Constructing Cospectral Graphs using Partitioned Tensor Product

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

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August, 2021

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Certificate

This is to certify that this dissertation entitled Constructing Cospectral Graphs using Partitioned Tensor Product, towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Hitesh Wankhede, under the supervision of M. Rajesh Kannan , Assistant Professor, Department of Mathematics, Indian Institute of Technology, Kharagpur, during the academic year 2020 - 2021.

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Declaration

I hereby declare that the matter embodied in the report entitled Constructing Cospectral Graphs using Partitioned Tensor Product are the results of the work carried out by me under the supervision of M. Rajesh Kannan, Assistant Professor, Department of Mathematics, Indian Institute of Technology, Kharagpur, and the same has not been submitted elsewhere for any other degree.

HWankhede

Hitesh Wankhede

This thesis is dedicated to the memories of my late father.

Acknowledgments

Firstly, I express my sincere gratitude to my supervisor Dr. M. Rajesh Kannan, for giving me this opportunity to work on this thesis under his supervision. I thank Dr. Chandrasheel Bhagwat for introducing me to Spectral Graph Theory through semester projects. I also thank him for evaluating this thesis and my mid-year presentation. In the past two years, I was motivated to think independently and give my best under their supervision.

Graphs, Matrices and Applications seminar series organized by my supervisor Dr. M. Rajesh Kannan and Algebraic Graph Theory seminar series organized by the University of Waterloo have introduced me to the current research as well as brought me closer to the community. I thank Shivaramkrishna Pragada for his help in preparing manuscripts. Discussions with him helped me improve the presentation of the thesis as well. I acknowledge my friends Harshavardhan and Arya for their help in access to the computational facility.

The work towards this thesis was carried out remotely from home. I thank IISER administration, my internet provider and my family for making it possible. I consider myself very fortunate to have an extended family who took care of me, my mother and my brother in difficult times. I am thankful to all my friends who have been available for me and reminded me that we are *together* and not really *alone together*.

Abstract

The goal in Spectral Graph Theory is to understand the structure of a graph using the spectrum of its associated matrices. This MS thesis is a contribution to the study of constructions of cospectral nonisomorphic graphs. We first generalize a construction based on partitioned tensor product introduced by Godsil and Mckay and discuss its particular cases. Then, we use the idea of taking partitioned tensor products to obtain new cospectral constructions from the existing ones. We also generalize the unfolding operation on the bipartite graph introduced by Butler, obtain its modifications, as well as introduce the notion of unfolding a multipartite graph to obtain cospectral nonisomorphic graphs.

Keywords: Graph, adjacency matrix, normalized Laplacian matrix, spectrum, unfolding, bipartite graph, partitioned tensor product

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Chapter 1

Introduction

1.1 Problem motivation

The goal in *Spectral Graph Theory* is to understand the structure of a graph using the spectrum of its associated matrices. The *spectrum* of a matrix is the set of its eigenvalues. A graph can be associated with various matrices and each matrix spectrum provides us diiferent information about the graph structure. Two graphs having the same spectrum of an associated matrix M are called *M-cospectral* graphs. There are limitation to the information the spectrum of a certain matrix M can provide, since two graphs with the same M-spectrum can be nonisomorphic. Let G be a graph on n vertices with adjacency matrix A and the degree matrix D which is a diagonal matrix with the degree of G as the diagonal entries. Let J and I be the all-one and the identity matrices of the same order as the graph. The associated matrices M can any of the following:

1. The adjacency matrix A
2. The adjacency matrix of the complement $\bar{A} = J - A - I$
3. The Laplacian matrix $L = D - A$
4. The signless Laplacian matrix $Q = D + A$
5. The normalized Laplacian matrix $\mathcal{L} = D^{-1/2} L D^{-1/2}$, defined when the corresponding graph has no isolated vertices.

6. The Seidel matrix $S = \bar{A} - A$

7. The distance matrix Δ

The spectrum of A together with the spectrum $\bar{A}$ is referred to as *generalized spectrum.* Suppose the associated matrix is the adjacency matrix. We call *A-cospectral* graphs as simply *cospectral* graphs. A graph is said to be *determined by its spectrum* (*DS* for short) if any other graph which is cospectral to it is also isomorphic. Otherwise we say that this graph has a *cospectral mate.* Haemers [7] conjectured the following,

Conjecture 1.1. *Almost all graph are DS.*

In other words, the fraction of DS graphs on n vertices $\to 1$ as $n \to \infty$. For more evidence for and against the conjecture see [7]. Only a very small number of graphs are known to be DS since this property is hard to prove. To show that a graph is not DS, we provide the construction of a cospectral mate. This conjecture suggests that examples of cospectral and nonisomorphic graphs are rare. Hence, given a graph G and an associated matrix M, try to answer the following two questions,

Problem 1.2. *Is G DS with respect to M?*

Problem 1.3. *Find all possible M-cospectral mates of G.*

Answer to either one gives us information about the graph structure. The matrix with respect to which there are less number of cospectral mates for a given graph is most suitable in understanding its structure.

Note that the *Graph Isomorphism Problem* for DS graphs reduces to the problem of checking whether they are cospectral.

1.2 Survey of existing results

Schwenk gave a construction to obtain cospectral trees based on which he proved

Theorem 1.4. *[19] Almost all trees are non-DS.*

Godsil and Mckay [4] proved it for the adjacency matrix of the complement $\bar{A}$ and Mckay [18] proved it for the Laplacian L (and hence for the signless Laplacian Q [22]) and the distance matrix Δ. *Seidel switching* introduced by Van Lint and Seidel [16] produces cospectral nonisomorphic graphs with respect to the Seidel matrix. Let S be the Seidel matrix for the graph G such that S is partitioned as $S = \begin{bmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{bmatrix}$. The Seidel switch $\widetilde{S}$ is given by $\widetilde{S} = \begin{bmatrix} S_1 & -S_2 \\ -S_2^T & S_3 \end{bmatrix}$ such that S and $\widetilde{S}$ are cospectral. It can be shown that

Theorem 1.5. *[20] No graph with more than one vertex is DS with respect to Seidel matrix.*

Seidel switching is a special case of *GM-switching* or *Godsil-Mckay switching.* Let G be a graph on n vertices and let $A(G)$ denote the corresponding adjacency matrix. Consider the simplest version of this switching and consider the orthogonal matrix $Q = diag(Q_0, I_{n-2m})$ where

$$Q_0 = \frac{1}{m} J_{2m} - I_{2m}.$$

Godsil and Mckay [5] investigated conditions on the graph G such that the matrix $Q^T A(G) Q$ is also an adjacency matrix of some graph G'. Then, it follows that the graphs G and G' are generalized cospectral. Wang, Qiu and Hu [23] consider another orthogonal matrix Q such that $Q = diag(U, I_{n-2p})$ where

$$U = \frac{1}{p} \begin{bmatrix} pI_p - J_p & J_p \\ J_p & pI_p - J_p \end{bmatrix}$$

and answer the same question. The corresponding construction is called *Generalized GM-switching.* Godsil and Mckay [5] gave another construction which is a generalization of this simplest version of GM-switching. Let A and B be two $m \times n$ *congruent* matrices (that is, $A^T A = B^T B$). Let H be an adjacency matrix of a graph on n vertices. Then the graphs corresponding to the following adjacency matrices are cospectral:

$$\begin{bmatrix} 0 & A \\ A^T & H \end{bmatrix} \text{ and } \begin{bmatrix} 0 & B \\ B^T & H \end{bmatrix}$$

Another cospectral construction introduced by Godsil and Mckay [3] is based on the idea of taking *partititioned tensor product* of a bipartitioned matrix whose diagonal blocks are identity matrices with any bipartitioned matrix. Let $L = \begin{bmatrix} I_m & V \\ W & I_n \end{bmatrix}$, $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $H^{\#} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ be partitioned matrices such that V is a $m \times n$ matrix. Let $\otimes_p$ denote the partitioned tensor product defined as

$$L \otimes_p H = \begin{bmatrix} I_m \otimes A & V \otimes B \\ W \otimes C & I_n \otimes D \end{bmatrix}, \; L \otimes_p H^{\#} = \begin{bmatrix} I_m \otimes D & V \otimes C \\ W \otimes B & I_n \otimes A \end{bmatrix}.$$

Then,

Theorem 1.6. *[3] The matrices $L \otimes_p H$ and $L \otimes_p H^{\#}$ are cospectral if and only if either $m = n$ or A and D are cospectral.*

When these matrices are taken to be adjacency matrices, the corresponding graphs are cospectral. This is one of the two constructions Godsil and Mckay introduced in [3] .

Butler [1] introduced an *unfolding* operation on a bipartite graph. Let us discuss this construction using the matrix forms. Let G be a bipartite graph with the adjacency matrix $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ where the biadjacency matrix B is a square matrix. Then, G can be *unfolded* in two ways to obtain bipartite graphs Γ_1 and Γ_2 with the adjacency matrices,

$$\begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}.$$

The bipartite graphs Γ_1 and Γ_2 are cospectral with respect to the adjacency as well as the normalized Laplacian matrix. We refer to [14] and [12] for some of it modifications and generalizations. Ji, Gong and Wang [12] gave equivalent conditions of isomorphism for the generalized case. Hence, Γ_1 and Γ_2 are isomorphic if and only if the block B is permutationally equivalent to its transpose.

Recently, Dutta and Adhikari [2] gave another cospectral construction motivated by GM-switching. Let A be a $m \times m$ partitioned block matrix such that the ij^{th} block for $1 \leq i, j \leq m$ is A_{ij}. The *partial transpose* of A is given by A^τ by replacing each block of A by its transpose. Then, the ij^{th} block of A^τ is A_{ij}^T. They show that if the blocks of A form a commuting family of normal matrices, then the matrices A and A^τ are cospectral. Hence, when A is taken to be an adjacency matrix of a graph, then the graphs corresponding to A and A^τ are cospectral.

1.3 Original contributions

Three forthcoming papers are planned. The first paper [15] will include the second idea below. The second paper will include the first and the fourth idea. The third paper will include the third idea.

1. Generalization of a cospectral construction based on partitioned tensor product given by Godsil and Mckay (Chapter 3, 4, 5)

Godsil and Mckay [3] gave two cospectral constructions (one of them described by Theorem 1.6). These constructions essentially involve taking partitioned tensor product of a bipartitioned matrix whose diagonal blocks are identity matrices with any bipartitioned matrix. We generalize this construction in Chapter 3 by showing that the bipartitioned matrix whose diagonal block are identity matrices can be replaced with any bipartitioned matrix satisfying a certain $C/M/T$ property. When these matrices are taken to be adjacency matrices of graphs, we get cospectral graphs. We give necessary and sufficient conditions for the corresponding graphs to be isomorphic. In chapter 4 and 5, we give more candidates for matrices that satisfy $C/M/T$ property and apply the isomorphism results on the corresponding cospectral constructions (see Constructions I-A, II-A, I-B, I-C, I-D and I-E).

2. Generalization of unfolding operation on a bipartite graph (Chapter 4)

The very important observation in generalizing unfolding of a bipartite graph is that the adjacency matrices corresponding to the unfoldings Γ_1 and Γ_2 can be written as partitioned tensor products. Let $J_{1,2}$ be the 1×2 all one matrix. Then,

$$\begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & J_{1,2} \otimes B \\ J_{2,1} \times B^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & J_{1,2} \\ J_{2,1} & 0 \end{bmatrix} \otimes_p \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & J_{1,2} \otimes B^T \\ J_{2,1} \otimes B & 0 \end{bmatrix} = \begin{bmatrix} 0 & J_{1,2} \\ J_{2,1} & 0 \end{bmatrix} \otimes_p \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix}$$

Generalizations of unfoldings considered by Kannan and Pragada [14] and Ji, Gong and Wang [12] are essentially replacement of the all one matrix $J_{1,2}$ to any all one matrix $J_{m,n}$. Since we realized that unfoldings can be expressed as partitioned tensor product we show that the block $J_{m,n}$ be be replaced by any $m \times n$ matrix V, hence generalizing the cospectral construction. We also show that the matrix $\begin{bmatrix} 0 & V \\ V^T & 0 \end{bmatrix}$ satisfies $C/M/T$ property. Then, we apply the the isomorphism results obtained for the generalized construction of Godsil and Mckay. This construction is known by I-A and produces cospectral bipartite graphs which are cospectral for the adjacency as well as the normalized Laplacian. For these graphs to be nonisomorphic, we introduce a certain property η_1 that the bipartite graphs corresponding to V and B have to satisfy. This property is satisfied when in one the following cases

1. the bipartite graph corresponding to V is biregular with distinct degrees

2. when the bipartite graphs corresponding to V and B are connected.

The former generalizes the cospectral nonisomorphic construction of Ji, Gong and Wang [12] and the latter relates with a complete different problem considered by Hammack [8] which is the investigation of isomorphism of the components of the direct product of two connected bipartite graphs. In other words, we unite the two different results from [12] and [8] under the property η_1.

3. Obtaining new cospectral graphs from the existing ones (Chapter 6)

In Chapter 6, we discuss how the idea of partitioned tensor product can be applied on some of the existing cospectral constructions to obtain new constructions.

The first candidate is a construction based on partial transpose introduced by Dutta and Adhikari [2]. We first give an alternate proof for their main result (Construction III-A) and fix an error in another result (Construction III-B). After applying the idea of partitioned tensor product, we are able to discuss a notion of unfolding a multipartite graph. In particular, we show how to unfold a tripartite graph to obtain cospectral nonisomorphic tripartite graphs.

The second candidate is GM-switching [5] (Construction IV). Since GM-switching produces generalized cospectral graphs, so does the new construction.

The third candidate is a construction based on congruence [5](Construction V). We first give equivalent conditions for the graphs to be isomorphic, we do the same for the new construction as well. Inspired by this construction, we show how a semi reflexive bipartite graph can be unfolded (Construction VI) to obtain cospectral nonisomorphic graphs.

4. Modifications of unfoldings (Chapter 5, 6)

Along with a generalization of unfolding operation, Kannan and Pragada [14] consider three modifications. The first modification is given by the following result

Theorem 1.7. *[14] Let* $A = \begin{bmatrix} K' & B & B \\ B^T & 0 & K \\ B^T & K & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} K & B^T & B^T \\ B & 0 & K' \\ B & K' & 0 \end{bmatrix}$. *Then, the matrices* $A \oplus \begin{bmatrix} 0 & K' \\ K' & 0 \end{bmatrix} \oplus K$ *and* $C \oplus \begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} \oplus K'$ *are cospectral.*

If we only consider the matrices A and C, then they can be written as partitioned tensor products such that $A = \begin{bmatrix} S & J_{1,2} \\ J_{2,1} & T \end{bmatrix} \otimes_p \begin{bmatrix} K' & B \\ B^T & K' \end{bmatrix}$ and $C = \begin{bmatrix} S & J_{1,2} \\ J_{2,1} & T \end{bmatrix} \otimes_p \begin{bmatrix} K & B^T \\ B & K \end{bmatrix}$ where S and T are the permutation matrices $S = \begin{bmatrix} 1 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ respectively. Surprisingly, Construction I-C generalizes this, that is, we show S and T can be replaced by any permutation matrices and give necessary and sufficient conditions for A and C be represent cospectral nonisomorphic graphs.

Now consider another modification,

Theorem 1.8. *[14] Let* $C = \begin{bmatrix} 0 & B & B & \dots & B \\ B^T & 0 & I & \dots & I \\ B^T & I & 0 & \dots & I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & I & I & \cdots & 0 \end{bmatrix}$ *and* $E = \begin{bmatrix} 0 & B^T & B^T & \dots & B^T \\ B & 0 & I & \dots & I \\ B & I & 0 & \dots & I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & I & I & \cdots & 0 \end{bmatrix}$ *be matrices of orders* $nq+p$ *and* $np+q$ *respectively, where* B *is a* $p \times q$ *matrix such that* $p \geq q$*. Then the matrices* $E \oplus 0_{p-q}$ *and* $C \oplus \underbrace{(J-I)_n \oplus \ldots \oplus (J-I)_n}_{(p-q)-times}$ *are cospectral.*

Here also observe C and E can be expressed as partitioned tensor products, $C = \begin{bmatrix} 0 & J_{1,n} \\ J_{n,1} & J_n - I_n \end{bmatrix} \otimes_p \begin{bmatrix} 0 & B \\ B^T & I \end{bmatrix}$ and $E = \begin{bmatrix} 0 & J_{1,n} \\ J_{n,1} & J_n - I_n \end{bmatrix} \otimes_p \begin{bmatrix} 0 & B^T \\ B & I \end{bmatrix}$. Surprisingly, C and E represent unfoldings of a semi reflexive bipatite graph given by the adjacency matrix $\begin{bmatrix} 0 & B \\ B^T & I \end{bmatrix}$ and this construction is a special case of Construction VI. Our motivation to this construction has come from applying the idea of partitioned tensor product on the congruence construction. We provide necessary and sufficient conditions for C and E to represent cospectral nonisomorphic graphs.

In Table 1.1., some of the modifications that we have obtained on the unfoldings of the bipartite graph are summarized (only the special cases). The conditions for the graphs to be nonisomorphic are obtained under the assumption that the matrix B has no zero rows or zero columns.

Construction	I-A	I-B	I-C	VI
Graph	$\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$	$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$	$\begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$	$\begin{bmatrix} 0 & B \\ B^T & I \end{bmatrix}$
Unfolding-1	$\begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} A & B & B \\ B^T & D & 0 \\ B^T & 0 & D \end{bmatrix}$	$\begin{bmatrix} A & B & B \\ B^T & 0 & D \\ B^T & D & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & B & B \\ B^T & 0 & I \\ B^T & I & 0 \end{bmatrix}$
Unfolding-2	$\begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} D & B^T & B^T \\ B & A & 0 \\ B & 0 & A \end{bmatrix}$	$\begin{bmatrix} D & B^T & B^T \\ B & 0 & A \\ B & A & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & B^T & B^T \\ B & 0 & I \\ B & I & 0 \end{bmatrix}$
Unfoldings are cospectral iff	B is square	A and D are cospectral	A and D are cospectral	B is square
Unfoldings are nonisomorphic if	B is non-PET	B is non-PET or G_A and G_D are nonisomorphic	B is non-PET or G_A and G_D are nonisomorphic	B is non-PET

Table 1.1: Unfoldings and modifications

Chapter 2

Preliminaries

In this chapter, we introduce some of the ideas and results that will be used in the further chapters.

2.1 Graphs and Matrices

A *digraph* G is a finite set $V(G)$ taken together with a binary relation $E(G)$ on $V(G)$. Elements of $V(G)$ and $E(G)$ are called *vertices* and *arcs* respectively. This binary relation is *symmetric* if $(u, v) \in E(G)$ implies $(v, u) \in E(G)$. Symmetric digraphs are called *undirected*. One vertex arcs, (v, v) for $v \in V(G)$, are called *loops*.

A *graph* is defined as an undirected digraph usually without loops. In case of graphs, the arcs (u, v) and (v, u) combined are referred to as an edge $\{u, v\}$, and the vertices u and v are called *adjacent*. A graph is called *reflexive* if every vertex has a loop, and called *simple* if no vertex has a loop.

The *adjacency* matrix $A(G)$ associated with a digraph G is an $n \times n$ 0-1 matrix where $|V(G)| = n$. The ij^{th} entry (for $1 \leq i, j \leq n$) of $A(G)$ is given by =

$$A(G)_{ij} = \begin{cases} 1 & (i,j) \in E(G) \\ 0 & otherwise \end{cases}$$

While the adjacency matrix of a digraph could be any 0-1 square matrix, the adjacency matrix of a graph is usually a symmetric 0-1 matrix with all diagonal entries 0. If the graph is reflexive, then all the diagonal entries of $A(G)$ are 1. Given a square 0-1 matrix A, the corresponding digraph or graph is denoted by G_A.

Let G and H be two digraphs. A *homomorphism* of G to H, written as $f : G \to H$, is a mapping $f : V(G) \to V(H)$ such that if $(u, v) \in E(G)$, then $(f(u), f(v)) \in E(H)$. Hence, homomorphisms preserve the directions of edges. In case of graphs, they preserve the adjacency. This homomorphic map is called *isomorphism* if it is also a bijection. In that case, G and H are called *isomorphic*.

An *automorphism* of a graph is an isomorphism from the graph to itself. The set of automorphism of a graph G is denoted by $Aut(G)$ and it forms a group. A graph is called *asymmetric* if its automorphism group is only the identity map.

The *Kronecker product* of two $m \times n$ matrices A and B is denoted by $A \otimes B$ which consists of m rows of n blocks where j^{th} block in the i^{th} row is the matrix $a_{ij}B$. The *direct product* of two digraphs G and H is denoted by $G \times H$. The resultant digraph is given by the adjacency matrix $A(G \times H) = A(G) \otimes A(H)$.

Consider a graph G. The *degree* of a vertex $v \in V(G)$ is the number of vertices it is adjacent to. A vertex is called *isolated* if it has degree 0. The *degree* matrix $D(G)$ is a $n \times n$ diagonal matrix whose ii^{th} entry is given by the degree corresponding to the i^{th} vertex or the i^{th} row sum of $A(G)$.

The *Laplacian* matrix $L(G)$ is defined as $L(G) = D(G) - A(G)$. If the graph G has no isolated vertices, then the *normalized Laplacian* matrix is defined as

$$\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2} = I_n - D(G)^{-1/2}A(G)D(G)^{-1/2}$$

where I_n is a $n \times n$ identity matrix. The adjacency, Laplacian and the normalized Laplacian matrices of a graph are symmetric matrices.

Two graphs are called *cospectral*, if the corresponding adjacency matrices have the same eigenvalues. Similarly, two graphs are called *Laplacian-cospectral*, if the corresponding Laplacian matrices have the same eigenvalues.

2.2 Matrix relations

Similarity

Two matrices A and B are called *similar* if there exists an invertible matrix Q such that $Q^{-1}AQ = B$. A matrix is called *diagonalizable* if it is similar to a diagonal matrix. Recall that symmetric matrices are diagonalizable, have real eigenvalues and,

Theorem 2.1. *[11][Corollary 2.5.11] Two real symmetric matrices are real orthogonally similar if and only if they have the same eigenvalues.*

Hence, the adjacency matrices of cospectral graphs are real orthogonally similar. Now the following proposition shows equivalence between permutation similarity of matrices and isomorphism of the corresponding digraphs. Every isomorphism map can be represented by a permutation matrix.

Proposition 2.2. *Two digraphs G and H are isomorphic if and only if the corresponding adjacency matrices are permutationally similar.*

Since we are interested in the construction of cospectral nonisomorphic graphs in this thesis, equivalently we want to find pairs of matrices which are orthogonally similar but not permutationally similar. Let us also discuss graph automorphisms. Without loss of generality, we can assume that $Aut(G)$ is the set of permutation matrices P such that $P^T A(G)P = A(G)$. Let S_n denote the set of all permutation matrices of order n, then $|S_n| = n!$.

Proposition 2.3. *Let $\mathcal{O}_n$, $\mathcal{I}_n$, K_n and $\mathcal{J}_n$ be the graphs corresponding to the adjacency matrices 0_n, I_n, $J_n - I_n$ and J_n, where 0_n, I_n and J_n are zero, identity and all-one matrices respectively. Then, the automorphism group of all these graphs is S_n.*

Lemma 2.4. *Let G and H be two graphs. Then,*

$$Aut(G \times H) \supseteq Aut(G) \times Aut(H)$$

If the direct product of graphs is asymmetric, the factors are necessarily asymmetric.

Lemma 2.5. *If A and B are two matrices, then there exists a permutation matrix P such that $P^T(A \otimes B)P = B \otimes A$.*

This shows that if G and H are two graphs, then $G \times H$ and $H \times G$ are isomorphic.

Theorem 2.6. *[21] A real square matrix is real similar to its transpose.*

The matrix similarity is directly related with the isomorphism of the graphs. But since we deal with bipartitioned graphs/matrices in this thesis, the matrix equivalence also appears in our discussion.

Equivalence

Two matrices A and B are called *equivalent* if there exists two invertible matrices P and Q such that $Q^{-1}AP = B$. Hence, if two matrices are similar, then they are automatically equivalent.

Lemma 2.7. *A real square matrix is real orthogonally equivalent to its transpose.*

Proof. Recall that the *Singular Value Decomposition (Corollary 2.6.7. [11])* of a real matrix A is given by $A = U\Sigma V^T$, where U and V are real orthogonal and Σ is a real diagonal matrix. Taking transposes on both sides, we have $A^T = V\Sigma U^T$. Substituting $\Sigma = U^T AV$, we have $A^T = VU^T AVU^T$. Let $Q = VU^T$, then Q is also real orthogonal matrix satisfying $QAQ = A^T$. $\square$

Lemma 2.8. *If two $m \times n$ matrices A and B are permutationally equivalent, then every row of A is some permuted row of B. Similarly, every column of A is some permuted column of B*

Proof. Suppose $P^T AQ = B$ for two permutation matrices P and Q. The left multiplication $(P^T A)$ by P^T permutes the rows of A and the right multiplication (AQ) by Q permutes the columns of A. Hence, i^{th} row and i^{th} column of $P^T AQ$ are permutations of j^{th} row and k^{th} column of A respectively, where $\sigma_P(i) = j$ and $\sigma_Q(i) = k$ and the permutations σ_P and σ_Q correspond to the permutation matrices P and Q respectively. $\square$

Corollary 2.9. *If A and B are two $m \times n$ permutationally equivalent 0-1 matrices, then*

1. *the set of row sums of A is same as the set of row sums of B*
2. *the set of column sums of A is same as the set of column sums of B*
3. *the sum of all entries of A is same as the sum of all entries of B.*
4. *the maximum row sum of A is same as the maximum row sum of B*
5. *the maximum column sum of A is same as the maximum column sum of B*

Proof. Let $\mathbf{1}$ be an all-one $n \times 1$ vector. Then $A\mathbf{1}$ and $B\mathbf{1}$ are $m \times 1$ vectors of row sums of A and B respectively. Since A are B are permutationally equivalent, every row of A is some permuted row of B. Also since, A and B are 0-1 matrices, the vector $A\mathbf{1}$ is a permutation of $B\mathbf{1}$. The set of row sums of the matrix A and B are the set of entries of the vector $A\mathbf{1}$ and $B\mathbf{1}$ respectively. Hence, set of row sums of A is the same as set of column sums of B. The second statement can be shown similarly by considering the equation $Q^T A^T P = B^T$. Remaining statements follow from the first two. $\square$

If any one of these five conditions does not hold, then A and B are not permutationally equivalent. A square matrix M is called PET if it is permutationally equivalent to its transpose, that is, if there exists two permutation matrices such that $P^T M Q = M^T$.

Corollary 2.10. *If a matrix M is PET, then*

1. *set of row sums of M is same as the set of column sums of M*
2. *maximum row sum of M is same as the maximum column sum of M*

Hence, if a matrix doesn't satisfy one of these two conditions, then we have a non-PET matrix. Non-PET matrices are very important in the construction of cospectral nonisomorphic graphs.

2.3 Cancellation Laws

In this section, we recall some of the cancellation laws for graphs/digraphs (see [17], [9]) and a cancellation law for matrices (see [8]).

Lemma 2.11. *[9][Corollary 9.8] Suppose A, B and C are bipartite graphs. If $A \times C$ and $B \times C$ are isomorphic, then A and B are.*

Theorem 2.12. *[17][Theorem 9] If K is a nonbipartite graph, then $G \times K$ and $H \times K$ are isomorphic if and only if G and H are.*

The above two results will be used later as cancellation law for graphs. To obtain a cancellation law for matrices, the following lemma will be useful.

Lemma 2.13. *[17][Theorem 6] Suppose A, B, C and D are digraphs and there is a homomorphism $D \to C$. If $A \times C$ and $B \times C$ are isomorphic, then $A \times D$ and $B \times D$ are.*

The following theorem will be used later as cancellation law for matrices.

Theorem 2.14. *[8][Lemma 3] Suppose A, B and C are 0-1 matrices for which $C \neq 0$, and A is square and has at least one nonzero entry in each row. Then, $C \otimes A$ and $C \otimes B$ are permutationally equivalent if and only if A and B are. Similarly, $A \otimes C$ and $B \otimes C$ are permutationally equivalent if and only if A and B are.*

Proof. Suppose $C \otimes A$ and $C \otimes B$ are permutationally equivalent, then there exists two permutation matrices P_1 and P_2 such that $P_2^T(C \otimes A)P_1 = C \otimes B$. Suppose C is an $m \times n$ matrix, then let $E = \begin{bmatrix} 0 & C \\ 0 & 0 \end{bmatrix}$ be the square matrix of order $(m+n)$, where the $(2,1)^{th}$ zero block has the same size as C^T. Suppose $P = \begin{bmatrix} P_2 & 0 \\ 0 & P_1 \end{bmatrix}$. Then,

$$\begin{aligned} P^T(E \otimes A)P &= \begin{bmatrix} P_2 & \\ & P_1 \end{bmatrix}^T \begin{bmatrix} 0 & C \otimes A \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_2 & \\ & P_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & P_2^T(C \otimes A)P_1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & C \otimes B \\ 0 & 0 \end{bmatrix} \\ &= E \otimes B \end{aligned}$$

This shows that the digraphs $G_{E\otimes A}$ and $G_{E\otimes B}$ are isomorphic. Let $K = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ be an adjacency matrix of a digraph on two vertices with one arc. Since $C \neq 0$, we have a homomorphism from G_K to G_E. Hence, from Lemma 2.13., $G_{K\otimes A}$ and $G_{K\otimes B}$ are isomorphic. Then, there exist a permutation matrix Q such that $Q^T(K \otimes A)Q = K \otimes B$, that is,

$$Q^T \begin{bmatrix} 0 & A \\ 0 & 0 \end{bmatrix} Q = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$$

Since A is a square matrix and has no zero rows, the rows of A must be permuted only among themselves. Hence, any such Q must be of the form $Q = \begin{bmatrix} Q_2 & 0 \\ 0 & Q_1 \end{bmatrix}$ where Q_1 and Q_2 are also permutation matrices. Then, $Q_2^T A Q_1 = B$ and A and B are permutationally equivalent. $\square$

The assumption that 'A has has no zero rows' can also be replaced with the assumption that 'A has no zero columns' (see proof of Lemma 3 in [9]). Hence, A cannot have both a zero row and a zero column. We will be stating this assumptions as 'A has no zero rows or zero columns'. It is also enough to make such an assumption for at least one of A or B, but we will be stating it for both A and B.

