dark Matter. Just this example suffices to show the level of trust we have in General Relativity. However, as it happens, not everybody agrees. Even though the Standard Model of Cosmology takes the side of Dark Matter, an enormous field of modified gravity is ever increasing since then. The constant battle between modified gravity and dark matter is not yet concluded.

Dark matter is a thing about which everybody may not agree, but there's one more theory that is perhaps more sacred to us than General Relativity. It is Quantum Theory. The blunder is that both theories are incompatible with each other. Since the scales at which we use these theories are very far from each other, so in everyday physics, we don't face any trouble due to their mutual inconsistencies, but on the scales when both theories matter, the necessity to have a consistent theory becomes apparent. The approach to solving the problem may differ, but almost everybody agrees that we require a quantum theory of gravity.

As General Relativity links gravity with spacetime, so we think that a theory of quantum gravity should necessarily be a quantum theory of spacetime; we think that the theory should talk about the nature of spacetime at the most fundamental level. However, what if we are making a mistake in interpreting General Relativity? Is it even possible for a theory that has passed numerous tests and still we do not understand it yet? If we talk about the possibility, then it is certainly possible. Quantum Mechanics is a prime example of this which has numerous interpretations. In the case of General Relativity, however, talking about the interpretation is less heard of. That is why the subject of this thesis is the interpretation of General Relativity. To state clearly, we are not going to question the validity of General Relativity. In fact, the primary assumption of the presented work is that General Relativity is correct in giving us the right numbers. What we will question is the meaning that General Relativity gives us about the world.

In this thesis, we will present two theories of gravity that build upon flat spacetime, just as Poincare said. Both of these theories are equivalent to General Relativity but differs in their conceptual understanding. By equivalent, we mean that these two theories will give us exactly the same numbers that General Relativity does, but the reason for giving those numbers will be different. That is where comes the interpretation part. In General Relativity, we think that spacetime gets curved in the presence of matter, but in these theories the spacetime is flat. The two facts can't be true at the same time. The spacetime out there must be either flat or curved. Just from the existence of these alternative formulations of gravity, we can conclude that the curved description of spacetime is just a mathematical convention, not an underlying reality of spacetime.

The two theories that we will discuss are Teleparallel Equivalent of General Relativity (TEGR) and Symmetric Teleparallel Equivalent of General Relativity (TEGR). In TEGR, the spacetime is curvature free, and gravity is mediated by torsion, and in STEGR, spacetime is both curvature and torsion-free, and gravity is mediated by non-metricity. The literature is filled with the modified versions of both of these theories, but their presentation is not that simple. Moreover, since they focus on a more generalised version of these theories, which make it even more complicated to appreciate the beauty contained within them, so my purpose in this thesis will be to present both theories in as simple as possible manner. The emphasis will be on conceptualisation rather than phenomenology. Interested readers who want to learn more details can refer to these references: [6] [7] [8] [9] [10].

Chapter 2

Preliminaries

In this chapter we will learn the meaning of some general terms, Tetrad Formalism, and then move on to work with tensor calculus and derive some general quantities which we will need in the next chapter to describe the Teleparallel and Symmetric Teleprallel Theories of Gravity. We will also learn the geometrical meaning of Torsion and Non-Metricity tensors that will be the base of those gravity theories. We will presume that the reader is familiar with basic differential geometry and general relativity, so we will only include what is generally not covered in an introductory course on General Relativity.

2.1 Meaning of some general terms

Before we begin, we will learn the meaning of some general terms that will often encounter in our discussion. Rather than giving a formal definitions which can be looked up in any textbook anyways, here we will try understand the meaning of these terms in a simpler manner.

1. **Invariance:** If F is a functional field of ϕ, then F is said to be invariant under transformation if $\phi \to \phi' \implies F[\phi] = F[\phi']$. That means an invariant quantity transform as a scaler, it does not change under a given transformation.

2. **Covariance:** If the form of equations describing a physical law remains invariant under a given transformation, we say that the equations are covariant.

3. **General Covariance:** lf the form of equations describing a physical law remains invariant under arbitrary differentiable coordinate transformations, we say that equations are generally covariant.

4. **Passive Transformation:** If a field $\phi(x)$ upon a coordinate transformation $x \to x'$ transforms as $\phi'(x) = \phi(x')$, we say that the field has transformed passively. A passive transformation is just a representation of old field in a new coordinate system. Both ϕ and ϕ' essentially represents the same field. Which means a point remains at the same point, we just relabel it.

5. **Active Transformation:** If a field $\phi(x)$ upon a coordinate mapping $x \to x' = h(x)$ transforms as $\phi'(x') = \phi(x)$, or equivalently $\phi'(x) = \phi(h^{-1}(x))$, we say that the field has transformed actively. In an active transformation ϕ' represents a completely new field. In this a points actually moves.

6. **Diffeomorphism:** If there exists a one-to-one, smooth, differentiable, and invertible map f from points on one manifold to points on another manifold then we say that the map f is diffeomorphism and the two manifolds are diffeomorphic. It is clear from invertible condition that both manifolds should have same dimension.

7. **Active Diffeomorphism:** If f is a diffeomorphism map from a manifold to itself, then a field $\phi(x)$ in this manifold transforms under active diffeomorphism as $\phi'(x) = \phi(f^{-1}(x))$.

8. **Metric Compatibility:** A connection is called metric compatible if the covariant derivative of the metric tensor gives zero.

2.2 Tetrad Formalism

Tetrad means a set of four. It is also known as *verbein* in German, which means four legs. The basic idea is to find a set of four orthonormal unit vectors at each point in spacetime. Since Riemannian (or Lorentzian) manifolds are locally flat, it is always possible to find a local coordinate at each point in the manifold that is orthonormal. Since an observer is always at some point in the manifold, working with tetrads means working in the observer's local frame. The applications of Tetrad formalism are endless, so learning a new method will be fruitful in the long run. Tetrad formalism is not only an alternative to metric formalism but essential for coupling gravity to spinor fields. That is because finite-dimensional spinor representation of the general covariance group does not exist. However, since the Lorentz group has finite-dimensional spinor representation, we can use tetrad formalism to incorporate spinors in gravity. It is not even possible in metric formalism, and that is why tetrad formalism is more fundamental than metric formalism. We can convert tetrads and metric formalism interchangeably for the bosonic field, but we can not do that for the fermionic field. In this thesis, we need tetrad formalism to learn teleparallel gravity. It is possible to do teleparallel gravity in metric formalism itself, but tetrads provide a more natural way to work in these theories. Since this formalism is very important for our purpose, now we will learn it in detail.

We all are familiar with coordinate basis vectors which are given by,

$$e_\mu = \frac{\partial}{\partial x^\mu} = \partial_\mu \tag{2.1}$$

and coordinate basis covectors which are given by,

$$e^\mu = dx^\mu \tag{2.2}$$

and the relationship between the two is given by the dot product,

$$e^\mu \cdot e_\nu = \delta^\mu_\nu \ . \tag{2.3}$$

Any vector v can be expanded into its basis vectors as,

$$v = v^\mu e_\mu \tag{2.4}$$

and any covector v^* can be expanded into its basis covectors as,

$$v^* = v^*_\mu e^\mu \tag{2.5}$$

Except cartesian coordinate, coordinate basis are in general not orthonormal. They are related by metric tenor $g_{\mu\nu}$ as,

$$e_\mu \cdot e_\nu = g_{\mu\nu} \tag{2.6}$$

and so the coordinate basis covectors are related by inverse metric tensor $g^{\mu\nu}$ as,

$$e^\mu \cdot e^\nu = g^{\mu\nu} \tag{2.7}$$

We know that Lorentzian manifolds are locally flat. We will use this fact to define an orthonormal vectors at every point in the manifold. We will enforce this following the metric definition $e_\mu \cdot e_\nu = g_{\mu\nu}$ such that,

$$\hat{e}_a \cdot \hat{e}_b = \eta_{ab} \tag{2.8}$$

Here $\hat{e}_a$ and $\hat{e}_b$ are not coordinate basis vectors, but they are just vectors that follow the above condition. We will call them non-coordinate basis vectors. We have put a hat in them to distinguish them from coordinate basis vectors. Notice that we have used Latin indices $(a, b, c...)$ instead of Greek indices $(\alpha,\ \beta, ...\ \mu,\ \nu...)$. That is to remind us that they are not coordinate basis vectors. From now on, we will reserve Greek indices for coordinates and Latin indices for non-coordinates. Since these are also vectors, we can expand them into basis vectors as,

