Exact Treatment of Quantum Quench in a Fermionic System and Time-Dependent Entanglement Entropy

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled "Exact Treatment of Quantum Quench in a Fermionic System and Time-Dependent Entanglement Entropy" submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Shruti Paranjape at the Tata Institute of Fundamental Research, Mumbai, under the supervision of Prof. Gautam Mandal during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled "Exact Treatment of Quantum Quench in a Fermionic System and Time-Dependent Entanglement Entropy" are the results of the investigations carried out by me at the Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai, under the supervision of Prof. Gautam Mandal and the same has not been submitted elsewhere for any other degree.

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Abstract

We consider a relativistic system of fermions with time-dependent mass, in $1 + 1$ space-time dimensions. We show that the state of this system immediately after a sudden quench, in which the mass is taken to zero, is a generalised Calabrese-Cardy state, as hypothesised by [1]. We then proceed to compute the exact post-quench time-dependent propagator and use it to obtain various correlators for a family of pre-quench states, including the ground state and squeezed states. We use these correlators, along with the replica trick and bosonisation methods, to compute the full entanglement entropy of a spatial region. We find that this entropy thermalises as expected.

Contents

Chapter 1

Introduction

Much of equilibrium thermodynamics presumes that generic systems, after a time-dependent excitation, equilibrate given enough time. For finite size systems, there are numerical experiments like the Fermi-Pasta-Ullam experiment [2] where it is shown that the system, instead, might show periodic revival. This means that though the system appears to "equilibrate" after some amount of time, it goes back to its initial state in another time-period and the process repeats thereafter.

For infinite size systems (or systems with periodic boundary conditions), the system may not equilibrate at all. This happens in systems exhibiting many-body localisation, i.e. systems without any extended states. The lack of extended states ensures that no mixing between different regions of the system occurs and hence no equilibration. Thus the question of whether or not a system will thermalise is a non-trivial one.

The statement of thermalisation or equilibration in a quantum system is, as it stands, ill-defined. Quantum systems described by Schrödinger evolution can be expressed as pure states, whereas by definition, a thermal quantum state is a mixed state with density matrix $e^{-\beta H}$. Following Mandal, Sinha and Sorokhaibam [1], we will use the term "thermalisation" to mean that the state describing the quantum system displays thermal expectation values asymptotically. This notion of thermalisation is called subsystem thermalisation [3].

We study the thermalisation of a system with a time-dependent Hamiltonian which settles down asymptotically. In particular, we study a system with a time-dependent mass $m(t)$ that settles down to 0. Though this might seem artificial, a fermionic field theory with a time-dependent mass can be mapped to the Ising model with a time-dependent external magnetic field [4]. Thus, the calculation can be thought of as describing how a chain of spins behaves asymptotically, when the external magnetic field is suddenly quenched to a critical value.

The main observables we will study are correlation functions and the entanglement entropy. Correlators are the traditional way of probing a quantum field theory. Entanglement entropy on the other hand, is a quantity which has only recently begun to be studied in quantum field theories [5]. In its own right, it provides a measure of the quantum entanglement between two regions (in real or momentum space).

In MSS [1], the authors hypothesise that the post-quench state is a generalised Calabrese-Cardy (gCC) state.

$$|\psi_{gCC}\rangle = e^{-\beta H - \sum_n \mu_n W_n} |D\rangle \tag{1.1}$$

where β is the inverse temperature, H is the Hamiltonian, μ_n are chemical potentials, W_n are conserved charges (for example, W_∞ charges) and $|D\rangle$ is a Dirichlet boundary state. This is a generalisation of the state considered by Calabrese and Cardy [6]. [1] prove subsystem thermalisation in this state in the perturbative regime of the chemical potentials μ_n.

In Sec 3.3, we find that after a quench, the ground state indeed is a gCC state. We also find in Sec 2.3, that the post-quench ground state correlator turns out to be

$$\langle \bar{\psi}^\dagger(0,t)\psi(r,t)\rangle = K_1(m(r+2t)) \xrightarrow{t\to\infty} \frac{e^{-2mt}}{\sqrt{t}} \tag{1.2}$$

This differs from the prediction of MSS [1] which states that it should go as a pure $e^{-\gamma t}$ at late times. This discrepancy arises from not being in the perturbative regime of the chemical potentials.

A natural question then, is how to go to the perturbative regime. We note that μ_n and β can be functions only of the initial mass m in the sudden limit, i.e. there are no other free parameters. So in the ground state, we have no way of tuning chemical potentials to make them small enough to be in the perturbative regime. Thus we look at a special class of excited states, squeezed states which also yield gCC states post-quench. These can give post-quench states with tunable chemical potentials, where the extra scale is introduced via the excited state.

We will show that by introducing squeezed states with small chemical potentials, we can reproduce MSS [1]'s results, thereby showing that actual post-quench ground states cannot be treated perturbatively in μ_n. These calculations may be found in Chapter 4.

Thus, the main new contributions in this thesis are :

- The fermionic analogue of the results presented in [7] : the post-quench state is a gCC state; the ground state correlators behave differently from as predicted in MSS [1]; squeezed states can be used to "remove" this discrepancy.

- The calculation of entanglement entropy in the post-quench state, which has been done for a CC state but not for a gCC state (we will see more about these states in Sec 3.3).

1.1 What is a Quantum Quench?

A quantum quench is a process in which a parameter in the Hamiltonian of a system changes with time. In addition, a critical quench is one in which the parameter passes through (or settles down to) its critical value.

In this thesis, we will study critical mass quenches, i.e. we will study field theories with a time-dependent mass $m(t)$ in which as $t \to -\infty$, $m(t) \to m$ and $m(t) \to 0$ as $t \to \infty$.

The role that the quench to criticality, in particular, plays is that the adiabatic theorem can no longer be used. The adiabatic theorem states that a system in a given eigenstate, will continue to be in the same eigenstate of the new Hamiltonian as long as the rate of change of the Hamiltonian is small when compared to the smallest energy gap in the energy levels of the system. The proof of this theorem may be found in App A.

For a critical mass quench, the energy gap vanishes, and there is no Hamiltonian whose rate of change is small as compared to 0. Thus we cannot use the adiabatic theorem. On the other hand, one can use the so-called 'sudden' approximation (which is equivalent to only studying the asymptotic behaviour of the system). This states that the state immediately after the quench is the same as the state immediately before.

In particular, when quenching a system in its ground state, the post-quench state is the ground state of the initial Hamiltonian. Further, that state evolves in time by action of the new Hamiltonian.

1.2 What is Entanglement Entropy?

Quantum entanglement is a well-known phenomenon. Making a measurement of some local observable affects the measurement of some other local observable instantaneously. A measure of this phenomenon is the entanglement entropy.

Before defining that, one can define a von Neumann entropy that gives us a measure of how mixed a state is.

$$S = -\mathrm{Tr}\,(\rho \log \rho) \tag{1.3}$$

Consider now a pure state of a system whose Hilbert space we have divided into two parts, i.e. $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$. We can think of A as some region (we will only consider spatial regions in this thesis) and $\bar{A}$ as its compliment. The entropy S is 0 for a pure state, even though the entanglement is non-zero. In this case we can consider the reduced density matrix ρ_A instead.

$$\rho_A = \mathrm{Tr}_{\bar{A}}\rho \tag{1.4}$$

Using this reduced density matrix (which is an operator on $\mathcal{H}_A$), we can now define the entanglement entropy of this region with the rest of the system as the von Neumann entropy of the reduced density matrix.

$$S_A = -\mathrm{Tr}_A\left(\rho_A \log \rho_A\right) \tag{1.5}$$

In practice, computing this quantity is not an easy task. As a result, one often resorts to using the quantity $\mathrm{Tr}_A\rho_A^n$. In terms of Renyi entropy S_n (defined below), the entanglement entropy is given by

$$S_A = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}_A\rho_A^n = \lim_{n\to 1} S_n \tag{1.6}$$

Thus, we will calculate $\mathrm{Tr}_A\rho_A^n$ in Chapter 5.

Chapter 2

Fermionic System with a Time-Dependent Mass

In this chapter, we aim to find the correlators of a theory of Dirac fermions in the post-quench state, by explicitly solving the Dirac equation, following the method in [8].

2.1 Solving the Dirac Equation

2.1.1 Conventions

$$
\begin{aligned}
\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^0 &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \gamma^1 &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
(\gamma^0)^2 &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\gamma^1)^2 &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
\tag{2.1}
$$

2.1.2 Solutions

The equation of motion for Dirac fermions is given by

$$\left[i\gamma^0\partial_t + i\gamma^1\partial_x - m(t)\right]\Psi(x,t) = 0 \tag{2.2}$$

Since Ψ is a spinor, there is no easy way to solve this equation. So we use the following ansatz

$$\Psi(x,t) = \left[\gamma^0\partial_t + \gamma^1\partial_x - im(t)\right] e^{\pm ikx}\Phi(t) \tag{2.3}$$

Substituting this ansatz into the equation of motion, we arrive at the equation that needs to be satisfied by Φ. This is a equation similar to a

Schrödinger equation.

$$\begin{aligned}\left[i\gamma^0\partial_t + i\gamma^1\partial_x - m(t)\right]\left[\gamma^0\partial_t + \gamma^1\partial_x - im(t)\right] e^{\pm ikx}\Phi(t) &= 0 \\ \left[i\partial_t^2 - i\partial_x^2 + im(t)^2 + \gamma^0\dot{m}(t)\right] e^{\pm ikx}\Phi(t) & \\ \Rightarrow \left[\partial_t^2 + k^2 + m(t)^2 - i\gamma^0\dot{m}(t)\right] e^{\pm ikx}\Phi(t) &= 0\end{aligned}$$

where $\dot{m}(t) = \partial_t m(t)$.

By choosing this particular representation of γ^0, we are in its eigenbasis. Thus, we can use the eigenvalues to label $\Phi(t)$'s components as $\Phi_+(t)$ and $\Phi_-(t)$ corresponding to the eigenvalues ± 1.

$$\begin{aligned}\left[\partial_t^2 + k^2 + m(t)^2 - i\dot{m}(t)\right]\Phi_+(t) = 0 \\ \left[\partial_t^2 + k^2 + m(t)^2 + i\dot{m}(t)\right]\Phi_-(t) = 0\end{aligned} \tag{2.4}$$

where $\Phi(t) = \begin{bmatrix}\Phi_+(t) \\ \Phi_-(t)\end{bmatrix}$.

We note that each of these components has two solutions since each individually satisfies a second-order differential equation. We will call these $\phi_{\pm,p}$ and $\phi_{\pm,n}$. In this thesis, we will consider two specific bases of solutions : the 'in' and 'out' bases. In these bases, p and n will denote the positive and negative energy solutions respectively[1].

We also note due to the symmetry in the equations, that $\Phi_\pm = \Phi^*_\mp$. By matching the asymptotic behaviour, we can make a more specific statement in the 'in' and 'out' bases : $\phi_{\pm,n} = \phi^*_{\mp,p}$. Thus, we will now drop the p or n subscripts with the understanding that $\phi_\pm = \phi_{\pm,p}$[2].

Similar to the usual Dirac field expansion, we can define our U spinor, which appears with the particle annihilation operator a_k as

$$U(x,t) = \left[\gamma^0\partial_t + \gamma^1\partial_x - im(t)\right] e^{ikx}\phi_+(t)\begin{bmatrix}1 \\ 0\end{bmatrix} \tag{2.5}$$

Note that Φ_+ has two parts. We chose ϕ_+. As noted before, this is the positive energy solution which in the 'in' basis, behaves as $\phi_{+,in} \to e^{-i\omega t}$ as $t \to -\infty$. Similarly, the V spinor that appears with the anti-particle creation operator $b_k^\dagger$, is

$$V(x,t) = \left[\gamma^0\partial_t + \gamma^1\partial_x - im(t)\right] e^{-ikx}\phi_+^*(t)\begin{bmatrix}0 \\ 1\end{bmatrix} \tag{2.6}$$

[1] Here, positive/negative energy solution means that the solution looks like a positive/negative energy plane wave at $t \to -\infty$ (for the 'in' basis) and $t \to \infty$ (for the 'out' basis).

[2] We can do this because throughout this thesis, we will always be in either the 'in' or the 'out' basis.

Here we chose the negative energy part (since it is the spinor associated to anti-particle creation) of Φ_-, i.e. ϕ_+^* which in the 'in' basis, behaves as $\phi_{+,in}^* \to e^{i\omega t}$ as $t \to -\infty$.

