AdS Instability

&

Nonlinear Perturbations in Confined Geometries

A thesis

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By

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To my Mom

Certificate

Certified that the work incorporated in the thesis entitled ("AdS instability & Nonlinear Perturbations in Confined Geometries"), submitted by Dhanya S. Menon was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or Institution.

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Abstract

In 2011, Bizoń and Rostworowski, through their seminal work, gave numerical evidence that Anti-de Sitter spacetime with reflecting boundary conditions is nonlinearly unstable against black hole formation for a wide range of initial conditions, under *arbitrarily small perturbations.* The results were particularly interesting because AdS has been known to be linearly stable. Numerical and perturbative analysis showed that the mechanism behind the instability was weak turbulence i.e. transfer of energy from low frequency to high frequency modes. The resonant nature of the spectrum was considered to be a crucial ingredient in driving this instability. Setups, which mimic AdS were also studied numerically, an example is—fields trapped in a spherical cavity in flat spacetime. While the mechanism which drives an instability in such systems were similar, there were important differences as well—specifically in those set-ups whose linear spectra were non-resonant. In such set-ups, a threshold amplitude was detected, below which the system remained stable for very long times. The value of the threshold amplitude would be too small, which would make its detection in numerical studies very difficult. The main objective of our thesis is to study such gravitational systems with confined geometries and obtain a deeper understanding of the nature of instabilities observed in them, by using the results in nonlinear dynamics. In the first part of the thesis, we study the necessary conditions for a

nonlinear instability to occur in similar gravity-scalar field systems. We also, take up the case of the AdS soliton and demonstrate that these results in nonlinear dynamics could be applied to (locally) asymptotically AdS spacetimes as well. In the second part, we study the gravitational perturbations of Minkowski enclosed in a spherical Dirichlet wall. We use the formalism by Ishibashi, Kodama and Seto to classify the metric perturbations according to their tensorial behavior on a sphere as the scalar, vector and tensor-type. We simplify the perturbation equations upto all orders of perturbation theory. We then apply the arguments developed in the first part to comment upon the nonlinear stability of the system. Finally, we study the gravitational perturbations of AdS spacetime in $(n+2)$ dimensions, with $n > 2$. We obtain the solutions to metric perturbations and render them asymptotically AdS at each order. We, for the first time, perform the higher order perturbative analysis of the tensor sector. As an example, we take a special case where the initial data contains only a single-mode tensor-type seed at the linear level. Interestingly, we find that there are no resonances at the second order.

Contents

Chapter I

Introduction

The past decade has seen exciting advancements in the area of nonlinear instability of Anti-de Sitter (AdS) spacetime. This thesis is motivated by these interesting developments, which prompted the question—what happens to small perturbations trapped in a spacetime with a 'box-like' or reflecting boundary condition?

AdS had an important role in theoretical physics, because of its application to the AdS/CFT conjecture, first proposed by Maldacena in [1]. The correspondence conjectured an equivalence between a gravity theory on asymptotically AdS spacetime and a conformally invariant field theory (CFT) living on the boundary of this spacetime. Small perturbations in Minkowski and de Sitter spacetimes remained small and in fact decayed to zero with time because of the dissipation of energy to infinity [2], [3]. However, the conformal infinity of AdS was timelike, hence in order to determine the evolution of fields in AdS, in addition to specifying the data on a $t = 0$ hypersurface, one also needed to impose suitable boundary conditions at the conformal infinity, so that the problem at hand made mathematical sense. One type of condition which ensured this were the reflecting boundary conditions, for which the flux of energy through the boundary was zero (the class of boundary conditions which made mathematical sense for vacuum Einstein equations with a negative cosmological constant was given by Friedrich in [4]). Although the linear stability of AdS, a much simpler problem, was well established [5], [6], [7], the question of the nonlinear stability still remained. This problem of nonlinear stability was non-trivial because of the following—massless fields could propagate to conformal infinity in finite time and get reflected back into the bulk. It was thus expected that perturbations, however small, could exhibit complex nonlinear gravitational interactions and strongly back-react on the AdS metric. The resultant long term behavior could have only been ascertained by numerical computations.

In 2011, Bizoń and Rostworowski, using numerical methods showed that, under the assumptions of spherical symmetry, the Einstein-massless scalar field system with a negative cosmological constant was unstable to the formation of a AdS-Schwarzschild black hole for a large class of initial conditions [8]. This held true for all spacetime dimensions equal to or greater than four [9], [10]. Moreover, in the case of AdS_3, it was seen that for small enough perturbations, although a black hole never formed, an instability was still observed [11]. These studies led the authors of [8] to conjecture that AdS is unstable against black hole formation for a large class of arbitrarily small perturbations. An analytical calculation in the same paper [8] considered weakly nonlinear perturbation theory and showed that the instability could be attributed to the linear level spectrum being commensurate (alternatively called "resonant spectrum"). For a general $n-$mode initial data with amplitude of perturbation ϵ, the commensurability of the frequencies led to the breakdown of perturbation theory at the third order, at time-scales of order ϵ^{-2}. This happened when the secular resonances which arose at the third order were irremovable. This resulted in solutions which grew linearly with time. From the energy spectrum, one could conclude that overall, there was a transfer of energy from low frequency modes to higher frequency modes (also called direct cascade of energy), which in turn led to concentration of energy to finer spatial scales. The transfer process then would eventually be cut off by black hole formation. This was the weakly turbulent instability and was very much analogous to the turbulent behavior observed in case of viscous fluids. (As pointed out by O. Dias and Jorge E. Santos [12], where nonlinear gravitational perturbations of pure AdS were considered, although the turbulent process involved both direct and inverse cascades of energy, the direct cascade tended to dominate.) Further studies by H.P. de Oliveira, Leopoldo A. Pando Zayas and E.L. Rodrigues [13] led to the observation that after the initial linear regime, the power spectrum of the Ricci scalar for this system followed the Kolmogorov-Zakharov spectrum, which was again an indicator of wave turbulence. A similar kind of numerical study was extended by Alex Buchel, Luis Lehner and Steven L. Liebling [14] to complex scalar fields and the results obtained were very similar to those of Bizoń and Rostworowski [8]. Note that, since the simulations were performed for small but finite perturbations, it was not possible to prove through numerical methods that a system will necessarily collapse at arbitrarily small amplitudes. (Interestingly, the rigorous proofs of the AdS instability conjecture, for set-ups different from the Einstein-scalar field system, were given by Moschidis [15], [16], where the instability in the Einstein-null dust system as well as the Einstein-massless Vlasov system were proved. While the former set-up by Moschidis [15] involved placing an inner mirror at a finite distance from the AdS origin, the latter set-up [16] made no such assumption.)

These observations led to a lot of questions. What was the role of AdS boundary in this kind of instability? Will the results be the same in any other gravitational system with reflecting boundary conditions? What was the role of the resonant nature of the linear eigenfrequencies in causing an 'AdS-type' instability? Several numerical studies were dedicated to probe these questions—for example, the numerical work [17], where Maliborski, under the assumptions of spherical symmetry, studied the evolution of a massless scalar field in a spherical cavity with perfectly reflecting walls. For both Dirichlet and Neumann boundary conditions, the results were very similar to that of P. Bizoń and A. Rostworowski [8], where the system exhibited turbulent behavior which led to eventual collapse to a black hole. But, the perturbative analysis showed the spectrum to be resonant in the Dirichlet case, and to be non-resonant in the Neumann case. This led to the assumption that the linear spectrum need not be fully resonant to trigger an instability. However, when the same experiment was repeated by Maliborski and Rostworowski in [18], it was found that for the Neumann case (which has a non-resonant spectrum), there exists a threshold amplitude below which no collapse takes place. Another numerical study by Hirotada Okawa, Vitor Cardoso and Paolo Pani [19] involved evolution of massive scalar fields in a spherical cavity with perfectly reflecting walls. For both Dirichlet and Neumann cases, the spectrum was non-resonant and yet a threshold amplitude was not detected.

In order to explain these results and obtain an understanding of the necessary conditions for an 'AdS-like' instability, we provided analytical arguments based on the results from nonlinear dynamics and KAM theory [20]. Our arguments explained the role of a resonant spectrum in triggering an instability and how this could be seen as an instability in a phase space. They also explained the occurrence of a threshold amplitude for asymptotically resonant spectrum, which could be too small to be detected by numerical methods.

We then discussed how these arguments can be applied to (locally) asymptotically spacetimes such as the AdS soliton as well. Since the numerical studies tell us that there is a direct cascade of energy from low modes to high modes, it was natural to deduce how the eigenfunctions associated with high mode number localize in space. We explored the same across various dimensions of spacetime. We found that the localization was least in case of AdS_3. This was consistent with the fact that in AdS_3, in spite of turbulent transfer of energy, a black hole does not form.

Although our arguments in [20] told us that a resonant spectrum was a necessary condition for an AdS-type instability to occur, it was not sufficient. For eg., studies by Nils Deppe, Allison Kolly, Andrew Frey and Gabor Kunstatter [21] (see also [22]) showed that the addition of Gauss-Bonnet terms to the Einstein action, prevented collapse for certain range of amplitudes, even though the linear spectrum was resonant.

Although initial investigations pointed towards AdS being nonlinearly unstable for generic perturbations, several non-collapsing solutions were also found. For example, in the earliest work of P. Bizoń and A. Rostworowski [8], it was observed that there existed initial conditions for which solutions to the Einstein-scalar field system did not collapse for a long time, when the perturbation was sufficiently small. In this case, it was shown that if the initial condition contained a single-mode data, one could perturbatively obtain solutions which were time-periodic till the third order (this is the order at which a multi-mode initial data will lead to secular growth). From this cue, the authors Maliborski and Rostworowski constructed time-periodic solutions (called 'oscillons') in [23] to the set-up in [8]. Such solutions were constructed starting from a single dominant mode at the linear order, both numerically and perturbatively. One could do a similar construction of time-periodic solutions with a complex scalar field. In this case, they were called 'boson stars'. Such boson stars were constructed by A. Buchel, L. Lehner and S. L. Liebling in [14] and it was shown that they are nonlinearly stable for sufficiently small perturbations [24]. Similarly, if one considered pure gravitational perturbations of AdS, then one could construct non-linear time-periodic solutions from a single mode. These were called geons and have been explored in context of the AdS instability problem in [12], [25], [26], [27], [28], [29], [30], [31]. Similarly, geons were also constructed from the AdS-soliton metric [32], [33]. A new family of black holes, called black resonators, were constructed by Dias, Santos and B. Way in [34]. These black holes were time-periodic and had a single Killing vector field. Their zero-size limit was connected to the geons. Since the black resonators were observed to have a higher entropy than the corresponding Kerr-AdS black holes they were expected to be possible candidates for the end point of superradiant instability. However, it was shown by B. E. Niehoff, J. E. Santos and Benson Way [35] that they were unstable. Many more non-collapsing solutions were found in [24], [36], [37] and [38]. For example, the initial data used in the AdS instability problem [8], [14] was a Gaussian-type wave-packet, with parameters being the width of the wave-packet as well as the amplitude of perturbation. It was observed by A. Buchel, S. L. Liebling and L. Lehner in [24] and Maliborski and Rostworowski in [36] that below a threshold amplitude of perturbation, the solutions remained stable if the width of the Gaussian wave-packet fell within a certain range. Likewise, Hirotada Okawa, Jorge C. Lopes and Vitor Cardoso [38] observed that if instead of a single Gaussian wave-packet, one used a multiple-Gaussian initial data to study the collapse of massless scalar fields in AdS, the solutions tended to be stable for long times. Such solutions not only featured a direct cascade of energy but featured an inverse cascade as well, which could partially explain their long-term stability. A spectral analysis of all such solutions showed that they were all, in fact, single-mode dominated [39]. Hence, the main feature of the non-collapsing solutions seemed to be that they were all single mode dominated. Further, Oscar J.C. Dias, Gary T. Horowitz, Don Marolf and Jorge E. Santos, using perturbative analysis, argued that the asymptotically AdS solutions such as boson stars, geons and AdS-Schwarzschild black hole were nonlinearly stable except possibly in higher dimensions. This was because the spectrum of these solutions approached the normal modes of AdS at large radial distances. It was precisely these modes which could potentially trigger an instability as they were close to the normal modes of AdS. Such modes were also associated with a large angular momentum as they were supported at large radial distances because of the centrifugal barrier. The authors used formal perturbation theory to argue that such modes were not strong enough to cause an instability, except possibly in higher dimensions.

In [41], Choptuik, Jorge E. Santos and Benson Way proposed a special kind of non-collapsing infinite-parameter family of solutions in AdS, to which boson stars belonged as well. These were called multi-oscillators. These solutions oscillated on any number of non-commensurate frequencies. The stability of double-oscillators (a multi-oscillator oscillating on two frequencies), when subjected to different kinds of boundary conditions, were explored by R. Masachs and B. Way [42] (see chapter II for a discussion).

As seen in [8], the standard perturbation theory would break down because of the commensurate frequencies of the linear perturbations around AdS. Hence, resummation techniques were used in addition to the standard perturbation theory. These were alternatively called the two-time framework (or TTF developed by V. Balasubramanian, A. Buchel, S. R. Green,

L. Lehner and S. L. Liebling [43]), renormalization group method [46] and resonant approximation [49] (see also [44], [45], [47], [48], [50], [51]). These techniques were all equivalent and captured energy transfer across modes upto time scales of the order ϵ^{-2}. The secular terms would get absorbed into the 'renormalized amplitudes and phases'. The renormalization flow/TTF equations not only allowed one to construct stable quasi-periodic (QP) solutions but the inherent scaling symmetry within these equations made it possible to probe the zero-amplitude limit, otherwise inaccessible to numerical simulations. This was especially important as numerical simulations could only consider small but finite amplitudes of perturbations and collapse at a particular amplitude did not guarantee that solutions would necessarily collapse at smaller amplitudes. Apart from [43], stable quasi-periodic (QP) solutions were constructed using TTF by Stephen R. Green, Antoine Maillard, Luis Lehner and Steven L. Liebling in [52] . A stability analysis of these QP solutions led to the conclusion that all the stable solutions, sufficiently close to these QP solutions formed 'islands of stability' [52], (see also [53] by Steven L. Liebling and Gaurav Khanna).

It is useful to see the behaviour of the interaction coefficients, which arise in the TTF equations, as they tell us how effective the resonant transfer of energy across modes is [48], [54], [55], [56]. For example, in case of a self-interacting probe ϕ^4 scalar field in AdS (where the backreaction from the metric was not considered), Pallab Basu, Chethan Krishnan and P.N. Bala Subramanian [54] observed through TTF that if one starts with only low-lying modes in the initial condition, then thermalization does not take place within the time-scales in which TTF is valid. In case of the full nonlinear theory, Ben Craps, Oleg Evnin, Puttarak Jai-akson and Joris Vanhoof [48] observed the behavior of the interaction coefficients in the large mode limit, from which it could be concluded that the turbulent instability was stronger in higher dimensions.

Note that, the collapse of solutions in AdS [8] was attributed to weak turbulence. This result was obtained on the assumption that the phases of eigenfunctions were randomly distributed. However, Ben Freivogel and I-Sheng Yang [57] showed that for the same power law $E_p \sim (p+1)^{-\alpha}$, the back reaction on the metric, which eventually leads to collapse, was much stronger in case the phases were fully coherent as compared to the situation where they were random. In fact, the two-mode initial data, which was prone to collapse, tended to have initially phase coherent energy cascade [57], (also [58] by Fotios V. Dimitrakopoulos, Ben Freivogel and Juan F. Pedraza).

The discovery of such a range of solutions prompted a question—is AdS generically stable or unstable? In other words, what happens to the set of collapsing and non-collapsing solutions as the amplitude of perturbation tends to zero [59], [60]? The authors F. Dimitrakopoulos and I-Sheng Yang [60] argued that the set of non-collapsing solutions continued to persist in the zero-amplitude limit (although a similar argument regarding collapsing solutions could not be made).

The collapse of massless scalar fields in flat spacetime was marked by a critical phenomenon (Choptuik [61]). Here, if ϵ referred to the amplitude of the wave-packet, then there existed a critical value of amplitude ϵ_*, such that if $\epsilon > \epsilon_*$, then a black hole formed immediately. Thus, the point ϵ_* acted as the threshold between black hole formation and dispersion. Additionally, near the critical solution, the mass of the black hole obeyed a power-law of the form, $M_{bh} \sim (\epsilon - \epsilon_*)^\gamma$. In case of AdS, there were a sequence of critical points ϵ_n, such that for all $\epsilon_{n+1} < \epsilon < \epsilon_n$, the collapse took place only after the scalar wave-packet made exactly n bounces from the AdS boundary. The points lying on the right and the left neighbourhood of these critical points followed certain scaling laws. The scaling laws near the critical solution were explored in [8], [62] and [63]. An interpretation on what happens on the CFT side at criticality was explored in [64] (see chapter II for details).

There are a few more systems of interest worth mentioning. For example, the hard-wall model [65], [66], [67] is used to study holographic thermalization in confining field theories. Here, a portion of the AdS spacetime is cut off by an infrared wall at a finite distance from the origin. The energy is then injected by introducing a massless scalar field. Ben Craps, Elias Kiritsis, Christopher Rosen, Anastasios Taliotis and Joris Vanhoof [65], using perturbative analysis showed that for sufficiently fast energy injection, the end result would be a black brane whereas for slow injection of energy, the scalar field would keep on scattering back and forth from the infrared wall to the AdS boundary. In order to study the long term evolution of the scattering solutions, this set-up was numerically studied by Ben Craps, E.J. Lindgren, Anastasios Taliotis, Joris Vanhoof and Hong-bao Zhang [66]. Through an analytical argument, they showed as to why the scattering solutions will never form a black brane. Another set-up was the collapse of self-interacting ϕ^4 field in the Einstein background with a negative cosmological constant, which was explored by Rong-Gen Cai, Li-Wei Ji and Run-Qiu Yang [68]. In this case, it was interesting to see that the interaction coefficient could either assist or inhibit collapse, depending on whether it was positive or negative. Eunseok Oh and Sang-Jin Sin [69] studied the collapse of dust particles in AdS. This model was interesting as it was seen as a viable candidate for explaining the early thermalization in RHIC. Richard Brito, Vitor Cardoso and Jorge V. Rocha [70] studied the evolution of two thin, concentric shells of matter in AdS (the evolution of such shells in a cavity in Minkowski was studied by Vitor Cardoso and Jorge V. Rocha [71]). It was found that depending upon various parameters, such shells either underwent prompt collapse, or collapsed after they crossed each other a finite number of times or oscillated for a long time. Hirotada Okawa, Vitor Cardoso and Paolo Pani [72] observed the evolution of massive fields in Minkowski and found that depending upon parameters like the width of the scalar pulse and the amplitude of perturbation, the system can exhibit two different kinds of behaviors—type-I and type-II collapse. While type-I collapse was similar to the collapse of massless scalar fields in AdS, type-II collapse was similar to the collapse of massless fields in flat spacetime. Another interesting set-up was the evolution of massive scalar fields in AdS. Brad Cownden, Nils Deppe and Andrew R. Frey [73] numerically studied the phase diagram for the stability of massive fields in AdS. Depending upon parameters like the mass of the scalar field and width of the initial data, the system could either have stable, unstable, meta-stable or chaotic behavior. Here, the meta-stable initial data were those which resulted in collapse in time scales much larger than suggested by the lowest order perturbation theory. Certain initial data which resulted in chaotic behavior were those which showed abrupt changes in horizon formation times. (more details in Chapter II).

Although a majority of the numerical works have been restricted to evolution of fields under the assumption of spherical symmetry, there have been some works which relaxed this assumption. In [74], H. Bantilan, P. Figueras, M. Kunesch and P. Romatschke studied the evolution of a massless scalar field in a non-spherically symmetric background with negative cosmological constant and compared the collapse with the spherically-symmetric case. Likewise in [75], Choptuik, O. J.C. Dias, J. E. Santos and B. Way studied the evolution of a complex doublet scalar field in a non-spherically symmetric asymptotically AdS background [77]. While these works still involved preserving symmetry in certain directions, the work by Bantilan, P. Figueras and L. Rossi [76] studied the evolution of a massless scalar field with no symmetry assumptions. While [74], [75] and [76] dealt with evolution of scalar fields in AdS, the work by Bizoń and Rostworowski [77] dealt with pure gravitational perturbation of AdS_5, with a cohomogeneity-two-biaxial Bianchi IX metric ansatz.

Besides numerics, perturbation theory was used to study the nonlinear perturbations of pure AdS. The earliest work in this direction was by O. J. C. Dias and Jorge E. Santos [25], which threw light upon the secular resonances arising at the various orders of perturbation theory. D. Hunik-Kostyra and A. Rostworowski [78] studied the resonant system for the gravitational perturbations of AdS_5 in the cohomogeneity-two biaxial Bianchi IX ansatz. Further discussions on the presence/absence of resonances as well as discussion on the perturbative construction of geons can be found in [12], [26], [27], [28], [29], [79]. In all these studies, the secular resonances entered only at the third order. In [80], we took up the gravitational perturbations of flat space enclosed in a Dirichlet wall. While the linear stability of this set-up was established in [81], we studied the nonlinear stability of the set-up by extending our analytical arguments in [20] for the pure gravity case as well. In [82], we used weakly nonlinear perturbation theory to study gravitational perturbations of AdS in dimensions higher than four. The key feature here is that unlike four dimensions, in higher dimensions along with the scalar and vector-type perturbations, the tensor-type perturbations are excited as well. We thus analyzed the tensor sector of the perturbation equations till the second order. For the example we took, we found that there were no secular resonances excited at the second order. The absence of certain resonant channels had been a key feature in the AdS instability problem. The discussion regarding absence of such resonant channels and their relation to the AdS isometries were explored by I-Sheng Yang in [83], Oleg Evnin and Chethan Krishnan in [84] and Oleg Evnin and Rongvoram Nivesvivat in [85]. The main objective of these works was to obtain the selection rules in a more elegant fashion rather than the brute force technique used before, for eg. in [46].

As mentioned earlier, one of the main applications of studying the stability of AdS is the AdS/CFT correspondence, as it is desirable to know how the discussion on nonlinear perturbations of AdS translate to the boundary theory. From the perspective of gauge/gravity correspondence, while the presence of collapsing solutions is expected, it is the presence of non-collapsing solutions which is more intriguing. This is because they are dual to CFTs states that never thermalize. Interesting CFT interpretations include [86] by Javier Abajo-Arrastia, Emilia da Silva, Esperanza Lopez, Javier Mas & Alexandre Serantes and [87] by Emilia da Silva, Esperanza Lopez, Javier Mas & Alexandre Serantes, where the authors studied the out of equilibrium dynamics of finite sized closed quantum systems in $2 + 1$ dimensions.

This thesis is dedicated to obtain an analytical understanding of the conditions required to trigger an instability in enclosed geometries. Much of the material is based on our work [20], [80] and [82]. In chapter II, we will first describe in detail the AdS instability problem and the subsequent works, which led to the questions regarding the role of a boundary and the linear spectra, in determining the stability of a system in a box. We then give the details of the analytical arguments we developed, to understand the numerical observations which led to these questions. These arguments are based on the KAM theory and help in understanding the role of the linear spectrum in triggering an instability. We will also review many interesting developments in the field of the AdS instability problem, which has happened since the work in [8], to obtain a general idea about the various caveats in this field. Although the framework developed in this chapter is in the context of the Einstein-scalar field system, in chapter III, we show that it can be applied to the pure gravity case as well. Here, we take up Minkowski with a Dirichlet wall and consider gravitational perturbations within this confined geometry. We then comment upon the nonlinear stability of this system, by applying the arguments developed in our previous work [20]. In chapter IV, we take up the study of pure gravitational perturbations of AdS in dimensions greater than four. In this case, apart from the scalar and vector type perturbations, we have the tensor-type perturbations as well. Once we obtain the perturbation equations, we do a perturbative analysis of the tensor sector of the equations till the second order of perturbation theory. We start with a single seed of tensor-type at the linear level, and show that there are no secular resonances at the second order. Chapter V contains the conclusions and the discussions.

Chapter II

AdS instability

In this chapter, we study the nonlinear perturbations of fields in spacetimes with reflecting boundary conditions, with special focus on Anti de-Sitter spacetime. As mentioned earlier, the interest in AdS mainly got piqued due to the AdS/CFT correspondence, first proposed by Maldacena in 1998 [1], which conjectures an equivalence between string theory on an asymptotically AdS spacetime and a conformally invariant quantum theory living on the boundary of spacetime. Prior to that, only the linear stability had been proved [5], [6]. The basic question about its nonlinear stability has only been addressed very recently.

Bizoń and Rostworowski, in their seminal work provided evidence that the spherically symmetric evolution of a massless scalar field in an Einstein background with a negative cosmological constant, collapsed and formed a black hole, for a large class of initial data [8]. Their work was numerical but they also used perturbative analysis to substantiate the numerical results. Since the work was numerical, the conjecture that the results hold true for arbitrarily small perturbations was based on extrapolating the data points to smaller amplitudes of perturbations.

The main theme of the chapter is to study the necessary conditions for an AdS-like instability. We first give an overview about maximally symmetric—AdS in particular in sections II.1-II.3. In sections II.4, we give a basic insight into the AdS instability problem. Here, we review the earliest works which involved studying the collapse of a scalar wave packet in AdS, under the assumptions of spherical symmetry. In section II.5, we review the instability observed in systems, where the AdS boundary has been replaced by an artificial wall in flat space. Once this background is set, in section II.6, we give the details of our work, which uses the results in nonlinear dynamics to gain insight into how the linear level spectrum can influence the nonlinearities subsequently. In sections II.7-II.15, we review the major developments that have taken place in the field of AdS instability. Finally, we conclude this chapter in section II.16.

II.1 Maximally symmetric spacetimes

The vacuum Einstein equation without a source given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0 \tag{II.1}$$

admits three maximally symmetric solutions, namely Minkowski ($\Lambda = 0$), de-Sitter ($\Lambda > 0$) and Anti-de Sitter ($\Lambda < 0$) spacetime. Out of this, the simplest one is the Minkowski or the flat spacetime, which finds application in many areas including General Relativity, Quantum field theory etc. The proof of its nonlinear stability was given by D. Christodoulou and S. Klainerman in 1993 [2].

The de-Sitter spacetime, which has a constant positive curvature, finds its application in the study of cosmological models. Its nonlinear stability was established in the year 1986 [3]. The main feature of both these spacetimes is that small perturbations tend to disperse away to infinity.

Unlike the the other two maximally symmetric solutions of (II.1), AdS is not globally hyperbolic, which means it has no Cauchy hypersurface and suitable boundary conditions must be specified at the timelike conformal boundary in order to ensure that the fields define sensible dynamics. The usual boundary condition imposed is the reflective boundary condition for which there is no flux of energy through the conformal boundary. In fact, field perturbations can reach the boundary in finite proper time. The perturbations, however small, bounce back to the bulk without getting dispersed. It was expected that this mechanism will lead to complex nonlinear interactions which can possibly grow in time.

II.2 Properties of maximally symmetric spacetimes

A manifold which is maximally symmetric possesses two key properties: isotropy and homogeneity. If a manifold is isotropic, it would look the same in all directions around a point. Moreover, if a manifold is isotropic everywhere, it is said to be homogeneous. Both these properties ensure that a spacetime has the maximum number of Killing vectors, which for a $d-$dimensional spacetime is equal to $\frac{d(d+1)}{2}$. Mathematically, the Riemann tensor of a maximally symmetric $d-$dimensional spacetime satisfies

$$R_{\alpha\beta\gamma\delta} = \frac{R}{d(d-1)}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{II.2}$$

where the Ricci scalar R is a constant throughout the spacetime. Note that, the solutions to (II.2) also satisfy (II.1). Depending upon the value of the cosmological constant, one gets three kinds of solutions to the above equation (here, without any loss of generality, we fix ($d = 4$)):

- For vanishing Λ, one obtains the Minkowski metric, given by

$$ds^2 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \tag{II.3}$$

- For $\Lambda > 0$, we have the de-Sitter spacetime. It is obtained by considering five-dimensional Minkowski spacetime

$$ds_5^2 = -du^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{II.4}$$

  and then embedding an hyperboloid of the form

$$-u^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = L^2. \tag{II.5}$$

  If one now induces the coordinates (t, χ, θ, ϕ) on it by defining

$$\begin{aligned}
u &= L \sinh(t/L), \\
x_4 &= L \cosh(t/L) \cos\chi, \\
x_1 &= L \cosh(t/L) \sin\chi \cos\theta \\
x_2 &= L \cosh(t/L) \sin\chi \sin\theta \cos\phi, \\
x_3 &= L \cosh(t/L) \sin\chi \sin\theta \sin\phi,
\end{aligned} \tag{II.6}$$

the metric (II.4) becomes

$$ds^2 = -dt^2 + L^2 \cosh^2(t/L) d\Omega_3^2 \tag{II.7}$$

where $d\Omega_3^2$ is the three-sphere metric given by

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{II.8}$$

The metric (II.7) is the de Sitter metric.

- On similar lines, one can obtain anti-de Sitter metric by considering a five-dimensional manifold of the form

$$ds_5^2 = -du^2 - dv^2 + dx_1^2 + dx_2^2 + dx_3^2 \tag{II.9}$$

and embedding a hyperboloid of the form

$$-u^2 - v^2 + x_1^2 + x_2^2 + x_3^2 = -L^2 \tag{II.10}$$

Then upon inducing the coordinates (t, ρ, θ, ϕ)

$$\begin{aligned} u &= L \sin t \cosh\rho \\ v &= L \cos t \cosh\rho \\ x_1 &= L \sinh\rho \cos\theta \\ x_2 &= L \sinh\rho \sin\theta \cos\phi \\ x_3 &= L \sinh\rho \sin\theta \sin\phi \end{aligned} \tag{II.11}$$

on the hyperboloid, the metric (II.9) becomes

$$ds^2 = L^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2\right) \tag{II.12}$$

This is the AdS metric. Here $t \in [0, 2\pi)$, $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. The two-sphere metric $d\Omega_2^2$ is given by $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$. Since the periodicity of the time coordinate t leads to closed timelike causal curves, one unwraps the time-like direction so that $-\infty < t < \infty$. In general $d = n + 2$ dimensions, the AdS metric becomes

$$ds^2 = L^2 \left(-\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_n^2\right) \tag{II.13}$$

where $d\Omega_n^2$ is the metric of the $n-$sphere.

II.3 AdS metric in alternate coordinate systems

The coordinates used in (II.13) are the **global coordinates** (t, ρ). We will now discuss a few other coordinate systems.

- **Static coordinates:** In the static coordinates (t, r), we define $r = \sinh\rho$, so that $r \geq 0$. If we make this substitution in (II.13), the AdS metric looks like

$$ds^2 = L^2 \left(-(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\Omega_n^2\right) \tag{II.14}$$

- **Conformal coordinates:** The conformal coordinates (t, x) are defined in terms of the static coordinates as $r = \tan x$, so that $x \in [0, \frac{\pi}{2})$. If we make this substitution in (II.14), the AdS metric looks like

$$\begin{aligned} ds^2 &= \frac{L^2}{\cos^2 x}(-dt^2 + dx^2 + \sin^2 x d\Omega_n^2) \\ &= \frac{L^2}{\cos^2 x} d\tilde{s}^2 \end{aligned} \qquad \text{(II.15)}$$

where the metric $d\tilde{s}^2$

$$d\tilde{s}^2 = (-dt^2 + dx^2 + \sin^2 x d\Omega_n^2) \qquad \text{(II.16)}$$

would have been the Einstein static universe if $0 \leq x < \pi$. Thus, anti-de Sitter is conformally related to the half of the Einstein static universe. Its infinity, at $x = \frac{\pi}{2}$, takes the form of a timelike hypersurface and is causally connected with the rest of the bulk, so that information can always flow in from infinity. Hence we cannot have a well-posed initial value problem by specifying information only on a spacelike slice. In other words, anti-de Sitter is not globally hyperbolic.

- **Poincaré coordinates:** The constraint (II.10) can also be satisfied by defining the coordinates in the following manner

$$\begin{aligned} u &= \frac{L^2}{2r}\left[1 + \frac{r^2}{L^4}(L^2 + \vec{y}^2 - t^2)\right] \\ v &= \frac{r}{L}t \\ x_i &= \frac{r}{L}y_i, \quad i = 1, 2 \\ x_3 &= \frac{L^2}{2r}\left[1 - \frac{r^2}{L^4}(L^2 - \vec{y}^2 + t^2)\right] \end{aligned} \qquad \text{(II.17)}$$

where $\vec{y}$ is a vector with components (y_1, y_2), which in d dimensions has components $y_1, ...y_{d-2}$. Moreover, it will have a Euclidean norm $|\vec{y}| \geq 0$. Hence, the AdS metric becomes

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}d\vec{y}^2 \tag{II.18}$$

These coordinates only cover a part of spacetime called the Poincaré patch, and find their application primarily in the AdS/CFT correspondence.

II.4 Weakly turbulent instability of AdS

We will first discuss the set-up by Bizoń and Rostworowski [8] which led to the conclusion that AdS might be unstable for large class of perturbations. Their work was primarily numerical and involved studying the spherically symmetric evolution of a massless scalar field ϕ in Einstein background with a negative cosmological constant in four dimensions, the results of which hold true for higher dimensions as well [9], [10]. Since this forms the core of the subsequent works done in this direction, we will discuss its set up and results in detail.

II.4.1 The set up

The dynamics of the Einstein-massless scalar field system in $(d+1)$ dimensions with a negative cosmological constant $\Lambda < 0$ is governed by two equations, namely, the Einstein's equation sourced by the stress-energy tensor of the scalar field ϕ

$$G_{\alpha\beta} + \Lambda g_{\alpha\beta} = 8\pi G \left(\partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} (\partial\phi)^2 \right) \tag{II.19}$$

as well as the Klein-Gordon equation

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\phi = 0 \tag{II.20}$$

Here $G_{\alpha\beta}$ is the Einstein's tensor, ϕ is the scalar field and the covariant derivative ∇ is associated with the metric $g_{\alpha\beta}$. Since the evolution was restricted to spherical symmetry, the following ansatz is chosen for the metric

$$ds^2 = \frac{L^2}{\cos^2 x}(-A(t,x)e^{-2\delta(t,x)}dt^2 + A(t,x)^{-1}dx^2 + \sin^2 x d\Omega^2_{d-1}) \tag{II.21}$$

where $L^2 = -3/\Lambda$ and $d\Omega^2_{d-1}$ is the standard $(d-1)$-sphere metric. The radial direction was compactified so that the $r = L\tan x$. Hence $x \in [0, \frac{\pi}{2})$ and $t \in (-\infty, \infty)$. The field equations were then written in terms of auxiliary variables, $\Phi = \phi'$ and $\Pi = A^{-1}e^{\delta}\dot{\phi}$:

$$\dot{\Phi} = (Ae^{-\delta}\Pi)', \quad \dot{\Pi} = \frac{1}{\tan^{d-1} x}(\tan^{d-1} x Ae^{-\delta}\Phi)' \tag{II.22}$$

The motive was to solve the first order PDEs (II.22) numerically, given the following constraints obtained from the Einstein equations (in units $4\pi G$=1):

$$A' = \frac{d-2+2\sin^2 x}{\sin x\cos x}(1-A) - \sin x\cos x A(\Phi^2 + \Pi^2) \tag{II.23}$$

$$\delta' = -\sin x\cos x(\Phi^2 + \Pi^2) \tag{II.24}$$

The overdots and primes stood for derivative w.r.t. t and x respectively. The numerics were carried out by choosing a Gaussian-type initial data of the form

$$\Phi(0,x) = 0, \quad \Pi(0,x) = \frac{2\epsilon}{\pi} \exp\left(-\frac{4\tan^2 x}{\pi^2\sigma^2}\right) \tag{II.25}$$

where the width of the wave-packet is fixed as $\sigma = 1/16$, while the amplitude of perturbation ϵ is varied.

II.4.2 Numerical results

Upon evolving the system with time, the following results were obtained. When the amplitude was large enough, the apparent horizon x_H formed almost immediately, signalling the formation of a black hole. This was indicated, when $A(t,x)$ drops to zero.

As the amplitude was decreased, the horizon radius also decreased and dropped to zero for some critical amplitude ϵ_0. Note that, for the points near to criticality that approach ϵ_0 from the right hand side (also called supercritical points), the influence of AdS boundary (or in other words, Λ itself) is irrelevant, since for these points, the horizon radius of the black hole is much less than the AdS boundary, i.e. $x_H \ll \pi/2$. Hence, for these points, the observation was similar to the one observed by Choptuik in the corresponding model with $\Lambda = 0$. In fact the scaling law for supercritical points went like $x_H \sim (\epsilon - \epsilon_0)^\gamma$ with $\gamma \simeq 0.37$. This exponent was the same for $\Lambda = 0$ case [61].

In asymptotically flat spacetime, there is no collapse below the critical amplitude and the field disperses away at asymptotic infinity. But in the present case, for $\epsilon < \epsilon_0$, initially there was no black hole formation. Instead, the wave packet traveled to the boundary in a finite time and got reflected back to the bulk. As it reached the the centre, the pulse peaked and became more focused and then eventually collapsed. This behavior continued till ϵ reached the next critical value. say ϵ_1, when once again the horizon radius x_H shrank to zero. As the amplitude was gradually decreased, one would get a decreasing sequence of such critical amplitudes ϵ_n, corresponding to $x_H = 0$.

The following were the additional key observations

- Around the neighborhood of each critical point ϵ_n, for which $\epsilon > \epsilon_n$, the horizon radius scaled according to $x_H \sim (\epsilon - \epsilon_n)^\gamma$ with $\gamma \simeq 0.37$

- If $T(\epsilon)$ denoted the time of collapse, then it satisfied $T(\epsilon_{n+1}) - T(\epsilon_n) \approx \pi$. This was consistent with the fact that massless particles take π time to make a round trip from the center and back to the bulk.

- For small initial data, the Ricci scalar at the center

$$R(t,0) = -2\Pi^2(t,0)/L^2 - 12/L^2 \tag{II.26}$$

  would remain almost constant in the first phase of the evolution and then grow exponentially after a time, $t \sim \epsilon^{-2}$. This behavior was what characterized the AdS instability, irrespective of whether a collapse occurred later or not.

Note that, numerical simulations have a limitation that one cannot perform the experiment up to arbitrarily small amplitudes. Nevertheless, it was expected that the collapse of the kind described above will continue indefinitely as amplitude of perturbation is decreased.

What is the exact mechanism behind this whole process? In order to gain some insight to this, a weakly nonlinear perturbative approach was used by Bizoń and Rostworowski [8], which we will discuss now.

II.4.3 Perturbative analysis

In perturbative analysis, the relevant variables were expanded in powers of the amplitude of perturbation $\epsilon \ll 1$. Thus the scalar field ϕ, and the metric variables A and δ were expanded in the following manner:

$$\phi(t,x) = \sum_{j=0}^{\infty} \phi_{2j+1}\epsilon^{2j+1}, \quad A(t,x) = 1 - \sum_{j=1}^{\infty} A_{2j}\epsilon^{2j}, \quad \delta(t,x) = \sum_{j=1}^{\infty} \delta_{2j}\epsilon^{2j} \tag{II.27}$$

where the initial data is of the form $(\phi, \dot{\phi})_{t=0} = (\epsilon f(x), \epsilon g(x))$. For $\epsilon = 0$, one would get back pure AdS with $A = 1$, $\delta = 0$ and $\phi = 0$. The expansion (II.27) was substituted in the system of equations (II.22-II.24) with $d = 3$. At first order in ϵ, the following equation was obtained

$$\ddot{\phi}_1 + \hat{L}\phi_1 = 0, \quad \hat{L} = -\frac{1}{\tan^2 x}\partial_x(\tan^2 x\partial_x) \tag{II.28}$$

From the results in [7], it is known that the operator $\hat{L}$ is essentially self adjoint on $\hat{L}^2([0, \frac{\pi}{2}], \tan^2 x dx)$ and the inner product $< f, g >$ is denoted by

$$< f, g >= \int_0^{\pi/2} f(x)g(x)\tan^2 x dx \tag{II.29}$$

The eigenvalues ω_p^2 and eigenfunctions $e_p(x)$ of $\hat{L}$, corresponding to a particular mode number p are

$$\begin{aligned} &\omega_p^2 = (2p+3)^2, \quad p = 0, 1, 2... \\ &e_p(x) = d_p \cos^2 x \, {}_2F_1(-p, 3+p, \frac{3}{2}; \sin^2 x) \end{aligned} \tag{II.30}$$

where $d_p = (16(p+1)(p+2)/\pi)^{1/2}$ is the normalization factor so that $< e_p, e_q >= \delta_{pq}$. The positivity of eigenvalues ensured that AdS was stable at the linear level. The completeness of the eigenfunctions allowed the solution ϕ_1 to be written as a infinite sum over the eigenbasis of functions $e_p(x)$

$$\phi_1 = \sum_{p=0}^{\infty} a_p \cos(\omega_p t + B_p) e_p(x) \tag{II.31}$$

where constants a_p and B_p were set by initial conditions. At order ϵ^2, the metric variables A_2 and δ_2 were obtained

$$A_2(t,x) = \frac{\cos^3 x}{\sin x} \int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2 \right) \tan^2 y dy \tag{II.32}$$

$$\delta_2(t,x) = -\int_0^x \left(\dot{\phi}_1(t,y)^2 + \phi_1'(t,y)^2 \right) \sin y \cos y dy \tag{II.33}$$

It was at the third order, that one got an insight to the nature of the instability Upon collecting terms of order ϵ^3, one would get an inhomogeneous equation:

$$\ddot{\phi}_3 + \hat{L}\phi_3 = S(\phi_1, A_2, \delta_2) \tag{II.34}$$

where $S = -2(A_2 + \delta_2)\ddot{\phi}_1 - (\dot{A}_2 + \dot{\delta}_2)\dot{\phi}_1 - (A_2' + \delta_2')\phi_1'$. Upon projecting this equation over $e_p(x)$, one would get the following equation in terms of the Fourier coefficients c_p defined as $c_p =< \phi_1, e_p >$.

$$\ddot{c}_p + \omega_p^2 c_p = S_p =< S, e_p > \tag{II.35}$$

Upon performing some straightforward (and yet tedious) calculations, one could observe that $< S, e_p >$ contained terms of the form $\cos(\omega_{p_1} + \omega_{p_2} - \omega_{p_3})$ and $\sin(\omega_{p_1} + \omega_{p_2} - \omega_{p_3})$. Hence, whenever $\omega_p = \omega_{p_1} + \omega_{p_2} - \omega_{p_3}$, the solution to (II.35) would have terms like $t\cos(\omega t)$ or $t\sin(\omega t)$. These secular terms which grew with time invalidated the perturbative expansion when $\epsilon^2 t = O(1)$.

Even though it was possible to remove some of these secular terms using tools like Poincaré-Lindstedt technique, for a general $n-$mode initial data, the number of these irremovable resonances grew rapidly with n. For eg: for a single mode initial data of the form $(\phi, \dot{\phi})_{t=0} = \epsilon(e_0(x), 0)$, only S_0 contained a secular term. This secular term could be removed by doing a frequency shift of the form

$$\omega_p \to \omega_p + \epsilon^2 \omega_p^{(2)} ... \tag{II.36}$$

and then by choosing an appropriate $\omega_p^{(2)}$ one could cancel out the secular terms. Hence, in case of a single mode data, it was possible to get rid of the resonances completely. But for a two mode data of the form, say $(\phi, \dot{\phi})_{t=0} = \epsilon(e_0(x) + e_1(x), 0)$, although the resonant terms in S_0 and S_1 could be eliminated, but the term in S_2 would lead to a secular term of the form $c_2(t) \sim t\sin(7t)$.

The presence of irremovable secular terms was attributed to higher order resonant mode mixing, which would then result in transfer of energy from low to higher frequencies.

II.4.4 Quantifying the resonant transfer across modes

The effect of resonant transfer of energy was captured by looking at the variation of energy contained in mode p with time. In order to do this, firstly, quantities Φ_p and Π_p were defined as

$$\Phi_p =< \sqrt{A}\Phi, e'_p >, \quad \Pi_p =< \sqrt{A}\Pi, e'_p > \tag{II.37}$$

Since $< e_p, e_q >= \delta_{pq}$ and $< e'_p, e'_q >= \omega_p^2 \delta_{pq}$, the total energy/mass M could be considered as a Parseval sum

$$M = \frac{1}{2}\int_0^{\pi/2} (A\Phi^2 + A\Pi^2)\tan^2 x dx = \sum_{p=0}^{\infty} E_p(t) \tag{II.38}$$

where

$$E_p = \Pi_p^2 + \omega_p^{-2}\Phi_p^2 \tag{II.39}$$

It was observed that for a two mode data of the form

$$\phi(0, x) = \epsilon\left(\frac{1}{d_0}e_0(x) + \frac{1}{d_1}e_1(x)\right) \tag{II.40}$$

the energy contained in the first 16 modes was almost constant, while those in the first few modes decreased with time [8]. This implied that there was indeed a cascade of energy from low to high modes.

II.4.5 Energy/Power spectrum

The turbulent nature of the evolution could also be seen from the fact that just before black hole formation, the spectrum for $d = 3, 4, 5$ seemed to exhibits a power law scaling of the form $E_p \sim p^{-\alpha}$ (p is the mode number) where

$$\alpha(d) = 6/5 + 4(d-3)/5 \tag{II.41}$$

was independent of the initial data [10]. Moreover, H.P. de Oliveira, Leopoldo A. Pando Zayas and E.L. Rodrigues [13] observed that after an initial linear regime with well defined frequencies (corresponding to AdS normal modes), the power spectrum of the Ricci scalar at the centre for AdS_4 and AdS_5 follows the Kolmogorov-Zakharov spectrum with a power law

$$P(\omega) = \omega^{-s} \tag{II.42}$$

where $s \sim 5/3$. This exponent was universal and seemed to be dimension-independent. Such a spectrum was a signature of turbulent behavior of strongly interacting waves.

Thus, it was concluded that the basic mechanism which drove AdS unstable was the turbulent cascade of energy from low to high frequencies which lead to concentration of energy on finer spatial scales, till this process was inevitably cut-off with the formation of an AdS-Schwarzschild black hole (which is itself nonlinearly stable [88]). This mechanism was very much analogous to the viscous cut-off for fluids observed in the study of fluid dynamics. In the language of AdS-CFT correspondence, this was equivalent to thermalization of the dual system on the CFT side.

II.4.6 Globally regular instability of AdS_3

The problem of instability was different in three dimensional AdS. In three dimensional AdS, there was an energy threshold below which a black hole cannot form. This could be seen by the fact that that the black hole solution for $d = 2$ took the form (Banados, Teitelboim, Zanelli [89])

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\phi^2, \quad f(r) = \left(1 - M + \frac{r^2}{L^2}\right) \tag{II.43}$$

Note that, the convention used here for the M in $f(r)$ is different from that in [89], where $1 - M \to -M$. As one can see, there is a mass gap between AdS_3 (corresponding to $M = 0$ in (II.43) and the lightest BTZ black hole (corresponding to $M = 1$ (II.43)). A BTZ black hole would only form if $M \geq 1$, whereas for higher dimensions there was no such restriction. Also, compared to higher dimensions, the rate of flow of energy to higher modes was very fast in three dimensions. This limited the time available for performing numerical simulation and also led loss of spatial resolution, as energy got transferred to finer and finer scales. To combat this issue, an alternate method called the analyticity strip method, first introduced in [90], was used by Bizoń and Jałmużna [11].