$$\hat{e}_a = e_a^\mu e_\mu \tag{2.9}$$

Here the vector coefficients e_a^μ are called tetrads. We can now put (2.9) in (2.8) to get,

$$\begin{aligned}
\eta_{ab} &= \hat{e}_a \cdot \hat{e}_b \\
&= e_a^\mu\ e_\mu \cdot e_b^\nu e_\nu \\
&= e_a^\mu\ e_b^\nu\ e_\mu \cdot e_\nu \\
&= e_a^\mu\ e_b^\nu\ g_{\mu\nu}
\end{aligned}$$

$$\boxed{\eta_{ab} = e_a^\mu\ e_b^\nu\ g_{\mu\nu}} \tag{2.10}$$

That is how we can write Lorentz metric in terms of metric tensor. Now we will see the opposite, how to write metric tensor in terms of Lorentz metric. For that we need to use the fact that as we can write any vector in terms of coordinate basis vectors, in the similar manner we can also write any vector in terms of non-coordinate basis vectors,

$$e_\mu = e_\mu^a \hat{e}_a \tag{2.11}$$

So we can write metric tensor $g_{\mu\nu}$ as,

$$\begin{aligned} g_{\mu\nu} &= e_\mu \cdot e_\nu \\ &= e^a_\mu \, \hat{e}_a \cdot e^b_\nu \, \hat{e}_b \\ &= e^a_\mu \, e^b_\nu \, \hat{e}_a \cdot \hat{e}_b \\ &= e^a_\mu \, e^b_\nu \, \eta_{ab} \end{aligned}$$

$$\boxed{g_{\mu\nu} = e^a_\mu \, e^b_\nu \, \eta_{ab}} \tag{2.12}$$

The meaning of indices is very important here. Lorentzian indices $\{a, b\}$ are like internal indices and will be contracted, raised or lowered with only Lotentz metric η_{ab} while coordinate or spacetime indices $\{\mu, \nu\}$ will be contracted, raised or lowered with only metric tensor $g_{\mu\nu}$. For example,

$$e^\mu_a = \eta_{ab} g^{\mu\nu} e^b_\nu \tag{2.13}$$

$$e^a_\mu = \eta^{ab} g_{\mu\nu} e^\nu_b \tag{2.14}$$

So far we were working with vectors. The process is same for covectors. We can write,

$$\hat{e}^a \cdot \hat{e}^b = \eta^{ab} \tag{2.15}$$

We can expand these covectors in terms of basis covectors as,

$$\hat{e}^a = e^a_\mu \hat{e}^\mu \tag{2.16}$$

Now we can put (2.16) into (2.15) to get,

$$\boxed{\eta^{ab} = e^a_\mu \, e^b_\nu \, g^{\mu\nu}} \tag{2.17}$$

Any covector can also be written in terms of non-coordinate basic covectors as,

$$e^\mu = e^\mu_a e^a \tag{2.18}$$

which can be used to get,

$$\boxed{g^{\mu\nu} = e^\mu_a \, e^\nu_b \, \eta^{ab}} \tag{2.19}$$

Tetrad e^a_μ is a 4 by 4 matrix, inverse of which is e^μ_a. Product of two tetrads gets contracted to give identity,

$$e^\mu_a e^a_\nu = \delta^\mu_\nu \tag{2.20}$$
$$e^\mu_a e^b_\mu = \delta^b_a \tag{2.21}$$

So far we were working in one frame of reference. Tetrads are not invariant under the the change of reference frame. If we change the reference frame from $x \to \tilde{x} = \Lambda x$, tetrads in new frame will be,

$$\tilde{e}^a_\mu = e^b_\mu \Lambda^a_b \tag{2.22}$$
$$\tilde{e}^\mu_a = e^\mu_b (\Lambda^{-1})^b_a \tag{2.23}$$

This is Lorentz transformation with $\Lambda^c_a \Lambda^d_b \eta^{ab} = \eta^{cd}$. Note that we used Latin indices which means this is local Lorentz Transformation. To learn more about tetrads, a reader should refer to [11] [12] [13].

2.3 General Affine Connection

In General Relativity we have studied Levi-Citiva connection. Levi-Citiva connection is the only connection that is both metric compatible and torsion free. What if we do not impose any prior condition on the connection? In this section we will study such a general connection which we will call general affine connection. A general affine connection can be written as a some of Levi-Civita Connection and some other quantities which will depend on torsion and non-metricity. We will define those quantities when we encounter them. Right now we will just focus on how to split the general affine connection. This is simple. We can start from the covariant derivative of metric tensor $g_{\mu\nu}$ in the same manner as we do in general relativity to find the Levi-Civita connection. Here however we have to be little careful. Since the connection has torsion, we can not assume connection to be symmetric and since we are also not assuming the connection to be metric compatible, so we can not put the covariant derivative of metric tensor to be zero.

The covariant derivative of metric tensor $g_{\mu\nu}$ can be written as,

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\beta_{\rho\mu} g_{\beta\nu} - \Gamma^\beta_{\rho\nu} g_{\mu\beta}. \tag{2.24}$$

We can find other two possible combination by changing indices,

$$\nabla_\mu g_{\rho\nu} = \partial_\mu g_{\rho\nu} - \Gamma^\beta_{\mu\rho} g_{\beta\nu} - \Gamma^\beta_{\mu\nu} g_{\rho\beta}. \tag{2.25}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\beta_{\nu\rho} g_{\beta\mu} - \Gamma^\beta_{\nu\mu} g_{\rho\beta}. \tag{2.26}$$

By adding (2.25) and (2.26) and substracting (2.24) we get,

$$\begin{aligned}
&\nabla_\mu g_{\rho\nu} + \nabla_\nu g_{\rho\mu} - \nabla_\rho g_{\mu\nu} \\
&\quad =(\partial_\mu g_{\rho\nu} - \Gamma^\beta_{\mu\rho} g_{\beta\nu} - \Gamma^\beta_{\mu\nu} g_{\rho\beta}) + (\partial_\nu g_{\rho\mu} - \Gamma^\beta_{\nu\rho} g_{\beta\mu} - \Gamma^\beta_{\nu\mu} g_{\rho\beta}) - (\partial_\rho g_{\mu\nu} - \Gamma^\beta_{\rho\mu} g_{\beta\nu} - \Gamma^\beta_{\rho\nu} g_{\mu\beta}) \\
&\quad =(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) - (\Gamma^\beta_{\mu\nu} + \Gamma^\beta_{\nu\mu}) g_{\rho\beta} + (\Gamma^\beta_{\rho\mu} - \Gamma^\beta_{\mu\rho}) g_{\beta\nu} + (\Gamma^\beta_{\rho\nu} - \Gamma^\beta_{\nu\rho}) g_{\beta\mu} \\
&\quad =(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) - (2\Gamma^\beta_{\mu\nu} + \Gamma^\beta_{\nu\mu} - \Gamma^\beta_{\mu\nu}) g_{\rho\beta} + (\Gamma^\beta_{\rho\mu} - \Gamma^\beta_{\mu\rho}) g_{\beta\nu} + (\Gamma^\beta_{\rho\nu} - \Gamma^\beta_{\nu\rho}) g_{\beta\mu}
\end{aligned} \tag{2.27}$$

If we look carefully, the only odd term in (2.27) is $2\Gamma^\beta_{\mu\nu} g_{\rho\beta}$. Taking this term to the left and rest of the terms to the right we get,

$$\begin{aligned}
2\Gamma^\beta_{\mu\nu} g_{\rho\beta} =&(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + (\nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\rho\nu} - \nabla_\nu g_{\rho\mu}) \\
&+ (\Gamma^\beta_{\rho\mu} - \Gamma^\beta_{\mu\rho}) g_{\beta\nu} + (\Gamma^\beta_{\rho\nu} - \Gamma^\beta_{\nu\rho}) g_{\beta\mu} + (\Gamma^\beta_{\mu\nu} - \Gamma^\beta_{\nu\mu}) g_{\rho\beta}
\end{aligned} \tag{2.28}$$