2.1.3 Normalising Spinors

In the 'in' basis, at $t \to -\infty$, we can find the exact expressions for these spinors using $\phi_{+,in}(t) = e^{-i\omega t}$ and $\phi_{+,in}^*(t) = e^{i\omega t}$.

$$\begin{aligned}
U(x,t) &= \left[\gamma^0\partial_t + \gamma^1\partial_x - im\right] e^{-i\omega t + ikx} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} -i(\omega+m) & -ik \\ ik & i(\omega-m) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-ik\cdot x} \\
&= i \begin{bmatrix} -(\omega+m) \\ k \end{bmatrix} e^{-ik\cdot x} \\
V(x,t) &= \left[\gamma^0\partial_t + \gamma^1\partial_x + im(t)\right] \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{ik\cdot x} \\
&= i \begin{bmatrix} k \\ -(\omega+m) \end{bmatrix} e^{ik\cdot x}
\end{aligned}$$

Hence, upto normalizations fixed by inner products, the 'in' spinors are

$$u(k,m) = i \begin{bmatrix} -(\omega+m) \\ k \end{bmatrix}, \quad v(k,m) = i \begin{bmatrix} k \\ -(\omega+m) \end{bmatrix}$$

$$\begin{aligned}
\bar{u}(k,m) &= u^\dagger(k,m)\gamma^0 = i\begin{bmatrix} -(\omega+m) & -k \end{bmatrix} \\
\bar{v}(k,m) &= v^\dagger(k,m)\gamma^0 = i\begin{bmatrix} k & (\omega+m) \end{bmatrix}
\end{aligned}$$

Now borrowing the conventions for spinors in Peskin and Schroeder [9], we want to fix the inner products $\bar{u}(k,m)u(k,m) = 2m$ and $\bar{v}(k,m)v(k,m) = -2m$.

$$\begin{aligned}
\bar{u}(k,m)u(k,m) &= -\begin{bmatrix} -(\omega+m) & -k \end{bmatrix} \begin{bmatrix} -(\omega+m) \\ k \end{bmatrix} = -2m(\omega+m) \\
\bar{v}(k,m)v(k,m) &= -\begin{bmatrix} k & (\omega+m) \end{bmatrix} \begin{bmatrix} k \\ -(\omega+m) \end{bmatrix} = 2m(\omega+m)
\end{aligned}$$

So the normalised spinors are

$$\begin{aligned}
u(k,m) &= \frac{1}{\sqrt{(\omega+m)}} \begin{bmatrix} -(\omega+m) \\ k \end{bmatrix}, \quad v(k,m) = \frac{1}{\sqrt{(\omega+m)}} \begin{bmatrix} k \\ -(\omega+m) \end{bmatrix} \\
\bar{u}(k,m) &= \frac{1}{\sqrt{(\omega+m)}} \begin{bmatrix} -(\omega+m) & -k \end{bmatrix}, \quad \bar{v}(k,m) = \frac{1}{\sqrt{(\omega+m)}} \begin{bmatrix} k & (\omega+m) \end{bmatrix}
\end{aligned}$$

These are the spinors at $t \to -\infty$ in the 'in' basis. At other times, $\phi_{+,in}$ will not be of the plane-wave form and the spinors will look more complicated. One aspect to be noted is that the normalisation changes for different spinors, but it is not a function of time, i.e. time-evolving u and v will leave the normalisation $\frac{1}{\sqrt{\omega+m}}$ unchanged. If this were not so, the spinors wouldn't continue being solutions of the corresponding Dirac equation at all times.

The transformation to a chiral basis is accomplished by using the transformation matrix $S = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$. The mode expansion in the 'out' basis (where $\phi_{+,out} \to e^{-i|k|t}$ as $t \to \infty$) is

$$\begin{aligned}\Psi(x,t) &= \int \frac{dk}{\sqrt{2|k|}}\left[a_{k,out}\frac{1}{\sqrt{|k|}}\begin{bmatrix}-|k|\\ k\end{bmatrix}e^{-ik\cdot x} + b^{\dagger}_{k,out}\frac{1}{\sqrt{|k|}}\begin{bmatrix}k\\ -|k|\end{bmatrix}e^{ik\cdot x}\right]\\ &= \int \frac{dk}{2}\begin{bmatrix}-a_{k,out}e^{-ik\cdot x} + \mathrm{sgn}(k)b^{\dagger}_{k,out}e^{ik\cdot x}\\ \mathrm{sgn}(k)a_{k,out}e^{-ik\cdot x} - b^{\dagger}_{k,out}e^{ik\cdot x}\end{bmatrix}\end{aligned}$$

On transforming to the chiral basis using S, we get

$$\begin{aligned}\Psi_c(x,t) &= S\cdot\Psi(x,t) = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\cdot\int \frac{dk}{2}\begin{bmatrix}-a_{k,out}e^{-ik\cdot x} + \mathrm{sgn}(k)b^{\dagger}_{k,out}e^{ik\cdot x}\\ \mathrm{sgn}(k)a_{k,out}e^{-ik\cdot x} - b^{\dagger}_{k,out}e^{ik\cdot x}\end{bmatrix}\\ &= \int \frac{dk}{2\sqrt{2}}\begin{bmatrix}(1+\mathrm{sgn}(k))(-a_{k,out}e^{-ik\cdot x} + b^{\dagger}_{k,out}e^{ik\cdot x})\\ (1-\mathrm{sgn}(k))(-a_{k,out}e^{-ik\cdot x} - b^{\dagger}_{k,out}e^{ik\cdot x})\end{bmatrix} \qquad (2.7)\end{aligned}$$

We can now write down our chiral fermion operators in the conformal field theory (CFT).

$$\psi(x,t) = \int \frac{dk}{2\sqrt{2}}(1+\mathrm{sgn}(k))(-a_{k,out}e^{-ik\cdot x} + b^{\dagger}_{k,out}e^{ik\cdot x}) \qquad (2.8)$$

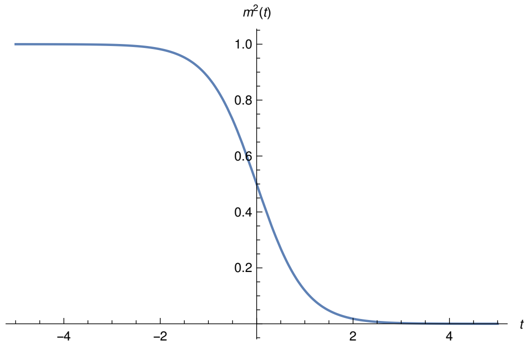
$$\bar{\psi}(x,t) = \int \frac{dk}{2\sqrt{2}}(1-\mathrm{sgn}(k))(-a_{k,out}e^{-ik\cdot x} - b^{\dagger}_{k,out}e^{ik\cdot x}) \qquad (2.9)$$

Note that they indeed are chiral, since ψ is a function of only z and has only $k > 0$ modes, and $\bar{\psi}$ is a function of only $\bar{z}$ and has only $k < 0$ modes.

2.1.4 Details

For a specific mass profile we can solve the Dirac equation.

$$m^2(t) = \frac{1}{2}\left(1 - \tanh(\rho t)\right) \qquad (2.10)$$



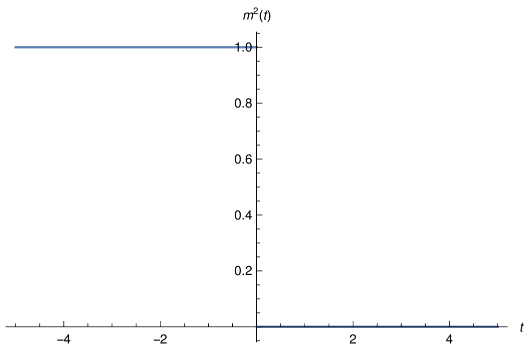
In practice, to solve Eq (2.4), we need to switch to $e^{\rho t}$ variables. This yields the 'in' solutions of the + equation as

$$\begin{aligned}\phi_{+,in} &= e^{-it(\omega+m)}\left(e^{-2\rho t}+1\right)^{-\frac{im}{2\rho}} \\ &\quad {}_2F_1\left(\frac{i\left(|k|-m-\omega\right)}{2\rho}, -\frac{i\left(|k|+m+\omega\right)}{2\rho}; 1-\frac{i\omega}{\rho}; -e^{2t\rho}\right) \quad (2.11)\\ \phi^*_{-,in} &= e^{it(\omega-m)}\left(e^{-2\rho t}+1\right)^{-\frac{im}{2\rho}} \\ &\quad {}_2F_1\left(\frac{i\left(-|k|-m+\omega\right)}{2\rho}, \frac{i\left(|k|-m+\omega\right)}{2\rho}; \frac{i\omega}{\rho}+1; -e^{2t\rho}\right) \quad (2.12)\end{aligned}$$

where $\omega = \sqrt{k^2+m^2}$.

Since the hypergeometric function ${}_2F_1$ has an expansion around 0 whose leading term is 1, it is easy to see that $\phi_{+,in} \to e^{-i\omega t}$ and $\phi^*_{-,in} \to e^{i\omega t}$ as $t \to -\infty$. (This is how we identify the two solutions as $\phi_{+,in}$ and $\phi^*_{-,in}$.)

It is also useful to note that in the $\rho \to \infty$, i.e. the sudden limit, the solutions assume their asymptotic plane wave form at all times and the mass profile $m(t)$ jumps from m to 0 at $t = 0$.



2.2 Bogoliubov Transformations

We note that at the moment we are dealing with two bases of solutions for the $\phi_\pm$ equations. These are $\phi_{\pm,in}$, $\phi^*_{\mp,in}$ and $\phi_{\pm,out}$, $\phi^*_{\mp,out}$. Since both of these are complete bases, we can write each in terms of the other.

$$\begin{aligned}\phi_{in,+p}(t,k) &= \alpha_+(k)\phi_{+,out}(t,k) + \beta_+(k)\phi^*_{-,out}(t,k)\\ \phi_{in,-p}(t,k) &= \alpha_-(k)\phi_{-,out}(t,k) + \beta_-(k)\phi^*_{+,out}(t,k)\end{aligned}$$

These $\alpha_\pm$ and $\beta_\pm$ are functions of $|k|$ known as Bogoliubov coefficients. We now aim to find out how U_{in} and V_{in} are connected to the 'out' spinors and consequently how the oscillators in these bases are connected.

$$\begin{aligned}U_{in} &= \begin{bmatrix}\partial_t - im(t)\\ \partial_x\end{bmatrix}(\alpha_+\phi_{+,out} + \beta_+\phi^*_{-,out})e^{ikx}\\ &\overset{t\to\infty}{=} \begin{bmatrix}-i|k|\\ ik\end{bmatrix}\alpha_+ e^{-ik\cdot x} + \begin{bmatrix}-i|k|\\ -ik\end{bmatrix}\beta_+ e^{ik\cdot x}\\ &\to (\alpha_+ U_{out}(k) - \mathrm{sign}(k)\beta_+ V_{out}(-k))\\ V_{in} &\to \left(\alpha^*_+ V_{out}(k) - \mathrm{sign}(k)\beta^*_+ U_{out}(-k)\right)\end{aligned}$$

Now, using this in the expression for Ψ in the 'in' basis and matching it with the expansion in the 'out' basis,

$$\Psi = \int dk \frac{N_{out}}{\sqrt{|k|}}\left(U_{out}(k)a_{k,out} + V_{out}(-k)b^\dagger_{-k,out}\right)$$

gives us the Bogoliubov transformation between the creation and annihilation operators.

$$a_{out,k} = \sqrt{\frac{\omega_{out}}{\omega_{in}}}\frac{N_{in}}{N_{out}}[\left[\alpha_+(k)a_{in,k} + b^\dagger_{in,-k}\,\chi(k)\beta^*_+(k)\right] \tag{2.13}$$

$$b^\dagger_{out,k} = \sqrt{\frac{\omega_{out}}{\omega_{in}}}\frac{N_{in}}{N_{out}}\left[\alpha^*_+(k)b^\dagger_{in,k} + a_{in,-k}\,\tilde{\chi}(k)\beta_+(k)\right] \tag{2.14}$$

Here, we want $\chi(k)$ and $\tilde{\chi}(k)$ to satisfy

$$\chi(k)u(k,\omega) = v(-k,-\omega) \quad \Rightarrow \chi(k)\begin{bmatrix}-|k|\\ k\end{bmatrix} = \begin{bmatrix}-k\\ |k|\end{bmatrix} \quad \Rightarrow \chi(k) = \mathrm{sign}(k)$$

$$\tilde{\chi}(k)v(k,\omega) = u(-k,-\omega) \quad \Rightarrow \tilde{\chi}(k)\begin{bmatrix}k\\ -|k|\end{bmatrix} = \begin{bmatrix}|k|\\ -k\end{bmatrix} \quad \Rightarrow \tilde{\chi}(k) = \mathrm{sign}(k)$$

2.2.1 Details

In the quench with a tanh mass profile, using the properties of confluent hypergeometric functions ${}_2F_1$ given in [10], we get the following Bogoliubov coefficients

$$\alpha_+ = \frac{\Gamma\left(-\frac{i|k|}{\rho}\right)\Gamma\left(1-\frac{i\omega}{\rho}\right)}{\Gamma\left(-\frac{i(|k|+m+\omega)}{2\rho}\right)\Gamma\left(\frac{-i|k|+im+2\rho-i\omega}{2\rho}\right)} \tag{2.15}$$

$$\beta_+ = \frac{\Gamma\left(\frac{i|k|}{\rho}\right)\Gamma\left(1-\frac{i\omega}{\rho}\right)}{\Gamma\left(1-\frac{i(-|k|-m+\omega)}{2\rho}\right)\Gamma\left(-\frac{i(-|k|+m+\omega)}{2\rho}\right)} \tag{2.16}$$

which matches the calculation by Duncan [8].

We will be finding correlators, calculating entanglement entropy and mapping our post-quench state to a gCC state only in the sudden ($\rho \to \infty$) limit. The sudden limit of our Bogoliubov coefficients with which we will be dealing henceforth, is

$$\alpha_+ = \frac{|k|+m+\omega}{2|k|} \tag{2.17}$$

$$\beta_+ = \frac{|k|-m-\omega}{2|k|} \tag{2.18}$$

2.2.2 Relations Between Bogoliubov Coefficients

Using the expressions for N and ω, we can write

$$\sqrt{\frac{\omega_{out}}{\omega_{in}}\frac{N_{in}}{N_{out}}} = \sqrt{\frac{|k|}{\omega}}\sqrt{\frac{|k|}{\omega+m}} = \sqrt{\frac{|k|}{\omega}}\sqrt{\frac{\omega-m}{|k|}} = \sqrt{\frac{\omega-m}{\omega}} \tag{2.19}$$

which gives us the Bogoliubov transformations between oscillators.

$$a_{out,k} = \sqrt{\frac{\omega-m}{\omega}}\left[\alpha_+(k)a_{in,k} + b^\dagger_{in,-k}\,\mathrm{sign}(k)\beta^*_+(k)\right] \tag{2.20}$$

$$b^\dagger_{out,k} = \sqrt{\frac{\omega-m}{\omega}}\left[\alpha^*_+(k)b^\dagger_{in,k} + a_{in,-k}\,\mathrm{sign}(k)\beta_+(k)\right] \tag{2.21}$$

Thus, the canonical relations between the Bogoliubov coefficients will be given by imposing $\{a_k, a^\dagger_{k'}\} = \delta_{kk'}$ for the 'out' oscillators.

$$\begin{aligned}
\{a_{k,out}, a^\dagger_{k',out}\} &= \frac{\omega - m}{\omega}\left(\alpha_+(k)\alpha^*_+(k')\delta_{kk'} + \text{sign}(k)\beta^*_+(k')\text{sign}(k')\beta_+(k')\delta_{-k,-k'}\right) \\
&= \frac{\omega - m}{\omega}\left(|\alpha_+(k)|^2 + \text{sign}(k)^2|\beta_+(k)|^2\right)\delta_{kk'} \qquad (2.22) \\
&\Rightarrow |\alpha_+(k)|^2 + |\beta_+(k)|^2 = \frac{\omega}{\omega - m} \qquad (2.23)
\end{aligned}$$

Thus we now have the canonical relations for our Bogoliubov coefficients which our α_+ and β_+ given in Eqs (2.15) and (2.16), satisfy.

