Relation between Algebra and Topology

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Relation between Algebra and Topology towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Patil Jatin Suresh at Indian Institute of Science Education and Research under the supervision of Dr. Rabeya Basu, Associate Professor, Department of Mathematics, during the academic year 2020-2021.

02.12.2021

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Dedicated to my family
and my peers at IISER

Declaration

I hereby declare that the matter embodied in the report entitled Relation between Algebra and Topology are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rabeya Basu and the same has not been submitted elsewhere for any other degree.

J. S. Patil

2.12.2021

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Abstract

In this thesis, we look at a few algebraic results about unimodular rows, some concering with the bounds on the numbers of generators. Then, we look at these results from the topological perspective, via the language of varieties and simplicial complexes.

Contents

Glossary of Notations

comm. - Commutative (ring, module)

$\langle a, b, \cdots \rangle$ - Ideal, module etc. generated by elements $a, b, \cdots$

$M_{m\times n}(A)$ - Set of $m \times n$ matrices with entries from A

$E_n(A)$ - Group of elementary $n \times n$ matrices with entries from A

$GL_n(A)$ - General linear group of $n \times n$ matrices with entries from A

$SL_n(A)$ - Special linear group of $n \times n$ matrices with entries from A

$\sqrt{I}$ - Radical of an ideal I of A

$dim(A)$ - Krull dimension of ring A

$ht(\mathfrak{p})$ - height of a prime ideal $\mathfrak{p}$

$\sqrt{I}$ - Radical of ideal I

$Nil(A)$ - Nilradical of ring A

$Jac(A)$ - Jacobson Radical of A

$A^\times$ - Group of units of A

Introduction

Projective modules can be defined as modules that are a direct summand to a free module. From the similarity, and as free modules tend to be easier to work with and are generally useful, the following question was asked around: *When is a projective module free?* Regarding the n-variable polynomial ring $A = k[X_1, X_2, \cdots, X_n]$, k a field, J. P. Serre had found that the finitely generated projective modules over this ring are stably free, i.e., the other direct summand is free. This led to the following question: *When is a stably free module free?* This statement would make sense for any commutative ring with identity.

In 1955, Serre asked if *finitely generated projective modules over the ring* $A = k[X_1, X_2, \cdots, X_n]$ *are all free.* This statement, known as 'Serre's Conjecture', was proved independently by D. Quillen and A. Suslin in 1976. Suslin, specifically, used the correspondence between the completion of unimodular rows and freeness of stably free modules to prove the statement.

An A-module P is stably free, in the sense that $P \oplus A^r = A^s$ for some $r, s \in \mathbb{N}$, $r \leq s$, iff we can write $P = ker(A^s \overset{f}{\twoheadrightarrow} A^r)$ for some epimorphism f. If $\alpha_{m\times n}$ represents f in matrix form, then the splitting of the corresponding exact sequence $0 \longrightarrow P \longrightarrow A^s \longrightarrow A^r \longrightarrow 0$ makes α right invertible. Conversely, given right invertible $\alpha_{m\times n}$ such that $\alpha\beta = I_r$, we can get a surjective $f : A^s \longrightarrow A^r$, $x \longmapsto \alpha x$ giving rise to the exact sequence $0 \longrightarrow P \longrightarrow A^s \xrightarrow{f} A^r \longrightarrow 0$, where $P = \{x \in A^s \,|\, \alpha x = 0\}$ is the solution set of α which is stably free. In this sense, the study of stably free A-modules is ends up being equivalent to the study of right invertible rectangular matrices. Accordingly, we also get that *for any right invertible matrix M, its solution space P is free iff M can be completed, using suitable rows, to a square invertible* *matrix.*

Now, the significance of these unimodular rows: Given a field k and an element $x = (x_1, x_2, \cdots, x_n) \in k^n$, x can be completed to a basis in k^n. For the analogous problem with rings, if, given $a = (a_1, a_2, \cdots, a_n) \in A^n$, if a can be completed to basis in A^n, then we can look at the basis elements as a matrix M instead, whose first row is a itself. As per the span of these elements, there exists another matrix N such that $MN = I_n$. Conversely, given an invertible matrix $M \in M_n(A)$ with first column being A, a basis can again be generated with a being a part. Thus, every completable row is unimodular. But, can the converse be said? The answer is generally no.