Taking the inverse of $g_{\rho\beta}$ we get,

$$\begin{aligned}
\Gamma^\lambda_{\mu\nu} =&\frac{1}{2} g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + \frac{1}{2} g^{\lambda\rho}(\nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\rho\nu} - \nabla_\nu g_{\rho\mu}) \\
&+ \frac{1}{2} g^{\lambda\rho}\{(\Gamma^\beta_{\rho\mu} - \Gamma^\beta_{\mu\rho}) g_{\beta\nu} + (\Gamma^\beta_{\rho\nu} - \Gamma^\beta_{\nu\rho}) g_{\beta\mu} + (\Gamma^\beta_{\mu\nu} - \Gamma^\beta_{\nu\mu}) g_{\rho\beta}\}
\end{aligned} \tag{2.29}$$

Now we need to define two terms present in (2.29). The first one is Torsion Tensor,

$$T^\lambda_{\mu\nu} \equiv \Gamma^\lambda_{\mu\nu} - \Gamma^\lambda_{\nu\mu} = -T^\lambda_{\nu\mu} \tag{2.30}$$

and, the second one is Non-Metricity Tensor,

$$Q_{\rho\mu\nu} \equiv \nabla_\rho g_{\mu\nu} = Q_{\rho\nu\mu} \tag{2.31}$$

These two tensors are the heart of this thesis. We will see in next chapter, how can we use these tensors to make two different theories of gravity. We will study the geometrical meaning of these terms in another section. Here we can write (2.29) in terms of Torsion and Non-Metricity tensor as,

$$\begin{aligned}\Gamma^{\lambda}_{\mu\nu} =&\frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) + \frac{1}{2}g^{\lambda\rho}(Q_{\rho\mu\nu} - Q_{\mu\rho\nu} - Q_{\nu\rho\mu}) \\ &+ \frac{1}{2}g^{\lambda\rho}(T_{\nu\rho\mu} + T_{\mu\rho\nu} + T_{\rho\mu\nu}).\end{aligned} \tag{2.32}$$

We can further see that in (2.32) we have three different types of quantities. The first one we can recognize as Levi-Civita Connection but what are other two quantities? We have not encountered them yet, so we will define them as, Disformation Tensor,

$$L^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\rho}(Q_{\rho\mu\nu} - Q_{\mu\rho\nu} - Q_{\nu\rho\mu}) \tag{2.33}$$

and Contorsion Tensor,

$$K^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\rho}(T_{\nu\rho\mu} + T_{\mu\rho\nu} + T_{\rho\mu\nu}). \tag{2.34}$$

Further since we are used to write Levi-Civita connection as $\Gamma^{\lambda}_{\mu\nu}$, but here we have reserved that symbol for general affine connection, so we will put a hat in Gamma to represent Levi-Civita connection,

$$\hat{\Gamma}^{\lambda}{}_{\mu\nu} \equiv \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{2.35}$$

From now on we put hat on any quantity that depends only on Levi-Civita connection.

Now putting all these defined quantities in (2.32) we get,

$$\begin{aligned}\Gamma^{\lambda}{}_{\mu\nu} =& \underbrace{\frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})}_{\text{Levi-Civita Connection, } \hat{\Gamma}^{\lambda}{}_{\mu\nu}} + \underbrace{\frac{1}{2}g^{\lambda\rho}(Q_{\rho\mu\nu} - Q_{\mu\rho\nu} - Q_{\nu\rho\mu})}_{\text{Disformation Tensor, } L^{\lambda}{}_{\mu\nu}} \\ &+ \underbrace{\frac{1}{2}g^{\lambda\rho}(T_{\nu\rho\mu} + T_{\mu\rho\nu} + T_{\rho\mu\nu}).}_{\text{Contorsion Tensor, } K^{\lambda}_{\mu\nu}}\end{aligned} \tag{2.36}$$

So we have shown that we can write the the general affine connection as a sum of Levi-Civita connection, Disformation Tensor and Contorsion Tensor,

$$\boxed{\Gamma^{\lambda}{}_{\mu\nu} = \hat{\Gamma}^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + L^{\lambda}{}_{\mu\nu}} \tag{2.37}$$

The benefit of writing the general affine connection in this form is that we can easily see the form of connection in the absence of some quantities. For example the general affine connection reduces to Levi-Civita connection in the absence of torsion and non-metricity tensors. The connection we use in general relativity is the Levi-Civita Connection. But this is not the only connection as we can see. As an another example when we set the general affine connection and torsion to zero, we get the Disformation Tensor as a minus of the Levi-Civita Connection. We will use this connection in symmetric teleparallel gravity.

2.4 Splitting of Curvature

We saw in (2.37) that the general affine connection can be written as a sum of Levi-Civita Connection $\Gamma^{\lambda}{}_{\mu\nu}$, Contortion Tensor $K^{\lambda}{}_{\mu\nu}$, and Disforamtion Tensor $L^{\lambda}{}_{\mu\nu}$. Now we want to see how Riemann Tensor, Ricci Tensor and Curvature Scalar splits when we write general affine connection as a sum of above (2.37). As we will not study a theory in which both torsion and non-metricity are present, so will only consider the special cases. In first case we will consider connection to be metric compatible, which will make Disforamtion Tensor $L^{\lambda}{}_{\mu\nu}$ to be zero. By studying this case we will get relationship between curvature and torsion. In second case will study a torsion-free case, which will make Contortion Tensor $K^{\lambda}{}_{\mu\nu}$ to be zero. By studying this case we get relationship between curvature and non-metricity.

2.4.1 Relationship between Curvature and Torsion

Curvature and Torsion are related in a very beautiful way. To see this we will consider the connection to be metric compatible. In that case general affine connection (2.37) reduces to,

$$\Gamma^{\alpha}_{\ \mu\nu} = \hat{\Gamma}^{\alpha}_{\ \mu\nu} + K^{\alpha}{}_{\mu\nu} \tag{2.38}$$

The Riemann curvature tensor is defined as usual by,

$$R^{\alpha}_{\ \beta\mu\nu} = \partial_\mu \Gamma^{\alpha}_{\ \nu\beta} - \partial_\nu \Gamma^{\alpha}_{\ \mu\beta} + \Gamma^{\alpha}_{\ \mu\lambda}\Gamma^{\lambda}_{\ \nu\beta} - \Gamma^{\alpha}_{\ \nu\lambda}\Gamma^{\lambda}_{\ \mu\beta} \tag{2.39}$$

We can put (2.38) in (2.39) to get,

$$R^{\alpha}_{\ \beta\mu\nu} = \hat{R}^{\alpha}_{\ \beta\mu\nu} + T^{\lambda}_{\ \ \mu\nu}K^{\alpha}_{\ \ \lambda\beta} + \nabla_\mu K^{\alpha}_{\ \ \nu\beta} - \nabla_\nu K^{\alpha}_{\ \ \mu\beta} + K^{\lambda}_{\ \ \mu\beta}K^{\alpha}_{\ \ \nu\lambda} - K^{\lambda}_{\ \ \nu\beta}K^{\alpha}_{\ \ \mu\lambda} \tag{2.40}$$

where $R^{\alpha}_{\ \beta\mu\nu}$ is Riemann tensor with General Affine Connection, and $\hat{R}^{\alpha}_{\ \beta\mu\nu}$ is Riemann tensor with Levi-civita connection.To remind again, all quantities with hat in them will be computed with Levi-civita connection. Needless to say that the covariant derivatives will be computed with General affine connection. Now we can find Ricci tensor $R_{\mu\nu} \equiv R^{\alpha}_{\ \mu\alpha\nu}$ as,

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + T^{\lambda}_{\ \ \alpha\nu}K^{\alpha}_{\ \ \lambda\mu} + \nabla_\alpha K^{\alpha}_{\ \ \nu\mu} - \nabla_\nu K^{\alpha}_{\ \ \alpha\mu} + K^{\lambda}_{\ \ \alpha\mu}K^{\alpha}_{\ \ \nu\lambda} - K^{\lambda}_{\ \ \nu\mu}K^{\alpha}_{\ \ \alpha\lambda} \tag{2.41}$$

and curvature scalar $R \equiv R_{\mu\nu}g^{\mu\nu}$ as,

$$R = \hat{R} + g^{\mu\nu}T^{\lambda}_{\ \ \alpha\nu}K^{\alpha}_{\ \ \lambda\mu} + \nabla_\alpha K^{\alpha\ \ \nu}_{\ \ \nu} - \nabla_\nu K^{\alpha\ \ \nu}_{\ \ \alpha} + g^{\mu\nu}(K^{\lambda}_{\ \ \alpha\mu}K^{\alpha}_{\ \ \nu\lambda} - K^{\lambda}_{\ \ \nu\mu}K^{\alpha}_{\ \ \alpha\lambda}) \tag{2.42}$$

$$= \hat{R} + \underbrace{\nabla_\alpha(K^{\alpha\ \ \nu}_{\ \ \nu} - K^{\nu\ \ \alpha}_{\ \ \nu})}_{\text{Boundry Term, } B_T} + \underbrace{g^{\mu\nu}(T^{\lambda}_{\ \ \alpha\nu}K^{\alpha}_{\ \ \lambda\mu} + K^{\lambda}_{\ \ \alpha\mu}K^{\alpha}_{\ \ \nu\lambda} - K^{\lambda}_{\ \ \nu\mu}K^{\alpha}_{\ \ \alpha\lambda})}_{\text{Torsion Scaler, } T} \tag{2.43}$$

where we have defined Torsion scalar as,

$$T \equiv g^{\mu\nu}(T^{\lambda}_{\ \ \alpha\nu}K^{\alpha}_{\ \ \lambda\mu} + K^{\lambda}_{\ \ \alpha\mu}K^{\alpha}_{\ \ \nu\lambda} - K^{\lambda}_{\ \ \nu\mu}K^{\alpha}_{\ \ \alpha\lambda}) \tag{2.44}$$