Relation from the Wronskian

In this section, we will follow the discussion in Duncan [8]. Consider the Wronskian of $\phi_{+,in}$ and $\phi^*_{-,in}$ (which is non-zero since they are linearly independent solutions to the same equation $\partial_t^2\phi = M(t)\phi$).

$$\begin{aligned}
W &= \phi_{+,in}\overleftrightarrow{\partial_t}\phi^*_{-,in} \qquad (2.24) \\
\Rightarrow \frac{\partial W}{\partial t} &= \phi_{+,in}\partial_t^2\phi^*_{-,in} - \phi^*_{-,in}\partial_t^2\phi_{+,in} \qquad (2.25) \\
&= \phi_{+,in}M(t)\phi^*_{-,in} - \phi^*_{-,in}M(t)\phi_{+,in} = 0 \qquad (2.26)
\end{aligned}$$

Thus the Wronskian is constant in time. At time $t \to -\infty$, $W = 2i\omega$. This needs to match the time $t \to \infty$ limit where $W = 2i|k|\left(\alpha_-\alpha^*_+ - \beta_-\beta^*_+\right)$. Thus

$$\alpha_-\alpha^*_+ - \beta_-\beta^*_+ = \alpha_+\alpha^*_- - \beta_+\beta^*_- = \frac{\omega}{|k|} \qquad (2.27)$$

Relation between α_+ and α_-

Consider the operator $\mathcal{D}_\pm = \partial_t \pm im(t)$. The equations for $\phi_\pm$ are given by

$$-\mathcal{D}_\pm\mathcal{D}_\mp\phi_\pm = -\left(\partial_t^2 + m^2(t) \mp i\partial_t m(t)\right)\phi_\pm = |k|^2\phi_\pm \qquad (2.28)$$

We now act $\mathcal{D}_\mp$ on the above equation to get

$$-\mathcal{D}_\mp\mathcal{D}_\pm(\mathcal{D}_\mp\phi_\pm) = |k|^2(\mathcal{D}_\mp\phi_\pm) \qquad (2.29)$$

Thus, we conclude that $\mathcal{D}_\mp\phi_\pm \sim \phi_\mp$. In particular, in the $t \to -\infty$ limit, we get $\mathcal{D}_\mp\phi_\pm = (-i\omega_{in} \mp im_{in})\phi_\mp$. Using a Bogoliubov transformation we can now take the $t \to \infty$ limit,

$$\begin{aligned}
\mathcal{D}_\mp(\alpha_\pm e^{-i|k|t} + \beta_\pm e^{i|k|t}) &= (-i\omega_{out} \mp im_{out})\alpha_\pm e^{-i|k|t} + (i\omega_{out} \mp im_{out})\beta_\pm e^{i|k|t} \\
&= (-i\omega_{in} \mp im_{in})(\alpha_\mp e^{-i|k|t} + \beta_\mp e^{i|k|t}) \qquad (2.30)
\end{aligned}$$

Matching the coefficients of the independent solutions $e^{\pm i|k|t}$, we get

$$\frac{\alpha_+}{\alpha_-} = \frac{\omega + m}{|k|} = \frac{|k|}{\omega - m} \tag{2.31}$$

$$\frac{\beta_+}{\beta_-} = -\frac{\omega + m}{|k|} = -\frac{|k|}{\omega - m} \tag{2.32}$$

2.3 Finding Correlators

We note that we are working in the Heisenberg picture where the operators, not the states have time-dependence (this can also be thought of as looking at asymptotics or using the sudden approximation). Thus, quenching the ground state $|0\rangle_{in}$ will not change the state.

Thus, the first post-quench correlator we want to find is $\langle 0_{in}|\psi^\dagger\psi|0_{in}\rangle$. But the expressions for the chiral ψ, $\bar{\psi}$ operators are given in terms of the 'out' operators. So we use the Bogoliubov transformations Eq (2.15) and Eq (2.16), to get

$$\begin{aligned}
\psi &= \int_0^\infty dk \left(-a_{out}(k)e^{-ik\cdot x} + b^\dagger_{out}(k)e^{ik\cdot x}\right) && (2.33)\\
&= \int_0^\infty dk \Big(-(\alpha_+(k)a_{in,k} + \text{sign}(k)\beta^*_+(k)b^\dagger_{in,-k})e^{-ik\cdot x} \\
&\quad +(\alpha^*_+(k)b^\dagger_{in,k} + \text{sign}(k)\beta_+(k)a_{in,-k})e^{ik\cdot x}\Big) && (2.34)
\end{aligned}$$

A similar expression for $\psi^\dagger$ gives us an expression for $\langle 0_{in}|\psi^\dagger(r,t)\psi(0,t)|0_{in}\rangle$

$$\begin{aligned}
&= \int_0^\infty dk dk' \left(\frac{\omega - m}{\omega}\right) \langle 0_{in}| \left(\alpha_+(k)b_{in,k}e^{-ik\cdot x} - \text{sign}(k)\beta_+(k)b_{in,-k})e^{ik\cdot x}\right) \\
&\quad \left(\alpha^*_+(k')b^\dagger_{in,k'}e^{ik'\cdot x'} - \text{sign}(k')\beta^*_+(k')b^\dagger_{in,-k'})e^{-ik'\cdot x'}\right) |0_{in}\rangle \\
&= \int_0^\infty dk dk' \left(\frac{\omega - m}{\omega}\right) \Big(\text{sign}(k)\beta_+(k)e^{ik\cdot x}\text{sign}(k')\beta^*_+(k')e^{-ik'\cdot x'}\delta_{-k,-k'} \\
&\quad +\alpha_+(k)e^{-ik\cdot x}\alpha^*_+(k')e^{ik'\cdot x'}\delta_{k,k'}\Big) \\
&= \int_0^\infty dk \left(\frac{\omega - m}{\omega}\right) \left(|\beta_+(k)|^2 e^{ik\cdot(x-x')} + |\alpha_+(k)|^2 e^{ik\cdot(x'-x)}\right) \\
&= \int_{-\infty}^\infty dk \left(\frac{\omega - m}{\omega}\right) e^{ikr} \left(\Theta(k)|\alpha_+(k)|^2 + \Theta(-k)|\beta_+(k)|^2\right)
\end{aligned}$$

Similarly, we can find formulae for the other correlators

$$\begin{aligned}
\langle 0_{in}|\bar{\psi}^\dagger(r,t)\bar{\psi}(0,t)|0_{in}\rangle &= \int_{-\infty}^{\infty} dk \left(\frac{\omega-m}{\omega}\right) e^{ikr}\left(\Theta(-k)|\alpha_+(k)|^2+\Theta(k)|\beta_+(k)|^2\right) \\
\langle 0_{in}|\bar{\psi}^\dagger(r,t)\psi(0,t)|0_{in}\rangle &= 4\int_{-\infty}^{\infty} dk \left(\frac{\omega-m}{\omega}\right) e^{ik(r+2t)}\alpha_+\beta_+ \\
\langle 0_{in}|\psi^\dagger(r,t)\bar{\psi}(0,t)|0_{in}\rangle &= 4\int_{-\infty}^{\infty} dk \left(\frac{\omega-m}{\omega}\right) e^{ik(r-2t)}\alpha_+\beta_+
\end{aligned}$$

2.3.1 Details

For the case with a tanh mass profile, we can perform the Fourier transforms above to get the following correlators in the sudden limit.

$$\langle 0_{in}|\psi^\dagger(r,t)\psi(0,t)|0_{in}\rangle = -imK_1(mr) \tag{2.35}$$
$$\langle 0_{in}|\bar{\psi}^\dagger(r,t)\bar{\psi}(0,t)|0_{in}\rangle = imK_1(mr) \tag{2.36}$$
$$\langle 0_{in}|\bar{\psi}^\dagger(r,t)\psi(0,t)|0_{in}\rangle = -mK_0(m(r+2t)) \tag{2.37}$$
$$\langle 0_{in}|\psi^\dagger(r,t)\bar{\psi}(0,t)|0_{in}\rangle = -mK_0(m|r-2t|) \tag{2.38}$$

It is easy to see that the first two have already equilibrated in the sense that they're independent of time. The second two tend to 0 as $t \to \infty$, which is what we would expect from holomorphic-anti holomorphic correlators as per the calculation in MSS [1].

Chapter 3

The Post-Quench State

Our aim in this chapter will be to prove that the post-quench state in a fermionic field theory, after a critical quench from the ground state, is a generalised Calabrese-Cardy (gCC) state.

$$|\psi\rangle = e^{-\beta H - \sum_n \mu_n W_n} |D\rangle \tag{3.1}$$

We will begin by understanding the elements in this state and then prove that it is equal to the post-quench state in Sec 3.3.

3.1 Boundary States

We will be following a discussion in the books by Blumenhagen and Plauschinn [11] and by Di Francesco, Mathieu and Senechal [12]. Consider a conformal field theory (CFT) that's living in a geometry with a boundary at $t = 0$, say. Varying the action and setting $\delta S = 0$ at the boundary gives us some possible boundary conditions. Quantum mechanically, we can enforce $\delta S = 0$ as an operator equation satisfied on a state. These states are called boundary states.

Another way of characterising boundary states is by using the stress-energy tensor. For a manifold with a boundary, we would want no energy to flow across the boundary. For a boundary in time, we would want the T_{xt} component to vanish. Let us switch to $z = t - x$ and $\bar{z} = t + x$ coordinates using

$$T_{zz} = \frac{\partial x^\lambda}{\partial z}\frac{\partial x^\rho}{\partial z} T_{\lambda\rho} = \frac{1}{4}\left(T_{xx} + T_{tt} - 2T_{xt}\right) \tag{3.2}$$

$$T_{\bar{z}\bar{z}} = \frac{\partial x^\lambda}{\partial \bar{z}}\frac{\partial x^\rho}{\partial \bar{z}} T_{\lambda\rho} = \frac{1}{4}\left(T_{xx} + T_{tt} + 2T_{xt}\right) \tag{3.3}$$

$$T_{z\bar{z}} = \frac{\partial x^\lambda}{\partial \bar{z}}\frac{\partial x^\rho}{\partial \bar{z}} T_{\lambda\rho} = \frac{1}{4}\left(T_{xx} - T_{tt}\right) \tag{3.4}$$

Here we have used the symmetry of the stress tensor i.e. $T_{xt} = T_{tx}$. Thus, the condition of no energy passing a t boundary becomes $\left(T - \bar{T}\right)|Bd\rangle = T_{xt}|Bd\rangle = 0$. Now we can use the usual mode expansion for T in terms of the Virasoro operators

$$T = \sum_n e^{-nz} L_n \quad \bar{T} = \sum_n e^{-n\bar{z}} \bar{L}_n \tag{3.5}$$

$$\begin{aligned} \Rightarrow \left(T - \bar{T}\right)|Bd\rangle &= \left(\sum_n e^{nx} L_n - \sum_n e^{-nx} \bar{L}_n\right)|Bd\rangle = 0 \\ &\Rightarrow \left(L_n - \bar{L}_{-n}\right)|Bd\rangle = 0 \end{aligned} \tag{3.6}$$

3.1.1 Fermions

Since we are in a massless theory after the quench, we can consider a chiral Dirac Lagrangian.

$$S = \int d^2x \frac{1}{2}\left(\psi^\dagger \bar{\partial}\psi + \psi\bar{\partial}\psi^\dagger + \bar{\psi}^\dagger \partial\bar{\psi} + \bar{\psi}\partial\bar{\psi}^\dagger\right) \tag{3.7}$$

On varying the action and collecting terms, we get the following

$$\delta S = \int d^2x \left(\delta\psi^\dagger \bar{\partial}\psi + \delta\psi\bar{\partial}\psi^\dagger + \delta\bar{\psi}^\dagger \partial\bar{\psi} + \delta\bar{\psi}\partial\bar{\psi}^\dagger\right) + \text{boundary terms} \tag{3.8}$$

Given a boundary at $t = 0$, δS will also have certain boundary terms, which we demand must vanish at $t = 0$ in order for $\delta S = 0$.

$$\psi^\dagger \delta\psi + \psi\delta\psi^\dagger + \bar{\psi}^\dagger \delta\bar{\psi} + \bar{\psi}\delta\bar{\psi}^\dagger\big|_{t=0} = 0 \tag{3.9}$$

We impose this as an operator equation on the boundary state $|B\rangle$. The conditions on the operators on the boundary state can then be achieved via two possible identifications [13].

$$\left(\psi - i\bar{\psi}^\dagger\right)|B\rangle \;= 0, \quad \left(\psi^\dagger - i\bar{\psi}\right)|B\rangle \;= 0 \quad \text{or} \tag{3.10}$$

$$\left(\psi - i\bar{\psi}\right)|B\rangle \;= 0, \quad \left(\psi^\dagger - i\bar{\psi}^\dagger\right)|B\rangle \;= 0 \tag{3.11}$$

In z, $\bar{z}$ coordinates, using the Dirac fermion creation and annihilation operators, we can write down the mode expansions as follows. This is nothing but the expansion we found in terms of the 'out' basis in the previous chapter in Eq (2.8) and Eq (2.9).

$$\psi = \sum_{k>0} -a_k e^{-ikz} + \sum_{k<0} b^\dagger_{-k} e^{-ikz} \tag{3.12}$$

$$\psi^\dagger = \sum_{k>0} b_k e^{-ikz} + \sum_{k<0} -a^\dagger_{-k} e^{-ikz} \tag{3.13}$$

$$\bar{\psi} = \sum_{k>0} -b^\dagger_{-k} e^{ik\bar{z}} + \sum_{k<0} -a_k e^{ik\bar{z}} \tag{3.14}$$

$$\bar{\psi}^\dagger = \sum_{k>0} -a^\dagger_{-k} e^{ik\bar{z}} + \sum_{k<0} -b_k e^{ik\bar{z}} \tag{3.15}$$

Now we impose the boundary conditions in terms of these modes :

1. $\psi = i\bar{\psi}^\dagger$ gives us $\sum\limits_{k>0} -a_k e^{ikx} + \sum\limits_{k<0} b^\dagger_{-k} e^{ikx} = \sum\limits_{k>0} -ia^\dagger_{-k} e^{ikx} + \sum\limits_{k<0} -ib_k e^{ikx}$. Since this needs to be valid on $|B\rangle$ for all x, we require $-a_k \sim -ia^\dagger_{-k}$ i.e. $\left(a_k - ia^\dagger_{-k}\right)|B\rangle = 0$ for $k > 0$ and $-b_k \sim -ib^\dagger_{-k}$ for $k < 0$ to be valid on $|B\rangle$.