Chapter 3

Partitioned tensor product

In this chapter, we first discuss the constructions of cospectral matrices introduced by Godsil Mckay in [3]. These constructions essentially involve taking partitioned tensor product of a bipartitioned matrix whose diagonal blocks are identity matrices with any bipartitioned matrix. We generalize these constructions by showing that the bipartitioned matrix whose diagonal block are identity matrices can be replaced with any bipartitioned matrix satisfying a certain $C/M/T$ property. When these matrices are taken to be adjacency matrices of graphs, we get cospectral graphs. We give necessary and sufficient conditions for the corresponding graphs to be isomorphic. In the further chapters, we apply these isomorphism results on the particular cases which satisfy $C/M/T$ property and result in a cospectral construction.

3.1 Construction of Godsil and Kckay

Let us first recall the definition and some of the properties of Kronecker products.

Definition 3.1. *Let $A = (a_{ij})$ be an $m \times n$ matrix. The Kronecker product of the matrices A and B is denoted by $A \otimes B$ which consists of m rows of n blocks where j^{th} block in the i^{th} row is the matrix $a_{ij}B$.*

Lemma 3.2. *Let A, B, C, D be matrices of appropriate order. Then,* *1.* $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

2. $(A \otimes B)^T = A^T \otimes B^T$.

3. If A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

4. If A is $n \times n$ square matrix and B is $m \times m$ square matrix have eigenvalues $\{\lambda_i\}$ where $i = \{1, 2, \ldots, n\}$ and $\{\mu_j\}$ where $j = \{1, 2, \ldots, m\}$ respectively, then $A \otimes B$ has eigenvalues $\{\lambda_i \mu_j\}$ where $i = \{1, 2, \ldots, n\}$, $j = \{1, 2, \ldots, m\}$.

Now, we introduce the notion of partitioned tensor product.

Definition 3.3. *[3] The partitioned tensor product of two partitioned matrices $K = \begin{bmatrix} U & V \\ W & X \end{bmatrix}$ and $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is denoted by $K \otimes_p H$. It is obtained by taking blockwise Kronecker products of the corresponding blocks, that is,*

$$K \otimes_p H = \begin{bmatrix} U \otimes A & V \otimes B \\ W \otimes C & X \otimes D \end{bmatrix}.$$

This product depends on the way K and H are partitioned.

Given the matrices U, V, W and X, define $\mathcal{I}(U, X)$ and $\mathcal{P}(V, W)$ to be the block matrices $\begin{bmatrix} U & 0 \\ 0 & X \end{bmatrix}$ and $\begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$ respectively, where 0 is the zero matrix of appropriate order. A 2×2 block matrix is *diagonal* (respectively *counter-diagonal*) block matrix if it is of the form $\mathcal{I}(U, X)$ (respectively $\mathcal{P}(V, W)$).

Proposition 3.4. *Let Q and R be of the form $\mathcal{I}(Q_1, Q_2)$ and $\mathcal{I}(R_1, R_2)$ respectively. Then for all matrices $K = \begin{bmatrix} U & V \\ W & X \end{bmatrix}$ and $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$,*

$$(Q \otimes_p R)(K \otimes_p H) = (QK) \otimes_p (RH).$$

The same holds true when Q and R are both of the form $\mathcal{P}(Q_1, Q_2)$ and $\mathcal{P}(R_1, R_2)$ respectively.

Proof. Suppose the matrices Q and R are of the form $\mathcal{I}(Q_1, Q_2)$ and $\mathcal{I}(R_1, R_2)$. Then,

$$\begin{aligned}
(Q \otimes_p R)(K \otimes_p H) &= \begin{bmatrix} Q_1 \otimes R_1 & 0 \\ 0 & Q_2 \otimes R_2 \end{bmatrix} \begin{bmatrix} U \otimes A & V \otimes B \\ W \otimes C & X \otimes D \end{bmatrix} \\
&= \begin{bmatrix} (Q_1 \otimes R_1)(U \otimes A) & (Q_1 \otimes R_1)(V \otimes B) \\ (Q_2 \otimes R_2)(W \otimes C) & (Q_2 \otimes R_2)(X \otimes D) \end{bmatrix} \\
&= \begin{bmatrix} Q_1U \otimes R_1A & Q_1V \otimes R_1B \\ Q_2W \otimes R_2C & Q_2X \otimes R_2D \end{bmatrix} \\
&= \begin{bmatrix} Q_1U & Q_1V \\ Q_2W & Q_2X \end{bmatrix} \otimes_p \begin{bmatrix} R_1A & R_1B \\ R_2C & R_2D \end{bmatrix} \\
&= (QK) \otimes_p (RH)
\end{aligned}$$

In the second step, Lemma 3.2. (1) is used. Hence, $(Q \otimes_p R)(K \otimes_p H) = (QK) \otimes_p (RH)$. Similarly, this equation can be shown to hold in case Q and R are of the form $\mathcal{P}(Q_1, Q_2)$ and $\mathcal{P}(R_1, R_2)$ □

Proposition 3.5. *[3] For $r = 1, 2, 3, \ldots$ we have*

1. $\mathcal{I}(A, D)^r = \mathcal{I}(A^r, D^r)$
2. $\mathcal{P}(B, C)^{2r} = \mathcal{I}((BC)^r, (CB)^r)$
3. $\mathcal{P}(B, C)^{2r+1} = \mathcal{P}((BC)^r B, (CB)^r C)$
4. $\mathcal{I}(A, D)\mathcal{P}(B, C) = \mathcal{P}(AB, DC)$
5. $\mathcal{P}(B, C)\mathcal{I}(A, D) = \mathcal{P}(BD, CA)$

Define $f(H) = f(\mathcal{I}(A, D), \mathcal{P}(B, C))$ and $g(H) = g_{ij}(A, B, C, D)$ for some monomials f and g. Whenever $f(H)$ is used, it is implied that f takes the variables $\mathcal{I}(A, D)$ and $\mathcal{P}(B, C)$ that appear in the decomposition $H = \mathcal{I}(A, D) + \mathcal{P}(B, C)$. Whenever, $g(H)$ is used, it is implied that g takes the variables A, B, C and D, the blocks in the partitioned matrix H .

Proposition 3.6. *[3]* $f(H) = \begin{bmatrix} g_{11}(H) & g_{12}(H) \\ g_{21}(H) & g_{22}(H) \end{bmatrix}$ *where f and $g_{ij}; 1 \leq i, j \leq 2$ are monomials.*

Proof. Let s be the degree of the term $\mathcal{P}(B,C)$ in the monomial $f(H)$ and let t be the total degree. If s is even, then from Proposition 3.5,

$$\begin{aligned} f(\mathcal{I}(A,D),\mathcal{P}(B,C)) &= \mathcal{I}(A,D)^{t-s}\mathcal{P}(B,C)^s \\ &= \mathcal{I}(A^{t-s},D^{t-s})\mathcal{I}((BC)^{s/2},(CB)^{s/2}) \\ &= \mathcal{I}(A^{t-s}(BC)^{s/2},D^{t-s}(CB)^{s/2}) \end{aligned}$$

If s is odd, then from Proposition 3.5,

$$\begin{aligned} f(\mathcal{I}(A,D),\mathcal{P}(B,C)) &= \mathcal{I}(A,D)^{t-s}\mathcal{P}(B,C)^s \\ &= \mathcal{I}(A^{t-s},D^{t-s})\mathcal{P}((BC)^{(s-1)/2}B,(CB)^{(s-1)/2}C) \\ &= \mathcal{P}(A^{t-s}(BC)^{(s-1)/2}B,D^{t-s}(CB)^{(s-1)/2}C) \end{aligned}$$

Hence, $f(H) = \begin{bmatrix} g_{11}(A,B,C,D) & g_{12}(A,B,C,D) \\ g_{21}(A,B,C,D) & g_{22}(A,B,C,D) \end{bmatrix}$ for some monomials g_{ij} for $1 \leq i,j \leq 2$. □

Lemma 3.7. *[3]* $f(K \otimes_p H) = f(K) \otimes_p f(H)$

Proof. Consider a monomial g_{ij}, then from Proposition 3.2, we get

$$g_{ij}(U \otimes A, V \otimes B, W \otimes C, Z \otimes D) = g_{ij}(U,V,W,X) \otimes g_{ij}(A,B,C,D)$$

Hence,

$$\begin{aligned} f(K \otimes_p H) &= f(\mathcal{I}(U \otimes_p A, Z \otimes_p D), \mathcal{P}(V \otimes_p B, W \otimes_p C)) \\ &= \begin{bmatrix} g_{11}(U \otimes A, V \otimes B, W \otimes C, Z \otimes D) & g_{12}(U \otimes A, V \otimes B, W \otimes C, Z \otimes D) \\ g_{21}(U \otimes A, V \otimes B, W \otimes C, Z \otimes D) & g_{22}(U \otimes A, V \otimes B, W \otimes C, Z \otimes D) \end{bmatrix} \\ &= \begin{bmatrix} g_{11}(U,V,W,Z) & g_{12}(U,V,W,Z) \\ g_{21}(U,V,W,Z) & g_{22}(U,V,W,Z) \end{bmatrix} \otimes_p \begin{bmatrix} g_{11}(A,B,C,D) & g_{12}(A,B,C,D) \\ g_{21}(A,B,C,D) & g_{22}(A,B,C,D) \end{bmatrix} \\ &= f(K) \otimes_p f(H) \end{aligned}$$

□

Define $H^{\#} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$ where A is $p \times p$ matrix and D is $q \times q$ matrix. Then $Q = \mathcal{P}(I_p, I_q)$ satisfies $Q^T H Q = H^{\#}$. Hence, H and $H^{\#}$ are permutationally similar

Lemma 3.8. *[3] $g_{11}(H) = g_{22}(H^{\#})$ and $g_{12}(H) = g_{21}(H^{\#})$*

Proof. Since $Q = \mathcal{P}(I_p, I_q)$ satisfies $Q^T H Q = H^{\#}$, then for a monomial f,

$$\begin{aligned} Q^T f(H) Q &= Q^T f(\mathcal{I}(A, D), \mathcal{P}(B, C)) Q \\ &= f(Q^T \mathcal{I}(A, D) Q, Q^T \mathcal{P}(B, C) Q) \\ &= f(Q^T H Q) \\ &= f(H^{\#}) \end{aligned}$$

To illustrate the second step, consider the example $f(X, Y) = X^2 Y X^3$ such that products are defined. Then,

$$\begin{aligned} Q^T f(X, Y) Q &= Q^T X^2 Y X^3 Q \\ &= (Q^T X Q)^2 (Q^T Y Q)(Q^T X Q)^3 \\ &= f(Q^T X Q, Q^T Y Q) \end{aligned}$$

Consider $f(H^{\#}) = \begin{bmatrix} g_{11}(H^{\#}) & g_{12}(H^{\#}) \\ g_{21}(H^{\#}) & g_{22}(H^{\#}) \end{bmatrix}$ and

$$\begin{aligned} Q^T f(H) Q &= \begin{bmatrix} 0 & I_p \\ I_q & 0 \end{bmatrix}^T \begin{bmatrix} g_{11}(H) & g_{12}(H) \\ g_{21}(H) & g_{22}(H) \end{bmatrix} \begin{bmatrix} 0 & I_p \\ I_q & 0 \end{bmatrix} \\ &= \begin{bmatrix} g_{22}(H) & g_{21}(H) \\ g_{12}(H) & g_{11}(H) \end{bmatrix} \end{aligned}$$

Since, $Q^T f(H) Q = f(H^{\#})$, we obtain $g_{11}(H) = g_{22}(H^{\#})$ and $g_{12}(H) = g_{21}(H^{\#})$. □

Lemma 3.9. *[3]*

$$tr[f(K \otimes_p H) - f(K \otimes_p H^{\#})] = tr[g_{11}(K) - g_{22}(K)] \times tr[g_{11}(H) - g_{22}(H)]$$

Proof. Using the Lemma 3.7., we can write,

$$\begin{aligned} tr[f(K \otimes_p H)] &= tr[f(K) \otimes_p f(H)] \\ &= tr[g_{11}(K)]tr[g_{11}(H)] + tr[g_{22}(K)]tr[g_{22}(H)] \end{aligned}$$

Similarly, using Lemma 3.7. and Lemma 3.8., we can write,

$$\begin{aligned} tr[f(K \otimes_p H^{\#})] &= tr[f(K) \otimes_p f(H^{\#})] \\ &= tr[g_{11}(K)]tr[g_{11}(H^{\#})] + tr[g_{22}(K)]tr[g_{22}(H^{\#})] \\ &= tr[g_{11}(K)]tr[g_{22}(H)] + tr[g_{22}(K)]tr[g_{11}(H)] \end{aligned}$$

Then,

$$\begin{aligned} tr[f(K \otimes_p H) - f(K \otimes_p H^{\#})] &= tr[g_{11}(K)]tr[g_{11}(H)] + tr[g_{22}(K)]tr[g_{22}(H)] \\ &\quad - tr[g_{11}(K)]tr[g_{22}(H)] - tr[g_{22}(K)]tr[g_{11}(H)] \\ &= tr[g_{11}(K) - g_{22}(K)] \times tr[g_{11}(H) - g_{22}(H)] \end{aligned}$$

$\square$

Let $K_i = \begin{bmatrix} U_i & V_i \\ W_i & X_i \end{bmatrix}; i = 1, 2.$

Lemma 3.10. *[3]*

$$\begin{aligned} tr[f(K_1 \otimes_p H) - f(K_2 \otimes_p H)] =& tr[g_{11}(H)](tr[g_{11}(K_1)] - tr[g_{11}(K_2)]) \\ &+ tr[g_{22}(H)](tr[g_{22}(K_1)] - tr[g_{22}(K_2)]) \end{aligned}$$

Proof. Using the Lemma 3.7., we can write,

$$\begin{aligned} tr[f(K_1 \otimes_p H)] &= tr[f(K_1) \otimes_p f(H)] \\ &= tr[g_{11}(K_1)]tr[g_{11}(H)] + tr[g_{22}(K_1)]tr[g_{22}(H)] \end{aligned}$$

Similarly, using Lemma 3.7., we can write,

$$\begin{aligned} tr[f(K_2 \otimes_p H)] &= tr[f(K_2) \otimes_p f(H)] \\ &= tr[g_{11}(K_2)]tr[g_{11}(H)] + tr[g_{22}(K_2)]tr[g_{22}(H)] \end{aligned}$$

Then,

$$\begin{aligned} tr[f(K_1 \otimes_p H)] - tr[f(K_2 \otimes_p H)] =& tr[g_{11}(K_1)]tr[g_{11}(H)] + tr[g_{22}(K_1)]tr[g_{22}(H)] \\ & - tr[g_{11}(K_2)]tr[g_{11}(H)] + tr[g_{22}(K_2)]tr[g_{22}(H)] \\ =& tr[g_{11}(H)](tr[g_{11}(K_1)] - tr[g_{11}(K_2)]) \\ & + tr[g_{22}(H)](tr[g_{22}(K_1)] - tr[g_{22}(K_2)]) \end{aligned}$$

□

Let us now turn to the applications of the above theory. Let the diagonal blocks in the matrix M_i be identity matrices. Let us call such matrices $L_i = \begin{bmatrix} I_{m_i} & V_i \\ W_i & I_{n_i} \end{bmatrix}$. If $f(X,Y)$ is monomial in X and Y, then let s be the degree of the variable Y in $f(X,Y)$ and t be the total degree. Then $f(L_i) = f(\mathcal{I}(I_{m_i}, I_{n_i}), \mathcal{P}(V_i, W_i))$. Since, $\mathcal{I}(I_{m_i}, I_{n_i})$ and $\mathcal{P}(V_i, W_i)$ commute, we have $f(L_i) = \mathcal{P}(V_i, W_i)^s$ when $s \neq 0$ and $f(L_i) = \mathcal{I}(I_{m_i}, I_{n_i})$ when $s = 0$.

Proposition 3.11. *Let V and W are $m \times n$ and $n \times m$ matrices respectively, then $tr[(VW)^r] = tr[(WV)^r]$ holds for all r.*

Proof. The statement holds trivially for $r = 0$. Let $r = 1$, then

$$\begin{aligned} tr[VW] &= \sum_{i=1}^{m} (VW)_{ii} \\ &= \sum_{i=1}^{m} \sum_{k=1}^{n} V_{ik} W_{ki} \\ &= \sum_{k=1}^{n} \sum_{i=1}^{m} W_{ki} V_{ik} \\ &= \sum_{k=1}^{n} (WV)_{kk} = tr[WV] \end{aligned}$$

Let $Z = (WV)^{r-1}W$, then $tr[VZ] = tr[ZV]$ and

$$\begin{aligned} tr[(VW)^r] &= tr[V(WV)^{r-1}W] \\ &= tr[VZ] \\ &= tr[ZV] \\ &= tr[(WV)^{r-1}WV] \\ &= tr[(WV)^r] \end{aligned}$$

□

Lemma 3.12. *[3] With above notations,*

1. *if $s \neq 0$, then $tr[g_{11}(L_i)] = tr[g_{22}(L_i)]$*
2. *if $s = 0$, then $tr[g_{11}(L_i)] = m_i$, $tr[g_{22}(L_i)] = n_i$*
3. *if $s \neq 0$, then $tr[g_{11}(L_1)] = tr[g_{22}(L_2)]$ for all monomials f if and only if L_1 and L_2 are cospectral.*

Proof. 1. Suppose s is odd, that is, $s = 2r + 1$. Then,

$$f(L_i) = \mathcal{P}(V_i, W_i)^s = \mathcal{P}((V_iW_i)^rV_i, (W_iV_i)^rW_i)$$

Hence, $tr[g_{11}(L_1)] = tr[g_{22}(L_2)] = 0$ and *1.* holds. Suppose s is even, that is, $s = 2r$ where $r \neq 0$. Then,

$$f(L_i) = \mathcal{P}(V_i, W_i)^s = \mathcal{I}((V_iW_i)^r, (W_iV_i)^r)$$

Hence, $tr[g_{11}(L_1)] = tr[(V_iW_i)^r]$ and $tr[g_{22}(L_1)] = tr[(W_iV_i)^r]$. Then, from Proposition 3.11, $tr[(V_iW_i)^r] = tr[(W_iV_i)^r]$, hence *1.* holds.

2. If $s = 0$, $f(L_i) = \mathcal{I}(I_{m_i}, I_{n_i})$. Hence, *2.* holds.

3. We have, $f(L_1) = g_{11}(L_1)+g_{22}(L_1)$ and $f(L_2) = g_{11}(L_2)+g_{22}(L_2)$. Suppose $s \neq 0$, then from *1.*, $tr[g_{11}(L_1)] = tr[g_{22}(L_1)]$ and $tr[g_{11}(L_2)] = tr[g_{22}(L_2)]$. Hence, having $tr[g_{11}(L_1)] = tr[g_{22}(L_2)]$ is equivalent to $tr[f(L_1)] = tr[f(L_2)]$. Since for any monomial f, we have $f(L_i) = \mathcal{P}(V_i, W_i)^s$. Hence, we want to show that $tr[\mathcal{P}(V_1, W_1)^t] = tr[\mathcal{P}(V_2, W_2)^t]$ for all $t \neq 0$ if and only if L_1 and L_2 are cospectral. Now consider

$$L_i^t = (I_{m_i+n_i} + \mathcal{P}(V_i, W_i))^t = I_{m_i+n_i} + \binom{t}{1}\mathcal{P}(V_i, W_i) + \ldots + \mathcal{P}(V_i, W_i)^t \tag{3.1}$$

Now suppose $tr[\mathcal{P}(V_1, W_1)^t] = tr[\mathcal{P}(V_2, W_2)^t]$ for all $t \neq 0$ then it follows that $tr[L_1^t] = tr[L_2^t]$ that is, L_1 and L_2 are cospectral.

Now conversely, suppose L_1 and L_2 are cospectral. Then, $tr[L_1^t] = tr[L_2^t]$ for $t = 0, 1, 2, \ldots$. Note that $tr[\mathcal{P}(V_1, W_1)^t] = tr[\mathcal{P}(V_2, W_2)^t]$ is true when the monomial corresponds to odd s since in that case trace is zero. Hence, we only need to show that it holds in the even cases. Let $t = 2$, then from equation 3.1, we have

$$tr[I_{m_1+n_1} + \mathcal{P}(V_1, W_1) + \mathcal{P}(V_1, W_1)^2] = tr[I_{m_2+n_2} + \mathcal{P}(V_2, W_2) + \mathcal{P}(V_2, W_2)^2]$$

Hence, $tr[\mathcal{P}(V_1, W_1)^2] = tr[\mathcal{P}(V_2, W_2)^2]$

Now suppose $tr[\mathcal{P}(V_1, W_1)^{t-1}] = tr[\mathcal{P}(V_2, W_2)^{t-1}]$ holds as the induction assumption. Then, from equation 3.1, trace of all the terms in $tr[L_i^t]$ except the last term $tr[\mathcal{P}(V_i, W_i)^t]$ are same for $i = 1$ and $i = 2$. Since $tr[L_1^t] = tr[L_2^t]$, then $tr[\mathcal{P}(V_1, W_1)^t] = tr[\mathcal{P}(V_2, W_2)^t]$

This shows $tr[f(L_1)] = tr[f(L_2)]$ for all monomials f if and only if L_1 and L_2 are cospectral. Hence, it follows if $s \neq 0$, $tr[g_{11}(L_1)] = tr[g_{22}(L_2)]$ for all monomials f if and only if L_1 and L_2 are cospectral. $\square$

Lemma 3.13. *[3]* $tr[(L \otimes_p H)^t] - tr[(L \otimes_p H^{\#})^t] = (m - n)(tr[A^t] - tr[D^t])$ *for* $t = 0, 1, 2, \ldots$

Proof. From Lemma 3.9, we have $tr[f(L \otimes_p H)] - tr[f(L \otimes_p H^{\#})] = tr[g_{11}(L) - g_{22}(L)] \times tr[g_{11}(H) - g_{22}(H)]$

Case 1: $s = 0$

From Lemma 3.12., we have $tr[g_{11}(L) - g_{22}(L)] = m - n$. Since $f(H) = \mathcal{I}(A, D)^t$, we have $tr[g_{11}(H) - g_{22}(H)] = tr[A^t] - tr[D^t]$. Also for any monomial $f(L \otimes_p H) = \mathcal{I}(I_m \otimes A, I_n \otimes D)^t = (L \otimes_p H)^t$, hence the statement follows.

Case 2: $s \neq 0$

From Lemma 3.12., we have $tr[g_{11}(L) - g_{22}(L)] = 0$. Hence, $tr[f(L \otimes_p H)] - tr[f(L \otimes_p H^{\#})] = 0$

for all monomials. Since, $(L\otimes_p H)^t$ and $(L\otimes_p H^{\#})^t$ both can be written as binomial expansions in which each term is a monomial, the LHS of the statement is zero. The statement trivially holds. □

Lemma 3.14. *[3] Let L_1 and L_2 be cospectral. Then, $tr[(L_1 \otimes_p H)^t] - tr[(L_2 \otimes_p H)^t] = (m_1 - m_2)(tr[A^t] - tr[D^t])$ for $t = 0, 1, 2, \ldots$*

Proof. From Lemma 3.10, we have

$$\begin{aligned} tr[f(L_1 \otimes_p H)] - tr[f(L_2 \otimes_p H)] &= tr[g_{11}(H)](tr[g_{11}(L_1)] - tr[g_{11}(L_2)] \\ &\quad + tr[g_{22}(H)](tr[g_{22}(L_1)] - tr[g_{22}(L_2)] \end{aligned}$$

Case 1: $s = 0$

From Lemma 3.12, we have $tr[g_{11}(L_1)] = m_1$, $tr[g_{22}(L_1)] = n_1$, $tr[g_{11}(L_2)] = m_2$ and $tr[g_{22}(L_2)] = n_2$. Since $f(H) = \mathcal{I}(A, D)^t$, we have $g_{11}(H) = A^t$ and $g_{22}(H) = D^t$. Also for any monomial f, $f(L \otimes_p H) = \mathcal{I}(I_m \otimes A, I_n \otimes D)^t = (L \otimes_p H)^t$. Hence, $tr[f(L_1 \otimes_p H)] - tr[f(L_2 \otimes_p H)] = (m_1 - m_2)tr[A^t] + (n_1 - n_2)tr[D^t]$. Since L_1 and L_2 are cospectral, $m_1 + n_1 = m_2 + n_2$, that is, $m_1 - m_2 = n_2 - n_1$. Hence, $tr[(L_1 \otimes_p H)^t] - tr[(L_2 \otimes_p H)^t] = (m_1 - m_2)(tr[A^t] - tr[D^t])$.

Case 2: $s \neq 0$

From Lemma 3.12, and since L_1 and L_2 are cospectral, we have $tr[g_{11}(L_1)] = tr[g_{22}(L_2)]$ and $tr[g_{11}(L_2)] = tr[g_{22}(L_1)]$. We also have $tr[g_{11}(L_1)] = tr[g_{22}(L_1)]$ and $tr[g_{11}(L_2)] = tr[g_{22}(L_2)]$. Hence, $tr[f(L\otimes_p H)] - tr[f(L\otimes_p H^{\#})] = 0$ for all monomials. Since, $(L\otimes_p H)^t$ can be written as sum of monomials, the LHS of the statement is zero. Hence, the statement trivially holds. □

Now we state the main results of Godsil and Mckay,

Theorem 3.15. *[3]Let*

$$L = \begin{bmatrix} I_m & V \\ W & I_n \end{bmatrix}, H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, H^{\#} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

where A and D are square matrices and I_m and I_n are $m \times m$ and $n \times n$ identity matrices. Then, $L \otimes_p H$ and $L \otimes_p H^{\#}$ are cospectral if and only if $m = n$ or A and D are cospectral.

Proof. Follows from Lemma 3.13. ☐

Theorem 3.16. *[3] Let*

$$L_i = \begin{bmatrix} I_{m_i} & V_i \\ W_i & I_{n_i} \end{bmatrix}, i = 1, 2; H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are square matrices and I_{m_i} and I_{n_i} are identity matrices. Let L_1 and L_2 be cospectral. Then, $L_1 \otimes_p H$ and $L_2 \otimes_p H$ are cospectral if and only if $m_1 = m_2$ or A and D are cospectral.

Proof. Follows from Lemma 3.14. ☐

3.2 Generalizing the construction of Godsil and Mckay

In this section, we generalize the construction we just discussed. We extend the results to matrices of the form other than $L = \begin{bmatrix} I_m & V \\ W & I_n \end{bmatrix}$. Such matrices are given by the following definition. Let $K = \begin{bmatrix} U & V \\ W & X \end{bmatrix}$ such that the blocks U and X are square matrices of the order m and n respectively. Similarly, let $K_i = \begin{bmatrix} U_i & V_i \\ W_i & X_i \end{bmatrix}$ for $i = 1, 2$ such that the blocks U_i and X_i are square matrices of the order m_i and n_i respectively.

Definition 3.17. *A matrix K is said to satisfy C/T property if it satisfies only the commuting and the trace property. Two matrices K_1 and K_2 are said to satisfy $C/M/T$ property if they satisfy the commuting, the monomial and the trace property.*

1. *(Commuting Property) $\mathcal{I}(U, X)$ and $\mathcal{P}(V, W)$ commute, that is, $UV = VX$ and $XW = WU$*

2. *(Monomial Property) $tr[f(K_1)] = tr[f(K_2)]$ for all monomials f if and only if $tr[K_1^t] = tr[K_2^t]$ for all $t = 0, 1, 2, \ldots$ (that is, K_1 and K_2 are cospectral).*

3. *(Trace Property) $tr[U^{t-s}(VW)^r] = tr[X^{t-s}(WV)^r]$ holds for all t and even s where $s = 2r$ and $s \neq 0$*

We can write $f(K_i) = f(\mathcal{I}(U_i, X_i), \mathcal{P}(V_i, W_i))$. Let s be the degree of $\mathcal{P}(V_i, W_i)$ in f and t be the total degree of f. The following proposition shows that the forward implication of the monomial property is true. Hence, in the further results when we need to show that monomial property holds, we will only need to show that the backward implication is true.

Proposition 3.18. *Let K_1 and K_2 be two matrices such that $tr[f(K_1)] = tr[f(K_2)]$ holds for all monomials f, then K_1 and K_2 are cospectral.*

Proof. Since, $f(K_i) = \mathcal{I}(U_i, X_i)^{t-s}\mathcal{P}(V_i, W_i)^s$ for all nonnegative t and s, the following holds for every t and s,

$$tr[\mathcal{I}(U_1, X_1)^{t-s}\mathcal{P}(V_1, W_1)^s] = tr[\mathcal{I}(U_2, X_2)^{t-s}\mathcal{P}(V_2, W_2)^s]$$

Now consider the following for $t = 0, 1, 2, \ldots$

$$\begin{aligned}
tr[K_1^t] &= tr[(\mathcal{I}(U_1, X_1) + \mathcal{P}(V_1, W_1))^t] \\
&= tr[\mathcal{I}(U_1, X_1)^t] + \binom{t}{1} tr[\mathcal{I}(U_1, X_1)^{t-1}\mathcal{P}(V_1, W_1)^1] + \ldots + tr[\mathcal{P}(V_1, W_1)^t] \\
&= tr[\mathcal{I}(U_2, X_2)^t] + \binom{t}{1} tr[\mathcal{I}(U_2, X_2)^{t-1}\mathcal{P}(V_2, W_2)^1] + \ldots + tr[\mathcal{P}(V_2, W_2)^t] \\
&= tr[(\mathcal{I}(U_2, X_2) + \mathcal{P}(V_2, W_2))^t] \\
&= tr[K_2^t]
\end{aligned}$$

Hence, K_1 and K_2 are cospectral. □

We now extend Lemma 3.12 to matrices not just of the form $\begin{bmatrix} I_m & V \\ W & I_n \end{bmatrix}$ but any matrices satisfying C/T and $C/M/T$ property respectively.