The basic idea behind the analyticity strip method was as follows. Suppose one needs to find the singularities of a real function $u(t, x)$, which is the solution to a nonlinear evolution equation. The idea then, was to look for the singularities of $u(t, z)$, its analytic extension to the complex plane instead. In general, the singularities of $u(t, z)$ would be complex numbers. Suppose the singularity which was closest to the real axis was $z_* = x_* + i\rho$. If during the course of evolution, the $\rho(t)$ became zero, then it would translate to $u(t, x)$ becoming singular. Thus by monitoring the evolution of $\rho(t)$, one could decide whether there is a blow up in finite time. The key to determining ρ was embedded in the behavior of Fourier coefficients $\hat{u}_k$ of $u(t, x)$ for large wavenumbers k, which went like [91]

$$\hat{u}_k \sim |k|^{\mu} e^{-\rho k} e^{i x_* k} \text{ as } k \to \infty. \tag{II.44}$$

Here k is the wavenumber and μ is related to the pole at z_* . Hence, the value of ρ could be obtained by fitting an exponential curve to the numerically computed Fourier coefficients.

The numerical results for the evolution equations in three dimensional Einstein-massless scalar field system were obtained for small amplitude perturbations corresponding to $M \ll 1$ [11]. Since, the aspects of weak turbulence was captured by the energy spectrum, the following ansatz was used for energy for high wavenumbers k

$$E_k(t) = C(t)k^{-\beta(t)}e^{-2\rho(t)k} \tag{II.45}$$

where $C(t)$, $\rho(t)$ and $\beta(t)$ were functions of time. Once the formula (II.45) was fitted with the numerical data, it was observed that $\rho(t)$ remained bounded away from zero and after some time decreased in an exponential manner as

$$\rho(t) = \rho_0 e^{-t/T} \tag{II.46}$$

where ρ_0 was a constant and T was proportional to ϵ^{-2}. Thus, the width of the analyticity strip $\rho(t)$ never became zero in finite time.

However, this did not imply that AdS_3 was stable. The higher Sobolev norms, like $\dot{H}_2 = ||\phi''(t,x)||_2$ of the scalar field ϕ, did grow exponentially fast. This showed that although small perturbations remained smooth forever, they did not remain small during the course of the evolution, implying that AdS_3 was indeed unstable. While in AdS_3, this turbulent process could happen forever, it would get cut off by black hole formation in higher dimensional AdS.

II.4.7 Complex Scalar fields in AdS

The numerical simulations were repeated by Buchel, Lehner & Liebling [14] , this time, by replacing a real scalar field with a massless complex scalar field of the form

$$\phi = \phi_1 + i\phi_2 \tag{II.47}$$

The action for this set-up was invariant under a global phase rotation $\phi \to \phi e^{-i\alpha}$ and hence, there was a conserved charge Q associated with this symmetry.

The evolution was done for both $Q = 0$ and $Q \neq 0$ and the results obtained were very similar to the one in massless scalar field of [8], [9], with the end result being black hole formation. The evolution of the CFT observable was determined by looking at the evolution of leading order behavior of the asymptotic form of the scalar field ϕ_i. This is given by

$$\phi_i(\rho, t) = \phi_3^{(i)}(t)\rho + \phi_5^{(i)}(t)\rho^3 \tag{II.48}$$

where $\rho = \frac{\pi}{2} - x$. It was seen that $\phi_3^{(i)}$ tends to sharpen with each bounce of the scalar field.

II.5 Instabilities in confined geometries

One of the key questions to the AdS instability problem has been: Is the turbulent behavior a feature of only asymptotically AdS spacetimes? We know that the AdS boundary, because of its timelike nature, acts as a mirror which reflects back the waves propagating outwards back into the bulk. The eventual collapse of fields in AdS has been attributed to the turbulent cascade of energy from low to high frequencies which translates to energy getting concentrated to increasingly finer spatial scales. Using perturbation theory, it was argued that the resonant nature of the spectrum has an important role to play in this mechanism. What then happens if the AdS boundary is replaced with an artificial mirror? Does the instability, as seen in case of AdS still occur? What if such a spacetime has non-resonant spectra? The key motivation in works like [17] by Maliborski and [18] by Maliborski and Rostworowski has been to probe into such aspects of confined geometries.

One way to create an artificial boundary is to place an artificial mirror at a distance, say R in the flat spactime. In [17]), Maliborski studied the evolution of a massless scalar field within a cavity of radius R in flat spacetime. The ansatz metric was parameterized as

$$ds^2 = -\frac{A}{N^2}dt^2 + A^{-1}dr^2 + r^2 d\Omega_2^2 \tag{II.49}$$

where $d\Omega_2^2$ is the metric of the two-sphere. In terms of the auxiliary variables defined as $\Phi = \phi'$ and $\Pi = A^{-1}N\dot{\phi}$, the wave equation $\nabla^\mu\nabla_\mu\phi = 0$ could then be written as

$$\dot{\Phi} = \left(\frac{A\Pi}{N}\right)', \quad \dot{\Pi} = \frac{1}{r^2}\left(r^2\frac{A\Phi}{N}\right)' \tag{II.50}$$

The Einstein equation in terms of the mass function $m = \frac{1}{2}r(1-A)$ then became

$$\frac{N'}{N} = -r(\Phi^2 + \Pi^2) \tag{II.51}$$

$$m' = \frac{1}{2}r^2 A(\Phi^2 + \Pi^2) \tag{II.52}$$

$$\dot{m} = r^2\frac{A}{N}\Phi\Pi \tag{II.53}$$

The objective of this work was to observe the effect of imposing Dirichlet boundary condition at $r = R$ instead of asymptotically flat conditions. This meant imposing

$$\Pi(t, R) = 0 \tag{II.54}$$

It should be noted that the linearized perturbations governed by the equations

$$\ddot{\phi} = \frac{1}{r^2}(r^2\phi')' \tag{II.55}$$

gave the following eigenfrequencies ω_p^2 and eigenmodes $e_p(x)$

$$\omega_p = \frac{p\pi}{R} \tag{II.56}$$

$$e_p(r) = \sqrt{\frac{2}{R}}\frac{\sin \omega_p r}{r}, \quad p \in \mathbb{N} \tag{II.57}$$

Thus, akin to AdS, this set up too had a resonant spectra. The system of equations (II.50-II.53) were evolved with a initial Gaussian data

$$\Phi(0, r) = 0, \quad \Pi(0, r) = \epsilon \exp\left(-32 \tan^2 \frac{\pi}{2} r\right) \tag{II.58}$$

The following results were obtained

- Similar to the AdS case, there existed a sequence of critical amplitude ϵ_n for which the horizon radius shrank to zero. For high enough amplitudes $\epsilon > \epsilon_0$, the collapse was immediate.
- Arbitrarily small perturbations grew exponentially in time scale $t \sim \epsilon^{-2}$, very similar to the AdS case. This was indicated by the growth of

the Ricci scalar at the center. In fact, just before horizon formation, the energy spectrum exhibited power-law scaling $E_p \sim p^{-\alpha}$, where $\alpha = 1.2 \pm 0.1$. This value of α was universal and did not depend on the different families of small initial data.

The experiment was repeated by imposing Neumann instead of Dirichlet boundary condition. This meant imposing

$$\Phi(t, R) = 0 \tag{II.59}$$

In this case the eigenfrequencies were given by solutions to the equation

$$\tan(R\omega_p) = R\omega_p, \tag{II.60}$$

which for large mode numbers p gave

$$\omega_p = \frac{\pi}{R}\left(p + \frac{1}{2}\right) + \mathcal{O}(p^{-1}) \tag{II.61}$$

This was a non-resonant spectra, however, for large p this spectra would asymptote to a resonant one. We will call such a spectrum, "asymptotically resonant". Despite having such a spectrum, the simulations showed similar turbulent behavior. Thus it was concluded in [17] that the spectrum of linearized perturbations need not be resonant to trigger an AdS-like instability.

Note that the above claim is not accurate. In [18], when the same experiment was repeated for Neumann boundary condition, it was noted that fields did not collapse for arbitrarily small perturbations. In fact for such a system, there existed a threshold amplitude below which the solutions remained stable.

In [19] (Okawa, Cardoso & Pani), similar numerics were performed with a massless scalar field being replaced with a massive one. For this case, the spectra look like

$$\omega_p = \sqrt{\mu^2 + (p\pi/R)^2}, \text{ Dirichlet}$$
$$\omega_p = \sqrt{\mu^2 + k_p^2}, \qquad \text{Neumann} \tag{II.62}$$

where μ is the mass of the scalar field and k_p are the roots of the equation $\tan(kR) = kR$. In spite of the asymptotically resonant nature of the spectra in both these cases, there was some confusion regarding the presence of a threshold amplitude, as it was challenging to perform numerical simulation at extremely low amplitudes.

II.6 Nonlinear dynamics and AdS instability

One of the key motivations in applying the results of nonlinear dynamics in the present context has been to explain the role of resonant spectrum in triggering an AdS-like instability. On one hand, the perturbative approach used in [8] gave an insight as to how a resonant spectra contributes in the energy cascade from low to high frequencies. On the other hand, both massless fields and massive fields subjected to reflecting boundary conditions in flat space [17], [18], [19] showed turbulent cascade of energy, in spite of a non-resonant (albeit asymptotically resonant spectra).

Apart from this, while the authors of [18] reported a threshold frequency below which the system remains stable, in [19], the authors claimed that instability is observed for arbitrarily small perturbations. How does one resolve this dichotomy?

In other words, what are the necessary conditions for an AdS-like instability? In order to answer these questions, we resorted to the tools developed in nonlinear dynamics. The discussions in this section will be based on our work in [20].

II.6.1 Basic set up

The basic set-up in our work, [20], was as follows. As we saw previously, the perturbative analysis of the Einstein-scalar field system of [8] shows that at the linear level, the scalar field can be written as $\sum_p a_p(t)e_p(x)$, where $a_p(t)$ obeys the harmonic oscillator equation. The third order corrections to the scalar field, on the other hand, are given by harmonic oscillator equation with the forcing terms coming from nonlinearities. Therefore, one can associate the scalar field with a set of decoupled harmonic oscillators at the linear level. Although, in the original system the number of oscillators were infinite, one can always assume that at any given time, only a finite number of them will hold significant energy. Hence, we can assume the number of oscillators to be n, where n is a large but finite number. The Hamiltonian associated with these oscillators will be integrable. The type of integrability, which we will be dealing with, is called Liouville integrability.

An integrable Hamiltonian system (M, H_0) having canonical coordinates $(q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$ will have n constants of motion, say $f_1, f_2, \ldots, f_n$ which are independent and in involution. Here M is the $2n-$dimensional phase space and H_0 is the associated Hamiltonian. Moreover, by involution, we mean that the Poisson bracket of f, g w.r.t coordinates (q, p) is zero i.e.

$$\{f_i, f_j\} = \sum_{k=1}^{n} \left(\frac{\partial f_i}{\partial q_k} \frac{\partial f_j}{\partial p_k} - \frac{\partial f_j}{\partial q_k} \frac{\partial f_i}{\partial p_k} \right) = 0 \tag{II.63}$$

Setting each of the $f_1, f_2, \ldots, f_n$ to a constant gives rise to an $n-$dimensional manifold in a $2n-$dimensional phase space, such that any solution beginning on the surface remains on it for all times. According to the well-known LMAJ (Liouville-Mineur-Arnold-Jost) theorem, if such a surface is bounded and connected, it is in fact an $n-$dimensional torus (here on denoted by, $\mathbb{T}^n$). Moreover, the theorem further states, that one can do a canonical transformation of the coordinates (q,p) to (I,θ) (also known as action-angle variables), such that the Hamiltonian is independent of θ, i.e. $H_0 = H_0(I)$ (note that, I and θ are short-hand notations for the $2n$ variables (I_i, θ_i), $i = 1, 2..., n$). Hence, the action-angle variables, take the form simple form

$$I_k(t) = I_k^0, \quad \theta_k(t) = \theta_k^0 + \omega_k t; \quad k = 1, \ldots, n \tag{II.64}$$

where I_k^0, θ_k^0 are the initial conditions and $\omega_k = \frac{\partial H_0(I)}{\partial I_k}$ are called frequencies. We see that the action variables are just some constants to be set by initial conditions. So we can say that each individual torus is basically parameterized by I^0.

How do the trajectories wrap around the $n-$dimensional torus ? In order to answer this, we fix the action $I(t) = I^0$ equal to some constant. Hence, different tori are characterized by different values of I^0. On a particular torus therefore, the trajectories of the flow wrap around it with frequency $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$. Now consider the following equation:

$$\mathbf{k} \cdot \boldsymbol{\omega} = k_1\omega_1 + k_2\omega_2 + \ldots + k_n\omega_n = 0 \tag{II.65}$$

where $\mathbf{k} \in \mathbb{Z}^n$.

- For a given spectrum, if this is only satisfied for $\mathbf{k} = 0$, then we say that the spectrum is non resonant.
- For a given spectrum, if it is satisfied for some $\mathbf{k} \in \mathbb{Z}^n - \{0\}$, then

we say that the spectrum is resonant. The characteristic feature of resonant spectrum is that the ratio of frequencies ω_i will be a rational number.

Analysis of the gravity-scalar field systems shows that they can be seen as perturbed integrable systems. Therefore, the pertinent question is, what happens when an integrable system is perturbed? Does the dynamics continue to remain confined on an $n-$torus? Or is it now on a lower dimensional torus? Or do all the tori get destroyed completely and the system ends up exploring any region of phase space? Finally, how do we relate these possible scenarios to the instability seen in our systems of interest? The last two scenarios, i.e. partial/complete destruction of torus is what we will refer to as an instability.

In the next section, we will try to address the effect of the addition of a small perturbation to the linearized system of oscillators by resorting to results in nonlinear dynamics (KAM OR Kolmogorov-Arnold-Moser theory).

II.6.2 The small denominator problem in hamiltonian perturbation theory

What happens upon the addition of a small perturbation, say $f(I,\theta)$ to the integrable hamiltonian $H_0(I)$? The new Hamiltonian $H(I,\theta)$ is now

$$H(I,\theta) = H_0(I) + \epsilon f(I,\theta) \tag{II.66}$$

If $H(I,\theta)$ were still integrable, the dynamics would again be confined on a $n-$torus. However, in general, this perturbed Hamiltonian need not be integrable. What one can try to do however, is to make it approximately integrable (up to the desired order in ϵ) by doing a suitable canonical transformation. That is, we go from (I, θ) to some (I', θ'). This is done via a generating function, say $\Psi(I', \theta)$ in the following manner

$$I = I' + \partial_\theta \Psi(I', \theta), \quad \theta' = \theta + \partial_{I'} \Psi(I', \theta) \tag{II.67}$$

Upon transformation, $H(I, \theta)$ takes the form

$$H(I, \theta) = h_\epsilon(I') + ... \tag{II.68}$$

where $h_\epsilon(I') = h_0(I') + \epsilon h_1(I') + \epsilon^2 h_2....$ The generating function is also expanded as $\Psi = \epsilon\Psi_1 + \epsilon^2\Psi_2 + ...$ Using (II.67) and (II.68) in (II.66), we can see that at order ϵ, the following relation holds

$$\boldsymbol{\omega}(I') \cdot \partial_\theta \Psi_1(I', \theta) + f(I', \theta) = h_1(I') \tag{II.69}$$

where $\boldsymbol{\omega}(I) = \partial_I h_0(I)$. Integrating this equation over $\theta \in \mathbb{T}^n$ yields

$$h_1(I') = \bar{f}(I') = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(I', \theta) d\theta \tag{II.70}$$

where the integration has been done with respect to $\theta \in \mathbb{T}^n$. Hence, one can rewrite (II.69) as

$$\boldsymbol{\omega}(I') \cdot \partial_\theta \Psi_1(I', \theta) = -[f(I', \theta) - \bar{f}(I')] \tag{II.71}$$

Further, we expand Ψ_1 and $(f - \bar{f})$ in a Fourier series:

$$f(I', \theta) - \bar{f}(I') = \sum_{\mathbf{k} \in \mathbb{Z}^n} \hat{f}_{\mathbf{k}}(I') e^{2\pi i \mathbf{k} \cdot \theta} \tag{II.72}$$

$$\Psi_1(I', \theta) = \sum_{\mathbf{k}\in\mathbb{Z}^n} \hat{\Psi}_{1\mathbf{k}}(I') e^{2\pi i \mathbf{k}\cdot\theta} \tag{II.73}$$

Upon substituting these expansions in (II.71), we see that

$$\hat{\Psi}_{1\mathbf{k}}(I') = -\frac{\hat{f}_{\mathbf{k}}(I')}{2\pi i(\boldsymbol{\omega}(I')\cdot\mathbf{k})} \tag{II.74}$$

Since $\mathbf{k} = 0$ is the trivial case, we will consider $\mathbf{k} \in \mathbf{Z}^n - \{0\}$. Note that, for the generating function $\Psi_1(I', \theta)$ to converge in the series solution, one needs to have a sufficiently large denominator. But what happens if the spectrum is resonant? In such a case, there will be some $\mathbf{k} \in \mathbf{Z}^n - \{0\}$ for which $\boldsymbol{\omega}(I')\cdot\mathbf{k} = 0$ gets satisfied and we obviously have a problem in carrying out this procedure. But $\Psi_{1\mathbf{k}}$ may also fail to converge, if the denominator is small. This happens if the spectrum is nearly resonant or is asymptotically resonant. In case the spectrum is asymptotically resonant, this difficulty can arise, if the initial condition has high mode numbers to begin with. This is called the "small denominator problem". Both these scenarios can prove to be an obstruction towards making the Hamiltonian integrable up to the desired order in ϵ.

What are the consequences of the small denominator problem? In case of resonant spectrum, this means even a minuscule perturbation could lead to the complete destruction of the $n-$torus, so that the motion is no longer restricted on it. We expect the energy to be transferred across modes via resonances, as the system samples more phase space. Thus, the resonant instability in the AdS-scalar field system could be seen as an instability in phase space. Similarly, if the frequencies are nearly resonant i.e the ratio of frequencies is close to a rational number, it could lead to the small denominator problem, and as a result, lack of integrability. This could again lead to destruction of torus and consequently transfer of energy across modes.

II.6.3 Diophantine condition on non resonant spectra

Non-resonant spectra satisfy a Diophantine condition i.e. for all $\mathbf{k} \in \mathbb{Z}^n - \{0\}$ and some $\gamma > 0$, they satisfy:

$$|\boldsymbol{\omega} \cdot \mathbf{k}| \geq \frac{\gamma}{|k|^n} \tag{II.75}$$

where $|k|$ denotes the supremum of $|k_i|$. By choosing sufficiently large values of k_i, one can try to get close to the resonant condition. The factor γ is a way of quantifying how close the frequencies are to being perfectly resonant. Therefore, asymptotically resonant or nearly resonant would be characterized by smaller γ.

II.6.4 Benettin-Gallavoti theorem and exponential stability

In order to understand the nature of instability seen in our systems of interest, we will resort to further results of nonlinear dynamics. Suppose the frequencies obey Diophantine condition. As stated before, the perturbed Hamiltonian is $H(I, \theta) = H_0(I) + \epsilon f(I, \theta)$. With some further analyticity conditions on $f(I, \theta)$, Benettin and Gallavotti showed that: For initial data $(I(0), \theta(0))$, and for amplitude of perturbation $\epsilon < \epsilon_0$, where $\epsilon_0 = \frac{D}{E^2}\gamma$ and $D > 0$ is a constant determined by n and analyticity parameters of f [92]

$$||I(t) - I(0)|| < C \left(\frac{\epsilon}{\epsilon_0} \right)^{1/2} \tag{II.76}$$

valid for times t such that

$$t \leq \frac{1}{\sqrt{\epsilon}E}(\frac{\epsilon}{\epsilon_0})^{-(1/\epsilon)^b} \tag{II.77}$$

Here C is a constant, $1/E$ is the timescale of oscillation of the linearized system, so that, $E = ||\boldsymbol{\omega}||$ and $b = \frac{1}{4(n+1)}$.

The above statement basically implies that the action variables will remain close to their initial values (implying stability) as long as the amplitude of perturbation is less than a certain $\epsilon_0(\propto \gamma)$ for exponentially long times in comparison to the time scale of oscillations of the linear system $\frac{1}{E}$. One important thing to note here is, that this theorem doesn't guarantee that an instability will necessarily occur if $\epsilon > \epsilon_0$. What it establishes is the stability of the system. As discussed earlier, for nearly or asymptotically resonant spectra, the value of γ is going to be small. Since the threshold amplitude ϵ_0 is proportional to γ, a small value of γ will mean a small threshold amplitude. Hence, the presence of high frequency modes in the initial conditions can trigger an instability for very small amplitudes (greater than ϵ_0). It also implies that the more the frequencies deviate from being resonant, higher the value of γ, and hence the system will exhibit stability for longer times.

II.6.5 Application of the results in nonlinear dynamics

Now we are in a position to apply these results to the systems of our interest.

- For scalar fields in Minkowski with Neumann boundary conditions [17], for large p values, $\omega_p \sim \frac{\pi}{R}\left(p+\frac{1}{2}\right) - \frac{1}{2\pi R}\left(p+\frac{1}{2}\right)^{-1} + O(\frac{1}{p^3})$. Since this spectrum is asymptotically resonant, we expect a threshold amplitude below which the system remains stable. This was indeed confirmed in [18].

- For massive scalar fields in enclosed in a cavity in Minkowski, for large p values, $\omega_p \sim \frac{p\pi}{R} + \frac{\mu^2 R^2}{2\pi p} + O(\frac{1}{p^3})$. Even for $p = 100$ and $\mu R = 10$ (which were the ranges considered in [19]), retaining the first (resonant) term is good enough approximation. If the initial data contains such frequencies, then the value of the threshold amplitude $\epsilon_0(\propto \gamma)$ would have indeed been hard to ascertain as it would be too small.

II.6.6 Localization properties of linear eigenfunctions

As we saw that the end state of the evolution of scalar field in AdS_{d+1} (for $d > 2$) is marked by the formation of black hole. On the other hand, small perturbations in AdS_3, in spite of causing weak turbulence, do not collapse to black hole formation. This motivates us to ask: Is there any way to capture this aspect at the linear level?

We know that an AdS-type instability is marked by transfer of energy to high frequency modes. As the energy flows into these high frequency modes, the fields get more and more localized into finer spatial scales. This means that key to understand this localization lies, at least partially, in the behavior of large p eigenfrequencies $e_p(x)$. Hence, in this section we will try to compare how the localization properties these modes vary with dimension.

The normalized eigenfunctions of the scalar field in a fixed AdS_{d+1} background is given by

$$e_p(x) = \frac{2\sqrt{p!(p+d-1)!}}{\Gamma(p+d/2)}(\cos x)^d P_p^{(\frac{d}{2}-1,\frac{d}{2})}(\cos 2x) \tag{II.78}$$

with corresponding eigenfrequencies ω_p

$$\omega_p = 2p + d \tag{II.79}$$

Since numerical studies show that the singularity, if formed is localized at $x = 0$, our focus will be to study the nature of eigenfunctions with high mode number p close to $x = 0$. According to Darboux formula, for large p and $x \in (0, \frac{\pi}{2})$

$$P_p^{(\frac{d}{2}-1,\frac{d}{2})}(\cos 2x) \sim A(x)\frac{\left(\frac{1}{2}\right)_p}{p!}2^d \cos\left(2px - (d-1)\frac{\pi}{4}\right) \tag{II.80}$$

where $(a)_p = a(a+1)...(a+p-1)$ refers to the Pochhamer symbol and

$$A(x) = 2^{-d}(\sin x)^{-(d-1)/2}(\cos x)^{-(d+1)/2} \tag{II.81}$$

Hence, by using the Stirling's formula, one can deduce, that for large p, $e_p(x)$ takes the form

$$e_p(x) = \frac{2}{\sqrt{\pi}}(\cos x)^d \frac{\left[\cos(2px)\cos\left((d-1)\frac{\pi}{4}\right) + \sin(2px)\sin\left((d-1)\frac{\pi}{4}\right)\right]}{(\sin x)^{(d-1)/2}(\cos x)^{(d+1)/2}} \tag{II.82}$$

For $d \geq 2$, clearly this quantity approaches zero as $x \to \frac{\pi}{2}$. Now let us see the variation of (II.82) across dimensions, when p is large and x is small, such that px is large.

(i) AdS_3 or $d = 2$: As $x \to 0$, $e_p(x)$ takes the form

$$e_p(x) \sim \frac{[C_1\cos(2px) - C_2\sin(2px)]}{\sqrt{x}} \tag{II.83}$$

The envelope function $\frac{1}{\sqrt{x}}$ of the rapidly oscillating $e_p(x)$, increases slowly as $x \to 0$.

(ii) AdS_4, or $d = 3$:As $x \to 0$, $e_p(x)$ approaches

$$e_p(x) = C\frac{\sin(2px)}{\sqrt{\pi x}} \tag{II.84}$$

which is in fact proportional to delta function $\delta(x)$.
(iii) For AdS_{d+1} with $d > 3$, the enveloping function $e_p(x)$ is $\frac{1}{x^{(d-1)/2}}$ as $x \to 0$. This localizes even better near $x = 0$.

Next we check the values of $e_p(x)$ at the end points $x = 0$ using the Mehler-Heine formula for Jacobi polynomials. Let $z = 2px$. Then near the neighborhood of $x = 0$

$$\lim_{p\to\infty} p^{-\alpha} P_p^{\alpha,\beta}(\cos\frac{z}{j}) = (z/2)^{-\alpha} J_\alpha(z) \tag{II.85}$$

where J_α is the Bessel function of order α. We consider the case where p is large and as $x \to 0$, $z \to 0$. In that case, $J_\alpha(z) \sim \frac{1}{\Gamma(\alpha+1)}(z/2)^\alpha$, where $\alpha = \frac{d}{2} - 1$. Again using the Sterling's formula for Gamma function, one can deduce that

$$e_p(0) \sim \frac{1}{\Gamma(\alpha+1)} p^{\frac{d-1}{2}} \tag{II.86}$$

which is least for $d = 2$. As we can see, the localization of the high frequency modes is the least for $d = 2$ case. This is indeed consistent with the fact that in spite of exhibiting turbulent cascade of energy, no black hole formation occurs for small enough perturbations.

Of course, this is a necessary condition and therefore, we cannot naively conclude that collapse will definitely happen, if fields tend to localize. For eg., if one starts with a initial condition with multiple scalar pulses in different locations, the interactions could be very complex [59].

II.6.7 An application to asymptotically AdS spacetimes

As an illustration, we now consider an example of a scalar field-gravity system. We consider a background which is (locally) asymptotically AdS—the AdS soliton.

The metric of the AdS-soliton in five dimensions takes the form [93]

$$ds^2 = \frac{r^2}{L^2}\left[\left(1-\frac{r_0^4}{r^4}\right)d\tau^2 + (dx^1)^2 + (dx^2)^2 - dt^2\right] + \left(1-\frac{r_0^4}{r^4}\right)^{-1}\frac{L^2}{r^2}dr^2. \tag{II.87}$$

The metric is only locally asymptotically AdS and is static with no horizons and singularities. The range of coordinate r is $[r_0, \infty)$ and the coordinate τ is chosen to be periodic so as to avoid conical singularity at $r = r_0$. Upon choosing period to be $\beta = \frac{4\pi L^2}{(p+1)r_0}$, the circle parameterized by τ smoothly shrinks to a point at $r = r_0$.

Now we consider the evolution of a scalar field Φ in a locally asymptotically AdS spacetime, which preserves the planar and circular symmetries. We assume the following ansatz for the metric

$$ds^2 = -e^{2a(t,r)}\frac{r^2}{L^2}dt^2 + e^{2b(t,r)}\frac{L^2}{f(r)}dr^2 + e^{2c(t,r)}\frac{f(r)}{l^2}d\tau^2 + e^{2d(t,r)}\frac{r^2}{L^2}\left[(dx^1)^2 + (dx^2)^2\right] \tag{II.88}$$

where $f(r) = r^2(1-\frac{r_0^4}{r^4})$. For $a = b = c = d = 0$, we will simply get back the AdS soliton metric (II.87). Looking at the equations of motion, one can deduce that the metric variables a, b, c, and d will only contribute at even orders of ϵ. For eg., $a(t,r)$ is expanded as

$$a(t,r) = a_2\epsilon^2 + a_4\epsilon^2 ... \tag{II.89}$$

On the other hand, the scalar field Φ will contain terms of only odd orders in ϵ

$$\Phi = \epsilon\Phi_1 + \epsilon^3\Phi_3 + ... \tag{II.90}$$

The field equations governing the evolution of equations are

$$R_{ab} + \frac{4}{L^2}g_{ab} = W_{ab} \tag{II.91}$$

$$\frac{1}{\sqrt{-g}}\partial_a(g^{ab}\sqrt{-g}\partial_b\Phi) = 0 \tag{II.92}$$

where $W_{ab} = \kappa(T_{ab} - \frac{1}{3}g_{ab}T)$ (with $\kappa = 8\pi G$) and $T_{ab} = \partial_a\Phi\partial_b\Phi - \frac{1}{2}g_{ab}(\partial\Phi)^2$. Upon simplifying (II.91), we get a set of four equations

$$\begin{aligned}&\frac{e^{-2b}}{L^4}\Big[e^{2a}rf'(1+ra') + e^{2a}f\{2 + r^2a'^2 - rb' + rc' + 2rd' + 4ra' - r^2a'b'\\&+ r^2a'c' + 2r^2a'd' + r^2a''\} + e^{2b}l^4\{\dot{a}\dot{b} + \dot{a}\dot{c} - \dot{b}^2 - \dot{c}^2 - 2\dot{d}^2 + 2\dot{a}\dot{d} - \ddot{b}\\&- \ddot{c} - 2\ddot{d}\}\Big] - \frac{4r^2e^{2a}}{L^4} = \kappa\dot{\Phi}^2\end{aligned} \tag{II.93}$$

$$\begin{aligned}&\frac{e^{-2a}}{2r^2f}\Big[-e^{2a}r^2f'' - e^{2a}rf'(3 + ra' - rb' + 3rc' + 2rd') - 2\{e^{2a}rf(ra'^2 + 2a'\\&- ra'b' + rc'^2 + 4d' + 2rd'^2 - 3b' - rb'c' - 2rb'd' + ra'' + rc'' + 2rd'')+\\&e^{2b}L^4(\dot{a}\dot{b} - \dot{b}\dot{c} - 2\dot{b}\dot{d} - \dot{b}^2 - \ddot{b})\}\Big] + \frac{4e^{2b}}{f} = \kappa\Phi'^2\end{aligned} \tag{II.94}$$

$$
\frac{-e^{-2a-2b+2c}f}{2L^4r^2}\Big[e^{2a}r^2f'' + e^{2a}rf'(3 + ra' - rb' + 3rc' + 2rd') + 2\{e^{2a}rf(rc'^2 + 3c' + ra'c' - rb'c' + 2rc'd' + rc'') + e^{2b}l^4(\dot{a}\dot{c} - \dot{b}\dot{c} - 2\dot{c}\dot{d} - \dot{c}^2 - \ddot{c})\}\Big] + 4e^{2c}f = 0 \tag{II.95}
$$

$$
- e^{-2a-2b+2d}\Big[e^{2a}rf'(1 + rd') + e^{2a}f\{2 + rc' + 5rd' + r^2c'd' + 2r^2d'^2 + ra' + r^2a'd' - rb' - r^2b'd' + r^2d''\} + e^{2b}L^4(\dot{a}\dot{d} - \dot{b}\dot{d} - \dot{c}\dot{d} - 2\dot{d}^2 - \ddot{d})\Big] + 4r^2e^{2d} = 0 \tag{II.96}
$$

The Klein Gordon equation (II.92) takes the form

$$
- rL^4e^{-a+b}\Big(\partial_t\partial_t\Phi + (-\dot{a} + \dot{b} + \dot{c} + 2\dot{d})\partial_t\Phi\Big) + e^{a-b}\Big((r^5 - r_0r)\partial_r\partial_r\Phi + (5r^4 - r_0^4)\partial_r\Phi + (r^5 - r_0^4r)(a' - b' + c' + 2d')\partial_r\Phi\Big) = 0 \tag{II.97}
$$

At the linear level, we obtain the dynamical equation which governs the propagation of Φ_1 in the background AdS-soliton field

$$
\ddot{\Phi}_1 + \hat{L}\Phi_1 = 0 \tag{II.98}
$$

where $\hat{L} = -\frac{1}{rL^4}\partial_r[(r^5 - r_0^4r)\partial_r]$. Without any loss of generality, we suppress the coordinates in x_1 and x_2 directions and assume an ansatz of the form $\Phi_1 = \phi_1(r)e^{i\omega t}$ to obtain a radial equation

$$
\hat{L}\phi_1 = \omega^2\phi_1 \tag{II.99}
$$

Although this equation cannot be solved exactly, one could use numerics or methods like the WKB approximation to obtain the frequencies under some approximations. Constable and Myers in [94] used the WKB approximation to calculate the eigenfrequencies, which are given by

$$\omega_p^2 \simeq p(p+1)\frac{56.67}{\beta^2} + O(n) \tag{II.100}$$

This is valid for $\frac{L^4\omega^2}{r_0^2} > 5$.

At the second order, one obtains equations which can be solved to get the second order metric coefficients a_2, b_2, c_2 and d_2. Here as a simplification, a special class of perturbations is considered in which $a = b = 0$. Putting, $a = b = 0$ in equations (II.93-II.96) yields,

$$\frac{rf'}{L^4} + \frac{2f}{L^4} + \frac{rfc'}{L^4} + \frac{2rfd'}{L^4} - \dot{c}^2 - 2\dot{d}^2 - \ddot{c} - 2\ddot{d} - 4\frac{r^2}{L^4} = \kappa\dot{\Phi}^2 \tag{II.101}$$

$$\begin{aligned} &-\frac{1}{2rf}\Big[rf'' + 3f' + 3rf'c' + 2f'rd' + 2f(rc'^2 + 4d' + 2rd'^2 + rc'' + 2rd'')\Big] \\ &+ \frac{4}{f} = \kappa\Phi'^2 \end{aligned} \tag{II.102}$$

$$\begin{aligned} &\frac{-1}{2r^2}\Big[r^2f'' + rf'(3 + 3rc' + 2rd') + 2rf(rc'^2 + 3c' + 2rc'd' + rc'') - L^4(2\dot{c}^2 \\ &+ 4\dot{c}\dot{d} + 2\ddot{c})\Big] + 4 = 0. \end{aligned} \tag{II.103}$$

$$\begin{aligned} &-rf' - r^2f'd' - 2f - 5frd' - 2fr^2d'^2 - frc' - fr^2c'd' - fr^2d'' + L^4(\dot{c}\dot{d} \\ &+ 2\dot{d}^2 + \ddot{d}) + 4r^2 = 0. \end{aligned} \tag{II.104}$$

Upon eliminating $\ddot{c}$ and $\ddot{d}$ from these equations, one gets

$$
-\frac{r^2 f''}{2} - \frac{5rf'}{2} - 4fr(c' + 2d') - \frac{3}{2}r^2 f'(c' + 2d') - 4fr^2 c'd' + 8r^2 + 2L^4(\dot{d})^2
$$
$$
- r^2 f(c'^2 + 4d'^2) - fr^2(c'' + 2d'') + 4l^4\dot{c}\dot{d} - 2f = L^4\kappa\dot{\Phi}^2 \qquad \text{(II.105)}
$$

$$
rf'' + 3f' + 3rf'c' + 2rf'd' + 2fr(c')^2 + 8fd' + 4rf(d')^2 + 2frc'' + 4frd''
$$
$$
-8r = -2rf\kappa\Phi'^2 \qquad \text{(II.106)}
$$

At second order, (II.105) and (II.106) become

$$
-4fr(c_2' + 2d_2') - \frac{3}{2}r^2 f'(c_2' + 2d_2') - fr^2(c_2'' + 2d_2'') = \kappa L^4(\dot{\Phi}_1)^2 \qquad \text{(II.107)}
$$

$$
r^2 f'(3c_2' + 2d_2') + 8rfd_2' + 2fr^2(c_2'' + 2d_2'') = -2\kappa r^2 f(\Phi_1')^2 \qquad \text{(II.108)}
$$

In order to simplify and solve the above set of equations further, let $c_2 + 2d_2 = X$. Then,

$$
-4frX' - \frac{3}{2}r^2 f'X' - fr^2 X'' = \kappa L^4(\dot{\Phi}_1)^2
$$
$$
X'' + \frac{4fr + \frac{3}{2}r^2 f'}{fr^2}X' = -\kappa L^4 \frac{(\dot{\Phi}_1)^2}{fr^2}
$$
$$
X'' + P(r)X' = -\kappa L^4 \frac{(\dot{\Phi}_1)^2}{fr^2}
$$

where $P(r) = \frac{4fr + \frac{3}{2}r^2 f'}{fr^2}$. In the above form, it is now easy to obtain X' (and hence X):

$$
X' = -\kappa L^4 e^{-\int^r P(r)dr} \int^r e^{\int^r P(r)dr} \frac{(\dot{\Phi}_1)^2}{fr^2} dr \qquad \text{(II.109)}
$$

Thus

$$c_2' = \frac{1}{4rf - 2f'r^2}\left[2\kappa(r^2 f(\Phi_1')^2 - L^4(\dot{\Phi}_1)^2) - X'(4rf + 2r^2 f')\right] \quad \text{(II.110)}$$

$$d_2' = \frac{X' - c_2'}{2} \quad \text{(II.111)}$$

At the third order, for the class of perturbations considered here, the equation takes the form

$$\partial_t \partial_t \Phi_3 + \hat{L}\Phi_3 = S(c_2, d_2, \Phi_1) \quad \text{(II.112)}$$

where $S = -(\dot{c_2} + 2\dot{d_2})\dot{\Phi}_1 + \frac{r^2 f}{L^4}(c_2' + 2d_2')\Phi_1'$.

Even though, further simplification requires one to find the eigenfunctions of the linear operator $\hat{L}$, this exercise shows that indeed, if one can find such an eigenbasis, one could repeat the entire perturbative analysis on the lines of [8], to static, diagonal, locally asymptotically AdS spacetimes as well. In such a case, because of the asymptotically resonant nature of the spectrum, the system will be stable, provided the amplitude of perturbation is below the threshold amplitude. In fact, Ben Craps, Erik Jonathan Lindgren and Anastasios Taliotis [95], did a fully backreacted numerical analysis of AdS soliton-scalar source system, where it was observed that if the energy injection to the system is small, it leads to scattering, while above a threshold, it leads to the formation of a black brane. In other words, there existed a minimum amplitude, below which the system remained stable.

The literature on AdS instability is much more vast and since the work by [8], a lot of interesting developments have taken place. Since, we feel that these topics are relevant, in the upcoming sections, we will be briefly reviewing some key areas.

II.7 Stable solutions

As mentioned earlier, the arguments based on nonlinear dynamics which we presented in the previous section were stability results. There is no guarantee that a system with resonant spectrum at the linear level will necessarily collapse for a given initial condition. For eg., in [24], [36], it was found that if the width of the initial Gaussian data fell within a certain range, then below a threshold amplitude, the system remains stable. Similarly, in [8], in case of the single-mode initial data, the secular resonances could be removed by the Poincaré-Lindstedt technique.

Taking cue from [8], Maliborski and Rostworowski [23] used nonlinear perturbative method and full numerical evolution to construct time-periodic solutions. The main idea was to expand the solutions to (II.22-II.24) in the following manner:

$$\phi = \epsilon \cos(\tau) e_\gamma + \sum_{\text{odd } \lambda \geq 3} \epsilon^\lambda \phi_\lambda(\tau, x) \tag{II.113}$$

$$\delta = \sum_{\text{even } \lambda \geq 2} \epsilon^\lambda \delta_\lambda(\tau, x), \quad 1 - A = \sum_{\text{even } \lambda \geq 2} \epsilon^\lambda A_\lambda(\tau, x) \tag{II.114}$$

where $e_p(x)$ are the linear level eigenfrequencies as defined by (II.30) and $e_\gamma(x)$ is the dominant mode in the solution in the limit $\epsilon \to 0$. $\tau = \Omega_\gamma t$ is the rescaled time with

$$\Omega_\gamma = \omega_\gamma + \sum_{\text{even } \lambda \geq 2} \epsilon^\lambda \omega_{\gamma,\lambda} \tag{II.115}$$

and

$$\phi_\lambda = \sum_j f_{\lambda,j}(\tau) e_j(x) \tag{II.116}$$

$$\delta_\lambda = d_{\lambda,-1}(\tau) + \sum_j d_{\lambda,j}(\tau) e_j(x), \quad A_\lambda = \sum_j a_{\lambda,j}(\tau) e_j(x) \tag{II.117}$$

where $f_{\lambda,j}(\tau)$, $d_\lambda(\tau)$ and $a_{\lambda,j}(\tau)$ are periodic in τ. Then one fixes $\omega_{\gamma,\lambda}$ so as to remove secular terms arising at $\epsilon^{\lambda+1}$ order. The stability of such a data was also indicated by the fact that the phase space generated by the time evolution of such an initial data formed closed curves. It was seen that if one perturbed these solutions slightly, they remained close to the periodic orbits and did not collapse.

One could also construct nonlinear time-periodic solutions by replacing the scalar field with a complex scalar field. Such solutions were called boson stars [14], [24] (the gravitational analogues of boson stars were called geons, which we will discuss in Chapter IV). In [24], Alex Buchel, Steven L. Liebling and Luis Lehner constructed boson star solutions in global AdS and showed that the ground state as well as the first three excited states of boson stars were perturbatively stable. In [37], Gyula Fodor, Péter Forgàcs and Philippe Grandclément constructed spatially localized, time-periodic solutions to scalar field in AdS, called scalar breathers, for various self-interacting potentials. These solutions were parameterized only by their amplitude ϵ, while their frequencies are functions of their amplitude. It was found that they were perturbatively stable.

Note that, the asymptotically AdS solutions like boson stars, geons and Schwarzschild black holes, which do not have a resonant spectrum like that of AdS, can approach normal mode frequencies of AdS in the large r limit (here, r is the radial coordinate in static coordinates). For example, for a $d-$ dimensional AdS-Schwarzschild black hole, such frequencies take the form (Oscar J.C. Dias, Gary T. Horowitz, Don Marolf and Jorge E. Santos [40])

$$\omega = \omega_{AdS} + \mathcal{O}(\ell^{-\frac{d-3}{2}}) \tag{II.118}$$

Such modes correspond to large angular momentum ℓ because the angular momentum provides a centrifugal barrier and the large ℓ modes tend to concentrate in the asymptotic region. There is indeed a possibility that these modes can trigger an instability. In [40], the authors, through formal perturbation theory, argued that except possibly in higher dimensions, the large ℓ frequencies were not strong enough to cause instabilities.

Another pertinent question to address was—what was the nature of such stable solutions? Both numerical and perturbative analysis indicated that a data which is single mode/oscillon dominated is stable. Spectral analysis of stable initial data by Nils Deppe and Andrew R. Frey [39] showed that they are in fact all single-mode dominated. This included Gaussian wave packet with their width in a certain range [24], [36], scalar breathers described in [37] as well as the two-mode initial data with the highest temperature [52], which had a dominant energy in $p = 0$ mode, with $p = 0$ mode having almost twice as energy than the $p = 1$ mode. The spectral decomposition of the multiple Gaussian initial data considered in [38] for massless scalar fields in AdS showed that approximately 82% of energy was concentrated in the lowest mode for all times.

The discovery of a wide range of non-collapsing solutions was surprising, since it implied CFT states which never thermalized. It was desirable to know further features of such non-collapsing solutions. For example, what was their power spectrum and how did it differ from that of collapsing solutions? How did the transfer of energy across various modes take place? We will now see how resummation methods like TTF have been employed to analyze small perturbations about AdS.

II.8 Resummation methods and their applications

In order to gain further insight into the nature of interactions in the small amplitude limit, a more comprehensive perturbative analysis was needed. As we saw previously in [8], the naive perturbation theory breaks at the third order because of the presence of irremovable resonances. Alternate resummation techniques were introduced as a way to systematically resum the such secular terms. These equivalent methods were called the two-time framework (TTF) [43], renormalization group method [46], and resonant approximation [49] in the literature. These methods had relevant applications. For eg., they were used to construct quasi-periodic solutions. Moreover, the inherent scaling symmetries in the equations allowed one to probe the the small amplitude limit, $\epsilon \to 0$, which were inaccessible in numerical simulations.

II.8.1 Two time Framework

The authors, V.Balasubramanian, A. Buchel, S. R. Green, L. Lehner & S. L. Liebling developed the two-time framework to capture the time-scale not captured in the naive perturbation theory in [43]. As the name suggests, in TTF one has two time scales, a slow time $\tau = \epsilon^2 t$ and a fast time t. While τ captures the energy transfer between modes, t characterizes the normal modes. Using this method, one effectively integrates out the fast time scale t, so that one can examine the transfer of energy across modes for long times.

The methodology in context of the Einstein-scalar field set-up is as follows. The scalar field was expanded as

$$\phi = \epsilon\phi_1(t, \tau, x) + \epsilon^3\phi_3(t, \tau, x) + \mathcal{O}(\epsilon^5) \tag{II.119}$$

On similar lines the metric variables were also expanded in even powers of ϵ. The partial derivative ∂_t was replaced with $\partial_t + \epsilon^2\partial_\tau$.

At the linear level, the general solution for ϕ_1 was written as a sum over the oscillon basis $e_p(x)$ in the following manner:

$$\phi_1(t, \tau, x) = \sum_{p=0}^{\infty}(\alpha_p(\tau)e^{-i\omega_p t} + \bar{\alpha}_p(\tau)e^{i\omega_p t})e_p(x) \tag{II.120}$$

Note that, here the key idea is to assume amplitudes $\alpha_p(\tau)$ to be functions of the slow-time instead of being constants. Then at third order in ϵ, one obtained an equation for ϕ_3. The secular growth at this level could be eliminated by choosing $\alpha(\tau)$ to satisfy the following equation

$$-2i\omega_p\partial_\tau\alpha_p = \sum_{klm} S^{(p)}_{klm}\bar{\alpha}_k\alpha_l\alpha_m \tag{II.121}$$

where $S^{(p)}_{klm}$ represents the different resonant channels. This equation could be solved for a given initial condition for ϕ.

The TTF and the full GR simulation converged in the limit $p_{max} \to \infty$ and $\epsilon \to 0$. But for numerical computations, it was required to truncate the solutions at a finite $p = p_{max}$. Moreover, the equations had a scaling symmetry $\alpha_p(\tau) \to \epsilon\alpha_p(\tau/\epsilon^2)$, This meant that if the solution with amplitude α_p at time τ, then it would do the same thing for amplitude $\epsilon\alpha_p$ at a longer time

τ/ϵ^2. This scaling symmetry, thus helps one to probe regimes of arbitrarily small perturbations.