   Similarly, the condition $\psi^\dagger = i\bar{\psi}$ gives us $-a_k \sim ia^\dagger_{-k}$ for $k < 0$ and $b_k \sim -ib^\dagger_{-k}$ for $k > 0$. Writing this more concisely, we have for all values of k, $a_k \sim i\ \mathrm{sign}(k) a^\dagger_{-k}$ and $b_k \sim -i\ \mathrm{sign}(k) b^\dagger_{-k}$. Thus[1]

$$|B_1\rangle = \exp\left(\sum_k i\ \mathrm{sign}(k)(a^\dagger_k a^\dagger_{-k} - b^\dagger_k b^\dagger_{-k})\right)|0\rangle \tag{3.16}$$

2. Following the same procedure for the condition $\psi = i\bar{\psi}$ gives us $-a_k \sim -ib^\dagger_{-k}$ for $k > 0$ and $-a_k \sim -ib^\dagger_{-k}$ for $k < 0$ to be identified on $|B\rangle$.

   Similarly, the condition $\psi^\dagger = i\bar{\psi}^\dagger$ gives us $-b_k \sim ia^\dagger_{-k}$ for $k < 0$ and $b_k \sim -ia^\dagger_{-k}$ for $k > 0$. Writing this more concisely, we have for all values of k, $a_k \sim ib^\dagger_{-k}$. Thus

$$|B_2\rangle = \exp\left(\sum_k ia^\dagger_k b^\dagger_{-k}\right)|0\rangle \tag{3.17}$$

[1] A boundary state is a state on which the above operator equations hold. To solve the operator equations, we note that a_k acts like $\frac{\partial}{\partial a^\dagger_k}$. This is because for any operator X, $[\partial_X, X] = 1$. Thus, $[a_k, a^\dagger_{k'}] = \delta_{kk'} \Rightarrow a_k = \partial_{a^\dagger_k}$. Now the operator equation becomes a simple differential equation $\frac{dB}{da^\dagger} = aB$ whose solution is an exponential $B = e^{aa^\dagger}$.

From the action S, we can find the components of the energy-momentum tensor $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$.

$$T =: \psi^\dagger \partial \psi + \psi \partial \psi^\dagger : \qquad \bar{T} =: \bar{\psi}^\dagger \bar{\partial} \bar{\psi} + \bar{\psi} \bar{\partial} \bar{\psi}^\dagger : \tag{3.18}$$

Using the mode expansion for ψ, we can expand T as follows.

$$\begin{aligned}
T &= \sum_{k,k'>0} (b_{k'} e^{-ik'z} - a^\dagger_{k'} e^{ik'z}) ik (a_k e^{-ikz} + b^\dagger_k e^{ikz}) \\
&+ (-a_{k'} e^{-ik'z} + b^\dagger_{k'} e^{ik'z}) ik (-b_k e^{-ikz} - a^\dagger_k e^{ikz}) \\
&= \sum_n \left(- \sum_{\substack{k>0 \\ k>-n}} ik \left(a^\dagger_{n+k} a_k + b^\dagger_{n+k} b_k \right) + \sum_{\substack{k>0 \\ k>n}} ik \left(b_{k-n} b^\dagger_k + a_{k-n} a^\dagger_k \right) \right. \\
&\quad \left. + \sum_{\substack{k>0 \\ k<n}} ik \left(-a^\dagger_{n-k} b^\dagger_k - b^\dagger_{n-k} a^\dagger_k \right) - \sum_{\substack{k>0 \\ k<-n}} ik \left(-b_{-k-n} a_k - a_{-k-n} b_k \right) \right) e^{inz} \\
&= \sum_n L_n e^{inz}
\end{aligned}$$

To find the expression for $\bar{L}_n$, we expand $\bar{T}$ in modes.

$$\begin{aligned}
\bar{T} &= \sum_n \left(- \sum_{\substack{k<0 \\ k>-n}} ik \left(a^\dagger_{-n-k} b^\dagger_k + b^\dagger_{-n-k} b^\dagger_k \right) + \sum_{\substack{k<0 \\ k>n}} ik \left(b_{n-k} a_k + a_{n-k} b_k \right) \right. \\
&\quad \left. + \sum_{\substack{k<0 \\ k<n}} ik \left(a^\dagger_{k-n} a_k + b^\dagger_{k-n} b_k \right) - \sum_{\substack{k<0 \\ k<-n}} ik \left(b_{n+k} b^\dagger_k + a_{n+k} a^\dagger_k \right) \right) e^{in\bar{z}} \\
&= \sum_n \bar{L}_n e^{in\bar{z}}
\end{aligned}$$

The condition for a boundary state is that $\left(L_{-n} - \bar{L}_n \right) |B\rangle = 0$ (see Eq (3.6)). We want to show that this definition agrees with our previous one. For this purpose we look at L_{-n} and use $-k$ instead of k as the summation index. This gives

$$\begin{aligned}
L_{-n} &= \sum_{\substack{k<0 \\ k<-n}} ik \left(a^\dagger_{-n-k} a_{-k} + b^\dagger_{-n-k} b_{-k} \right) - \sum_{\substack{k<0 \\ k<n}} ik \left(b_{-k+n} b^\dagger_{-k} + a_{n-k} a^\dagger_{-k} \right) \\
&\quad - \sum_{\substack{k<0 \\ k>n}} ik \left(-a^\dagger_{k-n} b^\dagger_{-k} - b^\dagger_{k-n} a^\dagger_{-k} \right) + \sum_{\substack{k<0 \\ k>-n}} ik \left(-b_{n+k} a_{-k} - a_{n+k} b_{-k} \right)
\end{aligned}$$

We notice that every oscillator index is positive. Thus, to check that the boundary state condition is satisfied for $|B_1\rangle$, we do the same identifications as earlier (for positive oscillator indices), $a_k \sim ia^\dagger_{-k}$, $b_k \sim -ib^\dagger_{-k}$, $a^\dagger_k \sim ia_{-k}$ and $b^\dagger_k \sim -ib_{-k}$. This gives us $L_{-n} = \bar{L}_n$ as desired.

For $|B_2\rangle$, we make the identifications $a_k \sim ib^\dagger_{-k}$, $b_k \sim -ia^\dagger_{-k}$, $a^\dagger_k \sim ib_{-k}$ and $b^\dagger_k \sim -ia_{-k}$. Again, this gives us $L_{-n} = \bar{L}_n$ as expected.

Thus $|B_{1,2}\rangle$ are boundary states. Note here that since all the indices on the oscillators were positive, we could have added/removed factors of sign(k) in the identifications and thus, in the expressions for the boundary states. This corresponds to adding a sign(k) in the ψ identifications, which also doesn't make a difference since δS is quadratic in ψ's.

Bosonised Boundary State

A theory of two Majorana (real) fermions (or one Dirac fermion) in $2d$ is dual to a bosonic theory via the following expressions [13]

$$\begin{aligned}
\psi &= e^{-i\frac{\pi}{2}\bar{p}} : e^{-i\sqrt{2}\phi(z)} : \\
\psi^\dagger &= e^{i\frac{\pi}{2}\bar{p}} : e^{i\sqrt{2}\phi(z)} : \\
\bar{\psi} &= e^{-i\frac{\pi}{2}p} : e^{i\sqrt{2}\bar{\phi}(\bar{z})} : \\
\bar{\psi}^\dagger &= e^{i\frac{\pi}{2}p} : e^{-i\sqrt{2}\bar{\phi}(\bar{z})} :
\end{aligned}$$

where ϕ and $\bar{\phi}$ are chiral bosons.

These two theories are dual in the sense that, the formulae above ensure that the central charge c of the theory, the correlators and other aspects of the theory are fully explained in terms of the fields ϕ and $\bar{\phi}$.

In bosonic theories, boundary states are of two types : Neumann boundary states where $\partial\varphi = 0$ at the boundary and Dirichlet states where $\varphi = 0$ at the boundary, where φ is the full boson field with both chiral components. Consider a Neumann boundary state $|N\rangle$. In terms of the chiral boson fields, the condition on the Neumann state becomes $\phi\,|N\rangle = \bar{\phi}\,|N\rangle$. To translate the bosonic Neumann condition into the fermionic one, we do the following.

$$\begin{aligned}
\psi\,|N\rangle &= e^{-i\frac{\pi}{2}\bar{p}} : e^{-i\sqrt{2}\bar{\phi}} : |N\rangle && (3.19)\\
&= e^{-i\frac{\pi}{2}\bar{p}} : e^{-i(\bar{x}-i\bar{p}+i\sum\limits_{n\neq 0}\alpha_n e^{-inz})} : |N\rangle && (3.20)\\
&= e^{-\frac{\pi}{2}[\bar{p},\bar{x}]}\bar{\psi}^\dagger e^{-i\frac{\pi}{2}(\bar{p}+p)}\,|N\rangle && (3.21)\\
&= e^{i\frac{\pi}{2}}\bar{\psi}^\dagger\,|N\rangle = i\bar{\psi}^\dagger\,|N\rangle && (3.22)
\end{aligned}$$

where we have used Baker-Campbell-Hausdorff formula to commute $e^{\bar{p}}$ through $e^{\bar{x}}$ using $[\bar{p},\bar{x}] = -i$ and using the fact that $(p+\bar{p})\,|N\rangle = 0$.

Similarly, we can show that $(\psi^\dagger - i\bar{\psi})\,|N\rangle$, $(\psi - i\bar{\psi})\,|D\rangle$ and $(\psi^\dagger - i\bar{\psi}^\dagger)\,|D\rangle$ vanish, where $|D\rangle$ is the Dirichlet state defined by $(\phi + \bar{\phi})\,|D\rangle = 0$.

Thus, the first boundary state $|B_1\rangle$, will be denoted as $|N\rangle$ and the second state $|B_2\rangle$, will be denoted as $|D\rangle$ (this matches the form given in [14]). Thus, $|B_2\rangle = |D\rangle$ is the state that appears in the gCC state, which will be discussed in the next section. Sometimes with a slight abuse of notation, we will also denote the sign(k) variants as $|N\rangle$ and $|D\rangle$.

3.2 Generalised Calabrese-Cardy State

For a sudden critical quench, it is only natural to assume that the post-quench state will be given by a time boundary state (the structure of which has been discussed in the previous section), since we have a boundary condition at $t = 0$ and we need it to be a conformal state as we are quenching to a CFT.

We note that being of the form $\sim e^{a^\dagger b^\dagger}$, the boundary state is not normalisable. Thus we need to add some cut-offs. This is why Calabrese and Cardy [6] proposed the following hypothesis for the post-quench state. The authors proved thermalisation in this state, which we will call the CC state.

$$|\psi_{CC}\rangle = e^{-\kappa H}\,|Bd\rangle \tag{3.23}$$

Later, MSS [1] extended this to a class of states with an infinite set of cut-offs and corresponding conserved charges, and proved thermalisation in this set of states. The generalised Calabrese-Cardy (gCC) state is thus given by

$$|\psi_{gCC}\rangle = e^{-\kappa_2 H - \sum\limits_n \kappa_n W_n}\,|Bd\rangle \tag{3.24}$$

In general the W_n's could be any operators that commute with the Hamiltonian. In the case of free fermions and bosons, one candidate set of charges presents itself : the W_∞ charges. These are of the form [15]

$$\begin{aligned}
T(z) &= \frac{1}{2}\left(\psi^*\partial\psi(z) - \partial\psi^*\psi(z)\right) \\
&= \sum_k |k| \left(a_k^\dagger a_k + b_k^\dagger b_k\right) \\
W_4(z) &= \frac{4}{5} q^2 \left(\partial^3\psi^*\psi(z) - 9\ \partial^2\psi^*\partial\psi(z) + 9\ \partial\psi^*\partial^2\psi(z) - \psi^*\partial^3\psi(z)\right) \\
&= \sum_k |k|^3 \left(a_k^\dagger a_k + b_k^\dagger b_k\right)
\end{aligned}$$

Thus, the gCC state can be written in k modes as

$$|gCC\rangle = e^{\sum\limits_k \kappa(k)\left(a_k^\dagger a_k + b_k^\dagger b_k\right)}\,|D\rangle \tag{3.25}$$

where $\kappa(k) = \sum_n \kappa_n |k|^n$.