We can also write Torsion Scalar in term of Torsion Tensors. For that we just have to directly put the controsion tensor (2.34) into the above Torsion Scalar. After simplifying we can see that the torsion scalar reduces to a simpler form as,

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\nu\mu\lambda} - T^{\mu}T_{\mu} \tag{2.45}$$

where we have defined T_μ and T^μ as,

$$T_\mu = T^{\alpha}_{\ \ \alpha\mu} \tag{2.46}$$

$$T^\mu = T^{\alpha\ \ \mu}_{\ \ \alpha} \tag{2.47}$$

For simplicity we can define one more quantity called superpotential $S_{\alpha}{}^{\mu\nu}$ in such a way that it gives torsion scaler T after contraction with Torsion Tensor,

$$T \equiv S_{\alpha}{}^{\mu\nu} T^{\alpha}{}_{\mu\nu} \tag{2.48}$$

From here it can be easily shown that the superpotential $S_{\alpha}{}^{\mu\nu}$ should be given by,

$$S_{\alpha}{}^{\mu\nu} \equiv K_{\alpha}{}^{\mu\nu} + \delta^{\mu}_{\alpha} T^{\nu} - \delta^{\nu}_{\alpha} T^{\mu} \tag{2.49}$$

In (2.43) we have also defined the boundary term as,

$$B_T \equiv \nabla_{\alpha}(K^{\alpha}{}_{\nu}{}^{\nu} - K^{\nu}{}_{\nu}{}^{\alpha}) \tag{2.50}$$

which upon simplification gives,

$$B_T = -2\nabla_{\alpha} T^{\alpha} \tag{2.51}$$

So we see that curvature scalar can be written as,

$$\boxed{R = \hat{R} + T - 2\nabla_{\alpha} T^{\alpha}} \tag{2.52}$$

2.4.2 Relationship between Curvature and Non-Metricity

The curvature and non-metricity are also related and the relationship between the two can be derived in a similar manner. To see this we will consider the connection to be torsion free. In that case, general affine connection (2.37) reduces to,

$$\Gamma^{\alpha}{}_{\mu\nu} = \hat{\Gamma}^{\alpha}{}_{\mu\nu} + L^{\alpha}{}_{\mu\nu} \tag{2.53}$$

The Riemann curvature tensor with general affine connection is given by,

$$R^{\alpha}{}_{\beta\mu\nu} = \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\mu\beta} \tag{2.54}$$

Putting (2.53) in the Riemann curvature tensor we get,

$$R^{\alpha}{}_{\beta\mu\nu} = \hat{R}^{\alpha}{}_{\beta\mu\nu} + \nabla_{\mu} L^{\alpha}{}_{\nu\beta} - \nabla_{\nu} L^{\alpha}{}_{\mu\beta} + L^{\lambda}{}_{\mu\beta} L^{\alpha}{}_{\nu\lambda} - L^{\lambda}{}_{\nu\beta} L^{\alpha}{}_{\mu\lambda} \tag{2.55}$$

where $R^{\alpha}_{\beta\mu\nu}$ is Riemann tensor with General Affine Connection, and $\hat{R}^{\alpha}_{\beta\mu\nu}$ is Riemann tensor with Levi-civita connection and the covariant derivatives are computed with General affine connection. Now using Ricci tensor $R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}$ we can find,

$$R_{\mu\nu} = \hat{R}_{\mu\nu} + \nabla_{\alpha} L^{\alpha}{}_{\nu\mu} - \nabla_{\nu} L^{\alpha}{}_{\alpha\mu} + L^{\lambda}{}_{\alpha\mu} L^{\alpha}{}_{\nu\lambda} - L^{\lambda}{}_{\nu\mu} L^{\alpha}{}_{\alpha\lambda} \tag{2.56}$$

and by the definition of curvature scalar $R \equiv R_{\mu\nu} g^{\mu\nu}$ we get,

$$R = \hat{R} + \nabla_{\alpha} L^{\alpha}{}_{\nu}{}^{\nu} - \nabla_{\nu} L^{\alpha}{}_{\alpha}{}^{\nu} + g^{\mu\nu}(L^{\lambda}{}_{\alpha\mu} L^{\alpha}{}_{\nu\lambda} - L^{\lambda}{}_{\nu\mu} L^{\alpha}{}_{\alpha\lambda}) \tag{2.57}$$

$$= \hat{R} + \underbrace{\nabla_{\alpha}(L^{\alpha}{}_{\nu}{}^{\nu} - L^{\nu}{}_{\nu}{}^{\alpha})}_{\text{Boundry Term, } B_Q} + \underbrace{g^{\mu\nu}(L^{\lambda}{}_{\alpha\mu} L^{\alpha}{}_{\nu\lambda} - L^{\lambda}_{\mu\nu} L^{\alpha}{}_{\alpha\lambda})}_{\text{Non-Mtericity Scaler, } Q} \tag{2.58}$$

where we have defined non-metricity scalar as,

$$Q = g^{\mu\nu}(L^{\lambda}{}_{\alpha\mu} L^{\alpha}{}_{\nu\lambda} - L^{\lambda}_{\mu\nu} L^{\alpha}{}_{\alpha\lambda}). \tag{2.59}$$

If we explicitly put Disformation tensor $L^{\lambda}{}_{\alpha\mu}$ in the above relation, then we can find the non-metricity scalar purely in terms of non-metricity tensor. After simplification we can find,

$$Q = \frac{1}{4} Q_{\mu\nu\lambda} Q^{\mu\nu\lambda} - \frac{1}{2} Q_{\mu\nu\lambda} Q^{\nu\lambda\mu} - \frac{1}{4} Q_{\mu} Q^{\mu} + \frac{1}{2} Q_{\mu} \tilde{Q}^{\mu} \tag{2.60}$$

where Q_{ν} and $\tilde{Q}_{\nu}$ are defined as,

$$Q_{\lambda} = Q_{\lambda\mu\nu} g^{\mu\nu} = Q_{\lambda}{}^{\nu}{}_{\nu} = Q_{\lambda\nu}{}^{\nu} \tag{2.61}$$

$$\tilde{Q}_{\nu} = Q_{\lambda\mu\nu} g^{\lambda\mu} = Q^{\mu}{}_{\mu\nu} = Q_{\lambda}{}^{\lambda}{}_{\nu} \tag{2.62}$$

Just as we did earlier in the case of torsion, for simplicity we can also define a superpotential $P^{\mu\nu\lambda}$ in such a what that upon contracting with non-metricity tensor, it gives non-metricity scalar,

$$Q = Q_{\mu\nu\lambda} P^{\mu\nu\lambda} \tag{2.63}$$

It can be easily checked that such superpotential $P^{\mu\nu\lambda}$ should be given by,

$$P^{\mu\nu\lambda} = -\frac{1}{2} L^{\mu\nu\lambda} + \frac{1}{4}(Q^{\mu} - \tilde{Q}^{\mu}) g^{\nu\lambda} - \frac{1}{4}(g^{\mu\nu} Q^{\lambda} - g^{\mu\lambda} Q^{\nu}) \tag{2.64}$$

In (2.58) we have also defined the boundary term as,

$$B_Q \equiv \nabla_\alpha (L^{\alpha}{}_{\nu}{}^{\nu} - L^{\nu}{}_{\nu}{}^{\alpha}) \tag{2.65}$$

which after simplification gives,

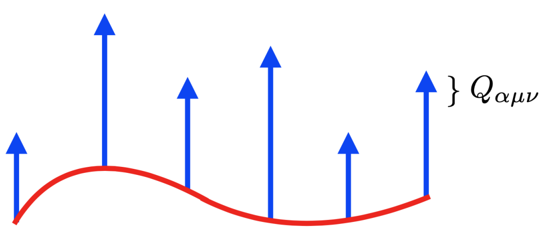
$$B_Q = \nabla_\mu (Q^\mu - \tilde{Q}^\mu) \tag{2.66}$$

So we see that curvature scalar can be written as,

$$\boxed{R = \hat{R} + Q + \nabla_\mu (Q^\mu - \tilde{Q}^\mu)} \tag{2.67}$$

2.5 Geometrical Meaning of Torsion, Curvature and Non-metricity

2.5.1 Geometrical Meaning of Non-Metricity



The Non-metricity $Q_{\lambda\mu\nu} \equiv \nabla_\lambda g_{\mu\nu}$ measures how much the length of a vector changes upon parallel transporting it along a curve.