Now, as per weak form of Hilbert's Nullstellensatz, If $A = k[X_1, X_2, \cdots, X_m]$, where k is an algebraically closed field, and if $a_1, a_2, \cdots, a_n \in A$, then $a_1, a_2, \cdots, a_n$ have a common zero iff $\langle a_1, a_2, \cdots, a_n \rangle \neq A$. Similarly, in the case of unimodular rows, we find that, for $A = \mathbb{R}[X_1, X_2, \cdots, X_n]/I$, if $\vec{a} = (a_1, a_2, \cdots, a_n)$ is unimodular over A^n, then the a_i $(1 \leq i \leq n)$ do not vanish simultaneously anywhere on $V(I)$.

Equipped with this, we can set up maps as follows: Given A to be a coordinate ring over polynomial ring over $\mathbb{R}$ and letting $a = (a_1, a_2, \cdots, a_n) \in A^n$, we can have maps of the kind $F_{\vec{a}} : x \in V(I) \longmapsto (a_1 x, \cdots, a_n x) \in \mathbb{R}^n$, and similar maps $G_{\vec{a}} : V(I) \longmapsto S^{n-1}$ etc. we also find that $\vec{a} \overset{E_n(A)}{\sim} \vec{b} \implies F_{\vec{a}} \sim F_{\vec{a}}$ and so on. Using these, we try to see the topological ideas hidden behind various algebraic results for unimodular rows.

In this thesis, the first chapter is devoted to the various algebraic preliminaries needed for this thesis, particularly some stuff about the unimodular rows which are pivotal for this thesis. The second chapter then looks at a few algebraic results about the unimodular rows, particularly about getting bounds about generators of projective modules under certain conditions with the help of these unimodular rows. Finally, in the last chapter, we first take a look at the topological background needed for the rest of the chapter, including a subsection given to the simplices. Using this machinery, we look at some topological analogues of some algebraic results regarding unimodular rows.

Chapter 1

Preliminaries

In this chapter, we go through all the algebraic preliminaries needed for the thesis. The main references for this section have been [1] and [5].

1.1 Basic Notions

First, we declare that throughout the thesis, when a ring A is mentioned, it is commutative with identity unless specified.

Definition 1.1.1 (Algebra on a ring)**.** Let A be a ring. Then a ring B is said to be an A-*algebra*, if the following are satisfied:

1. B is an A-module.
2. $a(b_1 \cdot b_2) = (ab_1)b_2 = b_1(a \cdot b_2)$ for all $a \in A, b_1, b_2 \in B$.

Alternatively, let A be commutative with an identity. Then by an A-algebra, we mean a ring B with a ring homomorphism $f : A \longrightarrow B$.

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2. Alternatively, let A be commutative with an identity. Then by an A-algebra, we mean a ring B with a ring homomorphism $f : A \longrightarrow B$.

3. An A-algebra B is said to be *finitely generated*, if all the elements of B can be obtained as polynomials in a finite number of elements $b_1, b_2, \cdots, b_n$ of B.

Example. 1. The n-variable polynomial ring on A, i.e., $A[X_1, X_2, \cdots, X_n]$ is a finitely generated A-algebra.

2. The power series ring $B = A[[X]]$ is an algebra that is not finitely generated.

Definition 1.1.2. Let k be a field. Then, a finitely generated commutative k-algebra is called an *affine algebra*.

Definition 1.1.3 (Generating Set)**.** For an algebraic object, the *generating set* is a subset of the object, such that all the elements of the object in question are obtainable by a finite number of operations allowed in the object. This is generally denoted within angled brackets ($\langle \cdots \rangle$).