3.3 gCC as the Post-Quench State

In the earlier chapter, we saw some properties of Bogoliubov coefficients that affected a transformation between the 'in' and 'out' bases. By inverting Eq (2.13) and using the properties in Sec 2.2.2, we get

$$a_{in,k} = \sqrt{\frac{\omega - m}{\omega}} \left(\alpha_+^* a_{k,out} + \mathrm{sign}(k)\beta_+^* b_{-k,out}^\dagger\right) \tag{3.26}$$

$$b_{in,k} = \sqrt{\frac{\omega - m}{\omega}} \left(\alpha_+^* b_{k,out} + \mathrm{sign}(k)\beta_+^* a_{-k,out}^\dagger\right) \tag{3.27}$$

Quenching the ground state $|0\rangle_{in}$, will not change the state, but now excitations will be in terms of the 'out' oscillators. Thus, to characterise the 'in' vacuum $|0\rangle_{in}$ in the 'out' basis, we need to solve the equation $a_{in,k}|0\rangle_{in} = b_{in,k}|0\rangle_{in} = 0$. Using Eq (3.26) and Eq (3.27), we can now write the post-quench ground state as

$$|0\rangle_{in} = \exp\left(\sum_k \gamma(k) a_{out}^\dagger(k) b_{out}^\dagger(-k)\right)|0\rangle_{out} \tag{3.28}$$

where $\gamma(k) = -\mathrm{sign}(k)\frac{\beta_+^*}{\alpha_+^*}$.

The only step remaining is to show that the post-quench state is a gCC state. We do this by a simple use of the Baker-Campbell-Hausdorff (BCH) formula. We will use the following form of the BCH formula

$$e^X e^Y e^{-X} = e^{\exp(s)Y} e^X \tag{3.29}$$

where $[X, Y] = sY$.

Consider the mode expansion of the gCC state.

$$\begin{aligned} |\psi_{gCC}\rangle &= \exp\left(\sum_{-\infty}^{\infty} \kappa(k)\left(a^\dagger(k)a(k) + b^\dagger(k)b(k)\right)\right)\exp\left(\sum_{-\infty}^{\infty} i a^\dagger(k) b^\dagger(-k)\right)|0\rangle \\ &= \exp\left(\sum_{-\infty}^{\infty} B(k)\right)\exp\left(\sum_{-\infty}^{\infty} A(k)\right)\exp\left(-\sum_{-\infty}^{\infty} B(k)\right)|0\rangle \end{aligned} \tag{3.30}$$

where we have defined $B(k) = \left(a^\dagger(k)a(k) + b^\dagger(k)b(k)\right)$ and $A(k) = i a^\dagger(k) b^\dagger(-k)$. Also note that $B(k)|0\rangle = 0$, so we can insert an additional factor of $\exp\left(-\sum_{-\infty}^{\infty} B(k)\right)$ acting on $|0\rangle$.

We find that $[B(l), A(k)] = 2\kappa(k)A(k)\delta_{kl}$. Thus, we can use the BCH formula with $s = 2\kappa(k)$. We can rearrange $|\psi_{gCC}\rangle$ as

$$\begin{aligned}
|\psi_{gCC}\rangle &= \prod_k e^{B(k)} \prod_l e^{A(l)} \prod_m e^{-B(m)} |0\rangle \\
&= \prod_k e^{B(k)} e^{A(k)} e^{-B(k)} |0\rangle = \prod_k \exp\left(e^{2\kappa(k)} A(k)\right) |0\rangle \\
&= \exp\left(\sum_{-\infty}^{\infty} i e^{2\kappa(k)} a^\dagger(k) b^\dagger(-k)\right) |0\rangle
\end{aligned}$$

where we have used the BCH formula in the second last equality.

For this to be a post-quench state we require that

$$\gamma(k) = \text{sign}(k)\frac{\beta_+^*}{\alpha_+^*} = ie^{2\kappa(k)} \tag{3.31}$$

$$\Rightarrow \kappa(k) = \frac{1}{2}\log\left(-i\ \text{sign}(k)\frac{\beta_+^*}{\alpha_+^*}\right) \tag{3.32}$$

Thus, as we saw in the previous section, the post-quench state will be a gCC state, as long as $\kappa(k)$ given by the formula above in terms of quench parameters, is analytic in k. [7] presents a more general discussion as to why we expect every post-quench ground state to be a gCC state.

3.3.1 Details

For our quench with tanh time-dependence, does γ give a gCC state? Indeed it does. The Taylor expansion of κ, calculated from γ in the sudden limit, is as follows

$$\kappa(k) = \frac{\text{sign}(k) i\pi}{4} - \frac{|k|}{2m} + \frac{|k|^3}{12m^3} + O\left(\frac{1}{m}\right)^4 \tag{3.33}$$

Since this expression is analytic, we can conclude that our post-quench state is a gCC state with chemical potentials $\mu_0 = \frac{\text{sign}(k)i\pi}{4}$, $\mu_2 = \beta = \frac{1}{2m}$, $\mu_4 = \frac{1}{12m^3}$, etc. Outside the sudden limit, we would have $\kappa(k)$ analytic in k but with coefficients dependent on both m and ρ.

Chapter 4

Quenching Squeezed States

In this chapter, we will deal with squeezed state correlators. The reason for this has been motivated in Chapter 1. We will be doing the fermionic analogue of the calculations in [7], where emphasis has been laid on the bosonic calculation.

4.1 Squeezed states

A squeezed state is one that satisfies the equation $\left(a_{in,k} - f(k)b^{\dagger}_{in,-k}\right)|f\rangle = 0$, where $f(k)$ is an appropriately normalisable squeezing function. One can make a Bogoliubov transformation to a basis $\tilde{a}_k$, $\tilde{b}_k$ such that $\tilde{a}_k|f\rangle = \tilde{b}_k|f\rangle = 0 \Rightarrow |f\rangle = |\tilde{0}\rangle$, i.e.

$$\begin{bmatrix} \tilde{a}_k \\ \tilde{b}^{\dagger}_{-k} \end{bmatrix} = \frac{1}{\sqrt{1+|f_k|^2}} \begin{bmatrix} 1 & -f(k) \\ f^*(k) & 1 \end{bmatrix} \begin{bmatrix} a_{in,k} \\ b^{\dagger}_{in,-k} \end{bmatrix}$$

The prefactor makes sure that $\{\tilde{a}_k, \tilde{a}^{\dagger}_{k'}\} = \delta_{k,k'}$ and $\{\tilde{b}_k, \tilde{b}^{\dagger}_{k'}\} = \delta_{k,k'}$. Again since we are in the Heisenberg picture, our state doesn't change with time. So to characterise the post-quench squeezed state, we can represent $|\tilde{0}\rangle$ in the $_{out}$ basis via a composite Bogoliubov transformation[1].

$$\frac{1}{\sqrt{1+|f_k|^2}} \begin{bmatrix} \alpha & \mathrm{sign}(k)\beta^* \\ -\mathrm{sign}(k)\beta & \alpha^* \end{bmatrix} \begin{bmatrix} 1 & f(k) \\ -f^*(k) & 1 \end{bmatrix} = \begin{bmatrix} A(k) & B^*(k) \\ -B(k) & A^*(k) \end{bmatrix} \tag{4.1}$$

Here, we have introduced $A(k) = \frac{1}{\sqrt{1+|f_k|^2}}(\alpha(k) - \mathrm{sign}(k)\beta^*(k)f^*(k))$ and

[1] Here we use $\alpha = \sqrt{\frac{\omega-m}{\omega}}\alpha_+$ and $\beta = \sqrt{\frac{\omega-m}{\omega}}\beta_+$ which give $\alpha^2+\beta^2 = 1$.

$B(k) = \frac{1}{\sqrt{1+|f_k|^2}}(\alpha^*(k)f^*(k) + \mathrm{sign}(k)\beta(k))$. These give

$$a_{out}(k) = A_k \tilde{a}_k + B_k^* \tilde{b}_{-k}^\dagger \tag{4.2}$$
$$b_{out}^\dagger(-k) = -B_k \tilde{a}_k + A_k^* \tilde{b}_{-k}^\dagger \tag{4.3}$$

Inverting this expression, we can write down the effective $\gamma(k)$ for this composite Bogoliubov transformation.

$$\gamma(k) = \frac{B^*}{A^*} = \frac{\alpha(k)f(k) + \mathrm{sign}(k)\beta^*(k)}{\alpha^*(k) - \mathrm{sign}(k)\beta(k)f(k)} \tag{4.4}$$

Note that $f = 0$ gives us our usual formula for the ground state quench Eq (3.28). Using the formula connecting the chemical potentials to γ Eq (3.32), an appropriately chosen f can give the CC state

$$\kappa(k) = -\kappa|k| \tag{4.5}$$

or the gCC state with one chemical potential

$$\kappa(k) = -\kappa|k| - \mu|k|^3 \tag{4.6}$$

It is important to note that from now on we will refer to the CC, gCC (with one chemical potential) and post-quench ground states separately, since as we saw before : the post-quench ground state is not the CC state or the gCC state with one chemical potential, contrary to earlier belief [6]. Only a gCC state with infinite chemical potentials (which are not necessarily small) mimics the post-quench ground state.

4.2 Squeezed Correlators

We want to find the fermion correlator in a squeezed state $|f\rangle$. To find correlators such as $\langle f|\psi^\dagger(r,t)\psi(0,t)|f\rangle$, we shift to the '$\sim$' basis. We will need to find ground state correlators in this basis, since $|f\rangle = |\tilde{0}\rangle$.

Now, we write down the chiral fermion fields in the '$\sim$' basis.

$$\begin{aligned}\psi(x,t) &= \int_0^\infty dk \left[-a_{out,k}e^{-ik\cdot x} + b_{out,k}^\dagger e^{ik\cdot x}\right] \qquad (4.7)\\ &= \int_0^\infty dk \left[-A(k)\tilde{a}_k e^{-ik\cdot x} - B(-k)\tilde{a}_{-k}e^{ik\cdot x} + A^*(-k)\tilde{b}_k^\dagger e^{ik\cdot x} - B^*(k)\tilde{b}_{-k}^\dagger e^{-ik\cdot x}\right]\end{aligned}$$

Similarly, we can write down the other chiral field as

$$
\begin{aligned}
\bar{\psi}(x,t) &= \int_{-\infty}^{0} dk \left[-A(k)\tilde{a}_k e^{-ik\cdot x} + B(-k)\tilde{a}_{-k} e^{ik\cdot x} - A^*(-k)\tilde{b}_k^\dagger e^{ik\cdot x} - B^*(k)\tilde{b}_{-k}^\dagger e^{-ik\cdot x}\right] \\
&= \int_0^\infty dk \left[-A(-k)\tilde{a}_{-k} e^{-i|k|t-ikx} + B(k)\tilde{a}_k e^{+i|k|t+ikx} - A^*(k)\tilde{b}_{-k}^\dagger e^{+i|k|t+ikx}\right. \\
&\quad \left. -B^*(-k)\tilde{b}_k^\dagger e^{-i|k|t-ikx}\right] \qquad (4.8)
\end{aligned}
$$

Now we are in a position to calculate the $\langle f|\psi^\dagger(r,t)\psi(0,t)|f\rangle$ correlator.

$$
\begin{aligned}
\langle \psi^\dagger(0,t)\psi(r,t)\rangle &= \int_0^\infty dk dk' \left\langle \tilde{0}\right| \left[A(-k)\tilde{b}_k e^{-i|k|t} - B(k)\tilde{b}_{-k} e^{i|k|t}\right] \\
&\quad \left[A^*(-k')\tilde{b}_{k'}^\dagger e^{i|k'|t-ik'r} - B^*(k')\tilde{b}_{-k'}^\dagger e^{-i|k'|t+ik'r}\right] \left|\tilde{0}\right\rangle \\
&= \int_0^\infty dk dk' \left[A(-k)e^{-i|k|t} A^*(-k') e^{i|k'|t-ik'r}\delta_{k,k'}\right. \\
&\quad \left. +B(k)e^{i|k|t} B^*(k') e^{-i|k'|t+ik'r}\delta_{-k,-k'}\right] \\
&= \int_0^\infty dk \left[|A(-k)|^2 e^{-ikr} + |B(k)|^2 e^{ikr}\right] \\
&= \int_{-\infty}^{\infty} dk e^{ikr}\left[\Theta(-k)|A(k)|^2 + \Theta(k)|B(k)|^2\right] \qquad (4.9)
\end{aligned}
$$

The calculation of the other correlators follows along similar lines, to give

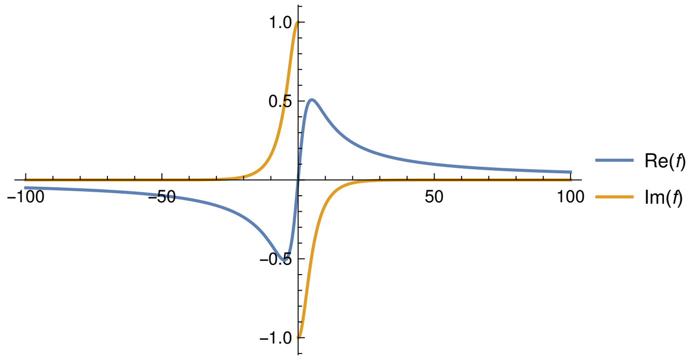
$$
\begin{aligned}
\langle \bar{\psi}^\dagger(0,t)\bar{\psi}(r,t)\rangle &= \int_{-\infty}^{\infty} dk e^{ikr}\left[\Theta(k)|A(k)|^2 + \Theta(-k)|B(k)|^2\right] \\
\langle \bar{\psi}^\dagger(0,t)\psi(r,t)\rangle &= \int_{-\infty}^{\infty} dk e^{ik(r-2t)}\left[\Theta(k)A(k)B^*(k) - \Theta(-k)A^*(k)B(k)\right] \\
\langle \psi^\dagger(0,t)\bar{\psi}(r,t)\rangle &= \int_{-\infty}^{\infty} dk e^{ik(r+2t)}\left[\Theta(k)A^*(k)B(k) - \Theta(-k)B^*(k)A(k)\right]
\end{aligned}
$$