In [44], various conservation laws were derived from the TTF equations. It included the conservation of energy E as well as conservation of particle number N. Each quasi-periodic TTF solution was characterized by these two conserved quantities. The consequence of the simultaneous conservation of both E and N was that the high p modes could not all get populated. As the energy flew to higher p modes, the number of particles populating the lower modes would accordingly increase, so as to conserve the particle number N. Thus, the simultaneous conservation of E and N implied that the quasi-periodic solutions exhibited dual cascades of energy.

II.8.2 Renormalization Group (RG) resummation

In [46], the authors, B. Craps, O. Evnin & J. Vanhoof, used a technique called Renormalization Group method, to study the AdS instability problem (this method was originally proposed in [96]). This resummation technique gives rise to, what is called, the Renormalization Flow (RF) equations.

The RG method agrees with the TTF method of [43] at the lowest non-trivial order. In fact, one can obtain the RF equations from the TTF equations by rewriting the latter in amplitude-phase representation

$$\alpha_p(\tau) = A_p(\tau)e^{iB_p(\tau)}, \tag{II.122}$$

where $\alpha_p(\tau)$ is the amplitude which comes in equation (II.121). The interaction coefficients in the renormalization flow equations govern the transfer of energy across modes. Ben Craps, Oleg Evnin and Joris Vanhoof calculated the ultraviolet asymptotics for these interaction coefficients in [48]. They found that in the ultraviolet regime, these interaction coefficients became more strong as one moves to higher dimensional AdS. This was another hint that the resonant transfer of energy among modes is more pronounced in higher dimensions.

II.8.3 Construction of Quasi-periodic solutions using TTF

One of the primary applications of TTF was construction of quasi-periodic (QP) solutions [43], [52]. Such solutions were mostly dominated by a single dominant mode $p = p_r$ but with non-zero energy in all other modes. For example, in [43], the initial data was of the form

$$\alpha_p(0) = \epsilon \frac{e^{-\mu p}}{2p+3} \tag{II.123}$$

which was a good approximation to the quasi-periodic solutions corresponding to $p_r = 0$. Upon adding small perturbations, these solutions were found to be stable. Moreover, there was no indication of turbulent behavior. Similarly, Stephen R. Green, Antoine Maillard, Luis Lehner and Steven L. Liebling constructed quasi-periodic solutions with $p_r > 0$ in [52]. Since E and N were two conserved quantities within TTF, one could parametrize these solutions by the quantity $T = E/N$, the temperature of the spectra. Only for small T, the energy spectra would approach exponentials, implying QP solutions. For large T, the solutions were highly deformed because of their dependence on mode truncation p_{max} and were therefore unphysical. The stability of such quasi-periodic solutions was checked by subjecting them to small perturbations. It was found that all of the physical solutions were indeed stable. This was also reflected in the fact that the spectra of perturbations about the

QP family were asymptotically resonant (which meant that there existed a minimum amplitude below which the perturbed solutions remained stable).

The reason behind the stability of quasi-periodic solutions could also be inferred from the conservation of E and N. Since each QP solution was parameterized by E and N, the conservation laws ensured that any energy which flowed into higher modes was balanced by simultaneous flow of energy to lower modes. This was in contrast to the collapsing solutions, where the direct cascade of energy dominated the inverse cascade of energy, which then led to concentration of energy to finer spatial scales.

How could one relate this to the initial data which lead to stable solutions in the AdS (in)stability problem? The authors of [52] studied the behavior of solutions for the Einstein-scalar field system of Bizon [8] using TTF. From their studies they inferred that all stable solutions of AdS_4 in the limit $\epsilon \to 0$ were the ones which were sufficiently close to the quasi-periodic solutions and these together formed "stability islands." In other words, stable solutions, despite not being quasi-periodic themselves, had their orbits centered around the QP solutions. Further, it was observed that unlike the collapsing solutions which had a power-law spectrum, the energy-spectra of the non-collapsing solutions were characterized by an exponential spectrum.

II.8.4 Two-mode initial data

As seen in section II.4.3, the two-mode data in [8] was found to exhibit irremovable resonances and was expected to collapse. The authors of [43] took the example of the two-mode initial data and analyzed their evolution in the TTF framework. They found that the scalar profile $\Pi^2(t, x = 0)$, which grows initially, instead of blowing up as in [8], decreased close to its initial value. This kind of behavior repeated itself and this recurrence phenomenon was observed for small enough ϵ during the entire course of the simulation. Thus, within the regime of validity of TTF, they found no signs of collapse for the solutions of the two mode initial data. This kind of behavior was also reflected in the way energy flows across the modes. Initially, the energy was distributed equally among $p = 0, 1$ mode. Then it flowed to the higher p modes. After some time, the energy started flowing back to the lower modes and then returned to its original configuration.

Note that, this claim was refuted by P. Bizoń and A. Rostworowski [45], where they claimed that the solutions for the two-mode initial data, after exhibiting recurrence phenomenon initially, the scalar profile $\Pi^2(t, x = 0)$ indeed blew up. They attributed the absence of blow-up in [43] to insufficient resolution of spatial scales. In order to further probe the instability related to the two-mode data, Bizoń, Maliborski and Rostworowski [49] studied the spherically symmetric evolution of massless scalar field in five dimensions for the same initial data, this time using the renormalization flow (RF) equations. They combined the RF equations with the analyticity method and provided evidence that the two-mode data is prone to collapse. This was reflected by the fact that if one fits the amplitudes $A_p(\tau)$ given by (II.122) as follows (see section-II.4.6 for more details on analyticity strip method)

$$A_j(\tau) = j^{-\gamma(\tau)} e^{-\rho(\tau) j}, \quad j >> 1 \tag{II.124}$$

then the analyticity radius $\rho(\tau)$ approached zero in finite time τ_*. This in turn meant that the solutions of the RF equations became singular, implying collapse. The inherent scaling symmetry $\alpha_p(\tau) \to \epsilon \alpha_p(\tau/\epsilon^2)$ of the TTF/RF equations would then allow one to conclude that this behavior should continue for arbitrarily small perturbations, ϵ.

Another relevant aspect in connection with two-mode data as well as collapsing solutions was phase-coherence. Note that previously, the association of weak turbulence with collapse was based on the assumption that the phases B_p of the eigenstates (II.31) were randomly distributed. But Freivogel & Yang [57] found that if the phases were coherent instead, then it led to solutions having a stronger back-reaction for the same power-law $E_p \sim (p+1)^{-\alpha}$. Mathematically phase coherence is defined as follows—suppose θ_p refers to the phase with mode p such that

$$\theta_p(\tau) = \omega_p t + B_p(\tau) \tag{II.125}$$

where $B_p(\tau)$ is the slow phase factor defined in (II.122). Then for large p, it should take the form

$$B_p(\tau) = p\gamma(\tau) + \delta(\tau) + ... \tag{II.126}$$

The authors of [57] tested for phase coherence in two-mode data and found that it leads to an initially phase-coherent energy cascade. This is consistent with the fact that such an initial data is prone to collapse [45], [49].

Further, in [58], Fotios V. Dimitrakopoulos, Ben Freivogel and Juan F. Pedraza observed that for the two-mode data (in AdS_4), the signature for phase coherence could be seen if one plots the variation of phases B_p with mode number p. For $60 \leq p < 140$, B_p varied with p in an almost linear fashion for different times $\tau \leq \tau_{max}$, where τ_{max} is the time till which the results from evolution is valid. This is consistent with the formula for phase coherence condition (II.126).

II.9 Multi-oscillators

In [41], M.Choptuik, J. E. Santos & B. Way proposed the existence of an infinite-parameter non-collpasing solutions called multi-oscillators, whose presence could imply the apparent stability of boson stars and oscillons. They explicitly constructed a two-parameter family called double-oscillators that branched from boson stars (constructed using massless scalar fields) and then showed that these solutions remained single-mode dominated. Moreover, R. Masachs & B. Way [42] studied the existence of double oscillators with equal energy distributed across the two modes, subjected to different kinds of boundary condition. Note that, in this case the double-oscillator was constructed from massive complex scalar field. Under Robin boundary condition, such a system of double-oscillators had a non-resonant spectra. It was verified that for this case, the double oscillators existed and were stable, which is consistent with the previous literature which states that a non-resonant spectrum is essential for a system to be stable. It was also seen that for Dirichlet case, such a double-oscillator does not exist. This is consistent with the idea that a two-mode data subjected to Dirichlet condition indeed collapses to form black holes.

The most interesting result was for the case of Neumann boundary condition, where it was found that in spite of the spectrum being resonant, the double oscillators existed. This was the first example of a non-collapsing multi-mode dominated system with a resonant spectrum. But it was also verified that the double-oscillator subjected to Neumann condition does not exactly describe a equal-energy two-mode data.

II.10 AdS selection rules

If we go back to the naive perturbation theory in [8], we see from (II.35), that a resonant condition, in principle can be satisfied, whenever the triad of frequencies $\{\omega_{p_1}, \omega_{p_2}, \omega_{p_3}\}$, coming from the initial seed satisfy

$$\omega_p = \pm\omega_{p_1} \pm \omega_{p_2} \pm \omega_{p_3} \tag{II.127}$$

And yet, only the $++-$ channel, namely, the condition

$$\omega_p = \omega_{p_1} + \omega_{p_2} - \omega_{p_3} \tag{II.128}$$

leads to secular growth of resonances. In fact, through a brute force analytical calculation, it was proved by B. Craps, O. Evnin & J. Vanhoof in [46] that the interaction coefficients of all, except that for the $++-$ channel vanish.

The fact that certain resonant channels has many implications. For eq., it enhances the submanifolds of quasi-periodic solutions as well as leads to conservation of certain quantities. These conserved quantities, in turn can lead to dual cascades of energy which can inhibit collapse. In order to further probe the vanishing resonant channels, a much simpler model of a self interacting probe scalar field in AdS, with a ϕ^N coupling was taken up in [83]. Upon carrying out the perturbative expansion, at the leading order the eigenvalues ω_{nl} were

$$\omega_{nl} = 2n + l + d \tag{II.129}$$

At the nonlinear level, the forced harmonic oscillator equation had several resonant channels of the form

$$\omega_{n_1 l_1} = \sum_{i=2}^{N} \pm\omega_{n_i l_i} \tag{II.130}$$

Out of these, the top resonant channel was one which is the most efficient in transferring energy from low to high modes, namely

$$\omega_{n_1 l_1} = \sum_{i=2}^{N} \omega_{n_i l_i} \tag{II.131}$$

I-Sheng Yang [83] proved that such channels are absent whenever Nd is even. Later, Oleg Evnin and Rongvoram Nivesvivat [85] pointed out that the same is true for odd Nd as well.

In both [46] and [83], brute force method was used to prove the absence of certain resonant channels. This led to a question-Is there a more natural and elegant way to derive these selection rules? Does the clue lie in the rich symmetries of the underlying AdS background?

Oleg Evnin and Chethan Krishnan [84] showed that the mode functions of the same frequency in global AdS_{d+1} formed multiplets of a hidden $SU(d)$ symmetry. Since the explicit construction of the $SU(d)$ generators was not known, it was deemed difficult to employ this knowledge in deriving the selection rules. In [85], the authors employed an alternate way to derive the selection rules for the case of a self-interacting probe scalar field in AdS. They did this by writing explicit formulas for the mode functions in terms of the isometry based raising operators. These then acted on modes of the lowest frequency. Hence, the selection rules were derived solely in terms of the AdS isometries.

II.11 Critical phenomenon in AdS

It is well known that flat spacetime exhibits a phenomenon called critical phenomenon [61]. In the seminal work by Choptuik, it was discovered that the spherically symmetric evolution of a massless scalar field in asymptotically flat background possessed a threshold between dispersion of fields and black hole formation. Thus such systems had a critical solution such that supercritical configurations formed black holes, with the black hole mass M_{bh} following the power-law $M_{bh} \propto (p - p^*)^\gamma$. Here, p parameterized a one-parameter family of initial data with p^* representing the critical solution. The scaling factor $\gamma \sim 0.37$ was universal and was same for any family of initial data.

On the other hand, the case with AdS was vastly different. If one looked at the variation of horizon radius (or black hole mass) with amplitude, one would observe that there was a sequence of critical amplitudes for which the apparent horizon (AH) shrank to zero. Corresponding to each such critical amplitude ϵ_n was a branch, such that each branch corresponded to AH formation after n number of bounces off the AdS boundary. The very first branch was associated with a direct collapse. The authors of [62] (D.S. Oliván & C. F. Sopuerta) drew the following conclusions regarding the points close to criticality.

- The supercritical configurations obeyed the same scaling power-law behavior as the asymptotically flat case. This was also first confirmed by [8].
- They also corroborated the presence of a mass gap between two branches.
- The upper part of a particular branch (corresponding to sub-critical

points) followed a power law

$$M_{AH} - M_g \propto (p^* - p)^\zeta \tag{II.132}$$

where M_g is the minimum mass and $\zeta \simeq 0.7$. This universal exponent ζ was the same for all branches and also for all families of initial data. Hence, this power-law was associated with points which approach a critical point from the left.

One may ask as to why the scaling laws for supercritical solutions remained the same as that for asymptotically flat case, while for the subcritical configurations different? The reason for this could be attributed to the fact that when one approached the critical point from the right, one was looking at black holes of very small horizon radius. In other words, one was looking at the scaling behavior in a infinitesimal region. Hence, the influence of cosmological constant was negligible in this case. On the other hand, for the later type, the Λ influence could not be ignored, so the critical behavior and exponent were different.

In order to further probe into the features of scaling laws for subcritical configurations, Rong-Gen Cai, Li-Wei Ji & Run-Qiu Yang [63] studied the spherically symmetrical evolution of massless scalar field confined within a reflecting wall in AdS. The authors confirmed the scaling law for subcritical configuration obeyed (II.132), but with the critical exponent ζ depending upon the value of Λr_d^2, where r_d refers to the position of the reflecting wall. This showed that unlike the supercritical exponents, the subcritical exponents were dependent on the position of the reflecting wall. Moreover, the subcritical solutions also obeyed a new time scaling for the formation time of the gapped collapse

$$T_{bh} - T_g \propto (\epsilon - \epsilon^*)^{\xi} \qquad \text{(II.133)}$$

where ξ also depends on Λr_d^2.

How does critical collapse manifest in the dual CFT? One could see this manifest in the expectation value of the QFT stress tensor $< T^{\mu\nu} >$ [64]. Paul M. Chesler and Benson Way [64] studied the dynamics of pure gravity in with a negative cosmological constant in five dimensions and found that $< T^{\mu\nu} >$ exhibited scale echoing near criticality i.e. $< T_{\mu\nu>}$ oscillated on increasingly finer scales as it approached criticality. In fact, as $\epsilon \to \epsilon_*$, the echo frequency became $f \sim |\epsilon - \epsilon_*|^{-\gamma}$ and the scale echoing terminated with $< T^{\mu\nu} >$ diverging. This meant that the formation of naked singularity on the gravity side near criticality manifested as singularity in the stress tensor on the dual CFT side.

II.12 Other system of interest

Effect of Gauss-Bonnet term: Often, when the short distance and higher curvature dynamics become relevant, one needs to add higher derivative terms to the gravitational and matter actions. The Gauss-Bonnet term is the only relevant higher curvature correction term in five dimensions and finds its application in the study of AdS_5/CFT_4 correspondence when the higher order curvature corrections are considered relevant. Therefore, it is desirable to know the effect on stability of adding such a term to the Einstein gravity. The authors of [21], [22] studied the spherically symmetric evolution of massless scalar field in five dimensional Einstein Gauss Bonnet (EGB) gravity. Its action is given by

$$S = \int d^5x\sqrt{-g}\Big\{ -\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + \frac{1}{2\kappa_5^2}\Big(R + 12\Lambda + \frac{\lambda_3}{2}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\Big)\Big\} \tag{II.134}$$

Despite this action being higher order, the EGB theory itself remain second order in derivative of the metric, which makes the study of AdS instability highly tractable.

One key feature of Gauss Bonnet gravity is that one cannot form a black hole if the ADM mass is less than a certain critical value. This is because in GB gravity, the Misner-Sharp mass contained within the black hole horizon r_h is given by

$$M(r_h) = \frac{1}{2}(\lambda r_h^4 + r_h^2 + \lambda_3) \tag{II.135}$$

Thus even as $r_h \to 0$, the mass instead of going to zero attains a finite critical mass say $M_{crit} = \lambda_3/2$. Below this critical value, a black hole cannot exist in GB gravity.

Nils Deppe, Allison Kolly, Andrew R. Frey and Gabor Kunstatter [22] studied the evolution for a range of amplitudes $\epsilon = 27 - 48$. It was seen that for high enough amplitudes, a black hole formed immediately. This behavior continued till a certain critical amplitude. Lower than that, the black hole formed only after multiple reflection as in case of Einstein gravity. But there were noticeable differences too. In case, of Einstein gravity, the time of horizon formation t_H remained approximately piecewise constant and appears as steps as the amplitude of perturbation ϵ is decreased. But in GB gravity, there was a transition region between these steps, when the number of reflections before collapse, increases by one. In some of these transition regions, t_H was very sensitive to initial conditions and varied very wildly. It was concluded that this could be a sign of possible chaotic behavior.

Additionally, the influence of GB term could also be seen in the critical phenomenon. The critical exponent for GB case was found to be $\gamma \simeq 0.42$ which was different from the value $\gamma \simeq 0.37$ for Einstein gravity.

It was also seen that the minimum amplitude below which no black hole formation was observed was $\epsilon = 32$, greater than the value of the critical amplitude, which in this case was $\epsilon_{cri} = 21.86$. Here critical amplitude refers to that value of perturbation below which theoretically no black hole can form because of the mass gap.

In order to probe the regions near ϵ_{cri}, the evolution was carried out for two amplitudes, one slightly less than the critical amplitude, one slightly more than that. For $\epsilon = 20, 22$, even though there was a growth in Ricci scalar, there was no horizon formation observed. This led the authors to conclude that the end result might be a naked singularity. Thus, the two key consequences because of the addition of Gauss-Bonnet term was the possibility of the formation of a naked singularity below a certain critical amplitude and also possible chaotic behavior between the transition regions.

Collapse of self interacting fields in AdS: Rong-Gen Cai, Li-Wei Ji and Run-Qiu Yang [68] studied the effect of adding a self interaction $\lambda\phi^4$ term in four dimensions. The evolution was kept spherically symmetric and the metric ansatz was chosen as (II.21) with $d = 3$. The action for such a theory is

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}(R - 2\Lambda) - \frac{1}{2}(\nabla\phi)^2 - \frac{\lambda}{4!}\phi^4\right] \tag{II.136}$$

The authors studied the evolution for both $\lambda > 0$ and $\lambda < 0$ (the nature of AdS boundary ensures that fields are stable even for $\lambda < 0$). A Gaussian profile of the form

$$\Pi(0,x) = 0, \quad \Pi(0,x) = \epsilon \exp\left(-\frac{\tan x}{\sigma^2}\right) \tag{II.137}$$

was used as the initial data with the width $\sigma = 1/16$. The effect of the self interaction term was studied by varying λ across a range of values. The key observation was that while a negative λ enhances instability, a positive one inhibits it. This gets reflected in collapse time near the critical amplitude where for a positive λ, black hole formation happens much later than the corresponding case with $\lambda = 0$ (with the opposite being true for $\lambda < 0$).

Upon varying the width σ, it was found that for narrow width, the influence of the self interaction term is lesser. This is because in this case the influence of self interaction term will only be in a narrow region and has less influence on the system. But for broad widths the influence is significant and this is reflected in the considerate difference in the black hole formation time. Moreover, upon observing energy transfer, it was seen that a positive λ delays energy transfer to high modes whereas a negative one assists it.

Collapse of dust particles in AdS: Eunseok Oh and Sang-Jin Sin [69] studied the collapse of a non-spherical shell made of dusts particles and showed that a black hole forms in one finite falling time. This set-up was studied as it was a more viable candidate to explain the early thermalization observed in RHIC. They attributed the collapse to the unique synchronization property of AdS, because of which all particles with zero initial velocity which start from random initial positions, reach the center simultaneously. Thus, even a non-spherical configuration of dust particles tends to synchronize and assume spherical symmetry as it falls towards center of AdS.

Collapsing shells: Another study by Richard Brito, Vitor Cardoso and Jorge V. Rocha involved spherically symmetric evolution of two concentric thin shells made of a perfect fluid, interacting only gravitationally in an asymptotically AdS background [70] (or in a cavity in flat spacetime [71]). The two-shells initially started from the same initial position R_i. Then the behavior of this two-shell system was observed for different values of R_i. The confinement of the shells within AdS ensured that they were free to cross each other repeatedly. It was found that depending on R_i, such shells either underwent prompt collapse, or collapse after they crossed each other a finite number of times or underwent oscillations for a long time. The oscillatory solutions belonged to the islands of stability. When the orbits of these oscillating stable solutions were analyzed for different ranges of initial conditions, it was observed that they exhibited mildly chaotic behavior.

The Hard wall model: Another example of a confined geometry is the hard wall model. In this, a portion of AdS_{d+1} is cut off at some radial value $r = r_0$. This location corresponds to the hard wall. The location of the hard wall is proportional to the confinement scale Λ.

Suppose a small perturbation of amplitude $\epsilon << 1$ is introduced in the space between r_0 to ∞ in time δt. What will be the end state of such a system? Ben Craps, Elias Kiritsis, Christopher Rosen, Anastasios Taliotis and Joris Vanhoof [65] studied the effect of injecting a massless scalar field ϕ into the hard wall system in four dimensions. For $\Lambda\delta t << 1$, they got the following results

- If $\epsilon^2 \geq (\lambda\delta t)^3$, then a AdS-Schawarzchild black brane with a horizon radius $r_h > r_0$.

- If $\epsilon^2 << (\Lambda\delta t)^5$ then the shell keeps on scattering between the hard wall and the UV boundary.

In [66], Ben Craps, E.J. Lindgren, Anastasios Taliotis, Joris Vanhoof and Hong-bao Zhang showed that the scattering solutions keep on oscillating for very long times and also provided an analytical argument as to why a black brane can never form out of scattering solutions.

Collapse of massive field in AdS: Brad Cownden, Nils Deppe and Andrew R. Frey [73] studied the evolution of massive scalar fields in AdS_5 under spherical symmetry in order to see the impact of varying mass μ of the scalar field and width σ of the Gaussian profile initial data on the behavior of the solutions. They found four kinds of initial data. Two of them were the regular ones: the stable class and the unstable class. While the stable class exhibited quasi-periodic behavior for a very long time, the unstable data collapsed in a time scale given by $t_h \sim \epsilon^{-2}$. Apart from these two, a third class was the metastable class, whose collapse time scales like $t_h \sim \epsilon^{-p}$ where $p > 2$. A fourth class was the irregular class, which did not follow any power law and showed non-monotonic and even chaotic behavior for various parameters (μ, σ). In this case, horizon formation could occur at any time leading to jumps in the collapse time as a function of amplitude. The chaotic behavior seen here was a reminiscence of the one observed for the Gauss Bonnet case [21], [22] and the collapse of two concentric shells [70].

Massive fields in Minkowski: The collapse of massive fields in flat spacetime was studied in [72] by Hirotada Okawa, Vitor Cardoso and Paolo

Pani. The action for this theory was

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi} - \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - \frac{1}{2}\mu^2\phi^2\right) \qquad \text{(II.138)}$$

where μ referred to the mass of field ϕ. There were three parameters in this equation which controlled the evolution—the Compton wavelength $1/\mu$, width of the initial scalar pulse σ and the amplitude of perturbation ϵ. The nature of collapse depended on the width to Compton length ratio $\sigma\mu$. This system tended to exhibit two kinds of behavior: Type-I and Type-II.

Type-II collapse was very similar to massless case and was seen in case $\sigma\mu$ is small. In this case, for amplitudes greater than the critical amplitude, the initial data gave rise to black hole formation. On the other hand, for amplitudes less than the critical value, the fields dispersed to infinity.

Type-I collapse occurred when $\sigma\mu >> 1$. This case was somewhat akin to the AdS case, though there were certain differences too. In this case, for an amplitude $\epsilon > \epsilon_* \sim \mu(\sigma\mu)^{-1.2}$, prompt collapse occurred. For $\epsilon < \epsilon_*$, collapse could still occur after multiple reflections from the potential barrier. Upon further decreasing the amplitude, one reached a threshold amplitude ϵ_{th}, at which point, the number of reflections became infinite. Finally, below ϵ_{th} the system approached a stable bound-state configuration called oscillations, which could belong to a family of oscillating soliton stars [104], [105], [106].

II.13 Non-spherically symmetric collapse

We will now discuss a few works where one deviates from spherical symmetry and hence introduces angular momentum in the system. Because of the complexities involved in the numerical code arising out of breaking all symmetries, it is much more challenging to evolve such systems. Some such studies include [74] and [75] where even though the assumptions of spherical symmetry is relaxed, some kind of symmetry is still preserved in some directions. The recent work by Bantilan, Figueras & Rossi [76], involves evolution of scalar field in aAdS background with no symmetry assumptions.

- **AdS-scalar field system:** Non-spherically symmetric collapse of massless scalar field in asymptotically AdS spacetime was studied in [74], where a certain kind of $SO(3)$ was preserved.

  The idea was to break the spherical symmetry in such a way, so as to preserve the $2-$sphere $d\Omega_2^2$. Thus, the full physical metric would be of the form

$$ds^2 = g_{tt}dt^2 + g_{xx}dx^2 + g_{yy}dy^2 + g_{\theta\theta}d\Omega_2^2 + 2(g_{tx}dtdx + g_{ty}dtdy + g_{xy}dxdy) \tag{II.139}$$

  such that each of the metric component $g_{\mu\nu}$ was only a function of (t, x, y).

  To such a geometry, a scalar field Ψ with a profile of the form

$$\phi = Af(\rho) + Bg(\rho)\cos\chi \tag{II.140}$$

  was coupled. The degree of spherical symmetry and non-spherical symmetry of the initial data was monitored by A and B respectively.

  The following were the results of the numerics:

  – It was seen that by keeping the mass M of the spacetime fixed, if one varied the parameter B, the collapse time also varied accordingly. As one decreased B (and proportionately decreased A so as to keep M fixed), the collapse time was also faster, suggesting that the more one deviates from spherical symmetry, faster the collapse time, which ultimately resulted in black hole formation.

– Similarly, for a fixed collapse time, the variation of critical mass was studied with varying non-spherical parameter B. Here, the critical mass was the minimum mass for which a black hole is formed after N bounces. It was observed the more system deviated from spherical symmetry, the lesser was the mass required for a given collapse time.

The overall conclusion was that the more system deviates from spherical symmetry, faster is the collapse.

- **AdS-Complex doublet scalar field:** The evolution of a complex doublet field Π in AdS_5 background was numerically studied in [75]. This system has the following action

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g}(R + 12 - 2|\nabla\Pi|^2) \tag{II.141}$$

such that the metric preserved an $SU(2)$ symmetry. In this way, one could deviate from spherical symmetry while keeping the numerics in $1+1$ dimensions. The evolution was performed for varying values of angular momentum J, which in turn was proportional to energy E.

For $J = 0$, the final state was simply Schwarzschild-Tangherlini solutions. For finite J, it was observed that whenever the energy $E > E_{extr}$, the end result was a Myers-Perry black hole, which was characterized by vanishing of the scalar field during late times. Whereas whenever $E < E_{extr}$, the end state was a hairy black hole, in which the scalar field approached a non-zero value at late times. Moreover, in contrast to [74], it was seen that the inclusion of angular momentum actually delayed collapse.

- **Scalar collapse in AdS with no symmetry assumptions:** In [76], the authors, Bantilan, Figueras & Rossi presented a Cauchy evolution scheme to study the evolution of a massless scalar field in asymptotically AdS background in four dimensions, with no symmetry assumptions. It was seen that the evolution has two phases. In the first phase, the scalar field would collapse to a black hole. It was then followed by a ringdown phase, where the system would settle down to a Schwarzschild-AdS black hole.

II.14 Is AdS generically stable/unstable?

In case of Minkowski spacetime, it is well known that small fluctuations about it are stable, no matter what kind of initial data one starts with. Hence, Minkowski is generically stable. In case of AdS, there are certain set of initial data which collapse while others which do not. In [59], the authors did a position space analysis to understand the mechanism of the instability. Here, the "generic instability" was defined as the case where the set of stable initial conditions which does not form a black hole, will shrink to measure-zero in the limit $\epsilon \to 0$. Whereas "mixed" meant that both unstable and stable initial conditions have non-zero measure in the limit $\epsilon \to 0$. In order to implement the position-space approach, the authors considered the evolution of a thin shell of energy $E \sim \epsilon^2$, thickness w and initial size r_0, such that $r_0 \gg w$. A "two-region" approximation was then implemented, where the relevant dynamics was the gravitational interaction of the thin-shell when it passes in the region $r < r_0$. Here, one could simply consider the back-reaction of the scalar field on Minkowski space. For $r > r_0$, the field simply propagated in the AdS background (the details of the set-up can be found in [59]). The main idea was to capture the two kinds of behaviors, one, which could in the limit $\epsilon \to 0$, potentially lead to black hole formation in time scale ϵ^{-2}, and another, where this may not happen. Through perturbative and heuristic arguments they proved that if the self interaction makes the shell thinner after each bounce into the region near the origin, more energy is squeezed into a smaller region. If this energy gets compressed into a region smaller than the Schwarzschild radius, it is very likely for the shell to evolve into a black hole. On the other hand, the gravitational effects could become strong if the shell became wider with each bounce, so that energy gets dispersed in a larger region. Since stability and instability were both likely scenarios, it prompted the question—was AdS generically stable or unstable? In other words, what happens to the set of collapsing and non-collapsing solutions as one approached the zero-amplitude limit?

F. Dimitrakopoulos and I-Sheng Yang [60] used the scaling symmetry of the TTF equations to show that non-collapsing solutions remained stable in the full nonlinear theory even as the amplitude $\epsilon \to 0$. This was in conflict with the conclusions of [40], which predicted shrinkage of such solutions to measure-zero in the zero-amplitude limit. The persistence of collapsing solutions would then rule out the generic stability hypothesis. This, in turn, would mean that AdS may neither be generically stable nor be generically unstable. Rather the mixed hypothesis could be true.

II.15 Interpretation in boundary theory through the AdS/CFT correspondence

From the perspective of AdS/CFT correspondence, it is desirable to know how the gravity dynamics translates on the field theory side. According to the duality, the formation and subsequent evaporation of black holes due to Hawking radiation is considered dual to the process of thermalization on the CFT side. The fact that even a small injection of energy can cause thermalization is not surprising. This is because the correspondence holds in the limit of large N_c (number of colors). This guarantees that there are enough number of particles to form a thermal state. The more interesting aspect is the existence of non-collapsing solutions, because they are dual to CFTs which never thermalize. Some of the studies which offer interpretation of the CFT side include [86], [87], [66].

Fo example in [86], Javier Abajo-Arrastia, Emilia da Silva, Esperanza Lopez, Javier Mas and Alexandre Serantes studied the dynamics of a finite size closed system upon injection of energy. It was modelled after the collapse of massless scalar fields in AdS_4. The fact that for small amplitudes, the scalar field collapsed only after multiple bounces off the AdS boundary, was related to the relaxation processes at strong coupling on the field theory, which delays thermalization. They also found that the entanglement entropy exhibits quasi-periodic oscillations before attaining ergodicity. In [87], Emilia da Silva, Esperanza Lopez, Javier Mas and Alexandre Serantes used the scalar collapse in AdS_3 and AdS_4 to interpret CFT on a circle and a sphere. While, for appropriate initial conditions, both circle and sphere could exhibit oscillations, on the circle it could happen for larger energy densities. These oscillations were reminiscent of revivals seen in quantum systems.

II.16 Conclusion

In this chapter, we provided various arguments to explain the numerical observations of instabilities seen in Einstein-scalar field system with reflecting boundary conditions. These arguments were based on the results in nonlinear dynamics. Since, one of the necessary conditions for black hole formation is the localization of fields in space, we analyzed the asymptotic properties of linear level eigenfunctions of the Einstein-scalar field set-up in various dimensions. We saw that the localization was minimum for AdS_3, which could explain, at least partially, the absence of black hole formation in this case. Thus, through our work, we discussed the necessary conditions for the cascade of energy to higher modes as well as the necessary condition for formation of black hole as the end state.

One can relate the above arguments to the literature on the nonlinear stability of AdS, particularly, to the class of quasi-periodic solutions which were constructed using TTF. As the stability analysis of these solutions showed, the linear spectra of such solutions were non-resonant, except for high mode numbers where they would approach a resonant one. This meant that these solutions, when perturbed, remained stable for long times provided the amplitude of perturbation was below a threshold amplitude.

One of the caveats we pointed out was that a resonant spectrum is only a necessary condition for an AdS-like instability. This means a solution can have a resonant spectrum and still be stable. One such example is the double-oscillator subjected to Neumann boundary condition, which has a resonant spectrum and still belongs to the islands of stability.

Additionally, we also reviewed the other interesting developments in the area of AdS instability like the resummation method to probe the small amplitude limit, the critical phenomenon in AdS as well as the evolution of fields in AdS when the assumptions of spherical symmetry are broken.

Chapter III

Gravitational perturbations in a cavity

In the previous chapter, our main focus was to develop a framework, where the results in nonlinear dynamics could be applied to Einstein-scalar field systems with reflecting boundary conditions. This helped us in distinguishing the nature of instability observed in works such as [8], [17], [18] and [19]. In this chapter, we will apply the results from nonlinear dynamics to gravitational perturbations confined in a box. More specifically, we consider the nonlinear stability of $(n+2)-$dimensional Minkowski spacetime, when it is enclosed by a spherical shell of radius r_0. It is based on our work in [80]. The linear stability of this set up was proved by T. Andrade, W.R. Kelly, D. Marolf & J. E. Santos in [81], where, the spherical symmetry of the background metric (Minkowski) allowed the use of Kodama-Ishibashi-Seto (KIS) formalism [99] to simplify the linear level equations. Here, since we will be considering nonlinearities, we will be extending the use of KIS formalism to higher orders of perturbation theory.

We start by giving a general overview as to how to obtain linearized solutions to vacuum Einstein equations in presence of a non-zero cosmological constant in $(m+n)-$dimensions. For this, we expand the solutions to Einstein equations about a background metric (Minkowski in this case). In section III.2, we give details of the KIS formalism in [99], applicable for dealing with linear perturbations of a $(m+n)$ background metric, which can be split into an $n-$ dimensional sphere part, γ_{ij} and a $m-$dimensional part g_{ab}, dependent on the time and radial coordinates. The metric perturbations are then classified as tensors, vectors and scalars, according to their tensorial behaviour on the $n-$sphere. Following the decomposition theorems applicable for one and two rank tensors, the metric perturbations are then expanded in the basis set of functions defined on an $n-$sphere, namely, the scalar, vector and tensor spherical harmonics. Since these harmonics are independent of each other, the linear level equations get completely decoupled, i.e. spherical harmonics of each type and associated with their respective individual quantum numbers, can be analyzed separately. Although the metric perturbations transform under infinitesimal gauge transformation, one can write linear level equations in terms of gauge-invariant variables constructed from combinations of the metric perturbations. In section III.3, we briefly discuss the results of [81], where the linear stability of this set-up has been proved. We then move on to section III.4, where we extend KIS formalism to higher orders in perturbation theory in general $(n+2)-$dimensions. In section III.5, we simplify the equations pertaining to different sectors and use them to obtain information about the dynamics of gravitational waves trapped in a shell in Minkowski, by applying the appropriate Dirichlet boundary condition. Section III.6 contains the conclusions.

III.1 Linearized gravity

Although in this chapter we will be dealing with gravitational perturbations of Minkowski, we will also be taking up the case of AdS perturbations in Chapter IV. Hence, here we will present the way to obtain perturbation equations for the general vacuum Einstein equations without source. We will first discuss how to obtain the equations governing the first-order perturbations by considering the solutions to the vacuum Einstein equations with a cosmological constant

$$R_{\mu\nu} + \frac{(n+1)}{L^2} g_{\mu\nu} = 0 \tag{III.1}$$

Here, the cosmological constant Λ is related to L as, $L^2 = \kappa \frac{n(n+1)}{2\Lambda}$, where $\kappa = 0$ for Minkowski and $\kappa = 1$ for Anti-de Sitter. We expand the solution to (III.1) around a background metric $\bar{g}_{\mu\nu}$ in the following manner

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \tag{III.2}$$

Note that, the background metric also obeys (III.1). From now on, any quantity with a "bar" over it will be related to $\bar{g}_{\mu\nu}$. Accordingly, the Ricci tensor too can be broken down into $R_{\mu\nu} = \bar{R}_{\mu\nu} + \delta R_{\mu\nu}$, where $\delta R_{\mu\nu}$ is given by

$$\begin{aligned} 2\delta R_{\mu\nu} = {} & 2\Delta_L h_{\mu\nu} \\ & - \bar{\nabla}^\alpha \bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h + \bar{\nabla}_\mu \bar{\nabla}_\alpha h^\alpha_\nu + \bar{\nabla}_\nu \bar{\nabla}_\alpha h^\alpha_\mu \\ & + \bar{R}_{\mu\alpha} h^\alpha_\nu + \bar{R}_{\nu\alpha} h^\alpha_\mu - 2\bar{R}_{\mu\alpha\nu\lambda} h^{\alpha\lambda} \end{aligned} \tag{III.3}$$

Here, the raising and lowering of indices is done with respect to the background metric $\bar{g}_{\mu\nu}$. Hence, the equation obeyed by perturbations $h_{\mu\nu}$ can be found by examining equation (III.1) at the linear order, which turns out be

$$\Delta_L h_{\mu\nu} + \frac{(n+1)}{L^2} h_{\mu\nu} = 0 \tag{III.4}$$

We will now take the trace of this equation and since the background metric also obeys (III.1), make the replacement $\frac{(n+1)}{L^2}\bar{g}^{\alpha\beta} = -\bar{R}^{\alpha\beta}$ in (III.4) to obtain

$$\bar{g}^{\alpha\beta}\Delta_L{}^{(i)}h_{\alpha\beta} - \bar{R}^{\alpha\beta\,(i)}h_{\alpha\beta} = 0 \tag{III.5}$$

We then do the operation: add $2\,\times$ (III.4) to $-\bar{g}_{\mu\nu}$(III.5) to obtain the working equation

$$G_{\mu\nu} = \tilde{\Delta}_L h_{\mu\nu} = 0 \tag{III.6}$$

where

$$2\tilde{\Delta}_L{}^{(i)}h_{\mu\nu} = 2\Delta_L{}^{(i)}h_{\mu\nu} + \frac{2(n+1)}{L^2}{}^{(i)}h_{\mu\nu} - \bar{g}_{\mu\nu}(\bar{g}^{\alpha\beta}\Delta_L{}^{(i)}h_{\alpha\beta} - {}^{(i)}h_{\alpha\beta}\bar{R}^{\alpha\beta}) \tag{III.7}$$

III.2 The KIS-formalism

III.2.1 The background quantities

We will now review the KIS formalism (Kodama, Ishibashi & Seto [99]) which originally dealt with simplifying the equations governing the first-order perturbations of a $(m+n)$ dimensional background spacetime, whose manifold

$\mathcal{M}$ is a cross product between two manifolds $\mathcal{N}^m$ and $\mathcal{K}^m$, i.e.

$$\mathcal{M} = \mathcal{N}^m \times \mathcal{K}^n \tag{III.8}$$

Therefore, in terms of the coordinates $z^\mu = (y^a, w^i)$, the background metric can be written as

$$ds^2 = \bar{g}_{\mu\nu}(z)dz^\mu dz^\nu = g_{ab}(y)dy^a dy^b + r^2(y)d\Omega_n^2 \tag{III.9}$$

where $g_{ab}dy^a dy^b$ is a Lorentzian metric associated with $\mathcal{N}^m$ and

$$d\Omega_n^2 = \gamma_{ij}dw^i dw^j \tag{III.10}$$

is a maximally symmetric Einstein metric associated with the manifold $\mathcal{K}^n$. Additionally, $d\Omega_n^2$ has a constant sectional curvature K. For the cases we consider, since $d\Omega_n^2$ is going to be the metric associated with an $n-$sphere, this sectional curvature is $K = 1$.

We will also denote the covariant derivative associated with $\bar{g}_{\mu\nu}$, g_{ab} and γ_{ij} as $\bar{\nabla}_\mu$, $\bar{D}_a$ and $\bar{D}_i$ respectively. Then the the various background quantities such as the Christoffel symbols $\bar{\Gamma}^\lambda_{\mu\nu}$ and curvature tensors $\bar{R}_{\mu\lambda\nu\delta}$ can be written in terms of the corresponding quantities on $g_{ab}dy^a dy^b$ and $\gamma_{ij}dw^i dw^j$. The forms of these quantities can be found in [99]. They are the following: (the superscript m on the top left of a quantity means it is solely associated with $\mathcal{N}^m$ whereas any quantity with a "hat" on top is solely defined on $\mathcal{K}^n$):

Christoffel symbols

$$\bar{\Gamma}^a_{bc} = {}^m\Gamma^a_{bc}, \quad \bar{\Gamma}^a_{ij} = -r\bar{D}^a r\gamma_{ij}$$
$$\bar{\Gamma}^i_{aj} = \frac{\bar{D}_a r}{r}\delta^i_j, \quad \bar{\Gamma}^i_{jk} = \hat{\Gamma}^i_{jk} \tag{III.11}$$

Curvature tensors

$$\bar{R}^a_{bcd} = {}^m R^a_{bcd}$$
$$\bar{R}^i_{ajb} = -\frac{\bar{D}_a\bar{D}_b r}{r}\delta^i_j$$
$$\bar{R}^i_{jkl} = [1-(\bar{D}r)^2](\delta^i_k\gamma_{jl} - \delta^i_l\gamma_{jk}) \tag{III.12}$$

Ricci tensors

$$\bar{R}_{ab} = {}^m R_{ab} - \frac{n}{r}\bar{D}_a\bar{D}_b r$$
$$\bar{R}_{ai} = 0$$
$$\bar{R}^i_j = \left(-\frac{\bar{D}^a\bar{D}_a r}{r} + (n-1)\frac{(1-(\bar{D}r)^2)}{r^2}\right)\delta^i_j$$
$$\bar{R} = {}^m R - 2n\frac{\bar{D}^a\bar{D}_a r}{r} + n(n-1)\frac{(1-(\bar{D}r)^2)}{r^2} \tag{III.13}$$

Moreover, the Lorentzian Lichnerowicz operator Δ_L defined by (III.3), when written in terms of the $\bar{D}_a$ and the $\bar{D}_i$ operators look like [99]

$$\begin{aligned} 2\triangle_L h_{ab} = &-\bar{D}^c\bar{D}_c h_{ab} + \bar{D}_b\bar{D}_c h^c_a + \bar{D}_a\bar{D}_c h^c_b + n\frac{\bar{D}^c r}{r}(-\bar{D}_c h_{ab} + \bar{D}_a h_{cb} + \bar{D}_b h_{ca}) \\ &+ {}^m R^c_a h_{cb} + {}^m R^c_b h_{ca} - 2{}^m R_{acbd}h^{cd} - \frac{1}{r^2}\hat{\triangle} h_{ab} + \frac{1}{r^2}(\bar{D}_a\bar{D}^i h_{bi} + \bar{D}_b\bar{D}^i h_{ai}) \\ &- \frac{\bar{D}_b r}{r^3}\bar{D}_a h_{ij}\gamma^{ij} - \frac{\bar{D}_a r}{r^3}\bar{D}_b h_{ij}\gamma^{ij} + \frac{4}{r^4}\bar{D}_a r\bar{D}_b r h_{ij}\gamma^{ij} - \bar{D}_a\bar{D}_b h \end{aligned} \tag{III.14}$$

$$
\begin{aligned}
2\triangle_L h_{ai} =& \bar{D}_i \bar{D}_b h^b_a + \frac{n-2}{r}\bar{D}^b r \bar{D}_i h_{ab} - r\bar{D}^c \bar{D}_c \left(\frac{1}{r}h_{ai}\right) \\
& - \frac{n}{r}\bar{D}^b r \bar{D}_b h_{ai} - \bar{D}_a r \bar{D}_b \left(\frac{1}{r}h^b_i\right) + \frac{n+1}{r}\bar{D}^b r \bar{D}_a h_{bi} \\
& + \left((n+1)\frac{(\bar{D}r)^2}{r^2} + (n-1)\frac{(1-\bar{D}r)^2}{r^2} - \frac{\bar{D}^c \bar{D}_c r}{r}\right) h_{ai} \\
& + r\bar{D}_a \bar{D}_b \left(\frac{1}{r}h^b_i\right) + \frac{1}{r^2}\bar{D}^b r \bar{D}_a r h_{bi} + (n+1) r \bar{D}_a \left(\frac{1}{r^2}\bar{D}^b r\right) h_{bi} \\
& - \frac{n+2}{r}\bar{D}_a \bar{D}^b r h_{bi} + {}^m R^b_a h_{ib} - \frac{1}{r^2}\hat{\triangle} h_{ai} \\
& + \frac{1}{r^2}\bar{D}_i \bar{D}^j h_{aj} + r\bar{D}_a \left(\frac{1}{r^3}\bar{D}^j h_{ij}\right) + \frac{1}{r^3}\bar{D}_a r \bar{D}^j h_{ij} \\
& - \frac{1}{r^3}\bar{D}_a r \bar{D}_i h_{jk}\gamma^{jk} - r\bar{D}_a \left(\frac{1}{r}\bar{D}_i h\right) \qquad \text{(III.15)}
\end{aligned}
$$

$$
\begin{aligned}
2\triangle_L h_{ij} =& [2r\bar{D}^a r \bar{D}_b h^b_a + 2(n-1)\bar{D}^a r \bar{D}^b r h_{ab} + 2r\bar{D}^a \bar{D}^b r h_{ab}]\gamma_{ij} \\
& + r\bar{D}_i \bar{D}_a \left(\frac{1}{r}h^a_j\right) + r\bar{D}_j \bar{D}_a \left(\frac{1}{r}h^a_i\right) + (n-1)\frac{\bar{D}^a r}{r}(\bar{D}_i h_{aj} + \bar{D}_j h_{ai}) \\
& + 2\frac{\bar{D}^a r}{r}\bar{D}^k h_{ka}\gamma_{ij} - r^2 \bar{D}^c \bar{D}_c \left(\frac{1}{r^2}h_{ij}\right) - n\frac{\bar{D}^a r}{r}\bar{D}_a h_{ij} \\
& - \frac{1}{r^2}\hat{\triangle} h_{ij} + 2\left(\frac{(n-1)}{r^2} + 2\frac{(\bar{D}r)^2}{r^2} - \frac{\bar{D}^c \bar{D}_c r}{r}\right) h_{ij} \\
& + \frac{1}{r^2}(\bar{D}_i \bar{D}^k h_{kj} + \bar{D}_j \bar{D}^k h_{ki}) - 2(\gamma^{kl} h_{kl}\gamma_{ij} - h_{ij})\frac{(1-(\bar{D}r)^2)}{r^2} \\
& - 2\frac{(\bar{D}r)^2}{r^2}\gamma_{ij}\gamma^{kl}h_{kl} - \bar{D}_i \bar{D}_j h - r\bar{D}^a r \bar{D}_a h \gamma_{ij} \qquad \text{(III.16)}
\end{aligned}
$$

III.2.2 Decomposition theorems

In order to reduce the linearized equations into simpler forms, the various perturbation variables are classified based on their tensorial behaviour on $\mathcal{K}^n$. Hence, h_{ab} are pure scalars, h_{ai} transform like vectors and h_{ij} transform like two-rank tensors on $\mathcal{K}^n$. We now quote the following two decomposition theorems given in [100] (H. Kodama & A. Ishibashi)

- The Hodge decomposition theorem states that any dual vector field v_i on a compact Riemannian manifold $(\mathcal{K}^n, \gamma_{ij})$ can be uniquely decomposed as

$$v_i = V_i + \bar{D}_i S \tag{III.17}$$

where $\bar{D}^i V_i = 0$. V_i and S are called vector and scalar type components of v_i respectively.