1. For rings, this means a set that can generate the entire ring with the addition and multiplication.

2. For ideals, this means a set that can generate the entire ideal with addition and products with ring elements.

3. For modules, this means a set that can generate the module with addition and scalar multiplication.

4. For A-algebras, this could be stated as a set that can generate the entire A-algebra as polynomials of the elements of the set.

A generating set S of an algebraic object R is called minimal if no proper subset of S can generate R.

Definition 1.1.4 (Noetherian Rings and Modules)**.** A module M is said to be *Noetherian* if any one of the equivalent conditions are satisfied:

1. Given any increasing sequence of submodules $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots,\ \ \exists\, n \in \mathbb{N}$ such that $I_n = I_{n+1} = I_{n+2} = \cdots$.
2. Every submodule of M is finitely generated.
3. Every nonempty collection of submodules of M has a maximal element.

Noetherian rings are just a special case where the ring A itself is viewed as an A-module, its ideals being its A-submodules.

Theorem 1.1.5 (Hilbert's Basis Theorem)**.** *[1, Thm. 7.5] If the ring A is Noetherian, then so is $A[X]$.*

Definition 1.1.6 (Operations on ideals)**.** For ideals $I,\ J \subseteq A$, we can have the following combinations:

1. (Sum of ideals) $I + J := \{a + b \,|\, a \in I, b \in J\}$
2. (Product of ideals) $IJ := \langle ab \,|\, a \in I, b \in J\rangle$
3. (Power of ideals) $I^n := \begin{cases} (0) & n = 0 \\ I & n = 1 \\ I^{n-1}I, & n > 1. \end{cases}$

Definition 1.1.7 (Prime Ideal)**.** Let A be a ring and let $\mathfrak{p} \subseteq A$ be an ideal. We say $\mathfrak{p}$ is a prime ideal of A, if $\mathfrak{P}$ is a proper ideal, and given $a, b \in A$ such that $ab \in \mathfrak{p}$, either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Equivalently, $\mathfrak{p}$ is a prime ideal of A iff $A/\mathfrak{p}$ is a domain.

Definition 1.1.8 (Maximal Ideal)**.** Let A be a ring and let $\mathfrak{m} \subseteq A$ be an ideal. We say I is a maximal ideal of A, if $\mathfrak{m}$ is a proper ideal and, given an ideal I such that $\mathfrak{m} \subseteq I \subseteq A$, either $\mathfrak{m} = I$ or $I = A$. Equivalently, $\mathfrak{m}$ is a maximal ideal of A if $A/\mathfrak{m}$ is a field. A maximal ideal is thus also a prime ideal.

Definition 1.1.9 (Radical of an Ideal)**.** Let A be a ring and let $I \subseteq A$ be an ideal. The radical of I, denoted as $\sqrt{I}$, is the set $\{a \in A \,|\, \exists\, n \in \mathbb{Z}^+ \text{ such that } a^n \in I\}$. It can be easily checked that the radical of any ideal is itself an ideal.

Definition 1.1.10 (Nilradical and Jacobson Radical)**.** 1. The *nilradical* of a ring A, denoted as $Nil(A)$, is the collection of all nilpotent elements of A, i.e., $\sqrt{(0)}$. Equivalently, it is also the intersection of all the prime ideals of A.

2. The *Jacobson radical* of a ring A, denoted by $Jac(A)$, is the intersection of all its maximal ideals.

As all maximal ideals are prime, intersection of all maximal ideals imposes fewer restrictions as compared to intersection of all prime ideals; as a result, $Nil(A) \subseteq Jac(A)$.

Lemma 1.1.11 (Prime Avoidance Lemma)**.** *Let A be a ring and let I be an ideal. Suppose $I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$, where $\mathfrak{p}_i \in Spec(A)$. Then $I \subseteq \mathfrak{p}_i$ for some i, where $1 \leq i \leq n$.*

Proof. We prove this with induction. The $n = 1$ case is obvious.