4.3 Details

Consider a squeezed state with an f that satisfies the equation

$$
\frac{1}{2}\log\left(i\frac{(\beta(k)\mathrm{sign}(k) + \alpha(k)f)}{\alpha(k) - \beta(k)f\mathrm{sign}(k)}\right) = -\kappa\,|k| \qquad (4.10)
$$

$$
\Rightarrow f(k) = \frac{\left(e^{2\kappa k} - i\right)\left(\sqrt{k^2+m^2}+m\right) + k\left(-e^{2\kappa k} - i\right)}{\left(e^{2\kappa k} + i\right)\left(\sqrt{k^2+m^2}+m\right) + k\left(e^{2\kappa k} - i\right)} \qquad (4.11)
$$



This squeezed state is a CC state in terms of the 'out' oscillators, i.e. it produces a CC state post-quench. Also, it has a valid squeezing function since f vanishes as $|k| \to \infty$ and is finite everywhere, i.e. the state is normalisable. Thus, using the expression for f in Eq (4.11) and the expressions for A and B, we can perform the Fourier transforms necessary to find the correlators.

$$\begin{aligned}
\langle \psi^\dagger(0,t)\psi(r,t)\rangle &= \int_{-\infty}^{\infty} dk e^{ikr} \frac{1}{|f(k)|^2+1}\left(\Theta(k)\left(\alpha^2+\beta^2\left|f(k)\right|^2-2\alpha\beta\mathrm{Re}(f)\mathrm{sign}(k)\right)\right. \\
&\quad \left. +\Theta(-k)\left(\alpha^2\left|f(k)\right|^2+\beta^2+2\alpha\beta\mathrm{Re}(f)\mathrm{sgn}(k)\right)\right) \\
&= \frac{1}{2}\int_{-\infty}^{\infty} dk e^{ikr}(\tanh(2\kappa k)+1) \\
&= \frac{i\pi\mathrm{csch}\left(\frac{\pi r}{4\kappa}\right)}{4\kappa} \qquad (4.12) \\
\langle \bar{\psi}^\dagger(0,t)\bar{\psi}(r,t)\rangle &= -\frac{1}{2}\int_{-\infty}^{\infty} dk e^{ikr}(\tanh(2\kappa k)+1) \\
&= -\frac{i\pi\mathrm{csch}\left(\frac{\pi r}{4\kappa}\right)}{4\kappa} \qquad (4.13)
\end{aligned}$$

In keeping with MSS [1]'s results, these correlators have no time dependence. The entire time-dependence comes from the 'mixed' correlators below.

$$\begin{aligned}
\langle \bar{\psi}^\dagger(0,t)\psi(r,t)\rangle &= \int_{-\infty}^{\infty} dk e^{ik(r-2t)} \frac{1}{|f(k)|^2+1}\left(\alpha(k)\beta(k)(1-|f(k)|^2)\right. \\
&\quad \left. +\Theta(k)(\alpha(k)^2 f(k)-\beta(k)^2 f^*(k))-\Theta(-k)(\alpha(k)^2 f^*(k)-\beta(k)^2 f(k))\right) \\
&= -\frac{1}{2} i\int_{-\infty}^{\infty} dk e^{ik(r-2t)}\mathrm{sech}(2\kappa k) \\
&= -\frac{i\pi\mathrm{sech}\left(\frac{\pi(r-2t)}{4\kappa}\right)}{4\kappa} \qquad (4.14)
\end{aligned}$$

$$\begin{aligned}
\langle\psi^\dagger(0,t)\bar{\psi}(r,t)\rangle &= \frac{1}{2}i\int_{-\infty}^{\infty} dke^{ik(r+2t)}\mathrm{sech}(2\kappa k) \\
&= \frac{i\pi\mathrm{sech}\left(\frac{\pi(r+2t)}{4\kappa}\right)}{4\kappa} \qquad (4.15)
\end{aligned}$$

Similarly, a post-quench gCC state with one chemical potential can be achieved using the squeezing function

$$f(k) = \frac{k\left(-e^{2k\left(\kappa+k^2\mu\right)}-i\right)+\left(\sqrt{k^2+m^2}+m\right)\left(e^{2k\left(\kappa+k^2\mu\right)}-i\right)}{k\left(e^{2k(\kappa+k^2\mu)}-i\right)+\left(\sqrt{k^2+m^2}+m\right)\left(e^{2k(\kappa+k^2\mu)}+i\right)} \qquad (4.16)$$

This f too, corresponds to a normalisable state and yields the correlators

$$\begin{aligned}
\langle\psi^\dagger(0,t)\psi(r,t)\rangle &= \frac{1}{2}\int_{-\infty}^{\infty} dke^{ikr}(\tanh(2k(\kappa+\mu k^2))+1) \qquad (4.17)\\
\langle\bar{\psi}^\dagger(0,t)\bar{\psi}(r,t)\rangle &= -\frac{1}{2}\int_{-\infty}^{\infty} dke^{ikr}(\tanh(2k(\kappa+\mu k^2))+1) \qquad (4.18)\\
\langle\bar{\psi}^\dagger(0,t)\psi(r,t)\rangle &= -\frac{1}{2}i\int_{-\infty}^{\infty} dke^{ik(r-2t)}\mathrm{sech}(2k(\kappa+\mu k^2)) \qquad (4.19)\\
\langle\psi^\dagger(0,t)\bar{\psi}(r,t)\rangle &= \frac{1}{2}i\int_{-\infty}^{\infty} dke^{ik(r+2t)}\mathrm{sech}(2k(\kappa+\mu k^2)) \qquad (4.20)
\end{aligned}$$

4.4 Residue Calculus for the gCC State Correlators

Let us consider tanh and sech. They have poles at $\frac{i(2n+1)\pi}{2}$, i.e. $\tanh(2k(\kappa+\mu k^2))$ and sech $(2k(\kappa+\mu k^2))$ have poles at the roots of the equation

$$2k(\kappa+\mu k^2) = \frac{i(2n+1)\pi}{2} \qquad (4.21)$$

The roots are

$$\begin{aligned}
k_1 &= \frac{-4\ 3^{2/3}\kappa\mu+\sqrt[3]{3}\left(\mu^{3/2}\sqrt{192\kappa^3-81\mu(2\pi n+\pi)^2}+9i\pi\mu^2(2n+1)\right)^{2/3}}{6\mu\sqrt[3]{\mu^{3/2}\sqrt{192\kappa^3-81\mu(2\pi n+\pi)^2}+9i\pi\mu^2(2n+1)}} \\
&\xrightarrow{\mu\to 0} \frac{i\pi(2n+1)}{4\kappa}+\frac{i\mu(2\pi n+\pi)^3}{64\kappa^4}+O\left(\mu^2\right)
\end{aligned}$$

$$
\begin{aligned}
k_2 &= \frac{8\sqrt[3]{-3}\kappa\mu + i\left(\sqrt{3}+i\right)\left(\mu^{3/2}\sqrt{192\kappa^3-81\mu(2\pi n+\pi)^2}+9i\pi\mu^2(2n+1)\right)^{2/3}}{4\ 3^{2/3}\mu\sqrt[3]{\mu^{3/2}\sqrt{192\kappa^3-81\mu(2\pi n+\pi)^2}+9i\pi\mu^2(2n+1)}} \\
&\xrightarrow{\mu\to 0} \frac{i\sqrt{\kappa}}{\sqrt{\mu}} - \frac{i(2\pi n+\pi)}{8\kappa} + O\left(\sqrt{\mu}\right) \\
k_3 &= \frac{-8(-3)^{2/3}\kappa\mu - \sqrt[3]{3}\left(1+i\sqrt{3}\right)\left(\mu^{3/2}\sqrt{192\kappa^3-81\mu(2\pi n+\pi)^2}+9i\pi\mu^2(2n+1)\right)^{2/3}}{12\mu\sqrt[3]{\mu^{3/2}\sqrt{192\kappa^3-81\mu(2\pi n+\pi)^2}+9i\pi\mu^2(2n+1)}} \\
&\xrightarrow{\mu\to 0} -\frac{i\sqrt{\kappa}}{\sqrt{\mu}} - \frac{i(2\pi n+\pi)}{8\kappa} + O\left(\sqrt{\mu}\right)
\end{aligned}
$$

Note that out of these, contributions from the poles k_2 and k_3 will give the non-perturbative effects (since they do not give a sensible $\mu \to 0$ limit). Thus, to stay in the perturbative regime of μ, we will only consider the contributions of k_1. The residue of tanh at k_1 is

$$
\frac{1}{4\kappa} + \frac{3\mu(2\pi n+\pi)^2}{64\kappa^4} + O\left(\mu^2\right)
$$

and that of sech is

$$
\frac{(-1)^n}{4\kappa} + \frac{(-1)^n 3\mu(2\pi n+\pi)^2}{64\kappa^4} + O\left(\mu^2\right)
$$

Thus, we note that upto linear order in μ, we can write down the correlators as

$$
\begin{aligned}
\langle\psi^\dagger(0,t)\psi(r,t)\rangle &= -\langle\bar{\psi}^\dagger(0,t)\bar{\psi}(r,t)\rangle \\
&= \pi i \sum_{n\geq 0}\left(\frac{1}{4\kappa} + \frac{3\mu(2\pi n+\pi)^2}{64\kappa^4}\right) e^{-\left(\frac{\pi(2n+1)}{4\kappa}+\frac{\mu(2\pi n+\pi)^3}{64\kappa^4}\right)r} \\
\langle\psi^\dagger(0,t)\bar{\psi}(r,t)\rangle &= \pi\sum_{n\geq 0}(-1)^{n+1}\left(\frac{3\mu(2\pi n+\pi)^2}{64\kappa^4} + \frac{1}{4\kappa}\right) e^{-\left(\frac{\pi(2n+1)}{4\kappa}+\frac{\mu(2\pi n+\pi)^3}{64\kappa^4}\right)(r+2t)} \\
\langle\bar{\psi}^\dagger(0,t)\psi(r,t)\rangle &= \pi\sum_{n\geq 0}(-1)^{n}\left(\frac{3\mu(2\pi n+\pi)^2}{64\kappa^4} + \frac{1}{4\kappa}\right) e^{-\left(\frac{\pi(2n+1)}{4\kappa}+\frac{\mu(2\pi n+\pi)^3}{64\kappa^4}\right)|r-2t|}
\end{aligned}
$$

Chapter 5

Entanglement Entropy

In this chapter, we will follow the discussions in Calabrese and Cardy [16] [5] and Casini, Huerta and Fosco [17]. As we saw in Sec 1.2, the first step toward calculating the entanglement entropy of a region A is to calculate $\mathrm{Tr}_A\rho_A^n$. In scalar field theory, we use the path integral formulation to characterise the density matrix ρ. The component connecting two field configurations ϕ_x and $\phi'_{x'}$ is

$$\rho(\phi_x|\phi'_{x'}) = \langle\phi_x|e^{-\beta H}|\phi'_{x'}\rangle \tag{5.1}$$

This can be thought of as an imaginary time evolution by β. In this time coordinate τ, we can write down a standard path integral which gives the amplitude of one field configuration time-evolving to another.

$$\begin{aligned}\rho(\phi_x|\phi'_{x'}) &= Z(\beta)^{-1}\int D\phi(y,\tau)\prod_{x'}\delta(\phi(y,0)-\phi'_{x'}) \\ &\quad \prod_x \delta(\phi(y,\beta)-\phi_x)e^{-S[\phi(y,\tau)]}\end{aligned} \tag{5.2}$$

Here, we have simply added two boundary conditions at times 0 and β to the usual path integral. Does this give us $\mathrm{Tr}\rho = 1$? Tracing, we get

$$\begin{aligned}\int D\phi_x\rho(\phi_x|\phi_x) &= Z(\beta)^{-1}\int D\phi_x\int D\phi(y,\tau)\prod_x\delta(\phi(y,0)-\phi_x) \\ &\quad \prod_x\delta(\phi(y,\beta)-\phi_x)e^{-S[\phi(y,\tau)]} \\ \Rightarrow \mathrm{Tr}\rho &= Z(\beta)^{-1}\int D\phi(y,\tau)\prod_x\delta(\phi(y,0)-\delta(\phi(y,\beta))e^{-S[\phi(y,\tau)]} \\ &= 1\end{aligned}$$

Here, $Z(\beta)$ is chosen such that $\mathrm{Tr}\rho = 1$. It is evident that we can think of this path integral as being in a cylindrical geometry with the edges $\tau = 0$ and

$\tau = \beta$ sewn together. What we are really interested in is partial traces, since this is what can give us the entanglement entropy. When we take a partial trace to find the reduced density matrix for a spatial region $A = (a_1, a_2)$, we trace over $\bar{A}$, i.e. we will let $\int D\phi_x$ only run over $x \in \bar{A}$[1].

$$\begin{aligned}\rho_A(\phi'_{x'}|\phi''_{x''}) &= Z(\beta)^{-1} \int D\phi_x D\phi(y,\tau) \prod_{x\in\bar{A}} \left(\delta(\phi(y,0)-\phi_x)\delta(\phi(y,\beta)-\phi_x)\right) \\ &\qquad \prod_{x'\in A} \delta(\phi(y,0)-\phi'_{x'}) \prod_{x''\in A} \delta(\phi(y,\beta)-\phi''_{x''}) e^{-S[\phi(y,\tau)]}\end{aligned}$$

We can think of this path integral as being in a cylindrical geometry with a cut along spatial region A. On either side of the cut i.e. $\tau = 0_+$ and $\tau = \beta_-$, we have specified boundary conditions ($\phi'_{x'}$ and $\phi''_{x''}$ respectively in the above example) according to which component of ρ_A we are looking at. We call this manifold $\mathcal{M}_1$. Thus, in short-hand, we can now write

$$\rho_A(\phi'(x')|\phi''(x'')) = Z(\beta)^{-1} \int_{\mathcal{M}_1, BC} D\phi(y,\tau) e^{-S[\phi(y,\tau)]} \tag{5.3}$$

where BC denotes the boundary conditions imposed by $\phi'_{x'}$ at $\tau = 0_+$ and $\phi''_{x''}$ at $\tau = \beta_-$.