- If $(\mathcal{K}^n, \gamma_{ij})$ is a compact Riemannian Einstein space with $\hat{R}_{ij} = c\gamma_{ij}$ for some constant c, any second rank symmetric tensor t_{ij} can be uniquely decomposed as

$$t_{ij} = t^{(2)}_{ij} + 2\bar{D}_{(i}t^{(1)}_{j)} + t_L\gamma_{ij} + \hat{L}_{ij}t_T \tag{III.18}$$

where

$$\hat{L}_{ij} = \bar{D}_i\bar{D}_j - \frac{1}{n}\gamma_{ij}\hat{\triangle} \tag{III.19}$$

and $\bar{D}^i t^{(2)}_{ij} = 0$, $t^{(2)i}_{i} = 0$ and $\bar{D}^i t^{(1)}_i = 0$. $t^{(2)}_{ij}$, $t^{(1)}_i$ and (t_L, t_T) as tensor, vector and scalar components of t_{ij} respectively. The operator $\hat{\triangle}$ is defined as $\hat{\triangle} = \bar{D}^i\bar{D}_i$.

Now the metric perturbations $\delta g_{\mu\nu} = h_{\mu\nu}$ can be projected relative to the manifold $\mathcal{K}^n$ as

$$h_{\mu\nu}dz^\mu dz^\nu = h_{ab}dy^a dy^b + 2h_{ai}dy^a dw^i + h_{ij}dw^i dw^j \tag{III.20}$$

As mentioned before, the component h_{ab} transforms like a pure scalar on $\mathcal{K}$, whereas decomposition theorems (III.17) and (III.18) can be applied to h_{ai} and h_{ij} respectively.

III.2.3 The spherical harmonics

For the spacetimes we consider in this chapter and the next, the manifold $\mathcal{K}^n$ is essentially the $n-$sphere . Hence, according to the decomposition theorems, one can write the perturbation variables in terms of the scalar, vector and tensor harmonics associated with a $n-$sphere. The various kinds of spherical harmonics are discussed below:

Scalar spherical harmonics: The scalar spherical harmonics $\mathbb{S}$ satisfy

$$(\hat{\Delta} + k_s^2)\mathbb{S}_{\mathbf{k}_s} = 0 \tag{III.21}$$

Here we will make a distinction between k_s^2 and $\mathbf{k}_s$. k_s^2 denotes the eigenvalue of equation (III.21) and is given by $k_s^2 = l(l+n-1)$, with $l = 0, 1, ...$ On the other hand, $\mathbf{k}_s$ is a multi-index of the form $\{l_s, l_s^{(1)}...l_s^{(n-1)} = m_s\}$, where $l_s, l_s^{(1)}...$ denote the various quantum numbers, such that $l_s \geq l_s^{(1)} \geq l_s^{(2)} \ldots \geq l_s^{(n-2)} \geq |m_s|$.

From $\mathbb{S}$, it is possible to construct scalar-type vector harmonics $\mathbb{S}_i$ as well as scalar-type tensor harmonics $\mathbb{S}_{ij}$. They are defined as

$$\mathbb{S}_i = -\frac{1}{k_s}\bar{D}_i\mathbb{S}; \quad \mathbb{S}_{ij} = \frac{1}{k_s^2}\bar{D}_i\bar{D}_j\mathbb{S} + \frac{1}{n}\gamma_{ij}\mathbb{S} \tag{III.22}$$

$\mathbb{S}_i$ and $\mathbb{S}_{ij}$ satisfy the following properties

$$\bar{D}^i\mathbb{S}_i = k_s\mathbb{S}; \quad \mathbb{S}_i^i = 0; \quad \bar{D}_j\mathbb{S}_i^j = \frac{(n-1)(k_s^2-n)}{nk_s}\mathbb{S}_i. \tag{III.23}$$

Vector spherical harmonics: Vector harmonics $\mathbb{V}_i$ obey the following equation

$$(\hat{\Delta} + k_v^2)\mathbb{V}_{\mathbf{k}_v i} = 0 \tag{III.24}$$

where $k_v^2 = l_v(l_v + n - 1) - 1$ and $l_v = 1, 2, ...$, such that

$$\bar{D}_i \mathbb{V}^i = 0. \tag{III.25}$$

Here, $\mathbf{k}_v$ is the multi-index associated with vector harmonics. From $\mathbb{V}_i$, one can construct vector-type tensors harmonics $\mathbb{V}_{ij}$:

$$\mathbb{V}_{ij} = -\frac{1}{2k_v}(\bar{D}_i \mathbb{V}_j + \bar{D}_j \mathbb{V}_i) \tag{III.26}$$

These tensors satisfy the following properties:

$$\mathbb{V}_i^i = 0; \quad \bar{D}_j \mathbb{V}_i^j = \frac{(k_v^2 - (n-1))}{2k_v}\mathbb{V}_i \tag{III.27}$$

Tensor spherical harmonics: Finally, the tensor type harmonics, $\mathbb{T}_{ij}$ are governed by

$$(\hat{\Delta} + k^2)\mathbb{T}_{\mathbf{k}ij} = 0 \tag{III.28}$$

where $k^2 = l(l + n - 1) - 2$ and $l = 2, 3, ...$ $\mathbf{k}$ denote the multi-index associated with tensor harmonics. They are traceless and divergence-free, i.e., they satisfy

$$\mathbb{T}_i^i = 0; \quad \bar{D}_j \mathbb{T}_i^j = 0 \tag{III.29}$$

III.2.4 Gauge transformation

Consider an infinitesimal coordinate (gauge) transformation of the form

$$z^\alpha \to z^\alpha + \zeta^\alpha \tag{III.30}$$

where ζ^α is of the same order as $h_{\mu\nu}$. Then, under such a gauge transformation $h_{\mu\nu}$ transforms as

$$h_{\mu\nu} \to h_{\mu\nu} - \bar{\nabla}_\mu \zeta_\nu - \bar{\nabla}_\nu \zeta_\mu \tag{III.31}$$

Although the metric perturbations themselves are gauge dependent, the final form of the working equations (III.6) can be written in a gauge independent way.

III.2.5 Gauge invariant perturbation equations

Since each of these sectors, namely the scalar, vector and tensor are completely independent of each other, the linearized Einstein equations pertaining to each of these sectors get completely decoupled and can be analyzed separately. The following equations are not applicable for $l_s = 0, 1$ and $l_v = 1$ cases.

Tensor perturbations

The tensor contribution comes only from h_{ij}. Hence, for this class of perturbations

$$h_{ab} = h_{ai} = 0, \quad h_{ij} = r^2 H_{T\mathbf{k}} \mathbb{T}_{\mathbf{k}ij} \tag{III.32}$$

Using the system of equations (III.14-III.16) in (III.6), and then substituting (III.32) in the resultant equation, one finds that the only non-trivial equation is $G_{ij} = 0$. Hence, one obtains a single equation in H_T which governs the tensor perturbations.

$$-r^2 \bar{D}^a \bar{D}_a H_T - nr \bar{D}^a r \bar{D}_a H_T + (k^2 + 2) H_T = 0 \tag{III.33}$$

Vector perturbations

In this case,

$$h_{ab} = 0, \quad h_{ai} = r f_a^{(v)} \mathbb{V}_i, \quad h_{ij} = 2r^2 H_T^{(v)} \mathbb{V}_{ij} \tag{III.34}$$

Unlike tensor case, vector perturbations change under gauge transformations. Let

$$\zeta_a = 0, \quad \zeta_i = r M^{(v)} \mathbb{V}_i \tag{III.35}$$

With the above choice of variables, one can see that the perturbation variables f_a and $H_T^{(v)}$ transform under the following way with a gauge change

$$\begin{aligned} f_a^{(v)} &\to f_a^{(v)} - r \bar{D}_a \left(\frac{M^{(v)}}{r} \right) \\ H_T^{(v)} &\to H_T^{(v)} + \frac{k_v}{r} M^{(v)} \end{aligned} \tag{III.36}$$

Knowing this, one can define a new variable Z_a

$$Z_a = f_a^{(v)} + \frac{r}{k_v} \bar{D}_a H_T^{(v)} \tag{III.37}$$

which is a gauge invariant quantity.

The non-trivial working equations in (III.6) are $G_{ai} = 0$ and $G_{ij} = 0$. In terms of Z_a, these equations are

$$-\frac{1}{r^n}\bar{D}^b\left\{r^{n+2}\left[\bar{D}_b\left(\frac{Z_a}{r}\right) - \bar{D}_a\left(\frac{Z_b}{r}\right)\right]\right\} + \frac{k_v^2 - (n-1)}{r}Z_a = 0 \qquad \text{(III.38)}$$

$$\bar{D}_a(r^{n-1}Z^a) = 0 \qquad \text{(III.39)}$$

Scalar perturbations

For the scalar sector, the perturbations are expanded in terms of the scalar harmonics

$$h_{ab} = f_{ab}\mathbb{S}, \quad h_{ai} = rf_a\mathbb{S}_i, \quad h_{ij} = 2r^2(H_L\gamma_{ij}\mathbb{S} + H_T^{(s)}\mathbb{S}_{ij}) \qquad \text{(III.40)}$$

Under infinitesimal gauge transformations of the form

$$\zeta_a = T_a\mathbb{S}, \quad \zeta_i = rM\mathbb{S}_i \qquad \text{(III.41)}$$

the perturbation variables transform as

$$f_{ab} \to f_{ab} - \bar{D}_aT_b - \bar{D}_bT_a \qquad \text{(III.42)}$$

$$f_a \to f_a - r\bar{D}_a\left(\frac{M}{r}\right) + \frac{k_s}{r}T_a \qquad \text{(III.43)}$$

$$H_L \to H_L - \frac{k_s}{nr}M - \frac{\bar{D}^a r}{r}T_a \qquad \text{(III.44)}$$

$$H_T^{(s)} \to H_T^{(s)} + \frac{k_s}{r}M \qquad \text{(III.45)}$$

$$X_a \to X_a + T_a \qquad \text{(III.46)}$$

where X_a is defined as

$$X_a = \frac{r}{k_s}\left(f_a + \frac{r}{k_s}\bar{D}_a{}^{(i)}H_T^{(s)}\right) \tag{III.47}$$

Hence, one can construct the following gauge invariant variables

$$F_{ab} = f_{ab} + \frac{1}{2}\bar{D}_{(a}X_{b)}; \quad F = H_L + \frac{H_T^{(s)}}{n} + \frac{1}{r}\bar{D}^a r X_a \tag{III.48}$$

In terms of these variables, the working equations $G_{ab} = 0$, $G_{ai} = 0$, as well as the traceless part of $G_{ij} = 0$ become

$$\begin{aligned}
&- \bar{D}^c\bar{D}_cF_{ab} + \bar{D}_a\bar{D}_cF_b^c + \bar{D}_b\bar{D}_cF_a^c + n\frac{\bar{D}^c r}{r}(-\bar{D}_cF_{ab} + \bar{D}_aF_{cb} + \bar{D}_bF_{ca})\\
&+ {}^mR_a^cF_{cb} + {}^mR_b^cF_{ca} - 2{}^mR_{acbd}F^{cd} + \left(\frac{k_s^2}{r^2} + \frac{2(n+1)}{L^2}\right){}^{(i)}F_{ab} - \bar{D}_a\bar{D}_bF_c^c\\
&- 2n\left(\bar{D}_a\bar{D}_bF + \frac{1}{r}\bar{D}_ar\bar{D}_bF + \frac{1}{r}\bar{D}_br\bar{D}_aF\right) - \Bigg[\bar{D}_c\bar{D}_dF^{cd}\\
&+ \frac{2n}{r}\bar{D}^cr\bar{D}^dF_{cd} + \left(-{}^mR_{cd} + \frac{2n}{r}\bar{D}^c\bar{D}^dr + \frac{n(n-1)}{r^2}\bar{D}^cr\bar{D}^dr\right)F_{cd}\\
&- 2n\bar{D}^c\bar{D}_cF - \frac{2n(n+1)}{r}\bar{D}^cr\bar{D}_cF + 2(n-1)\frac{(k_s^2-n)}{r^2}F - \bar{D}^c\bar{D}_cF_d^d\\
&- \frac{n}{r}\bar{D}^cr\bar{D}_cF_d^d + \frac{k_s^2}{r^2}F_d^d\Bigg]g_{ab} = 0
\end{aligned} \tag{III.49}$$

$$\left(\frac{1}{r^{n-2}}\bar{D}_b(r^{n-2}F_a^b) - r\bar{D}_a\left(\frac{1}{r}F_b^b\right) - 2(n-1)\bar{D}_aF\right) = 0 \tag{III.50}$$

$$[2(n-2)F + F_c^c] = 0 \tag{III.51}$$

III.3 Gravity in presence of a Dirichlet wall

The linear stability of vacuum gravity in the presence of various kinds of Dirichlet walls was studied by Andrade, Kelly, Marolf and Santos in [81]. Among the various kinds of Dirichlet wall considered in [81], includes gravitational perturbations of the flat spacetime subjected to a spherical cavity of fixed radius r_0. Their analysis made use of the KIS formalism discussed in the previous section where the gravitational perturbations pertaining to the three sectors, namely, scalar, vector and tensor sectors, were written in terms of a single master variable. Since our work relies on this analysis, we review this set-up in [81] briefly.

III.3.1 Master equation for each sector

The tensor as well as the set of coupled equations in vector and scalar sectors, were simplified so that each sector was now governed by a single master variable. The equation governing these master variables had the form [81].

$$\bar{D}^a \bar{D}_a \Phi_{(I)}(t,r) = V_{(I)} \Phi_{(I)}(t,r) \tag{III.52}$$

where $I = T, v, s$ for tensor, vector and scalar respectively. These master variables were related to the gauge invariant variables in the following manner :

$$\Phi_T = r^{n/2} H_T \tag{III.53}$$

$$\epsilon^{ab} \bar{D}_b (r^{n/2} \Phi_v) = r^{n-1} Z^a \tag{III.54}$$

$$\Phi_s = \frac{2nr^{n-1}F - nr^{n-2}\dot{F}_t^r}{r^{n/2-1}(k_s^2 - n)} \tag{III.55}$$

where, the potential $V_I(r)$ took the following form, in case the background was Minkowski:

$$V_T(r) = \frac{1}{r^2}\left[2 + k^2 + \frac{n(n-2)}{4}\right] \tag{III.56}$$

$$V_v(r) = \frac{1}{r^2}\left[1 + k_v^2 + \frac{n(n-2)}{4}\right] \tag{III.57}$$

$$V_s(r) = \frac{1}{4r^2}\left[n(n+2) + 4(k_s^2 - n)\right] \tag{III.58}$$

Equation (III.52) could be further simplified by assuming an ansatz for Φ_I of the form $\Phi_I = e^{-i\omega t}\phi_I$

$$\partial_r \partial_r \phi_I + \omega^2 \phi_I = V_I \phi_I \tag{III.59}$$

III.3.2 Boundary condition

The next step once the master equations for each sector had been obtained was to determine how the Dirichlet conditions on the metric translated in terms of the master fields. As shown by in [81], a Dirichlet wall at $r = r_0$ was equivalent to fixing the induced metric at $r = r_0$. Mathematically this meant that any perturbation $h_{\mu\nu}$ with both $\mu, \nu \neq r$ should be zero, i.e.

$$H_T^{(s)} = H_L = f_t = f_{tt} = H_T^{(v)} = f_t^{(v)} = H_T = 0\Big|_{r=r_0} \tag{III.60}$$

How did these boundary conditions translate in terms of the master variables pertaining to each sector?

Tensor modes : From (III.53) and (III.60), one could see, that for tensor sector this simply translated to

$$\phi_T(r_0) = 0 \tag{III.61}$$

Vector modes : The boundary condition (III.60) demanded that at $r = r_0$, $f_t^{(v)}(r)$ and $H_T^{(v)}(r)$. This in fact translated to the gauge invariant variable Z_t vanishing at the boundary because

$$\begin{aligned} Z_t(r_0) &= f_t^{(v)}(r_0) - i\omega \frac{r H_T^{(v)}(r_0)}{k_v} \\ &= 0 \end{aligned} \tag{III.62}$$

From (III.54), one could see that this is equivalent to

$$\partial_r(r^{n/2}\phi_v) = 0\Big|_{r=r_0} \tag{III.63}$$

Scalar modes : In case of scalar modes, the only gauge-invariant quantity which vanished at $r = r_0$ was F_{tt}. So in order to get boundary condition in terms of ϕ_s, one needed to know the relation between the master variable and F_{tt}, which can be found in [101]. This translated to

$$\alpha(r)\partial_r\phi_s + \beta(r)\phi_s\Big|_{r=r_0} = 0 \tag{III.64}$$

where

$$\alpha(r) = -\frac{(n-1)}{r} \tag{III.65}$$

$$\beta(r) = \omega^2 - \frac{(n-1)(2k_s^2 + n(n-2))}{2nr} \tag{III.66}$$

III.3.3 Stability analysis

Here, we will review the stability analysis done in [81] for the gravitational perturbations of Minkowski under the influence of the Dirichlet wall. The authors of [81] did this by showing that the linear spectrum for each of the tensor, vector and scalar sectors is always positive since in such a case, the linearized solutions will be periodic in time. In order to do so, they considered master equation (III.59),

$$-\frac{d^2}{dr^2}\phi_I + V_I\phi_I = \omega^2\phi_I \tag{III.67}$$

where V_I was given by (III.56-III.58). Multiplying this equation overall with the conjugate of ϕ_I, $\bar{\phi}_I$ and then integrating from $r = 0$ to $r = r_0$ gives

$$\int_0^{r_0} (|\partial_r\phi_I|^2 + V_I|\phi_I|^2)dr - \bar{\phi}_I\partial_r\phi_I\Big|_0^{r_0} = \omega^2 \int_0^{r_0} |\phi_I|^2 dr \tag{III.68}$$

The behavior of ϕ_I at boundary points was as follows:

- Equation (III.66), had two linearly independent solutions. One went like $r^{l_I+n/2}$ close to $r = 0$ and another like $r^{1-l_I-n/2}$, where $l_I = l, l_v, l_s$. To ensure the regularity of ϕ_I at the origin, one would need to choose the former solution for ϕ_I. It would then satisfy $\phi_I(r = 0) = 0$.

- The boundary conditions (III.61), (III.63) and (III.64) could be written as

$$a_I(r, k_I)\partial_r\phi_I + b_I(r, k_I)\phi_I - \omega^2 c_I(r, k_I)\phi_I\Big|_{r=r_0} = 0 \tag{III.69}$$

where $a_I(r, k_I)$, $b_I(r, k_I)$ and $c_I(r, k_I)$ were functions of the radial coordinate r and $k_I = k, k_v, k_s$.

For tensor modes, the term $\bar{\phi}_I \partial_r \phi_I \Big|_0^{r_0}$ in (III.68) would vanish and this would imply $\omega^2 > 0$. For vector and scalar modes, (III.68) would take the following form

$$\int_0^{r_0} (|\partial_r \phi_I|^2 + V_I |\phi_I|^2) dr + \frac{b_I(r_0, k_I)}{a_I(r_0, k_I)} |\phi_I(r_0)|^2 = \omega^2 \left(\int_0^{r_0} |\phi_I|^2 dr + \frac{c_I(r_0, k_I)}{a_I(r_0, k_I)} |\phi_I(r_0)|^2 \right) \quad \text{(III.70)}$$

Taking note of the fact that for vector modes

$$a_v = r, \quad b_v = \frac{n}{2}, \quad c_v = 0 \quad \text{(III.71)}$$

one could see from (III.69) that $\omega^2 > 0$ for vector modes as well. Similarly, by noting that for scalar modes,

$$a_s = \frac{(n-1)}{r}, \quad b_s = \frac{(n-1)}{2nr^2}[2k_s^2 + n(n-2)], \quad c_s = 1 \quad \text{(III.72)}$$

one could deduce that the scalar sector was linearly stable as well.

III.4 Extending KI-formalism to higher orders

The main motive of our work in [80] was to study the *nonlinear* stability of gravitational perturbations of (n+2) dimensional Minkowski spacetime within a spherical cavity, by applying the framework developed in our work [20]. In order to assist its study, we need to extend the KIS formalism to higher orders. The KIS formalism was first extended to higher orders in perturbation theory in [28] to study nonlinear interactions of gravitational waves of AdS in four dimensions. Similarly, a detailed and systematic approach to nonlinear gravitational perturbations of vacuum spacetimes was given by Rostworowski in [29], [79] using the formalism by Regge-Wheeler [97] and Zerilli [98]. In the subsections that follow, we will use the KIS formalism to study nonlinear gravitational perturbations trapped in a cavity in $(n + 2)$ dimensional Minkowski spacetime. Before that, we need to obtain the structure of equations governing the higher order perturbations.

III.4.1 Higher order equations

In this section, we will follow Rostworowski's notation in [29] while considering nonlinear perturbations. To accommodate nonlinearities, we will write the solutions of (III.1) as $g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu}$, where

$$\delta g_{\mu\nu} = \sum_{1\leq i} {}^{(i)}h_{\mu\nu}\epsilon^i \tag{III.73}$$

Here, ϵ^i is the expansion parameter and the superscript (i) on the left hand side of a quantity tells us at what order of perturbation theory we are in. Accordingly, the inverse metric $g^{\mu\nu}$, the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ as well as the Ricci tensor $R_{\mu\nu}$ can be decomposed in the following manner [29]:

<u>**Inverse metric**</u>:

$$\begin{aligned} g^{\alpha\beta} &= (\bar{g}^{-1} - \bar{g}^{-1}\delta g\bar{g}^{-1} + \bar{g}^{-1}\delta g\bar{g}^{-1}\delta g\bar{g}^{-1} - ...)^{\alpha\beta} \\ &= \bar{g}^{\alpha\beta} + \delta g^{\alpha\beta}. \end{aligned} \tag{III.74}$$

Christoffel symbol:

$$\Gamma^{\alpha}_{\mu\nu} = \bar{\Gamma}^{\alpha}_{\mu\nu} + \frac{1}{2}(\bar{g}^{-1} - \bar{g}^{-1}\delta g \bar{g}^{-1} + ...)^{\alpha\beta}(\bar{\nabla}_{\mu}\delta g_{\beta\nu} + \bar{\nabla}_{\nu}\delta g_{\beta\mu} - \bar{\nabla}_{\beta}\delta g_{\mu\nu})$$
$$= \bar{\Gamma}^{\alpha}_{\mu\nu} + \delta\Gamma^{\alpha}_{\mu\nu}, \tag{III.75}$$

Ricci tensor:

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_{\alpha}\delta\Gamma^{\alpha}_{\mu\nu} - \bar{\nabla}_{\nu}\delta\Gamma^{\alpha}_{\alpha\mu} + \delta\Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\mu\nu} - \delta\Gamma^{\lambda}_{\mu\alpha}\delta\Gamma^{\alpha}_{\lambda\nu}$$
$$= \bar{R}_{\mu\nu} + \delta R_{\mu\nu}. \tag{III.76}$$

If we denote $\delta R_{\mu\nu} = {}^{(i)}R_{\mu\nu}\epsilon^{i}$, the ${}^{(i)}R_{\mu\nu}$ has the following form

$${}^{(i)}R_{\mu\nu} = \Delta_{L}{}^{(i)}h_{\mu\nu} - {}^{(i)}A_{\mu\nu} \tag{III.77}$$

where ${}^{(i)}A_{\mu\nu}$ is given by

$$\begin{aligned}{}^{(i)}A_{\mu\nu} =&[\epsilon^{i}]\Big\{ - \bar{\nabla}_{\alpha}\big[(-\bar{g}^{-1}\delta g \bar{g}^{-1} + ...)^{\alpha\lambda}(\bar{\nabla}_{\mu}\delta g_{\lambda\nu} + \bar{\nabla}_{\nu}\delta g_{\lambda\nu} - \bar{\nabla}_{\lambda}\delta g_{\mu\nu})\big] \\ &+ \bar{\nabla}_{\nu}\big[(-\bar{g}^{-1}\delta g \bar{g}^{-1} + ...)^{\alpha\lambda}(\bar{\nabla}_{\mu}\delta g_{\lambda\alpha} + \bar{\nabla}_{\alpha}\delta g_{\lambda\mu} - \bar{\nabla}_{\lambda}\delta g_{\mu\alpha})\big] \\ &- 2\delta\Gamma^{\alpha}_{\alpha\lambda}\delta\Gamma^{\lambda}_{\mu\nu} + 2\delta\Gamma^{\lambda}_{\mu\alpha}\delta\Gamma^{\alpha}_{\lambda\nu}\Big\}\end{aligned} \tag{III.78}$$

Here $[\epsilon^{i}]\ f$ denotes the coefficient of ϵ^{i} in the expansion of the power series $\sum_{i}\epsilon^{i}f_{i}$. Hence, by examining (III.1), one can see that the the perturbed Einstein equation at order i is

$$2\triangle_{L}{}^{(i)}h_{\mu\nu} + \frac{2(n+1)}{L^{2}}{}^{(i)}h_{\mu\nu} = {}^{(i)}A_{\mu\nu} \tag{III.79}$$

Similar to the steps followed for linearized equations, we first take the trace of the above equation, then make the replacement $\frac{(n+1)}{L^{2}}\bar{g}^{\alpha\beta} = -\bar{R}^{\alpha\beta}$ and then divide it overall by two to obtain

$$\bar{g}^{\alpha\beta}\triangle_L{}^{(i)}h_{\alpha\beta} - \bar{R}^{\alpha\beta(i)}h_{\alpha\beta} = \frac{1}{2}\bar{g}^{\alpha\beta(i)}A_{\alpha\beta} \tag{III.80}$$

Subtracting $\bar{g}_{\mu\nu}$(III.80) from (III.79) gives us

$$2\triangle_L{}^{(i)}h_{\mu\nu} + \frac{2(n+1)}{L^2}{}^{(i)}h_{\mu\nu} - \bar{g}_{\mu\nu}(\bar{g}^{\alpha\beta}\triangle_L{}^{(i)}h_{\alpha\beta} - {}^{(i)}h_{\alpha\beta}\bar{R}^{\alpha\beta}) = {}^{(i)}A_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta(i)}A_{\alpha\beta} \tag{III.81}$$

Let us define a operator $\tilde{\triangle}_L{}^{(i)}h_{\mu\nu}$, given by

$$2\tilde{\triangle}_L{}^{(i)}h_{\mu\nu} = 2\triangle_L{}^{(i)}h_{\mu\nu} + \frac{2(n+1)}{L^2}{}^{(i)}h_{\mu\nu} - \bar{g}_{\mu\nu}(\bar{g}^{\alpha\beta}\triangle_L{}^{(i)}h_{\alpha\beta} - {}^{(i)}h_{\alpha\beta}\bar{R}^{\alpha\beta}) \tag{III.82}$$

and also a quanity ${}^{(i)}S_{\mu\nu}$, which is defined in terms of ${}^{(i)}A_{\mu\nu}$ as

$${}^{(i)}S_{\mu\nu} = {}^{(i)}A_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta(i)}A_{\alpha\beta} \tag{III.83}$$

Then (III.81) can be written as

$${}^{(i)}G_{\mu\nu} = 2\tilde{\triangle}_L{}^{(i)}h_{\mu\nu} - {}^{(i)}S_{\mu\nu} = 0 \tag{III.84}$$

The higher order metric perturbations too are decomposed on the basis of their behavior on $n-$sphere. Like the linear case, each individual perturbation can be expanded in the basis set of the spherical harmonic functions.

$${}^{(i)}h_{ab} = \sum_{\mathbf{k_s}} {}^{(i)}f_{ab\mathbf{k_s}}\mathbb{S}_{\mathbf{k_s}}; \quad {}^{(i)}h_{ai} = r\left(\sum_{\mathbf{k_s}} {}^{(i)}f_{a\mathbf{k_s}}\mathbb{S}_{\mathbf{k}_s i} + \sum_{\mathbf{k_v}} {}^{(i)}f^{(v)}_{a\mathbf{k_v}}\mathbb{V}_{\mathbf{k}_v i}\right)$$

$$
\begin{aligned}
{}^{(i)}h_{ij} =& r^2 \Bigg(\sum_{\mathbf{k}} {}^{(i)}H_{T\mathbf{k}} \mathbb{T}_{\mathbf{k}ij} + 2 \sum_{\mathbf{k}_v} {}^{(i)}H^{(v)}_{T\mathbf{k}_v} \mathbb{V}_{\mathbf{k}_v ij} \\
& + 2 \sum_{\mathbf{k}_s} ({}^{(i)}H^{(s)}_{T\mathbf{k}_s} \mathbb{S}_{\mathbf{k}_s ij} + {}^{(i)}H_{L\mathbf{k}_s} \gamma_{ij} \mathbb{S}_{\mathbf{k}_s}) \Bigg)
\end{aligned} \tag{III.85}
$$

III.4.2 Gauge transformation

The gauge transformation for ${}^{(i)}h_{\mu\nu}$ is subtle. Consider an infinitesimal gauge transformation $z^\alpha \to z^\alpha + \bar{\delta} z^\alpha$, where

$$
\bar{\delta} z^\alpha = \sum_i {}^{(i)}\zeta^\alpha \epsilon^i \tag{III.86}
$$

Then ${}^{(i)}h_{\mu\nu}$ gets affected by gauge tranformations at orders $j < i$. If we now absorb the effect of previous order perturbations in the ${}^{(i)}h_{\mu\nu}$, then the infinitesimal gauge transformation at any order i becomes

$$
{}^{(i)}h_{\mu\nu} \to {}^{(i)}h_{\mu\nu} - \bar{\nabla}_\mu {}^{(i)}\zeta_\nu - \bar{\nabla}_\nu {}^{(i)}\zeta_\mu \tag{III.87}
$$

Thus, each individual component of ${}^{(i)}h_{\mu\nu}$ transforms as

$$
\begin{aligned}
{}^{(i)}h_{ab} \to& {}^{(i)}h_{ab} - \bar{D}_a {}^{(i)}\zeta_b - \bar{D}_b {}^{(i)}\zeta_a \\
{}^{(i)}h_{ai} \to& {}^{(i)}h_{ai} - \bar{D}_i {}^{(i)}\zeta_a - r^2 \bar{D}_a \left(\frac{{}^{(i)}\zeta_i}{r^2} \right) \\
{}^{(i)}h_{ij} \to& {}^{(i)}h_{ij} - \bar{D}_i {}^{(i)}\zeta_j - \bar{D}_j {}^{(i)}\zeta_i - 2r \bar{D}^a r \, {}^{(i)}\zeta_a \gamma_{ij}
\end{aligned} \tag{III.88}
$$

Let ${}^{(i)}\zeta_a = {}^{(i)}T_a \mathbb{S}$ and ${}^{(i)}\zeta_i = r \, {}^{(i)}M \mathbb{S}_i + r \, {}^{(i)}M^{(v)} \, \mathbb{V}_i$. Then using (III.85) and (III.88), one can see how the various expansion coefficients pertaining to scalar, vector and tensor sector transform under change of gauge:

$$ {}^{(i)}f_{ab} \to {}^{(i)}f_{ab} - \bar{D}_a {}^{(i)}T_b - \bar{D}_b {}^{(i)}T_a \tag{III.89} $$

$$ {}^{(i)}f_a \to {}^{(i)}f_a - r\bar{D}_a \left(\frac{{}^{(i)}M}{r} \right) + \frac{k_s}{r} {}^{(i)}T_a \tag{III.90} $$

$$ {}^{(i)}H_L \to {}^{(i)}H_L - \frac{k_s}{nr} {}^{(i)}M - \frac{\bar{D}^a r}{r} {}^{(i)}T_a \tag{III.91} $$

$$ {}^{(i)}H_T^{(s)} \to {}^{(i)}H_T^{(s)} + \frac{k_s}{r} {}^{(i)}M \tag{III.92} $$

$$ {}^{(i)}f_a^{(v)} \to {}^{(i)}f_a^{(v)} - r\bar{D}_a \left(\frac{{}^{(i)}M^{(v)}}{r} \right) \tag{III.93} $$

$$ {}^{(i)}H_T^{(v)} \to {}^{(i)}H_T^{(v)} + \frac{k_v}{r} {}^{(i)}M^{(v)} \tag{III.94} $$

$$ {}^{(i)}H_T \to {}^{(i)}H_T \tag{III.95} $$

Thus, with the exception of $l_s = 0, 1$ and $l_v = 1$ modes, similar to linear order, one can define the following 'gauge invariant variables'.

$$ {}^{(i)}Z_a = {}^{(i)}f_a^{(v)} + \frac{r}{k_v} \bar{D}_a {}^{(i)}H_T^{(v)} \tag{III.96} $$

$$ {}^{(i)}F_{ab} = {}^{(i)}f_{ab} + \frac{1}{2} \bar{D}_{(a} {}^{(i)}X_{b)}; \quad {}^{(i)}F = {}^{(i)}H_L + \frac{{}^{(i)}H_T^{(s)}}{n} + \frac{1}{r} \bar{D}^a r {}^{(i)}X_a \tag{III.97} $$

where

$$ {}^{(i)}X_a = \frac{r}{k_s} \left({}^{(i)}f_a + \frac{r}{k_s} \bar{D}_a {}^{(i)}H_T^{(s)} \right) \tag{III.98} $$

III.4.3 Equations in terms of gauge invariant variables

We simplify the working equation (III.84) further, by writing it in terms of the $\bar{D}_a$ and the $\bar{D}_i$ operators. In doing so, we may use system of equations

(III.14-III.16). Then we substitute for ${}^{(i)}h_{\mu\nu}$ from (III.85). In terms of the gauge invariant variables, ${}^{(i)}G_{ij}=0$ is

$$\sum_{\mathbf{k}}\left[-r^2\bar{D}^a\bar{D}_a{}^{(i)}H_T - nr\bar{D}^a r\bar{D}_a{}^{(i)}H_T + (k^2+2K){}^{(i)}H_T\right]_{\mathbf{k}}\mathbb{T}_{\mathbf{k}ij}$$
$$+\sum_{\mathbf{k}_v}\left[-\frac{2k_v}{r^{n-2}}\bar{D}_a(r^{n-1(i)}Z^a)\right]_{\mathbf{k}_v}\mathbb{V}_{\mathbf{k}_v ij}+\sum_{\mathbf{k}_s}\left[-k_s^2[2(n-2){}^{(i)}F+{}^{(i)}F^c_c]\right]_{\mathbf{k}_s}\mathbb{S}_{\mathbf{k}_s ij}$$
$$={}^{(i)}S_{ij}-\sum_{\mathbf{k}_s}[Q_4]_{\mathbf{k}_s}\gamma_{ij}\mathbb{S}_{\mathbf{k}_s}. \qquad \text{(III.99)}$$

where, the contribution to $[Q_4]$ comes from the trace of the equation ${}^{(i)}G_{ij}=0$. Since this term will not be used in the calculations, we don't give its explicit form

From ${}^{(i)}G_{ai}=0$, one obtains

$$\sum_{\mathbf{k}_v}\left[-\frac{1}{r^n}\bar{D}^b\left\{r^{n+2}\left[\bar{D}_b\left(\frac{{}^{(i)}Z_a}{r}\right)-\bar{D}_a\left(\frac{{}^{(i)}Z_b}{r}\right)\right]\right\}+\frac{k_v^2-(n-1)K}{r}{}^{(i)}Z_a\right]_{\mathbf{k}_v}\mathbb{V}_{\mathbf{k}_v i}$$
$$+\sum_{\mathbf{k}_s}\left[-k_s\left(\frac{1}{r^{n-2}}\bar{D}_b(r^{n-2(i)}F^b_a)-r\bar{D}_a\left(\frac{1}{r}{}^{(i)}F^b_b\right)-2(n-1)\bar{D}_a{}^{(i)}F\right)\right]_{\mathbf{k}_s}\mathbb{S}_{\mathbf{k}_s i}$$
$$={}^{(i)}S_{ai}. \qquad \text{(III.100)}$$

In order to decouple all the sectors belonging to different kind of tensors, we use the following property:

$$\int\mathbb{T}^{ij}\mathbb{V}_{ij}d^n\Omega=\int\mathbb{T}^{ij}\mathbb{S}_{ij}d^n\Omega=\int\mathbb{V}^{ij}\mathbb{S}_{ij}d^n\Omega=\int\mathbb{V}^i\mathbb{S}_i d^n\Omega=0. \qquad \text{(III.101)}$$

Upon projecting equation (III.99) on $\mathbb{T}^{ij}$, one obtains

$$-r^2\bar{D}^a\bar{D}_a{}^{(i)}H_T-nr\bar{D}^a r\bar{D}_a{}^{(i)}H_T+(k^2+2){}^{(i)}H_T=\int\mathbb{T}^{ij}_{\mathbf{k}}\,{}^{(i)}S_{ij}d^n\Omega. \qquad \text{(III.102)}$$

Similarly, projecting (III.99) and (III.100) on $\mathbb{V}^{ij}$ and $\mathbb{V}^{i}$, one obtains

$$-\frac{1}{r^n}\bar{D}^b\left\{r^{n+2}\left[\bar{D}_b\left(\frac{{}^{(i)}Z_a}{r}\right)-\bar{D}_a\left(\frac{{}^{(i)}Z_b}{r}\right)\right]\right\}+\frac{k_v^2-(n-1)}{r}{}^{(i)}Z_a$$
$$=\int\mathbb{V}^i_{\mathbf{k_v}}\,{}^{(i)}S_{ai}d^n\Omega,\quad\text{(III.103)}$$

$$-\frac{2k_v}{r^{n-2}}\bar{D}_a(r^{n-1(i)}Z^a)=\int\mathbb{V}^{ij}_{\mathbf{k_v}}\,{}^{(i)}S_{ij}d^n\Omega.\quad\text{(III.104)}$$

For scalar sector, we obtain one equation from ${}^{(i)}G_{ab}=0$, and two by projecting $\mathbb{S}_{ij}$ and $\mathbb{S}_i$ on (III.99) and (III.100). Hence, the relevant equations are

$$\begin{aligned}&-\bar{D}^c\bar{D}_c{}^{(i)}F_{ab}+\bar{D}_a\bar{D}_c{}^{(i)}F^c_b+\bar{D}_b\bar{D}_c{}^{(i)}F^c_a+n\frac{\bar{D}^cr}{r}(-\bar{D}_c{}^{(i)}F_{ab}+\bar{D}_a{}^{(i)}F_{cb}+\bar{D}_b{}^{(i)}F_{ca})\\&+{}^mR^{c(i)}_aF_{cb}+{}^mR^{c(i)}_bF_{ca}-2{}^mR_{acbd}{}^{(i)}F^{cd}+\left(\frac{k_s^2}{r^2}+\frac{2(n+1)}{L^2}\right){}^{(i)}F_{ab}-\bar{D}_a\bar{D}_b{}^{(i)}F^c_c\\&-2n\left(\bar{D}_a\bar{D}_b{}^{(i)}F+\frac{1}{r}\bar{D}_ar\bar{D}_b{}^{(i)}F+\frac{1}{r}\bar{D}_br\bar{D}_a{}^{(i)}F\right)-\Big[\bar{D}_c\bar{D}_d{}^{(i)}F^{cd}\\&+\frac{2n}{r}\bar{D}^cr\bar{D}^{d(i)}F_{cd}+\Big(-{}^mR_{cd}+\frac{2n}{r}\bar{D}^c\bar{D}^dr+\frac{n(n-1)}{r^2}\bar{D}^cr\bar{D}^dr\Big){}^{(i)}F_{cd}\\&-2n\bar{D}^c\bar{D}_c{}^{(i)}F-\frac{2n(n+1)}{r}\bar{D}^cr\bar{D}_c{}^{(i)}F+2(n-1)\frac{(k_s^2-n)}{r^2}{}^{(i)}F-\bar{D}^c\bar{D}_c{}^{(i)}F^d_d\\&-\frac{n}{r}\bar{D}^cr\bar{D}_c{}^{(i)}F^d_d+\frac{k_s^2}{r^2}{}^{(i)}F^d_d\Big]g_{ab}=\int\mathbb{S}_{\mathbf{k_s}}\,{}^{(i)}S_{ab}d^n\Omega,\end{aligned}\quad\text{(III.105)}$$

$$-k_s\Big(\frac{1}{r^{n-2}}\bar{D}_b(r^{n-2(i)}F^b_a)-r\bar{D}_a\left(\frac{1}{r}{}^{(i)}F^b_b\right)-2(n-1)\bar{D}_a{}^{(i)}F\Big)$$
$$=\int\mathbb{S}^i_{\mathbf{k_s}}\,{}^{(i)}S_{ai}d^n\Omega,\quad\text{(III.106)}$$

$$-k_s^2[2(n-2)^{(i)}F + {}^{(i)}F_c^c] = \int \mathbb{S}_{\mathbf{k_s}}^{ij}\,{}^{(i)}S_{ij}d^n\Omega. \tag{III.107}$$

Note that, we recover the linear equations of section III.2.5 by putting ${}^{(1)}S_{\mu\nu} = 0$ in (III.102-III.107).

III.5 Gravitational perturbations in a box: Nonlinearities

In the following subsections, we will simplify the perturbation equations upto all orders in ϵ for the various sectors and see how associating an integrable Hamiltonian at the linear level helps in immediate application of the arguments we developed in chapter II. We start by revisiting the linear perturbations and give details as to how to simplify the equations in terms of a single master variable. For this, we have used the method by Takahashi & Soda [102] rather than that of Ishibashi and Kodama [110], used in [81]. We then discuss the properties of eigenfunctions and eigenmodes which will enable as to associate a integrable Hamiltonian (that of decoupled harmonic oscillators) at the linear level. Then using these properties, we will present a formal way of writing higher order equations.

Note that, the Dirichlet boundary condition demands that the induced metric should be fixed at $r = r_0$. This translates to

$${}^{(i)}H_T^{(s)} = {}^{(i)}H_L = {}^{(i)}f_t^{(s)} = {}^{(i)}f_{tt} = {}^{(i)}H_T^{(v)} = {}^{(i)}f_t = {}^{(i)}H_T = 0\Big|_{r=r_0} \tag{III.108}$$

We will now turn our attention to linear level equations. Note that, while considering linear perturbations, we will drop the superscript (1) on the perturbation variables.

III.5.1 Tensor sector: Linear level

The tensor perturbations are easiest to deal with as they are simply governed by one equation, Let $H_{T\mathbf{k}} = \Phi_{T\mathbf{k}}$ in equation (III.33)

$$-r^2\ddot{\Phi}_T + r^2\Phi_T'' + nr\Phi_T' - l(l+n-1)\Phi_T = 0. \tag{III.109}$$

We will rewrite this equation in terms of an operator $\hat{L}$ defined as

$$\hat{L} = -\frac{1}{r^n}\partial_r(r^n\partial_r) + \frac{l(l+n-1)}{r^2}. \tag{III.110}$$

so that equation (III.109) takes the form

$$\ddot{\Phi}_T + \hat{L}\Phi_T = 0, \tag{III.111}$$

The general solution to (III.111) is given as a superposition of eigenfunctions $e_{p,l}(r)$ over all the modes p.

$$\Phi_{T\mathbf{k}} = \sum_{p=1}^{\infty} a_{p,\mathbf{k}} \cos(\omega_{p,l}t + b_{p,\mathbf{k}})e_{p,l}(r), \tag{III.112}$$

where $e_{p,l}(r)$ is given by

$$e_{p,l}(r) = d_{p,l}\frac{J_\nu(\omega_{p,l}r)}{r^{(n-1)/2}}; \quad \nu = l + \frac{(n-1)}{2}, \tag{III.113}$$

The constants $a_{p,\mathbf{k}}$ and $b_{p,\mathbf{k}}$ in (III.112) are fixed by the initial conditions and $d_{p,l}$ is the normalization constant given by $\frac{\sqrt{2}}{r_0 J_\nu'(\omega_{p,l}r_0)}$. The eigenfrequencies $\omega_{p,l}$ are discrete and associated with a mode number p for each l. They are determined by solving the Dirichlet boundary condition: $\Phi_T = 0$ at $r = r_0$.

$$\Rightarrow \omega_{p,l} = \frac{j_{\nu,p}}{r_0}, \tag{III.114}$$

where $j_{\nu,p}$ is the p^{th} zero of a Bessel function of order ν. By inspecting the properties of zeroes of Bessel function for large mode numbers, one can see that as $p \to \infty$, the eigenfrequencies approach $(p+\nu/2-1/4)\frac{\pi}{r_0}$. This implies that the spectrum in this case is asymptotically resonant spectrum.