Lemma 3.19. *If a matrix K satisfies C/T property, then it satisfies the first two conditions below. If two matrices K_1 and K_2 satisfy $C/M/T$ property, then the satisfy all three conditions below.*

1. *If $s \neq 0$, $tr[g_{11}(K)] = tr[g_{22}(K)]$,*

2. *If $s = 0$, $tr[g_{11}(K)] = tr[U^t]$ and $tr[g_{22}(K)] = tr[X^t]$,*

3. If $s \neq 0$, $tr[g_{11}(K_1)] = tr[g_{22}(K_2)]$ *for all monomials* f *if and only if* K_1 *and* K_2 *are cospectral.*

Proof. 1. If $s \neq 0$, then $f(K) = \mathcal{I}(U,X)^{t-s}\mathcal{P}(V,W)^s$. We have two cases:

Case 1: s is odd and $s = 2r+1$
Then $\mathcal{P}(V,W)^s = P[(VW)^rV, (WV)^rW]$ and $\mathcal{I}(U,X)^{t-s} = \mathcal{I}(U^{t-s}, X^{t-s})$. Hence,

$$\begin{aligned} f(K) &= \mathcal{I}(U^{t-s}, X^{t-s})P[(VW)^rV, (WV)^rW] \\ &= P[U^{t-s}(VW)^rV, X^{t-s}(WV)^rW] \end{aligned}$$

Then, $tr[g_{11}(K)] = tr[g_{22}(K)] = 0$.

Case 2: s is even and $s = 2r$
Then $\mathcal{P}(V,W)^s = I[(VW)^r, (WV)^r]$. Hence,

$$\begin{aligned} f(K) &= \mathcal{I}(U^{t-s}, X^{t-s})I[(VW)^r, (WV)^r] \\ &= I[U^{t-s}(VW)^r, X^{t-s}(WV)^r] \end{aligned}$$

$tr[g_{11}(K)] = tr[U^{t-s}(VW)^r]$ and $tr[g_{22}(K)] = tr[X^{t-s}(WV)^r]$. Since, K satisfies trace property, then $tr[g_{11}(K)] = tr[g_{22}(K)]$, hence *1.* holds.

2. If $s = 0$, $f(K) = \mathcal{I}(U^t, X^t)$. Then, $tr[g_{11}(K)] = tr[U^t]$ and $tr[g_{22}(K)] = tr[X^t]$.

3. If $s \neq 0$, and suppose K_1 and K_2 be cospectral, then $tr[f(K_1)] = tr[f(K_2)]$ for all monomials f because the matrices K_1 and K_2 satisfy monomial property. We can write, $tr[f(K_i)] = tr[g_{11}(K_i)] + tr[g_{22}(K_i)]$ and also from *1.* we have $tr[g_{11}(K_i)] = tr[g_{22}(K_i)]$. Hence, when $s \neq 0$ having $tr[f(K_1)] = tr[f(K_2)]$ is equivalent to $tr[g_{11}(K_1)] = tr[g_{22}(K_2)]$ for all monomials f. Conversely suppose $tr[g_{11}(K_1)] = tr[g_{22}(K_2)]$ holds for all monomials

f. From *1.*, we have $tr[g_{11}(K_i)] = tr[g_{22}(K_i)]$. Then,

$$\begin{aligned} tr[f(K_1)] &= tr[g_{11}(K_1)] + tr[g_{22}(K_1)] \\ &= tr[g_{22}(K_2)] + tr[g_{11}(K_2)] \\ &= tr[f(K_2)] \end{aligned}$$

Hence, K_1 and K_2 are cospectral. □

Theorem 3.20. *Let the matrix K satisfy C/T property, and let*

$$K = \begin{bmatrix} U & V \\ W & X \end{bmatrix}, H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, H^{\#} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

Then, $K \otimes_p H$ and $K \otimes_p H^{\#}$ are cospectral if and only if U and X are cospectral or A and D are cospectral.

Proof. From Lemma 3.9, we have

$$tr[f(K \otimes_p H)] - tr[f(K \otimes_p H^{\#})] = (tr[g_{11}(K)] - tr[g_{22}(K)])(tr[g_{11}(H)] - tr[g_{22}(H)])$$

Case 1: $s = 0$

Then for some t, $f(K) = \mathcal{I}(U^t, X^t)$, $f(H) = \mathcal{I}(A^t, D^t)$ and

$$f(K \otimes_p H) = \mathcal{I}((U \otimes_p A)^t, (X \otimes_p D)^t) = (K \otimes_p H)^t$$

Similarly, $f(K \otimes_p H^{\#}) = (K \otimes_p H^{\#})^t$. Hence, $tr[g_{11}(K)] = tr[U^t]$, $tr[g_{22}(K)] = tr[X^t]$, $tr[g_{11}(H)] = tr[A^t]$, $tr[g_{22}(H)] = tr[D^t]$. Then, we obtain

$$tr[(K \otimes_p H)^t] - tr[(K \otimes_p H^{\#})^t] = (tr[U^t] - tr[X]^t)(tr[A^t] - tr[D^t])$$

for $t = 0, 1, 2, \ldots$. Hence, $K \otimes_p H$ and $K \otimes_p H^{\#}$ are cospectral if and only if U and X are cospectral or A and D are cospectral

Case 2: $s \neq 0$

From Lemma 3.19 (1), we have $tr[g_{11}(K)] = tr[g_{22}(K)]$. Then,

$$tr[f(K \otimes_p H)] - tr[f(K \otimes_p H^{\#})] = 0$$

then it follows from Proposition 3.18 that $K \otimes_p H$ and $K \otimes_p H^{\#}$ are cospectral. $\square$

Theorem 3.21. *Let K_1 and K_2 be cospectral and satisfy $C/M/T$ property. Let*

$$K_i = \begin{bmatrix} U_i & V_i \\ W_i & X_i \end{bmatrix}, for\ i = 1, 2, H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

Then $K_1 \otimes_p H$ and $K_2 \otimes_p H$ are cospectral if and only if U_1 and U_2 are cospectral or A and D are cospectral.

Proof. From Lemma 3.10, we have

$$\begin{aligned} tr[f(K_1 \otimes_p H) - f(K_2 \otimes_p H)] &= tr[g_{11}(H)](tr[g_{11}(K_1)] - tr[g_{11}(K_2)]) \\ &\quad + tr[g_{22}(H)](tr[g_{22}(K_1)] - tr[g_{22}(K_2)]) \end{aligned}$$

Case 1: $s = 0$

Then for some t, $f(K_i) = \mathcal{I}(U_i^t, X_i^t)$, $f(H) = \mathcal{I}(A^t, D^t)$ and

$$f(K_i \otimes_p H) = \mathcal{I}((U_i \otimes_p A)^t, (X_i \otimes_p D)^t) = (K_i \otimes_p H)^t$$

Hence, $tr[g_{11}(K_i)] = tr[U_i^t]$, $tr[g_{22}(K_i)] = tr[X_i^t]$, $tr[g_{11}(H)] = tr[A^t]$, $tr[g_{22}(H)] = tr[D^t]$. Then,

$$tr[(K_1 \otimes_p H)^t] - tr[(K_2 \otimes_p H)^t] = tr[A^t](tr[U_1^t] - tr[U_2^t]) + tr[D^t](tr[X_1^t] - tr[X_2^t])$$

Since K_1 and K_2 are cospectral, $tr[f(K_1)] = tr[f(K_2)]$ holds for all monomials f. We have $tr[f(K_i)] = tr[U_i^t] + tr[X_i^t]$. Hence, $tr[U_1^t] + tr[X_1^t] = tr[U_2^t] + tr[X_2^t]$, that is, $tr[U_1^t] - tr[U_2^t] = -tr[X_1^t] + tr[X_2^t]$. We obtain

$$tr[(K_1 \otimes_p H)^t] - tr[(K_2 \otimes_p H)^t] = (tr[U_1^t] - tr[U_2^t])(tr[A^t] - tr[D^t])$$

for all $t = 0, 1, 2, \ldots$. Hence, $K_1 \otimes_p H$ and $K_2 \otimes_p H$ are cospectral if and only if U_1 and U_2 are cospectral or A and D are cospectral.

Case 2: $s \neq 0$

Since K_1 and K_2 are cospectral, from Lemma 3.19 (3), we have $tr[g_{11}(K_1)] = tr[g_{22}(K_2)]$ and $tr[g_{11}(K_2)] = tr[g_{22}(K_1)]$ for all monomials. Then, $tr[f(K_1 \otimes_p H) - f(K_2 \otimes_p H)] = 0$

for all monomials. Then, it follows from Proposition 3.18 that $K_1 \otimes_p H$ and $K_2 \otimes_p H$ are cospectral. $\square$

3.3 Isomorphism of the corresponding graphs

If the matrices in the construction we just obtained are taken to be adjacency matrices of graph or digraphs, then we get conspectral graphs or digraphs. Let us assume the matrices to be adjacency matrices of graphs, that is, let the matrices be symmetric 0-1 matrices. We will not assume that the graphs don't have loops, that is, some matrices might have nonzero diagonal entries. In this section, we investigate necessary and sufficient conditions for the corresponding graphs to be isomorphic. A similar analysis can also be done for the digraphs.

Let H be a symmetric partitioned matrix such that $H = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$ and the diagonal blocks A and D are square symmetric.

Definition 3.22. *A graph G_H is said to have interchanging automorphism with respect to its bipartition, if it interchanges the induced graphs G_A and G_D.*

Proposition 3.23. *If a graph G_H admits an interchanging automorphism with respect to its bipartition, then the corresponding permutation matrix has the form $\mathcal{P}(Q_1, Q_2)$ such that Q_1 and Q_2 are permutation matrices. Then, $Q = \mathcal{I}(Q_1, Q_2)$ satisfies $Q^T HQ = H^\#$.*

Proof. It can be easily seen that the permutation matrix corresponding to such an automorphism has the form $Q' = \mathcal{P}(Q_1, Q_2)$ such that Q_1 and Q_2 are permutation matrices of appropriate orders. Then from $Q'^T HQ' = H$, we have $Q_1^T BQ_2 = B^T$, $Q_1^T AQ_1 = D$, and $Q_2^T DQ_2 = A$. Let $Q = \mathcal{I}(Q_1, Q_2)$, then it satisfies $Q^T HQ = H^\#$. $\square$

The following proposition gives a sufficient condition for a graph to not admit an interchanging automorphism.

Proposition 3.24. *A graph G_H does not admit an interchanging automorphism with respect to its bipartition if one of the following holds:*

1. *B is not PET (in particular, B is not square)*

2. the graphs G_A and G_D are not isomorphic, (in particular, they are are not cospectral)

Proof. If G_H admits such an automorphism, then the corresponding permutation matrix has the form $\mathcal{P}(Q_1, Q_2)$ such that Q_1 and Q_2 are permutation matrices satisfying $Q_1^T B Q_2 = B^T$, $Q_1^T A Q_1 = D$, and $Q_2^T D Q_2 = A$, that is, B is PET and G_A and G_D are isomorphic. When G_A and G_D are not cospectral, the graphs G_A and G_D are nonisomorphic. If B is not square, then B is not PET. Hence, if any one of the two conditions hold, the automorphism cannot interchange G_A and G_D. □

3.3.1 Construction-I

Recall the construction given by Theorem 3.20. Let us assume that the matrix $K = \begin{bmatrix} U & V \\ V^T & X \end{bmatrix}$ satisfies C/T property, and at least one of the pairs U and X or A and D is cospectral. Hence, the graphs $G_{K \otimes_p H}$ and $G_{K \otimes_p H^\#}$ corresponding to the matrices $K \otimes_p H$ and $K \otimes_p H^\#$ as adjacency matrices are cospectral. This also implies that least one of G_K or G_H admits equal bipartition size.

The following lemma gives a sufficient condition for the graphs to be isomorphic.

Lemma 3.25. *If at least one of G_K or G_H admits an interchanging automorphism with respect to its bipartition, then $G_{K \otimes_p H}$ and $G_{K \otimes_p H^\#}$ are isomorphic.*

Proof. **Case 1:** Suppose G_K admits such as automorphism
Then, there exists permutation matrices R_2 and R_3 such that $R = \mathcal{P}(R_2, R_3)$ satisfies $R^T K R = K$. Since $Q^T H Q = H^\#$ for $Q = \mathcal{P}(I_p, I_q)$, the partitioned tensor product $P = R \otimes_p Q$ is also a permutation matrix.

Case 2: Suppose G_H has such an automorphism
Then there exists permutation matrices Q_1 and Q_4 such that $Q = \mathcal{I}(Q_1, Q_4)$ satifies $Q^T H Q = H^\#$. Let R be an identity matrix, then the partitioned tensor product $P = R \otimes_p Q$ is also a permutation matrix.

In any case, it follows that

$$\begin{aligned} P^T(K \otimes_p H)P &= (R \otimes_p Q)^T(K \otimes_p H)(R \otimes_p Q) \\ &= (R^T \otimes_p Q^T)(K \otimes_p H)(R \otimes_p Q) \\ &= (R^T K R) \otimes_p (Q^T H Q) \\ &= K \otimes_p H^{\#} \end{aligned}$$

Note that Proposition 3.4 is used in the second step. Hence, $G_{K\otimes_p H}$ and $G_{K\otimes_p H^{\#}}$ are isomorphic, since the corresponding adjacency matrices are permutationally similar. $\square$

Let us define a property that will help us showing the sufficient condition for the isomorphism to be also a necessary condition.

Definition 3.26. *The graphs G_K and G_H are said to satisfy property η_1, if whenever $G_{K\otimes_p H}$ and $G_{K\otimes_p H^{\#}}$ are isomorphic, the induced subgraph $G_{U\otimes A}$ of $G_{K\otimes_p H}$ is isomorphic to at least one of the induced subgraphs $G_{U\otimes D}$ and $G_{X\otimes A}$ of $G_{K\otimes_p H^{\#}}$.*

Lemma 3.27. *Let the graphs G_K and G_H satisfy property η_1 and let $G_{K\otimes_p H}$ and $G_{K\otimes_p H^{\#}}$ be isomorphic. Then at least one of the following holds:*

1. *$P_1^T(U \otimes A)P_1 = U \otimes D$, $P_1^T(V \otimes B)P_4 = V \otimes B^T$, $P_4^T(X \otimes D)P_4 = X \otimes A$ for some permutation matrices P_1 and P_4. Then,*

   If G_U is nonbipartite, then G_A and G_D are isomorphic.

   If G_A, G_D and G_U are bipartite, then G_A and G_D are isomorphic.

   If $V \neq 0$ and B has no zero rows or zero columns, then B is PET.

   If G_U is nonbipartite, then G_A and G_D are isomorphic.

   If G_A, G_D and G_X are bipartite, then G_A and G_D are isomorphic.

2. *$P_2^T(U \otimes A)P_2 = X \otimes A$, $P_3^T(V^T \otimes B^T)P_2 = V \otimes B^T$ and $P_3^T(X \otimes D)P_3 = U \otimes D$ for some permutation matrices P_2 and P_3. Then,*

   If G_A is nonbipartite, then G_U and G_X are isomorphic.

   If G_U, G_X and G_A are bipartite, then G_U and G_X are isomorphic.

   If $B \neq 0$ and V has no zero rows or zero columns, then V is PET.

If G_D is nonbipartite, then G_U and G_X are isomorphic.

If G_U, G_X and G_D are bipartite, then G_U and G_X are isomorphic.

Proof. Suppose $G_{K\otimes_p H}$ and $G_{K\otimes_p H^\#}$ are isomorphic and $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ is a permutation matrix such that $P^T(K \otimes_p H)P = K \otimes_p H^\#$. Then from,

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}^T \begin{bmatrix} U \otimes A & V \otimes B \\ V^T \otimes B^T & X \otimes D \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} U \otimes D & V \otimes B^T \\ V^T \otimes B & X \otimes A \end{bmatrix}$$

Then from property η_1, P has either the form $\mathcal{I}(P_1, P_4)$ or $\mathcal{P}(P_2, P_3)$. If $P = \mathcal{I}(P_1, P_4)$, we have, $P_1^T(U \otimes A)P_1 = U \otimes D$, $P_1^T(V \otimes B)P_4 = V \otimes B^T$, $P_4^T(X \otimes D)P_4 = X \otimes A$. If $P = \mathcal{P}(P_2, P_3)$, we have, $P_3^T(X \otimes D)P_3 = U \otimes D$, $P_3^T(V^T \otimes B^T)P_2 = V \otimes B^T$, $P_2^T(U \otimes A)P_2 = X \otimes A$. The further statements follow from the cancellation law for graphs (Lemma 2.11, Theorem 2.12) and matrices (Lemma 2.14). $\square$

We will apply these isomorphism results on the particular constructions corresponding to K's that satisfy $C/M/T$ property in the further chapters.

3.3.2 Construction-II

Recall the construction given by Theorem 3.21. Let us assume that the matrices $K_1 = \begin{bmatrix} U_1 & V_1 \\ V_1^T & X_1 \end{bmatrix}$ and $K_2 = \begin{bmatrix} U_2 & V_2 \\ V_2^T & X_2 \end{bmatrix}$ are cospectral and satisfy $C/M/T$ property. Let at also assume that least one of the pair U_1 and U_2 or A and D is cospectral. Hence, the graphs $G_{K_1\otimes_p H}$ and $G_{K_2\otimes_p H}$ corresponding to the matrices $K_1 \otimes_p H$ and $K_2 \otimes_p H$ as adjacency matrices are cospectral. This also implies that either G_H admits equal bipartition size or G_{K_1} and G_{K_2} are partitioned similarly.

Definition 3.28. *Let the graphs G_{K_1} and G_{K_2} be isomorphic. An isomorphism between them is called a Type-1 (Type-2) isomorphism, if it maps the induced subgraph G_{U_1} of G_{K_1} to the induced subgraph G_{U_2} (G_{X_2}) of G_{K_2}.*

The permutation matrices corresponding to *Type-1* and *Type-2* isomorphisms are of the form $\mathcal{I}(Q_1.Q_2)$ and $\mathcal{P}(Q_1, Q_2)$ respectively for permutation matrices Q_1 and Q_2 of appropriate orders. The following proposition gives a sufficient condition for the isomorphic graphs G_{K_1} and G_{K_2} to not admit such a isomorphism.

Proposition 3.29. *Let the graphs G_{K_1} and G_{K_2} be isomorphic. They do not admit isomorphism of Type-1 (Type-2) if one of the following holds:*

1. *G_{U_1} is not isomorphic with G_{U_2} (G_{X_2})*

2. *G_{X_1} is not isomorphic with G_{X_2} (G_{U_2})*

3. *V_2 is not permutationally equivalent to V_1 (V_1^T).*

Proof. Since G_{K_1} and G_{K_2} are isomorphic, there exists a permutation matrix P such that $P^T K_1 P = K_2$. Let $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ such that the blocks are of appropriate orders. If case the isomorphism is *Type-1*, we have $P = \mathcal{I}(P_1, P_3)$, and hence $P_1^T U_1 P_1 = U_2$, $P_1^T V_1 P_3 = V_2$, and $P_4^T X_1 P_4 = X_2$. In case the isomorphism is *Type-2*, we have $P = \mathcal{P}(P_2, P_3)$, and hence $P_3^T X_1 P_3 = U_2$, $P_3^T V_1^T P_2 = V_2$, and $P_2^T U_1 P_2 = X_2$. The proposition follows by contrapositive. □

The next lemma gives a sufficient condition for the graphs constructed to be isomorphic.

Lemma 3.30. *The graphs $G_{K \otimes_p H}$ and $G_{K \otimes_p H\#}$ are isomorphic if at least one of the following holds:*

1. *K_1 and K_2 are isomorphic via a Type-1 isomorphism*

2. *K_1 and K_2 are isomorphic via a Type-2 isomorphism, and G_H admits an interchanging automorphism with respect to its bipartition*

Proof. **Case 1:** Suppose K_1 and K_2 ar isomorphic via a *Type-1* isomorphism
If R is the corresponding permutation matrix such that $R^T K_1 R = K_2$, then R has form $R = \mathcal{I}(R_1, R_4)$ for two permtuation matrices R_1 and R_4 of appropriate orders. Let $Q = I_{p+q}$ be an identity matrix, then the partitioned tensor product $P = R \otimes_p Q$ is also a permutation matrix.

Case 2: Suppose K_1 and K_2 ar isomorphic via a *Type-2* isomorphism, and G_H admits an interchanging automorphism with respect to its bipartition
If R is the corresponding permutation matrix such that $R^T K_1 R = K_2$, then R has form $R = \mathcal{P}(R_2, R_3)$ for two permutationa matrices R_2 and R_3 of appropriate orders. Since G_H has such an interchanging automorphism with respect to its bipartition, there exists permutation matrices Q_2 and Q_3 such that $Q = \mathcal{I}(Q_2, Q_3)$ and $Q^T H Q = H$. Then, the partitioned tensor product $P = R \otimes_p Q$ is also a permutation matrix.

In any case,

$$\begin{aligned}
P^T(K_1 \otimes_p H)P &= (R \otimes_p Q)^T (K_1 \otimes_p H)(R \otimes_p Q) \\
&= (R^T \otimes_p Q^T)(K_1 \otimes_p H)(R \otimes_p Q) \\
&= (R^T K_1 R) \otimes_p (Q^T H Q) \\
&= K_2 \otimes_p H
\end{aligned}$$

Note that Proposition 3.4 is used in the second step. Hence, $G_{K_1 \otimes_p H}$ and $G_{K_2 \otimes_p H}$ are isomorphic since the corresponding adjacency matrices are permutationally similar. $\square$

Let us define a property that will be helpful in showing the sufficient condition for isomorphism to be also a necessary condition.

Definition 3.31. *The graphs G_{K_1}, G_{K_2} and G_H are said to satisfy property η_2, if whenever $G_{K_1 \otimes_p H}$ and $G_{K_2 \otimes_p H}$ are isomorphic, the induced subgraph $G_{U_1 \otimes A}$ of $G_{L_1 \otimes_p H}$ is isomorphic to at least one of the induced subgraphs $G_{U_2 \otimes A}$ and $G_{X_2 \otimes D}$ of $G_{K_2 \otimes_p H}$.*

Lemma 3.32. *Suppose G_{K_1}, G_{K_2} and G_H satisfy property η_2 and suppose $G_{K_1 \otimes_p H}$ and $G_{K_2 \otimes_p H}$ are isomorphic. Then at least one of the following holds:*

1. *$P_1^T(U_1 \otimes A)P_1 = U_2 \otimes A$, $P_1^T(V_1 \otimes B)P_4 = V_2 \otimes B$, $P_4^T(X_1 \otimes D)P_4 = X_2 \otimes D$ for some permutation matrices P_1 and P_4. Then,*

   If G_A is nonbipartite, then G_{U_1} and G_{U_2} are isomorphic.

   If G_A, G_{U_1} and G_{U_2} are bipartite, then G_{U_1} and G_{U_2} are isomorphic.

   If $B \neq 0$ and V_1 and V_2 have no zero rows or zero columns, then V_1 is permutationally equivalent to V_2.

   If G_D is nonbipartite, then G_{X_1} and G_{X_2} are isomorphic.

If G_D, G_{X_1} and G_{X_2} are bipartite, then G_{X_1} and G_{X_2} are isomorphic.

2. $P_3^T(X_1 \otimes D)P_3 = U_2 \otimes A$, $P_3^T(V_1^T \otimes B^T)P_2 = V_2 \otimes B$, $P_2^T(U_1 \otimes A)P_2 = X_2 \otimes D$ *for some permutation matrices* P_2 *and* P_3. *Suppose* $A = D$ *and* $B = B^T$. *Then,*

   If G_A is nonbipartite, then G_{X_1} and G_{U_2} are isomorphic and G_{U_1} and G_{X_2} are isomorphic

   If G_A, G_{X_1} and G_{U_2} are bipartite, then G_{X_1} and G_{U_2} are isomorphic.

   If G_A, G_{U_1} and G_{X_2} are bipartite, then G_{U_1} and G_{X_2} are isomorphic.

   If $B \neq 0$ and V_1 and V_2 have no zero rows or zero columns, then V_1^T is permutationally equivalent to V_2.

Proof. Suppose $G_{K_1 \otimes_p H}$ and $G_{K_2 \otimes_p H}$ are isomorphic, the there exists a permutation matrix P such that $P^T(K_1 \otimes_p H)P = K_2 \otimes_p H$. Let $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$, then

$$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}^T \begin{bmatrix} U_1 \otimes A & V_1 \otimes B \\ V_1^T \otimes B^T & X_1 \otimes D \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} = \begin{bmatrix} U_2 \otimes A & V_2 \otimes B \\ V_2^T \otimes B^T & X_2 \otimes D \end{bmatrix}$$

From property η_2, P is either of the form $\mathcal{I}(P_1, P_4)$ or $\mathcal{P}(P_2, P_3)$. In case $P = \mathcal{I}(P_1, P_4)$, we have, $P_1^T(U_1 \otimes A)P_1 = U_2 \otimes A$, $P_1^T(V_1 \otimes B)P_4 = V_2 \otimes B$, $P_4^T(X_1 \otimes D)P_4 = X_2 \otimes D$. In case $P = \mathcal{P}(P_2, P_3)$, we have $P_3^T(X_1 \otimes D)P_3 = U_2 \otimes A$, $P_3^T(V_1^T \otimes B^T)P_2 = V_2 \otimes B$, $P_2^T(U_1 \otimes A)P_2 = X_2 \otimes D$. The further statements follow from the cancellation law for graphs (Lemma 2.11, Theorem 2.12) and matrices (Lemma 2.14). $\square$

Chapter 4

Unfolding a bipartite graph

In this chapter, we first discuss a construction based on unfolding a bipartite graph to obtain bipartite graphs which are cospectral for the adjacency as well as the normalized Laplacian (see [1]) and some of its existing generalizations (see [14], [12]). We then introduce the idea of partitioned tensor products to obtain the most generalized version of this construction. We show it is in fact a particular case of Theorem 3.20 discussed in the previous chapter, in other words, we discuss a candidate for a matrix satisfying $C/M/T$ property. We then apply the isomorphism results from the previous chapter to obtain equivalent conditions for the cospectral graphs to be nonisomorphic. We give partial characterization of property η_1 required in the investigation of the isomorphism and show how a result of Ji, Gong and Wang [12] can be generalized and also show a complete different problem considered by Hammack [9] is related to unfolding.

4.1 Butler's construction of unfolding a bipartite graph

Let G be a bipartite graph with vertex partitioning $V(G) = X \cup Y$ such that $|X| = p$ and $|Y| = q$. The adjacency matrix of graph G can be given by

$$A(G) = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

where the biadjacency matrix B is a $p \times q$ matrix. The operation of unfolding a partition can be done in following way: Consider two copies Y_1 and Y_2 of partition Y. If there is an edge between $x_i \in X$ and $y_j \in Y$, then draw an edge between x_i and y_j^1, and between x_i and y_j^2, where $y_j^1 \in Y_1$ and $y_j^2 \in Y_2$. Note that the vertices $y_j^1 \in Y_1$ and $y_j^2 \in Y_2$ corresponds to the vertex $y_j \in Y$. Call this resultant graph Γ_1. Similarly Γ_2 can be obtained by unfolding the partition X twice in similar way. The adjacency matrices of Γ_1 and Γ_2 are given by:

$$A(\Gamma_1) = \begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix}, \ A(\Gamma_2) = \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}$$

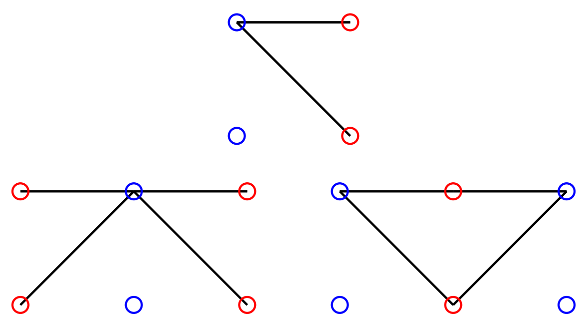


Figure 4.1: Smallest unfolding example: G, Γ_1, Γ_2

The square matrices $A(\Gamma_1)$ and $A(\Gamma_2)$ have orders $(p + 2q)$ and $(2p + q)$ respectively. The graphs G, Γ_1 and Γ_2 corresponding to the matrix $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ are shown in Figure 4.1. Vertices from the same partite sets are coloured using the same colour in G. The colours of the new vertices in Γ_1 and Γ_2 denote the new unfolded partite sets.

The following theorems discusses the eigenvalues of the unfoldings in terms of the base bipartite graph. Let $\sigma(A)$ denote the eigenvalues of the matrix A.

Theorem 4.1. *[1]If $p \geq q$, then*

$$\sigma(A(\Gamma_1)) = \sqrt{2} \times \sigma(A(G)) \cup \{\underbrace{0 \ldots 0}_{q-times}\}$$

$$\sigma(A(\Gamma_2)) = \sqrt{2} \times \sigma(A(G)) \cup \{\underbrace{0 \ldots 0}_{p-times}\}$$

and if $p = q$, then $A(\Gamma_1)$ and $A(\Gamma_2)$ are cospectral.

Recall that if a graph G has no isolated vertices, then the normalized Laplacian is given by $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$ where $D(G)$ is the degree matrix of G.

Theorem 4.2. *[1]Suppose G has no isolated vertices. If $p \geq q$, then*

$$\sigma(\mathcal{L}(\Gamma_1)) = \sqrt{2} \times \sigma(\mathcal{L}(G)) \cup \{\underbrace{1 \dots 1}_{q-times}\}$$

$$\sigma(\mathcal{L}(\Gamma_2)) = \sqrt{2} \times \sigma(\mathcal{L}(G)) \cup \{\underbrace{1 \dots 1}_{p-times}\}$$

and if $p = q$, then $\mathcal{L}(\Gamma_1)$ and $\mathcal{L}(\Gamma_2)$ are cospectral.

Hence, if G has no isolated vertices and has equal partition sizes ($p = q$), the bipartite graphs Γ_1 and Γ_2 are cospectral for the adjacency matrix as well as for the normalized Laplacian. These two results are also be generalized by unfolding each partition n-times instead of twice. Adjacency matrices of such unfoldings are given by:

$$A(\Gamma_1) = \begin{bmatrix} 0 & B & B & \dots & B \\ B^T & 0 & 0 & \dots & 0 \\ B^T & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & 0 & 0 & \dots & 0 \end{bmatrix}$$

$$A(\Gamma_2) = \begin{bmatrix} 0 & B^T & B^T & \dots & B^T \\ B & 0 & 0 & \dots & 0 \\ B & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & 0 & 0 & \dots & 0 \end{bmatrix}$$

In this case, $A(\Gamma_1)$ and $A(\Gamma_2)$ are square matrices of orders $p+nq$ and $np+q$ respectively. The following theorems discusses the eigenvalues of these new graphs.