5.1 Replica Trick

The replica trick involves introducing a geometry which is a generalisation of $\mathcal{M}_1$ to describe ρ_A^n as a path integral. An easy description of ρ_A^n is

$$\rho_A^n(\phi'_{x'}|\phi''_{x''}) = \int D\phi_1(x_1)\cdots D\phi_{n-1}(x_{n-1})\rho_A(\phi'_{x'}|\phi_1(x_1))\cdots\rho_A(\phi_{n-1}(x_{n-1})|\phi''_{x''}) \tag{5.4}$$

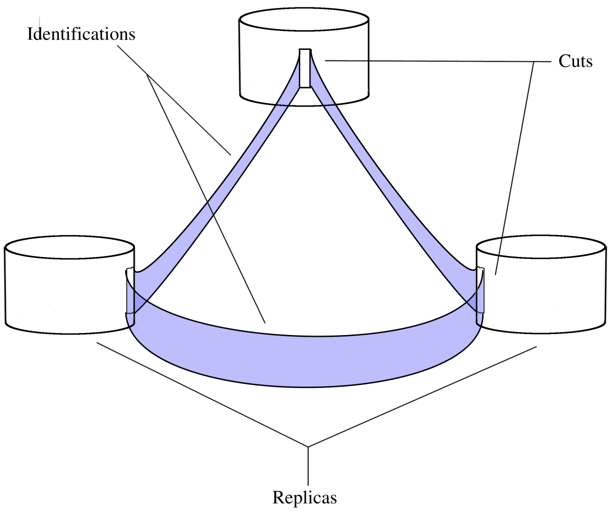
Further, tracing over ρ_A^n adds a further integral over ϕ_x and the condition that $\phi'_{x'} = \phi''_{x''} = \phi_x$. On performing these integrals $\int D\phi_i(x_i)$ and $\int D\phi_x$, we are left with n path integrals over $\phi_i(y_i)$, each corresponding to an insertion of ρ_A. From Eq (5.3), we see that this corresponds to having the following boundary conditions on the fields $\phi_i(y_i)$

$$\phi_{i+1}(y, 0_+) = \phi_i(y, \beta_-) \tag{5.5}$$

with the understanding that $n + 1 \sim 1$ and $y \in A$.

[1] This is analogous to the matrix partial trace where the sum over diagonal elements is restricted to diagonal elements lying in some part of the vector space.

Can this also be thought of as a simple path integral over a complicated manifold, let's say $\mathcal{M}_n$? Indeed it can. A representation of this manifold (which automatically imposes the above mentioned boundary conditions for $n = 3$) is



Here, each cylinder represents one of the spaces over which $\phi_i(y_i)$ is being integrated. They are called replicas, each corresponding to a single insertion of ρ_A in the trace.

Each blue path enforces the identifications in Eq (5.5). Lastly, each rectangle represents the cut of which we spoke earlier. The value of the fields at the cuts are again given by the boundary conditions in Eq (5.5).

The explicit calculation for $n = 2$ is as follows

$$\begin{aligned}
\rho_A^2(\phi''_{x''}|\phi'_{x'}) &= \int D\phi_1(x_1)D\phi_1(y_1)D\phi_2(y_2)\prod_{\bar{A}}(\delta(\phi_1(y_1,0)-\phi_1(y_1,\beta)) \\
&\quad \delta(\phi_2(y_2,0)-\phi_2(y_2,\beta)))\prod_{A}(\delta(\phi_1(y_1,0)-\phi''_{x''})\delta(\phi_1(y_1,\beta)-\phi_1(x_1)) \\
&\quad \delta(\phi_2(y_2,0)-\phi_1(x_1))\delta(\phi_2(y_2,\beta)-\phi'_{x'}))\, e^{-S[\phi_1(y_1)]-S[\phi_2(y_2)]} \\
&= \int D\phi_1(y_1)D\phi_2(y_2)\prod_{\bar{A}}(\delta(\phi_1(y_1,0)-\phi_1(y_1,\beta)) \\
&\quad \delta(\phi_2(y_2,0)-\phi_2(y_2,\beta)))\prod_{A}(\delta(\phi_1(y_1,0)-\phi''_{x''}) \\
&\quad \delta(\phi_1(y_1,\beta)-\phi_2(y_2,0))\delta(\phi_2(y_2,\beta)-\phi'_{x'}))\, e^{-S[\phi_1(y_1)]-S[\phi_2(y_2)]}
\end{aligned}$$

Diagrammatically, we now have two cylinders and one blue path joining one of the edges of each cut with the other. The second blue path comes from taking the trace.

$$
\begin{aligned}
\mathrm{Tr}\rho_A^2 \;&=\; \int D\phi_x \rho_A^2(\phi_x|\phi_x) \\
&\int D\phi_1(y_1)D\phi_2(y_2)\prod_{\bar{A}}\left(\delta(\phi_1(y_1,0)-\phi_1(y_1,\beta))\delta(\phi_2(y_2,0)-\phi_2(y_2,\beta))\right) \\
&\prod_{A}\left(\delta(\phi_1(y_1,0)-\phi_2(y_2,\beta))\delta(\phi_1(y_1,\beta)-\phi_2(y_2,0))\right)e^{-S[\phi_1(y_1)]-S[\phi_2(y_2)]}
\end{aligned}
$$

This now affects the full geometry with $n = 2$ replicas, 2 identifications and 2 cuts.

For fermions, we need to take the trace a little more carefully. Consider fermionic states $|\psi\rangle$ which form the eigenbasis of operator $\hat{\psi}|\psi\rangle = |\psi\rangle\psi$. We normalise the eigenbasis such that an inner product between different eigenvectors gives the Grassman delta function $\psi - \psi' = \langle\psi|\psi'\rangle$.

For this to happen, we need to define $\langle\psi|\hat{\psi} = -\psi\langle\psi|$ [18]. This can be verified by taking the inner product with $|\psi'\rangle$

$$
\begin{aligned}
\langle\psi|\hat{\psi}|\psi'\rangle = &\quad -\psi\langle\psi|\psi'\rangle &= \langle\psi|\psi'\rangle\psi' \qquad (5.6)\\
= &\; -\psi(\psi-\psi') &= (\psi-\psi')\psi' \qquad (5.7)
\end{aligned}
$$

Thus, though written as $\langle\psi|$, the bra eigenstate has a negative eigenvalue. Thus, the trace of ρ for a fermionic theory is

$$
\int D\psi_x \rho(\psi_x| - \psi_x) \tag{5.8}
$$

There are other signs to be taken care of as well. Consider the identifications in the $n = 2$ case.

$$
\begin{aligned}
\rho_A^2(\psi_x|\psi'_{x'}) \;\to\;& \prod_{\bar{A}}\left(\delta(\psi_1(y_1,0)+\psi_1(y_1,\beta))\delta(\psi_2(y_2,0)+\psi_2(y_2,\beta))\right) \\
& \prod_{A}\left(\delta(\psi_1(y_1,0)-\psi_x)\delta(\psi_1(y_1,\beta)-\psi_2(y_2,0))\delta(\psi_2(y_2,\beta)-\psi'_{x'})\right) \\
\to\;& \prod_{\bar{A}}\left(\delta(\psi_1(y_1,0)-\psi_1(y_1,\beta))\delta(-\psi_2(y_2,0)+\psi_2(y_2,\beta))\right) \\
& \prod_{A}\left(\delta(\psi_1(y_1,0)-\psi_x)\delta(-\psi_1(y_1,\beta)+\psi_2(y_2,0))\delta(\psi_2(y_2,\beta)-\psi'_{x'})\right)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \mathrm{Tr}_A\rho_A^2 &= \int D\psi_x \rho(\psi_x|-\psi_x) \\
&\rightarrow \prod_{\bar{A}} (\delta(\psi_1(y_1,0)-\psi_1(y_1,\beta))\delta(\psi_2(y_2,0)-\psi_2(y_2,\beta))) \\
&\quad \prod_{A} (\delta(\psi_1(y_1,0)+\psi_2(y_2,\beta))\delta(\psi_1(y_1,\beta)-\psi_2(y_2,0)))
\end{aligned} \tag{5.9}
$$

Similarly, for n replicas, it can be shown that the final boundary condition between $\psi_1(y_1)$ and $\psi_n(y_n)$ picks up a factor of $(-1)^{n+1}$, while the others are as given in Eq (5.5).

5.1.1 Diagonalisation

We will continue with our presentation of the discussion in [17]. $\mathrm{Tr}_A\rho_A^n$ is now given by a path integral over a bunch of fields $\psi_i(y_i)$, on a complicated manifold $\mathcal{M}_n$. We can instead think of each of these fields as living on planes with cuts at A, such that on going from the top to bottom of the cut, the vector Ψ transforms under the matrix

$$
T = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & \ddots & 1 & 0 \\ 0 & 0 & \ddots & 1 \\ (-1)^{n+1} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} \tag{5.10}
$$

We call the vector space in which this is a linear transformation the replica space. We can now go to the eigenbasis of T in replica space where we will call our fields ψ_k corresponding to the eigenvalues $e^{i\frac{k}{n}2\pi}$, where $k = -\frac{n-1}{2}, \cdots, \frac{n-1}{2}$.

In this space, we can now write our Lagrangian on the plane as

$$
\mathcal{L}_0(k) = \bar{\psi}_k\gamma^\mu\partial_\mu\psi_k + m\bar{\psi}_k\psi_k \tag{5.11}
$$

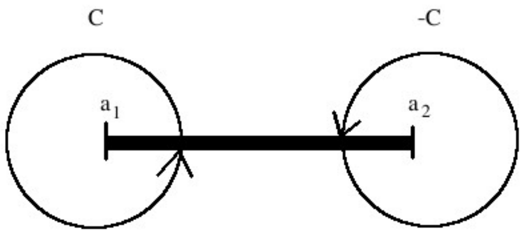
where the fields ψ_k are now decoupled. Each picks up a phase $e^{i\frac{k}{n}2\pi}$ on going from the top to the bottom and an opposite phase while going from the bottom to the top of the cut.

This presents a problem, as it suggests that the fields ψ_k are multi-valued on our region A. This can be taken care of by introducing gauge fields A_μ^k [17], which satisfy

$$
\int_C dx^\mu A_\mu^k = \frac{-2\pi k}{n} \tag{5.12}
$$

$$
\int_{-C} dx^\mu A_\mu^k = \frac{2\pi k}{n} \tag{5.13}
$$

where C is a closed path going from the top to bottom of the cut, while $-C$ runs from bottom to top.



These conditions (being valid for any such paths), ensure that

$$\epsilon^{\mu\nu}\partial_\nu A_\mu^k = 2\pi\frac{k}{n}\left(\delta(x-a_1)-\delta(x-a_2)\right) \tag{5.14}$$

where our region $A = (a1, a2)$.

This condition on A_μ makes sure that, when the fields go around the cut, they pick up the appropriate phase and that this phase cannot be removed via any gauge fixing conditions. Thus, these fields are single-valued and they satisfy the right boundary conditions.

$$\begin{aligned}\mathcal{L}(k) &= \bar{\psi}_k\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi_k + m\bar{\psi}_k\psi_k &(5.15)\\ &= \mathcal{L}_0(k) + iA_\mu^k j_k^\mu &(5.16)\end{aligned}$$

where j^μ is the Dirac current $\bar{\psi}_k\gamma^\mu\psi_k$.

Thus, the partition function on the manifold $\mathcal{M}_n$ can finally be written as follows, where all fields ψ_k are single-valued and decoupled.

$$Z = \prod_k Z_k = \prod_k \langle e^{i\int d^2x A_\mu^k j_k^\mu}\rangle \tag{5.17}$$

5.2 Formula via Bosonisation

Via the bosonisation formulae mentioned in Sec 3.1.1, we find that the Dirac current can be written in terms of bosonic operators as [19]

$$j_k^\mu = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\varphi \tag{5.18}$$

Thus, the partition function can now be written as

$$\begin{aligned} Z_k &= \left\langle e^{i\int d^2x\ A^k_\mu \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\partial_\nu\varphi_k}\right\rangle & (5.19)\\ &= \left\langle e^{i\int d^2x\ \partial_\nu A^k_\mu \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\varphi_k}\right\rangle & (5.20)\\ &= \left\langle e^{i\int d^2x\ 2\pi\frac{k}{n}(\delta(x-a_1)-\delta(x-a_2))\frac{1}{\sqrt{\pi}}\varphi_k}\right\rangle & (5.21)\\ &= \left\langle e^{i\sqrt{4\pi}\frac{k}{n}(\varphi_k(a_1)-\varphi_k(a_2))}\right\rangle & (5.22)\end{aligned}$$

As long as our state is Gaussian (which gCC states are), we can use Wick's theorem to bring down the φ fields. Re-summing this expression[2], gives us

$$Z[\mathcal{M}_n] = \prod_k e^{\frac{4\pi k^2}{n^2}\langle\varphi_k(a_1)\varphi_k(a_2)\rangle - \langle\varphi_k^2(a_1)\rangle} \tag{5.23}$$

As we showed in the beginning of this chapter $Z[\mathcal{M}_n] = \mathrm{Tr}_A\rho_A^n$. Thus, by using the formula for entanglement entropy in terms of $\mathrm{Tr}_A\rho_A^n$ given in Chapter 1, we finally arrive at the formula

$$S_A = \frac{2\pi}{3}\left(\langle\varphi(a_1)\varphi(a_2)\rangle - \langle\varphi^2(a_1)\rangle\right) \tag{5.24}$$

5.3 The Post-Quench Entanglement Entropy

According to the calculation in the previous section, the entanglement entropy of a region extending from x to y is given by the full $\langle\varphi(x)\varphi(y)\rangle - \langle\varphi(0)^2\rangle$ correlator. This is simply $\langle\phi(x)\phi(y)\rangle + \langle\bar{\phi}(x)\bar{\phi}(y)\rangle + \langle\phi(x)\bar{\phi}(y)\rangle + \langle\bar{\phi}(x)\phi(y)\rangle -$ (coincident terms), where ϕ and $\bar{\phi}$ are the chiral bosons. The first two terms are independent of time and will give the equilibrium value, while the last two give the decay.