Since the eigenfunctions $e_{p,l}$ form a complete set and are orthonormal in the space of functions $\hat{L}^2([0, r_0], r^n dr)$. The inner product $\langle f, g\rangle_T$ is defined as

$$\langle f, g\rangle_T = \int_0^{r_0} f(r) g(r) r^n dr. \tag{III.115}$$

III.5.2 Vector sector: Linear level

Upon expanding the vector equations (III.38) and (III.39) (for $a = r$), one gets the following two eqautions

$$\dot{Z}_t = (n-1)\frac{Z_r}{r} + Z_r'. \tag{III.116}$$

$$-\partial_t^2 Z_r + r\partial_t\partial_r\left(\frac{Z_t}{r}\right) - \frac{(k_v^2 - (n-1))}{r^2} Z_r = 0. \tag{III.117}$$

In order to get a second order equation in Z_r, we substitute for $\dot{Z}_t$ from (III.116) in (III.117) and obtain

$$-\ddot{Z}_r + Z_r'' + \frac{(n-2)}{r} Z_r' - \frac{(l_v(l_v + n - 1) + (n-2))}{r^2} Z_r = 0. \tag{III.118}$$

We define the master variable Φ_v as

$$Z_{r\mathbf{k}_v} = r\Phi_{v\mathbf{k}_v}. \tag{III.119}$$

In terms of Φ_v, equation (III.118) becomes

$$\ddot{\Phi}_v + \hat{L}_v\Phi_v = 0, \tag{III.120}$$

where $\hat{L}_v$ is defined as:

$$\hat{L}_v = -\frac{1}{r^n}\partial_r(r^n\partial_r) + \frac{l_v(l_v+n-1)}{r^2}. \tag{III.121}$$

In order to get an equation solely in r, we assume an ansatz for Φ_v of the form $\Phi_{v\mathbf{k}_v} = \cos(\omega t + b)\phi_{v\mathbf{k}_v}(r)$. Then the resultant radial equation in ϕ_v becomes

$$\hat{L}_v\phi_v = \omega^2\phi_v. \tag{III.122}$$

Next we apply the Dirichlet condition $Z_t(r_0) = 0$.

$$\begin{aligned} Z_t &= \int^t (r\Phi_v' + n\Phi_v)dt\Big|_{r=r_0}. \\ &= \frac{1}{\omega}\{r\phi_v' + n\phi_v\}\sin(\omega t + b)\Big|_{r=r_0} = 0 \end{aligned} \tag{III.123}$$

Since these needs to be valid for all times, we have

$$r\phi_v' + n\phi_v = 0\Big|_{r=r_0} \tag{III.124}$$

The solutions to this equation sets the value for the eigenfrequencies. Although, one cannot solve this equation exactly, atleast for large ω, one can deduce the behavior of the frequencies by noting that $\phi_v \sim \frac{J_{\nu_v}(\omega r)}{r^{(n-1)/2}}$, where $\nu_v = l_v + (n-1)/2$. Substituting this in (III.124) gives

$$\omega r J'_{\nu_v}(\omega r) + \frac{(n+1)}{2} J_{\nu_v}(\omega r) = 0\big|_{r=r_0}. \tag{III.125}$$

Next we use the large argument approximation for Bessel function

$$J_{\nu_v}(z) \sim \sqrt{\frac{2}{z\pi}} \cos\left(z - \frac{\nu_v \pi}{2} - \frac{\pi}{4}\right) \text{ as } z \to \infty. \tag{III.126}$$

to deduce that the condition (III.125), for large modes becomes

$$\tan\left(z - \frac{\nu_v \pi}{2} - \frac{\pi}{4}\right) = \frac{(n+1)/2}{z}, \tag{III.127}$$

where $z = \omega r_0$. It can be seen that the frequencies tend to $\left(p + \frac{\nu_v}{2} - \frac{3}{4}\right)\frac{\pi}{r_0}$.

We will denote the inner product $\langle f, g \rangle_v$ as

$$\langle f, g \rangle = \int_0^{r_0} f(r) g(r) r^n dr \tag{III.128}$$

Let $e^{(v)}_{p,l_v}$ denote the eigenfunctions of (III.122) given by

$$\phi_{v\mathbf{k}_v} = e^{(v)}_{p,l_v} = d^{(v)}_{p,l_v} \frac{J_{\nu_v}(\omega_{p,l_v} r)}{r^{(n-1)/2}}; \quad \nu_v = l_v + \frac{(n-1)}{2}, \tag{III.129}$$

where $d^{(v)}_{p,l_v}$ is the normalization constant. Since these eigenfunctions are complete and we demand them to be a orthonormal basis in the space of functions $\hat{L}^2_v([0, r_0], r^n dr)$, using (III.128), we can deduce the normalization constant to be

$$d^{(v)}_{p,l_v} = \frac{\sqrt{2}\omega_{p,l_v}}{J_{\nu_v}(\omega_{p,l_v} r_0)} \left[(n+1)^2/4 + (\omega_{p,l_v} r_0)^2 - \nu_v^2\right]^{-1/2}. \tag{III.130}$$

Hence, the general solution to (III.120) can be written as is given by

$$\Phi_{v\mathbf{k}_v} = \sum_{p=1}^{\infty} a^{(v)}_{p,\mathbf{k}_v} \cos(\omega_{p,l_v} t + b_{p,\mathbf{k}_v}) e^{(v)}_{p,l_v}(r), \tag{III.131}$$

where $a^{(v)}_{p,\mathbf{k}_v}$ and $b_{p,\mathbf{k}_v}$ are determined from initial conditions

III.5.3 Scalar sector: Linear level

The equation of this sector is governed by equations (III.49-III.51). To derive the scalar master equation, we define a master variable Φ_s on the lines of [102]. We assume

$$F_{rt} = 2r(\dot{\Phi}_s + \dot{F}), \tag{III.132}$$

Consider $G_{rt} = 0$ equation

$$\frac{n}{r}\dot{F}_{rr} + \frac{k_s^2}{r^2} F_{rt} - 2n\dot{F}' - \frac{2n}{r}\dot{F} = 0. \tag{III.133}$$

Upon substituting (III.132) in (III.133), and integrating the resultant equation w.r.t time, one obtains

$$F_{rr} = 2rF' + 2F - \frac{2k_S^2}{n} F - \frac{2k_s^2}{n}\Phi_s. \tag{III.134}$$

In the above equation, any extra $r-$dependent integration factor is absorbed in the definition of Φ_s. From $G_{tt} = 0$ one gets

$$-2nF'' + \frac{n}{r}F'_{rr} + \left(\frac{k_s^2}{r^2} + \frac{n(n-1)}{r^2}\right)F_{rr} - \frac{2n(n+1)}{r}F' + \left(\frac{2k_s^2(n-1)}{r^2} - \frac{2n(n-1)}{r^2}\right)F = 0. \tag{III.135}$$

Substituting the expression for F_{rr} from (III.134) into (III.135) gives us an expression for F solely in terms of Φ_s and its derivative:

$$F = -\frac{n}{k_s^2 - n}\left[r\Phi'_s + \left(\frac{k_s^2}{n} + n - 1\right)\Phi_s\right]. \qquad \text{(III.136)}$$

Thus, so far we have obtained F_{rr}, F_{rt} and F solely in terms of Φ_s. Only F_{tt} is left. In order to obtain F_{tt}, we use the traceless part of $G_{ij} = 0$

$$[2(n-2)F + F_c^c] = 0, \qquad \text{(III.137)}$$

Expanding this, one gets

$$\begin{aligned} F_{tt} &= F_{rr} + 2(n-2)F \\ &= 2rF' + 2(n-1)F - \frac{2k_s^2}{n}F - \frac{2k_s^2}{n}\Phi_s \\ &= -\frac{n}{k_s^2 - n}\left[2r^2\Phi''_s + 2r(2n-1)\Phi'_s + 2\left((n-1)^2 - \frac{k_s^2}{n}\right)\Phi_s\right] \end{aligned} \qquad \text{(III.138)}$$

Finally, in order to derive the master equation for Φ_s, we consider $G_{ir} = 0$ and obtain

$$-\frac{(n-2)}{r}F_{rr} + \dot{F}_{rt} - F'_{rr} + 2F' + \frac{2(n-2)}{r}F = 0. \qquad \text{(III.139)}$$

From $G_{rr} = 0$, if we eliminate F_{tt} by using (III.137), we will obtain

$$\begin{aligned} -2n\ddot{F} + \frac{2n}{r}F' - \frac{2k_s^2}{r^2}F + \frac{2n(n-1)}{r^2}F - \frac{n}{r}F'_{rr} \\ + \left(\frac{k_s^2}{r^2} - \frac{n(n-1)}{r^2}\right)F_{rr} + \frac{2n}{r}\dot{F}_{rt} = 0. \end{aligned} \qquad \text{(III.140)}$$

Upon, eliminating F'_{rr} from (III.140) by using (III.139), one obtains

$$-2n\ddot{F} + \frac{(k_s^2 - n)}{r^2}F_{rr} - \frac{2(k_s^2 - n)}{r^2}F + \frac{n}{r}\dot{F}_{rt} = 0. \tag{III.141}$$

In (III.141), we substitute for F_{rt}, F_{rr} and F from (III.132), (III.134) and (III.136) respectively, to obtain a second order master equation in Φ_s:

$$\ddot{\Phi}_s - \Phi''_s - \frac{n}{r}\Phi'_s + \frac{l_s(l_s + n - 1)}{r^2}\Phi_s = 0. \tag{III.142}$$

In terms of the operator $\hat{L}_s$ defined as

$$\hat{L}_s = -\frac{1}{r^n}\partial_r(r^n\partial_r) + \frac{l_s(l_s + n - 1)}{r^2} \tag{III.143}$$

we can rewrite (III.142) as

$$\ddot{\Phi}_s + \hat{L}_s\Phi_s = 0, \tag{III.144}$$

To get an eigenvalue equation, we assume an ansatz of the form

$$\hat{L}_s\phi_s = \omega^2\phi_s \tag{III.145}$$

The boundary condition requires that f_{tt} and $H_T^{(s)}$ or equivalently F_{tt} should vanish at the boundary $r = r_0$. These translates to

$$(n-1)r\phi'_s + \left(-\omega^2 r^2 + \frac{(n-1)}{n}(k_s^2 + n(n-1))\right)\phi_s = 0\Big|_{r=r_0}. \tag{III.146}$$

Solution to these equation sets the frequencies of the system. Though this equation can't be solved exactly by analytical means, one can still ascertain the asymptotic behavior of the frequencies. Taking note of the fact that the solution to (III.145) which is regular at origin is $\phi_s \sim \frac{J_{\nu_s}}{r^{(n-1)/2}}$, one can see that the frequencies are asymptotically resonant and approach $(p + \frac{\nu_s}{2} - \frac{5}{4})\frac{\pi}{r_0}$ for large mode numbers p. In general, the solutions to (III.146) are discrete and each ω is associated with mode no. p for a particular l_s.

Let the eigensolutions to (III.145) be

$$\phi_{s\mathbf{k}_s} = e^{(s)}_{p,l_s}(r) = d^{(s)}_{p,l_s} \frac{J_{\nu_s}(\omega_{p,l_s} r)}{r^{(n-1)/2}}; \quad \nu_s = l_s + \frac{(n-1)}{2}. \tag{III.147}$$

where we choose $d^{(s)}_{p,l_s}$ to be

$$d^{(s)}_{p,l_s} = \left[\int_0^{r_0} |J_{\nu_s}(\omega_{p,l_s} r)|^2 r dr + \frac{r_0^2}{(n-1)} |J_{\nu_s}(\omega_{p,l_s} r_0)|^2 \right]^{-1/2} \tag{III.148}$$

the eigenfunctions $e^{(s)}_{p,l_s}$ are not orthogonal, they satisfy the modified orthogonality relation, $< e^{(s)}_{p,l_s}, e^{(s)}_{q,l_s} >_s$, given by

$$< e^{(s)}_{p,l_s}, e^{(s)}_{q,l_s} >_s = \int_0^{r_0} e^{(s)}_{p,l_s} e^{(s)}_{q,l_s} r^n dr + \frac{r_0^{n+1}}{(n-1)} e^{(s)}_{p,l_s}(r_0) e^{(s)}_{q,l_s}(r_0) = \begin{cases} 0 & \text{for } p \neq q \\ 1 & \text{for } p = q \end{cases} \tag{III.149}$$

We now refer to the work by Zecca [103] which deals with Bessel equation in a finite interval with singularity at the origin and a eigenvalue dependent boundary condition at the other regular point, which is very much similar to our case. It is shown that the general solution can be expanded in a series of Bessel functions within this finite interval. Thus applying the results of [103] to our case, we see that the general solution can be written as

$$\Phi_{s\mathbf{k}_s} = \sum_{p=0}^{\infty} a^{(s)}_{p,\mathbf{k}_s} \cos(\omega_{p,l_s} t + b_{p,\mathbf{k}_s}) e^{(s)}_{p,l_s}(r), \tag{III.150}$$

where a_{p,l_s} and b_{p,l_s} are constants set by initial conditions.

Further, an expansion theorem by the same author states that a function $f(r)$ in $C^1[0,1]$ with square integrable second derivative and which satisfies the same boundary condition as $J_{\nu_s}(\omega_{p,l_s} r)$ (or $e^{(s)}_{p,l_s}$), can be expanded in a series in Bessel functions (eigenfunctions $e^{(s)}_{p,l_s}$). Hence such a function $f(r)$ can be written as

$$f = \sum_p \sigma_p e^{(s)}_{p,l_s} \qquad \text{(III.151)}$$

where

$$\sigma_p =< f, e^{(s)}_{p,l_s} >_s= \int_0^{r_0} f r^n e^{(s)}_{p,l_s} dr + \frac{r_0^{n+1}}{(n-1)} f(r_0) e^{(s)}_{p,l_s}(r_0) \qquad \text{(III.152)}$$

III.5.4 Tensor sector: Higher level

We let ${}^{(i)}H_{T\mathbf{k}} = {}^{(i)}\Phi_{T\mathbf{k}}$ in (III.102), to obtain

$${}^{(i)}\ddot{\Phi}_{T\mathbf{k}} + \hat{L}^{(i)}\Phi_{T\mathbf{k}} = \frac{1}{r^2} \int \mathbb{T}^{ij}_{\mathbf{k}} \, {}^{(i)}S_{ij} d^n\Omega. \qquad \text{(III.153)}$$

Using the completeness property of $e_{p,l}(r)$, we formally expand $\Phi_{T\mathbf{k}}$ as

$${}^{(i)}\Phi_{T\mathbf{k}} = \sum_{p=1}^{\infty} {}^{(i)}c_{p,\mathbf{k}}(t) e_{p,l}(r). \qquad \text{(III.154)}$$

Note that, this expansion is not valid at the cavity wall as the behavior of eigenfunctions $e_{p,l}(r)$ at the wall is not compatible with the source term on the right hand side of (III.153). We can further simplify (III.153) and get an equation for the dynamics of coefficients $c_{p,\mathbf{k}}(t)$, by taking the projection of

(III.153) on $e_{p,l}(r)$

$$ {}^{(i)}\ddot{c}_{p,\mathbf{k}}(t) + \omega_{p,l}^2\,{}^{(i)}c_{p,\mathbf{k}}(t) = \left\langle \frac{1}{r^2}\int \mathbb{T}_{\mathbf{k}}^{ij}\,{}^{(i)}S_{ij}, e_{p,l}(r)\right\rangle_T . \qquad \text{(III.155)} $$

III.5.5 Vector sector: Higher level

The relevant equations which we use to simplify our equations are

$$ {}^{(i)}\dot{Z}_t = (n-1)\frac{{}^{(i)}Z_r}{r} + {}^{(i)}Z_r' + \frac{1}{2k_v r}\,{}^{(i)}V_{s1}. \qquad \text{(III.156)} $$

which we obtain by expanding (III.104), as well as

$$ -{}^{(i)}\ddot{Z}_r + r\partial_t\partial_r\left(\frac{{}^{(i)}Z_t}{r}\right) - \frac{(k_v^2-(n-1))}{r^2}\,{}^{(i)}Z_r = -\frac{{}^{(i)}V_{s2}}{r}. \qquad \text{(III.157)} $$

which we obtain by making the substitution $a = r$ in (III.103). ${}^{(i)}V_{s1}$ and ${}^{(i)}V_{s2}$ are just source depended terms which are defined as follows:

$$ {}^{(i)}V_{s1} = \int \mathbb{V}_{\mathbf{k}_v}^{ij}\,{}^{(i)}S_{ij}d^n\Omega, \qquad \text{(III.158)} $$

$$ {}^{(i)}V_{s2} = \int \mathbb{V}_{\mathbf{k}_v}^{i}\,{}^{(i)}S_{ri}d^n\Omega. \qquad \text{(III.159)} $$

On the same lines as linear level equations, we substitute for ${}^{(i)}\dot{Z}_t$ from (III.156) in (III.157) to obtain a second order equation in ${}^{(i)}Z_r$

$$ -{}^{(i)}\ddot{Z}_r + {}^{(i)}Z_r'' + \frac{(n-2)}{r}\,{}^{(i)}Z_r' - \frac{(l_v(l_v+n-1)+(n-2))}{r^2}\,{}^{(i)}Z_r = -\left[\frac{{}^{(i)}V_{s2}}{r} + r\left(\frac{{}^{(i)}V_{s1}}{2k_v r^2}\right)'\right] \qquad \text{(III.160)} $$

We define a new variable ${}^{(i)}Z_r$ which is related to ${}^{(i)}\Phi_{v\mathbf{k}_v}$ as

$$ {}^{(i)}Z_{r\mathbf{k}_v} = r\left({}^{(i)}\Phi_{v\mathbf{k}_v} - \frac{1}{2k_v r^n}\int {}^{(i)}V_{s1} r^{(n-2)} dr\right). \tag{III.161} $$

Substituting for ${}^{(i)}Z_{r\mathbf{k}_v}$ equation (III.161) in (III.160) gives us

$$ {}^{(i)}\ddot{\Phi}_{v\mathbf{k}_v} + \hat{L}_v {}^{(i)}\Phi_{v\mathbf{k}_v} = \frac{1}{r}\left[\frac{{}^{(i)}V_{s2}}{r} + r\left(\frac{{}^{(i)}V_{s1}}{2k_v r^2}\right)'\right] + \frac{\int {}^{(i)}\ddot{V}_{s1} r^{(n-2)} dr}{2k_v r^n} + \hat{L}_v\left[\frac{\int {}^{(i)}V_{s1} r^{(n-2)} dr}{2k_v r^n}\right]. \tag{III.162} $$

Making use of the completeness properties of e_{p,l_v}, we expand ${}^{(i)}\Phi_{v\mathbf{k}_v}$ as

$$ {}^{(i)}\Phi_{v\mathbf{k}_v} = \sum_{p=1}^{\infty} {}^{(i)}c^{(v)}_{p,\mathbf{k}_v}(t) e^{(v)}_{p,l_v}(r). \tag{III.163} $$

Again, the expansion (III.163) is not valid at the wall $r = r_0$. By projecting (III.162) on $e^{(v)}_{p,l_v}$, we get an equation for ${}^{(i)}c^{(v)}_{p,\mathbf{k}_v}$:

$$ {}^{(i)}\ddot{c}^{(v)}_{p,\mathbf{k}_v} + \omega^2_{p,l_v} {}^{(i)}c^{(v)}_{p,\mathbf{k}_v} = \left\langle \left[\frac{{}^{(i)}V_{s2}}{r^2} + \left(\frac{{}^{(i)}V_{s1}}{2k_v r^2}\right)'\right] + \frac{\int {}^{(i)}\ddot{V}_{s1} r^{(n-2)} dr}{2k_v r^n} + \hat{L}_v\left[\frac{\int {}^{(i)}V_{s1} r^{(n-2)} dr}{2k_v r^n}\right], e^{(v)}_{p,l_v}\right\rangle_v. \tag{III.164} $$

Finally, imposing boundary condition (III.108) requires ${}^{(i)}Z_t(r_0) = 0$, i.e.

$$ {}^{(i)}Z_{t\mathbf{k}_v} = \int \{r {}^{(i)}\Phi'_{v\mathbf{k}_v} + n {}^{(i)}\Phi_{v\mathbf{k}_v}\} dt\Big|_{r=r_0} \tag{III.165} $$

In the above equation any $r-$dependent integration constant is put to zero. Then, if we write ${}^{(i)}\Phi_v$ using (III.163), we see that this condition is automatically satisfied at $r = r_0$ because of (III.124)

$$
\begin{aligned}
{}^{(i)}Z_{t\mathbf{k}_v} &= \sum_{p=1}^{\infty} \int {}^{(i)}c_{p,\mathbf{k}_v}(t)dt\{r\phi'_{v\mathbf{k}_v}(r_0) + n\phi_{v\mathbf{k}_v}(r_0)\} \\
&= 0
\end{aligned}
\tag{III.166}
$$

III.5.6 Scalar sector: Higher level

Expanding the set of equations (III.105-III.107) gives us the equations for scalar sector. The equations pertaining to traceless part of ${}^{(i)}G_{ij}$=0 as well as ${}^{(i)}G_{rt}$=0, ${}^{(i)}G_{tt}$=0, ${}^{(i)}G_{rr} = 0$ and ${}^{(i)}G_{ri} = 0$ are given below

$$
-k_s^2[{}^{(i)}F^c_c + 2(n-2){}^{(i)}F] = {}^{(i)}S_{s0} \tag{III.167}
$$

$$
\frac{n}{r}{}^{(i)}\dot{F}_{rr} + \frac{k_s^2}{r^2}{}^{(i)}F_{rt} - 2n{}^{(i)}\dot{F}' - \frac{2n}{r}{}^{(i)}\dot{F} = {}^{(i)}S_{s1} \tag{III.168}
$$

$$
\begin{aligned}
\frac{n}{r}{}^{(i)}F'_{rr} + \Big(\frac{k_s^2}{r^2} + \frac{n(n-1)}{r^2}\Big){}^{(i)}F_{rr} - 2n{}^{(i)}F'' - \frac{2n(n+1)}{r}{}^{(i)}F' \\
+ 2(n-1)\frac{(k_s^2-n)}{r^2}{}^{(i)}F = {}^{(i)}S_{s2}
\end{aligned}
\tag{III.169}
$$

$$
\begin{aligned}
\frac{2n}{r}{}^{(i)}\dot{F}_{rt} - 2n{}^{(i)}\ddot{F} + \frac{2n(n-1)}{r}{}^{(i)}F' - \frac{n(n-1)}{r^2}{}^{(i)}F_{rr} + \frac{n}{r}({}^{(i)}F^t_t)' - \frac{k_s^2}{r^2}{}^{(i)}F^t_t \\
- \frac{2(n-1)(k_s^2-n)}{r^2}{}^{(i)}F = {}^{(i)}S_{s3}
\end{aligned}
\tag{III.170}
$$

$$ {}^{(i)}\dot{F}_{rt} + ({}^{(i)}F^t_t)' - \frac{1}{r}{}^{(i)}F^t_t + 2(n-1){}^{(i)}F' - \frac{(n-1)}{r}{}^{(i)}F^r_r = {}^{(i)}S_{s4} \qquad \text{(III.171)} $$

where

$$ {}^{(i)}S_{s0\mathbf{k}_s} = \int \mathbb{S}^{ij}_{\mathbf{k_s}}\,{}^{(i)}S_{ij}d^n\Omega, \qquad \text{(III.172)} $$

$$ {}^{(i)}S_{s1\mathbf{k}_s} = \int \mathbb{S}_{\mathbf{k_s}}\,{}^{(i)}S_{rt}d^n\Omega, \qquad \text{(III.173)} $$

$$ {}^{(i)}S_{s2\mathbf{k}_s} = \int \mathbb{S}_{\mathbf{k_s}}\,{}^{(i)}S_{tt}d^n\Omega, \qquad \text{(III.174)} $$

$$ {}^{(i)}S_{s3\mathbf{k}_s} = \int \mathbb{S}_{\mathbf{k}_s}\,{}^{(i)}S_{rr}d^n\Omega \qquad \text{(III.175)} $$

$$ {}^{(i)}S_{s4\mathbf{k}_s} = \frac{1}{k_s}\int \mathbb{S}^i_{\mathbf{k}_s}\,{}^{(i)}S_{ir}d^n\Omega \qquad \text{(III.176)} $$

$$ {}^{(i)}S_{s5} = \left(\frac{k_s^2}{nr} + \frac{(n-1)}{r}\right)\int^t {}^{(i)}S_{s1}dt - {}^{(i)}S_{s2} + \frac{1}{r}\left(r\int^t {}^{(i)}S_{s1}dt\right)', \qquad \text{(III.177)} $$

We first start by defining a new variable ${}^{(i)}\Psi_{s\mathbf{k}_s}$

$$ {}^{(i)}F_{rt} = 2r({}^{(i)}\dot{\Psi}_s + {}^{(i)}\dot{F}), \qquad \text{(III.178)} $$

where ${}^{(i)}\Psi$ itself is defined in terms of a master variable ${}^{(i)}\Phi$

$$ {}^{(i)}\Psi_s = {}^{(i)}\Phi_s - {}^{(i)}S_{s8}. \qquad \text{(III.179)} $$

The term ${}^{(i)}S_{s8}$ is defined as

$$ {}^{(i)}S_{s8} = -\frac{1}{2}\left(\frac{k_s^2}{n}-1\right) r^{-(\frac{k_s}{\sqrt{n}}+n-1)} \int^r \left[r^{(\frac{2k_s}{\sqrt{n}}-1)} \int^r r'^{(-\frac{k_s}{\sqrt{n}}+n-2)}\, {}^{(i)}\mathcal{B}\, dr' \right] dr, \tag{III.180} $$

where

$$ {}^{(i)}\mathcal{B}(t,r) = \frac{n}{k_s^2 - n}\left[r\left(\frac{r^2}{k_s^2}{}^{(i)}S_{s5}\right)' + \frac{r^2}{k_s^2}\left((n-1) - \frac{k_s^2}{n}\right){}^{(i)}S_{s5}\right] + \frac{r}{n}\int^t {}^{(i)}S_{s1} dt + \frac{{}^{(i)}S_{s0}}{k_s^2}. \tag{III.181} $$

Upon substituting (III.178) in (III.168) and integrating the resultant expression w.r.t time, one obtains an expression for ${}^{(i)}F_{rr}$:

$$ {}^{(i)}F_{rr} = 2r{}^{(i)}F' + 2{}^{(i)}F - \frac{2k_S^2}{n}{}^{(i)}F - \frac{2k_s^2}{n}{}^{(i)}\Psi_s + \frac{r}{n}\int^t {}^{(i)}S_{s1} dt. \tag{III.182} $$

To get ${}^{(i)}F$ in terms of ${}^{(i)}\Psi_s$ and ${}^{(i)}\Psi_s'$, we substitute (III.182) in (III.169), so that

$$ {}^{(i)}F = -\frac{n}{k_s^2 - n}\left[r{}^{(i)}\Psi_s' + \left(\frac{k_s^2}{n} + n - 1\right){}^{(i)}\Psi_s - \frac{r^2}{2k_s^2}{}^{(i)}S_{s5}.\right] \tag{III.183} $$

where ${}^{(i)}S_{s5}$ is as defined in (III.177).

Next, we obtain ${}^{(i)}F_{tt}$ from (III.167) by substituting for ${}^{(i)}F_{rr}$ and ${}^{(i)}F$ from (III.182) and (III.183):

$$ {}^{(i)}F_{tt} = 2r{}^{(i)}F' + 2(n-1){}^{(i)}F - \frac{2k_s^2}{n}{}^{(i)}F - \frac{2k_s^2}{n}{}^{(i)}\Psi + \frac{r}{n}\int^t {}^{(i)}S_{s1} dt + \frac{{}^{(i)}S_{s0}}{k_s^2} \tag{III.184} $$

We now eliminate ${}^{(i)}F'_{tt}$ from (III.170) by using (III.167), to get

$$-2n{}^{(i)}\ddot{F} + \frac{2n}{r}{}^{(i)}F' - \frac{2k_s^2}{r^2}{}^{(i)}F + \frac{2n(n-1)}{r^2}{}^{(i)}F - \frac{n}{r}{}^{(i)}F'_{rr} + \left(\frac{k_s^2}{r^2} - \frac{n(n-1)}{r^2}\right){}^{(i)}F_{rr} + \frac{2n}{r}{}^{(i)}\dot{F}_{rt} = {}^{(i)}S_{s3\mathbf{k}_s} + \frac{n}{rk_s^2}{}^{(i)}S'_{s0} - \frac{{}^{(i)}S_{s0}}{r^2}, \tag{III.185}$$

Next, by eliminating ${}^{(i)}F_{tt}$ from (III.171) by using (III.167)

$$-2n{}^{(i)}\ddot{F} + \frac{(k_s^2-n)}{r^2}{}^{(i)}F_{rr} - \frac{2(k_s^2-n)}{r^2}{}^{(i)}F + \frac{n}{r}{}^{(i)}\dot{F}_{rt} = {}^{(i)}S_{s4\mathbf{k}_s} + \left(\frac{{}^{(i)}S_{s0}}{k_s^2}\right)' - \left(\frac{{}^{(i)}S_{s0}}{rk_s^2}\right). \tag{III.186}$$

Subtracting (III.186) from (III.185) gives

$$-2n{}^{(i)}\ddot{F} + \frac{(k_s^2-n)}{r^2}{}^{(i)}F_{rr} - \frac{2(k_s^2-n)}{r^2}{}^{(i)}F + \frac{n}{r}{}^{(i)}\dot{F}_{rt} = {}^{(i)}S_{s6}. \tag{III.187}$$

where ${}^{(i)}S_{s6}$ is defined as

$${}^{(i)}S_{s6} = {}^{(i)}S_{s3} - \frac{n}{r}{}^{(i)}S_{s4} + \left(\frac{n}{k_s^2} - 1\right)\frac{{}^{(i)}S_{s0}}{r^2} \tag{III.188}$$

and ${}^{(i)}S_{s8}$ is as defined by (III.180).

Finally, to obtain a second order equation solely in terms of ${}^{(i)}\Phi_s$, we substitute for ${}^{(i)}F_{rt}$, ${}^{(i)}F_{rr}$ and ${}^{(i)}F$ from (III.178), (III.182) and (III.183) in (III.187), to obtain

$${}^{(i)}\ddot{\Phi}_s + \hat{L}_s{}^{(i)}\Phi_s = {}^{(i)}S_{s9}, \tag{III.189}$$

where ${}^{(i)}S_{s9}$ is defined as

$$ {}^{(i)}S_{s9} = {}^{(i)}S_{s7} + {}^{(i)}\ddot{S}_{s8} + \hat{L}_s {}^{(i)}S_{s8}. \tag{III.190} $$

where ${}^{(i)}S_{s7}$ is given by

$$ {}^{(i)}S_{s7} = \frac{{}^{(i)}S_{s5}}{2n} - \frac{(k_s^2 - n)}{2n^2 r} \int^t {}^{(i)}S_{s1} dt + \frac{{}^{(i)}S_{s6}}{2n} - \frac{1}{2rk_s^2} \left[r^{2(i)}S_{s5} \right]', \tag{III.191} $$

We now expand ${}^{(i)}\Phi_s$ in the basis of the eigenfunctions $e^{(s)}_{p,l_s}$ as follows:

$$ {}^{(i)}\Phi_s = \sum_{p=0}^{\infty} c^{(s)}_{p,\mathbf{k}_s}(t) e^{(s)}_{p,l_s}(r) \tag{III.192} $$

Once again, this expansion is not valid at $r = r_0$. The boundary condition (III.108) requires ${}^{(i)}F_{tt}$ to vanish at $r = r_0$. This translates to

$$ r^{2(i)}\Phi''_s + r(2n-1){}^{(i)}\Phi'_s + \left((n-1)^2 - \frac{k_s^2}{n} \right) {}^{(i)}\Phi_s = 0 \Big|_{r=r_0} \tag{III.193} $$

If we substitute for ${}^{(i)}\Phi_s$ from (III.192) in (III.193), we obtain

$$ {}^{(i)}F_{tt} = -\frac{2n}{k_s^2 - n} \sum_{p=0}^{\infty} {}^{(i)}c^{(s)}_{p,\mathbf{k}_s} \left[r^2 e^{(s)}_{p,l_s}{}'' + (2n-1) r e^{(s)}_{p,l_s}{}' + \left((n-1)^2 - \frac{k_s^2}{n} \right) e^{(s)}_{p,l_s} \right]. \tag{III.194} $$

Since $e^{(s)}_{p,l_s}$ satisfy (III.145), $r^2 e^{(s)}_{p,l_s}{}'' = (-r^2\omega^2 + k_s^2) e^{(s)}_{p,l_s} - nr e^{(s)}_{p,l_s}{}'$, the following is holds true

$$ {}^{(i)}F_{tt} = \sum_{p=0}^{\infty} \frac{2n^{(i)}c^{(s)}_{p,\mathbf{k}_s}}{n - k_s^2} \left[(n-1) r e^{(s)}_{p,l_s}{}' + \left(-\omega^2 r^2 + \frac{(n-1)}{n} (k_s^2 + n(n-1)) \right) e^{(s)}_{p,l_s} \right], \tag{III.195} $$

which vanishes at $r = r_0$ because of (III.146).

Finally, by projecting (III.189) on the eigenfunctions $e^{(s)}_{p,l_s}$, we obtain the following equation governing the $c^{(s)}_{p,\mathbf{k}_s}(t)$ coefficients

$$ {}^{(i)}\ddot{c}^{(s)}_{q,\mathbf{k}_s} + \omega^2_{q,l_s}\, {}^{(i)}c^{(s)}_{q,\mathbf{k}_s} =< {}^{(i)}S_{s7}, e^{(s)}_{q,l_s} >_s \qquad \text{(III.196)} $$

III.5.7 Constructing metric perturbations

The source terms ${}^{(i)}S_{\mu\nu}$ at any order of perturbation theory depend on the metric perturbations ${}^{(j)}h_{\mu\nu}$ at lower orders, i.e. when for $j < i$. As we saw in the preceding sections, once we solve for the master variables at each order, we can get back each of the gauge invariant variables. In order to construct metric perturbations ${}^{(i)}h_{\mu\nu}$, we need to fix the gauge as well.

Tensor components:

By definition, ${}^{(i)}H_{T\mathbf{k}} = {}^{(i)}\Phi_{\mathbf{k}}$, so once ${}^{(i)}H_{T\mathbf{k}}$ is determined, one can obtain tensor component of ${}^{(i)}h_{ij}$.

Vector components:

${}^{(i)}Z_{r\mathbf{k}_v}$ can be computed from ${}^{(i)}\Phi_{v\mathbf{k}_v}$ through equations (III.119) and (III.161) for linear order and higher orders respectively.

${}^{(i)}Z_{t\mathbf{k}_v}$ is obtained from ${}^{(i)}Z_{r\mathbf{k}_v}$ through (III.116) and (III.156) for linear and higher orders respectively.

By making use of the single gauge freesom in the vector case, one can put $H^{(v)}_T$ to sero. Then, the vector components are given by ${}^{(i)}Z_a = {}^{(i)}f^{(v)}_a$.

Scalar components:

Once the quantities ${}^{(i)}F_{\mathbf{k}_s}$ and components of ${}^{(i)}F_{ab\mathbf{k}_s}$ are determined in terms of ${}^{(i)}\Phi_{s\mathbf{k}_s}$, we make use of the gauge freedom to set ${}^{(i)}f_t$, ${}^{(i)}H_L$ and ${}^{(i)}H^{(s)}_T$ to zero. Then the various expansion coefficients in terms of the gauge invariant variables are:

$$ {}^{(i)}f_r = k_s {}^{(i)}F, \tag{III.197} $$

$$ {}^{(i)}f_{tt} = {}^{(i)}F_{tt}, \tag{III.198} $$

$$ {}^{(i)}f_{rr} = {}^{(i)}F_{rr} - \frac{1}{k_s}(r\,{}^{(i)}f_r)'. \tag{III.199} $$

$$ {}^{(i)}f_{rt} = {}^{(i)}F_{rt} - \frac{r}{k_s}{}^{(i)}\dot{f}_r \tag{III.200} $$

III.5.8 Special modes

The $l_s = 0, 1$ modes for scalar perturbations and $l_v = 1$ for vector perturbations are gauges at the linear order. But in subsequent orders they are physical perturbation and need to be dealt separately, since certain harmonics are not defined for these cases. In this section we will derive the expressions for these modes.

Scalar perturbations $l_s = 0$ mode: The scalar harmonic $\mathbb{S}$ is constant in this case and hence, $\mathbb{S}_i$ and $\mathbb{S}_{ij}$ are undefined. This means, we only need to solve for ${}^{(i)}f_{ab}$ and ${}^{(i)}H_L$. We will use gauge freedom to put

$$ {}^{(i)}H_L = {}^{(i)}f_{tt} = 0 \tag{III.201} $$

Let ${}^{(i)}\tilde{S}_{0\,\mu\nu} = \int \mathbb{S}_{l_s=0}{}^{(i)}S_{0\,\mu\nu} d^n\Omega$. We get the following equations from ${}^{(i)}G_{rt} = 0$, ${}^{(i)}G_{tt} = 0$ and ${}^{(i)}G_{rr} = 0$ respectively.

$$\frac{n}{r}{}^{(i)}\dot{f}_{rr} = {}^{(i)}\tilde{S}_{0\,rt} \tag{III.202}$$

$$\frac{n}{r}{}^{(i)}f'_{rr} + \frac{n(n-1)}{r^2}{}^{(i)}f_{rr} = {}^{(i)}\tilde{S}_{0\,tt} \tag{III.203}$$

$$\frac{2n}{r}{}^{(i)}\dot{f}_{rt} - \frac{n(n-1)}{r^2}{}^{(i)}f_{rr} = {}^{(i)}\tilde{S}_{0\,rr} \tag{III.204}$$

From (III.202), we can obtain ${}^{(i)}f_{rr}$ as:

$$ {}^{(i)}f_{rr} = \int_{t_1}^{t} \frac{r}{n}{}^{(i)}\tilde{S}_{0\,rt} + {}^{(i)}f_{rr}(t_1, r) \tag{III.205}$$

${}^{(i)}f_{rr}(t_1, r)$ can be obtained from (III.203):

$$ {}^{(i)}f_{rr}(t_1, r) = \frac{1}{r^{n-1}} \int_0^r \frac{\tilde{r}^n}{n}{}^{(i)}\tilde{S}_{0\,tt}(t_1, r) d\tilde{r} \tag{III.206}$$

Finally, from (III.204), ${}^{(i)}f_{rt}$ is given by

$$ {}^{(i)}f_{rt} = \int^t \left[\frac{(n-1)}{2r}{}^{(i)}f_{rr} + \frac{r}{2n}{}^{(i)}\tilde{S}_{0\,rr} \right] dt \tag{III.207}$$

Scalar perturbations $l_s = 1$ ($k_s^2 = n$) mode: Let ${}^{(i)}\tilde{S}_{1\,\mu\nu} = \int \mathbb{S}_{l_s=1}{}^{(i)}S_{1\,\mu\nu} d^n\Omega$. Since $\mathbb{S}_{ij}$ vanishes for this mode, only ${}^{(i)}f_{ab}$, ${}^{(i)}f_a$ and ${}^{(i)}H_L$ exist. We will use gauge freedom to put ${}^{(i)}f_{tt}$, ${}^{(i)}f_t$ and ${}^{(i)}H_L$ to zero. Now we define the following quantities, composed solely of source terms.

$$ {}^{(i)}S_1 = \frac{1}{r^n} \int_0^r \tilde{r}^n \left[\frac{\tilde{r}}{\sqrt{n}} {}^{(i)}\tilde{S}_{1\,tt} - \frac{1}{\sqrt{n}} \left(r \int^t {}^{(i)}\tilde{S}_{1\,rt} dt \right)' - \sqrt{n} \left(\int^t {}^{(i)}\tilde{S}_{1\,rt} dt \right) \right] d\tilde{r} \tag{III.208} $$

$$ {}^{(i)}S_2 = \left[\frac{1}{2\sqrt{n}} {}^{(i)}\tilde{S}_{1\,rr} - \frac{(n-1)}{2nr} \int^t {}^{(i)}\tilde{S}_{1\,rt} dt + \left[1 + \frac{1}{2\sqrt{n}} \right] \frac{(n-1)}{r^2} {}^{(i)}S_1 \right] \tag{III.209} $$

We will use the following four equations, namely ${}^{(i)}G_{rt} = 0$, ${}^{(i)}G_{tt} = 0$, ${}^{(i)}G_{rr} = 0$ and ${}^{(i)}G^i_i = 0$:

$$ \frac{n}{r} {}^{(i)}\dot{f}_{rr} + \frac{n}{r^2} {}^{(i)}f_{rt} + \frac{\sqrt{n}}{r} {}^{(i)}\dot{f}_r = {}^{(i)}\tilde{S}_{1\,rt} \tag{III.210} $$

$$ \frac{n}{r} {}^{(i)}f'_{rr} + \frac{n^2}{r^2} {}^{(i)}f_{rr} + \frac{2\sqrt{n}}{r} {}^{(i)}f'_r + \frac{n^{3/2}}{r^2} {}^{(i)}f_r = {}^{(i)}\tilde{S}_{tt} \tag{III.211} $$

$$ \frac{2n}{r} {}^{(i)}\dot{f}_{rt} - \frac{n(n-1)}{r^2} {}^{(i)}f_{rr} - \frac{2\sqrt{n}(n-1)}{r^2} {}^{(i)}f_r = {}^{(i)}\tilde{S}_{rr} \tag{III.212} $$

$$ \begin{aligned} {}^{(i)}\dot{f}'_{rt} + \frac{(n-1)}{r} {}^{(i)}\dot{f}_{rt} - \frac{1}{2} {}^{(i)}\ddot{f}_{rr} - \frac{(n-1)}{2r} {}^{(i)}f'_{rr} - \frac{(n-1)}{\sqrt{n}r} {}^{(i)}f'_r - \frac{(n-1)^2}{2r^2} {}^{(i)}f_{rr} \\ + \frac{(n-1)^2}{\sqrt{n}r^2} {}^{(i)}f_r = \frac{1}{n} {}^{(i)}\tilde{S}^i_{1\,i} \end{aligned} \tag{III.213} $$

We will redefine ${}^{(i)}f_{rt}$ as

$$ {}^{(i)}f_{rt} = \frac{r}{\sqrt{n}}\,{}^{(i)}\dot{\phi}_0 \tag{III.214} $$

Substituting this ansatz in (III.210) and then integrating w.r.t to t gives,

$$ {}^{(i)}f_{rr} = -\frac{1}{\sqrt{n}}\,{}^{(i)}\phi_0 - \frac{1}{\sqrt{n}}\,{}^{(i)}f_r + \int^t \frac{r}{n}\,{}^{(i)}\tilde{S}_{1\,rt}dt \tag{III.215} $$

The extra $r-$dependent integration function can be absorbed in the definition of ${}^{(i)}\phi_0$. Substituting the expression for ${}^{(i)}f_{rr}$ from (III.215) in (III.211) allows us to obtain ${}^{(i)}f_r$ in terms of ${}^{(i)}\phi_0$:

$$ {}^{(i)}f_r = {}^{(i)}\phi_0 + {}^{(i)}S_1 \tag{III.216} $$

Now, by substituting (III.216) in (III.212), one obtains:

$$ {}^{(i)}\ddot{\phi}_0 = {}^{(i)}S_2 \tag{III.217} $$

Hence from (III.213), we can obtain the following expression for ${}^{(i)}\phi_0$.

$$ \begin{aligned} {}^{(i)}\phi_0 = \frac{\sqrt{n}r^2}{2(n-1)^2}\Bigg[& \frac{1}{n}\,{}^{(i)}\tilde{S}^i_{1\,i} + \frac{r}{2n}\,{}^{(i)}\dot{\tilde{S}}_{1\,rt} + \frac{(n-1)}{2r}\Big(\frac{r}{n}\int^t {}^{(i)}\tilde{S}_{1\,rt}dt\Big)' \\ & + \frac{(n-1)^2}{2nr}\int^t {}^{(i)}\tilde{S}_{1rt}dt + \frac{(n-1)}{2\sqrt{n}r}\,{}^{(i)}S_1' - \frac{3(n-1)^2}{2\sqrt{n}r^2}\,{}^{(i)}S_1 \\ & - \frac{r}{\sqrt{n}}\,{}^{(i)}S_2' - \frac{(n+1)}{\sqrt{n}}\,{}^{(i)}S_2 - \frac{1}{2\sqrt{n}}\ddot{S}_1\Bigg] \end{aligned} \tag{III.218} $$

Once, ${}^{(i)}\phi_0$ is obtained, ${}^{(i)}f_r$, ${}^{(i)}f_{rr}$ and ${}^{(i)}f_{rt}$ can be determined using (III.216), (III.215) and (III.214) respectively.

Vector perturbations $l_v = 1$ $(k_v^2 = n-1)$ **mode:** Let ${}^{(i)}\tilde{S}^{(v)}_{1\,ia} = \int \mathbb{V}^i_{l_v=1}\, {}^{(i)}S_{1\,ia} d^n\Omega$ be the source associated with these modes. Since $\mathbb{V}_{ij}$ vanishes, only ${}^{(i)}f^{(v)}_a$ exist. Through a suitable gauge choice, one can put ${}^{(i)}f^{(v)}_t$ to zero. Thus, from ${}^{(i)}G_{ir} = 0$ one can obtain ${}^{(i)}f^{(v)}_r$ as

$$\partial_t {}^{(i)}f^{(v)}_r = \frac{1}{r}\int_{t_1}^{t} {}^{(i)}\tilde{S}^{(v)}_{1\,ir} dt' + {}^{(i)}\dot{f}^{(v)}_r(t_1, r) \tag{III.219}$$

where ${}^{(i)}\dot{f}^{(v)}_r(t_1, r)$ is obtained from ${}^{(i)}G_{it} = 0$ equation,

$${}^{(i)}\dot{f}^{(v)}_r(t_1, r) = \frac{1}{r^{n+1}}\int_0^r \bar{r}^n\, {}^{(i)}\tilde{S}^{(v)}_{1\,it}(t_1, r) d\bar{r} \tag{III.220}$$

III.6 Conclusions

In this chapter, we studied the nonlinear interactions of gravitational perturbations within a shell of finite radius r_0 in Minkowski spacetime. Note that, in general, without such a Dirichlet wall, the outgoing boundary condition at infinity would have ensured that small perturbations eventually disperse to infinity. The presence of a Dirichlet wall, on the other hand, would fix the induced metric at $r = r_0$.

We first discussed the linear perturbations for each of the scalar, vector and tensor sectors and analyzed the properties of eigenvalues and the eigenfunctions. Using the completeness properties of the linear level eigenfunctions, we simplified the higher order equations, so that the expansion coefficients ${}^{(i)}c_{p,l_I}(t)$ obeyed the forced harmonic oscillator equation. The completeness and orthogonality properties, even though were simpler for tensor and vector sector, in case of the scalar sector, they were not very evident. This is because, the boundary condition for scalar modes was frequency dependent and this made the analysis of the eigenvalue problem non-trivial. Moreover, the associated eigenfunctions e_{p,l_s} did not exhibit the standard orthogonality property (although one could still define a modified inner product in this case).

Since the associated spectrum is asymptotically resonant, and satisfied the diophantine condition (II.75), based on the arguments developed in the previous chapter, one can readily conclude that this system is nonlinearly stable for arbitrarily small perturbations.

Chapter IV

Gravitational perturbations of Anti-de Sitter spacetime

In this chapter, we study weakly nonlinear perturbations of Anti-de Sitter spacetime. As we shall see, the linear spectrum is commensurate in this case as well. Hence, one can expect a turbulent instability mechanism. However, considering generalized gravitational degrees of freedom involves breaking spherical symmetry and introducing angular momentum in the system. The angular momentum can then provide a centrifugal barrier. There are few pertinent questions one can ask. How does the addition of angular momentum affect the turbulent instability? Suppose one starts with a single-mode data or two-mode data as the linear seed. At what order in perturbation theory do secular resonances arise and how many of them exist? Finally, what will be the end state of the system? It is difficult to answer the last question only using perturbative analysis and one needs to resort to numerical simulations to know the exact end point of the system. Even though, it is more challenging to perform numerical simulations in this case, very recently, a Cauchy evolution scheme was proposed for the Einstein-scalar field system without any assumptions of symmetry [76]. Few other numerical studies which includes non-spherical symmetric evolution, albeit with symmetries preserved along some directions, includes [74], [75] and [77] (we have already reviewed them in chapter II). Coming back to the pure gravity case, perturbative analysis allows one to construct the gravitational analogs of boson stars, called geons, which are nonlinear generalizations of individual perturbative modes. They are time-periodic, non-singular and asymptotically AdS. In the context of the AdS instability problem, they were first constructed perturbatively in [25].