Theorem 4.3. *[14]If $p \geq q$, then*

$$\sigma(A(\Gamma_1)) = \sqrt{n} \times \sigma(A(G)) \cup \underbrace{\{0 \ldots 0\}}_{(n-1)q-times}$$

$$\sigma(A(\Gamma_2)) = \sqrt{n} \times \sigma(A(G)) \cup \underbrace{\{0 \ldots 0\}}_{(n-1)p-times}$$

and if $p = q$, then $A(\Gamma_1)$ and $A(\Gamma_2)$ are cospectral.

Theorem 4.4. *[14]Suppose G has no isolated vertices. If $p \geq q$, then*

$$\sigma(\mathcal{L}(\Gamma_1)) = \sqrt{n} \times \sigma(\mathcal{L}(G)) \cup \underbrace{\{1 \ldots 1\}}_{(n-1)q-times}$$

$$\sigma(\mathcal{L}(\Gamma_2)) = \sqrt{n} \times \sigma(\mathcal{L}(G)) \cup \underbrace{\{1 \ldots 1\}}_{(n-1)p-times}$$

and if $p = q$, then $\mathcal{L}(\Gamma_1)$ and $\mathcal{L}(\Gamma_2)$ are cospectral.

Hence, similarly when G has no isolated vertices and has equal partition sizes ($p = q$), the bipartite graphs Γ_1 and Γ_2 are cospectral for the adjacency matrix as well as the normalized Laplacian matrix.

Remark 4.5. *The matrix B is a 0-1 matrix. In both of these constructions in [1] and [14], if B is chosen in such a way that the maximum row sum of B is different than the maximum column sum of B, then the bipartite graphs Γ_1 and Γ_2 are non-isomorphic. The maximum row sum and the maximum column sum of B corresponds to the vertex with maximum degree in partition X and in the partition Y of the graph G respectively.*

Let us discuss how to generalize this idea further. Let n be a positive integer, k be any divisor of n and $\sigma(n)$ be the number of divisors of n. Consider a bipartite graph G such that the vertex set is partitioned as $V(G) = X \cup Y$. Take $\frac{n}{k}$ copies of the partition X and form a set $W = X_1 \cup X_2 \cup \cdots \cup X_{\frac{n}{k}}$. Take k copies of Y and form a set $Z = Y_1 \cup Y_2 \cup \cdots \cup Y_k$. Use the independent sets W and Z as partitions to construct a bipartite graph Γ_k as follows: For all $1 \leq i \leq \frac{n}{k}$ and $1 \leq j \leq k$, draw edges between X_i and Y_j as given by the edges between X and Y. For a fixed partition X_i, the partition Y is '*unfolded*' k times. Since there are $\frac{n}{k}$ such i's, *the total number of unfoldings* is $k \times \frac{n}{k} = n$. Let $F_k = A(\Gamma_k)$ be the adjacency matrix of the unfolding Γ_k of graph G. Then F_k is a square matrix of order $\frac{n}{k}p + kq$ given by:

$$F_k = \begin{bmatrix} 0 \cdots 0 & B \cdots B \\ \vdots \ddots \vdots & \vdots \ddots \vdots \\ 0 \cdots 0 & B \cdots B \\ B^T \cdots B^T & 0 \cdots 0 \\ \vdots \ddots \vdots & \vdots \ddots \vdots \\ B^T \cdots B^T & 0 \cdots 0 \end{bmatrix}$$

There are $\frac{n}{k}$ columns whose first entry is a zero block of order p and k columns whose first entry is the block B. For a fixed n, there are $\sigma(n)$ such possible F_k's.

Theorem 4.6. *[14] Let $p \geq q$. Consider the family of adjacency matrices F_k constructed above. Then,*

$$\sigma(A(\Gamma_k)) = \sigma(F_k) = \sqrt{n} \times \sigma(A(G)) \cup \underbrace{\{0, 0, \ldots 0\}}_{[(\frac{n}{k}-1)p+(k-1)q]-times}$$

then $A(\Gamma_k) \oplus 0_{(n-\frac{n}{k})p+(1-k)q}$, with k varying over the set of all divisors of n, are cospectral.

Similarly for normalized Laplacian matrix, we can write,

Theorem 4.7. *Suppose G has no isolated vertices. Let $p \geq q$. Consider the family of adjacency matrices F_k constructed above. Then*

$$\sigma(\mathcal{L}(\Gamma_k)) = \sqrt{n} \times \sigma(\mathcal{L}(G)) \cup \underbrace{\{1, 1, \ldots 1\}}_{[(\frac{n}{k}-1)p+(k-1)q]-times}$$

If $p = q$, then for any divisior k of n, the pair $\mathcal{L}(\Gamma_k)$ and $\mathcal{L}(\Gamma_{\frac{n}{k}})$ is cospectral.

Example 4.8. *Let $n = 6$, then divisors of n are $\{1, 2, 3, 6\}$. Hence, four such matrices F_k are possible. Note that the nodes in the figures of Table 4.1 represent the copies of the partite sets of the base bipartite graph. We have not specified the graph G.*

In particular,

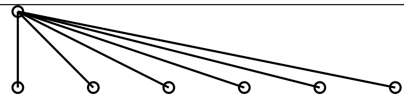
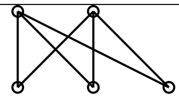
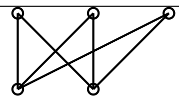
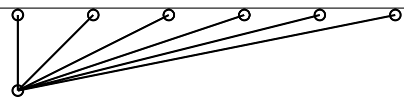
k	Γ_k	$\sigma(F_k)$
1		$\sqrt{6} \times \sigma(A(G)) \cup \underbrace{\{0, 0, \ldots 0\}}_{5p-times}$
2		$\sqrt{6} \times \sigma(A(G)) \cup \underbrace{\{0, 0, \ldots 0\}}_{3p-times}$
3		$\sqrt{6} \times \sigma(A(G)) \cup \underbrace{\{0, 0, \ldots 0\}}_{3p-times}$
6		$\sqrt{6} \times \sigma(A(G)) \cup \underbrace{\{0, 0, \ldots 0\}}_{5p-times}$

Table 4.1: Unfoldings Γ_1, Γ_2, Γ_3 and Γ_6 for $n = 6$

Theorem 4.9. *If G has no isolated vertices, then the graphs Γ_k and $\Gamma_{n/k}$ are cospectral for the adjacency matrix as well as the normalized Laplacian.*

Proof. The result follows from Theorem 4.6 and Theorem 4.7. $\square$

So far, the condition 'maximum row sum is not same as maximum column sum of B' was sufficient to have the constructed cospectral graphs to be nonisomorphic. The following theorem gives the equivalent conditions for the isomorphism. Let k and l to be any two natural numbers and let $n = kl$ to be the fixed total number of unfoldings. Let $p = q$. Since k and l both are divisors of n, consider the graphs Γ_k and Γ_l which are cospectral for the adjacency matrix as well the normalized Laplacian. We assume the matrix B to have no zero rows as well as no zero columns. The reason for it is that if B has a zero row and no zero columns, then Γ_k and Γ_l have l and k isolated vertices respectively and they might not be cospectral with respect to the normalized Laplacian.

Theorem 4.10. *[12] Suppose B is a square matrix without any zero rows or zero columns and $k \neq l$. Then, the graphs Γ_k and Γ_l are nonisomorphic if and only if B non PET.*

We refer to [12] for the proof of this theorem which is based on Hall's theorem. We will give an alternate proof in the general setting in the next section. Two of the properties of the unfoldings are:

Proposition 4.11. *The unfolding preserves the diameter, $diam(\Gamma_k) = diam(G)$.*

Proposition 4.12. *If G has no isolated vertices, then the unfolding preserves the number of connected components, $n(F_k) = n(G)$.*

Proof. If G has no isolated vertices, then from Theorem 4.7, the multiplicity of eigenvalue zero in $\sigma(\mathcal{L}(\Gamma_k))$ is same as in $\sigma(\mathcal{L}(G))$. The multiplicity of eigenvalue zero of the Laplacian matrix corresponds to the number of connected components. The multiplicity of eigenvalue zero is same for the Laplacian and the normalized Laplacian. Hence, unfolding preserves the number of connected components. $\square$

4.2 Generalization of the unfolding operation

Consider the matrices of the form $L = \begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$ such V and W are $m \times n$ and $n \times m$ matrices respectively.

Lemma 4.13. *The matrix $L = \mathcal{P}(V, W)$ satisfies C/T property. Two matrices $L_1 = \mathcal{P}(V_1, W_1)$ and $L_2 = \mathcal{P}(V_2, W_2)$ satisfy $C/M/T$ property.*

Proof. The commuting property is trivially satisfies since $\mathcal{I}(U, X)$ is a zero matrix which commutes with $\mathcal{P}(V, W)$.

Let f be a monomial, then for $i = 1, 2$ $f(L_i) = 0^{t-s}\mathcal{P}(V_i, W_i)^s$. Hence, $f(L_i)$ is either $\mathcal{P}(V_i, W_i)^t = L_i^t$ for some t or a zero matrix. Hence, $tr[f(L_1)] = tr[f(L_2)]$ holds for all monomials if and only if $tr[L_1^t] = tr[L_2^t]$ holds for all t. Hence, the monomial property is satisfied.

Suppose $s = 2r$ and $s \neq 0$. If $t \neq s$, then $tr[U^{t-s}(VW)^r] = 0$ and $tr[X^{t-s}(WV)^r] = 0$. If $t = s$, then $tr[U^{t-s}(VW)^r] = tr[(VW)^r]$ and $tr[X^{t-s}(WV)^r] = tr[(WV)^r]$. But from Proposition 3.11, we have $tr[(VW)^r] = tr[(WV)^r]$. Hence, the trace property is also satisfied. $\square$

Now that we've shown that the matrices of the form $L = \mathcal{P}(V, W)$ satisfy $C/M/T$ property, we can apply the isomorphism results from the previous chapter.

4.2.1 Construction I-A: Bipartite graph

Let $L = \begin{bmatrix} 0 & V \\ V^T & 0 \end{bmatrix} = \mathcal{P}(V, V^T)$, $H = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} = \mathcal{P}(B, B^T)$ and $H^{\#} = \begin{bmatrix} 0 & B^T \\ B & 0 \end{bmatrix} = \mathcal{P}(B^T, B)$ be the adjacency matrices of the bipartite graphs such that V and B are $m \times n$ and $p \times q$ 0-1 matrices respectively.

Theorem 4.14. *The bipartite graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ are cospectral if and only at least one of G_L or G_H admits equal partition sizes.*

Proof. From Lemma 4.13, L satisfies C/T property. Then the result follows as a corollary of Theorem 3.20. $\square$

The next lemma gives a relation between the normalized Laplacian of partitioned tensor product and its individual components.

Lemma 4.15. *Let G_1 and G_2 be two graphs with no isolated vertices such that the corresponding adjacency matrices are bipartitioned. Then, $\mathcal{L}(G_{A(G_1) \otimes_p A(G_2)}) = 2I - \mathcal{L}(G_1) \otimes_p \mathcal{L}(G_2)$.*

Proof. Let $D(G_1)$ and $D(G_2)$ denote the degree matrices corresponding to the graphs G_1 and G_2 respectively. Since G_1 and G_2 do not have any isolated vertices, $D(G_1)^{-1/2}$ and $D(G_2)^{-1/2}$ exists. Then,

$$\begin{aligned}
\mathcal{L}(G_{A(G_1) \otimes_p A(G_2)}) &= I - D(G_{A(G_1) \otimes_p A(G_2)})^{-1/2}(A(G_1) \otimes_p A(G_2))D(G_{A(G_1) \otimes_p A(G_2)})^{-1/2} \\
&= I - (D(G_1)^{-1/2} \otimes_p D(G_2)^{-1/2})(A(G_1) \otimes_p A(G_2))(D(G_1)^{-1/2} \otimes_p D(G_2)^{-1/2}) \\
&= I - (D(G_1)^{-1/2}A(G_1)D(G_1)^{-1/2}) \otimes_p (D(G_2)^{-1/2}A(G_2)D(G_2)^{-1/2}) \\
&= I - [(I - \mathcal{L}(G_1)) \otimes_p (I - \mathcal{L}(G_2))] \\
&= I \otimes_p \mathcal{L}(G_2) + \mathcal{L}(G_1) \otimes_p I - \mathcal{L}(G_1) \otimes_p \mathcal{L}(G_2) \\
&= 2I - \mathcal{L}(G_1) \otimes_p \mathcal{L}(G_2)
\end{aligned}$$

The third step in the above computations uses Proposition 3.4 and the matrices $I \otimes \mathcal{L}(G_2)$ and $\mathcal{L}(G_1) \otimes_p I$ are identity matrices. $\square$

Theorem 4.16. *Let G_L and G_H be bipartite graphs with no isolated vertices, then the graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H\#}$ are cospectral for the normalized Laplacian if and only if at least one of G_L or G_H admits equal partition sizes.*

Proof. Equivalently, we must show that the corresponding normalized Laplacian matrices $\mathcal{L}(G_{L\otimes_p H})$ and $\mathcal{L}(G_{L\otimes H\#})$ have the same eigenvalues if and only if $m = n$ or $p = q$. Let $D(G_L)$, $D(G_H)$ and $D(G_{H\#})$ denote the degree matrices for the graphs G_L, G_H and $G_{H\#}$ respectively. Suppose either $m = n$ or $p = q$ holds. Then,

Case 1: Suppose $m = n$

Let $D(G_L) = \mathcal{I}(C_1, C_2)$ where C_1 and C_2 are diagonal matrices. Since G_L does not have any isolated vertices, $C_1^{-1/2}$ and $C_2^{-1/2}$ exists. Let $E = C_1^{-1/2}VC_2^{-1/2}$. Then from Lemma 2.7. and the assumption that $m = n$, the matrix E is orthogonally equivalent to its transpose. Hence, there exists two orthogonal matrices R_1 and R_2 such that $E = R_2^T E^T R_1$. Let $R = \mathcal{P}(R_1, R_2)$. Then,

$$\begin{aligned}
\mathcal{L}(G_L) &= I - D(G_L)^{-1/2} L D(G_L)^{-1/2} \\
&= I - \begin{bmatrix} 0 & C_1^{-1/2}VC_2^{-1/2} \\ C_2^{-1/2}V^TC_1^{-1/2} & 0 \end{bmatrix} \\
&= I - \begin{bmatrix} 0 & R_2^T(C_2^{-1/2}V^TC_1^{-1/2})R_1 \\ R_2(C_1^{-1/2}VC_2^{-1/2})R_1^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix}^T \left(I - \begin{bmatrix} 0 & C_1^{-1/2}VC_2^{-1/2} \\ C_2^{-1/2}V^TC_1^{-1/2} & 0 \end{bmatrix} \right) \begin{bmatrix} 0 & R_1 \\ R_2 & 0 \end{bmatrix} \\
&= R^T \mathcal{L}(G_L) R
\end{aligned}$$

The permutation matrix $Q = \mathcal{P}(I_p, I_q)$ satisfies $Q^T\mathcal{L}(G_H)Q = \mathcal{L}(G_{H\#})$. The partitioned tensor product $P = R \otimes_p Q$ is also an orthogonal matrix and from Proposition 3.4 it satisfies $P^T(\mathcal{L}(G_L) \otimes_p \mathcal{L}(G_H))P = \mathcal{L}(G_L) \otimes_p \mathcal{L}(G_{H\#})$.

Case 2: Suppose $p = q$

Let $D(G_H) = \mathcal{I}(D_1, D_2)$ where D_1 and D_2 are diagonal matrices. Then, $D(G_{H\#}) = \mathcal{I}(D_2, D_1)$. Since G_H does not have any isolated vertices, $D_1^{-1/2}$ and $D_2^{-1/2}$ exists. Let $F = D_1^{-1/2}BD_2^{-1/2}$. Then from Lemma 2.7. and the assumption that $p = q$, the matrix F is orthogonally equivalent to its transpose. Hence, there exists two orthogonal matrices

Q_1 and Q_2 such that $Q_1^T F Q_2 = F^T$, that is, $Q_1^T(D_1^{-1/2} B D_2^{-1/2})Q_2 = D_2^{-1/2} B^T D_1^{-1/2}$. Let $Q = \mathcal{I}(Q_1, Q_2)$. Then,

$$\begin{aligned}
\mathcal{L}(G_{H\#}) &= I - D(H^{\#})^{-1/2} H^{\#} D(G_{H\#})^{-1/2} \\
&= I - \begin{bmatrix} 0 & D_2^{-1/2} B^T D_1^{-1/2} \\ D_1^{-1/2} B D_2^{-1/2} & 0 \end{bmatrix} \\
&= I - \begin{bmatrix} 0 & Q_1^T(D_1^{-1/2} B D_2^{-1/2})Q_2 \\ Q_1(D_2^{-1/2} B^T D_1^{-1/2})Q_2^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}^T \left(I - \begin{bmatrix} 0 & D_1^{-1/2} B D_2^{-1/2} \\ D_2^{-1/2} B^T D_1^{-1/2} & 0 \end{bmatrix} \right) \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \\
&= Q^T (I - D(G_H)^{-1/2} H D(G_H)^{-1/2}) Q \\
&= Q^T \mathcal{L}(G_H) Q.
\end{aligned}$$

Let R be the identity matrix such that $R^T \mathcal{L}(G_L) R = \mathcal{L}(G_L)$. The partitioned tensor product $P = R \otimes_p Q$ is also an orthogonal matrix and from Proposition 3.4 it satisfies $P^T(\mathcal{L}(G_L) \otimes_p \mathcal{L}(G_H))P = \mathcal{L}(G_L) \otimes_p \mathcal{L}(G_{H\#})$.

In both of the cases, we have shown that the matrices $\mathcal{L}(G_L) \otimes_p \mathcal{L}(G_H)$ and $\mathcal{L}(G_L) \otimes_p \mathcal{L}(G_{H\#})$ are orthogonally similar. Since G_L and G_H have no isolated vertices, we have, $\mathcal{L}(G_{L \otimes_p H}) = 2I - \mathcal{L}(G_L) \otimes_p \mathcal{L}(G_H)$ as well as $\mathcal{L}(G_{L \otimes_p H\#}) = 2I - \mathcal{L}(G_L) \otimes_p \mathcal{L}(G_{H\#})$ from Lemma 4.15. Then, the matrices $\mathcal{L}(G_{L \otimes_p H})$ and $\mathcal{L}(G_{L \otimes_p H\#})$ are orthogonally similar, and hence have the same eigenvalues.

Now conversely assume that the graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are cospectral for normalized Laplacian, then the corresponding normalized Laplacian matrices have the same order. Then, $mp + nq = mq + np$, that is, $(m - n)(p - q) = 0$. Hence, either $m = n$ or $p = q$ holds. □

We have seen that the condition on the bipartite graphs G_L and G_H to have no isolated vertices is required for the cospectrality with respect to the normalized Laplacian. This condition is not required for the cospectrality with respect to the adjacency matrix. In this chapter, let us assume that both G_L and G_H have no isolated vertices. This is equivalent to assuming that both V and B have no zero rows as well as no zero columns.

Let G be a bipartite graph whose vertex set is partitioned as $V(G) = X \cup Y$. We say that an automorphism f of G *fixes* the partite sets if $f(X) = X$ and $f(Y) = Y$, and *interchanges* the partite sets if $f(X) = Y$ and $f(Y) = X$.

Proposition 4.17. *The bipartite graph G corresponding to the adjacency matrix $\begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ admits an automorphism that interchanges its partite sets if and only if the biadjacency matrix B is PET.*

Proof. The permutation matrix corresponding to such an automorphism is of the form $P = \mathcal{P}(P_1, P_2)$ for some permutation matrices P_1 and P_2 of same size. Since P satisfies $P^T A(G) P = A(G)$, equivalently $P_1^T B P_2 = B^T$ holds, that is, B is PET. $\square$

Let us restate property η_1 for the bipartite graphs.

Definition 4.18. *The bipartite graphs G_L and G_H are said to satisfy property η_1, if whenever the bipartite graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are isomorphic, there exists an isomorphism between them that respects the partite sets.*

The next theorem gives equivalent conditions for the graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ to be isomorphic.

Theorem 4.19. *Let the bipartite graphs G_L and G_H satisfy property η_1, then $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are isomorphic if and only if at least one of G_L and G_H admits an automorphism that interchanges its partite sets.*

Proof. Suppose at least one of G_L and G_H admits an automorphism that interchanges its partite sets, then from Lemma 3.25, the bipartite graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are isomorphic. Now conversely, suppose they are isomorphic, since G_L and G_H satisfy property η_1, then from Lemma 3.27 at least one of V or B is PET. Hence, from Proposition 4.17 at least one of G_L and G_H admit an automorphism that interchanges its partite sets.

$\square$

Now let us discuss when bipartite graphs G_L and G_H satisfy property η_1 and how our construction relates with some of the existing results.

Connected bipartite graphs

Let G_1 and G_2 be two isomorphic bipartite graphs whose vertex sets are partitioned as $V(G_1) = X_1 \cup Y_1$ and $V(G_2) = X_2 \cup Y_2$. If f is an isomorphism from G_1 to G_2, we say that f *respects* the partite sets if it satisfies either $f(X_1) = X_2$ and $f(Y_1) = Y_2$ or $f(X_1) = Y_2$ and $f(Y_1) = X_2$.

Lemma 4.20. *If G_1 and G_2 are two connected isomorphic bipartite graphs, then any isomophism between them respects the partite sets.*

Proof. Let $x_1, x_2 \in X_1$. Then, there exists a path between x_1 and x_2 due to connectedness of G_1. Since, these vertices belong to the same partition, this path has even length. The isomorphism f preserves the distance between vertices. Hence, the path between $f(x_1)$ and $f(x_2)$ has even length. Since G_2 is bipartite, $f(x_1)$ and $f(x_2)$ belong to either X_2 or Y_2. Since, for a given $x_1 \in X_1$, all $x_2 \in X_1$ are at even distance from x_1, then either $f(X_1) = X_2$ or $f(X_1) = Y_2$. The result follows. □

Corollary 4.21. *If G is a connected bipartite graph, then its every automorphism repects the partite sets.*

Next, we provide a lemma which is useful for proving the theorem 4.23.

Lemma 4.22. *[24] If G_L and G_H are two connected bipartite graphs, then $G_L \times G_H$ has exactly two connected bipartite components.*

We observe that the disjoint union of the bipartite graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H\#}$ is in fact same as the direct product $G_{L\otimes H} = G_L \times G_H$. Hence, if G_L and G_H are connected, then from Lemma 4.22, the two connected components of $G_{L\otimes H}$ are precisely $G_{L\otimes_p H}$ and $G_{L\otimes_p H\#}$.

Theorem 4.23. *If the bipartite graphs G_L and G_H are connected, then they satisfy property η_1.*

Proof. Suppose G_L and G_H are connected and $G_{L\otimes_p H}$ and $G_{L\otimes_p H\#}$ are isomorphic. Since $G_{L\otimes_p H}$ and $G_{L\otimes_p H\#}$ are connected, it follows from Lemma 4.20 that any isomorphism between them respects the partite sets. Hence, G_L and G_H satisfy property η_1. □

Hammack [9] had already given an equivalent condition for the isomorphism of the components of direct product of two connected bipartite graphs. We state it here as a corollary.

Corollary 4.24. *[9] Let G_1 and G_2 be two connected bipartite graphs. The two components of $G_1 \times G_2$ are isomorphic if and only if if at least one of G_1 or G_2 admits an automorphism that interchanges its partite sets.*

Proof. Follows from the Theorems 4.19, and 4.23. □

Since we are interested in the construction of cospectral nonisomorphic graphs, we use this result as a cospectral construction.

Theorem 4.25. *Let G_L and G_H be connected bipartite graphs and let at least one of them have equal partition sizes. Then, the bipartite graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^\#}$ are cospectral for the adjacency as well as the normalized Laplacian and they are nonisomorphic if and only if both G_L and G_H do not admit an automorphism that interchanges its partite sets.*

Proof. Cospectrality follows from Theorem 4.14 and Theorem 4.16,. The condition for nonisomorphism follows from Theorem 4.19 and Theorem 4.23. □

Example 4.26. *Suppose G_L admits equal partition sizes and G_H does not. Then, G_H automatically does not admit an automorphism that interchanges its partite sets, since B is non-square and hence non-PET. Then, finding cospectral nonisomorphic graphs using the theorem 4.25 requires finding*

1. *A square 0-1 non-PET matrix V which corresponds to a connected bipartite graph G_L*

2. *A non-square 0-1 matrix B which corresponds to a connected bipartite graph G_H.*

The smallest size candidates for such a V are the following 3×3 matrices:

$$V_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; V_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; \textit{and } V_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

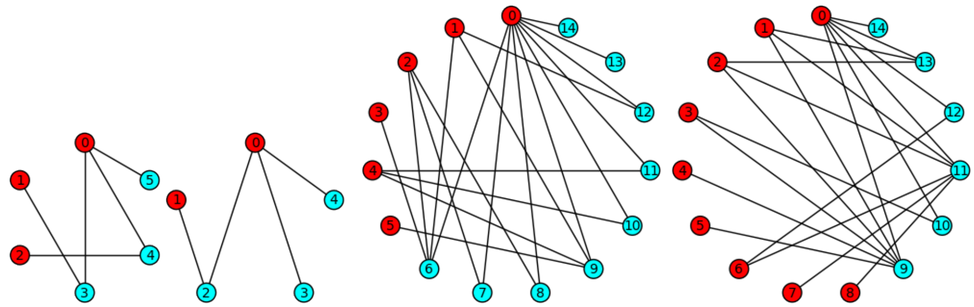


Figure 4.2: Example 1 of construction I-A: G_{L_1}, G_{H_1}, $G_{L_1 \otimes H_1}$ and $G_{L_1 \otimes H_1^\#}$

Let us denote the corresponding bipartite graph by G_{L_1}, G_{L_2}, and G_{L_3} respectively. The smallest size candidates for such a B (which cannot be obtained by the construction in the next subsection) are the following 2×3 matrices:

$$B_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \ \textit{and } B_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Let us denote the corresponding bipartite graph by G_{H_1}, G_{H_2}, and G_{H_3} respectively. Hence, we can obtain the following nine different pairs of bipartite graphs $G_{L_i \otimes H_j}$ and $G_{L_i \otimes H_j^\#}$ for $1 \leq i, j \leq 3$ whose biadjacency matrices are $V_i \otimes B_j$ and $V_i \otimes B_j^T$ respectively. Note that these are the smallest sized graph pairs possible using this construction and they are of order 15. It can be verified that all 9 pairs of graphs are nonisomorphic and cospectral for the adjacency as well as the normalized Laplacian but not for the Laplacian matrix. Figure 4.2 shows the graphs G_{L_1}, G_{H_1}, $G_{L_1 \otimes H_1}$ and $G_{L_1 \otimes H_1^\#}$ respectively.

Biregular bipartite graphs

The following theorem gives another partial characterization of property η_1. A *biregular* bipartite graph is a bipartite graph such that the vertices from the same partite sets have the same degrees.

Theorem 4.27. *If G_L is a biregular bipartite graph with distinct degrees, then G_L and G_H satisfy property η_1.*

Proof. Let G_L be a biregular bipartite graph and let the biadjacency matrix V have constant row sum l and constant column sum k such that $l \neq k$, where $1 \leq k \leq m$ and $1 \leq l \leq n$.

Suppose $k < l$. Let $\Gamma_1 = G_{L\otimes_p H}$ and $\Gamma_2 = G_{L\otimes_p H^{\#}}$ be isomorphic. Then, there exists a permutation matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ such that $P^T(L \otimes_p H)P = L \otimes_p H^{\#}$. Consider,

$$L \otimes_p H = \begin{bmatrix} 0 & V \otimes B \\ V^T \otimes B^T & 0 \end{bmatrix} \ and \ L \otimes_p H^{\#} = \begin{bmatrix} 0 & V \otimes B^T \\ V^T \otimes B & 0 \end{bmatrix}.$$

Let the vertex sets be partitioned as $V(\Gamma_1) = X_1 \cup Y_1$ and $V(\Gamma_2) = X_2 \cup Y_2$ as shown by the corresponding partitioned adjacency matrices. Let f be an isomorphism from Γ_1 to Γ_2. Let b_i and b_i' denote the i^{th} row sum of the matrices B and B^T respectively. Let $x_i \in X_1$ be the vertex of maximum degree in this set and suppose $f(x_i) \in Y_2$. Then, $d_{\Gamma_1}(x_i) = lb_i$ for some $1 \leq i \leq p$. Then, $d_{\Gamma_2}(f(x_i)) = kb_j$ for some $1 \leq j \leq p$. Since the isomorphism preserves the degrees, we have $lb_i = kb_j$. Since x_i has maximum degree in X_1, $b_i \geq b_j$ for any $1 \leq j \leq p$. Then, $kb_j \geq lb_j$. Since G_H has no isolated vertices, $b_j \neq 0$. Hence, $k \geq l$ which is a contradiction to the initial assuption that $k < l$. Hence, if $x_i \in X_1$, then $f(x_i) \in X_2$. Now removing the vertex x_i of the maximum degree in X_1 from Γ_1 and $f(x_i)$ in X_2 from Γ_2 respectively, we apply the same argument on the induced graphs to conclude that $f(X_1) = X_2$ and hence $f(Y_1) = Y_2$.

In case $k > l$, consider a vertex y_i of maximum degree in Y_1. Similarly, we can show that $f(X_1) = X_2$ and $f(Y_1) = Y_2$. Hence, G_L and G_H satisfy property η_1. □

Since we are interested in the construction of cospectral nonisomorphic graphs, we use this result as a cospectral construction.