If we parallel transport a vector v^μ along a curve $\mathcal{C} : x^\mu = x^\mu(\lambda)$, it holds that their absolute derivative vanishes [14] ,

$$\frac{D}{D\lambda} v^\mu = 0 \tag{2.68}$$

$$\frac{dx^\nu}{d\lambda} \nabla_\nu v^\mu = 0 \tag{2.69}$$

Suppose the inner product of two vectors u and v are,

$$u \cdot v = u^\mu v^\nu g_{\mu\nu} \tag{2.70}$$

Now suppose we parallel transport both vectors along a curve $\mathcal{C} : x^\mu = x^\mu(\lambda)$. If we were working in torsion free and metric compatible space, we would expect that their inner product will remain same. In other words their absolute derivative would vanish. However, if we do not assume the space to be metric compatible then their inner product will change upon parallel transporting them along a curve. We can show it by taking the absolute derivative of the inner products of the vectors u and v,

$$\frac{D}{D\lambda}(u \cdot v) = \frac{D}{D\lambda}(u^\mu v^\nu g_{\mu\nu}) \tag{2.71}$$

$$= \frac{dx^\alpha}{d\lambda} \nabla_\alpha (u^\mu v^\nu g_{\mu\nu}) \tag{2.72}$$

$$= \frac{dx^\alpha}{d\lambda} (v^\nu g_{\mu\nu} \nabla_\alpha u^\mu + u^\mu g_{\mu\nu} \nabla_\alpha v^\nu + u^\mu v^\nu \nabla_\alpha g_{\mu\nu}) \tag{2.73}$$

The first two terms vanish by the virtue of (2.69). So we are left with,

$$\frac{D}{D\lambda}(u \cdot v) = \frac{dx^\alpha}{d\lambda}(u^\mu v^\nu \nabla_\alpha g_{\mu\nu}) \tag{2.74}$$

$$= u^\mu v^\nu Q_{\alpha\mu\nu} \frac{dx^\alpha}{d\lambda} \tag{2.75}$$

Since the absolute derivative of inner product of the vectors u and v does not vanish, it implies that the inner product of two vectors changes if we parallel transport those vectors along a curve. What about parallel transporting a vector along a curve? To see this, let's assume $u = v$,

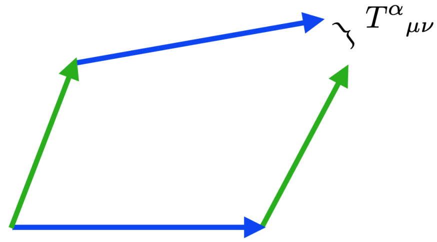
$$\frac{D}{D\lambda}(u \cdot u) = u^\mu u^\nu Q_{\alpha\mu\nu} \frac{dx^\alpha}{d\lambda} \tag{2.76}$$

$$\frac{D}{D\lambda}||u||^2 = u^\mu u^\nu Q_{\alpha\mu\nu} \frac{dx^\alpha}{d\lambda} \tag{2.77}$$

Which means just as inner products, the length of the vector also changes upon parallel transporting them along a curve.

2.5.2 Geometrical Meaning of Torsion

If our space has torsion, it is impossible to form a parallelogram by parallel transporting two vectors along each other. Torsion $T^{\lambda}_{\mu\nu} \equiv \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}{}_{\nu\mu}$, measure how much it will take for the vectors to form a closed parallelogram. To see this, let us consider two curves $\mathcal{C}_1 : x_1^{\mu} = x_1^{\mu}(\lambda)$



and $\mathcal{C}_2 : x_2^{\mu} = x_2^{\mu}(\lambda)$ whose tangent vectors are u_1^{μ} and u_2^{μ} respectively. They can be given by,

$$u_1^{\mu} = \frac{dx_1^{\mu}}{d\lambda} \tag{2.78}$$

$$u_2^{\mu} = \frac{dx_2^{\mu}}{d\lambda} \tag{2.79}$$

If we parallel transport u_1^{μ} along the curve $\mathcal{C}_2$ by the amount dx_2^{μ}, we get,

$$u_1'^{\mu} = u_1^{\mu} + \partial_{\nu} u_1^{\mu} dx_2^{\nu} \tag{2.80}$$

Since we have parallel transported u_1^{μ} along the curve $\mathcal{C}_2$, so by the virtue of (2.69) its absolute derivative must vanish,

$$\frac{dx_2^{\alpha}}{d\lambda} \nabla_{\alpha} u_1^{\mu} = 0 \tag{2.81}$$

$$\frac{dx_2^{\alpha}}{d\lambda} (\partial_{\alpha} u_1^{\mu} + \Gamma^{\mu}{}_{\beta\alpha} u_1^{\beta}) = 0 \tag{2.82}$$

$$(\partial_{\alpha} u_1^{\mu}) dx_2^{\alpha} = -\Gamma^{\mu}{}_{\beta\alpha} u_1^{\beta} \frac{dx_2^{\alpha}}{d\lambda} d\lambda \tag{2.83}$$

Putting back (2.83) into (2.80) gives,

$$u_1'^{\mu} = u_1^{\mu} - \Gamma^{\mu}{}_{\beta\alpha} u_1^{\beta} \frac{dx_2^{\alpha}}{d\lambda} d\lambda \tag{2.84}$$

$$= u_1^{\mu} - \Gamma^{\mu}{}_{\beta\alpha} u_1^{\beta} u_2^{\alpha} d\lambda \tag{2.85}$$

Now if we parallel transport u_2^μ along the curve $\mathcal{C}_1$ by the amount dx_1^μ, following the similar calculation we get,

$$u_2'^\mu = u_2^\mu - \Gamma^\mu{}_{\beta\alpha} u_2^\beta u_1^\alpha d\lambda \tag{2.86}$$

An infinitesimal parallelogram can exist only if vectors $(u_1^\mu + u_2'^\mu)$ and $(u_2^\mu + u_1'^\mu)$ comes out to be equal. However, they are not in general equal. Their difference is,

$$(u_1^\mu + u_2'^\mu) - (u_2^\mu + u_1'^\mu) \tag{2.87}$$

$$=(u_1^\mu + u_2^\mu - \Gamma^\mu{}_{\beta\alpha} u_2^\beta u_1^\alpha d\lambda) - (u_2^\mu + u_1^\mu - \Gamma^\mu{}_{\beta\alpha} u_1^\beta u_2^\alpha d\lambda) \tag{2.88}$$

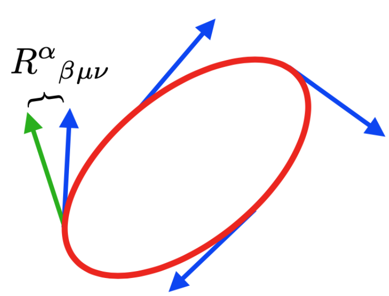
$$=\Gamma^\mu{}_{\alpha\beta} u_2^\alpha u_1^\beta d\lambda - \Gamma^\mu{}_{\beta\alpha} u_1^\beta u_2^\alpha d\lambda \tag{2.89}$$

$$=(\Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}) u_1^\beta u_2^\alpha d\lambda \tag{2.90}$$

$$=T^\mu{}_{\alpha\beta} u_1^\beta u_2^\alpha d\lambda \tag{2.91}$$

So we can see how the torsion comes into the picture. Two vectors can only form a parallelogram by parallel transporting them along each other if the space is torsion free. If the space has torsion, they will fail to form a parallelogram. $T^\mu{}_{\alpha\beta} u_1^\beta u_2^\alpha$ measures the amount by which vectors u_1 and u_2 fails to form a parallelogram. Torsion is the measure of twisting.