We will first review the linear mode analysis of gravitational perturbations of AdS in general $(n+2)$ dimensions, which has been extensively discussed by Ishibashi & Wald in [7]. In section IV.2, we show, in detail, how to obtain the perturbation equations at each order and render the solutions in asymptotically AdS form. In section IV.3, we do a perturbative analysis of the perturbation equations. Here, we will briefly review the work done by previous authors, where the resonant system has been analyzed order by order for AdS_4 perturbations [25], [12], [27], [28], [29] and also for the Biaxial case [78]. Unlike four dimensions, the higher dimensions have tensor-mode as well. Hence, for us the tensor-sector holds special interest. Starting from a single-mode tensor-type initial data, we analyze the perturbation equations till the second order. We show that in this particular case, there are no secular resonances at the second order, consistent with the previous works as well. In section IV.4, we give a brief review of geons and black hole resonators. We conclude this chapter in section IV.5.

IV.1 Linear stability of AdS

The linear stability of AdS was established in [5], [6] . The generalization for linear perturbations of AdS in any $(d+1)-$dimension has been presented in [7], in which gravitational perturbations were studied using the Kodama-Ishibashi-Seto formalism [99]. Here we will first give a brief summary of the same, before moving onto presenting the detailed methodology to construct asymptotically AdS solutions at linear as well as higher orders of perturbation theory.

We know that the KIS formalism is applicable whenever one wants to study linear perturbations of an $(m+n)-$dimensional spherically symmetric spacetime, where n is the dimension of the sphere. Here, the background is the AdS metric, which in Schwarzchild coordinates is given by

$$ds^2 = -\left(1+\frac{r^2}{L^2}\right)L^2dt^2 + \frac{dr^2}{\left(1+\frac{r^2}{L^2}\right)} + r^2 d\Omega_n^2 \tag{IV.1}$$

where $d\Omega_n^2 = \gamma_{ij}dw^i dw^j$ is the $n-$sphere metric. In terms of the conformal coordinates obtained by defining $r = L\cot x$, the metric becomes

$$ds_{n+2}^2 = \frac{L^2}{\sin^2 x}\left(-dt^2 + dx^2 + \cos^2 x d\Omega_n^2\right) \tag{IV.2}$$

Here, $x \in [\frac{\pi}{2}, 0)$, with $x = \frac{\pi}{2}$ being the origin. Following the KI formalism, it was shown in [7] that each of the tensor, vector and scalar sector can be written in terms of a single master variable Φ, where $\Phi = \Phi_s, \Phi_v, \Phi_T$ for scalar, vector and tensor respectively (to see how these various master variables are related to the gauge-invariant variables, see [7]). It was shown that the equation governing the master variables take the following form:

$$\ddot{\Phi} - \Phi'' + \left(\frac{\nu^2 - 1/4}{\sin^2 x} + \frac{\sigma^2 - 1/4}{\cos^2 x} \right) \Phi = 0 \tag{IV.3}$$

where the over-dots and the primes denote partial derivative w.r.t. t and x coordinates respectively.

$$\sigma = l_I + \frac{n-1}{2}, \tag{IV.4}$$

with $l_I = l, l_v, l_s$ denoting the angular quantum number associated with tensor, vector and scalar modes respectively. The expression for ν^2 varies as

$$\nu^2 - 1/4 = \begin{cases} \frac{n(n+2)}{4} & \text{for tensor} \\ \frac{n(n-2)}{4} & \text{for vector} \\ \frac{(n-2)(n-4)}{4} & \text{for scalar} \end{cases} \tag{IV.5}$$

Upon doing a Fourier transform of equation (IV.3) in time, with Fourier parameter ω, the equation then reduced to

$$-\Phi'' + \left(\frac{\nu^2 - 1/4}{\sin^2 x} + \frac{\sigma^2 - 1/4}{\cos^2 x} \right) \Phi = \omega^2 \Phi \tag{IV.6}$$

As discussed in [7], the general solution to the above equation is written in terms of the hypergeometric function as

$$\Phi = (\sin x)^{\nu+1/2} (\cos x)^{\sigma+1/2} \left\{ A \cdot {}_2F_1(\zeta^{\omega}_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}, 1+\nu; \sin^2 x) + B \cdot \tilde{F}(\sin^2 x) \right\} \tag{IV.7}$$

where A and B are constants and

$$\zeta^{\nu,\sigma}_{\omega} = \frac{\nu + \sigma + 1 + \omega}{2} \tag{IV.8}$$

The form of the function $\tilde{F}(\sin^2 x)$ depends on whether one is working in even or odd dimensions. In even dimensions, ν is not an integer, in which case

$$\tilde{F}(\sin^2 x) = (\sin x)^{-2\nu}{}_2F_1(\zeta^{\omega}_{-\nu,\sigma}, \zeta^{-\omega}_{-\nu,\sigma}, 1 - \nu; \sin^2 x) \tag{IV.9}$$

whereas, for odd dimensions, when ν is even, we have

$$\begin{aligned} \tilde{F}(\sin^2 x) &= {}_2F_1(\zeta^{\omega}_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}, 1 + \nu; \sin^2 x) \log(\sin^2 x) \\ &+ \sum_{k=1}^{\infty} \frac{(\zeta^{\omega}_{\nu,\sigma})_k (\zeta^{-\omega}_{\nu,\sigma})_k}{(1+\nu)_k k!} \{h(k) - h(0)\} (\sin x)^{2k} \\ &- \sum_{k=1}^{\sigma} \frac{(k-1)!(-\nu)_k}{(\zeta^{\omega}_{-\nu,-\sigma})_k (\zeta^{-\omega}_{-\nu,-\sigma})_k} (\sin x)^{-2k} \end{aligned} \tag{IV.10}$$

where

$$(\zeta)_k = \frac{\Gamma(\zeta + k)}{\Gamma(\zeta)} \tag{IV.11}$$

$$\psi(z) = \frac{d}{dz} \log \Gamma(z) \tag{IV.12}$$

$$h(k) = \psi(\zeta^{\omega}_{\nu,\sigma} + k) + \psi(\zeta^{-\omega}_{\nu,\sigma} + k) - \psi(1 + \nu + k) - \psi(k + 1) \tag{IV.13}$$

As discussed in [7], for the function Φ to behave regularly at both the boundary points $x = 0, \frac{\pi}{2}$, one needs to set $B = 0$ in (IV.7) and also have

$$\omega = 2p + \nu + \sigma + 1, \quad p = 0, 1, 2... \tag{IV.14}$$

This sets Φ to be

$$\begin{aligned}\Phi = e_{p,l_I} &= A \cdot (\sin x)^{\nu+1/2} (\cos x)^{\sigma+1/2} {}_2F_1(-p, p+\nu+\sigma+1, 1+\nu; \sin^2 x) \\ &= \tilde{A} \cdot (\sin x)^{\nu+1/2} (\cos x)^{\sigma+1/2} P_p^{(\nu,\sigma)}(\cos 2x) \end{aligned} \tag{IV.15}$$

where we have made use of the fact that [112]

$${}_2F_1(-p, p+\nu+\sigma+1, 1+\nu; \sin^2 x) = \frac{p!}{(\nu+1)_p} P_p^{(\nu,\sigma)}(\cos 2x) \tag{IV.16}$$

IV.2 Gravitational perturbations of AdS_{n+2}

In this section, we will illustrate in detail as to how to construct asymptotically AdS solutions in any general $(n+2)-$dimension, with $n > 2$. (The case $n = 2$, which corresponds to AdS_4 was explored in [25], [28],[29], [79] and we will be discussing them in the later sections.) In order to facilitate the study of perturbations, we rely heavily on sections III.1, III.2 and III.4 of the previous chapter, where we have illustrated how one can obtain linear and higher order equations in perturbation theory. Since it has already been discussed in these sections in great detail, we will not repeat it here.

IV.2.1 Asymptotically AdS conditions and nature of source terms

It has been established that the metric perturbations $\delta g_{\mu\nu}$ satisfying asymptotically AdS conditions will have the following leading order behaviour, as $r \to \infty$ [107], [108], [109]:

$$\delta g_{rr} \sim \frac{1}{r^{n+3}} \quad ; \delta g_{r\gamma} \sim \frac{1}{r^{n+2}} \quad ; \delta g_{\gamma\sigma} \sim \frac{1}{r^{n-1}} \tag{IV.17}$$

where $\sigma, \gamma \neq r$. Now since,

$$^{(i)}h^{\mu\nu} = \bar{g}^{\mu\alpha}\bar{g}^{\nu\beta\,(i)}h_{\alpha\beta} \tag{IV.18}$$

the leading order behavior of $^{(i)}h^{\mu\nu}$ in the limit $r \to \infty$ should be

$$^{(i)}h^{rr} \sim \frac{1}{r^{n-1}} \quad ; {}^{(i)}h^{r\gamma} \sim \frac{1}{r^{n+2}} \quad ; {}^{(i)}h^{\gamma\sigma} \sim \frac{1}{r^{n+3}} \tag{IV.19}$$

On similar lines it can be deduced that a term like $^{(i)}h^{\mu}_{\nu}$ will fall off at least like $\frac{1}{r^n}$.

Suppose $^{(i)}f^{\mu\nu}$ denotes the $i-$th order component of $\delta g^{\mu\nu}$. Now, since the full metric $g_{\mu\nu}$ needs to obey $g^{\mu\lambda}g_{\lambda\nu} = \delta^{\mu}_{\nu}$, one obtains

$$^{(i)}f^{\mu\nu} = -{}^{(i)}h^{\mu\nu} - \sum_{x=1}^{(i-1)} {}^{(x)}h^{\mu\lambda_1\,(i-x)}f^{\nu}_{\lambda_1} \tag{IV.20}$$

Our aim is now to show that the asymptotic behavior of $^{(i)}f^{\mu\nu}$ is same as $^{(i)}h^{\mu\nu}$. For $i = 1$, $^{(1)}f^{\mu\nu} = -{}^{(1)}h^{\mu\nu}$ (from here on, we omit the label (1) on linear perturbations), and hence (IV.19) holds true. For $i = 2$,

$$ {}^{(2)}f^{\mu\nu} = -{}^{(2)}h^{\mu\nu} + h^{\mu\lambda_1} h^{\nu}_{\lambda_1} \tag{IV.21} $$

Since the products of the metric perturbations tend to fall off at a faster rate, from (IV.21), one can easily see that the leading order term is contributed only by ${}^{(2)}h_{\mu\nu}$.

In general, at any order i, the leading order behavior of ${}^{(i)}f^{\mu\nu}$ is the same as ${}^{(i)}h^{\mu\nu}$, because the rest of the terms in (IV.20) tend to fall off faster than ${}^{(i)}h^{\mu\nu}$. Let us tackle the terms individually.

In case of ${}^{(i)}f^{rr}$, the terms in $\sum_{x=1}^{(i-1)} {}^{(x)}h^{r\lambda_1}\, {}^{(i-x)}f^{r}_{\lambda_1}$ are at least of the order $\frac{1}{r^{2n}}$. in case of ${}^{(i)}f^{rt}$, while ${}^{(i)}h^{rt}$ falls off like $\frac{1}{r^{n+2}}$, the rest of the terms in $\sum_{x=1}^{i-1} {}^{(x)}h^{r\lambda_1 (i-1)}f^{t}_{\lambda_1}$ fall off at least like $\frac{1}{r^{2n+3}}$. On the same lines, one can deduce that the leading order fall off of ${}^{(i)}f^{tt}$ is same as that of ${}^{(i)}h^{tt}$, which is $\frac{1}{r^{n+3}}$, because the rest of the terms fall off like $\frac{1}{r^{2n+4}}$ or at a faster rate.

In conclusion, the leading order behavior of $\delta g^{\mu\nu}$ would be:

$$ \delta g^{rr} \sim \frac{1}{r^{n-1}} \;\; ; \delta g^{r\gamma} \sim \frac{1}{r^{n+2}} \;\; ; \delta g^{\gamma\sigma} \sim \frac{1}{r^{n+3}} \tag{IV.22} $$

When we solve the perturbation equations, the metric perturbations ${}^{(i)}h_{\mu\nu}$ need to be rendered asymptotically AdS at each order i. We note that ${}^{(i)}h_{\mu\nu}$ are dependent on the sources ${}^{(i)}S_{\mu\nu}$ as well. Since ${}^{(i)}S_{\mu\nu}$ is made of lower order metric perturbations, their behavior at any order is fixed. We wish to deduce their asymptotic behavior as $r \to \infty$. This is to ensure that through suitable gauge fixing we can render the $i-$th order metric perturbations in aAdS form.

To do this, we simplify (III.78) and then use (IV.17) and (IV.22) in the resultant expression. We will also require the behavior of background Christoffel quantities. They are:

$$\bar{\Gamma}^t_{tr} = \frac{f'}{2f} \sim \frac{1}{r} \qquad \bar{\Gamma}^r_{rr} = -\frac{f'}{2f} \sim \frac{1}{r}$$
$$\bar{\Gamma}^k_{kr} = \frac{n}{r} \sim \frac{1}{r} \tag{IV.23}$$

If we do a detailed calculation, we see that the asymptotic behavior of ${}^{(i)}A_{\mu\nu}$ is of the following form

$${}^{(i)}A_{rr} \sim \frac{1}{r^{2n+4}} \;\; ; {}^{(i)}A_{r\gamma} \sim \frac{1}{r^{2n+3}} \;\; ; {}^{(i)}A_{\gamma\sigma} \sim \frac{1}{r^{2n}} \tag{IV.24}$$

where $\gamma, \sigma \neq r$ Here we will give some of the terms which contribute to the leading order behavior in each of the ${}^{(i)}A_{\mu\nu}$. Consider a term of the following form in the expansion of ${}^{(i)}A_{\mu\nu}$

$$-[\epsilon^i]\bar{\Gamma}^\alpha_{\alpha\lambda_1}(-\bar{g}^{-1}\delta g\bar{g}^{-1} + ...)^{\lambda_1\lambda}(-\bar{\nabla}_\lambda\delta g_{\mu\nu} + \bar{\nabla}_\mu\delta g_{\nu\lambda} + \bar{\nabla}_\nu\delta g_{\mu\lambda})$$

One of the components of this term in ${}^{(i)}A_{tt}$ is

$$\begin{aligned} & - [\epsilon^i]\bar{\Gamma}^t_{tr}[(-\bar{g}^{-1}\delta g\bar{g}^{-1} + ...)^{rr}(-\bar{\nabla}_r\delta g_{tt} + ...)] \\ & \sim \frac{1}{r} \times \frac{1}{r^{n-1}} \times \frac{1}{r^n} \\ & \sim \frac{1}{r^{2n}} \end{aligned} \tag{IV.25}$$

Here, we have used the fact that $(-\bar{g}^{-1}\delta g\bar{g}^{-1} + ...)^{\mu\nu}$ will have the same leading order behaviour as that of $\delta g^{\mu\nu}$. Similarly, for ${}^{(i)}A_{ij}$, one obtains,

$$\begin{aligned}
&-[\epsilon^i]\bar{\Gamma}^k_{kr}(-\bar{g}^{-1}\delta g\bar{g}^{-1}+...)^{rr}(-\bar{\nabla}_r\delta g_{ij}+...)\\
&\sim\frac{1}{r}\times\frac{1}{r^{n-1}}\times\frac{1}{r^n}\\
&\sim\frac{1}{r^{2n}}
\end{aligned} \tag{IV.26}$$

Next, consider another term in ${}^{(i)}A_{\mu\nu}$ of the form

$$-[\epsilon^i]\partial_\alpha[(-\bar{g}^{-1}\delta g\bar{g}^{-1}+...)^{\alpha\lambda}(-\bar{\nabla}_\lambda\delta g_{\mu\nu}+\bar{\nabla}_\mu\delta g_{\lambda\nu}+\bar{\nabla}_\nu\delta g_{\lambda\mu})]$$

In ${}^{(i)}A_{ir}$, the above term contains the following part

$$\begin{aligned}
&-[\epsilon^i]\partial_t[(-\bar{g}^{-1}\delta g\bar{g}^{-1}+...)^{tt}(\partial_r\delta g_{it})]\\
&\sim\frac{1}{r^{n+3}}\times\frac{1}{r^n}\\
&\sim\frac{1}{r^{2n+3}}
\end{aligned} \tag{IV.27}$$

Similarly, in ${}^{(i)}A_{rt}$, following component is present

$$\begin{aligned}
&-[\epsilon^i]\partial_t[(-\bar{g}^{-1}\delta g\bar{g}^{-1}+...)^{tt}\bar{\nabla}_r\delta g_{tt}]\\
&\sim\frac{1}{r^{n+3}}\times\frac{1}{r^n}\\
&\sim\frac{1}{r^{2n+3}}
\end{aligned} \tag{IV.28}$$

Lastly, one of the terms in ${}^{(i)}A_{rr}$ is

$$[\epsilon^i]\bar{\Gamma}^{\lambda_1}_{\alpha r}(-\bar{g}^{-1}\delta g\bar{g}^{-1})^{\alpha\lambda}[-\bar{\nabla}_\lambda\delta g_{r\lambda_1}+\bar{\nabla}_r\delta g_{\lambda\lambda_1}+\bar{\nabla}_{\lambda_1}\delta g_{r\lambda}]$$

From this, the component which contributes to the leading order term is

$$
\begin{aligned}
&[\epsilon^i]\bar{\Gamma}^r_{rr}(-\bar{g}^{-1}\delta g\bar{g}^{-1}+...)^{rr}\bar{\nabla}_r\delta g_{rr} \\
&\sim \frac{1}{r}\times\frac{1}{r^{n-1}}\times\frac{1}{r^{n+4}} \\
&\sim \frac{1}{r^{2n+4}}
\end{aligned}
\tag{IV.29}
$$

Hence, by using (IV.24), we get the following behavior for ${}^{(i)}S_{\mu\nu}$ as $r\to\infty$

$$
{}^{(i)}S_{rr}\sim\frac{1}{r^{2n+4}}\quad ;\, {}^{(i)}S_{r\gamma}\sim\frac{1}{r^{2n+3}}\quad ;\, {}^{(i)}S_{\gamma\sigma}\sim\frac{1}{r^{2n}} \tag{IV.30}
$$

where $\sigma,\gamma\neq r$

The next step is to simplify and solve the equations pertaining to each of the tensor, vector and scalar sectors with the background AdS metric being the AdS_{n+2} metric (IV.1). For each sector, we will be discussing the linearized perturbations (of [7]) first, before moving on to higher order in perturbation theory. For further simplification of vector and scalar sector particularly, we will be using a method similar to that of Takahashi and Soda [102].

IV.2.2 Tensor sector: Linear level

Here, we give the details for the linear level tensor perturbations. We let $H_{T\mathbf{k}}=r^{-n/2}\Phi_{T\mathbf{k}}$ in III.33. In r coordinate, the equation governing the master variable $\Phi_{T\mathbf{k}}$ for tensor perturbations at linear level defined as becomes

$$
\ddot{\Phi}_T-f^2\Phi_T''-f'f\Phi_T'+\left(\frac{n(n-2)}{4}\frac{f^2}{r^2}+\frac{n}{2r}f'f+\frac{l(l+n-1)}{r^2}\right)\Phi_T=0 \tag{IV.31}
$$

If we substitute the ansatz $\Phi_T = \cos(\omega t + b)\phi$ in (IV.31) we get:

$$\hat{L}\phi = \omega^2 \phi \tag{IV.32}$$

where $\hat{L}$ is given by

$$\hat{L} = -f^2 \partial_r^2 - f' f \partial_r + \left(\frac{n(n-2)}{4} \frac{f^2}{r^2} + \frac{n}{2r} f' f + \frac{l(l+n-1)}{r^2} \right) \tag{IV.33}$$

Equation (IV.31) has two linearly independent solutions. In case we are in even dimensions of spacetime, they exhibit the following behavior as $r \to \infty$.

$$\Phi_T \sim A(t) \frac{1}{r^{\frac{n}{2}+1}} (1 + \mathcal{O}(r^{-2})) + B(t) r^{\frac{n}{2}} (1 + \mathcal{O}(r^{-2})) \tag{IV.34}$$

$$\Phi \sim A(t) \frac{1}{r^{\frac{n}{2}+1}} (1 + \mathcal{O}(r^{-2})) + B(t) r^{\frac{n}{2}} (1 + \mathcal{O}(r^{-2})) \tag{IV.35}$$

Since $h_{ij} = \sum_{\mathbf{k}} r^{2-\frac{n}{2}} \Phi_{\mathbf{k}}$ the only way h_{ij} will have the correct $\frac{1}{r^{n-1}}$ fall off is if $B = 0$. For odd dimensions, the asymptotic behavior is different, but similar arguments follow. Hence the solution to (IV.32) which satisfies the correct asymptotically AdS condition at the boundary is given by

$$\phi_T = e_{p,l} = d_{p,l} \frac{L^{\frac{1}{2}+\nu} r^{\frac{1}{2}+\sigma}}{(r^2 + L^2)^{\frac{(\nu+\sigma+1)}{2}}} {}_2F_1 \left(\zeta^{\omega}_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}, 1 + \nu; \frac{L^2}{(r^2 + L^2)} \right) \tag{IV.36}$$

where $\zeta^{\omega}_{\nu,\sigma} = \frac{\nu+\sigma+\omega L+1}{2}$, $\nu = \frac{(n+1)}{2}$ and $\sigma = l + \frac{(n-1)}{2}$ The eigenfrequencies ω are determined by imposing regularity of ϕ at the origin, which gives us

$$\omega L = 2p + l + n + 1; \quad p = 0, 1, 2... \tag{IV.37}$$

The eigenfunctions $e_{p,l}$ form a complete orthogonal set w.r.t the inner product

$$< e_{p,l}, e_{p',l} >= \int_0^\infty e_{p,l} e_{p',l} w(r) dr = \delta_p^{p'} \tag{IV.38}$$

where $w(r)$ is the appropriate weight function given by

$$w(r) = \frac{1}{f} \tag{IV.39}$$

Hence the normalization constant $d_{p,l}$ is given by

$$d_{p,l} = \left[\frac{2}{L} \frac{(2p+l+n+1)\Gamma(p+l+n+1)}{p!\Gamma(p+l+\frac{n+1}{2})\Gamma(p+\frac{n+3}{2})} \right]^{1/2} \left(\frac{n+3}{2} \right)_p \tag{IV.40}$$

IV.2.3 Tensor sector: Higher orders

As seen earlier, the higher order tensor sector equation is governed by (III.102). Therefore, we let ${}^{(i)}H_{T\mathbf{k}} = r^{-\frac{n}{2}}\,{}^{(i)}\Phi_{T\mathbf{k}}$ in (III.102), which leads to

$${}^{(i)}\ddot{\Phi}_T + \hat{L}\,{}^{(i)}\Phi_T = r^{\frac{n}{2}-2} f \int \mathbb{T}^{ij}\,{}^{(i)}S_{ij} d^n\Omega \tag{IV.41}$$

The solution to the above equation can be written as

$${}^{(i)}\Phi_T = {}^{(i)}\Phi_T^{(H)} + {}^{(i)}\Phi_T^{(P)} \tag{IV.42}$$

where ${}^{(i)}\Phi_T^{(H)}$ satisfies the homogeneous part of equation (IV.41). The behavior of ${}^{(i)}\Phi_T^{(P)}$ part, depends on the R.H.S. of equation (IV.41). In order to determine the asymptotic behavior of ${}^{(i)}\Phi_T^{(P)}$ in the limit $r \to \infty$, we assume the following ansatz for ${}^{(i)}\Phi_T^{(P)}$

$${}^{(i)}\Phi_T^{(P)} = \frac{{}^{(i)}a_k}{r^k} + \frac{{}^{(i)}a_{k+1}}{r^{k+1}} + ... \tag{IV.43}$$

Using the fact that (see (IV.30))

$$
{}^{(i)}S_{ij} \sim \frac{1}{r^{2n}}, \tag{IV.44}
$$

one can deduce the leading order behavior of R.H.S of (IV.41) in the limit $r \to \infty$ as $\frac{1}{r^{3n/2}}$. Hence, in this limit, ${}^{(i)}\Phi_T$ behaves as

$$
{}^{(i)}\Phi_T \sim \frac{{}^{(i)}A(t)}{r^{\frac{n}{2}+1}} + \mathcal{O}(r^{-(\frac{n}{2}+3)}) + {}^{(i)}B(t) r^{\frac{n}{2}} (1 + \mathcal{O}(r^{-2})) \tag{IV.45}
$$

If we plug this in the expression for ${}^{(i)}h_{ij}$, we see that

$$
\begin{aligned}
{}^{(i)}h_{ij} &= \sum_{\mathbf{k}} r^2 H_{T\mathbf{k}} \mathbb{T}_{\mathbf{k}ij} \\
&= \sum_{\mathbf{k}} r^{2-\frac{n}{2}} \Phi_{T\mathbf{k}} \mathbb{T}_{\mathbf{k}ij} \\
&\sim \frac{{}^{(i)}A(t)}{r^{n-1}} + \mathcal{O}(r^{-(n+1)}) + {}^{(i)}B(t) r^2 (1 + \mathcal{O}(r^{-2})) \text{ as } r \to \infty
\end{aligned} \tag{IV.46}
$$

The only way tensor perturbations at higher order can be rendered aAdS is by putting ${}^{(i)}B = 0$.

Finally, one can further simplify (IV.41) by using the orthonormality and completeness of eigenfunctions. The completeness of $e_{p,l}$ allows one to write ${}^{(i)}\Phi_T$ as ${}^{(i)}\Phi_{T\mathbf{k}} = \sum_{p=0}^{\infty} {}^{(i)}c_{p,\mathbf{k}}(t) e_{p,l}(r)$, so that ${}^{(i)}c_{p,\mathbf{k}}$ satisfies:

$$
{}^{(i)}\ddot{c}_{p,\mathbf{k}}(t) + \omega^{2\,(i)} c_{p,\mathbf{k}}(t) = < r^{\frac{n}{2}-2} f \int \mathbb{T}^{ij}_{\mathbf{k}}\, {}^{(i)}S_{ij} d^n\Omega, e_{p,l} > \tag{IV.47}
$$

IV.2.4 Vector sector: Linear level

In the following sections, we will illustrate the detailed procedure for simplifying the equations pertaining to the vector sector. At linear level, vector equations are governed by two independent equations, one which we obtain by expanding (III.39), the other by putting $a = r$ in (III.38). Thus we obtain

$$\dot{Z}_t - f^2 Z_r' - f' f Z_r - \frac{(n-1)f^2}{r} Z_r = 0 \tag{IV.48}$$

$$\frac{r}{f}\ddot{Z}_r - \frac{r}{f}\dot{Z}_t' + \frac{1}{f}\dot{Z}_t + \left(\frac{k_v^2 - (n-1)}{r}\right) Z_r = 0 \tag{IV.49}$$

These two equations can be used to obtain a single master equation by defining $Z_{r\mathbf{k}_v} = f^{-1} r^{-\frac{(n-2)}{2}} \Phi_{v\mathbf{k}_v}$, so that,

$$\ddot{\Phi}_v - f^2 \Phi_v'' - f' f \Phi_v' + \left(\frac{n(n+2)}{4}\frac{f^2}{r^2} + (l_v(l_v + n - 1) - n)\frac{f}{r^2} - \frac{n}{2}\frac{f' f}{r}\right) \Phi_v = 0 \tag{IV.50}$$

By letting $\Phi_v = \cos(\omega t + b)\phi_v$, (IV.50) becomes

$$\hat{L}_v \phi_v = \omega^2 \phi_v \tag{IV.51}$$

Here, $\hat{L}_v$ is given by

$$\hat{L}_v = -f^2 \partial_r^2 - f f' \partial_r + \left(\frac{n(n+2)}{4}\frac{f^2}{r^2} + (l_v(l_v + n - 1) - n)\frac{f}{r^2} - \frac{n}{2}\frac{f f'}{r}\right) \tag{IV.52}$$

For even n, as $r \to \infty$, the solution approaches

$$\Phi_v \sim \frac{A(t)}{r^{\frac{n}{2}}}(1 + \mathcal{O}(r^{-2})) + B(t) r^{\frac{n}{2}-1}(1 + \mathcal{O}(r^{-2})) \tag{IV.53}$$

In order to set the correct constant to zero, we will first construct metric perturbations $h_{\mu\nu}$. If we consider a class of perturbations where $H_T^{(v)} = 0$, then the metric perturbations h_{ij} and h_{ri} can be constructed as (summation over $\mathbf{k}_v$ is implied on the R.H.S. of the following equations)

$$h_{ij} = 2rk_v M^{(v)}\,\mathbb{V}_{ij} \tag{IV.54}$$

$$h_{ri} = \left[r^{2-\frac{n}{2}} f^{-1}\Phi_v - r^2 \bar{D}_r \left(\frac{M^{(v)}}{r} \right) \right] \mathbb{V}_i \tag{IV.55}$$

In order to ensure that h_{ij} has the correct $\frac{1}{r^{n-1}}$ fall off, we must choose $M^{(v)}$ to be of the form

$$M^{(v)} = \frac{m_n}{r^n} + \mathcal{O}(r^{-(n+1)}) \tag{IV.56}$$

as $r \to \infty$. Now if we try to fix $M^{(v)}$ by substituting (IV.56) in the large r limit of (IV.55), then there is no way we can achieve the required $\frac{1}{r^{n+2}}$ for h_{ri} without putting $B = 0$. Note that for $n = 2$ vector (axial) perturbations, $\Phi_v \sim B + \frac{A}{r} + ...$ and similar conclusions follow [29],[28].

Hence, the solution to (IV.51) which has such a desirable fall off is given by

$$\phi_v = e^{(v)}_{p,l_v} = d^{(v)}_{p,l_v} \frac{L^{1/2+\nu_v} r^{1/2+\sigma_v}}{(r^2+L^2)^{\frac{1}{2}(1+\nu_v+\sigma_v)}} {}_2F_1\left(\zeta^{\omega_v}_{\nu_v,\sigma_v}, \zeta^{-\omega_v}_{\nu_v,\sigma_v}, 1+\nu_v; \frac{L^2}{(r^2+L^2)} \right) \tag{IV.57}$$

where $\sigma_v = l_v + \frac{(n-1)}{2}$, $\nu_v = \frac{(n-1)}{2}$ and $\zeta^{\omega_v}_{\nu_v,\sigma_v} = \frac{\nu_v+\sigma_v+\omega_v L+1}{2}$. Regularity of the eigensolution at origin sets the eigenfrequencies to be

$$\omega_v L = 2p + l_v + n; \quad p = 0, 1, 2... \tag{IV.58}$$

The vector modes also form a complete orthogonal set with an inner product

$$< e^{(v)}_{p,l_v}, e^{(v)}_{p',l_v} >_v = \int_0^\infty e^{(v)}_{p,l_v} e^{(v)}_{p',l_v} w_v(r) dr = \delta_p^{p'} \qquad \text{(IV.59)}$$

where the weight function $w_v(r)$ is given by

$$w_v(r) = \frac{1}{f} \qquad \text{(IV.60)}$$

Hence the normalization constant $d^{(v)}_{p,l_v}$ is fixed as

$$d^{(v)}_{p,l_v} = \left[\frac{2}{L} \frac{(2p + l_v + n)\Gamma(p + l_v + n)}{p!\Gamma(p + l_v + \frac{n+1}{2})\Gamma(p + \frac{n+1}{2})} \right]^{1/2} \left(\frac{n+1}{2} \right)_p \qquad \text{(IV.61)}$$

IV.2.5 Vector sector: Higher orders

On the same lines as linearized equations, the higher order equations are obtained from (III.104) and (III.103):

$$^{(i)}\dot{Z}_t = f^{2\,(i)}Z_r' + f' f\,^{(i)}Z_r + (n-1)\frac{f^2}{r}\,^{(i)}Z_r + \frac{f\,^{(i)}V_{s1}}{2k_v r} \qquad \text{(IV.62)}$$

$$\frac{r}{f}\,^{(i)}\ddot{Z}_r - \frac{r}{f}\,^{(i)}\dot{Z}_t' + \frac{1}{f}\,^{(i)}\dot{Z}_t + \frac{k_v^2 - (n-1)}{r}\,^{(i)}Z_r = \,^{(i)}V_{s2} \qquad \text{(IV.63)}$$

where

$$^{(i)}V_{s1\mathbf{k}_v} = \int \mathbb{V}^{ij}_{\mathbf{k}_v}\,^{(i)}S_{ij} d^n\Omega \qquad \text{(IV.64)}$$

$$^{(i)}V_{s2\mathbf{k}_v} = \int \mathbb{V}^{i}_{\mathbf{k}_v}\,^{(i)}S_{ir} d^n\Omega \qquad \text{(IV.65)}$$

If we substitute for ${}^{(i)}\dot{Z}_t$ given by (IV.62) in (IV.63), we get the following equation solely in terms of ${}^{(i)}Z_r$:

$$\frac{r}{f}{}^{(i)}\ddot{Z}_r - rf{}^{(i)}Z_r'' - (3rf' + (n-2)f)\,{}^{(i)}Z_r' + \Big(-rf'' - \frac{r}{f}(f')^2 - (2n-3)f' + 2(n-1)\frac{f}{r} + \frac{k_v^2 - (n-1)}{r}\Big){}^{(i)}Z_r = {}^{(i)}V_{s3} \tag{IV.66}$$

where

$${}^{(i)}V_{s3\mathbf{k}_v} = \left[{}^{(i)}V_{s2\mathbf{k}_v} - \frac{{}^{(i)}V_{s1\mathbf{k}_v}}{k_v r} + \frac{1}{2k_v f}(f{}^{(i)}V_{s1\mathbf{k}_v})' \right] \tag{IV.67}$$

Letting ${}^{(i)}Z_{r\mathbf{k}_v} = f^{-1}r^{-\frac{(n-2)}{2}}{}^{(i)}\Phi_{v\mathbf{k}_v}$ in (IV.66) gives us a master equation for ${}^{(i)}\Phi_v$:

$${}^{(i)}\ddot{\Phi}_v + \hat{L}_v{}^{(i)}\Phi_v = r^{\frac{n}{2}-2}f^{2}{}^{(i)}V_{s3} \tag{IV.68}$$

The solution to above equation can be written as

$${}^{(i)}\Phi_v = {}^{(i)}\Phi_v^{(H)} + {}^{(i)}\Phi_v^{(P)} \tag{IV.69}$$

where ${}^{(i)}\Phi_v^{(H)}$ satisfies the homogeneous part of equation (IV.68) whereas ${}^{(i)}\Phi_v^{(P)}$ is determined by its R.H.S. From (IV.30) we know that in the asymptotic limit ${}^{(i)}S_{ij}$ and ${}^{(i)}S_{ir}$ behave like

$${}^{(i)}S_{ij} \sim \frac{1}{r^{2n}}, \quad {}^{(i)}S_{ir} \sim \frac{1}{r^{2n+3}} \tag{IV.70}$$

we deduce that the R.H.S. of (IV.68) has a leading order behavior of the form $\frac{1}{r^{\frac{3n}{2}-1}}$ in the same limit. If we assume ${}^{(i)}\Phi_v^{(P)}$ to have the following form as $r \to \infty$,

$$ {}^{(i)}\Phi_v^{(P)} = \frac{{}^{(i)}a_k}{r^k} + \frac{{}^{(i)}a_{k+1}}{r^{k+1}} ... \tag{IV.71} $$

and plug it back in (IV.68), we see that the leading order behavior in this asymptotic limit needs to be ${}^{(i)}\Phi_v^{(P)} \sim \frac{1}{r^{\frac{3n}{2}+1}}$. The complete solution will thus have the following form

$$ {}^{(i)}\Phi_v \sim \frac{{}^{(i)}A(t)}{r^{n/2}} + \mathcal{O}(r^{-(\frac{n}{2}+2)}) + {}^{(i)}B(t) r^{\frac{n}{2}-1}(1 + \mathcal{O}(r^{-2})), \quad r \to \infty \tag{IV.72} $$

The next step is to construct asymptotically AdS solutions upto all orders. In order to so we consider the class of metric perturbations where ${}^{(i)}H_T^{(v)} = 0$. For such a class, ${}^{(i)}f_a$ in (III.85) is simply: ${}^{(i)}f_a = {}^{(i)}Z_a$. Hence the metric perturbations, along with their gauge transformations take the following form (summation over $\mathbf{k}_v$ on the R.H.S. of the following equations is implied):

$$ {}^{(i)}h_{ri} = \left[r^{2-\frac{n}{2}} f^{-1\,(i)}\Phi_v - r^2 \bar{D}_r \left(\frac{{}^{(i)}M^{(v)}}{r} \right) \right] \mathbb{V}_i \tag{IV.73} $$

$$ {}^{(i)}h_{ti} = \left[\int^t \left(\frac{f}{r^{n-2}} \left(r^{\frac{n}{2}\,(i)}\Phi_v \right)' + \frac{f}{2k_v}\,{}^{(i)}V_{s1} \right) dt - r\,{}^{(i)}\dot{M}^{(v)} \right] \mathbb{V}_i \tag{IV.74} $$

$$ {}^{(i)}h_{ij} = 2 r k_v\,{}^{(i)}M^{(v)}\, \mathbb{V}_{ij} \tag{IV.75} $$

What should the choice of ${}^{(i)}M^{(v)}$ be? From (IV.75) we notice that since ${}^{(i)}h_{ij}$ needs to fall off like $r^{-(n-1)}$, ${}^{(i)}M^{(v)}$ should have an expansion of the following form as $r \to \infty$:

$$ {}^{(i)}M^{(v)} = \frac{{}^{(i)}m_n}{r^n} + \dots \qquad \text{(IV.76)} $$

Note that in order to render the rest of the metric perturbations aAdS, one needs to set ${}^{(i)}B = 0$ in (IV.72). Hence, in the limit $r \to \infty$, ${}^{(i)}\Phi_v$ is of the form:

$$ {}^{(i)}\Phi_v = \frac{{}^{(i)}\phi_{\frac{n}{2}}}{r^{\frac{n}{2}}} + \frac{{}^{(i)}\phi_{\frac{n}{2}+2}}{r^{\frac{n}{2}+2}} + \dots \qquad \text{(IV.77)} $$

We expand (IV.73) in the large r limit and substitute (IV.76) and (IV.77) in it to obtain

$$ \begin{aligned} {}^{(i)}h_{ri} =& \frac{L^2}{r^{\frac{n}{2}}}\left(1 + \frac{L^2}{r^2}\right)^{-1}\left[\frac{{}^{(i)}\phi_{\frac{n}{2}}}{r^{\frac{n}{2}}} + \mathcal{O}(r^{-(\frac{n}{2}+2)})\right] - r^2\left[\frac{{}^{(i)}m_n}{r^{n+1}} + \dots\right]' \\ =& L^2\left[\frac{{}^{(i)}\phi_{\frac{n}{2}}}{r^n} + \mathcal{O}(r^{-(n+2)})\right] + (n+1)\frac{{}^{(i)}m_n}{r^n} + \dots \end{aligned} \qquad \text{(IV.78)} $$

If we choose

$$ {}^{(i)}m_n = -\frac{L^2}{(n+1)}{}^{(i)}\phi_{\frac{n}{2}} \qquad \text{(IV.79)} $$

then it will ensure that ${}^{(i)}h_{ri}$ has the correct leading order behavior of the form $\frac{1}{r^{n+2}}$ as $r \to \infty$. Hence, we can choose ${}^{(i)}M^{(v)}$ to be:

$$ {}^{(i)}M^{(v)} = -\frac{L^2}{(n+1)}r^{-n/2}\,{}^{(i)}\Phi_v \qquad \text{(IV.80)} $$

The above expression is similar to that given by [29] for $n = 2$ and is applicable for linearized perturbations (where ${}^{(1)}S_{\mu\nu} = 0$) as well. One can see that the source dependent term in (IV.74) falls off like $r^{-(2n-2)}$ and hence doesn't spoil the aAdS boundary condition for ${}^{(i)}h_{ti}$ for the given choice of ${}^{(i)}M^{(v)}$ (even in the lowest possible $n = 2$ case).

Finally, in order to further simplify (IV.68), we make use of the completeness of $e^{(v)}_{p,l_v}$, so that we can write,

$$ {}^{(i)}\Phi_{v\mathbf{k}_v} = \sum_{p=0}^{\infty} {}^{(i)}c_{p,\mathbf{k}_v}(t) e^{(v)}_{p,l_v}(r) \tag{IV.81} $$

where $c_{p,\mathbf{k}_v}(t)$ satisfies,

$$ {}^{(i)}\ddot{c}_{p,\mathbf{k}_v}(t) + \omega_v^{2}\,{}^{(i)}c_{p,\mathbf{k}_v}(t) = < r^{\frac{n}{2}-2} f^{2}\,{}^{(i)}V_{s3\mathbf{k}_v}, e^{(v)}_{p,l_v} >_v \tag{IV.82} $$

IV.2.6 Scalar sector: Linear level

The equations governing scalar perturbations are obtained from (III.49-III.51). They are:

$$ F^c_c + 2(n-2)F = 0 \tag{IV.83} $$

$$ \frac{nf}{r}\dot{F}_{rr} + \frac{k_s^2}{r^2}F_{rt} - 2n\dot{F}' + \frac{nf'}{f}\dot{F} - \frac{2n}{r}\dot{F} = 0 \tag{IV.84} $$

$$ \begin{aligned} \frac{nf^2}{r}F'_{rr} + \Big(\frac{k_s^2}{r^2}f + \frac{2n}{r}f'f + n(n-1)\frac{f^2}{r^2}\Big)F_{rr} - 2nfF'' \\ - \Big(nf' + 2n(n+1)\frac{f}{r}\Big)F' + \frac{2(n-1)(k_s^2 - n)}{r^2}F = 0 \end{aligned} \tag{IV.85} $$

$$\frac{2n}{rf}\dot{F}_{rt} - \frac{2n}{f^2}\ddot{F} - \frac{nf'}{r}F_{rr} + \frac{nf'}{f}F' + \frac{2n(n-1)}{r}F' - \frac{n(n-1)f}{r^2}F_{rr} + \frac{n}{r}(F^t_t)' \\ - \frac{k_s^2}{fr^2}F^t_t - \frac{2(n-1)(k_s^2-n)}{fr^2}F = 0 \tag{IV.86}$$

$$\frac{1}{f}\dot{F}_{rt} + (F^t_t)' - \frac{1}{r}F^t_t + 2(n-1)F' - \frac{(n-1)}{r}F^r_r - \frac{f'}{2}F_{rr} + \frac{f'}{2f}F^t_t = 0 \tag{IV.87}$$

Following the method used by [102], we obtain a single master equation from the above set of equations by defining the master variable Φ_s as

$$F_{rt} = \frac{2r}{f}(\dot{\Phi}_s + \dot{F}) \tag{IV.88}$$

The master equation then becomes

$$\ddot{\Phi}_s - f^2\Phi_s'' - \left(f'f + \frac{nf^2}{r}\right)\Phi_s' + \left(-(n-1)\frac{f'f}{r} + \frac{k_s^2}{r^2}f\right)\Phi_s = 0 \tag{IV.89}$$

Letting in $\Phi_s = \cos(\omega_s t + b)\phi_s$ (IV.89) gives us

$$\hat{L}_s\phi_s = \omega_s^2\phi_s \tag{IV.90}$$

Here, $\hat{L}_s$ is defined as

$$\hat{L}_s = -f^2\partial_r^2 - \left(f'f + \frac{nf^2}{r}\right)\partial_r + \left(-(n-1)\frac{f'f}{r} + \frac{k_s^2}{r^2}f\right) \tag{IV.91}$$

For even n, the asymptotic structure of the general solution looks like

$$\Phi_s \sim \frac{A(t)}{r^2}(1 + \mathcal{O}(r^{-2})) + \frac{B(t)}{r^{n-1}} + \mathcal{O}(r^{-(n+1)}), \quad r \to \infty \tag{IV.92}$$

As we argued for the tensor and vector case, in order to ensure the correct aAdS behavior of the metric perturbations, one needs to have $A = 0$ (this is true for odd n as well, although Φ_s will have a different asymptotic behavior in this case. Since we have used the Takahashi-Soda formalism [102] to simplify the equations, our master variable Φ_s is defined differently than in [29],[28], [7]. Hence, in four dimension ($n = 2$), the asymptotic structure of the general solution Φ_s is $\frac{B}{r} + \frac{A}{r^2}$ and not $B + \frac{A}{r}$ as in [29], [28].