Theorem 4.28. *Let G_L be a biregular bipartite graph with distinct degrees and let G_H have equal partition sizes. Then, the bipartite graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^{\#}}$ are cospectral for adjacency as well as normalized Laplacian matrix and they are nonisomorphic if and only if G_H does not admit an automorphism that interchanges its partite sets.*

Proof. Since G_L is a biregular bipartite graph with distinct degrees, suppose the corresponding $m \times n$ biadjacency matrix V has constant row sums k and constant column sums l. Since, the sum of row sums must be the same as the sum of column sums, we have $km = ln$. But $k \neq l$, hence $m \neq n$. Hence, G_L has unequal partition sizes. Since G_H has equal partition sizes, then cospectrality follows from Theorem 4.14 and Theorem 4.16,. Now since G_L has unequal partitions sizes, it doesn't admit an automorphism that interchanges its partite sets. Hence, the condition for nonisomorphism follows from Theorem 4.19 and Lemma 4.27. □

Example 4.29. *Unlike Theorem 4.25., finding cospectral nonisomorphic graphs using Theorem 4.28 requires finding:*

1. *a non-square 0-1 matrix V with constant row sums (different than) and constant column sums with no zero rows and no zero columns*

2. *a square 0-1 non-PET matrix B with no zero rows and no zero columns.*

The smallest size candidates for such a V (which cannot be obtained by the construction in the previous subsection or the special case we discuss next) are the following matrices:

$$V_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \textit{and} \;\; V_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We observe that V_4 and V_5 correspond to union of complete bipartite graphs (trivial extension of the special case we discuss next). Excluding such cases, the smallest such matrix V is a 3×6 *matrix:*

$$V_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let us denote the corresponding bipartite graph by G_{L_4}, G_{L_5} and G_{L_6} respectively. The smallest size candidates for such a B are the following 3×3 *matrices:*

$$B_4 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B_5 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; \textit{and} \; B_6 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

Let us denote the corresponding bipartite graph by G_{H_4}, G_{H_5} and G_{H_6}. Hence, we can obtain the following nine different pairs of graphs $G_{L_i \otimes H_j}$ and $G_{L_i \otimes H_j^{\#}}$ for $4 \leq i, j \leq 6$ whose biadjacency matrices are $V_i \otimes B_j$ and $V_i \otimes B_j^T$ respectively. Note that these are the smallest size graph pairs possible using this construction, the one in the figure 4.3 is of order 27 and shows the graphs G_{L_6}, G_{H_4}, $G_{L_6 \otimes H_4}$ and $G_{L_6 \otimes H_4^{\#}}$ respectively. It can be verified that all nine

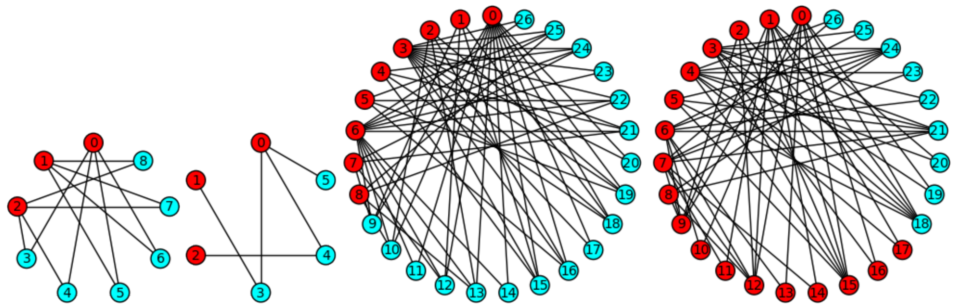


Figure 4.3: Example 2 of construction I-A: G_{L_6}, G_{H_4}, $G_{L_6 \otimes H_4}$ and $G_{L_6 \otimes H_4^{\#}}$

pairs of graphs are nonisomorphic and cospectral for the adjacency as well as the normalized Laplacian but not for the Laplacian matrix.

Now as a corollary, we obtain the main result of Ji, Gong and Wang.

Corollary 4.30. *[12] Let $V = J_{m,n}$ such that $m \neq n$ and let B is a square matrix. Then, the bipartite graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ are cospectral for adjacency as well as the normalized Laplacian matrix and they are isomorphic if and only if B is PET.*

Proof. Since $V = J_{m,n}$ and $m \neq 0$, the corresponding bipartite graph G_L is a biregular bipartite graph with distinct degrees. Hence, the result follows as a corollary of Theorem 4.28. ☐

Note that this corollary follows from the most general result Theorem 4.19 which uses cancellation law for matrices (Theorem 2.14). The cancellation idea of Ji Gong and Wang is different. They show that when $V = J_{m,n}$ and B has no zero rows or zero columns, $V \otimes B$ and $V \otimes B^T$ are permutationally equivalent if and only if B is PET using another approach based on Hall's Theorem.

Example 4.31. *For this construction, the smallest such V's are $V_7 = J_{1,2}$, $V_8 = J_{1,3}$ and $V_9 = J_{2,3}$. Let us denote the corresponding bipartite graph by G_{L_7}, G_{L_8} and G_{L_9} respectively. The smallest such B's can be taken to be B_4, B_5 and B_6. Hence, we can obtain the following nine different pairs of graphs $G_{L_i \otimes H_j}$ and $G_{L_i \otimes H_j^{\#}}$ for $7 \leq i \leq 9$ and $4 \leq j \leq 6$ whose biadjacency matrices are $V_i \otimes B_j$ and $V_i \otimes B_j^T$ respectively. Note that these are the smallest size graph pairs possible using this construction, the one in the Figure 4.4 is of order 9 and shows the graphs G_{L_7}, G_{H_7}, $G_{L_7 \otimes H_4}$ and $G_{L_7 \otimes H_4^{\#}}$ respectively (the original unfolding*

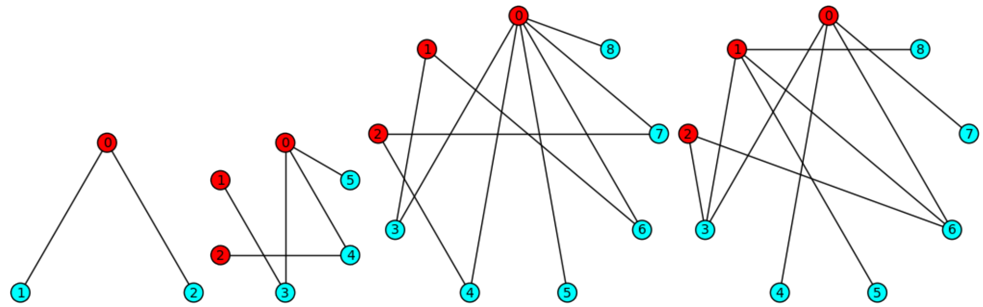


Figure 4.4: Example 3 of construction I-A: G_{L_7}, G_{H_7}, $G_{L_7 \otimes H_4}$ and $G_{L_7 \otimes H_4^{\#}}$

construction of Butler). It can be verified that all nine pairs of graphs are nonisomorphic and cospectral for the adjacency as well as the normalized Laplacian but not for the Laplacian matrix.

Chapter 5

Some particular cospectral constructions based on partitioned tensor product

In the previous chapter, we showed that the matrices of the form $\begin{bmatrix} 0 & V \\ W & 0 \end{bmatrix}$ satisfy $C/M/T$ property and discussed the corresponding Construction - I (Theorem 3.20). We showed that this construction produces cospectral bipartite graphs which are cospectral for the normalized Laplacian as well.

In this chapter, we first discuss the corresponding Construction - II (Theorem 3.21) for the bipartite case and give a partial characterization of property η_2. We then give four more candidates for matrices which satisfy $C/M/T$ property and discuss only the corresponding Construction - I . We give partial characterization of property η_1 required in the investigation of the isomorphism.

Similarly, Construction - II can also be discussed for these candidates and the isomorphism results obtained in the previous chapter can be applied. But characterization of property η_2 is required which we leave it as an open problem. Also note that all the constructions in this chapter are cospectral only for the adjacency matrix. We have not yet shown if they are cospectral with respect to some other matrices.

5.1 Construction II-A: Bipartite graph

Let $L_i = \begin{bmatrix} 0 & V_i \\ V_i^T & 0 \end{bmatrix}$; $i = 1, 2$ and $H = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$ be the adjacency matrices of the bipartite graphs such that V_i and B are $m_i \times n_i$ and $p \times q$ nonzero 0-1 matrices respectively.

Theorem 5.1. *Let L_1 and L_2 be cospectral. Then, the bipartite graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are cospectral if and only if $m_1 = m_2$ or $p = q$.*

Proof. From Lemma 4.13, L_1 and L_2 satisfy $C/M/T$ property. Then from Theorem 3.21, the result follows. □

The following proposition states the statements about Type-1 and Type isomorphism for the bipartite graph case.

Proposition 5.2. *G_{L_1} and G_{L_2} are isomorphic via Type-1 (Type-2) isomorphism if and only if V_2 is permutationally equivalent to V_1 (V_1^T).*

Proof. Follows from Definition 3.28. □

Theorem 5.3. *Let G_{L_1}, G_{L_2} and G_H satisfy property η_2 and let G_{L_1} and G_{L_2} have no isolated vertices. Then,*

1. *Let $p \neq q$, then $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic if and only if V_2 is permutationally equivalent to V_1.*
2. *Let $B = B^T$, then $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic if and only if V_2 is permutationally equivalent to either V_1 or V_1^T.*

Proof. **Case 1:** Suppose V_2 is permutationally equivalent to V_1
Then from Proposition 5.2, it follows that G_{L_1} and G_{L_2} are isomorphic via Type-1 isomorphism. From Lemma 3.30, the graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic.

Case 2: Suppose V_2 is permutationally equivalent to V_1^T and B is PET
Then from Proposition 5.2, it follows that G_{L_1} and G_{L_2} are isomorphic via Type-2 isomorphism. Also, the graph G_H admits an automorphism that interchanges its partite sets. From Lemma 3.30, the graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic.

Now conversely, suppose the graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic. Since G_{L_1}, G_{L_2} and G_H satisfy property η_2, from Theorem 3.32 at least one of the following two cases occurs:

Case 1: If V_1 and V_2 have no zero rows or zero columns, then V_1 is permutationally equivalent to V_2

Hence, if G_{L_1} and G_{L_2} have no isolated vertices, the graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic if and only if V_2 and V_1 are permutationally equivalent.

Case 2: Suppose $B = B^T$ and V_1 and V_2 have no zero rows or zero columns, then V_1^T is permutationally equivalent to V_2.

Since $B = B^T$, B is PET. Hence, if G_{L_1} and G_{L_2} have no isolated vertices, the graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic if and only if V_2 is permutationally equivalent to either V_1 or V_1^T.

□

We now discuss a partial characterization of property η_2.

Lemma 5.4. *If G_{L_i} for $i = 1, 2$ and G_H are connected, then they satisfy property η_2.*

Proof. Suppose G_{L_1}, G_{L_2} and G_H are connected bipartite graphs, then Lemma 4.22, the two components of the $G_{L_1 \times H}$ and $G_{L_2 \times H}$ are also connected. The bipartite graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are one of the two components of $G_{L_1 \times H}$ and $G_{L_2 \times H}$ respectively. Now suppose $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are isomorphic, from Lemma 4.20, any isomorphism between them respects the partite sets. Hence, G_{L_1}, G_{L_2} and G_H satisfy property η_2. □

The following theorem shows how to obtain cospectral nonisomorphic graphs.

Theorem 5.5. *Let G_{L_1} and G_{L_2} be cospectral bipartite graphs with no isolated vertices. Suppose G_{L_1}, G_{L_2} and G_H are connected. Then*

1. *Let $p \neq q$ and $m_1 = m_2$, then $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are cospectral. They are nonisomorphic if and only if V_2 is not permutationally equivalent to V_1.*

2. *Let $B = B^T$, then $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ are cospectral. They are nonisomorphic if and only if V_2 is permutationally equivalent to neither V_1 nor V_1^T.*

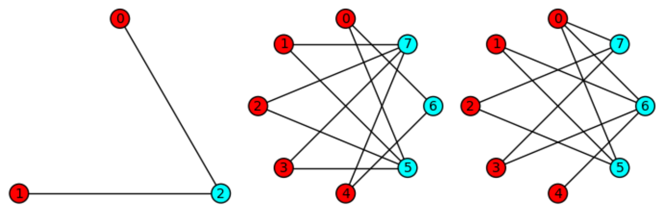


Figure 5.1: Example 1 of construction II-A: G_H, G_{L_1} and G_{L_2}

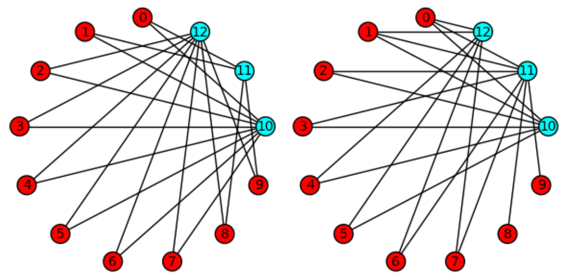


Figure 5.2: Example 1 of construction II-A: $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$

Proof. Follows from Theorem 5.1 and Theorem 5.3. $\square$

Example 5.6. *Let* $V_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$*, then the corresponding bipartite graphs* G_{L_1} *and* G_{L_2} *are connected, cospectral and nonisomorphic.* V_2 *is permutationally equivalent to neither* V_1 *nor* V_1^T*. Let* $B = \begin{bmatrix} 1 & 1 \end{bmatrix}$*, then* $p \neq q$ *and* $m_1 = m_2$*. Figure 5.1 shows the corresponding graphs* G_H*,* G_{L_1} *and* G_{L_2} *and Figure 5.2 shows the cospectral nonisomorphic graphs* $G_{L_1 \otimes_p H}$ *and* $G_{L_2 \otimes_p H}$*. This demonstrates Theorem 5.5.(1). Note that there are 3 pairs of cospectral nonisomorphic bipartite graphs on 8 vertices having the same partitioning and the example of* G_{L_1} *and* G_{L_2} *we have considered is only one of the 3. Hence, 2 more cospectral nonisomorphic graphs* $G_{L_1 \otimes_p H}$ *and* $G_{L_2 \otimes_p H}$ *can be generated for the same matrix* B*.*

Example 5.7. *Let* $V_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ *and* $V_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$*, then the corresponding bipartite graphs* G_{L_1} *and* G_{L_2} *are connected, cospectral and nonisomorphic.* V_2 *is permutationally equivalent to neither* V_1 *nor* V_1^T*. Let* $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$*, then* $B = B^T$*. Figure 5.3 shows*

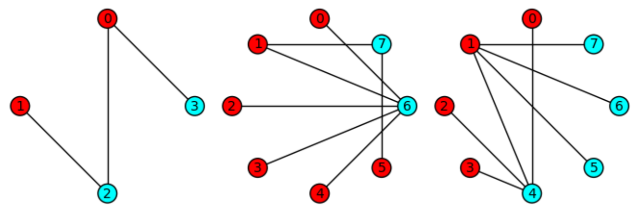


Figure 5.3: Example 2 of construction II-A: G_H, G_{L_1} and G_{L_2}

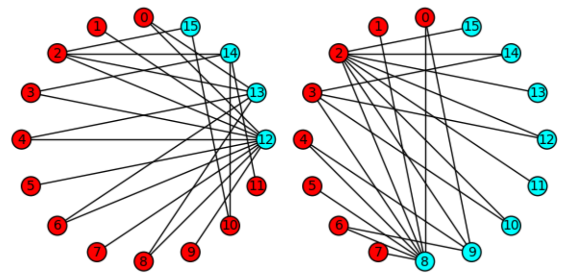


Figure 5.4: Example 2 of construction II-A: $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$

the corresponding graphs G_H, G_{L_1} and G_{L_2} and Figure 5.4 shows the cospectral nonisomorphic graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$. This demonstrates Theorem 5.5.(2). Note that there are 8 pairs of cospectral nonisomorphic bipartite graphs on 8 vertices and the example of G_{L_1} and G_{L_2} we have considered is only one of the 8. Hence, 7 more cospectral nonisomorphic graphs $G_{L_1 \otimes_p H}$ and $G_{L_2 \otimes_p H}$ can be generated for the same matrix B.

5.2 Construction I-B: Reflexive bipartite graph

In this section we apply the isomorphism results on the original construction of Godsil and Mckay. Let $L = \begin{bmatrix} I_m & V \\ V^T & I_n \end{bmatrix}$ $H = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$ and $H^{\#} = \begin{bmatrix} D & B^T \\ B & A \end{bmatrix}$ be the adjacency matrices of graphs such that V and B are $m \times n$ and $p \times q$ matrices respectively. We will be assuming that the blocks V, A, B, D are nonzero to distinguish this construction from the others.

Theorem 5.8. *The graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ are cospectral if and only if $m = n$ or A and D are cospectral.*

Proof. Follows from Theorem 3.15. $\square$

The matrix L corresponds to the adjacency matrix of a reflexive bipartite graph. Similar to the bipartite case, we have

Proposition 5.9. *The reflexive bipartite graph G_L corresponding to the adjacency matrix $L = \begin{bmatrix} I_m & V \\ V^T & I_n \end{bmatrix}$ admits an automorphism that interchanges its partite sets if and only if V is PET.*

The following theorem gives sufficient condition for the graphs constructed to be nonisomorphic.

Theorem 5.10. *Let G_L and G_H satisfy property η_1, and let V and B have no zero rows or zero columns. The graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are nonisomorphic if at least one of the following holds:*

1. *V is non-PET*

2. *B is non-PET*

3. *G_A and G_D are nonisomorphic.*

In case $p \neq q$, the graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are nonisomorphic if and only if V is non-PET.

Proof. Let V is non-PET and G_H does not admit an automorphism that interchanges G_A and G_D. Suppose on the contrary, the graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H\#}$ are isomorphic. Since G_L and G_H satisfy property η_1, then from Lemma 3.27, at least one of the following holds:

Case 1: $P_1^T(I_m \otimes A)P_1 = I_m \otimes D$, $P_1^T(V \otimes B)P_4 = V \otimes B^T$, $P_4^T(I_n \otimes D)P_4 = I_n \otimes A$ for some permutation matrices P_1 and P_4
Since both G_{I_m} and G_{I_n} are nonbipartite, G_A and G_D are isomorphic. Since $V \neq 0$ and B has no zero rows or zero columns, B is PET.

Case 2: $P_2^T(I_m \otimes A)P_2 = I_n \otimes A$, $P_3^T(V^T \otimes B^T)P_2 = V \otimes B^T$ and $P_3^T(I_n \otimes D)P_3 = I_m \otimes D$ for some permutation matrices P_2 and P_3
Since $B \neq 0$ and V has no zero rows or zero columns, V is PET.

The result follows by taking contrapositive. Now suppose the graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^\#}$ are nonisomorphic, then from Lemma 3.25, V is non-PET and G_H doesn't admit an automorphism that interchanges G_A and G_D. Now suppose $p \neq q$, then the first case does not occur. In that case, $G_{L\otimes_p H}$ and $G_{L\otimes_p H^\#}$ are nonisomorphic iff V is non-PET. □

This theorem gives sufficient condition to construct nonisomorphic graphs and equivalent condition in case when $p \neq n$. It would be interesting to see if the condition 'V is non-PET and G_H doesn't admit an automorphism that interchanges G_A and G_D' is an equivalent condition for the isomorphism. In particular, it remains to see if the three equalities in Case 1 imply that G_H admits such an automorphism.

Now we give a partial characterization of property η_1 for this construction.

Lemma 5.11. *The graphs G_L and G_H satisfy property η_1 in the following cases:*

1. *B has no zero rows, V has constant row sum l and constant column sum k and $k < l$*
2. *B has no zero columns, V has constant row sum l and constant column sum k and $k > l$*
3. *B has no zero rows, $V = J_{m,n}$ and $m < n$*
4. *B has no zero columns, $V = J_{m,n}$ and $m > n$.*

Proof. Suppose V has constant row sum l and constant column sum k, where $1 \leq k \leq m$ and $1 \leq l \leq n$. Let $\Gamma_1 = G_{L\otimes_p H}$ and $\Gamma_2 = G_{L\otimes_p H^\#}$ be isomorphic. Then, there exists a permutation matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ such that $P^T(L \otimes_p H)P = L \otimes_p H^\#$. We have,

$$L \otimes_p H = \begin{bmatrix} I_m \otimes A & V \otimes B \\ V^T \otimes B^T & I_n \otimes D \end{bmatrix} \text{ and } L \otimes_p H^\# = \begin{bmatrix} I_m \otimes D & V \otimes B^T \\ V^T \otimes B & I_n \otimes A \end{bmatrix}$$

Let f be an isomorphism from Γ_1 to Γ_2. Let the vertex sets be partitioned as $V(\Gamma_1) = X_1 \cup Y_1$ and $V(\Gamma_2) = X_2 \cup Y_2$ as shown by the corresponding adjacency matrices. Let b_i and b_i' denote the i^{th} row sum and column sum of B respectively. Let a_i and d_i denote the i^{th} row sums of A and D respectively.

1. Let $k < l$ and B has no zero rows

Let $x_i \in X_1$ be the vertex of maximum degree in this set and suppose $f(x_i) \in Y_2$. Then, $d_{\Gamma_1}(x_i) = a_i + lb_i$ for some $1 \leq i \leq p$. This explains the subscript i in x_i. Then, $d_{\Gamma_2}(f(x_i)) = kb_j + a_j$ for some $1 \leq j \leq p$. Since the isomorphism preserves the degrees, we have $a_i + lb_i = kb_j + a_j$. Since x_i has maximum degree in X_1, $a_i + lb_i \geq a_{j'} + lb_{j'}$ for any $1 \leq j' \leq p$ in particular for j. Hence, $kb_j + a_j \geq a_j + lb_j$ and since B has no zero row, we have $k \geq l$. This is a contradiction to the initial assumption that $k < l$. Hence, if $x_i \in X_1$, then $f(x_i) \in X_2$. Now removing the vertex x_i of the maximum degree in X_1 from Γ_1 and $f(x_i)$ in X_2 from Γ_2 respectively, we apply the same argument on the induced graphs to conclude that $f(X_1) = X_2$ and hence $f(Y_1) = Y_2$.

2. Let $k > l$ and B has no zero columns

Let $y_i \in Y_1$ be the vertex of maximum degree in this set and suppose $f(x_i) \in X_2$. Then, $d_{\Gamma_1}(y_i) = d_i + kb'_i$ for some $1 \leq i \leq q$. This explains the subscript i in y_i. Then, $d_{\Gamma_2}(f(y_i)) = d_j + lb'_j$ for some $1 \leq j \leq q$. Since the isomorphism preserves the degrees, we have $d_i + kb'_i = d_j + lb'_j$. Since y_i has maximum degree in Y_1, $d_i + kb'_i \geq d_{j'} + kb'_{j'}$ for any $1 \leq j' \leq q$ in particular for j. Hence, $d_j + lb'_j \geq d_j + kb'_j$ and since B has no zero column, we have $l \geq k$. This is a contradiction to the initial assumption that $k > l$. Hence, if $y_i \in Y_1$, then $f(x_i) \in Y_2$. Now removing the vertex y_i of the maximum degree in Y_1 from Γ_1 and $f(y_i)$ in Y_2 from Γ_2 respectively, we apply the same argument on the induced graphs to conclude that $f(Y_1) = Y_2$ and hence $f(X_1) = X_2$.

In both cases, the isomorphism between Γ_1 and Γ_2 maps the induced subgraph $G_{I_m \otimes A}$ to $G_{I_m \otimes D}$ and $G_{I_n \otimes D}$ to $G_{I_n \otimes A}$. Hence, G_L and G_H satisfy property η_1 in the first two cases. The remaining two cases follow as a corollary. $\square$

A particular case

Let $L = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 1 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]$, $H = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$, $H^{\#} = \begin{bmatrix} D & B^T \\ B & A \end{bmatrix}$ where H is an adjacency matrix of a graph. Then $L \otimes_p H = \begin{bmatrix} A & B & B \\ B^T & D & 0 \\ B^T & 0 & D \end{bmatrix}$, $L \otimes_p H^{\#} = \begin{bmatrix} D & B^T & B^T \\ B & A & 0 \\ B & 0 & A \end{bmatrix}$. In this case, $m < n$.

Corollary 5.12. *The graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^\#}$ are cospectral if and only if A and D are. Suppose B does not have any zero rows, then $G_{L\otimes_p H}$ and $G_{L\otimes_p H^\#}$ are nonisomorphic if one of the following holds:*

1. *B is non-PET*

2. *G_A and G_D are nonisomorphic.*

Example 5.13. *Let $A = D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then B is a non-PET matrix and has no zero rows. Although, the graphs G_A and G_D are cospectral as well as isomorphic, the graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^\#}$ are cospectral and nonisomorphic as shown in Figure 5.5.*

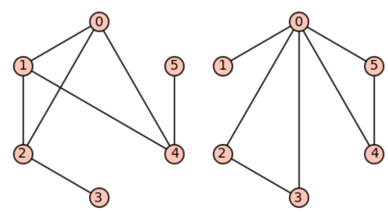


Figure 5.5: Example of construction I-B

5.3 Construction I-C: Modified complete bipartite graph

Consider the matrices of the form $L = \begin{bmatrix} S & J_{m,n} \\ J_{n,m} & T \end{bmatrix}$ where S and T are permutation matrices of orders m and n respectively and $J_{m,n}$ is a $m \times n$ all-one matrix. $L_i = \begin{bmatrix} S_i & J_{m_i,n_i} \\ J_{n_i,m_i} & T_i \end{bmatrix}$ for $i = 1, 2$.

Lemma 5.14. *L satisfies C/T property and L_1 and L_2 satisfy $C/M/T$ property.*

Proof. Since $SJ_{m,n} = J_{m,n}$ and $J_{m,n}T = J_{m,n}$ for any permutation matrices S and T, we have $SJ_{m,n} = J_{m,n}T$. Similarly, $TJ_{n,m} = J_{n,m}S$ holds. Hence, the commuting property is satisfied.

Suppose L_1 and L_2 are cospectral, then we need to show that $tr[f(L_1)] = tr[f(L_2)]$ holds for all monomials f. Let f be some monomial, then we have $f(L_i) = \mathcal{I}(S_i, T_i)^{t-s}\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})^s$ for some t and s.

Case 1: s is even and $s = 2r$

Then from Proposition 3.5, we have

$$\begin{aligned}\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})^s &= \mathcal{I}((J_{m_i,n_i}J_{n_i,m_i})^r, (J_{n_i,m_i}J_{m_i,n_i})^r) \\ &= (m_in_i)^{r-1}\mathcal{I}(n_iJ_{m_i,m_i}, m_iJ_{n_i,n_i})\end{aligned}$$

Hence,

$$\begin{aligned}tr[f(L_i)] &= (m_in_i)^{r-1}tr[\mathcal{I}(n_iS_i^{t-s}J_{m_i,m_i}, m_iT_i^{t-s}J_{n_i,n_i})] \\ &= (m_in_i)^{r-1}tr[\mathcal{I}(n_iJ_{m_i,m_i}, m_iJ_{n_i,n_i})] \\ &= 2(m_in_i)^r\end{aligned}$$

Case 2: s is odd and $s = 2r + 1$

Then from Proposition 3.5, we have

$$\begin{aligned}\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})^s &= \mathcal{P}((J_{m_i,n_i}J_{n_i,m_i})^rJ_{m_i,n_i}, (J_{n_i,m_i}J_{m_i,n_i})^rJ_{n_i,m_i}) \\ &= (m_in_i)^r\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})\end{aligned}$$

Hence,

$$\begin{aligned}tr[f(L_i)] &= (m_in_i)^rtr[\mathcal{I}(S_i, T_i)^{t-s}\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})] \\ &= (m_in_i)^rtr[\mathcal{P}(S_i^{t-s}J_{m_i,n_i}, T_i^{t-s}J_{n_i,m_i})] \\ &= (m_in_i)^rtr[\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})] \\ &= 0\end{aligned}$$

Suppose L_1 and L_2 are cospectral, then $tr[L_1^t] = tr[L_2^t]$ holds for all $t = 0, 1, 2, \ldots$. Suppose

$t = 0$, then we have $m_1 + n_2 = m_2 + n_2$. Suppose $t = 2$, then we have

$$\begin{aligned} tr[L_i^2] &= tr[\mathcal{I}(S_i, T_i)^2 + 2\mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i}) + \mathcal{P}(J_{m_i,n_i}, J_{n_i,m_i})^2] \\ &= tr[\mathcal{I}(I_{m_i}, I_{n_i})] + 0 + tr[\mathcal{I}(n_i J_{m_i,m_i}, m_i J_{n_i,n_i})] \\ &= m_i + n_i + 2m_i n_i \end{aligned}$$

Since $tr[L_1^2] = tr[L_2^2]$, we have $m_1 + n_1 + 2m_1n_1 = m_2 + n_2 + 2m_2n_2$. Hence, $m_1n_1 = m_2n_2$. It follows that $tr[f(L_1)] = tr[f(L_2)]$ when f corresponds to even s as well as odd s. Hence, the monomial property is satisfied

Now consider,

$$\begin{aligned} tr[U^{t-s}(VW)^r] &= tr[S^{t-s}(J_{m,n}J_{n,m})^r] \\ &= n^r tr[S^{t-s}J_{m,m}^r] \\ &= n^r m^{r-1} tr[S^{t-s}J_{m,m}] \\ &= n^r m^{r-1} tr[J_{m,m}] \\ &= (mn)^r \end{aligned}$$

Similarly, we have $tr[X^{t-s}(WV)^r] = (mn)^r$. This shows that $tr[U^{t-s}(VW)^r] = tr[X^{t-s}(WV)^r]$ holds for all t and s. Hence, the trace property is satisfied. □

Let us first show that the graphs corresponding to two cospectral symmetric permutation matrices are isomorphic.

Proposition 5.15. *Eigenvalues of a symmetric permutation matrix can only be -1 or 1.*

Proof. Let S be a permutation matrix and let λ be an eigenvalue of S with eigenvector x. Then, $Sx = \lambda x$ and $\|Sx\|^2 = \lambda^2\|x\|^2$. But $\|Sx\| = \|x\|$, hence $\lambda^2 = 1$. Since S is symmetric, all its eigenvalues are real. Hence, λ can either be 1 or -1. □

A *reversal* matrix is a permutation matrix whose counterdiagonal entries are 1. Let rvs_n denote a $n \times n$ reversal matrix. A symmetric permutation matrix S of order n is permutationally similar to the direct sum of reversal matrices

$$S' = \underbrace{rvs_1 \oplus \ldots \oplus rvs_1}_{s_1-times} \oplus \underbrace{rvs_2 \oplus \ldots \oplus rvs_2}_{s_2-times} \ldots \oplus \underbrace{rvs_n \oplus \ldots \oplus rvs_n}_{s_n-times}$$

where $\sum_{i=1}^{i=n} s_i i = n$ and $s_i \geq 0$ for each i. Hence, any symmetric permutation matrix is determined by some parameters $s_1, s_2, \ldots, s_n$ upto permutational similarity.