2.5.3 Geometrical Meaning of Curvature



Since we all are familiar with the curvature, we will not get into much detail. However, for comparison, the presence of curvature changes the direction of a vector upon parallel transporting it along a closed curve.

Chapter 3

Teleparallel Equivalent of General Relativity

In 1928, Einstein in his attempts to unify gravity and electromagnetism, introduced the notion of teleparallelism [15]. Even though Einstein's unification program did not work out well but his efforts introduced another elegant idea of representing gravity with torsion instead of curvature. In 1960s, Møller reviewed Einstein's original idea in the pursuit of a gauge theory for gravitation. In 1967, Hayashi et al formulated a gauge theory for the spacetime translation group which they unified with teleparallel gravity in 1979 [16]. Just like other fundamental interactions of nature, gravity too can be described by the gauge theory. In General Relativity, gravity is geometrized, and the particle trajectories are not determined by the gravitational force but by the geodesics. In Teleparallel gravity, there are no geodesics. Gravity is attributed entirely to torsion but not with geometrisation, but with the force equation. This alternate description of gravity is known as New General Relativity or Teleparallel Equivalent of General Relativity.(TEGR) [6] [17]. In this chapter we will discuss the theory briefly and show that TEGR is indeed equivalent to General Relativity up to a boundary term.

In General Relativity if we proceed from Einstein-Hilbert action, and do the variation of action with respect to metric, we do not get Einstein Field Equation until we make two postulates namely,

- Connection is metric compatible, $\nabla_\lambda g_{\mu\nu} = 0$.

- Connection is symmetric, or torsion free, $T^{\alpha}{}_{\mu\nu} = 0$.

These two postulates makes sure that the connection is necessarily Levi-Civita connection. If we do not want to impose these conditions by hand, then we can make use of Lagrange Multiplier, and add an additional term in the Einstein-Hilbert action. The term will give us the necessary condition that will fix the connection to be Levi-Civita connection. This However will unnecessarily make our life complicated, and most importantly take out the elegance of Einstein-Hilbert action. There is one more way known by Palatini's name. We can take metric and connection both to be independent variables and do the variation of Einstein-Hilbert action with respect to both variables. Doing variation with respect to connection will give us the condition of metric compatibility, which we had to assume earlier. However even in this way, we will have to assume that the connection is symmetric, or torsion free. Then only we will be able to get Levi-Civita connection [14]. So we see that there is no escape. We can make things more complicated but even then the number of postulates will not decrease. It will just turn our focus from one to another. That is why we explicitly stated the postulates in the more direct way, without coming from anywhere else.

So, in the same manner to proceed from the action of Teleparallel Gravity we will make two postulates namely,

- Connection is metric compatible, $\nabla_{\lambda} g_{\mu\nu} = 0$.

- Spacetime is globally flat, or curvature free, $R^{\alpha}{}_{\lambda\mu\nu} = 0$.

The connection which satisfies both of these properties is known as Weitzenböck Connection, and is given by,

$$\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{a} \partial_{\mu} e^{a}_{\nu} \tag{3.1}$$

here e^{α}_{a} is tetrad with whom we met in detail in the first chapter. Now we will check Riemann curvature tensor, torsion tensor and metric compatibility condition with Weitzenböck connection.

Riemann Curvature Tensor with Weitzenböck connection: Putting Weitzenböck Connection, $\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{a}\partial_{\mu}e^{a}_{\nu}$, into the Riemann Tensor gives,

$$
\begin{aligned}
R^{\alpha}_{\beta\mu\nu} &= \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\gamma}_{\nu\beta}\Gamma^{\alpha}_{\mu\gamma} - \Gamma^{\gamma}_{\mu\beta}\Gamma^{\alpha}_{\nu\gamma} \\
&= \partial_{\mu}(e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta}) - \partial_{\nu}(e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta}) + (e^{\gamma}_{a}\partial_{\nu}e^{a}_{\beta})(e^{\alpha}_{b}\partial_{\mu}e^{b}_{\gamma}) - (e^{\gamma}_{a}\partial_{\mu}e^{a}_{\beta})(e^{\alpha}_{b}\partial_{\nu}e^{b}_{\gamma}) \\
&= \partial_{\mu}e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta} + \cancel{e^{\alpha}_{a}\partial_{\mu}\partial_{\nu}e^{a}_{\beta}} - \partial_{\nu}e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta} - \cancel{e^{\alpha}_{a}\partial_{\nu}\partial_{\mu}e^{a}_{\beta}} + e^{\gamma}_{a}\partial_{\nu}e^{a}_{\beta}e^{\alpha}_{b}\partial_{\mu}e^{b}_{\gamma} - e^{\gamma}_{a}\partial_{\mu}e^{a}_{\beta}e^{\alpha}_{b}\partial_{\nu}e^{b}_{\gamma} \\
&= \partial_{\mu}e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta} - \partial_{\nu}e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta} + e^{\gamma}_{a}\partial_{\nu}e^{a}_{\beta}e^{\alpha}_{b}\partial_{\mu}e^{b}_{\gamma} - e^{\gamma}_{a}\partial_{\mu}e^{a}_{\beta}e^{\alpha}_{b}\partial_{\nu}e^{b}_{\gamma} \\
&= \partial_{\mu}e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta} - \partial_{\nu}e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta} - e^{\gamma}_{a}\partial_{\nu}e^{a}_{\beta}e^{b}_{\gamma}\partial_{\mu}e^{\alpha}_{b} + e^{\gamma}_{a}\partial_{\mu}e^{a}_{\beta}e^{b}_{\gamma}\partial_{\nu}e^{\alpha}_{b} \\
&= \partial_{\mu}e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta} - \partial_{\nu}e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta} - (e^{\gamma}_{a}e^{b}_{\gamma})\partial_{\nu}e^{a}_{\beta}\partial_{\mu}e^{\alpha}_{b} + (e^{\gamma}_{a}e^{b}_{\gamma})\partial_{\mu}e^{a}_{\beta}\partial_{\nu}e^{\alpha}_{b} \\
&= \partial_{\mu}e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta} - \partial_{\nu}e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta} - (\delta^{b}_{a})\partial_{\nu}e^{a}_{\beta}\partial_{\mu}e^{\alpha}_{b} + (\delta^{b}_{a})\partial_{\mu}e^{a}_{\beta}\partial_{\nu}e^{\alpha}_{b} \\
&= \cancel{\partial_{\mu}e^{\alpha}_{a}\partial_{\nu}e^{a}_{\beta}} - \cancel{\partial_{\nu}e^{\alpha}_{a}\partial_{\mu}e^{a}_{\beta}} - \cancel{\partial_{\nu}e^{a}_{\beta}\partial_{\mu}e^{\alpha}_{a}} + \cancel{\partial_{\mu}e^{a}_{\beta}\partial_{\nu}e^{\alpha}_{a}} \\
&= 0
\end{aligned}
\tag{3.2}
$$

Non-Metricity Tensor with Weitzenböck Connection: Putting the Weitzenböck Connection, $\Gamma^{\alpha}_{\mu\nu} = e^{\alpha}_{a}\partial_{\mu}e^{a}_{\nu}$ into the covariant derivative of metric tensor gives,

$$
\begin{aligned}
\nabla_{\rho}g_{\mu\nu} &= \partial_{\rho}g_{\mu\nu} - \Gamma^{\beta}_{\rho\mu}g_{\beta\nu} - \Gamma^{\beta}_{\rho\nu}g_{\mu\beta} \\
&= \eta_{ab}(\partial_{\rho}(e^{a}_{\mu}e^{b}_{\nu}) - \Gamma^{\beta}_{\rho\mu}(e^{a}_{\beta}e^{b}_{\nu}) - \Gamma^{\beta}_{\rho\nu}(e^{a}_{\mu}e^{b}_{\beta})) \\
&= \eta_{ab}(\partial_{\rho}e^{a}_{\mu}e^{b}_{\nu} + e^{a}_{\mu}\partial_{\rho}e^{b}_{\nu} - (\Gamma^{\beta}_{\rho\mu}e^{a}_{\beta})e^{b}_{\nu} - (\Gamma^{\beta}_{\rho\nu}e^{b}_{\beta})e^{a}_{\mu}) \\
&= \eta_{ab}(\cancel{\partial_{\rho}e^{a}_{\mu}e^{b}_{\nu}} + \cancel{e^{a}_{\mu}\partial_{\rho}e^{b}_{\nu}} - \cancel{(\partial_{\rho}e^{a}_{\mu})e^{b}_{\nu}} - \cancel{(\partial_{\rho}e^{b}_{\nu})e^{a}_{\mu}}) \\
&= 0
\end{aligned}
\tag{3.3}
$$