Hence, the solution which is regular at the origin and has the correct asymptotic behavior is given by

$$\phi_s = e^{(s)}_{p,l_s} = d^{(s)}_{p,l_s} \frac{L^{\frac{1}{2}+\nu_s} r^{\frac{1}{2}+\sigma_s-\frac{n}{2}}}{(r^2+L^2)^{\frac{1}{2}(\nu_s+\sigma_s+1)}} {}_2F_1\left(\zeta^{\omega_s}_{\nu_s,\sigma_s}, \zeta^{-\omega_s}_{\nu_s,\sigma_s}, 1+\nu_s; \frac{L^2}{(r^2+L^2)}\right) \tag{IV.93}$$

where $\nu_s = \frac{(n-3)}{2}$, $\sigma_s = l_s + \frac{(n-1)}{2}$ and $\zeta^{\omega_s}_{\nu_s,\sigma_s} = \frac{\nu_s+\sigma_s+\omega_s L+1}{2}$. The eigenfrequencies ω_s are obtained by imposing the regularity condition at the origin, which gives

$$\omega_s L = 2p + l_s + n - 1 \quad ; p = 0, 1, 2... \tag{IV.94}$$

The associated eigenfunctions $e^{(s)}_{p,l_s}$ form a complete orthogonal set and the inner product is given by

$$< e^{(s)}_{p,l_s}, e^{(s)}_{p',l_s} >_s = \int_0^\infty e^{(s)}_{p,l_s} e^{(s)}_{p',l_s} w_s(r) dr \tag{IV.95}$$

where the weight function $w_s(r)$ is given by

$$w_s(r) = \frac{r^n}{f} \tag{IV.96}$$

Hence the normalization constant is given by

$$d^{(s)}_{p,l_s} = \left(\frac{2}{L}\frac{(2p+l_s+n-1)\Gamma(p+l_s+n-1)}{p!\Gamma(p+l_s+\frac{n+1}{2})\Gamma(p+\frac{n-1}{2})}\right)^{1/2}\left(\frac{n-1}{2}\right)_p \tag{IV.97}$$

IV.2.7 Scalar sector: Higher orders

The equations governing the scalar perturbations at higher order can be obtained from the system of equation (III.105-III.107). they are:

$$-k_s^2[{}^{(i)}F^c_c + 2(n-2){}^{(i)}F] = {}^{(i)}S_{s0} \tag{IV.98}$$

$$\frac{n}{r}f{}^{(i)}\dot{F}_{rr} + \frac{k_s^2}{r^2}{}^{(i)}F_{rt} - 2n{}^{(i)}\dot{F}' + n\frac{f'}{f}{}^{(i)}\dot{F} - \frac{2n}{r}{}^{(i)}\dot{F} = {}^{(i)}S_{s1} \tag{IV.99}$$

$$\begin{aligned}&\frac{nf^2}{r}{}^{(i)}F'_{rr} + \left(\frac{k_s^2}{r^2}f + \frac{2n}{r}f'f + n(n-1)\frac{f^2}{r^2}\right){}^{(i)}F_{rr} - 2nf{}^{(i)}F''\\ &\qquad - \left(nf' + 2n(n+1)\frac{f}{r}\right){}^{(i)}F' + 2(n-1)\frac{(k_s^2-n)}{r^2}{}^{(i)}F = \frac{{}^{(i)}S_{s2}}{f}\end{aligned} \tag{IV.100}$$

$$\begin{aligned}&\frac{2n}{rf}{}^{(i)}\dot{F}_{rt} - \frac{2n}{f^2}{}^{(i)}\ddot{F} - \frac{nf'}{r}{}^{(i)}F_{rr} + \frac{nf'}{f}{}^{(i)}F' + \frac{2n(n-1)}{r}{}^{(i)}F' - \frac{n(n-1)}{r^2}f{}^{(i)}F_{rr}\\ &\qquad + \frac{n}{r}({}^{(i)}F^t_t)' - \frac{k_s^2}{fr^2}{}^{(i)}F^t_t - \frac{2(n-1)(k_s^2-n)}{fr^2}{}^{(i)}F = {}^{(i)}S_{s3}\end{aligned} \tag{IV.101}$$

$$\begin{aligned}&\frac{1}{f}{}^{(i)}\dot{F}_{rt} + ({}^{(i)}F^t_t)' - \frac{1}{r}{}^{(i)}F^t_t + 2(n-1){}^{(i)}F' - \frac{(n-1)}{r}{}^{(i)}F^r_r - \frac{f'}{2}{}^{(i)}F_{rr} + \frac{f'}{2f}{}^{(i)}F^t_t\\ &\qquad = {}^{(i)}S_{s4}\end{aligned} \tag{IV.102}$$

where

$$ {}^{(i)}S_{s0\mathbf{k}_s} = \int \mathbb{S}^{ij}_{\mathbf{k_s}} {}^{(i)}S_{ij} d^n\Omega, \tag{IV.103} $$

$$ {}^{(i)}S_{s1\mathbf{k}_s} = \int \mathbb{S}_{\mathbf{k_s}} {}^{(i)}S_{rt} d^n\Omega, \tag{IV.104} $$

$$ {}^{(i)}S_{s2\mathbf{k}_s} = \int \mathbb{S}_{\mathbf{k_s}} {}^{(i)}S_{tt} d^n\Omega, \tag{IV.105} $$

$$ {}^{(i)}S_{s3\mathbf{k}_s} = \int \mathbb{S}_{\mathbf{k}_s} {}^{(i)}S_{rr} d^n\Omega \tag{IV.106} $$

$$ {}^{(i)}S_{s4\mathbf{k}_s} = \frac{1}{k_s} \int \mathbb{S}^{i}_{\mathbf{k}_s} {}^{(i)}S_{ir} d^n\Omega \tag{IV.107} $$

$$ {}^{(i)}S_{s5\mathbf{k}_s} = \left(\frac{k_s^2}{nr} + 2f' + \frac{(n-1)f}{r} \right) \int^t {}^{(i)}S_{s1} dt - \frac{{}^{(i)}S_{s2}}{f} + \frac{f^2}{r} \left(\frac{r}{f} \int^t {}^{(i)}S_{s1} dt \right)', \tag{IV.108} $$

$$ {}^{(i)}S_{s6\mathbf{k}_s} = \frac{f^2}{2n} \Bigg\{ {}^{(i)}S_{s3} - \frac{n}{r} {}^{(i)}S_{s4} - \left(1 - \frac{n}{k_s^2} \right) \frac{{}^{(i)}S_{s0}}{r^2 f} - \left(\frac{(k_s^2 - n)}{nfr} - \frac{f'}{2f^2} \right) \int^t {}^{(i)}S_{s1} dt + \frac{1}{f} {}^{(i)}S_{s5} - \frac{1}{k_s^2 r f} \left(nr^2 f {}^{(i)}S_{s5} \right)' \Bigg\} \tag{IV.109} $$

The above set of equations are then used to obtained a single equation in terms of a master variable ${}^{(i)}\Phi_s$ defined as

$$ {}^{(i)}F_{rt} = \frac{2r}{f}({}^{(i)}\dot{\Phi}_s + {}^{(i)}\dot{F}) \qquad \text{(IV.110)} $$

Following the procedure in [102], we plug (IV.110) in (IV.99) and integrate w.r.t t to obtain ${}^{(i)}F_{rr}$ in terms of ${}^{(i)}F$ and ${}^{(i)}\Phi_s$:

$$ {}^{(i)}F_{rr} = \frac{2r}{f}{}^{(i)}F' + \frac{(-k_s^2+n)}{nf^2}{}^{(i)}F - \frac{2k_s^2}{nf^2}{}^{(i)}\Phi_s + \frac{r}{nf}\int {}^{(i)}S_{rt}dt \qquad \text{(IV.111)} $$

Plugging this expression for ${}^{(i)}F_{rr}$ in (IV.100), we obtain an expression for ${}^{(i)}F$ in terms of ${}^{(i)}\Phi_s$ and ${}^{(i)}\Phi'_s$, which is

$$ {}^{(i)}F = \frac{1}{(-k^2+n)}\left[nrf{}^{(i)}\Phi'_s + (k_s^2 + n(n-1)f){}^{(i)}\Phi_s - \frac{nr^2 f}{2k_s^2}{}^{(i)}S_{s5}\right] \qquad \text{(IV.112)} $$

Once ${}^{(i)}F_{rr}$ and ${}^{(i)}F$ are determined, one can easily obtain ${}^{(i)}F_{tt}$ from (IV.98).

The final equation governing ${}^{(i)}\Phi_s$ can be obtained by eliminating $({}^{(i)}F^t_t)'$ from (IV.102) using (IV.101), and substituting for ${}^{(i)}F$ and ${}^{(i)}F_{rr}$ in the resultant equation. This gives us

$$ {}^{(i)}\ddot{\Phi}_s + \hat{L}_s{}^{(i)}\Phi_s = {}^{(i)}S_{s6} \qquad \text{(IV.113)} $$

where ${}^{(i)}S_{s6}$ is as defined by (IV.109). the solution to the above equation can be written as

$$ {}^{(i)}\Phi_s = {}^{(i)}\Phi_s^{\mathcal{H}} + {}^{(i)}\Phi_s^{\mathcal{P}} \qquad \text{(IV.114)} $$

where the ${}^{(i)}\Phi_s^{\mathcal{H}}$ satisfies the homogeneous part of equation (IV.113). ${}^{(i)}\Phi_s^{\mathcal{P}}$ on the other depends on the R.H.S of (IV.113). Upon inspection we find that the leading order behavior of R.H.S of (IV.113) goes like $\frac{1}{r^{2n-2}}$ as $r \to \infty$. This implies that in the same limit, ${}^{(i)}\Phi_s^{\mathcal{P}}$ should behave like

$$ {}^{(i)}\Phi_s^{\mathcal{P}} = \frac{{}^{(i)}b_{2n}}{r^{2n}} + O(r^{-(2n+1)}) \tag{IV.115} $$

Thus, as $r \to \infty$, ${}^{(i)}\Phi_s$ attains the form

$$ {}^{(i)}\Phi_s = \frac{{}^{(i)}A(t)}{r^2}(1 + \mathcal{O}(r^{-2})) + \frac{{}^{(i)}B(t)}{r^{n-1}}(1 + \mathcal{O}(r^{-(n+1)})) \tag{IV.116} $$

In order to fix these constants, we need to construct asymptotically AdS metric perturbations. To do so, we consider a class of perturbations where ${}^{(i)}H_L = {}^{(i)}f_a = 0$ at each order. For this choice, ${}^{(i)}f_{ab} = {}^{(i)}F_{ab}$ and ${}^{(i)}H_L^{(s)} = {}^{(i)}F$. Hence the metric perturbations along with the gauge transformations are given as (summation over $\mathbf{k}_s$ on the R.H.S. of each of the equations is implied):

$$ {}^{(i)}h_{tt} = \left[{}^{(i)}F_{tt} - 2\,{}^{(i)}\dot{T}_t + f' f\,{}^{(i)}T_r\right]\mathbb{S} \tag{IV.117} $$

$$ {}^{(i)}h_{rr} = \left[{}^{(i)}F_{rr} - 2\,{}^{(i)}T_r' - \frac{f'}{f}\,{}^{(i)}T_r\right]\mathbb{S} \tag{IV.118} $$

$$ {}^{(i)}h_{rt} = \left[{}^{(i)}F_{rt} - {}^{(i)}T_t' - {}^{(i)}\dot{T}_r + \frac{f'}{f}\,{}^{(i)}T_t\right]\mathbb{S} \tag{IV.119} $$

$$ {}^{(i)}h_{ti} = \left[-r\,{}^{(i)}\dot{M} + k_s\,{}^{(i)}T_t\right]\mathbb{S}_i \tag{IV.120} $$

$$^{(i)}h_{ri} = \left[-r^2 \left(\frac{^{(i)}M}{r} \right)' + k_s T_r \right] \mathbb{S}_i \tag{IV.121}$$

$$^{(i)}h_{ij} = 2 \left[r^{2\,(i)}F - \frac{k_s r}{n}{}^{(i)}M - rf\,^{(i)}T_r \right] \gamma_{ij}\mathbb{S} + 2k_s r\ ^{(i)}M\mathbb{S}_{ij} \tag{IV.122}$$

Now we will do a detailed construction of the gauge terms, such that (IV.117-IV.122) are rendered asymptotically AdS. The first step involved in this is to expand the various metric perturbations as defined by (IV.117-IV.122) in the large r limit and then substitute (IV.116) in each of these equations. Then we will see that the only way (IV.117-IV.122) can be rendered aAdS is if we put $^{(i)}A = 0$.

Hence, $^{(i)}\Phi_s$ behaves in the following in the limit $r \to \infty$

$$^{(i)}\Phi_s = \frac{^{(i)}\phi_{n-1}}{r^{n-1}} + \frac{^{(i)}\phi_{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+3)}) \tag{IV.123}$$

where the ellipsis denote the terms with lower powers of r. Now we expand $^{(i)}F$ and $^{(i)}F_{rr}$ (as given by (IV.112) and (IV.111) respectively) in the large r limit. They take the following form

$$^{(i)}F = \frac{^{(i)}f_{n-1}}{r^{n-1}} + \frac{^{(i)}f_{n+1}}{r^{n+1}} + \dots \tag{IV.124}$$

$$^{(i)}F_{rr} = \frac{^{(i)}f_{rr}^{(n+1)}}{r^{n+1}} + \frac{^{(i)}f_{rr}^{(n+3)}}{r^{n+3}} + \dots \tag{IV.125}$$

where

$$^{(i)}f_{n-1} = \frac{1}{(-k_s^2 + n)} \left(k_s^{2\,(i)}\phi_{n-1} - \frac{2n}{L^2}{}^{(i)}\phi_{n+1} \right) \tag{IV.126}$$

and

$$ {}^{(i)}f_{rr}^{(n+1)} = -2L^2(n-1)\,{}^{(i)}f_{n-1} \tag{IV.127} $$

We first assume that as $r \to \infty$, ${}^{(i)}T_r$ should have the following behavior

$$ {}^{(i)}T_r = \frac{{}^{(i)}T_r^{(n)}}{r^n} + \dots \tag{IV.128} $$

Now we put the expansions (IV.125) and (IV.128) in (IV.118) and take the large r limit to obtain

$$ \begin{aligned} {}^{(i)}h_{rr} =& \frac{{}^{(i)}f_{rr}^{(n+1)}}{r^{n+1}} + \frac{{}^{(i)}f_{rr}^{(n+3)}}{r^{n+3}} + \dots - 2\left[\frac{{}^{(i)}T_r^{(n)}}{r^n} + \dots\right]' \\ & - \frac{2}{r}\left(1 + \frac{L^2}{r^2}\right)^{-1}\left[\frac{{}^{(i)}T_r^{(n)}}{r^n} + \dots\right] \end{aligned} \tag{IV.129} $$

If we choose

$$ {}^{(i)}T_r^{(n)} = -\frac{1}{2(n-1)}\,{}^{(i)}f_{rr}^{(n+1)}, \tag{IV.130} $$

then we can kill the terms which go like $\frac{1}{r^{n+1}}$, so that leading order behavior of ${}^{(i)}h_{rr}$ is now the desired $\frac{1}{r^{n+3}}$ fall off.

Next, in order to ensure the correct aAdS behaviour of ${}^{(i)}h_{rt}$, we assume the following expansion for ${}^{(i)}T_t$ in the limit $r \to \infty$

$$ {}^{(i)}T_t = \frac{{}^{(i)}T_t^{(n-1)}}{r^{n-1}} + \dots \tag{IV.131} $$

Then we expand ${}^{(i)}h_{rt}$ (as given by (IV.119)) in the large r limit and substitute (IV.131) in it:

$$
\begin{aligned}
{}^{(i)}h_{rt} =& 2L^2\left(1+\frac{L^2}{r^2}\right)^{-1}\left[\frac{{}^{(i)}\phi_{n-1}}{r^n}+\frac{{}^{(i)}\dot{f}_{n-1}}{r^n}+\dots\right]-\left[\frac{{}^{(i)}T_r^{(n)}}{r^n}+\dots\right] \\
&-\left[\frac{{}^{(i)}T_t^{(n-1)}}{r^{n-1}}+\dots\right]'+2\left(1+\frac{L^2}{r^2}\right)^{-1}\left[\frac{{}^{(i)}T_t^{(n-1)}}{r^n}+\dots\right] \qquad \text{(IV.132)}
\end{aligned}
$$

By choosing

$$
{}^{(i)}T_t^{(n-1)} = \frac{1}{(n+1)}\left[{}^{(i)}\dot{T}_r^{(n)} - 2L^{2\,(i)}\dot{f}_{n-1} - 2L^{2\,(i)}\dot{\phi}_{n-1}\right] \qquad \text{(IV.133)}
$$

we can kill off all the terms with r^{-n} fall off, so that ${}^{(i)}h_{rt}$ has correct fall off. Lastly, for ${}^{(i)}M$, we assume an expansion of the form

$$
{}^{(i)}M = \frac{{}^{(i)}m_n^{(s)}}{r^n}+\dots \qquad \text{(IV.134)}
$$

and then substitute this in the expansion of (IV.121) in the large r limit to obtain

$$
{}^{(i)}h_{ri} = -r^2\left[\frac{{}^{(i)}m_n^{(s)}}{r^{n+1}}+\dots\right]' + k_s\left[\frac{{}^{(i)}T_r^{(n)}}{r^n}+\dots\right] \qquad \text{(IV.135)}
$$

Choosing

$$
{}^{(i)}m_n^{(s)} = -\frac{k_s}{(n+1)}\,{}^{(i)}T_r^{(n)} \qquad \text{(IV.136)}
$$

will put (IV.121) in aAdS form. Hence taking cue from (IV.130), (IV.133) and (IV.136), we make the following constructions for ${}^{(i)}T_r$, ${}^{(i)}T_t$ and ${}^{(i)}M$:

$$ {}^{(i)}T_r = \frac{1}{(-k_s^2+n)}\left[\frac{k_s^2L^2}{r}{}^{(i)}\Phi_s + \frac{n}{r^{n-3}}\partial_r(r^{n-1(i)}\Phi_s)\right] \tag{IV.137} $$

$$ {}^{(i)}T_t = -\frac{L^2}{(n+1)(-k_s^2+n)}\left[(-k_s^2+2n){}^{(i)}\dot{\Phi}_s + \frac{n}{L^2r^{n-4}}\partial_r(r^{n-1(i)}\dot{\Phi}_s)\right] \tag{IV.138} $$

$$ {}^{(i)}M = -\frac{k_s}{(n+1)}{}^{(i)}T_r \tag{IV.139} $$

These gauge choices are also valid for linearized perturbations (${}^{(i)}S_{\mu\nu}=0$).

Finally, since $e^{(s)}_{p,l_s}$ form a complete orthonormal set, one can write ${}^{(i)}\Phi_s$ as ${}^{(i)}\Phi_{s\mathbf{k}_s} = \sum_{p=0}^{\infty}{}^{(i)}c_{p,\mathbf{k}_s}(t)e^{(s)}_{p,l_s}(r)$. From (IV.113), we see ${}^{(i)}c_{p,\mathbf{k}_s}(t)$ satisfies:

$$ {}^{(i)}\ddot{c}_{p,\mathbf{k}_s} + \omega_s^{2(i)}c_{p,\mathbf{k}_s} = < {}^{(i)}S_{s6\mathbf{k}_s}, e^{(s)}_{p,l_s} >_s \tag{IV.140} $$

IV.2.8 Special modes

The analysis of the modes is still incomplete, since so far we have only dealt with gravitational degrees of freedom. We also have special set of modes, namely, $l_s = 0, 1$ and $l_v = 1$, which are gauge terms at the linear level but need to be treated carefully at subsequent orders. We will discuss them below. We first start with scalar $l_s = 0, 1$ modes.

Scalar mode, $l_s = 0$ mode: Let ${}^{(i)}\tilde{S}_{0\,\mu\nu}$ be the source terms associated with $l_s = 0$ modes. Since $\mathbb{S}$ is just a constant here, only ${}^{(i)}f_{ab}$ and ${}^{(i)}H_L$ exist.

We make a gauge choice where

$$ {}^{(i)}H_L = {}^{(i)}f_{rt} = 0 \tag{IV.141} $$

The Einstein equations ${}^{(i)}G_{rt} = 0$, ${}^{(i)}G_{tt} = 0$ and ${}^{(i)}G_{rr} = 0$ respectively take the form:

$$ \frac{nf}{r}{}^{(i)}\dot{f}_{rr} = {}^{(i)}\tilde{S}_{0\,rt} \tag{IV.142} $$

$$ \frac{nf}{r}({}^{(i)}f^r_r)' + \left(\frac{n(n-1)f}{r^2} + \frac{nf'}{r}\right){}^{(i)}f^r_r = \frac{1}{f}{}^{(i)}\tilde{S}_{0\,tt} \tag{IV.143} $$

$$ \frac{n}{r}({}^{(i)}f^t_t)' - \left(\frac{nf'}{fr} + \frac{n(n-1)}{r^2}\right){}^{(i)}f^r_r = {}^{(i)}\tilde{S}_{0\,rr} \tag{IV.144} $$

Upon solving (IV.142), one obtains the following expression for ${}^{(i)}f_{rr}$

$$ {}^{(i)}f_{rr} = \int_{t_1}^{t} \frac{r}{nf}{}^{(i)}\tilde{S}_{0\,rt}dt + {}^{(i)}f_{rr}(t_1, r) \tag{IV.145} $$

Here, ${}^{(i)}f_{rr}(t_1, r)$ can be obtained from (IV.143)):

$$ {}^{(i)}f_{rr}(t_1, r) = \frac{1}{f^2 r^{n-1}} \int_0^r \frac{r^n}{nf}{}^{(i)}\tilde{S}_{0\,tt}(t_1, r)dr \tag{IV.146} $$

Similarly ${}^{(i)}f_{tt}$ is obtained from (IV.144)

$$ {}^{(i)}f_{tt} = f^{2(i)}f_{rr} - \frac{f}{n}\int_0^r dr\left({}^{(i)}\tilde{S}_{0\,rr} + \frac{1}{f^2}{}^{(i)}\tilde{S}_{0\,tt}\right)r \tag{IV.147} $$

Scalar mode, $l_s = 1$ mode: We let ${}^{(i)}\tilde{S}_{1\,\mu\nu} = \int \mathbb{S}^{(i)}S_{\mu\nu} d^n\Omega$ to be the source associated with $l_s = 1$ We will choose a gauge, where ${}^{(i)}H_L$ and ${}^{(i)}f_a$ is zero. Hence, we need to solve for each of the ${}^{(i)}f_{ab}$. From the ${}^{(i)}G_{tt} = 0$ equation, we obtain

$$ {}^{(i)}f'_{rr} + \left(\frac{1}{rf} + \frac{2f'}{f} + \frac{(n-1)}{r}\right){}^{(i)}f_{rr} = \frac{r}{nf^3}{}^{(i)}\tilde{S}_{1\,tt} \tag{IV.148} $$

Hence, we can obtain an expression for ${}^{(i)}f_{rr}$ from (IV.148)

$$ {}^{(i)}f_{rr} = \frac{1}{r^n f^{3/2}}\left(\int_0^r \frac{r^{n+1}}{nf^{3/2}}{}^{(i)}\tilde{S}_{1\,tt}dr\right) \tag{IV.149} $$

From ${}^{(i)}G_{rt} = 0$ and ${}^{(i)}G_{it} = 0$, we obtain

$$ \frac{nf}{r}{}^{(i)}\dot{f}_{rr} + \frac{n}{r^2}{}^{(i)}f_{rt} = {}^{(i)}\tilde{S}_{1\,rt} \tag{IV.150} $$

$$ -f\sqrt{n}\left[{}^{(i)}f'_{rt} + \left(\frac{(n-2)}{r} + \frac{f'}{f}\right){}^{(i)}f_{rt} - {}^{(i)}\dot{f}_{rr}\right] = {}^{(i)}\tilde{S}_{1\,it} \tag{IV.151} $$

Hence, by eliminating ${}^{(i)}\dot{f}_{rr}$ from the two equations, we can obtain an expression for ${}^{(i)}f_{rt}$:

$$ {}^{(i)}f_{rt} = \frac{1}{f^{1/2}r^{n-1}}\left\{\int_0^r \frac{r^{n-1}}{f^{1/2}}\left(\frac{r}{n}{}^{(i)}\tilde{S}_{1\,rt} - \frac{1}{\sqrt{n}}{}^{(i)}\tilde{S}_{1\,it}\right)dr\right\} \tag{IV.152} $$

Similarly from ${}^{(i)}G_{ir} = 0$ equation we get ${}^{(i)}f_{tt}$:

$$ {}^{(i)}f_{tt} = rf^{1/2}\left\{\int^r f^{1/2}\left(\frac{{}^{(i)}\dot{f}_{tr}}{rf} - \frac{(n-1)f}{r^2}{}^{(i)}f_{rr} - \frac{{}^{(i)}\tilde{S}_{ir}}{\sqrt{n}r} - \frac{f'}{2r}{}^{(i)}f_{rr}\right)dr\right\} \tag{IV.153} $$

Vector modes $l_v = 1$ mode: In this case, $\mathbb{V}_{ij}$ is undefined, only ${}^{(i)}f_a^{(v)}$ exist. We will choose a gauge where ${}^{(i)}f_t^{(v)} = 0$. Let ${}^{(i)}\tilde{S}^{(v)}_{1\,ia} = \int \mathbb{S}^{i\,(i)}S_{ia} d^n\Omega$. Then from ${}^{(i)}G_{ir} = 0$, one obtains

$$ {}^{(i)}\dot{f}_r^{(v)} = \frac{f}{r}\int_{t_1}^{t} {}^{(i)}\tilde{S}^{(v)}_{1\,ir} dt + {}^{(i)}\dot{f}_r^{(v)}(t_1, r) \tag{IV.154} $$

where ${}^{(i)}\dot{f}_r^{(v)}(t_1, r)$ can be determined from ${}^{(i)}G_{it} = 0$:

$$ {}^{(i)}\dot{f}_r^{(v)}(t_1, r) = \frac{1}{r^{n+1}}\int_{0}^{r} \frac{r^n}{f} {}^{(i)}\tilde{S}^{(v)}_{1\,it}(t_1, r) dr \tag{IV.155} $$

IV.3 Perturbative Analysis

Once the perturbation equations have been obtained, one can check secular resonances at each order. The special feature of spacetimes with dimensions greater than four is the presence of tensor-type perturbations. Most studies involve analysis of scalar and vector sectors in AdS_4 [25], [12], [27], [28], [29]. Pure gravitational perturbations for a metric ansatz of the cohomogeneity-two biaxial Bianchi IX was studied in [78]. Although such a system breaks spherical symmetry, it still involves preservation of some kind of symmetry in certain directions. We will review these works briefly first. Then using the perturbation equations discussed in the previous section, we will study the resonant system for tensor sectors of AdS_5 perturbations. This, of course involves breaking all symmetries. Our focus will be to study the resonant system for a single mode tensor-type data upto the second order in perturbation theory.

IV.3.1 Perturbation theory in four dimensions

The action in four dimensions is given by

$$S = \int d^4x\sqrt{-g}\left(R + \frac{6}{L^2}\right) \tag{IV.156}$$

where L if the AdS length. Following the expansion of the metric as $g = \bar{g} + \sum_i {}^{(i)}h\,\epsilon^i$, where $\bar{g}$ is the AdS metric in four dimensions. At each order of perturbation theory, one can obtain the Einstein equation

$$\triangle_L {}^{(i)}h_{\mu\nu} = {}^{(i)}T_{\mu\nu} \tag{IV.157}$$

where $\triangle_L$ is as defined in (III.3) and ${}^{(i)}T_{\mu\nu}$ is made up of lower order metric perturbations. As discussed in chapter III, it is possible to reduce (IV.157), into two decoupled PDEs, for scalar and vector sector each (tensor sector is absent in four dimensions).

$$\bar{D}^a\bar{D}_a {}^{(i)}\Phi_I(t,r) + V_I(r){}^{(i)}\Phi_I(t,r) = {}^{(i)}\tilde{T}(t,r) \tag{IV.158}$$

where operator $\bar{D}_a$ is solely defined on the metric

$$g_{ab}dy^a dy^b = -\left(1 + \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{\left(1 + \frac{r^2}{L^2}\right)} \tag{IV.159}$$

The perturbative analysis of (IV.158) involved considering single-mode and two-mode scalar or vector-type initial data [25], [12], [27], [28], [29]. We will review these works briefly now.

Linear mode analysis: Single mode data

One can assume that only a single seed is present in the initial mode data. It could be a scalar or a vector mode (as tensor modes are absent in four dimensions). Let such a mode be characterized by four labels, $\{l, m, p, \tilde{\omega}\}_I$ where the subscript $I = s, v$ stands for scalar mode or a vector mode respectively. The eigenfrequency corresponding to scalar sector would be

$$\omega L = 2p + l + 1, \quad p = 0, 1, 2... \tag{IV.160}$$

whereas for vector sector, it would be

$$\omega L = 2p + l + 2, \quad p = 0, 1, 2... \tag{IV.161}$$

In the earliest work [25], the authors took $\{2, 2, 0, \frac{3}{L}\}_s$ as the initial seed. Then the study of single-mode initial data was extended to include many more examples in [12], [27], [28], [29].

Second order analysis: Single mode data

Then next step was to check for resonances at the second order in ϵ. Given a single mode data with source frequency $\tilde{\omega}$, there were two kinds of excited frequencies at this order, namely $\{0, 2\tilde{\omega}\}$. From (IV.160) and (IV.161), it was obvious that there was no value of l_s/l_v or p for which a scalar or vector mode would have an eigenfrequency equal to zero. Moreover, for the examples considered in [25], [12], [28], there was no scalar or vector frequency which is equal to $2\tilde{\omega}$. This is because, for all the single mode data considered in these works, the polar quantum number l_s, excited at the second order were all even. From (IV.160), it was clear that if l_s is even, the resonant frequency for the scalar sector had to be odd, which can never be equal to $2\tilde{\omega}$. Similarly, the polar quantum number l_v, excited at second order were all odd. From (IV.161), it was clear that any excited resonant frequency has to be odd. But then, such a frequency can never be equal to $2\tilde{\omega}$. Hence, there are no resonances at this order.

Before moving on to the third order, one may observe that since the background metric is changing, there is no reason why the frequency itself should remain unchanged. This warrants a shift in the frequency of the form

$$\omega \to \omega + \epsilon^2 \, {}^{(2)}\omega \tag{IV.162}$$

Note that, because of the structure of Einstein equations, the frequency corrections will only have even powers in ϵ.

Third order analysis: Single mode data

For the earliest of examples considered in [25] for the single mode data $\{2, 2, 0, \frac{3}{L}\}_s$, it was found that one could remove the secular resonances arising because of naive perturbation theory through the frequency shift (IV.162). Such a mode which can be nonlinearly extended is called geon. Such a geon was also constructed numerically in [111].

Many more examples of single-mode initial data were considered [12], [28]. It was stated that unlike the spherically symmetric, there are several single-mode data which develop irremovable resonances. For eg., if one starts with a single normal mode $\{2, 0, 1, \frac{5}{L}\}_s$ at the third order, two kinds of modes are excited which have the same frequency as the normal mode frequency. They are $\{2,0,1,\frac{5}{L}\}_s$ and $\{4,0,0,\frac{5}{L}\}_s$. It was argued in [28], that while the former is removable through a frequency shift of the form (IV.162), the later leads to a secular resonance.

Note that, in [27], this claim was tempered with the argument that this is merely a technical difficulty arising because of degeneracy of the spectrum. One could, in principle remove the resonances at third order by taking a linear combination of these modes in the following manner

$$^{(1)}\Phi_s(t,r) = (\eta\, e_{2,1}(r) + (1-\eta)e_{4,0}(r))\cos\left((5/L + {}^{(2)}\omega\epsilon^2)t\right) \qquad \text{(IV.163)}$$

where η is the mixing parameter. The scalar field Φ_s is expanded in the following manner

$$\Phi_s(t,r) = {}^{(1)}\Phi_s\epsilon + {}^{(2)}\Phi_s(t,r)\epsilon^2 + {}^{(3)}\Phi_s(t,r)\epsilon^3 + \mathcal{O}(\epsilon^4) \qquad \text{(IV.164)}$$

The values of η and ${}^{(2)}\omega$ for which a nonlinear extension to the original time-periodic seed existed was determined by a set of two equations, which was cubic in η and linear in ${}^{(2)}\omega$. It turned out that for this particular case there were two real solutions for the pair $(\eta, {}^{(2)}\omega)$. This implied that there were two geons which bifurcate from source frequency $\omega = \frac{5}{L}$. Many more examples like these were illustrated in [29]. Moreover, such solutions were also numerically constructed in [30]. Further, it was stated in [29] that the same procedure could be applied for a frequency with multiplicity k. In such a case, one would need to take a linear combination of k corresponding linear seeds and in order to remove resonances at the third order, one would need to solve a system of k equations, which were cubic in $\eta_1, \eta_2 ... \eta_{k-1}$ and linear in ${}^{(2)}\omega$. Another interesting observation was that for all the individual examples considered, the number of bifurcating time-periodic solutions was equal to the multiplicity of the eigenfrequency.

In [31], the authors took the superposition of modes with $l = 2$ even parity and $\omega = \frac{3}{L}$ and they found that at the fifth order of the perturbation theory, one obtains five one-parameter family of geons—One constructed from $(l = 2, m = 0)$ which has axial symmetry, two of them namely, those constructed from $(l = 2, m = \pm 1)$ and $(l = 2, m = \pm 2)$ having helical symmetry as well as two others which have continuous symmetries. One is constructed from $(l = 2, m = 2)$ and corresponds to non-rotating case and the other one which oscillates between $(l = 2, m = 2)$ and $(l = 2, m = 0)$.

Two-mode data

The earliest example of two-mode data studied was in [25], where a linear combination of $\{2, 2, 0, \frac{3}{L}\}_s$ and $\{4, 4, 0, \frac{5}{L}\}_s$ were considered. As usual, the solutions remain regular at the second order. At the third order, out of the four resonant modes, only the highest $\{6, 6, 0, \frac{7}{L}\}_s$ mode gives rise to a irremovable secular resonance.

Many other two-mode data which were studied in [12], [28] also lead to irremovable resonances. For eq., if one takes $\{4, 4, 0, \frac{5}{L}\}$ and $\{6, 6, 0, \frac{7}{L}\}$, the modes with frequencies $\frac{3}{L}$ and $\frac{9}{L}$, which do not match with any of the linear seed are irremovable. In fact, apart from satisfying the following resonant condition

$$\omega = \tilde{\omega}_1 + \tilde{\omega}_2 - \tilde{\omega}_3, \quad m = m_1 + m_2 - m_3 \tag{IV.165}$$

(here $\tilde{\omega}_i$ are the frequencies present in the linear seed), only when the frequency ω has the same quantum numbers as that of the initial seed, can a resonance be removed through a frequency shift of the form (IV.162).

IV.3.2 Biaxial case

A set up, in which one could break the assumptions about spherical symmetry, while still retaining some symmetries is that of the cohomogeneity-two biaxial Bianchi IX ansatz in five dimensions [77], [78]

$$ds^2 = \frac{L^2}{\cos^2 x}\left(-Ae^{-2\delta}dt^2 + A^{-1}dx^2 + \frac{1}{4}\sin^2 x(e^{2B}(\sigma_1^2 + \sigma_2^2) + e^{-4B}\sigma_3^2)\right) \tag{IV.166}$$

where x was related to the radial coordinate as $r = L\tan x$ and A, δ and B were functions of (t, x). The domain of the coordinates were $t \in (-\infty, \infty)$ and $x \in (0, \pi/2]$. The σ_k were given by

$$\sigma_1 + i\sigma_2 = e^{i\psi}(\cos\vartheta d\varphi + id\vartheta), \quad \sigma_3 = d\psi - \sin\vartheta d\varphi \tag{IV.167}$$

where $\vartheta \in [0, \pi]$ and $\psi, \varphi \in [0, 2\pi]$. One can get back the 3−sphere metric by putting $B = 0$. The evolution of the system is governed by vacuum Einstein equation in five dimensions,

$$R_{\mu\nu} + \frac{4}{L^2}g_{\mu\nu} = 0 \tag{IV.168}$$

Note that, in this case, the auxiliary variable B acted like the massless scalar field in [8].

In [78], a perturbative approach was used to study the resonant system of this system upto the third order in perturbation theory. It was seen that there were no secular resonances at the second order. The detailed recurrence relations for the interaction coefficients at the third order can be found in

[78].

Additionally, because of the symmetries involved in this set-up, it was still possible to treat it as a problem in $1+1-$dimensions in (t,x) coordinates . This problem was studied numerically by Bizon and Rostworowski in [77]. It was seen that the Kretschmann scalar at the origin $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ grew exponentially with time. Thus, it was conjectured that AdS_5 is unstable against black hole formation for arbitrarily small purely gravitational perturbations for cohomogeneity-two biaxial Bianchi IX ansatz.

IV.3.3 Analysis of secular resonances at second order in AdS_5

In this section, we will partially analyze the nature of resonances at the second order for a single mode data in five dimensions. The key feature in five and higher dimensions is that now we have tensor-type of perturbations as well. Since the scalar and the vector sector has alredy been explored somewhat in four dimension, this time we will check for resonances in the tensor sector.

Suppose we assume that at the linear order only a single mode is present with a frequency $\tilde{\omega}$. We know that, in this case, the only nontrivial frequency which gets excited is $2\tilde{\omega}$. The reason this doesn't lead to a secular resonance in four dimensions is because, no matter what kind of single mode data we take, there is no value of ω at the second order for which $\omega = 2\tilde{\omega}$. In five dimensions it is not that straightforward. Since this argument fails in five dimensions, we need a detailed analysis similar to that done in [78].

Five dimensions corresponds to $n = 3$. This means that any harmonic would have three quantum numbers associated with it. We will denote it as $\mathbf{k} = \{l, l^{(1)}, m\}$. We will consider the simplest type of tensor-type single mode data, which is

$$\tilde{\mathbf{k}} = \{l = 2, l^{(1)} = 2, m = 0\} \tag{IV.169}$$

From (IV.37) we see that the frequency corresponding to this harmonic is going to have the value $\tilde{\omega}L = 2p' + 6$. Note that if we start with tensor-type perturbations at linear level, at subsequent levels, all three-tensor, vector and scalar get excited as well. Here, we will be restricting ourselves to tensor sector though.

In such a specialized case, many harmonic excitations are possible at the second order in the tensor sector. We will specifically check for the presence of a tensor harmonic, whose quantum numbers are same as that of the initial mode data. Which means we will take the projection of (IV.41) on a tensor harmonic $\mathbb{T}^{ij}_{\mathbf{k}}$ with $\mathbf{k} = \tilde{\mathbf{k}}$. Then the R.H.S of (IV.41) is given by

$$< r^{-1/2} f \int \mathbb{T}^{ij}_{\tilde{\mathbf{k}}} \, {}^{(2)}S_{ij} d^3\Omega, e_{p,2} > = < r^{-1/2} f \int \mathbb{T}^{ij}_{\tilde{\mathbf{k}}} \, {}^{(2)}A_{ij} d^3\Omega, e_{p,2} > \tag{IV.170}$$

The above replacement of ${}^{(2)}S_{ij}$ with ${}^{(2)}A_{ij}$ is possible because of the traceless property of the tensor harmonics, $\mathbb{T}^i_i = 0$. We can simolify the expression for ${}^{(2)}A_{ij}$ by writing it in terms of the $\bar{D}_a$ and $\bar{D}_i$ operators and substituting for h_{ij} in it. The detailed calculations for calculating ${}^{(2)}A_{\mu\nu}$, when only tensor type perturbations are present at the linear level is given in Appendix A. From there, we use (A.9), which gives us the expression for ${}^{(2)}A_{ij}$ when only tensor type perturbations are present at linear level. This is as follows:

$$
\begin{aligned}
{}^{(2)}A_{ij} = H_{T\mathbf{k_1}} H_{T\mathbf{k_2}} \Big(& \mathbb{T}^{kl}_{\mathbf{k_1}} (-\bar{D}_i \bar{D}_j \mathbb{T}_{kl_{\mathbf{k_2}}} + \bar{D}_k \bar{D}_i \mathbb{T}_{jl_{\mathbf{k_2}}} + \bar{D}_k \bar{D}_j \mathbb{T}_{li_{\mathbf{k_2}}} \\
& - \bar{D}_k \bar{D}_l \mathbb{T}_{ij_{\mathbf{k_2}}}) - \frac{\bar{D}_i \mathbb{T}^{kl}_{\mathbf{k_1}} \bar{D}_j \mathbb{T}_{kl_{\mathbf{k_2}}}}{2} + \bar{D}_k \mathbb{T}^{l}_{i_{\mathbf{k_1}}} \bar{D}_l \mathbb{T}^{k}_{j_{\mathbf{k_2}}} - \bar{D}^k \mathbb{T}_{il_{\mathbf{k_1}}} \bar{D}_k \mathbb{T}^{l}_{j_{\mathbf{k_2}}} \Big) \\
& - r \bar{D}^a r \bar{D}_a H_{T\mathbf{k_1}} H_{T\mathbf{k_2}} \gamma_{ij} \mathbb{T}^{kl}_{\mathbf{k_1}} \mathbb{T}_{kl_{\mathbf{k_2}}} - r^2 \bar{D}^a H_{T\mathbf{k_1}} \bar{D}_a H_{T\mathbf{k_2}} \mathbb{T}_{ik_{\mathbf{k_1}}} \mathbb{T}^{k}_{j_{\mathbf{k_2}}}
\end{aligned} \tag{IV.171}
$$

(Since we are considering a single mode data, we can omit any summation sign in the above expression. Moreover, here we have $\mathbf{k}_1 = \mathbf{k}_2 = \tilde{\mathbf{k}}$). We will first tackle the kind of integral, arising as a result of a term like $H_{\mathbf{k}_1} H_{\mathbf{k}_2}$ in ${}^{(2)}A_{ij}$. If the angular integral multiplying this term vanishes, then of course, the whole term will be zero. We will assume here that the angular integral does not vanish. In such a case, we show that the $r-$integral will definitely vanish. For this we define a new coordinate y, which is defined as

$$
r^2 = L^2 \frac{(1+y)}{(1-y)} \tag{IV.172}
$$

Then our integral of interest is

$$
\begin{aligned}
\int_0^\infty r^{-1/2} (H_{T_{p',l=2}})^2 e_{p,2} dr \sim \int_{-1}^{1} (1-y)^2 [P^{2,3}_{p'}(y)]^2 \frac{d^p}{dy^p} [(1-y)^{p+2} (1+y)^{p+3}] dy \\
\times \ \cos^2(\tilde{\omega} t)
\end{aligned} \tag{IV.173}
$$

where we have used relation (IV.16). Accordingly,

$$
H_{T\tilde{\mathbf{k}},p}(y) \sim (1-y)^2 (1+y) P^{(2,3)}_p(y) \cos \tilde{\omega} t \tag{IV.174}
$$

$$
e_{2p+3,2}(y) \sim (1+y)^{7/4} (1-y)^{5/4} P^{(2,3)}_{2p+3}(y) \tag{IV.175}
$$

and the Jacobi polynomial in (IV.173) is rewritten using the formula

$$P_p^{(\alpha,\beta)}(y) \sim (1-y)^{-\alpha}(1+y)^{-\beta}\frac{d^p}{dy^p}\left[(1-y)^{p+\alpha}(1+y)^{p+\beta}\right] \tag{IV.176}$$

As discussed before, the only frequency excited at the second order is $2\tilde{\omega}$. Hence, the frequencies ω, for which $\omega_{p,l=2} = 2\tilde{\omega}$ satisfy

$$p = 2p' + 3 \tag{IV.177}$$

Now if in (IV.173), we perform integration by parts, we find that each time we get boundary terms which vanish at the boundaries $y = \pm 1$. Moreover, in the resultant integral left, the derivative operator acts $p = 2p' + 3$ times on the term

$$(1-y)^2[P_{p'}^{(2,3)}]^2$$

which is a polynomial of degree $2p' + 2$. Thus, this integral as a whole vanishes. This method was also used in [78] to prove that the interaction coefficients at second order are zero.

The second kind of integral arises because of the term

$$-r\bar{D}^a r\bar{D}_a H_{T\mathbf{k}_1} H_{T\mathbf{k}_2}\gamma_{ij}\mathbb{T}^{kl}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2 kl}$$

in ${}^{(2)}A_{ij}$. Since this proportional to γ_{ij}, hence when we take projection of $\mathbb{T}_{\mathbf{k}^{ij}}$ on it, the integrand will have $\mathbb{T}^i_i$, which is zero. Hence, the contribution of this term to (IV.170) is also zero.

Finally, we are left with the contribution due to the term $r^2\bar{D}^a H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}$.

Hence, (IV.47) now becomes

$$ {}^{(2)}\ddot{c}_{p,\tilde{\mathbf{k}}} + \omega_{p,l=2}^{2} {}^{(2)}c_{p,\tilde{\mathbf{k}}} = \int_{S^3} \mathbb{T}^{ij}_{\tilde{\mathbf{k}}} \mathbb{T}_{\tilde{\mathbf{k}}ik} \mathbb{T}^{k}_{\tilde{\mathbf{k}}j} d^3\Omega \, \left\langle r^{3/2} f \bar{D}^a H_{T\tilde{\mathbf{k}},p} \bar{D}_a H_{T\tilde{\mathbf{k}},p}, e_{2p+3,2}(r) \right\rangle \tag{IV.178} $$

The explicit forms of the tensor harmonics in five dimensions which has been used to evaluate the angular integral is given in appendix B. When evaluated, this integral yields a non-zero quantity:

$$ \int_{S^3} \mathbb{T}^{ij}_{\tilde{\mathbf{k}}} \mathbb{T}_{ik\tilde{\mathbf{k}}} \mathbb{T}^{k}_{j\tilde{\mathbf{k}}} d^3\Omega = -\frac{379}{210\sqrt{6}\pi} \tag{IV.179} $$

Hence, a harmonic with $\mathbf{k} = \tilde{\mathbf{k}}$ indeed gets excited. Next we consider the term

$$ \begin{aligned} &\left\langle r^{3/2} f \bar{D}^a H_{T\tilde{\mathbf{k}},p} \bar{D}_a H_{T\tilde{\mathbf{k}},p}, e_{2p+3,2}(r) \right\rangle \\ &\qquad = \int_0^\infty r^{3/2} e_{2p+3,2} \left(\frac{1}{f} (\partial_t H_{T\tilde{\mathbf{k}},p})^2 - f (\partial_r H_{T\tilde{\mathbf{k}},p})^2 \right) dr \end{aligned} \tag{IV.180} $$

By writing this integral in y coordinate and by using (IV.174) and (IV.175), we obtain

$$
\begin{aligned}
&\int_0^\infty r^{3/2} e_{2p+3,2}\left(\frac{1}{f}(\partial_t H_{T\tilde{\mathbf{k}},p})^2 - f(\partial_r H_{T\tilde{\mathbf{k}},p})^2\right) dr \\
&= a_1 \int_{-1}^{1} (1+y)^3(1-y) P_{2p+3}^{(2,3)}(y) \Big[(p+3)^2(1+y)(1-y)^3 (P_p^{(2,3)})^2 \sin^2\tilde{\omega}t \\
&\quad - \left(\partial_y[(1-y)^2(1+y)P_p^{(2,3)}]\right)^2 \cos^2\tilde{\omega}t\Big] dy \\
&= a_1 \int_{-1}^{1} (1+y)^3(1-y) P_{2p+3}^{(2,3)}(y) \Big[(p+3)^2(1+y)(1-y)^3 (P_p^{(2,3)})^2 \sin^2(\tilde{\omega}t) \\
&\quad - B(y)\cos^2(\tilde{\omega}t)\Big] dy \\
&= a\sin^2\tilde{\omega}t - b\cos^2\tilde{\omega}t \qquad \text{(IV.181)}
\end{aligned}
$$

where a_1 denotes the common constant factors in the integral. a and b are

$$
a = a_1 \int_{-1}^{1} (p+3)^2(1+y)^4(1-y)^4 P_{2p+3}^{(2,3)}(y)(P_p^{(2,3)})^2 \qquad \text{(IV.182)}
$$

$$
b = a_1 \int_{-1}^{1} (1+y)^3(1-y) P_{2p+3}^{(2,3)}(y) B(y) dy \qquad \text{(IV.183)}
$$

with

$$
B(y) = (\partial_y[(1-y)^2(1+y)P_p^{(2,3)}])^2. \qquad \text{(IV.184)}
$$

Now we will use the following relations for Jacobi polynomials to show that $a = -b$. The relevant relations are

$$
\frac{d}{dy} P_p^{(\alpha,\beta)}(y) = \frac{1}{2}(p+\alpha+\beta+1) P_{p-1}^{(\alpha+1,\beta+1)} \qquad \text{(IV.185)}
$$

$$P_p^{(\alpha,\beta-1)}(y) - P_p^{(\alpha-1,\beta)}(y) = P_{p-1}^{(\alpha,\beta)}(y) \tag{IV.186}$$

$$(1-y)P_p^{(\alpha+1,\beta)}(y) + (1+y)P_p^{(\alpha,\beta+1)}(y) = 2P_p^{(\alpha,\beta)}(y) \tag{IV.187}$$

$$(2p+\alpha+\beta+1)P_p^{(\alpha,\beta)}(y) = (p+\alpha+\beta+1)P_p^{(\alpha,\beta+1)}(y) + (p+\alpha)P_{p-1}^{(\alpha,\beta+1)}(y) \tag{IV.188}$$

We can now simplify the function $B(y)$ in the following way:

$$\begin{aligned}
B(y) &= \left\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + (1+y)(1-y)^2\frac{d}{dy}P_p^{(2,3)}\right\}^2 \\
&= \left\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + \frac{1}{2}(p+6)(1+y)(1-y)^2P_{p-1}^{(3,4)}\right\} \\
&= \left\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + \frac{1}{2}(p+6)(1+y)(1-y)^2[P_p^{(3,3)} - P_p^{(2,4)}]\right\}^2 \\
&= \Big\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + \frac{1}{2}(p+6)(1-y^2)[(1-y)P_p^{(3,3)} \\
&\quad - (1-y)P_p^{(2,4)}]\Big\}^2 \\
&= \Big\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + \frac{1}{2}(p+6)(1-y^2)[2P_p^{(2,3)} \\
&\quad - (1+y)P_p^{(2,4)} - (1-y)P_p^{(2,4)}]\Big\}^2 \\
&= \left\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + (p+6)(1-y^2)[P_p^{(2,3)} - P_p^{(2,4)}]\right\}^2 \\
&= \Bigg\{[-2(1-y^2)+(1-y)^2]P_p^{(2,3)} + (p+6)(1-y^2)\Bigg[P_p^{(2,3)} - \frac{(2p+6)}{(p+6)}P_p^{(2,3)} \\
&\quad + \frac{(p+2)}{(p+6)}P_{p-1}^{(2,4)}\Bigg]\Bigg\}^2 \\
&= \left\{(1-y)\left[2-(p+3)(1+y)\right]P_p^{(2,3)} + (p+2)(1-y^2)P_{p-1}^{(2,4)}\right\}^2
\end{aligned} \tag{IV.189}$$

where we have used identities (IV.185), (IV.186), (IV.187) and (IV.188) in the second, third, fifth and seventh steps of (IV.189) respectively.