Lemma 5.16. *1. Multiplicities of the eigenvalues -1 and 1 of rvs_n are $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ respectively.*

2. Let S be a symmetric permutation matrix of order n determined by the parameters $s_1, s_2, \ldots, s_n$ upto permutational similarity. The multiplicities of the eigenvalues -1 and 1 of S are $\sum_{i=1}^{i=n} s_i \lfloor i/2 \rfloor$ and $\sum_{i=1}^{i=n} s_i \lceil i/2 \rceil$ respectively.

Proof. 1. If n is even and $n = 2m$, then the eigenvalues -1 and 1 of rvs_n both have multiplicities m. If n is odd and $n = 2m+1$, then the eigenvalues -1 and 1 of rvs_n have multiplicities m and $m+1$ respectively. Hence, the result follows.

2. The result follows from the fact that S is permutationally similar to the direct sum of reversal matrices given by the parameters $s_1, s_2, \ldots s_n$. □

Let G_{rvs_n} and G_S denote the graphs corresponding to the matrices rvs_n and S respectively.

Lemma 5.17. *1. A graph G_{rvs_n} is disjoint union of $\lfloor n/2 \rfloor$ edges and $n - 2\lfloor n/2 \rfloor$ loops*

2. The multiplicity of eigenvalue -1 for the graph G_{rvs_n} is same as the number of edges.

3. A graph G_S is disjoint union of $\sum_{i=1}^{i=n} s_i \lfloor i/2 \rfloor$ edges and $n - 2\sum_{i=1}^{i=n} s_i \lfloor i/2 \rfloor$ loops.

4. The multiplicity of eigenvalue -1 for the graph G_S is same as the number of edges.

Proof. 1. This result follows trivially since rvs_n is a permutation matrix whose counterdiagonal entries are all one.

2. Follows from Lemma 5.16.(1) and the Lemma 5.17.(1).

3. Follows from Lemma 5.17.(1). and the observation that S is permutationally similar to the direct sum of reversal matrices.

4. Follows from Lemma 5.17.(3). and Lemma 5.16.(2). □

Lemma 5.18. *The graphs corresponding to two cospectral symmetric permutation matrices are isomorphic.*

Proof. Consider two cospectral symmetric permutation matrices, then they must have the same multiplicity for the eigenvalues -1. Hence, from Lemma 5.17.(4), the corresponding graphs must have the same number of disjoint edges and from Lemma 5.17.(3), they must also have the same number of loops. Then, these graphs are isomorphic.

$\square$

Now let S and T to be symmetric permutation matrices of orders m and n respectively. Let $H = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$ and $H^{\#} = \begin{bmatrix} D & B^T \\ B & A \end{bmatrix}$ be two partitioned matrices such that A and D are square symmetric matrices of orders p and q respectively. We assume that the blocks A, B, D are nonzero to distinguish this construction from the others.

Theorem 5.19. *The graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^{\#}}$ are cospectral if and only if S and T are cospectral or A and D are cospectral.*

Proof. From Lemma 5.14, the matrix L satisfies C/T property, then the result follows from Theorem 3.20. $\square$

Proposition 5.20. *The graph G_L corresponding to the adjacency matrix $L = \begin{bmatrix} S & J_{m,n} \\ J_{n,m} & T \end{bmatrix}$ admits an interchanging automorphism with respect to its bipartition if and only if the graphs G_S and G_T are isomorphic.*

Proof. Suppose G_L admits such an automorphism, then $m = n$ and there exists a permutation matrix $P = \mathcal{P}(P_1, P_2)$ such that P_1 and P_2 are permutation matrices of size $m \times m$. Suppose $P^T LP = L$, then,

$$\begin{aligned} P^T LP &= \begin{bmatrix} 0 & P_2^T \\ P_1^T & 0 \end{bmatrix} \begin{bmatrix} S & J_{m,n} \\ J_{n,m} & T \end{bmatrix} \begin{bmatrix} 0 & P_1 \\ P_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} P_2^T T P_2 & P_2^T J_{n,m} P_1 \\ P_1^T J_{m,n} P_2 & P_1^T S P_1 \end{bmatrix} \end{aligned}$$

Then, $P_2^T J_{n,m} P_1 = J_{m,n}$ and hence $m = n$. Also, $P_1^T S P_1 = T$, that is, S and T are permutationally similar. The equation $P_2^T T P_2 = S$ also says the same. Hence, G_S and G_T are isomorphic. Conversely, if G_S and G_T are isomorphic, then $P_1^T S P_1 = T$ for some permutation matrix P_1. Let $P_2 = P_1^T$ and $P = \mathcal{P}(P_1, P_1^T)$. Then, the isomorphism corresponding to P is an automorphism such that it interchanges G_S and G_T. □

The following lemma gives a sufficient condition for the graphs to be isomorphic.

Lemma 5.21. *If S and T are cospectral or if G_H admits an interchanging automorphism with respect to its bipartition, then $G_{L \otimes_p H}$ and $G_{L \otimes_p H^\#}$ are isomorphic.*

Proof. If S and T are cospectral, then from Lemma 5.18., the graphs G_S and G_T are isomorphic. Hence, from Proposition 5.20., the graph G_L admits an interchanging automorphism with respect to its bipartition. Recall Lemma 3.25 that if at least one of G_L or G_H admits an interchanging automorphism with respect to its bipartition, then $G_{L \otimes_p H}$ and $G_{L \otimes_p H^\#}$ are isomorphic. Hence, the result follows. □

The following theorem gives sufficient condition for the graphs to be nonisomorphic.

Theorem 5.22. *Let the graphs G_L and G_H satisfy property η_1 and let B have no zero rows or zero columns. Then, $G_{L \otimes_p H}$ and $G_{L \otimes_p H^\#}$ are nonisomorphic if*

1. *B is non-PET*
2. *Either one of G_S or G_T is nonbipartite and G_A and G_D are nonisomorphic*
3. *Either one of G_A and G_D is nonbipartite and G_S and G_T are nonisomorphic.*

Proof. Suppose the graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^\#}$ are isomorphic. Since G_L and G_H satisfy property η_1, then from Lemma 3.27, we have

Case 1: $P_1^T(S \otimes A)P_1 = S \otimes D$, $P_1^T(J_{m,n} \otimes B)P_4 = J_{m,n} \otimes B^T$, $P_4^T(T \otimes D)P_4 = T \otimes A$ for some permutation matrices P_1 and P_4
Since $J_{m,n} \neq 0$ and B has no zero rows or zero columns, B is PET. If either G_S or G_T is nonbipartite, then G_A and G_D are isomorphic.

Case 2: $P_2^T(S\otimes A)P_2 = T\otimes A$, $P_3^T(J_{n,m}\otimes B^T)P_2 = J_{m,n}\otimes B^T$ and $P_3^T(T\otimes D)P_3 = S\otimes D$ for some permutation matrices P_2 and P_3
Since $B \neq 0$ and $J_{m,n}$ has no zero rows or zero columns, $m = n$. If either G_A or G_D is nonbipartite, then G_S and G_T are isomorphic.

The result follows by taking contrapositive. $\square$

Next, we give a partial characterization of property η_1 for this construction.

Lemma 5.23. *The graphs G_L and G_H satisfy property η_1 if one of the following occurs:*

1. *$m < n$ and B has no zero rows*

2. *$m > n$ and B has no zero columns*

Proof. Suppose $\Gamma_1 = G_{L\otimes_p H}$ and $\Gamma_2 = G_{L\otimes_p H^\#}$ be isomorphic. Then, there exists a permutation matrix $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ such that $P^T(L\otimes_p H)P = L\otimes_p H^\#$. We have,

$$L\otimes_p H = \begin{bmatrix} S\otimes A & J_{m,n}\otimes B \\ J_{n,m}\otimes B^T & T\otimes D \end{bmatrix} \textit{ and } L\otimes_p H^\# = \begin{bmatrix} S\otimes D & J_{m,n}\otimes B^T \\ J_{n,m}\otimes B & T\otimes A \end{bmatrix}$$

Let f be an isomorphism from Γ_1 to Γ_2. Let the vertex sets be partitioned as $V(\Gamma_1) = X_1\cup Y_1$ and $V(\Gamma_2) = X_2\cup Y_2$ as shown by the corresponding adjacency matrices. Let b_i and b_i' denote the i^{th} row sum and column sum of B respectively. Let a_i and d_i denote the i^{th} row sums of A and D respectively.

1. Suppose $m < n$ and B has no zero rows
Let $x_i \in X_1$ be the vertex of maximum degree in this set and suppose $f(x_i) \in Y_2$. Then, $d_{\Gamma_1}(x_i) = a_i + nb_i$ for some $1 \leq i \leq p$. This explains the subscript i in x_i. Then, $d_{\Gamma_2}(f(x_i)) = mb_j + a_j$ for some $1 \leq j \leq p$. Since the isomorphism preserves the degrees, we have $a_i + nb_i = mb_j + a_j$. Since x_i has maximum degree in X_1, $a_i + nb_i \geq a_{j'} + nb_{j'}$ for any $1 \leq j' \leq p$ in particular for j. Hence, $mb_j + a_j \geq a_j + nb_j$ and since B has no zero row, we have $m \geq n$. Since $m \neq n$, this implies $m > n$. This is a contradiction to the initial assumption that $m < n$. Hence, if $x_i \in X_1$, then $f(x_i) \in X_2$. Now removing the vertex x_i

of the maximum degree in X_1 from Γ_1 and $f(x_i)$ in X_2 from Γ_2 respectively, we apply the same argument on the induced graphs to conclude that $f(X_1) = X_2$ and hence $f(Y_1) = Y_2$.

2. Suppose $m > n$ and B has no zero columns

Then consider a vertex y_i of maximum degree in Y_1. Similarly, we can show that $f(Y_1) = Y_2$ and $f(X_1) = X_2$.

In both cases, the isomorphism between Γ_1 and Γ_2 maps the induced subgraphs $G_{S\otimes A}$ and $G_{T\otimes D}$ of Γ_1 to $G_{S\otimes D}$ and $G_{T\otimes A}$ of Γ_2 respectively. Hence, G_L and G_H satisfy property η_1. □

Now we show how to construct cospectral nonisomorphic graphs using this construction.

Theorem 5.24. *Suppose $m < n$ and B has no zero rows or $m > n$ and B has no zero columns. Let A and D be cospectral, then $G_{L\otimes_p H}$ and $G_{L\otimes_p H\#}$ are cospectral. They are nonisomorphic if*

1. *B is non-PET*

2. *Either one of G_S or G_T has a loop and G_A and G_D are nonisomorphic*

Proof. Since $m < n$ and B has no zero rows or $m > n$ and B has no zero columns, then from Theorem 5.23, the graphs G_L and G_H satisfy property η_1. Hence, from Theorem 5.22, the conditions for nonisomorphism follows.

Note that since $m \neq n$, the graphs G_S and G_S are noncospectral and hence nonisomorphic. Hence, the third condition for nonisomorphism from Theorem 5.22 disappears. Also the graphs G_S and G_T are nonbipartite if and only if they allow loops. □

A particular case

Let $L = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right]$, $H = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$, $H^{\#} = \begin{bmatrix} D & B^T \\ B & A \end{bmatrix}$ where H is an adjacency matrix of a graph. Then $L \otimes_p H = \begin{bmatrix} A & B & B \\ B^T & 0 & D \\ B^T & D & 0 \end{bmatrix}$, $L \otimes_p H^{\#} = \begin{bmatrix} D & B^T & B^T \\ B & 0 & A \\ B & A & 0 \end{bmatrix}$. In this case, $m < n$.

Corollary 5.25. *The graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^{\#}}$ are cospectral if and only if A and D are. Suppose B has no zero rows, then $G_{L\otimes_p H}$ and $G_{L\otimes_p H^{\#}}$ are nonisomorphic if*

1. *B is non-PET*

2. *G_A and G_D are nonisomorphic.*

Example 5.26. *Let $A = D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then B is a non-PET matrix and has no zero rows. Although, the graphs G_A and G_D are cospectral as well as isomorphic, the graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^{\#}}$ are cospectral and nonisomorphic as shown in Figure 5.6.*

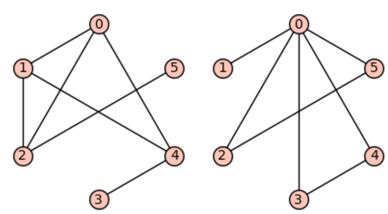


Figure 5.6: Example of construction I-C

Observe that this graph pair is the same as the one in Figure 5.5.

5.4 Construction I-D: Disjoint union of graphs

Consider the matrices of the form $L = \begin{bmatrix} U & 0 \\ 0 & X \end{bmatrix} = \mathcal{I}(U, X)$ where U and X are square matrices of orders m and n respectively.

Lemma 5.27. *The matrix $L = \mathcal{I}(U, X)$ satisfies C/T property. Two matrices $L_1 = \mathcal{I}(U_1, X_1)$ and $L_2 = \mathcal{I}(U_2, X_2)$ satisfy $C/M/T$ property.*

Proof. The commuting property is trivially satisfies since $\mathcal{P}(V, W)$ is a zero matrix which commutes with $\mathcal{I}(U, X)$.

Let f be a monomial, then for $i = 1, 2$ $f(L_i) = \mathcal{I}(U_i, X_i)^{t-s}0^s$. Hence, $f(L_i)$ is either $\mathcal{I}(U_i, X_i)^t = L_i^t$ for some t or a zero matrix. Hence, $tr[f(L_1)] = tr[f(L_2)]$ holds for all monomials if and only if $tr[L_1^t] = tr[L_2^t]$ holds for all t. Hence, the monomial property is satisfied.

If $s = 2r$ and $s \neq 0$, then $tr[U^{t-s}(VW)^r] = 0$ and $tr[X^{t-s}(WV)^r] = 0$. Hence, the trace property is also satisfied. □

Theorem 5.28. *Let $L = \mathcal{I}(U, X)$, $H = \mathcal{I}(A, D)$ and $H^{\#} = \mathcal{I}(D, A)$ such that U, X, A and D symmetric matrices of orders m, n, p and q respectively. The graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ are cospectral if and only if U and X are or A and D are.*

Proof. From Lemma 5.27, L satisfies C/T property. Then the results follows as a corollary of Theorem 3.20. □

Theorem 5.29. *Suppose G_L and G_H satisfy property η_1.*

1. *Suppose $m \neq n$, and G_U and G_X are nonbipartite. Then, $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ cospectral nonisomorphic if and only if G_A and G_D are cospectral nonisomorphic*

2. *Suppose $m \neq n$, and G_U, G_X, G_A and G_D are bipartite. Then, $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ cospectral nonisomorphic if and only if G_A and G_D are cospectral nonisomorphic*

3. *Suppose $p \neq q$, and G_A and G_D are nonbipartite. Then, $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ cospectral nonisomorphic if and only if G_U and G_X are cospectral nonisomorphic*

4. *Suppose $p \neq q$, and G_A, G_D, G_U and G_D are bipartite. Then, $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ cospectral nonisomorphic if and only if G_U and G_X are cospectral nonisomorphic*

Proof. Follows from Theorem 5.28, Lemma 3.25 and Lemma 3.27. □

This result is only a trivial extension of the cancellation law for graphs (Lemma 2.11 and Theorem 2.12) that had been used in the proof. Let us look at the implication of the cancellation law given by Theorem 2.12.

Lemma 5.30. *Let G_A and G_D be two nonempty graphs, and G_U be a nonbipartite graph. Then, $G_U \times G_A$ and $G_U \times G_D$ are cospectral and nonisomorphic if and only if G_A and G_D are.*

Example 5.31. *Suppose $U_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, then G_{U_1} is nonbipartite since it has a loop. The graphs G_{A_i} and G_{D_i} in Figure 5.7 and Figure 5.8 are cospectral and nonisomorphic. Hence, the corresponding graphs $G_{U_1} \times G_{A_i}$ and $G_{U_1} \times G_{D_i}$ are also cospectral and nonisomorphic for $i = 1, 2$.*

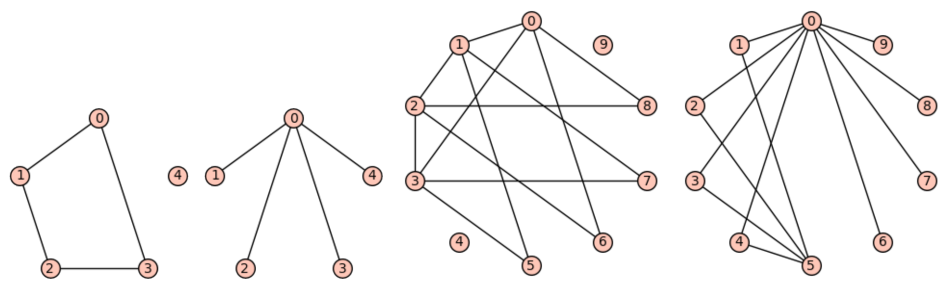


Figure 5.7: Example 1 of construction I-D: G_{A_1} and G_{D_1}, $G_{U_1} \times G_{A_1}$ and $G_{U_1} \times G_{D_1}$

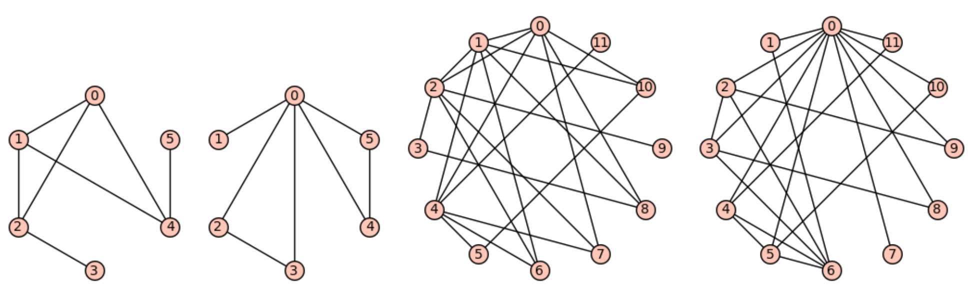


Figure 5.8: Example 2 of construction I-D: G_{A_2} and G_{D_2}, $G_{U_1} \times G_{A_2}$ and $G_{U_1} \times G_{D_2}$

Now let us look at the implication of the cancellation law given by Lemma 2.11.

Lemma 5.32. *Let G_A, G_D and G_U be bipartite graph. Then, $G_U \times G_A$ and $G_U \times G_D$ are cospectral and nonisomorphic if and only if G_A and G_D are.*

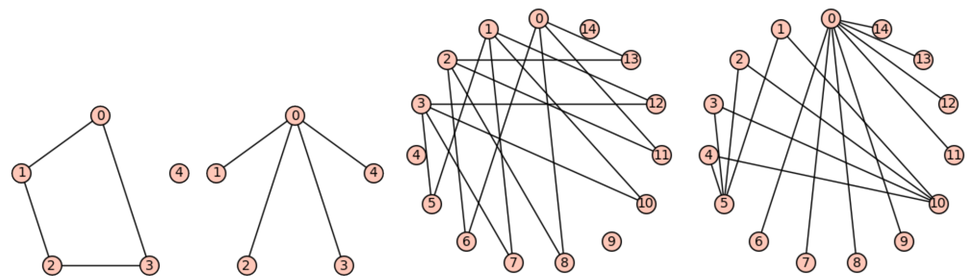


Figure 5.9: Example 3 of construction I-D: G_{A_1} and G_{D_1}, $G_{U_2} \times G_{A_1}$ and $G_{U_2} \times G_{D_1}$

Example 5.33. *Let* $U_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$*, then* G_{U_2} *is a bipartite graph. Since the graphs* G_{A_1} *and* G_{D_1} *shown in Figure 5.9 are cospectral nonisomorphic bipartite graphs, the corresponding graphs* $G_{U_2} \times G_{A_1}$ *and* $G_{U_2} \times G_{D_1}$ *are also cospectral and nonisomorphic.*

5.5 Construction I-E: Two matched cospectral graphs

Consider the matrices of the form $L = \begin{bmatrix} U & E \\ E^T & X \end{bmatrix}$ where E is a permutation matrix such that $E^TUE = X$. L_1 and L_2 are similarly defined using corresponding subscripts.

Lemma 5.34. *L satisfies C/T property and L_1 and L_2 satisfy $C/M/T$ property.*

Proof. The commuting property is satisfied since $UE = EX$ and $XE^T = E^TU$.

Suppose L_1 and L_2 are cospectral, then we need to show that $tr[f(L_1)] = tr[f(L_2)]$ holds for all monomials f. Let f be some monomial, then we have $f(L_i) = \mathcal{I}(U_i, X_i)^{t-s}\mathcal{P}(E_i, P_i^T)^s$ for some t and s.

Case 1: Let s be odd

Then from Proposition 3.5, we have $\mathcal{P}(E_i, E_i^T)^s = \mathcal{P}(E_i, E_i^T)$ and $f(L_i) = \mathcal{P}(U^{t-s}E_i, X^{t-s}E_i^T)$. Hence, we have $tr[f(L_1)] = tr[f(L_2)] = 0$ when the monomial f corresponds to an odd s.

Case 2: Let s be even

Then from Proposition 3.5, we have $\mathcal{P}(E_i, E_i^T)^s = I$ and $f(L_i) = \mathcal{I}(U_i, X_i)^{t-s}$. We need to show that $tr[\mathcal{I}(U_1, X_1)^{t-s}] = tr[\mathcal{I}(U_2, X_2)^{t-s}]$ holds for any even s. Now consider $tr[L_i^t] = tr[(\mathcal{I}(U_i, X_i) + \mathcal{P}(E_i, E_i^T))^t]$ for some t. If $t = 1$, then $tr[L_i^1] = tr[\mathcal{I}(U_i, X_i)]$. Hence, $tr[L_1] = tr[L_2]$ implies $tr[\mathcal{I}(U_1, X_1)] = tr[\mathcal{I}(U_2, X_2)]$. If $t = 2$, then

$$\begin{aligned} tr[L_i^2] &= tr[(\mathcal{I}(U_i, X_i) + \mathcal{P}(E_i, E_i^T))^2] \\ &= tr[\mathcal{I}(U_i, X_i)^2 + 2\mathcal{I}(U_i, X_i)\mathcal{P}(E_i, E_i^T) + \mathcal{P}(E_i, E_i^T)^2] \\ &= tr[\mathcal{I}(U_i, X_i)^2 + I] \end{aligned}$$

Hence, $tr[L_1^2] = tr[L_2^2]$ implies $tr[\mathcal{I}(U_1, X_1)^2] = tr[\mathcal{I}(U_2, X_2)^2]$. Now consider

$$\begin{aligned} tr[L_i^z] &= tr[(\mathcal{I}(U_i, X_i) + \mathcal{P}(E_i, E_i^T))^z] \\ &= tr[\mathcal{I}(U_i, X_i)^z] + \binom{z}{1} tr[\mathcal{I}(U_i, X_i)^{z-1}\mathcal{P}(E_i, E_i^T)^1] + \ldots + tr[\mathcal{P}(E_i, E_i^T)^z] \end{aligned}$$

For every odd power of $\mathcal{P}(E_i, E_i^T)$ in the above expansion, the corresponding trace term is 0. Suppose z is odd, then

$$\begin{aligned} tr[L_i^z] &= tr[(\mathcal{I}(U_i, X_i) + \mathcal{P}(E_i, E_i^T))^z] \\ &= tr[\mathcal{I}(U_i, X_i)^z] + \binom{z}{2} tr[\mathcal{I}(U_i, X_i)^{z-2}\mathcal{P}(E_i, E_i^T)^2] + \ldots + \binom{z}{z-1} tr[\mathcal{I}(U_i, X_i)^1\mathcal{P}(E_i, E_i^T)^{z-1}] \\ &= tr[\mathcal{I}(U_i, X_i)^z] + \binom{z}{2} tr[\mathcal{I}(U_i, X_i)^{z-2}] + \ldots + \binom{z}{z-1} tr[\mathcal{I}(U_i, X_i)^1] \end{aligned}$$

Hence, finally we have only the odd powers of $\mathcal{I}(U_i, X_i)$ in the expansion. Suppose z is even, then

$$\begin{aligned} tr[L_i^z] &= tr[(\mathcal{I}(U_i, X_i) + \mathcal{P}(E_i, E_i^T))^z] \\ &= tr[\mathcal{I}(U_i, X_i)^z] + \binom{z}{2} tr[\mathcal{I}(U_i, X_i)^{z-2}\mathcal{P}(E_i, E_i^T)^2] + \ldots + tr[\mathcal{P}(E_i, E_i^T)^z] \\ &= tr[\mathcal{I}(U_i, X_i)^z] + \binom{z}{2} tr[\mathcal{I}(U_i, X_i)^{z-2}] + \ldots + tr[\mathcal{I}(U_i, X_i)^0] \end{aligned}$$

Hence, finally we have only the even powers of $\mathcal{I}(U_i, X_i)$ in the expansion. By the induction assumption, suppose we have $tr[\mathcal{P}(V_1, W_1)^t] = tr[\mathcal{P}(V_2, W_2)^t]$ for $t = 0, 1, 2, \ldots, z-1$. Since $tr[L_1^z] = tr[L_2^z]$, all the terms except the first in the expansion (take any odd or even case) of $tr[L_1^z]$ and $tr[L_2^z]$ are equal. Hence, $tr[\mathcal{I}(U_1, X_1)^z] = tr[\mathcal{I}(U_2, X_2)^z]$. This shows L_1 and L_2

are cospectral implies $tr[\mathcal{I}(U_1, X_1)^{t-s}] = tr[\mathcal{I}(U_2, X_2)^{t-s}]$ for all $t-s$. Hence, the monomial property is satisfied.

Now to show trace property, consider

$$tr[U^{t-s}(VW)^r] = tr[U^{t-s}(PP^T)^r] = tr[U^{t-s}]$$

$$tr[X^{t-s}(WV)^r] = tr[X^{t-s}(P^TP)^r] = tr[X^{t-s}]$$

Since, $P^TUP = X$, U and X are cospectral. Then, $tr[U^{t-s}] = tr[X^{t-s}]$ for all $t-s$. Hence, $tr[U^{t-s}(VW)^r] = tr[X^{t-s}(WV)^r]$ for, in particular, all t and s such that $s = 2r$ and $s \neq 0$. Hence, the trace property is also satisfied. $\square$

Let $H = \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$ and $H^{\#} = \begin{bmatrix} D & B^T \\ B & A \end{bmatrix}$ be two partitioned matrices such that A and D are square symmetric matrices of orders p and q respectively.

Theorem 5.35. *The graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ are cospectral.*

Proof. From Lemma 5.34, the matrix L satisfies C/T property. Since, U and X are permutationally similar, they are cospectral. Hence, from Theorem 3.20, $L \otimes_p H$ and $L \otimes_p H^{\#}$ are also cospectral. $\square$

Proposition 5.36. *The graph G_L admits an interchanging automorphism with respect to its bipartition.*

Proof. Since $E^TUE = X$ for the permutation matrix E, let $P = \mathcal{P}(E, E^T)$, then

$$\begin{aligned} P^TLP &= \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \begin{bmatrix} U & E \\ E^T & X \end{bmatrix} \begin{bmatrix} 0 & E \\ E^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} EXE^T & E \\ E^T & E^TUE \end{bmatrix} \\ &= \begin{bmatrix} U & E \\ E^T & X \end{bmatrix} \\ &= L \end{aligned}$$

This proves the proposition.

□

Theorem 5.37. *The graphs $G_{L \otimes_p H}$ and $G_{L \otimes_p H^{\#}}$ are isomorphic as well.*

Proof. From Proposition G_L admits an interchanging automorphism with respect to its bipartition, then from Lemma 3.25, the isomorphism follows. □

Although L satisfies C/T property, we show that the cospectral graphs constructed using Construction-I corresponding to this particular L are isomorphic.

Chapter 6

Application of partitioned tensor product

In this chapter, we apply the idea of partitioned tensor product on some of the existing cospectral constructions to obtain new cospectral constructions. The following theorem show how:

Theorem 6.1. *Let H_1 and H_2 be two multipartioned $(m \times m)$ cospectral block matrices such that the similarity matrix Q satisfying $Q^{-1}H_1Q = H_2$ is a diagonal block matrix. Let L be a multipartitioned $(m \times m)$ block matrix such that $L \otimes_p H_1$ and $L \otimes_p H_2$ are defined, then $L \otimes_p H_1$ and $L \otimes_p H_2$ are cospectral.*

Proof. Let the diagonal block matrix Q be such that $Q = diag(Q_1, Q_2, \ldots, Q_m)$ where Q_i's are nonsingular matrices of appropriate orders. Let I be an $m \times m$ partitioned identity matrix such that $R = I \otimes_p Q$ is a diagonal block nonsingular matrix. Then, R satisfies $R^{-1}(L \otimes_p H_1)R = L \otimes_p H_2$. Hence, $L \otimes_p H_1$ and $L \otimes_p H_2$ are cospectral.

□

Suppose H_1 and H_2 are adjacency matrices of cospectral graphs G_{H_1} and G_{H_2}, then the graphs $G_{L \otimes_p H_1}$ and $G_{L \otimes_p H_2}$ are cospectral. If at most one of G_{H_1} and G_{H_2} is allowed loops, then the graphs $G_{L \otimes_p H_1}$ and $G_{L \otimes_p H_2}$ don't have loops.

In the first section, we discuss a candidate for H_1 and H_2 which is given by the construction based on partial transpose (III-A, III-B) introduced by Dutta and Adhikari (see [2]). We give alternate proof for the main theorem (III-A) and fix an error of the second (III-B). The new constructions based on partitioned tensor product help us discuss the notion of unfolding a multipartite graphs. We also give examples of how a tripartite graph can be unfolded to obtain cospectral nonisomorphic graphs.