3.1 The Field Equation of Teleparallel Gravity

The action of Teleparallel Equivalent of General Relativity (TEGR) is given by torsion scalar T as [18],

$$\mathcal{S}_{TEGR} = -\frac{c^4}{16\pi G}\int \sqrt{-g}T d^4x \ + \ \mathcal{S}_m \tag{3.4}$$

where $\mathcal{S}_m$ is the action of matter, g=det$(g_{\mu\nu})$ and The torsion scalar T as defined in (2.45) is given by,

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\nu\mu\lambda} - T^{\mu}T_{\mu} \tag{3.5}$$

where T_μ and T^μ are defined as $T^\alpha{}_{\alpha\mu}$ and $T^\alpha{}_\alpha{}^\mu$ respectively. It can be shown that,

$$e = det(e^a_\mu) = \sqrt{-g} \tag{3.6}$$

Since we are working with tetrads, we should have written e instead of $\sqrt{-g}$ in in the above action. However, we will stick to the notation $\sqrt{-g}$ just because introducing a new notation might be confusing to readers who are used to using $\sqrt{-g}$ in General Relativity.

Now taking the variation of the action (3.4) with respect to tetrads e^a_λ to be zero, we can find the field equations of TEGR to be,

$$\frac{2}{\sqrt{-g}}\partial_\mu(\sqrt{-g}S_a{}^{\mu\lambda}) - 2T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu} - \frac{1}{2}Te^\lambda_a = \frac{8\pi G}{c^4}\mathcal{T}^\lambda_a \tag{3.7}$$

where the energy-momentum tensor $\mathcal{T}^\lambda_a$ is defined as,

$$\mathcal{T}^\lambda_a = \frac{1}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta e^a_\lambda} \tag{3.8}$$

and Superpotential as defined in (2.49) is given by,

$$S_\alpha{}^{\mu\nu} \equiv K_\alpha{}^{\mu\nu} + \delta^\mu_\alpha T^\nu - \delta^\nu_\alpha T^\mu \tag{3.9}$$

For easier comparison with General Relativity, we can write the TEGR field equation (3.7) that looks more like Einstein field equation. What we can do is, we can introduce a new quantity,

$$T^\lambda_a \equiv \frac{2}{\sqrt{-g}}\partial_\mu(\sqrt{-g}S_a{}^{\mu\lambda}) - 2T^\sigma{}_{\mu a}S_\sigma{}^{\lambda\mu} \tag{3.10}$$

Using the definition we can re-write the TEGR field equation as,

$$T^\lambda_a - \frac{1}{2}Te^\lambda_a = \frac{8\pi G}{c^4}\mathcal{T}^\lambda_a \tag{3.11}$$

which can also be written as,

$$\boxed{T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu}} \tag{3.12}$$

3.2 Equivalence of Teleprallel Gravity with General Relativity

The action of General Relativity, Einstein-Hilbert action is given by,

$$\mathcal{S}_{GR} = \frac{c^4}{16\pi G}\int \sqrt{-g}\hat{R}d^4x \; + \; \mathcal{S}_m \tag{3.13}$$

and action of Teleprallel Gravity is given by,

$$\mathcal{S}_{TEGR} = -\frac{c^4}{16\pi G}\int eT \; d^4x \; + \; \mathcal{S}_m \tag{3.14}$$

Subtracting (3.14) from (3.13) gives,

$$\mathcal{S}_{GR} - \mathcal{S}_{TEGR} = \frac{c^4}{16\pi G}\int d^4x\sqrt{-g}(\hat{R} + T) \tag{3.15}$$

We saw in (2.52) that the for a metric compatible connection, curvature and torsion are related by,

$$R = \hat{R} + T - 2\nabla_\mu T^\mu \tag{3.16}$$

Since we also saw in (3.2) that Riemann Tensor with Weitzenböck connection vanishes. So we get,

$$\hat{R} + T = 2\nabla_\mu T^\mu \tag{3.17}$$

which is nothing but a covariant derivative of torsion vector. Putting it in (3.15) we get,

$$\mathcal{S}_{GR} - \mathcal{S}_{TEGR} = \frac{2c^4}{16\pi G}\int d^4x\sqrt{-g}(\nabla_\mu T^\mu) \tag{3.18}$$

So we can see that action of General Relativity and Teleparallel Gravity are same up to a boundary term. Since the boundary term does not contribute in the field equations, so in Teleparallel Equivalent of General Relativity (TRGR) we get precisely the same field equation as of General Relativity. To learn more about TEGR, a reader should refer to the book [6]. The most recent and very detailed revive of the generalisation of teleparallel theories and their applications to cosmology can be found here [19].

Chapter 4

Symmetric Teleparallel Equivalent of General Relativity

In 1998, Nester and Yo presented Symmetric Teleparallel Equivalent of General Relativity in which both curvature and torsion was assumed to be zero and the gravitational interaction was attributed to non-metricity [20]. The formation of this theory is quite similar to general relativity and teleparallel gravity. This theory is too equivalent to general relativity and gives more insight to look at the gravity. This theory is also known as Coincident General Relativity or The Newer General Relativity [21] [22].

Starting from the action, we needed two postulates in General Relativity as well as in Teleparallel Gravity. Symmetric Teleparallel Gravity is no different. The two postulates of Symmetric Teleparallel Gravity are,

- Connection is curvature free, $R^{\alpha}{}_{\beta\mu\nu} = 0$
- Connection is torsion free, $T^{\alpha}{}_{\mu\nu} = 0$

The connection which satisfies both of these properties is surprisingly very simple and can be given by,

$$\Gamma^{\alpha}{}_{\mu\nu} = 0 \tag{4.1}$$

That's right. Because of its simplicity we will name it Trivial connection. Trivial connection is an interesting connection which is both curvature and torsion free but metric incompatible.

Riemann Tensor and Torsion gives trivially zero with trivial connection since both directly depends on the connection. However unlike the Weitzenböck connection and Levi-Civita connection, the covarient derivative of metric tensor with trivial connection does not give us zero. It just reduces to simple partial derivative of the metric. We can see it by,

$$\begin{aligned} Q_{\rho\mu\nu} &= \nabla_\rho g_{\mu\nu} \\ &= \partial_\rho g_{\mu\nu} - \Gamma^\beta_{\rho\mu} g_{\beta\nu} - \Gamma^\beta_{\rho\nu} g_{\mu\beta} \\ &= \partial_\rho g_{\mu\nu} \end{aligned} \tag{4.2}$$

Now we have found our desired connection which is torsion and curvature free. We saw in the first chapter that General Affine Connection can be decomposed into Levi-Civita connection, Controsion tensor and Disformation tensor. Since Contorsion tensor depends only on torsion tensor, and that is zero with trivial connection, General Affine Connection reduces to,

$$\Gamma^\lambda{}_{\mu\nu} = \hat{\Gamma}^\lambda{}_{\mu\nu} + L^\lambda{}_{\mu\nu} \tag{4.3}$$

where $\hat{\Gamma}^\lambda{}_{\mu\nu}$ is Levi-Civita connection, and $L^\lambda{}_{\mu\nu}$ is Disformation tensor. Since the connection is zero, we fine that disformation tensor can be written as a minus of Levi-Civita connection,

$$L^\lambda{}_{\mu\nu} = -\hat{\Gamma}^\lambda{}_{\mu\nu} \tag{4.4}$$

It is no surprise. It was clear from the definition of disformation tensor itself. To see this, we write the disformation tensor as defined in (2.33) is given by,

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (Q_{\rho\mu\nu} - Q_{\mu\rho\nu} - Q_{\nu\rho\mu}) \tag{4.5}$$

In (4.2) we saw that with trivial connection $Q_{\rho\mu\nu}$ reduces to $\partial_\rho g_{\mu\nu}$. Putting this fact into the definition of contorsion tensor we get,