We will now substitute this final form of $B(y)$ in (IV.183) and also make the substitution for $P^{(2,3)}_{2p+3}$ using (IV.176), to obtain the following form for b:

$$
\begin{aligned}
b = a_1 \int_{-1}^{1} (1+y)^3(1-y)P^{(2,3)}_{2p+3}B(y)dy \\
= a_1 \int_{-1}^{1} (1-y)^{-1}B(y)\frac{d^{2p+3}}{dy^{2p+3}}[(1-y)^{2p+5}(1+y)^{2p+6}]dy
\end{aligned}
\tag{IV.190}
$$

We use the fact that performing integration by parts in (IV.190) only those polynomials in $(1-y)^{-1}B(y)$ which have a degree more than or equal to $2p+3$ will have non-zero contribution to the integral. Then, integral b can be written as

$$
\begin{aligned}
b &= a_1 \int_{-1}^{1} (1+y)^3(1-y)P^{(2,3)}_{2p+3}B(y)dy \\
&= a_1 \int_{-1}^{1} (1+y)^3(1-y)P^{(2,3)}_{2p+3}\left[(p+3)^2(1-y)^2y^2(P^{(2,3)}_p)^2\right]dy \\
&= a_1 \int_{-1}^{1} (1+y)^3(1-y)^3P^{(2,3)}_{2p+3}(p+3)^2y^2(P^{(2,3)}_p)^2dy \\
&\quad - a_1 \int_{-1}^{1} (1+y)^3(1-y)^3P^{(2,3)}_{2p+3}(p+3)^2(P^{(2,3)}_p)^2dy \\
&= a_1 \int_{-1}^{1} (1+y)^3(1-y)^3P^{(2,3)}_{2p+3}(p+3)^2(y^2-1)(P^{(2,3)}_p)^2dy \\
&= -a_1 \int_{-1}^{1} (1+y)^4(1-y)^4P^{(2,3)}_{2p+3}(p+3)^2(P^{(2,3)}_p)^2dy \\
&= -a
\end{aligned}
\tag{IV.191}
$$

the addition of an extra term in the third step doesn't affect the original integral as this additional term is essentially zero. We can see it by using

(IV.176), and rewriting this term in the following manner:

$$-a_1\int_{-1}^{1}(1+y)^3(1-y)^3P^{(2,3)}_{2p+3}(p+3)^2(P^{(2,3)}_p)^2dy$$
$$= -a_1\int_{-1}^{1}(p+3)^2(1-y)(P^{(2,3)}_p(y))^2\frac{d^{2p+3}}{dy^{2p+3}}\left[(1-y)^{2p+5}(1+y)^{2p+6}\right] \tag{IV.192}$$

Using the argument we previously used, since the term $(p+3)^2(1-y)(P_p^{(2,3)}(y))^2$ has a degree $2p+1$, less than $2p+3$, doing integration by parts will make this integral vanish. Hence, finally, equation (IV.178) becomes

$$^{(2)}\ddot{c}_{p,\tilde{\mathbf{k}}} + \omega^2_{p,l=2}\,{}^{(2)}c_{p,\tilde{\mathbf{k}}} = a \tag{IV.193}$$

Therefore, for the example we considered, there are no secular resonances at the third order. This is consistent with the previous results in literature where so far no resonances have been found at the second order. A full analysis, of course needs to include perturbative analysis of scalar and vector sector, for various kinds of initial data.

But as we can see, in higher dimensions, the labels attached to spherical harmonics also increase proportionately. This makes such a brute force approach very cumbersome and unappealing. If it is true that the resonances at second order are always absent, then we hope to have a more elegant derivation for the same in future.

IV.4 Geons and Black Resonators

We will now review the work by various authors, which deals with construction of geons and black hole resonators. As we saw in the previous sections, even though the perturbation theory breaks down for generic initial data, it is possible to sustain the theory to higher orders, if one starts with a single mode. The solutions so obtained are horizonless and time-periodic and are called oscillons in case of a real field, boson stars in case of a complex scalar field and geons for the pure gravity case. One feature of the geon is that it is invariant under a Killing vector $K = \partial_t + \frac{\omega}{m}\partial_\phi$, where ω is an AdS normal mode, m is the azimuthal quantum number and t and ϕ are the time and azimuthal coordinates. respectively. At the lowest order, the energy E and angular momentum J are related as $\frac{E}{J} = \frac{\omega}{Lm}$

In [34], a new family of black holes were constructed, which joins the onset of superradiant instability of Kerr-AdS to the geons. Similar to geons, black resonators were also time-periodic and possessed a single killing vector. While the only Kerr-AdS black holes which were stable were the ones with $\Omega_H L \leq 1$, the black resonators constructed in [34] all had $\Omega_H L > 1$ (here Ω_H is the angular velocity of the horizon). Although resonators had more entropy than the Kerr-AdS black holes, they still could not have been the possible end state of the superradiant instability as they were themselves unstable. Only those black resonators with $l = m = 2$ modes as the progenitors were stable. But even they are unstable to higher m modes. The only way that a black resonator could be a possible end point to superradiant instability is if in a certain limit $\Omega_H L \to 1$. This would happen if a black resonator with small energy and an azimuthal number m approached the limit $m \to \infty$. In [35], it was proved that such limiting resonator did not exist.

Black resonators and geons have also been constructed in five dimensions under certain symmetry considerations [113], [114]. Geons can also be constructed from the AdS solition, which is a locally asymptotically AdS spacetime [32], [33]. Note that, even though AdS soliton does not have a resonant spectrum, one does need to add higher order correction to its normal modes so as to make the solutions regular and normalizable upto all orders of perturbation theory.

IV.5 Conclusions

In this chapter, we have done a systematic study of the nonlinear theory of gravitational perturbations of AdS_{n+2}, with $n > 2$. We see that at each order, one needs to make a suitable gauge choice, so as to render the metric perturbations asymptotically AdS. Once, we obtain the perturbation equations for each order in ϵ, we are ready to study the resonant system. Since most of the previous studies have been centred around AdS_4 perturbations, where the focus was to study scalar and vector-type sectors, we go one dimension higher, and study the tensor sector instead. We take a single-mode tensor-type initial data and show that for such a linear seed, there are no secular resonances at the second order. This is consistent with the previous studies, where so far, no secular growth is seen at the second order, the irremovable resonances only enter at the third order. Since, computations become much more tedious as we move to higher dimensions, we have restricted the study only till the second order.

Chapter V

Conclusions

In this thesis, we have studied the AdS instability as well as nonlinear instability of similar gravitational systems. Our primary motivation came from the very important work by Bizon and Rostworowski [8], where it was conjectured that AdS might be unstable against black hole formation for arbitrarily small perturbations. Subsequent works involved mimicking a setting similar to AdS and studying the evolution of scalar fields in such a background. For eg., in [17] and [18], the massless scalar fields were evolved within a cavity in Minkowski under Dirichlet and Neumann boundary conditions. Similar studies were performed for massive scalar fields in [19]. The results from these numerical results tell us that the primary mechanism of instability was due to cascade of energy from low frequency to high frequency modes. And yet there were differences as well. Such systems usually possessed either a resonant spectrum or an asymptotically resonant one. The primary difference between these two cases seems to be that in the latter, there was a minimum threshold amplitude, below which the system remained stable. The numerical observations in these works prompted us to study the necessary conditions required for an AdS-like instability to take place in bounded domains and to ask what the relation between the linear level spectra and nonlinear (in)stability is.

In the first part, which is chapter II, we studied the Einstein-scalar field system in the weakly nonlinear perturbation theory. At the linear level, the system could be described by an integrable Hamiltonian H_0 of n decoupled harmonic oscillators. Here n is large and denotes the modes which participate significantly in the energy cascade. As it is well known, by the Liouville-Arnold theorem, this system could then be described by action-angle variables which are $2n$ in number. However, the dynamics in phase space would be restricted to a $n-$dimensional torus and H_0 would be a function of the action variables only. When we go to higher orders in perturbation theory, any nonlinear perturbation in the gravitational system which couples the harmonic oscillators, could be viewed as perturbing the Hamiltonian of these harmonic oscillators. The resulting Hamiltonian does not need to be integrable. We could, in principle, try to make the Hamiltonian integrable through a suitable canonical transformation of the action-angle variables to an alternate set of variables, upto the desired order of perturbation theory. However, this has its own challenges. In order to perform a canonical transformation, one requires a generating function. We can fail to construct a generating function because of the "small denominator problem". This happens in two cases— a) when the frequencies are either resonant, in which case they don't satisfy a Diophantine condition (II.75) or b) when the Diophantine (non-resonant) frequencies are nearly resonant. Inability to obtain an integrable Hamiltonian would mean that the dynamics will no longer be restricted on an $n-$torus. This manifests as transfer of energy across the modes, which is a characteristic feature in systems exhibiting an AdS-type instability.

The next question we addressed was—what can we say about the stability of a system whose spectrum satisfied the Diophantine condition? For this, we used the results in nonlinear dynamics which implied that the action variable of a system remained close to its initial value for exponentially long times, if the amplitude of perturbation was smaller than a threshold amplitude ϵ_{th} [92]. This threshold amplitude would in turn be dependent on the Diophantine frequencies. These arguments can also explain as to why in systems with non-resonant, albeit asymptotically resonant spectrum, there is a threshold amplitude below which the system remains stable. This also explains why if the initial condition contains large mode frequencies, this threshold amplitude could be too small to be detected in numerical simulations.

We also pointed out that the statements above were stability results rather than instability results. This meant that a system with non-resonant spectrum would be stable for exponentially long times, provided the amplitude of perturbation was below a minimum threshold amplitude as given by the theorem by Benettin and Gallavotti. However, there was no guarantee that a system with a resonant spectrum would necessarily be unstable. However, because of the small denominator problem, instability was more likely. An example where the linear level spectrum was resonant and yet no instability was observed was the presence of non-collapsing solutions for certain initial conditions in case of the Einstein-massless scalar field system [8], [23] (these initial conditions are mostly single-mode dominated).

The next step was to look for the necessary conditions for black hole formation. The resonant energy transfer causes the energy to transfer to finer spatial scales. It was hence desirable to see how the linear eigenfunctions corresponding to high modes localize in space. Our analysis included using the Mehler-Heine formula and the Darboux formula for Jacobi polynomials, to study how high mode eigenfunctions localize across various dimensions of spacetime. We found that the localization came out to be least for AdS_3, which is consistent with the fact that there is no black hole formation in AdS_3. We also gave an illustrative demonstration as to how one can apply these arguments for locally asymptotically AdS spacetimes as well.

In the second part (chapter III), we gave a systematic way to analyze and apply the results in KAM theory to the pure gravity case, where all the gravitational degrees of freedom are excited. We took up a region in flat space which is enclosed by a Dirichlet wall and considered small gravitational perturbations trapped inside this cavity. Our motive was to study the nonlinear stability of the system (the linear stability was proved in [81]). The spherical symmetry of the background Minkowski metric allows one to use the Ishibashi-Kodama-Seto formalism [99] to considerably simplify the perturbation equations. Although the KIS formalism was developed to simplify linear perturbations, we extended it to higher orders of perturbation theory as well. The idea is to classify the metric perturbations on the basis of their tensorial behavior on $n-$sphere—scalar-type, vector-type and the tensor-type. It turns out that the linear level eigenfunctions for each sector are complete and the linear spectrum is asymptotically resonant. We simplified the higher order equations for this system. We demonstrated that the results of the previous paper containing arguments from nonlinear dynamics could be applied to this system to comment on its nonlinear stability. Hence, this system is stable for small enough perturbations.

Finally, in chapter IV, we did a systematic study of gravitational perturbations of AdS in $(n+2)$ dimensions with $n > 2$. As shown by Henneaux and Teitelboim [107], [108], [109], metric of asymptotically AdS spacetimes satisfy certain fall off and we want our perturbation to also satisfy the same.

Thus, we make use of the gauge freedom to render the metric perturbations aAdS at each order. The source terms ${}^{(i)}S_{\mu\nu}$ which are made of lower order metric perturbations, too will exhibit a certain fall off as $r \to 0$. Once we determined this fall off, we proceeded to simplify the linear level and the higher order equations pertaining to each sector. Once we had obtained the perturbation equation, the next step was to do a perturbative analysis of these equations order by order. The main feature of spacetime dimensions greater than four is the presence of tensor-type perturbations, along with scalar-type and vector-type. Previous studies involved studying the AdS_4 perturbations, where one started with a single-mode or a two-mode initial data which is of scalar and/or vector-type. In all these cases, it was observed there are no secular resonances at the second order. For eg., for a single-mode data with a source frequency $\tilde{\omega}$, the excited frequency at second order is $2\tilde{\omega}$. But for all the examples of single-mode initial data taken up in [28], one can see that there is no scalar frequency or vector frequency equal to $2\tilde{\omega}$. In case of AdS_5 perturbations for the cohomogeneity biaxial Bianchi IX ansatz [78], the secular resonances corresponding to $2\tilde{\omega}$ are absent because of vanishing $r-$dependent interaction coefficients. Here, we analyzed the tensor sector of the pure gravity system, with no assumptions of any symmetries. We took a specific case of a single-mode tensor type initial data, and analytically proved that there are indeed no resonances at the second order. One of the main shortcomings of this method is the brute force technique one has to employ to check for presence/absence of resonances at various orders for different kinds of initial data. This brute force technique is not only impractical but becomes more tedious as we move to higher spacetime dimensions. Moreover, as mentioned earlier, so far no resonances have arisen at the second order of perturbation theory. Hence, it would be very interesting to rigorously prove the absence of resonances at second order as well as gain a better understanding as to why it happens.

Appendix A

Calculations for second order source terms for a single-mode tensor-type initial data

In this section we will evaluate the source terms ${}^{(2)}A_{\mu\nu}$ as defined in (III.78) in terms of the $\bar{D}_a$ and $\bar{D}_i$ operators for a case where only tensor-type perturbations are excited at the linear order. Upon collecting terms of order ϵ^2 in (III.78), one obtains

$$
\begin{aligned}
{}^{(2)}A_{\mu\nu} = & \frac{1}{2}\bar{\nabla}_\alpha h \left(-\bar{\nabla}^\alpha h_{\mu\nu} + \bar{\nabla}_\nu h^\alpha_\mu + \bar{\nabla}_\mu h^\alpha_\nu\right) \\
& - h^{\lambda\alpha}\Bigg(- \bar{\nabla}_\lambda \bar{\nabla}_\alpha h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h_{\lambda\alpha} + \bar{\nabla}_\lambda \bar{\nabla}_\mu h_{\alpha\nu} + \bar{\nabla}_\alpha \bar{\nabla}_\nu h_{\mu\lambda}\Bigg) \\
& + \frac{\bar{\nabla}_\nu h^{\lambda\alpha} \bar{\nabla}_\mu h_{\lambda\alpha}}{2} - \bar{\nabla}_\lambda h^\alpha_\mu \bar{\nabla}_\alpha h^\lambda_\nu + \bar{\nabla}^\alpha h_{\mu\lambda} \bar{\nabla}_\alpha h^\lambda_\nu \\
& - \bar{\nabla}_\lambda h^{\lambda\alpha}\Bigg(- \bar{\nabla}_\alpha h_{\mu\nu} + \bar{\nabla}_\nu h_{\mu\alpha} + \bar{\nabla}_\mu h_{\alpha\nu}\Bigg)
\end{aligned}
\tag{A.1}
$$

If the source terms only have tensor type perturbations, then since $h = 0$, the terms in first line of (A.1) don't contribute. Similarly, the term in the fourth line of the above expression, with a $\bar{\nabla}_\lambda h^{\lambda\alpha}$ will also be zero. The contribution of the rest of the terms in each of the components of ${}^{(2)}A_{\mu\nu}$ in terms of the operators $\bar{D}_a$ and $\bar{D}_i$ are as follows:

<u>**For** ${}^{(2)}A_{ij}$:</u>

$$
\begin{aligned}
-h^{kl}\bar{\nabla}_k\bar{\nabla}_l h_{ij} &= -h^{kl}\Big[\bar{D}_k\bar{D}_l h_{ij} - \bar{\Gamma}^a_{kl}\bar{\nabla}_a h_{ij} - \bar{\Gamma}^a_{ki}\bar{\nabla}_l h_{aj} - \bar{\Gamma}^a_{kj}\bar{\nabla}_l h_{ia}\Big] \\
&= -h^{kl}\Big[\bar{D}_k\bar{D}_l h_{ij} + r\bar{D}^a r\gamma_{kl}\bar{D}_a h_{ij} - 2(\bar{D}r)^2\gamma_{kl}h_{ij} - (\bar{D}r)^2\gamma_{ki}h_{jl} \\
&\quad - (\bar{D}r)^2\gamma_{jk}h_{il}\Big] \\
&= -\mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_k\bar{D}_l\mathbb{T}_{\mathbf{k}_2 ij}H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} + 2(\bar{D}r)^2\mathbb{T}^k_{\mathbf{k}_1 i}\mathbb{T}_{\mathbf{k}_2 jk}H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
h^{kl}\bar{\nabla}_k\bar{\nabla}_i h_{lj} &= h^{kl}\Big[\bar{D}_k\bar{D}_i h_{lj} + r\bar{D}^a r\gamma_{ik}\bar{D}_a h_{lj} - 2(\bar{D}r)^2\gamma_{ki}h_{lj} - (\bar{D}r)^2\gamma_{kl}h_{ij} \\
&\quad - (\bar{D}r)^2\gamma_{kj}h_{il}\Big] \\
&= \mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_k\bar{D}_i\mathbb{T}_{\mathbf{k}_2 lj}H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} + r\bar{D}^a r H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}\mathbb{T}^k_{\mathbf{k}_1 i}\mathbb{T}_{\mathbf{k}_2 jk} \\
&\quad - (\bar{D}r)^2 H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\mathbb{T}^k_{\mathbf{k}_1 i}\mathbb{T}_{\mathbf{k}_2 jk}
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
-h^{kl}\bar{\nabla}_i\bar{\nabla}_j h_{kl} &= -h^{kl}\Big[\bar{D}_i\bar{D}_j h_{kl} + r\bar{D}^a r\gamma_{ij}\bar{D}_a h_{kl} - 2(\bar{D}r)^2\gamma_{ij}h_{kl} - 2(\bar{D}r)^2 h^k_i h_{jk}\Big] \\
&= -\mathbb{T}^{kl}_{\mathbf{k}_1}D_iD_j\mathbb{T}_{\mathbf{k}_2 kl}H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} - r\bar{D}^a r\gamma_{ij}H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}\mathbb{T}^{kl}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2 kl} \\
&\quad + 2(\bar{D}r)^2 H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\mathbb{T}^k_{\mathbf{k}_1 i}\mathbb{T}_{\mathbf{k}_2 jk}
\end{aligned}
\tag{A.4}
$$

$$
\begin{aligned}
h^{kl}\nabla_k\nabla_j h_{li} = h^{kl}\Big[&\bar{D}_k\bar{D}_j h_{li} + r\bar{D}^a r\gamma_{jk}\bar{D}_a h_{li} - 2(\bar{D}r)^2\gamma_{kj}h_{li} - (\bar{D}r)^2\gamma_{kl}h_{ij} \\
&- (\bar{D}r)^2\gamma_{ki}h_{jl}\Big] \\
= \ &\mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_k\bar{D}_j\mathbb{T}_{\mathbf{k}_2 li}H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} + r\bar{D}^a r H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}\mathbb{T}^k_{\mathbf{k}_1}i\mathbb{T}_{\mathbf{k}_2 jk} \\
&- (\bar{D}r)^2 H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\mathbb{T}_{\mathbf{k}_1 ki}\mathbb{T}^k_{\mathbf{k}_2 j}
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
-\frac{1}{2}\bar{\nabla}_i h^{\lambda\sigma}\bar{\nabla}_j h_{\lambda\sigma} &= -\frac{1}{2}\left[2\bar{\nabla}_i h^{ak}\bar{\nabla}_j h_{ak} + \bar{\nabla}_i h^{kl}\bar{\nabla}_j h_{kl}\right] \\
&= -\frac{1}{2}\left[-2\bar{\Gamma}^a_{mi}h^{km}\bar{\Gamma}^n_{ja}h_{nk} + \bar{D}_i h^{kl}\bar{D}_j h_{kl}\right] \\
&= -\frac{1}{2}\left[\frac{2}{r^2}(\bar{D}r)^2 h^k_i h_{jk} + \bar{D}_i h^{kl}\bar{D}_j h_{kl}\right] \\
&= -\left[\frac{1}{2}\bar{D}_i\mathbb{T}^{\mathbf{k}_1 kl}D_j\mathbb{T}_{\mathbf{k}_2 kl} + (\bar{D}r)^2\mathbb{T}^k_{\mathbf{k}_1 i}\mathbb{T}_{\mathbf{k}_2 jk}\right]H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
-\bar{\nabla}_\sigma h^\lambda_i\bar{\nabla}^\sigma h_{j\lambda} = -&\left[\bar{\nabla}_a h^k_i\bar{\nabla}^a h_{jk} + \bar{\nabla}_k h^a_i\bar{\nabla}^k h_{ja} + \bar{\nabla}_k h^l_i\bar{\nabla}^k h_{jl}\right] \\
= -\Big[&g^{ab}\bar{D}_a h^k_i(\bar{D}_b h_{jk} - \bar{\Gamma}^m_{bj}h_{mk} - \bar{\Gamma}^m_{bk}h_{mj}) - \frac{1}{r^2}\gamma^{nk}\bar{\Gamma}^a_{km}h^m_i\Gamma^l_{na}h_{jl} \\
&+ \frac{1}{r^2}\bar{D}_k h^l_i\bar{D}^k h_{jl}\Big] \\
= -\Big[&\bar{D}^a H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}r^2\mathbb{T}^k_{\mathbf{k}_1 i}\mathbb{T}_{\mathbf{k}_2 jk} + (\bar{D}r)^2 H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\mathbb{T}_{\mathbf{k}_1 ik}\mathbb{T}^k_{\mathbf{k}_2 j} \\
&+ H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\bar{D}_k\mathbb{T}^l_{\mathbf{k}_1 i}D^k\mathbb{T}_{\mathbf{k}_2 jl}\Big]
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
\bar{\nabla}_\lambda h_i^\mu \bar{\nabla}_\mu h_j^\lambda &= \bar{\nabla}_a h_i^k \bar{\nabla}_k h_j^a + \bar{\nabla}_k h_i^a \bar{\nabla}_a h_j^k + \bar{\nabla}_k h_i^l \bar{\nabla}_l h_j^k \\
&= \bar{D}_a h_i^k \bar{\Gamma}^a_{km} h_j^m + \bar{D}_a h_j^k \bar{\Gamma}^a_{km} h_i^m + \bar{D}_k h_i^l \bar{D}_l h_j^k \\
&= -\frac{1}{r} \bar{D}^a r \bar{D}_a h_i^k h_{jk} - \frac{1}{r} \bar{D}^a r \bar{D}_a h_j^k h_{ik} + \bar{D}_k h_i^l \bar{D}_l h_j^k \\
&= -\mathbb{T}^k_{\mathbf{k_1} i} \mathbb{T}_{\mathbf{k_2} jk} r \bar{D}^a r H_{T\mathbf{k_2}} \bar{D}_a H_{T\mathbf{k_1}} - \mathbb{T}^k_{\mathbf{k_1} i} \mathbb{T}_{\mathbf{k_2} jk} r \bar{D}^a r H_{T\mathbf{k_1}} \bar{D}_a H_{T\mathbf{k_2}} \\
&\quad + \bar{D}_k \mathbb{T}^l_{\mathbf{k_1} i} \bar{D}_l \mathbb{T}^k_{\mathbf{k_2} j} H_{T\mathbf{k_1}} H_{T\mathbf{k_2}} \qquad \text{(A.8)}
\end{aligned}
$$

Adding all the individual pieces (A.2-A.8) together, we obtain the following expression for ${}^{(2)}A_{ij}$

$$
\begin{aligned}
{}^{(2)}A_{ij} = &\sum_{\mathbf{k_1}} \sum_{\mathbf{k_2}} H_{T\mathbf{k_1}} H_{T\mathbf{k_2}} \Big(T^{kl}_{\mathbf{k_1}} (-D_i D_j \mathbb{T}_{kl_{\mathbf{k_2}}} + D_k D_i \mathbb{T}_{jl_{\mathbf{k_2}}} + D_k D_j \mathbb{T}_{li_{\mathbf{k_2}}} \\
&- D_k D_l \mathbb{T}_{ij_{\mathbf{k_2}}}) - \frac{D_i \mathbb{T}^{kl}_{\mathbf{k_1}} D_j \mathbb{T}_{kl_{\mathbf{k_2}}}}{2} + D_k \mathbb{T}^l_{i_{\mathbf{k_1}}} D_l \mathbb{T}^k_{j_{\mathbf{k_2}}} - D^k \mathbb{T}_{il_{\mathbf{k_1}}} D_k \mathbb{T}^l_{j_{\mathbf{k_2}}} \Big) \\
&- r D^a r D_a H_{T\mathbf{k_1}} H_{T\mathbf{k_2}} \gamma_{ij} \mathbb{T}^{kl}_{\mathbf{k_1}} \mathbb{T}_{kl_{\mathbf{k_2}}} - r^2 D^a H_{T\mathbf{k_1}} D_a H_{T\mathbf{k_2}} \mathbb{T}_{ik_{\mathbf{k_1}}} \mathbb{T}^k_{j_{\mathbf{k_2}}} \qquad \text{(A.9)}
\end{aligned}
$$

For ${}^{(2)}A_{ai}$:

$$
\begin{aligned}
-h^{kl} \bar{\nabla}_k \bar{\nabla}_l h_{ai} &= -h^{kl} \left[\partial_k (\bar{\nabla}_l h_{ai}) - \bar{\Gamma}^\lambda_{kl} \bar{\nabla}_\lambda h_{ai} - \bar{\Gamma}^\lambda_{ka} \bar{\nabla}_l h_{\lambda i} - \bar{\Gamma}^\lambda_{ki} \bar{\nabla}_l h_{a\lambda} \right] \\
&\quad - h^{kl} \left[\partial_k (-\bar{\Gamma}^m_{la} h_{mi}) + \bar{\Gamma}^m_{kl} \bar{\Gamma}^n_{ma} h_{ni} - \bar{\Gamma}^m_{ka} \bar{D}_l h_{mi} + \bar{\Gamma}^m_{ki} \bar{\Gamma}^n_{la} h_{nm} \right] \\
&= h^{kl} \left[\frac{1}{r} \bar{D}_a r \bar{D}_l h_{ik} + \frac{1}{r} \bar{D}_a r \bar{D}_k h_{il} \right] \\
&= 2 h^{kl} \left[\frac{1}{r} \bar{D}_a r \bar{D}_l h_{ik} \right] \\
&= \frac{2}{r} \bar{D}_a r H_{T\mathbf{k_1}} H_{T\mathbf{k_2}} \mathbb{T}^{kl}_{\mathbf{k_1}} D_l \mathbb{T}_{\mathbf{k_2} ik} \qquad \text{(A.10)}
\end{aligned}
$$

$$\begin{aligned}
-h^{kl}\nabla_a\nabla_i h_{kl} &= -h^{kl}\left[\partial_a(\bar{\nabla}_i h_{kl}) - \bar{\Gamma}^{\lambda}_{ai}\nabla_\lambda h_{kl} - \bar{\Gamma}^{\lambda}_{ak}\bar{\nabla}_i h_{\lambda l} - \bar{\Gamma}^{\lambda}_{al}\bar{\nabla}_i h_{k\lambda}\right] \\
&= -h^{kl}\left[\bar{D}_a\bar{D}_i h_{kl} - \bar{\Gamma}^{m}_{ai}\bar{\nabla}_m h_{kl} - \bar{\Gamma}^{m}_{ak}\bar{\nabla}_i h_{ml} - \bar{\Gamma}^{m}_{ia}\bar{\nabla}_i h_{km}\right] \\
&= -h^{kl}\left[\bar{D}_a\bar{D}_i h_{kl} - \frac{3}{r}\bar{D}_a r\bar{D}_i h_{kl}\right] \\
&= -\mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_i\mathbb{T}_{\mathbf{k}_2 kl}H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2} + \frac{1}{r}\bar{D}_a r H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_i\mathbb{T}_{\mathbf{k}_2 kl}
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
h^{kl}\bar{\nabla}_k\bar{\nabla}_a h_{li} &= h^{kl}\left[\partial_k(\bar{D}_a h_{li} - 2\frac{1}{r}\bar{D}_a r h_{li}) - \bar{\Gamma}^{\lambda}_{ka}\bar{\nabla}_\lambda h_{li} - \bar{\Gamma}^{\lambda}_{kl}\bar{\nabla}_a h_{\lambda i} - \bar{\Gamma}^{\lambda}_{ki}\bar{\nabla}_a h_{l\lambda}\right] \\
&= h^{kl}\left[\bar{D}_k\bar{D}_a h_{li} - \frac{2}{r}\bar{D}_a r\bar{D}_k h_{li} - \frac{1}{r}\bar{D}_a r\bar{D}_k h_{li}\right] \\
&= h^{kl}\left[\bar{D}_k\bar{D}_a h_{li} - \frac{3}{r}\bar{D}_a r\bar{D}_k h_{li}\right] \\
&= -\mathbb{T}^{kl}_{\mathbf{k}_1}\bar{D}_k\mathbb{T}_{\mathbf{k}_2 il}\left[-H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2} + \frac{1}{r}\bar{D}_a r H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}\right]
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
h^{kl}\nabla_k\nabla_i h_{la} &= h^{kl}\left[\partial_k(-\bar{\Gamma}^{m}_{ia}h_{lm}) - \bar{\Gamma}^{\lambda}_{ki}\bar{\nabla}_\lambda h_{la} - \bar{\Gamma}^{\lambda}_{kl}\bar{\nabla}_i h_{a\lambda} - \bar{\Gamma}^{\lambda}_{ka}\bar{\nabla}_i h_{l\lambda}\right] \\
&= h^{kl}\left[-\frac{1}{r}\bar{D}_a r\partial_k h_{li} + \bar{\Gamma}^{m}_{ki}\bar{\Gamma}^{n}_{ma}h_{ln} + \bar{\Gamma}^{m}_{kl}\bar{\Gamma}^{n}_{ia}h_{mn} - \frac{1}{r}\bar{D}_a r\bar{D}_i h_{kl}\right] \\
&= -\frac{1}{r}\bar{D}_a r h^{kl}\left[\bar{D}_k h_{li} + \bar{D}_i h_{kl}\right] \\
&= -\mathbb{T}^{kl}_{\mathbf{k}_1}(\bar{D}_k\mathbb{T}_{\mathbf{k}_2 li} + \bar{D}_i\mathbb{T}_{\mathbf{k}_2 kl})\frac{1}{r}\bar{D}_a r H_{T\mathbf{k}_1}H_{T\mathbf{k}_2}
\end{aligned} \tag{A.13}$$

$$
\begin{aligned}
-\frac{1}{2}\bar{\nabla}_a h^{\lambda\sigma}\bar{\nabla}_i h_{\lambda\sigma} &= -\frac{1}{2}\bar{\nabla}_a h^{jk}\bar{\nabla}_i h_{jk} \\
&= -\frac{1}{2}\left(\partial_a h^{jk} + 2\bar{\Gamma}^j_{al} h^{lk}\right)\bar{D}_i h_{jk} \\
&= -\frac{1}{2}\left(\bar{D}_a h^{jk} + \frac{2}{r}\bar{D}_a r h^{jk}\right)\bar{D}_i h_{jk} \\
&= -\frac{1}{2}\mathbb{T}^{\mathbf{k}_1 jk}\bar{D}_i \mathbb{T}_{\mathbf{k}_2 jk} H_{T\mathbf{k}_2}\bar{D}_a H_{T\mathbf{k}_1} \qquad \text{(A.14)}
\end{aligned}
$$

$$
\begin{aligned}
-\bar{\nabla}^\sigma h_{a\lambda}\bar{\nabla}_\sigma h^\lambda_i &= -\bar{\nabla}^j h_{ak}\bar{\nabla}_j h^k_i \\
&= \bar{g}^{jl}\bar{\Gamma}^m_{la} h_{mk}\bar{D}_j h^k_i \\
&= \bar{g}^{jl}\frac{1}{r}\bar{D}_a r h_{kl}\bar{D}_j h^k_i \\
&= \frac{1}{r}\bar{D}_a r h^j_k \bar{D}_j h^k_i \\
&= \frac{1}{r}\bar{D}_a r H_{T\mathbf{k}_1} H_{T\mathbf{k}_2}\mathbb{T}^{jk}_{\mathbf{k}_1}\bar{D}_j \mathbb{T}_{\mathbf{k}_2 ik} \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\nabla}_\lambda h^\sigma_a \nabla_\sigma h^\lambda_i &= \bar{\nabla}_j h^k_a \bar{\nabla}_k h^j_i \\
&= -\bar{\Gamma}^m_{aj} h^k_m \bar{D}_k h^j_i \\
&= -\frac{1}{r}\bar{D}_a r h^k_j \bar{D}_k h^j_i \\
&= -\mathbb{T}^{jk}_{\mathbf{k}_1}\bar{D}_k \mathbb{T}_{\mathbf{k}_2 ij}\frac{1}{r}\bar{D}_a r H_{T\mathbf{k}_1} H_{T\mathbf{k}_2} \qquad \text{(A.16)}
\end{aligned}
$$

Putting all the terms from (A.10) to (A.16) together we get

$$
\begin{aligned}
{}^{(2)}A_{ai} = \sum_{\mathbf{k}_1,\mathbf{k}_2} \Bigg\{ & H_{T\mathbf{k}_1} \bar{D}_a H_{T\mathbf{k}_2} \mathbb{T}^{kl}_{\mathbf{k}_1} \Big(- \bar{D}_i \mathbb{T}_{\mathbf{k}_2 kl} + \bar{D}_k \mathbb{T}_{\mathbf{k}_2 il} \Big) \\
& - \frac{1}{2} \bar{D}_a H_{T\mathbf{k}_1} H_{T\mathbf{k}_2} \mathbb{T}^{kl}_{\mathbf{k}_1} \bar{D}_i \mathbb{T}_{\mathbf{k}_2 kl} \Bigg\}
\end{aligned} \tag{A.17}
$$

For ${}^{(2)}A_{ab}$:

$$
\begin{aligned}
& - h^{kl} \bar{\nabla}_k \bar{\nabla}_l h_{ab} = h^{kl} \left[\Gamma^m_{ka} \bar{\nabla}_l h_{bm} + \bar{\Gamma}^m_{kb} \bar{\nabla}_l h_{am} \right] \\
& = - h^{kl} \left[\bar{\Gamma}^n_{ka} \bar{\Gamma}^n_{lb} h_{mn} + \bar{\Gamma}^m_{kb} \bar{\Gamma}^n_{la} h_{mn} \right] \\
& = - \frac{2}{r} \bar{D}_a r \bar{D}_b r h^{kl} h_{kl} \\
& = -2 \mathbb{T}^{ij}_{\mathbf{k}_1} \mathbb{T}_{\mathbf{k}_2 ij} \frac{1}{r^2} \bar{D}_a r \bar{D}_b r H_{T\mathbf{k}_1} H_{T\mathbf{k}_2}
\end{aligned} \tag{A.18}
$$

$$
\begin{aligned}
-h^{kl} \bar{\nabla}_a \nabla_b h_{kl} = & -h^{kl} \Bigg[\partial_a (\bar{D}_b h_{kl} - \bar{\Gamma}^m_{bk} h_{lm} - \bar{\Gamma}^m_{bl} h_{mk}) - \bar{\Gamma}^c_{ab} \bar{\nabla}_c h_{kl} - \bar{\Gamma}^m_{ak} \bar{\nabla}_b h_{ml} \\
& - \bar{\Gamma}^m_{al} \bar{\nabla}_b h_{mk} \Bigg] \\
= & -h^{kl} \Bigg[\bar{D}_a \bar{D}_b h_{kl} - 2 \bar{D}_a (\frac{\bar{D}_b r}{r}) h_{kl} - \frac{2}{r} \bar{D}_b r \partial_a h_{kl} - \bar{\Gamma}^m_{ak} \Big[\bar{D}_b h_{ml} \\
& - \frac{2}{r} \bar{D}_b r h_{ml} \Big] - \bar{\Gamma}^m_{al} \Big[\bar{D}_b h_{mk} - \frac{2}{r} \bar{D}_b r h_{mk} \Big] \Bigg] \\
= & -h^{kl} \Bigg[\bar{D}_a \bar{D}_b h_{kl} - \frac{2}{r} \bar{D}_a \bar{D}_b r h_{kl} - \frac{2}{r} \bar{D}_a r \bar{D}_b h_{kl} - \frac{2}{r} \bar{D}_b r \bar{D}_a h_{kl} \\
& + \frac{6}{r^2} \bar{D}_a r \bar{D}_b r h^{kl} h_{kl} \Bigg] \\
= & -\mathbb{T}^{kl}_{\mathbf{k}_1} \mathbb{T}_{\mathbf{k}_2 kl} H_{T\mathbf{k}_1} \bar{D}_a \bar{D}_b H_{T\mathbf{k}_2}
\end{aligned} \tag{A.19}
$$

$$
\begin{aligned}
h^{ij}\bar{\nabla}_i\bar{\nabla}_a h_{bj} &= h^{ij}\left[-\bar{\Gamma}^k_{ia}\bar{\nabla}_k h_{bj} - \bar{\Gamma}^k_{ib}\bar{\nabla}_a h_{kj} - \bar{\Gamma}^k_{ij}\bar{\nabla}_a h_{b\lambda}\right] \\
&= h^{ij}\left[-\frac{1}{r}\bar{D}_b r\bar{D}_a h_{ij} + \frac{3}{r^2}\bar{D}_a r\bar{D}_b r h_{ij}\right] \\
&= \mathbb{T}^{ij}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2 ij}\left[\frac{1}{r^2}\bar{D}_a r\bar{D}_b r H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} - \frac{1}{r}\bar{D}_b r H_{T\mathbf{k}_1}\bar{D}_a H_{T\mathbf{k}_2}\right] \quad \text{(A.20)}
\end{aligned}
$$

Similarly,

$$
h^{ij}\bar{\nabla}_i\bar{\nabla}_b h_{aj} = \mathbb{T}^{ij}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2 ij}\left[\frac{1}{r^2}\bar{D}_a r\bar{D}_b r H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} - \frac{1}{r}\bar{D}_a r H_{T\mathbf{k}_1}\bar{D}_b H_{\mathbf{k}_2}\right] \quad \text{(A.21)}
$$

$$
\begin{aligned}
-\frac{1}{2}\bar{\nabla}_a h^{\lambda\sigma}\bar{\nabla}_b h_{\lambda\sigma} &= -\frac{1}{2}\bar{\nabla}_a h^{ij}\bar{\nabla}_b h_{ij} \\
&= -\frac{1}{2}\left(\bar{D}_a h^{ij} + 2\bar{\Gamma}^i_{ak}h^{kj}\right)\left(\bar{D}_b h_{ij} - 2\bar{\Gamma}^k_{bi}h_{kj}\right) \\
&= -\frac{1}{2}\left(\bar{D}_a h^{ij} + \frac{2}{r}\bar{D}_a r h^{ij}\right)\left(\bar{D}_b h_{ij} - \frac{2}{r}\bar{D}_b h_{ij}\right) \\
&= -\frac{1}{2}\mathbb{T}^{ij}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2 ij}\left(\bar{D}_a H_{T\mathbf{k}_1}\bar{D}_b H_{T\mathbf{k}_2}\right) \quad \text{(A.22)}
\end{aligned}
$$

$$
\begin{aligned}
-\bar{\nabla}_\sigma h^\lambda_a\bar{\nabla}^\sigma h_{b\lambda} &= -\bar{\nabla}_i h^j_a\bar{\nabla}^i h_{bj} \\
&= -\bar{g}^{li}\bar{\Gamma}^k_{ia}h^j_k\bar{\Gamma}^n_{lb}h_{nj} \\
&= -\frac{1}{r^2}\bar{D}_a r\bar{D}_b r h^{ij}h_{ij} \\
&= -\mathbb{T}^{ij}_{\mathbf{k}_1}\mathbb{T}_{\mathbf{k}_2 ij}\frac{1}{r^2}\bar{D}_a r\bar{D}_b r H_{T\mathbf{k}_1}H_{T\mathbf{k}_2} \quad \text{(A.23)}
\end{aligned}
$$

$$
\begin{aligned}
\bar{\nabla}_\lambda h^\mu_a \bar{\nabla}_\mu h^\lambda_b &= \bar{\nabla}_i h^j_a \bar{\nabla}_j h^i_b \\
&= \bar{\Gamma}^k_{ia} h^j_k \bar{\Gamma}^n_{bj} h^j_n \\
&= \frac{1}{r^2} \bar{D}_a r \bar{D}_b r h^{ij} h_{ij} \\
&= \mathbb{T}^{ij}_{\mathbf{k}_1} \mathbb{T}_{\mathbf{k}_2 ij} \frac{1}{r^2} \bar{D}_a r \bar{D}_b r H_{T\mathbf{k}_1} H_{T\mathbf{k}_2} \qquad \text{(A.24)}
\end{aligned}
$$

Hence, gathering together all the contributions from (A.18) to (A.24) gives us the following:

$$
\begin{aligned}
{}^{(2)}A_{ab} = \sum_{\mathbf{k}_1,\mathbf{k}_2} \Bigg\{ \Bigg(& - H_{T\mathbf{k}_1} \bar{D}_a \bar{D}_b H_{T\mathbf{k}_2} - \bar{D}_a H_{T\mathbf{k}_1} \bar{D}_b H_{T\mathbf{k}_2} \\
& - \frac{1}{r} \bar{D}_a r H_{T\mathbf{k}_1} \bar{D}_b H_{T\mathbf{k}_2} - \frac{1}{r} \bar{D}_b r \bar{D}_a H_{T\mathbf{k}_1} H_{T\mathbf{k}_2} \Bigg) \mathbb{T}^{ij}_{\mathbf{k}_1} \mathbb{T}_{\mathbf{k}_2 ij} \Bigg\} \qquad \text{(A.25)}
\end{aligned}
$$

Appendix B

Tensor mode corresponding to $\mathbf{k} = \{2, 2, 0\}$

The following convention has been used for the 3−sphere metric.

$$d\Omega_3^2 = \gamma_{ij} dw^i dw^j = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) \tag{B.1}$$

We use the formulae given in [115] to evaluate the two classes of various traceless, divergence-free tensors. For our case, since $l = l^{(1)}$, the following expressions hold true

$$\mathbb{T}^{ll0}_{(1)ij} = \sqrt{\frac{(l-1)}{2l}} \Bigg\{ \frac{1}{2} E^{ll} (\bar{D}_i F^{l0}_j + \bar{D}_j F^{l0}_i) + \csc^2\chi \left[\frac{1}{2}(l-1)\cos\chi E^{ll} + C^{ll} \right] (F^{l0}_i \bar{D}_j \cos\chi + F^{l0}_j \bar{D}_i \cos\chi) \Bigg\} \tag{B.2}$$

$$\mathbb{T}^{ll0}_{(2)ij} = \frac{1}{2(l+1)} \left(\epsilon_i^{\ k\ell} \bar{D}_k \mathbb{T}^{ll0}_{(1)\ell j} + \epsilon_j^{\ k\ell} \bar{D}_k \mathbb{T}^{ll0}_{(1)\ell i} \right) \tag{B.3}$$

where for $l = l^{(1)}$,

$$C^{ll} = (-1)^{l+1} 2^l l! \sqrt{\frac{2(l+1)}{\pi(2l+1)!}} \tag{B.4}$$

$$E^{ll} = -\frac{2C^{ll}}{l-1} \cos\chi \tag{B.5}$$

$$F_i^{lm} = \frac{1}{\sqrt{l(l+1)}} \epsilon_i^{\ jk} \bar{D}_j (\sin^l \chi \, Y^{lm}) \bar{D}_k \cos\chi \tag{B.6}$$

Here, we have used the convention $\epsilon_{\chi\theta\phi} = -\sin^2\chi \sin\theta$. Thus, the various traceless, divergence-free tensors are

$$\mathbb{T}_{\phi\chi} = \frac{\sqrt{6}}{2\pi} \sin^2\chi \sin^2\theta \cos\theta \tag{B.7}$$

$$\mathbb{T}_{\theta\phi} = -\frac{\sqrt{6}}{2\pi} \sin^3\chi \cos\chi \sin^3\theta \tag{B.8}$$

$$\mathbb{T}_{\chi\theta} = \sqrt{\frac{2}{3}} \frac{1}{\pi} \sin\chi \cos\chi \sin\theta \cos\theta \tag{B.9}$$

$$\mathbb{T}_{\theta\theta} = -\frac{1}{\sqrt{6}\pi} \sin^2\chi \left[3\cos^2\chi \sin^2\theta - 1\right] \tag{B.10}$$

$$\mathbb{T}_{\chi\chi} = -\frac{1}{\sqrt{6}\pi} [3\cos^2\theta - 1] \tag{B.11}$$

$$\mathbb{T}_{\phi\phi} = \frac{1}{\sqrt{6}\pi} \sin^2\chi \sin^2\theta \left[\sin^2\theta(1 - 3\sin^2\chi) + \cos^2\theta\right] \tag{B.12}$$

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