In the next sections, we discuss GM switching (IV) and a construction based on congruence (V) both introduced by Godsil and Mckay (see [5]) as candidates for H_1 and H_2. Inspired by the construction V, we introduce another construction VI which is based on unfolding a semi reflexive bipartite graph. Since GM switching produces cospectral graphs with cospectral complement, the new construction using partitioned tensor product also produces cospectral graphs with cospectral complements. We give conditions for obtaining cospectral nonisomorphic graphs for the constructions V and VI as well as the new constructions obtained using partitioned tensor product

6.1 Construction III: Partial transpose

A partitioned block matrix A is a block matrix of order mn whose ij^{th} block is the matrix A_{ij} of order m where $1 \leq i, j \leq n$. Let A^τ be the matrix whose ij^{th} block is A_{ij}^T. The matrix A^τ is called the partial transpose of matrix A.

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}, \; A^\tau = \begin{bmatrix} A_{11}^T & A_{12}^T & \dots & A_{1m}^T \\ A_{21}^T & A_{22}^T & \dots & A_{2m}^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}^T & A_{m2}^T & \dots & A_{mm}^T \end{bmatrix}$$

6.1.1 Construction III-A: Commuting family of normal matrices

Recall the following result on commuting family of diagonalizable matrices.

Lemma 6.2. *[11][Theorem 1.3.21.] Let $F \subset M_n$ be a family of diagonalizable matrices. Then F is a commuting family if and only if it is a simultaneously diagonalizable family.*

The following lemma shows that the commuting family of diagonalizable matrices are simultaneouly similar to their transposes.

Lemma 6.3. *Let $\{A_i, i = 1, \ldots, k\}$ be a commuting family of diagonalizable matrices of order m. Then, there exists a real nonsingular matrix X such that $A_i^T = X^{-1}A_iX$ for $i = 1, \ldots, k$.*

Proof. From Lemma 6.2., there exists a nonsingular matrix V such that $A_i = V^{-1}D_iV$ for each $i = 1, \ldots, k$, where D_i is a diagonal matrix of eigenvalues of A_i. Then, we have $VA_iV^{-1} = D_i$. Since V is nonsingular, we have $VV^{-1} = I$. Then, $(V^{-1})^TV^T = I$. Since V^T is also nonsingular, $(V^{-1})^T = (V^T)^{-1}$. By taking transposes on both sides of the original equation,

$$\begin{aligned}
A_i^T &= V^TD_i(V^{-1})^T \\
&= V^T(VA_iV^{-1})(V^{-1})^T \\
&= (V^TV)A_i(V^{-1}(V^T)^{-1}) \\
&= (V^TV)A_i(V^TV)^{-1}
\end{aligned}$$

Let $X = (V^TV)^{-1}$, then it is a nonsingular matrix and $A_i^T = X^{-1}A_iX$ for $i = 1, 2, \ldots, k$. □

Now we a give a construction of cospectral matrices.

Theorem 6.4. *Let A be a partitioned matrix block matrix of order mn. If the blocks of matrix A form a commuting family of diagonalizable matrices, then A and A^τ are cospectral.*

Proof. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1m} \\ A_{21} & A_{22} & \ldots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \ldots & A_{mm} \end{bmatrix}$$

Since the blocks of A form a commuting family of diagonalizable matrices, from Lemma 6.3., it follows that there exists a real nonsingular matrix X such that $A_{ij}^T = X^{-1}A_{ij}X$ for $1 \leq i, j, \leq m$. Now consider the block diagonal orthogonal matrix $Q = diag(X, X, \ldots, X)$.

Then,

$$
\begin{aligned}
Q^{-1}AQ &= \begin{bmatrix} X^{-1}A_{11}X & X^{-1}A_{12}X & \dots & X^{-1}A_{1m}X \\ X^{-1}A_{21}X & X^{-1}A_{22}X & \dots & X^{-1}A_{2m}X \\ \vdots & \vdots & \ddots & \vdots \\ X^{-1}A_{m1}X & X^{-1}A_{m2}X & \dots & X^{-1}A_{mm}X \end{bmatrix} \\
&= \begin{bmatrix} A_{11}^T & A_{12}^T & \dots & A_{1m}^T \\ A_{21}^T & A_{22}^T & \dots & A_{2m}^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}^T & A_{m2}^T & \dots & A_{mm}^T \end{bmatrix} \\
&= A^\tau
\end{aligned}
$$

Hence A and A^τ are cospectral. □

Now let A be an adjacency matrix of a graph such that its blocks form a commuting family of diagonalizable matrices. Since A is symmetric, consider the off diagonal blocks A_{ij} and A_{ij}^T. A_{ij} and A_{ij}^T commute, hence all the off diagonal blocks are normal. The diagonal blocks A_{ii} are symmetric, hence they are also normal. This shows if A is symmetric and blocks of A form commuting diagonalizable family, then it is in fact commuting normal family. Now we do the similar analysis by considering results on commuting family of normal matrices and we show that the similarity matrix is an orthogonal matrix. Recall,

Lemma 6.5. *[11][Theorem 2.5.5.] Let $N \subseteq M_n$ be a nonempty family of normal matrices. Then N is a commuting family if and only if it is a simultaneously unitarily diagonalizable family.*

The following is the main result given by Dutta and Adhikari [2] which uses Lemma 6.5 in its proof.

Lemma 6.6. *[2] Let $\{A_i, i = 1, \dots, k\}$ be a commuting family of normal matrices of order m. Then, there exists a nonsingular matrix X such that $A_i^T = X^{-1}A_iX$ for $i = 1, \dots, k$.*

We give an alternate proof for this theorem to show that the similarity matrix is an orthogonal matrix. The original proof is based on vectorization of matrices and solving Lyapunov equations. The following lemma shows, in particular, that two real unitarily similar matrices are real orthogonally similar.

Lemma 6.7. *[11][Theorem 2.5.21.] Let* $F = \{A_\alpha : \alpha \in I\} \subset M_n(R)$ *and* $G = \{B_\alpha : \alpha \in I\} \subset M_n(R)$ *be given families of real matrices. If there is a unitary* $U \in M_n$ *such that* $A_\alpha = UB_\alpha U^*$ *for every* $\alpha \in I$*, then there is a real orthogonal* $Q \in M_n(R)$ *such that* $A_\alpha = QB_\alpha Q^T$ *for every* $\alpha \in I$*.*

Similarly, we now show that commuting family of normal matrices is simultaneously orthogonally similar to their transposes.

Lemma 6.8. *Let* $\{A_i, i = 1, \dots, k\}$ *be a commuting family of normal matrices of order* m*. Then, there exists a real orthogonal matrix* Y *such that* $A_i^T = Y^{-1}A_iY$ *for* $i = 1, \dots, k$*.*

Proof. From Lemma 6.5., there exists a unitary matrix U such that $U^*A_iU = D_i$ for each $i = 1, \dots, k$, where D_i is a diagonal matrix of eigenvalues of A_i. Then, $A_i = UD_iU^*$. By taking transposes on both sides,

$$\begin{aligned} A_i^T &= (U^*)^T D_i^T U^T \\ &= \overline{U} D_i U^T \\ &= \overline{U}(U^* A_i U)U^T \\ &= (UU^T)^* A_i (UU^T) \end{aligned}$$

Let $X = UU^T$, then X is an unitary matrix and $A_i^T = X^*A_iX$ for $i = 1, 2, \dots, k$. Since A_i's are real matrices, it follows from Lemma 6.7., that there exists a real orthogonal matrix Y such that $A_i^T = Y^TA_iY$. $\square$

Now we give construction of cospectral matrices.

Theorem 6.9. *[2] Let* A *be a partitioned matrix block matrix of order* mn*. If the blocks of matrix* A *form a commuting family of normal matrices, then* A *and* A^τ *are cospectral.*

Proof. Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{bmatrix}$$

Since the blocks of A form a commuting normal family, from Lemma 6.8., it follows that there exists a real orthogonal matrix Y such that $A_{ij}^T = Y^{-1}A_{ij}Y$ for $1 \leq i, j, \leq m$. Now consider the block diagonal orthogonal matrix $Q = diag(Y, Y, \ldots, Y)$. Then,

$$\begin{aligned} Q^{-1}AQ &= \begin{bmatrix} Y^{-1}A_{11}Y & Y^{-1}A_{12}Y & \ldots & Y^{-1}A_{1m}Y \\ Y^{-1}A_{21}Y & Y^{-1}A_{22}Y & \ldots & Y^{-1}A_{2m}Y \\ \vdots & \vdots & \ddots & \vdots \\ Y^{-1}A_{m1}Y & Y^{-1}A_{m2}Y & \ldots & Y^{-1}A_{mm}Y \end{bmatrix} \\ &= \begin{bmatrix} A_{11}^T & A_{12}^T & \ldots & A_{1m}^T \\ A_{21}^T & A_{22}^T & \ldots & A_{2m}^T \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}^T & A_{m2}^T & \ldots & A_{mm}^T \end{bmatrix} \\ &= A^\tau \end{aligned}$$

Hence A and A^τ are cospectral. □

Now let A to be a symmetric 0-1 matrix, then the graphs corresponding to A and A^τ are cospectral. In that case, diagonal blocks needs to be symmetric and $A_{ji} = A_{ij}^T$ for all $1 \leq i, j \leq m$. It should be noted that the construction based on commuting family of diagonalizable matrices is a construction of cospectral digraphs, but only the construction based on commuting family of normal matrices is useful in construction of cospectral graphs.

Now we apply the idea of partitioned tensor product on this construction. Let L and A be two symmetric partitioned matrices whose partitioning is the same, that is, let L and A be two $m \times m$ block matrices such that the partitioned tensor product $L \otimes_p A$ exists. Let the blocks of A form a commuting family of normal matrices. Hence, from Theorem 6.9., A and A^τ are cospectral. Also note that the partitioned tensor product $L \otimes_p A^\tau$ is the partial transpose of $L \otimes_p A$.

Theorem 6.10. *The graphs $G_{L\otimes_p A}$ and $G_{L\otimes_p A^\tau}$ are cospectral.*

Proof. Since the blocks of A forms a commuting family of normal matrices, from the proof of Lemma 6.9, the orthogonal matrix Q satisfying $Q^T AQ = A^\tau$ is a diagonal block matrix. Let I be an identity matrix, then $I^T LI = L$. The partitioned tensor product $R = I \otimes_p Q$ is also an orthogonal matrix such that $R^T(L \otimes_p A)R = L \otimes_p A^\tau$. □

In case the diagonal blocks of L and A are zero blocks, then the correspond to adjacency matrix of multipartite graphs.

6.1.2 Construction III-B: Unfolding a multipartite graph

Let us now consider a particular case of multipartite graphs whose adjacency matrices are constructed using only one nontrivial block. The idea for the following theorem comes from Theorem 6.9. Let A be a partitioned symmetric matrix of order mn such that each of its blocks is of order m. Let the diagonal blocks of A to be zero matrices and the off diagonal blocks of A are either B, B^T, I_m or 0_m where B is a $m \times m$ matrix.

Theorem 6.11. *A and A^τ are cospectral if any of the following holds:*

1. *B is normal*
2. *B is similar to its transpose via an involutory matrix*[1]
3. *B is similar to its transpose via an orthogonal matrix*

Proof. 1. Then, $\{B, B^T, I_m, 0_m\}$ forms a commuting family of normal matrices. Hence, the result follows as a corollary of Theorem 6.9.

2. Since B is similar to its transpose, $S_0^{-1}BS_0 = B^T$ holds for some invertible matrix S_0. Since S_0 is involutory, we have $S_0 = S_0^{-1}$, and hence $S_0^{-1}B^TS_0 = B$. Let $S = diag(\underbrace{S_0, S_0, \ldots, S_0}_{n-times})$. Let X be ij^{th} block of A for $1 \leq i, j \leq n$ where $X \in \{B, B^T, I_m, 0_m\}$. Then the ij^{th} blocks of $S^{-1}AS$ are $S_0^{-1}XS_0 \in \{B^T, B, I_m, 0_m\}$ respectively. Hence, $S^{-1}AS = A^\tau$ and A and A^τ are cospectral.

3. Since B is similar to its transpose, $Q^{-1}BQ = B^T$ holds for some orthogonal matrix Q. Since Q is orthogonal, we have $Q^T = Q^{-1}$, and hence by taking transposes, we have $Q^{-1}B^TQ = B$. The rest follows similar to the proof of the second statement. □

[1] this result is obtained by fixing an error in Theorem 7 of [2] where only the fact 'B is similar to its transpose' is used which is insufficient.

Now let us apply the idea of partitioned tensor product on this construction too. Let A and L be two symmetric matrices partitioned similarly such that the partitioned tensor product $L \otimes_p A$ exists.

Theorem 6.12. *Let A and A^τ be cospectral under the assumptions of Theorem 6.11. Then, the graphs $G_{L\otimes_p H}$ and $G_{L\otimes_p H^\tau}$ are cospectral.*

Proof. The proof is similar to the proof of Theorem 6.10. $\square$

Next, we give sufficient condition for the graphs to be isomorphic.

Lemma 6.13. *If B is permutationally similar to its transpose, then G_A and G_{A^τ} are isomorphic as well as $G_{L\otimes_p A}$ and $G_{L\otimes_p A^\tau}$ are.*

Proof. If B is permutationally similar to its transpose, then $P_0^T B P_0 = B^T$ for some permutation matrix P_0. Since P is also orthogonal, we have $P_0^T B^T P_0 = B$. Let $P = diag(P_0, P_0, \ldots, P_0)$, then $P^T A P = A^\tau$. Hence, G_A and G_{A^τ} are isomorphic. Since this similarity matrix P is diagonal block matrix, $G_{L\otimes_p A}$ and $G_{L\otimes_p A^\tau}$ are also isomorphic. $\square$

Let us now discuss how Theorem 6.12 is related to unfoldings of a bipartite graphs. Let G be a bipartite graph with an adjacency matrix $A(G) = \mathcal{P}(B, B^T)$. Then, the adjacency matrices of its two unfoldings Γ_1 and Γ_2 as defined by Butler [1] are given by:

$$A(\Gamma_1) = \begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \text{ and } A(\Gamma_2) = \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}.$$

Let $L = \mathcal{P}(J_{1,2}, J_{2,1})$, the $A(\Gamma_1) = L \otimes_p A(G)$ and $A(\Gamma_2) = L \otimes_p A(G)^\tau$. The matrices $A(\Gamma_1)$ and $A(\Gamma_2)$ are, in fact, partial transposes of each other. Recall that if B is a square matrix, then $A(\Gamma_1)$ and $A(\Gamma_2)$ are cospectral. No other condition on B is required. This is bacause of the bipartite graph case and the fact that the similarity matrix between $A(\Gamma_1)$ and $A(\Gamma_2)$ is not a diagonal block matrix. In case of multipartite graphs, we can use Theorem 6.10 and Theorem 6.12 to develope the notion unfolding a multipartite graphs.

We say that the graph G_A is *unfolded* with respect to the graph G_L to obtain the *unfoldings* $G_{L\otimes_p A}$ and $G_{L\otimes_p A^\tau}$.

Unfolding a tripartite graph

In this subsection, we demonstrate how to use Theorem 6.12. on a tripartite graph G_A to obtain cospectral nonisomorphic unfoldings $G_{L\otimes_p A}$ and $G_{L\otimes_p A^\tau}$. Let

$$A = \begin{bmatrix} 0 & B & B \\ B^T & 0 & B \\ B^T & B^T & 0 \end{bmatrix} \text{ and } A^\tau = \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & B^T \\ B & B & 0 \end{bmatrix}$$

where B be a $n \times n$ 0-1 matrix. Observe that A and A^τ are permutationally similar via the permutation matrix $P = \begin{bmatrix} 0 & 0 & I_n \\ 0 & I_n & 0 \\ I_n & 0 & 0 \end{bmatrix}$. Let $J_{m,n}$ be all-one matrix of size $m \times n$ and let

$$L(p,q,r) = \begin{bmatrix} 0 & J_{p,q} & J_{p,r} \\ J_{q,p} & 0 & J_{q,r} \\ J_{r,p} & J_{r,q} & 0 \end{bmatrix}.$$

If B is permutationally similar to its transpose, then from Lemma 6.13, the graphs $G_{L(p,q,r)\otimes_p A}$ and $G_{L(p,q,r)\otimes_p A^\tau}$ are isomorphic. Here is another sufficient condition for the isomorphism.

Proposition 6.14. *If $p = r$, then the graphs $G_{L(p,q,r)\otimes_p A}$ and $G_{L(p,q,r)\otimes_p A^\tau}$ are isomorphic.*

Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then B is similar to its transpose via real orthogonal symmetric involutary matrix $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Note that B is not permutationally similar to its transpose. For this particular B, we give examples of cospectral non-isomorphic graphs $G_{L(p,q,r)\otimes_p A}$ and $G_{L(p,q,r)\otimes_p A^\tau}$ for different pairs of (p,q,r) such that $p \neq r$. Each pair (p,q,r) corresponds to a different way of unfolding the given tripartite graph G_A. The matrix $L(p,q,r)$ need not be the adjacency matrix of the complete multipartite graph in general.

Figure 6.1. shows the tripartite graphs G_A and G_{A^τ} corresponding to the triadjacency matrix B. Vertices from the same partite sets of the graphs G_A and G_{A^τ} are coloured using the same colour. Table 6.1 gives cospectral nonisomorphic graph pairs $G_{L(p,q,r)\otimes_p A}$

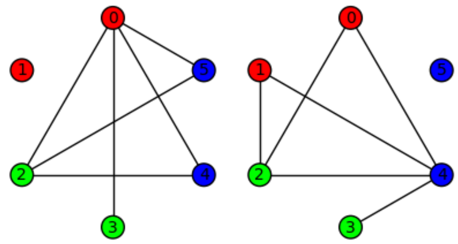


Figure 6.1: Tripartite graphs G_A and G_{A^τ}

and $G_{L(p,q,r)\otimes_p A^\tau}$ for each (p,q,r). Vertices of the same colour in the graphs $G_{L(p,q,r)\otimes_p A}$ and $G_{L(p,q,r)\otimes_p A^\tau}$ indicate the appropriate number of unfolded partite sets. Note that $G_{L(p,q,r)\otimes_p A}$ and $G_{L(p,q,r)\otimes_p A^\tau}$ are tripartite graphs.

6.2 Construction IV: Godsil-Mckay switching

Let us discuss the famous Godsil-Mckay switching (see [5]). Let G be a graph with partitioned adjacency matrix,

$$A(G) = \begin{bmatrix} C_1 & C_{12} & \dots & C_{1k} & D_1 \\ C_{12}^T & C_2 & \dots & C_{2k} & D_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1k}^T & C_{2k}^T & \dots & C_k & D_k \\ D_1^T & D_2^T & \dots & D_k^T & D \end{bmatrix}$$

such that each C_i and C_{ij} have constant row sums and constant column sums, and each column of each D_i has either $0, \frac{n_i}{2}$ or n_i ones, where n_i is the number of rows of the block D_i. Each C_i is an $n_i \times n_i$ block. Let D be an $n_0 \times n_0$ block. Define $Q_m = \frac{2}{m}J_m - I_m$ and $Q = diag(Q_{n_1}, Q_{n_2}, \dots, Q_{n_k}, I_{n_0})$. Then, $Q^T A(G) Q$ is also an adjacency matrix of some graph, Let G' be a graph with adjacency matrix $A(G') = Q^T A(G) Q$. The graph G and G' are cospectral.

An orthogonal matrix is said to be *regular* if all its row sums and column sums are 1. A *rational* orthogonal matrix has all its entries rational.

Lemma 6.15. *[13]Let G_1 and G_2 be two cospectral graphs. Then TFAE:*

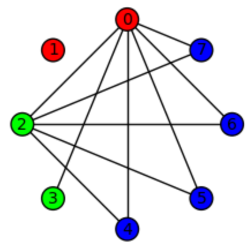
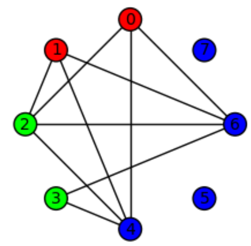
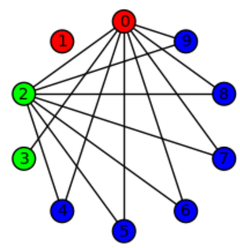
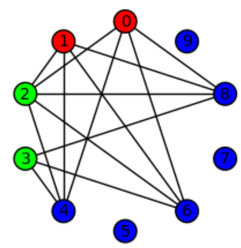
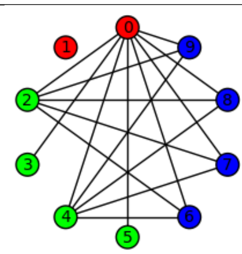
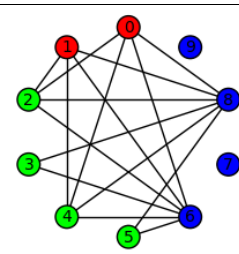
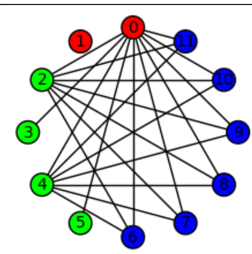
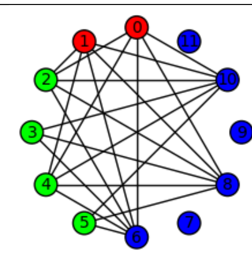
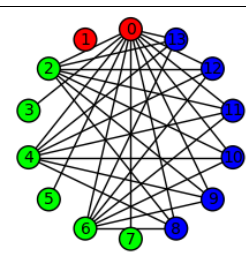
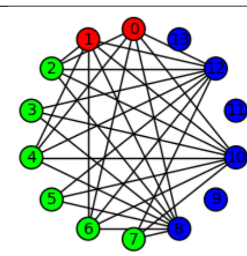
(p,q,r)	$G_{L(p,q,r)\otimes_p A}$	$G_{L(p,q,r)\otimes_p A^\tau}$
(1,1,2)		
(1,1,3)		
(1,2,2)		
(1,2,3)		
(1,3,3)		

Table 6.1: Unfoldings of tripartite graphs $G_{L(p,q,r)\otimes_p A}$ and $G_{L(p,q,r)\otimes A^\tau}$

1. G_1 and G_2 has cospectral complements

2. the orthogonal matrix of similarity Q such that $Q^T A(G_1)Q = A(G_2)$ is rational and regular.

Since the orthogonal similarity matrix in the Godsil-Mckay switching is a rational regular matrix, from Lemma 6.15., the construction produces cospectral graphs with cospectral complements. Let us now apply the idea of partitioned tensor product. Let L be a symmetric partitioned $(k+1)\times(k+1)$ block matrix such that the partitioned tensor products $L\otimes_p A(G)$ and $L\otimes_P A(G')$ exists, where the graphs G and G' are cospectral through the Godsil-Mckay switching. Then,

Theorem 6.16. *The graphs $G_{L\otimes_p A(G)}$ and $G_{L\otimes_p A(G')}$ are cospectral with cospectral complements.*

Proof. Since G and G' are cospectral via Godsil-Mckay switching, the orthogonal similarity matrix $Q = diag(Q_{n_1}, Q_{n_2}, \ldots, Q_{n_k}, I_{n_0})$ which satisfies $A(G') = Q^T A(G)Q$ is a rational regular matrix. Let I be an identity matrix of the order same as L. Then, $R = I \otimes_p Q$ is also a rational regular orthogonal matrix such that $R^T(L \otimes_p A(G))R = L \otimes_p A(G')$. Hence, from Lemma 6.15. the graphs $G_{L\otimes_p A(G)}$ and $G_{L\otimes_p A(G')}$ are cospectral with cospectral complements. □

6.3 Construction V: Congruence

Suppose the blocks C_i's and C_{ij}'s are zero blocks and $k = 1$. Then, $A(G) = \begin{bmatrix} 0 & D_1 \\ D_1^T & D \end{bmatrix}$ and $A(G') = \begin{bmatrix} 0 & D_1' \\ D_1'^T & D \end{bmatrix}$ are cospectral such that the orthogonal similarity matrix is $Q = diag(Q_{n_1}, I_{n_0})$ where $Q_{n_1} = \frac{2}{m} J_m - I_m$, D_1 is a $n_1 \times n_0$ matrix and D_0 is square matrix of order n_0. In this section, we consider the generalization of this special case we described.

Construction

Two $m \times n$ matrices A and B are called *congruent* if $A^T A = B^T B$.

Lemma 6.17. *Two real $m \times n$ matrices A and B are congruent if and only if there exists an orthogonal matrix Q such that $A = QB$.*

Proof. Suppose we have an orthogonal matrix Q such that $A = QB$. Then, $A^TA = (QB)^T(QB) = B^TQ^TQB = B^TB$. Now suppose we have $A^TA = B^TB$. Let $x \in \mathbb{R}^n$, then $\langle Ax, Ax\rangle = \langle x, A^TAx\rangle = \langle x, B^TBx\rangle = \langle Bx, Bx\rangle$. This shows $\|Ax\| = \|Bx\|$ for all $x \in \mathbb{R}^n$. Then, $ker(A) = ker(B)$ and hence $range(A) = range(B)$. Suppose $rank(A) = k$ and let $range(A) = span\{v_1, v_2, \ldots, v_k\}$ such that $\{Av_1, Av_2, \ldots, Av_k\}$ and $\{Bv_1, Bv_2, \ldots, Bv_k\}$ are orthonormal basis for $range(A)$ and $range(B)$ respectively. Since $range(A) = range(B)$, there exists an orthogonal matrix Q such that $Av_i = QBv_i$ for each $1 \leq i \leq k$. This shows $Av = QBv$ for any $v \in range(A)$. Let $x \in \mathbb{R}^n$, then $x = v + w$ for $v \in range(A)$ and $w \in ker(A)$. Now $Ax = Av + Aw = QBv = QBv + QBw = QBx$. Hence, $Ax = QBx$ for all $x \in \mathbb{R}^n$. This shows $A = QB$. □

Hence, if A and B are two $m \times n$ real matrices, then $A^TA = B^TB$ holds if and only if $A = QB$ for some orthogonal matrix Q. Similarly, $AA^T = BB^T$ holds if and only if $A = BR$ for some orthogonal matrix R.

Let H be an adjacency matrix of a graph on n vertices and let A be an $m \times n$ 0-1 matrix. Then $H(A)$ is defined to be the matrix $\begin{bmatrix} 0 & A \\ A^T & H \end{bmatrix}$. The following theorem generalizes the special case of Godsil-Mckay switching we described earlier.

Theorem 6.18. *[5] Let A and B be two congruent matrices, then $H(A)$ and $H(B)$ are cospectral.*

Proof. Suppose A and B are congruent, then there exists an orthogonal matrix Q such that $B = QA$. Consider an orthogonal matrix $R = \mathcal{I}(Q, I)$, then $RH(A)R^T = H(B)$. Hence, $H(A)$ and $H(B)$ are similar and cospectral. □

One source of congruent matrices comes from GM-switching. Let $R = \frac{1}{m}J_{2m} - I_{2m}$ and let C be a $2m \times (n-2m)$ 0-1 matrix such any columns of C has exactly 0, m or $2m$ nonzero entries. Then RC is also a 0-1 matrix. Let $D = RC$. Since, R is an orthogonal matrix, D is a 0-1 matrix and C and D are congruent.

Now we apply the idea of partitioned tensor product. Let X and H be an adjacency matrices of graphs on n and q vertices respectively, and A and B be two congruent $p \times q$ 0-1 matrices. Let U be an $m \times n$ 0-1 matrix. (Note the notational change in the sizes of the matrices). Then, $X(U) = \begin{bmatrix} 0 & U \\ U^T & X \end{bmatrix}$.

Lemma 6.19. *If A and B are congruent, then $X(U) \otimes_p H(A)$ and $X(U) \otimes_p H(B)$ are cospectral.*

Proof. Since A and B are congruent, there exists an orthogonal matrix Q such that $B = QA$. Let $R = \mathcal{I}(Q, I)$, then $RH(A)R^T = H(B)$. Since R is block diagonal, the partitioned tensor product $Q_0 = I \otimes_p R$ is also an orthogonal matrix, where I is an identity matrix. Then, $Q_0(X(U) \otimes_p H(A))Q_0^T = X(U) \otimes_p H(B)$, and $X(U) \otimes_p H(A)$ and $X(U) \otimes_p H(B)$ are cospectral. □

Isomorphism

Let us first discuss the concept of weak permutation matrices which will be useful in the investigation of the isomorphism of the corresponding graphs. Consider a $2n \times 2n$ partitioned permutation matrix $P = \begin{bmatrix} * & P_0 \\ * & * \end{bmatrix}$, where the block P_0 has size $n \times n$. If the matrix P_0 has a nonzero entry, then all other entries in that row and column are zero, but P_0 can also admit zero rows and zero columns. In this way, the matrices P_0 are different that the permutation matrices, let's call such matrices P_0 weak permutation matrices. We assume that a weak permutation matrix is nonzero. A permutation matrix is a weak permutation matrix, but a weak permutation need not be a permutation matrix.

Lemma 6.20. *Let P_0 be a weak permutation matrix of order n and let G be a graph on n vertices. Then, $P_0^T A(G) P_0 = 0$ implies $P_0 = 0$ if and only if G is reflexive.*

Proof. The matrix P_0 is permutationally similar to $\begin{bmatrix} P_0' & 0 \\ 0 & 0 \end{bmatrix}$, where P_0' is either a permutation matrix or a zero matrix. Hence, there exists a permutation matrix R such that $P_0 =$

$R^T \begin{bmatrix} P_0' & 0 \\ 0 & 0 \end{bmatrix} R$. Suppose $RA(G)R^T = \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_4 \end{bmatrix}$. Then,

$$\begin{aligned} P_0^T A(G) P_0 &= R^T \begin{bmatrix} P_0'^T & 0 \\ 0 & 0 \end{bmatrix} RA(G)R^T \begin{bmatrix} P_0' & 0 \\ 0 & 0 \end{bmatrix} R \\ &= R^T \begin{bmatrix} P_0'^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_2^T & A_4 \end{bmatrix} \begin{bmatrix} P_0' & 0 \\ 0 & 0 \end{bmatrix} R \\ &= R^T \begin{bmatrix} P_0'^T A_1 P_0' & 0 \\ 0 & 0 \end{bmatrix} R. \end{aligned}$$

Since R is a permutation matrix, $P_0^T A(G) P_0 = 0$ implies $P_0'^T A_1 P_0' = 0$. If P_0' is a permutation matrix, then $A_1 = 0$. If P_0' is a zero matrix, then there are no conditions on A_1. Since $P_0' = 0$ implies $P_0 = 0$, if $P_0^T A(G) P_0 = 0$ for some weak permutation matrix P_0, then either $A_1 = 0$ or $P_0 = 0$.