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}) \tag{4.6}$$

$$= -\frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{4.7}$$

$$= -\hat{\Gamma}^\lambda{}_{\mu\nu} \tag{4.8}$$

This is interesting. If we remember, in (2.59) we have defined non-metricity scalar in terms of disformation tensor,

$$Q = g^{\mu\nu}(L^{\lambda}{}_{\alpha\mu}L^{\alpha}{}_{\nu\lambda} - L^{\lambda}_{\mu\nu}L^{\alpha}{}_{\alpha\lambda}) \tag{4.9}$$

Since we already found the disfomation tensor in terms of Levi-Civita connection, we can write the non-metricity scalar as,

$$Q = g^{\mu\nu}(\hat{\Gamma}^{\lambda}{}_{\alpha\mu}\hat{\Gamma}^{\alpha}{}_{\nu\lambda} - \hat{\Gamma}^{\lambda}_{\mu\nu}\hat{\Gamma}^{\alpha}{}_{\alpha\lambda}) \tag{4.10}$$

4.1 Field Equation of Symmetric Teleparallel Gravity

The action of Symmetric Teleparallel Equivalent of General Relativity (STEGR) is given by non-metricity scaler Q as [18],

$$\mathcal{S} = -\frac{c^4}{16\pi G}\int \sqrt{-g}Q d^4x \ + \ \mathcal{S}_m \tag{4.11}$$

where $\mathcal{S}_m$ is the action of matter, $g = \det(g_{\mu\nu})$, and non-metricity scaler Q as defined in (2.60) is given by,

$$Q = \frac{1}{4}Q_{\mu\nu\lambda}Q^{\mu\nu\lambda} - \frac{1}{2}Q_{\mu\nu\lambda}Q^{\nu\lambda\mu} - \frac{1}{4}Q_{\mu}Q^{\mu} + \frac{1}{2}Q_{\mu}\tilde{Q}^{\mu} \tag{4.12}$$

where Q_μ and $\tilde{Q}_\mu$ are defined as, $Q_{\mu}{}^{\nu}{}_{\nu}$ and $Q^{\nu}{}_{\nu\mu}$ respectively.

Taking the variation of action with respect to metric to be zero, we can find the field equations of STEGR to be [23],

$$-\frac{2}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}P^{\alpha}_{\mu\nu}) - P_{\mu\alpha\beta}Q^{\alpha\beta}_{\nu} + 2Q^{\alpha\beta}_{\mu}P_{\alpha\beta\nu} - \frac{1}{2}Qg_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu} \tag{4.13}$$

where $P^{\mu\nu\lambda}$ is superpotential as defined in (2.64) as,

$$P^{\mu\nu\lambda} = -\frac{1}{2}L^{\mu\nu\lambda} + \frac{1}{4}(Q^{\mu} - \tilde{Q}^{\mu})g^{\nu\lambda} - \frac{1}{4}(g^{\mu\nu}Q^{\lambda} - g^{\mu\lambda}Q^{\nu}) \tag{4.14}$$

and $\mathcal{T}_{\mu\nu}$ is energy-momentum tensor,

$$\mathcal{T}_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \tag{4.15}$$

For easier comparison with GR, we can further define,

$$Q_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \partial_\alpha(\sqrt{-g}P^\alpha_{\mu\nu}) - P_{\mu\alpha\beta}Q_\nu^{\alpha\beta} + 2Q^{\alpha\beta}_\mu P_{\alpha\beta\nu} \tag{4.16}$$

Putting this back into the STEGR Field equation we get,

$$\boxed{Q_{\mu\nu} - \frac{1}{2}Qg_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu}} \tag{4.17}$$

4.2 Equivalence of Symmetric Teleprallel Gravity with General Relativity:

The action of General Relativity, Einstein-Hilbert action is given by,

$$\mathcal{S}_{GR} = \frac{c^4}{16\pi G} \int \sqrt{-g}\hat{R} d^4x \; + \; \mathcal{S}_m \tag{4.18}$$

and action of Symmetric Teleprallel Gravity is given by,

$$\mathcal{S}_{STEGR} = -\frac{c^4}{16\pi G} \int \sqrt{-g}Q \; d^4x \; + \; \mathcal{S}_m \tag{4.19}$$

Subtracting (4.19) from (4.18) gives,

$$\mathcal{S}_{GR} - \mathcal{S}_{STEGR} = \frac{c^4}{16\pi G} \int d^4x\sqrt{-g}(\hat{R} + Q) \tag{4.20}$$

We saw in (2.67) that for a torsion free connection, curvature and non-metricity are related by,

$$R = \hat{R} + Q + \nabla_\mu(Q^\mu - \tilde{Q}^\mu) \tag{4.21}$$

Since in case of trivial connection, Riemann Tensor also identically vanishes, so we get,

$$\hat{R} + Q = -\nabla_\mu (Q^\mu - \tilde{Q}^\mu) \tag{4.22}$$

which is nothing but a covariant derivative of difference of non-metricity vectors. By putting this result in (4.20) we get,

$$\mathcal{S}_{GR} - \mathcal{S}_{STEGR} = \frac{-c^4}{16\pi G} \int d^4x \sqrt{-g} \nabla_\mu (Q^\mu - \tilde{Q}^\mu) \tag{4.23}$$

So we can see that action of General Relativity and Symmetric Teleparallel Gravity are same up to a boundary term. Since the boundary term does not contribute in the field equations, so we get precisely the same field equation as of General Relativity. It should also be noted that Einstein-Hilbert action also gives non-zero boundary term. We need to add Gibbons–Hawking–York boundary term in Einstein-Hilbert action in order to make it zero. The beauty of non-metricity formulation of gravity is that we do not get any boundary term upon variation of its action. [21].

Chapter 5

Conclusion

General Relativity is one of the most beautiful and well-tested theories humankind has ever discovered. The most radical feature of the theory was its identification of gravity with spacetime curvature. Since the inception of General Relativity, it has passed numerous theoretical and empirical tests, all indicating that the theory has at least a grain of truth into it. Even though the theory works exceptionally well in most of the scenarios, there are some theoretical indications such as its incompatibility with quantum theory, the existence of singularity, the black hole information paradox, and some empirical indications such as the problem of dark matter and dark energy, all of which suggests that the theory may not represent the whole reality. There have been numerous attempts at solving all these problems, and as a result, countless ideas are floating in the literature in which some are more popular than others. However, a solution or an approach upon which the whole community of reasonable physicists can agree has yet to be found.

Einstein once said, "I want to know how God created this world. I'm not interested in this or that phenomenon, in the spectrum of this or that element. I want to know His thoughts; the rest are just details." Following the same pursuit, the primary question of the thesis was not to explain any of the yet unexplained natural phenomenon, but to ponder about, do we really require a curved description of spacetime to explain gravity? To answer the question, we discussed two alternate formulations of gravity in which spacetime was flat and either torsion or non-metricity mediated gravity. These conceptually different formulations of gravity show us that General Relativity does not tell us about the actual nature of spacetime.

Instead, the curvature interpretation of spacetime is just a mathematical convention that does not exist beyond the theory of General Relativity. When we do empirical measurements, we never actually measure the curvature of spacetime. What we measure is the effect that the curved interpretation of spacetime makes in observable quantities. Be it deflection of light, perihelion precession of Mercury, black holes, gravitational waves, gravitational redshift or gravitational lensing. All of these phenomena existed in Newtonian Gravity itself. What Newtonian gravity lacking was in giving the precise numbers, the shortfall of which was successfully taken away by General Relativity. However, General Relativity did not only gave us the correct numbers, but it also completely changed our understanding of gravity by identifying it with the curvature of spacetime. There have been numerous attempts at modifying gravity since then for various reasons, but most of these theories have in common that they also identify gravity with spacetime curvature. Our purpose in this thesis was not to explain any new empirical phenomenon but to question whether the understanding of spacetime given by General Relativity actually represents reality. If we think of curved spacetime as an analogy, then it works great, but the question is, is it only an analogy? We picked up two theories as an example which took completely different analogy, but reproduces the same numbers as of General Relativity. These alternate examples shows us that we do not yet understand the classical gravity itself, let alone quantum gravity. Sure we can explain the numbers, but the purpose of making a theory that can make falsifiable predictions and can give us the right numbers is that so we can go, check and be sure that we are in the right direction. Understand that the goal was not the details, the goal was to understand the nature of reality, but to be sure if we were going in the right direction, we needed to work out the details. The numbers are details. We have worked out the details and getting the right numbers. But now the trouble is that we are getting the right numbers from three conceptually different ways, which describe three different types of the reality. Which one if any actually represents the nature of reality, we don't know.

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