Since we can think of the gCC state as a Gaussian state, we can use Wick's theorem to arrive at the expression (see App B)

$$\left\langle e^{i\alpha\phi(x)}e^{i\beta\phi(y)}\right\rangle = e^{-\alpha\beta\langle\phi(x)\phi(y)\rangle - \alpha^2\langle\phi(x)^2\rangle - \beta^2\langle\phi(y)^2\rangle} \tag{5.25}$$

Using the bosonisation formulae given in Sec 3.1.1

$$\begin{aligned} \psi &= e^{-i\frac{\pi}{2}\bar{p}} : e^{-i\sqrt{2}\phi(z)} : \\ \psi^\dagger &= e^{i\frac{\pi}{2}\bar{p}} : e^{i\sqrt{2}\phi(z)} : \\ \bar{\psi} &= e^{-i\frac{\pi}{2}p} : e^{i\sqrt{2}\bar{\phi}(\bar{z})} : \\ \bar{\psi}^\dagger &= e^{i\frac{\pi}{2}p} : e^{-i\sqrt{2}\bar{\phi}(\bar{z})} : \end{aligned}$$

[2] The exact nature of this re-summation is given in App B.

the $\langle\phi(r)\phi(0)\rangle - \langle\phi^2(0)\rangle$ correlator is given by

$$\begin{aligned}\log\langle\psi^\dagger(r)\psi(0)\rangle &= -\log\langle e^{i\phi(r)}e^{-i\phi(0)}\rangle &= -\log e^{-\langle\phi(r)\phi(0)\rangle+\langle\phi^2(0)\rangle} \\ && = \langle\phi(r)\phi(0)\rangle - \langle\phi^2(0)\rangle \end{aligned} \tag{5.26}$$

A similar formula for the $\langle\bar{\phi}(r)\bar{\phi}(0)\rangle - \langle\bar{\phi}^2(0)\rangle$ correlator gives the equilibrium value. What we are really interested in is the decay to this equilibrium value. This will be given by $-\log\langle\bar{\psi}^\dagger(r)\psi(0)\rangle$ and $-\log\langle\psi^\dagger(r)\bar{\psi}(0)\rangle$, since these are the only time-dependent correlators. Thus, finally we get the formula for the total $\langle\varphi(r)\varphi(0)\rangle$ correlator at separated points,

$$\begin{aligned}\langle\varphi(r)\varphi(0)\rangle &= \log\left(\frac{\langle\psi^\dagger(r)\psi(0)\rangle\langle\bar{\psi}^\dagger(r)\bar{\psi}(0)\rangle}{\langle\psi^\dagger(r)\bar{\psi}(0)\rangle\langle\bar{\psi}^\dagger(r)\psi(0)\rangle}\right) \\ &= \langle\phi(r)\phi(0)\rangle + \langle\bar{\phi}(r)\bar{\phi}(0)\rangle + \langle\phi(r)\bar{\phi}(0)\rangle + \langle\bar{\phi}(r)\phi(0)\rangle\end{aligned}$$

We also need the coincident parts. These are given entirely by

$$-\log\left(\langle\bar{\psi}^\dagger(0)\psi(0)\rangle\langle\psi^\dagger(0)\bar{\psi}(0)\rangle\right) = \langle\phi^2(0)\rangle + \langle\bar{\phi}^2(0)\rangle + 2\langle\phi(0)\bar{\phi}(0)\rangle \tag{5.27}$$

Thus, the full entanglement entropy formula is given by

$$\log\left(\frac{\langle\bar{\psi}^\dagger(0)\psi(0)\rangle\langle\psi^\dagger(0)\bar{\psi}(0)\rangle\langle\psi^\dagger(r)\psi(0)\rangle\langle\bar{\psi}^\dagger(r)\bar{\psi}(0)\rangle}{\langle\psi^\dagger(r)\bar{\psi}(0)\rangle\langle\bar{\psi}^\dagger(r)\psi(0)\rangle}\right) \tag{5.28}$$

5.3.1 For a CC State

In the squeezed state that mimics a CC state post-quench, we got the following correlators (Eq (4.12), (4.13), (4.14) and (4.15)):

$$\langle\psi^\dagger(r)\psi(0)\rangle = -\langle\bar{\psi}^\dagger(r)\bar{\psi}(0)\rangle \sim \text{cosech}\left(\frac{\pi r}{4\kappa}\right) \tag{5.29}$$

$$\langle\bar{\psi}^\dagger(r)\psi(0)\rangle \sim \text{sech}\left(\frac{\pi(r+2t)}{4\kappa}\right) \tag{5.30}$$

$$\langle\psi^\dagger(r)\bar{\psi}(0)\rangle \sim \text{sech}\left(\frac{\pi(r-2t)}{4\kappa}\right) \tag{5.31}$$

Taking the $\kappa \to 0$, i.e. the large temperature limit, for $2t < r$, we get the entanglement entropy as

$$\log\left(\frac{e^{-\frac{2\pi t}{4\kappa}}e^{-\frac{2\pi t}{4\kappa}}e^{-\frac{\pi r}{4\kappa}}e^{-\frac{\pi r}{4\kappa}}}{e^{-\frac{\pi(r+2t)}{4\kappa}}e^{-\frac{\pi(r-2t)}{4\kappa}}}\right) \tag{5.32}$$

$$= \log\left(e^{-\frac{\pi t}{\kappa}}\right) = -\frac{\pi t}{\kappa} \tag{5.33}$$

For $2t > r$, we get

$$\log\left(\frac{e^{-\frac{2\pi t}{4\kappa}}e^{-\frac{2\pi t}{4\kappa}}e^{-\frac{\pi r}{4\kappa}}e^{-\frac{\pi r}{4\kappa}}}{e^{-\frac{\pi(r+2t)}{4\kappa}}e^{-\frac{\pi(2t-r)}{4\kappa}}}\right) \tag{5.34}$$

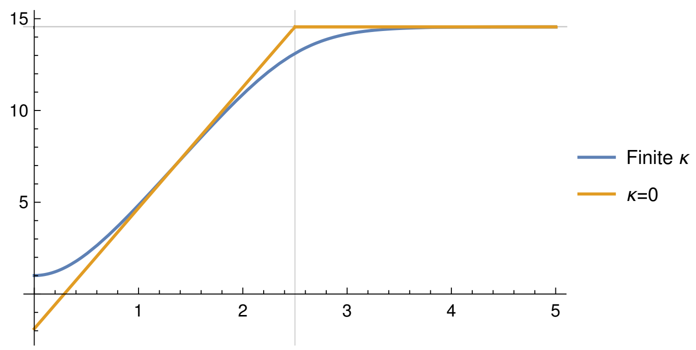
$$= \log\left(e^{-\frac{\pi r}{2\kappa}}\right) = -\frac{\pi r}{2\kappa} \tag{5.35}$$

which is exactly what we expect from a CC state in the large temperature limit [6]. We can get the sub-leading corrections by expanding log(sech) and log(cosech) in the $\kappa \to 0$ limit.

In fact, we can do better than high temperature or long-time limits. Using the formula in Eq (5.28), we can write down an expression for the full time-dependent entanglement entropy.

$$S_A = \frac{2\pi}{3}\log\left(\frac{\pi^2\left(\text{csch}^2\left(\frac{\pi r}{4\kappa}\right)+\text{sech}^2\left(\frac{\pi t}{2\kappa}\right)\right)}{16\kappa^2}\right) \tag{5.36}$$

We can plot this and the well-known high temperature limit as below :



where the horizontal line gives the asymptotic value $-\frac{2\pi}{3}\log\left(\frac{\pi^2}{16\kappa^2}\text{cosech}^2\left(\frac{\pi r}{4\kappa}\right)\right)$ and the vertical line is at $t = \frac{r}{2}$.

Chapter 6

Discussion

6.1 Results

We quote here the results of the calculations.

1. On quenching a system in the ground state or in a squeezed state, we arrive at a gCC state (which is the ansatz used by MSS [1] in their proof of subsystem thermalisation).

2. The post-quench correlators for a ground state quench with tanh time dependence, do not behave as predicted by MSS [1] or Cardy and Sotiriadis [20] because of the large chemical potentials μ_n.

3. Prepared post-quench CC and gCC states thermalise as expected by the aforementioned works, because the chemical potentials have been tuned to the perturbative regime.

4. Entanglement entropy for fermions after a quench can be calculated since the twist operators are known and their correlators are easily calculated [17].

5. Entanglement entropy in a prepared CC state matches earlier results.

6.2 Future Directions

The entanglement entropy calculation for the CC state, is a check of the method we have established in the course of this thesis. We can further this method by calculating and understanding the time-dependent entanglement entropy, following a ground state quench.

We can also use this method to calculate the entanglement entropy of a prepared post-quench gCC state with a single chemical potential, via the residue calculus method discussed in Sec 4.4. Here we will need to understand the various limits to be taken to get a sensible answer, since it is impossible to compute the infinite sum of residues.

This thesis is a continuation of [7] (where emphasis is laid on the bosonic calculation) to the fermionic case. Another possible generalisation of this thesis or of [7], would be to understand quenches in higher dimensions. The post-quench state will remain a gCC state (since the discussions in Chapter 3 can be easily generalised), but calculating correlators will not be as simple. Also, calculating the entanglement entropy via our method will not be possible, since the bosonisation formula is special to $2d$.

Lastly, it would be interesting to understand the holographic dual of this thesis and of [7]. In MSS [1], it is observed that the relaxation rate of various correlators matches the quasi-normal modes (QNM's) of a higher spin black hole. It would be interesting to see whether or not all the branches (perturbative and non-perturbative) of the relaxation rate in our calculation match all the branches of these QNM's as well.

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Appendix A

Proof of Adiabatic Theorem

Consider the usual solution of a time-independent Schrödinger equation.

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-iE_n t/\hbar} \tag{A.1}$$

where $c_n = \langle \Psi | \psi_n \rangle$.

In a system with a time-dependent Hamiltonian, we can simply generalise this solution.

$$\Psi(x,t) = \sum_n c_n(t) \psi_n(x) e^{-\frac{i}{\hbar}\int_0^t E_n(t')dt'} \tag{A.2}$$

To find the equation governing $c_n(t)$, we plug this back into Schrödinger equation and take its inner product with ψ_m, to get

$$\dot{c}_m(t) = -c_m \langle \psi_m | \dot{\psi}_n \rangle - \sum_{n \neq m} c_n \frac{\langle \psi_m | \dot{\hat{H}} | \psi_n \rangle}{E_n - E_m} e^{-\frac{i}{\hbar}\int_0^t (E_n(t') - E_m(t'))dt'} \tag{A.3}$$

The adiabatic theorem states that if $\dot{\hat{H}} << E_n - E_m$, then the second term can be ignored and the equation becomes

$$\begin{aligned} \dot{c}_m(t) &= -c_m \langle \psi_m | \dot{\psi}_n \rangle && \text{(A.4)} \\ \Rightarrow c_m(t) &= c_m(0) e^{-\int_0^t \langle \psi_m(t') | \dot{\psi}_n(t') \rangle dt'} && \text{(A.5)} \end{aligned}$$

Thus, in this limit, time evolution of the time-dependent system just adds an additional phase to the state, i.e. eigenstates of the old Hamiltonian remain in the same eigenstate (of the new Hamiltonian) upto a phase.

Appendix B

Vertex Operator Correlators

In conformal field theory, the operators $e^{i\alpha\phi}$ are known as vertex operators. They have conformal weight $\frac{\alpha^2}{8\pi}$ [12]. In the body of this thesis, we have used a formula for vertex operator correlators Eq. (5.25) that expresses them in terms of $\langle\phi\phi\rangle$ correlators. For this, we need Wick's theorem

$$\langle\varphi_1\varphi_2\cdots\varphi_n\rangle = \sum \text{all contractions} \tag{B.1}$$

This statement of Wick's theorem is valid in all Gaussian states of free field theories. One easy thing to note is that when n is odd, the expectation values vanish.

Applying Wick's theorem to our exponential operator correlator, will give us the formula in Eq. (5.25).

$$\begin{aligned}\langle\psi|e^{i\alpha\phi(0)}e^{i\beta\phi(x)}|\psi\rangle &= \langle\psi|\left(1 + i\alpha\phi(0) - \frac{\alpha^2}{2}\phi(0)^2\cdots\right) \\ &\quad \left(1 + i\beta\phi(x) - \frac{\beta^2}{2}\phi(x)^2\cdots\right)|\psi\rangle\end{aligned} \tag{B.2}$$

Expanding this out, we get

$$= \langle\psi|\left(1 - \frac{\beta^2}{2}\phi(x)^2 - \alpha\beta\phi(0)\phi(x) - \frac{\alpha^2}{2}\phi(0)^2 + \cdots\right)|\psi\rangle \tag{B.3}$$

$$= \exp\left(-\frac{\beta^2}{2}\langle\phi(x)^2\rangle - \alpha\beta\langle\phi(0)\phi(x)\rangle - \frac{\alpha^2}{2}\langle\phi(0)^2\rangle\right) \tag{B.4}$$