Suppose the graph G is reflexive, then all the diagonal entries of $A(G)$ are 1. Hence, A_1 is never zero and $P_0 = 0$. Now suppose the graph G is not reflexive and there is one vertex which does not have a loop. Then, the adjacency matrix can be written as $A(G) = \begin{bmatrix} 0 & A_2 \\ A_2^T & A_4 \end{bmatrix}$ where the block A_4 has size $n-1 \times n-1$. Let $P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0_{n-1} \end{bmatrix}$ where 0_{n-1} is a $n-1 \times n-1$ zero matrix. Then, $P_0^T A(G) P_0 = 0$ but $P_0 \neq 0$. This proves the result. □

The following theorem gives equivalent condition for the isomorphism of the congruence construction given in Theorem 6.18.

Theorem 6.21. *Let G_H be a reflexive graph. Then, the graphs $G_{H(A)}$ and $G_{H(B)}$ are isomorphic if and only if there exists two permutation matrices P_1 and P_2 such that $P_1 A P_2^T = B$ and $P_2 H P_2^T = H$.*

Proof. Let $G_{H(A)}$ and $G_{H(B)}$ be isomorphic. Then there exists a permutation matrix $P =$

$\begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ such that $PH(A)P^T = H(B)$. Consider,

$$\begin{aligned} H(B) &= PH(A)P^T \\ &= \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} 0 & A \\ A^T & H \end{bmatrix} \begin{bmatrix} P_1^T & P_3^T \\ P_2^T & P_4^T \end{bmatrix} \\ \begin{bmatrix} 0 & B \\ B^T & H \end{bmatrix} &= \begin{bmatrix} P_2A^TP_1^T + P_1AP_2^T + P_2HP_2^T & P_2A^TP_3^T + P_1AP_4^T + P_2HP_4^T \\ P_4A^TP_1^T + P_3AP_2^T + P_4HP_2^T & P_4A^TP_3^T + P_3AP_4^T + P_4HP_4^T \end{bmatrix} \end{aligned}$$

Comparing both sides, we have $P_2A^TP_1^T + P_1AP_2^T + P_2HP_2^T = 0$. All the matrices on the left side are 0-1 matrices, hence $P_2HP_2^T = 0$. Since the graph G_H is reflexive, from Lemma 6.20, it cannot be weak permutationally similar to a zero matrix. Hence, P_2 is a zero matrix. It follows P_3 is also a zero matrix, since P needs to be a permutation matrix. Then P_1 and P_4 are permutation matrices and we have, $P_1AP_4^T = B$ and $P_4HP_4^T = H$.

Now conversely, suppose $P_1AP_2^T = B$ and $P_2HP_2^T = H$ holds for some permutation matrices P_1 and P_2. Let $P = \mathcal{I}(P_1, P_2)$, then P is a permutation matrix satisfying $PH(A)P^T = H(B)$. Hence, $H(A)$ and $H(B)$ are isomorphic. □

The following lemma gives a sufficient condition for the isomorphism of the graphs constructed in Theorem 6.19.

Lemma 6.22. *If there exists two permutation matrices P_1 and P_2 such that $P_1^TAP_2 = B$ and $P_2^THP_2 = H$, then $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are isomorphic.*

Proof. Suppose there exists two permutation matrices P_1 and P_2 such that $P_1^TAP_2 = B$ and $P_2^THP_2 = H$. Let $P = \mathcal{I}(P_1, P_2)$, then $P^TH(A)P = H(B)$. Since P is block diagonal, the partitioned tensor product $P_0 = I \otimes_p P$ is also a permutation matrix, where I is an identity matrix whose order is same as $X(U)$. Then $P_0^T(X(U) \otimes_p H(A))P_0 = X(U) \otimes_p H(B)$, hence $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are isomorphic. □

Now to obtain a necessary condition for the isomorphism, we need to recall property η_1. We restate the definition with the new terms.

Definition 6.23. *The graphs $G_{X(U)}$ and $G_{H(A)}$ are said to satisfy property η_1 if the graphs $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are isomorphic, then there exists an isomorphism between them* *such that it takes the copy of $G_{X\otimes H}$ in $G_{X(U)\otimes_p H(A)}$ to the copy of $G_{X\otimes H}$ in $G_{X(U)\otimes_p H(B)}$, in other words, the induced isomorphism is an automorphism for the induced subgraph $G_{X\otimes H}$.*

Now we give the necessary condition for the isomorphism.

Lemma 6.24. *Let U and H be nonzero, and let A and B have no zero rows or zero columns. Let $G_{X(U)}$ and $G_{H(A)}$ satisfy property η_1. If $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are isomorphic, then A and B are permutationally equivalent.*

Proof. Suppose $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are isomorphic. Then there exists a permutation matrix P such that $P^T(X(U)\otimes_p H(A))P = X(U)\otimes_p H(B)$. From property η_1, P must be block diagonal matrix with form $\mathcal{I}(P_1, P_2)$. Then, we have $P_1^T(U\otimes A)P_2 = U\otimes B$ and $P_2^T(X\otimes H)P_2 = X\otimes H$. Using cancellation law (2.14.) in the first equation, there exists permutation matrices R_1 and R_2 such that $R_1^T A R_2 = B$. $\square$

Hence, under the assumptions of this lemma, if A and B are congruent but not permutationally equivalent, then $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are nonisomorphic and cospectral. In case, automorphism group $Aut(G_H)$ the graph G_H is the group S_q (the set of all permutation matrices of order q), then the necessary condition is also the sufficient one. Examples of those cases are $H = I_q$ and $H = J_q$.

Theorem 6.25. *Let G_H be a reflexive graph on q vertices such that $Aut(G_H) = S_q$. Let U be nonzero and A and B have no zero rows and zero columns. Let $G_{X(U)}$ and $G_{H(A)}$ satisfy property η_1. Then, the graphs $G_{X(U)\otimes_p H(A)}$ and $G_{X(U)\otimes_p H(B)}$ are nonisomorphic if and only if A and B are not permutationally equivalent.*

Hence, we must find matrices A and B such that they are congruent but not permutationally equivalent, to obtain cospectral nonisomorphic graphs using this construction. Note that if X has no diagonal entries, the construction produces graphs with no loops. This justifies why the matrix H was allowed nonzero diagonal entries. We leave characterization of property η_1 for this construction as an open problem, but investigate it for a modified construction in the next section.

6.4 Construction VI: Unfolding a semi-reflexive bipartite graph

In this section, we discuss the case when H to be an identity matrix. We replace A and B by B and B^T. The next theorem shows how the assumption of congruence of B and B^T, that is, the normality of B can be dropped for this particular case. Let B be a square 0-1 matrix and let I be the identity matrix of the same order p such that $I(B) = \begin{bmatrix} 0 & B \\ B^T & I \end{bmatrix}$. Let X be an adjacency matrix of a graph on n vertices and let U be some $m \times n$ 0-1 matrix such $X(U) = \begin{bmatrix} 0 & U \\ U^T & X \end{bmatrix}$.

Theorem 6.26. *The matrices $I(B)$ and $I(B^T)$ are cospectral. The matrices $X(U) \otimes_p I(B)$ and $X(U) \otimes_p I(B^T)$ are also cospectral.*

Proof. Since B is a square matrix, from Lemma 2.7. there exists orthogonal matrices Q_1 and Q_2 such that $Q_1^T B Q_2 = B^T$. Taking transposes on both sides, we have $Q_2^T B^T Q_1 = B$. The diagonal block matrix $Q = \mathcal{I}(Q_1, Q_2)$ is also orthogonal. Consider

$$\begin{aligned} Q^T I(B) Q &= \begin{bmatrix} Q_1^T & 0 \\ 0 & Q_2^T \end{bmatrix} \begin{bmatrix} 0 & B \\ B^T & I \end{bmatrix} \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & Q_1^T B Q_2 \\ Q_2^T B^T Q_1 & Q_2^T Q_2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & B^T \\ B & I \end{bmatrix} \\ &= I(B^T) \end{aligned}$$

This shows $I(B)$ and $I(B^T)$ are similar, hence cospectral.

Since Q is orthogonal, the partitioned tensor product $R = I_{m+n} \otimes_p Q$ is also orthogonal.

Consider

$$\begin{aligned}
R^{-1}(X(U)\otimes_p I(B))R &= (I_{m+n}\otimes_p Q)^{-1}(X(U)\otimes_p I(B))(I_{m+n}\otimes_p Q)\\
&= (I_{m+n}^{-1}X(U)I_{m+n})\otimes_p (Q^{-1}I(B)Q)\\
&= X(U)\otimes_p I(B^T)
\end{aligned}$$

This show $X(U)\otimes_p I(B)$ and $X(U)\otimes_p I(B^T)$ are similar, hence cospectral. □

Lemma 6.27. *The graphs $G_{I(B)}$ and $G_{I(B^T)}$ are isomorphic if and only if B is PET.*

Proof. Since G_I is a reflexive graph, the result follows as a corollary of Theorem 6.21. □

Theorem 6.28. *Suppose U is nonzero and B does not have zero row or column. Suppose $G_{X(U)}$ and $G_{I(B)}$ satisfy property η_1. Then, the graphs $G_{X(U)\otimes_p I(B)}$ and $G_{X(U)\otimes_p I(B^T)}$ are nonisomorphic if and only if B is non-PET.*

Proof. Since G_I is a reflexive bipartite graph, the result follows a corollary of Theorem 6.25. □

We now give a partial characterization of property η_1 for this construction. Let G be a semi reflexive bipartite graph with the adjacency matrix $A(G) = I(B)$ and with vertex partitioning given by $V(G) = X \cup Y$ such that every vertex in Y has a loop. Let $\Gamma_1 = G_{X(U)\otimes_p I(B)}$ and $\Gamma_2 = G_{X(U)\otimes_p I(B^T)}$ be graphs with vertex partitioning given by $V(\Gamma_i) = X_i \cup Y_i$ for $i = 1, 2$ as indicated in the adjacency matrices below,

$$X(U)\otimes_p I(B) = \begin{bmatrix} 0 & U\otimes B \\ U^T\otimes B^T & X\otimes I \end{bmatrix}$$

$$X(U)\otimes_p I(B^T) = \begin{bmatrix} 0 & U\otimes B^T \\ U^T\otimes B & X\otimes I \end{bmatrix}$$

Let $u_i, u_i', b_i, b_i', x_i$ denote the i^{th} row sum of the matrices U, U^T, B, B^T, X respectively.

Lemma 6.29. *Suppose $U = J_{m,n}$, $X = J_n - I_n$, then $X(U)$ and $I(B)$ satisfy property η_1 in the following cases:*

1. *$m = 1$, $n > 1$ and at least one row B has row sum at least 2.*

2. m divides n, $m \neq 1$

Proof. Let Γ_1 and Γ_2 be isomorphic and let f be an isomorphism from Γ_1 to Γ_2. Let $x \in X_1$ be the vertex of maximum degree in this set. Then, we will show that $f(x) \in X_2$. Suppose on the contrary $f(x) \in Y_2$. Then, $d_{\Gamma_1}(x) = u_i b_j$ for some $1 \leq i \leq m$, $1 \leq j \leq p$ and $d_{\Gamma_2}(f(x)) = u'_k b_l + x_k$ for some $1 \leq k \leq n$, $1 \leq l \leq p$. Since the isomorphism preserves the degrees, we have $u_i b_j = u'_k b_l + x_k$. Also since $U = J_{m,n}$, $X = J_n - I_n$, we have $nb_j = mb_l + n - 1$. Since x has maximum degree in X_1, $b_j \geq b_l$ for any $1 \leq l \leq p$.

Case 1: Suppose $m = 1$, $n > 1$ and at least one row of B has row sum at least 2. Then consider $(b_j - b_l) + (n-1)(b_j - 1) = 0$. Since $n \neq 1$ and $b_j \geq b_l$, we have $b_j - 1 \leq 0$. Then, $1 \geq b_j \geq b_l$. This is a contradiction since at least one row of B has row sum at least 2.

Case 2: Suppose $n = km$ for some k and $m \neq 1$. Then, $kmb_j = mb_l + km - 1$, that is, $kb_j - b_l = k - \frac{1}{m}$. Since the LHS is an integer and the RHS is not, this gives a contradition.

Hence, $f(x) \in X_2$ in any of the cases above. Removing the vertices x and $f(x)$ respectively from Γ_1 and Γ_2 and repeating the same procedure for other vertices in set X_1, we show $f(X_1) = X_2$ and hence $f(Y_1) = Y_2$.

□

Now the following example demonstrates how Theorem 6.28 can be thought of as unfolding a semi reflexive bipartite graph or a modification of the unfolding operation on the bipartite graphs.

Example 6.30. *Let $U = j_n^T$ be the all-one vector of length $n > 1$ and X be the adjacency matrix of a complete graph on n vertices, that is, $X = J - I$. Let the matrix B have no zero rows or zero columns and at least one row with row sum at least 2. Then,*

$$X(U) \otimes_p I(B) = \begin{bmatrix} 0 & B & B & \dots & B \\ B^T & 0 & I & \dots & I \\ B^T & I & 0 & \dots & I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^T & I & I & \cdots & 0 \end{bmatrix};$$

$$X(U) \otimes_p I(B^T) = \begin{bmatrix} 0 & B^T & B^T & \dots & B^T \\ B & 0 & I & \dots & I \\ B & I & 0 & \dots & I \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B & I & I & \cdots & 0 \end{bmatrix}$$

From the Theorem 6.26., the graphs $G_{X(U)\otimes_p I(B)}$ and $G_{X(U)\otimes_p I(B^T)}$ are cospectral and from Theorem 6.28. they are nonisomorphic if and only if B is PET. This specific construction also appears as the Construction-III in [14], but here we give the equivalent condition for its isomorphism.

Let $n = 2$, then $U = j_2^T$ and $X = J_2 - I_2$. Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then B has a row with row sum 2 but also a zero row. Recall that the condition 'B has no zero rows or zero columns' was required in proving equivalent conditions of isomorphism. Hence, this example will show that this assuption can be dropped in some cases and we can still obtain cospectral nonisomorphic graphs.

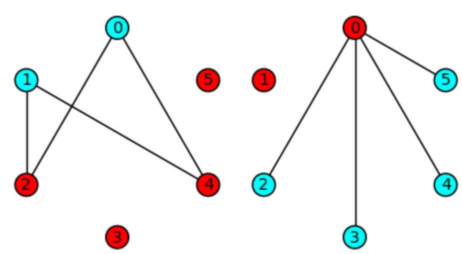


Figure 6.2: Unfoldings of a bipartite graph

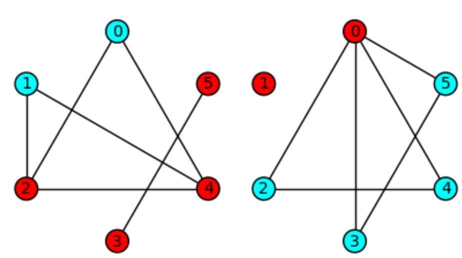


Figure 6.3: Unfoldings of a semi reflexive bipartite graph

Figure 6.2. shows the unfoldings of a bipartite graph corresponding to B and given by the

adjacency matrices

$$\begin{bmatrix} 0 & B & B \\ B^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & 0 \\ B & 0 & 0 \end{bmatrix}$$

Figure 6.3. shows the unfoldings of a bipartite graph corresponding to B and given by the adjacency matrices

$$\begin{bmatrix} 0 & B & B \\ B^T & 0 & I \\ B^T & I & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & B^T & B^T \\ B & 0 & I \\ B & I & 0 \end{bmatrix}$$

Note that the unfoldings in both cases are cospectral and nonisomorphic since B is non-PET.

Remark 6.31. *We can also allow all the vertices in the graph $G_{I(B)}$ to have loops, but then the diagonal blocks will be the same, hence such construction would be the same as Construction I-B. The identity matrix in $\begin{bmatrix} 0 & B \\ B^T & I_n \end{bmatrix}$ can be replaced by $\mathcal{I}(I_{n-1}, 0)$ if B admits Q_2 such that $Q_2BQ_2 = B^T$ and $Q_2 = \mathcal{I}(Q, 1)$, where Q is orthogonal matrix of order $n-1$.*

Chapter 7

Other results and open problems

7.1 Natural number network

In this section, we first collect some properties of a Natural Number Network and give some computational evidence that this graph could be a DS graph, that is, any graph which is cospectral to it is also isomorphic.

Natural Number Network (NNN) is a Divisibility Graph on first n natural numbers denoted by G_n. The vertex set $V(G_n)$ is a set of first n natural numbers. If i divides j or j divides i, then (i, j) is an edge, that is, $(i, j) \in E(G_n)$. Since $Diam(G_n) = 2, \forall n \geq 3$, the distance matrix is $\Delta = 2J - 2I - A$.

Seidel switching on NNN implies,

Lemma 7.1. *Consider the graph G_n. The graph obtained by removing all the edges adjacent to the vertex 1 is Seidel-cospectral with the original graph.*

Proof. Since, the vertex 1 is adjacent to all other vertices, The first row of the adjacency matrix is $(0, 1, 1, \dots, 1)^T$. The adjacency matrix of G_n can be written as $A(G_n) = \begin{bmatrix} 0 & 1^T \\ 1 & A' \end{bmatrix}$ where A' is a square matrix of size $n-1$ and 1^T is a vector of length $n-1$. Then the Seidel matrix is given by $S(G_n) = \begin{bmatrix} 0 & -1^T \\ -1 & S' \end{bmatrix}$. The Seidel switch $S(\widetilde{G}_n) = \begin{bmatrix} 0 & 1^T \\ 1 & S' \end{bmatrix}$ is cospectral with $S(G_n)$. The adjacency matrix of the Seidel switch is $A(\widetilde{G}_n) = \frac{1}{2}(J - I - S(\widetilde{G}_n)) = \begin{bmatrix} 0 & 0^T \\ 0 & A' \end{bmatrix}$. The graph $\widetilde{G}_n$ is essentially obtained by removing all the edges adjacent to the vertex 1. □

Theorem 7.2. *Automorphism group of G_n is non-trivial for $n > 1$.*

Proof. The number of degree-one vertices in G_n is same as the number of primes i such that $n/2 < i \leq n$. Only when $n \in \{2, 4, 6, 10\}$, the number of degree-one vertices is exactly 1. If $n \notin \{2, 4, 6, 10\}$, there are at least 2 vertices with degree 1. Let f be an automorphism on $V(G)$ such that if (i, j) is an edge then $(f(i), f(j))$ is also an edge. Let p_1 and p_2 be two primes, then $(1, p_1)$ and $(1, p_2)$ are edges of G_n. Hence, under automorphism, $(f(1), f(p_1))$ and $(f(1), f(p_2))$ are also edges. Construct another automorphism g such that

$$g(x) = \begin{cases} f(p_2) & if \quad x = p_1 \\ f(p_1) & if \quad x = p_2 \quad x \\ f(x) & if \quad otherwise \end{cases}$$

If f is a trivial automorphism, then g is a non-trivial. Hence, automorphism group of G_n is non trivial for $n \notin \{2, 4, 6, 10\}$. The automorphism groups of G_2, G_4, G_6 and G_{10} are permutations groups defined using generators $(1, 2)$, $(2, 4)$, $(2, 6)(3, 4)$ and $(4, 8)$ respectively. This shows automorphism group of G_n is non-trivial for $n > 1$. □

We observe that the number of pendant vertices, that is, the number of degree one vertices in G_n is the number of primes p such that $n/2 < p \leq n$. Let $L(G_n)$ denote the Laplacian matrix of G_n and $\sigma(L(G_n))$ denote its Laplacian eigenvalues. Suppose $|\{$p is a prime : $n/2 < p \leq n\}| = k$. Then,

Theorem 7.3. *[10] Multiplicity of eigenvalue 1 in $L(G_n) = k$.*

Proposition 7.4. *Suppose $\{0, \lambda_2, \lambda_3, \ldots, \lambda_n\}$ are the Laplacian eigenvalues of a graph G on n vertices in the nondecreasing order. Construct a graph G' by adding a vertex v in G and drawing an edge between v and every other vertex in G. Then, the Laplacian eigenvalues of G' are $\{0, \lambda_2 + 1, \lambda_3 + 1, \ldots, \lambda_n + 1, n + 1\}$*

Proof. Let v be a vector of size n all of whose entries are 1 and let I_n be an $n \times n$ identity matrix. Let $L(G_n)$ and $L(G'_n)$ denote the Laplacian matrices of the graphs G_n and G'_n

respectively. Then,

$$L(G') = \begin{bmatrix} L(G_n) + I_n & -v \\ -v^T & n \end{bmatrix}$$

Let λ' be an eigenvalue of $L(G')$ with eigenvector Y where $Y = \begin{bmatrix} X^T & x_{n+1} \end{bmatrix}^T$ and $X = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$. Then,

$$\begin{aligned} \lambda' Y &= L(G')Y \\ &= \begin{bmatrix} L(G_n) + I_n & -v \\ -v^T & n \end{bmatrix} \begin{bmatrix} X \\ x_{n+1} \end{bmatrix} \\ \begin{bmatrix} \lambda' X \\ \lambda' x_{n+1} \end{bmatrix} &= \begin{bmatrix} L(G_n)X + X - v x_{n+1} \\ -v^T X + n x_{n+1} \end{bmatrix} \end{aligned}$$

Case 1 : $X = v$ and $x_{n+1} = 1$
That is, Y is an all one vector. Hence $\lambda' = 0$ and also $L(G_n)v = 0$

Case 2: $x_{n+1} = 0$
Then, $L(G_n)X = (\lambda' - 1)X$ and $v^T X = 0$. Then, $\sum_{i=1}^{n} x_n = 0$. Hence, if λ is an eigenvalue of $L(G_n)$ whose corresponding eigenvector X is orthogonal to the all-one vector v, then $\lambda + 1$ is an eigenvalue for $L(G')$. Since $v^T X = 0$, λ cannot be 0. Hence, if $\lambda_2, \lambda_3, \dots, \lambda_n$ are eigenvalues of $L(G_n)$, then $\lambda_2 + 1, \lambda_3 + 1, \dots, \lambda_n + 1$ are eigenvalues of $L(G_n')$.

The remaining one eigenvalue is obtained using the following corollary.

Corollary*[13.14 [6]] Let X be a graph on n vertices, then $\lambda_n(X) \leq n$. If the complement graph $\bar{X}$ has c components, then the multiplicity of n as an eigenvalue of the Laplacian $L(X)$ is $c - 1$.*

The complement $\bar{G}_n'$ has two connected components. Hence, the multiplicity of $n+1$ as an eigenvalue of $L(G_n')$ is 1. $\square$

Theorem 7.5. *Let H_n be an induced subgraph formed by removing the vertices $\{1\} \cup \{p \text{ is a prime} : n/2 < p \leq n\}$ from the G_n. Let $\sigma(L(H_n)) = \{0, \lambda_2, \lambda_3, \dots, \lambda_{n-k-1}\}$, then $\sigma(L(G_n)) = \{n, 0, \underbrace{1, 1, \dots, 1}_{k \text{ times}}, \lambda_2 + 1, \lambda_3 + 1, \dots, \lambda_{n-k-1} + 1\}$.*

Proof. If $\{0, \lambda_2, \lambda_3, \ldots, \lambda_{n-k-1}\}$ are the eigenvalues of $L(H_n)$, then the Laplacian eigenvalues of the disjoint union of the graph H_n and k isolated vertices in $\{\text{p is a prime} : n/2 < p \leq n\}$ is $\{\underbrace{0, 0, \ldots, 0}_{\text{k+1 times}}, \lambda_2, \lambda_3, \ldots, \lambda_{n-k-1}\}$. Then add the vertex 1 to this disjoint union and draw an edge between 1 and every other vertex. From Proposition 7.4., it follows that the eigenvalues of $L(G_n)$ are $\{n, 0, \underbrace{1, 1, \ldots, 1}_{\text{k times}}, \lambda_2 + 1, \lambda_3 + 1, \ldots, \lambda_{n-k-1} + 1\}$. $\square$

Not only Theorem 7.5. gives a proof for Theorem 7.3., but also shows that the Laplacian spectrum of G_n is completely determined by the induced subgraph H_n. Hence, the problem of finding Laplacian cospectral graphs to G_n reduces to finding Laplacian cospectral graphs to H_n. Suppose H_n and H'_n are Laplacian cospectral, then add a vertex v to H'_n which is adjacent to all the vertices. Then add k pendant vertices which are adjacent to only v. Call the new graph G'_n, then G_n and G'_n are Laplacian cospectral.

Based on the direct SageMath computations, we have the following propositions,

Proposition 7.6. *For $n = 1, 2, \ldots, 8$, G_n is DS for the adjacency, signless Laplacian and the normalized Laplacian matrix.*

Proposition 7.7. *For $n = 1, 2, \ldots, 11$, G_n is DS for the Laplacian matrix.*

That is, for $n = 1, 2, \ldots, 11$ any graph which is Laplacian cospectral with G_n is also isomorphic to it.

The adjacency matrices of G_n can be shown to be permutationally similar to the matrices of the form $\begin{bmatrix} 0 & B_n \\ B_n^T & H_n \end{bmatrix}$. Hence, construction V can be applied to find cospectral graphs as follows:

1. Find C_n's which are congruent with B_n, that is, $B_n^T B_n = C_n^T C_n$. Then the graph G'_n corresponding to the adjacency matrix $\begin{bmatrix} 0 & C_n \\ C_n^T & H_n \end{bmatrix}$ is cospectral with G_n.
2. Now from among these C_n's find the ones for which $P_1^T B_n P_2 = C_n$ and $P_2^T H_n P_2 = H_n$ doesn't hold for some permutation matrices P_1 and P_2. Then, G'_n and G_n will be nonisomorphic graphs.

7.2 PET matrices

In this thesis, construction of a non-PET matrix is necessary in producing cospectral nonisomorphic graphs. Let $\mathcal{M}_n$ be the set of all 0-1 matrices of order n which are PET and let $\mathcal{P}_n$ be the set of all 0-1 matrices of order n for which the set of row sums is the same as the set of column sums. Then Corollary 2.10 implies $\mathcal{M}_n \subseteq \mathcal{P}_n$, that is, $\mathcal{P}_n^c \subseteq \mathcal{M}_n^c$. Hence, $|\mathcal{P}_n^c|$ is a lower bound for $|\mathcal{M}_n^c|$, that is, for the number of $n \times n$ 0-1 non-PET matrices.

n	$\lvert\mathcal{M}_n\rvert$	$\lvert\mathcal{P}_n\rvert$	$\lvert\mathcal{M}_n^c\rvert$	$\lvert\mathcal{P}_n^c\rvert$	frac. of $\lvert\mathcal{P}_n^c\rvert$	frac. of $\lvert\mathcal{M}_n^c\rvert$
1	2	2	0	0	0	0
2	12	12	4	4	0.25	0.25
3	248	248	264	264	0.52	0.52
4	15428	18884	50108	46652	0.71	0.76
5	-	5651872	-	27902560	0.83	at least 0.83

Table 7.1: non-PET matrices

The numbers in Table 7.1) are obtained by direct SageMath computations. Since $\mathcal{M}_n^c = \mathcal{P}_n^c$ for $n = 1, 2, 3$, we have

Proposition 7.8. *Let M be a 0-1 matrix of order $n \leq 3$. Then, M is non-PET if and only if the set of row sums of M is different that the set of column sums of M.*

The following lemma gives a relation between a non-PET matrix and its submatrices.

Lemma 7.9. *Let M be a square matrix of order n and let i^{th} row of M be a permutation of j^{th} column for some $1 \leq i, j \leq n$. If the submatrix $M[i, j]$ is non-PET, then M is non-PET.*

Proof. Suppose on the contrary, that M is *PET*. Then, $P^T MQ = M^T$ holds for some permutation matrices P and Q. Then every row of M is a permutation of some column. Consider the permutation association $\{(i, \sigma_P(i))\}_{i=1}^n$ where i^{th} row is a permutation of $\sigma_P(i)^{th}$ column. The way they are permuted is given by Q. For some index i_0, remove the row-column pair $(i_0, \sigma(i_0))$ to obtain a submatrix $M[i_0, \sigma(i_0)]$. This submatrix still carries the permutation association $\{(i, \sigma(i))\}_{i \neq i_0}$, that is, for every $i \neq i_0$ and $i \in \{1, \ldots, n\}$ every row of $M[i_0, \sigma(i_0)]$ is a permutation of some column given by the matrix Q. Hence, $M[i_0, \sigma(i_0)]$ is *PET*.

Hence, there exists exactly n submatrices of order $n-1$ which are PET. This proves the lemma. □

7.3 Open problems

Problem 7.10. *Prove or disprove:*

1. *Construction V cannot produce a cospectral mate for Natural Number Network.*
2. *Natural Number Network is DS for adjacency, Laplacian, signless Laplacian and the normalized Laplacian.*

Problem 7.11.

1. *Give a combinatorial characterization for non-PET square matrices. Do the same for matrices with no zero rows and also for the matrices with no zero rows as well as no zero columns.*
2. *Show that the fraction of non-PET matrices of order $n \to 1$ as $n \to \infty$.*

We gave a few candidates for the matrices satisfying $C/M/T$ property which resulted in the constructions I-A, II-A, I-B, I-C, I-D and I-E. We showed construction I-D is trivial extension of the cancellation law and construction I-E produces cospectral but isomorphic graphs. The isomorphism results can be applied II-B, II-C and II-E which we left as an open problem, since characterization of property η_2 was not obtained for these constructions.

Problem 7.12.

1. *Give complete characterization of bipartitioned matrices satisfying $C/M/T$ property.*
2. *Give complete characterizations of property η_1 and η_2 for each construction.*

We also discussed the idea of unfolding a multipartite graph to obtain cospectral nonisomorphic graphs and gave some sufficient conditions for their isomorphism.

Problem 7.13.

1. *Find necessary and sufficient conditions for the graphs constructed using construction III-B to be isomorphic.*
2. *Study 0-1 matrices that are similar to their transpose via an involutory or orthogonal matrix*

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