Suppose that the statement is true for $n-1$, and that $I \nsubseteq \mathfrak{p}_i$, $1 \leq i \leq n$. Then, for each such $\mathfrak{p}_i$, we get an element $x_i \in I$ such that $x_i \notin \mathfrak{p}_j, i \neq j$. Now we look at the element $a := \sum_{i=1}^{n} x_1 \cdots x_{i-1}x_{i+1} \cdots x_{n-1}x_n \in I$. If a were in $\mathfrak{p}_i$, then as $a = (\sum_{i\neq j} x_1 \cdots x_{i-1}x_{i+1} \cdots x_{j-1}x_{j+1} \cdots x_n)x_i + x_1 \cdots x_{i-1}x_{i+1} \cdots x_n$, we get that $x_1 \cdots x_{i-1}x_{i+1} \cdots x_n \in \mathfrak{p}_i$, which is impossible as $x_i \notin \mathfrak{p}_j, i \neq j$ for prime $\mathfrak{p}_i$, which means that $a \notin \mathfrak{p}_i, 1 \leq i \leq n$, i.e., $I \ni a \notin \bigcup_{i=1}^{n} \mathfrak{p}_i$. This proves that $I \nsubseteq \mathfrak{p}_i \Rightarrow I \nsubseteq \bigcup_{i=1}^{n} \mathfrak{p}_i$, $1 \leq i \leq n$, whose contrapositive is indeed the original statement for n. □

Definition 1.1.12 (Localisation)**.**

1. Let A be a ring and let $S \subseteq A$. We say that S is *multiplicatively closed*, if $1 \in S$, $0 \notin S$ and $a, b \in S \Rightarrow ab \in S$.

2. Let A be a ring and let $S \subseteq A$ be a multiplicatively closed set. Then we can define a ring denoted as $S^{-1}A$ or A_S, with as elements pairs of the form (a/s), $a \in A, s \in S$, the binary operations being $(a/s) + (b/t) = (at + bs/st)$, $(a/s)(b/t) = (ab/st)$; and the notion of equality being $(a/s) = (b/t)$ iff $\exists\, r \in S(r(at - bs) = 0)$.

Example.

1. By definition, for a domain A, $S := A \backslash \{0\}$ is multiplicatively closed. Then the set, $S^{-1}A$, is called the field of fractions of A.

2. Given an element $a \in A$, $a \neq 0$, the set $S := \{a, a^2, a^3, \cdots\}$ is multiplicatively closed and the corresponding localisation is denoted by A_x.

3. An ideal $\mathfrak{p} \subseteq A$ is prime iff $A \backslash \mathfrak{p}$ is multiplicatively closed, as the two statements are basically contrapositives of each other. In this case, with $S = A \backslash \mathfrak{p}$, the corresponding localisation is denoted by $A_{\mathfrak{p}}$.

Definition 1.1.13 (Local and Semilocal rings)**.** A ring is called *local* if it has a unique maximal ideal, and *semilocal* if it has finitely many maximal ideals.

Example.

1. All localised rings $A_\mathfrak{p}$ with the prime ideal $\mathfrak{p} \in A$ are local, the only maximal ideal being $\mathfrak{p}A_\mathfrak{p}$.

2. $\mathbb{Z}/m\mathbb{Z}$, $m \neq 0$ is a semilocal ring, local when m is a prime power.

Definition 1.1.14 (Krull dimension of a ring)**.** We say that a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ has a *length n*. We define the *height of* $\mathfrak{p}$ as the supremum of lengths of all the chains of the form $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$. The *Krull dimension of ring A* can now be defined as $dim(A) = \sup\{ht(\mathfrak{p}) \,|\, \mathfrak{p} \subset Spec(A)\}$.

Example.

1. As fields have no proper ideals, their Krull dimension is 0.

2. All PIDs have Krull dimension 1, as all the nonzero prime ideals become maximal just by their primality. An example would be $k[X]$, where k is a field.

Lemma 1.1.15 (Nakayama's Lemma)**.** *[1, Coro. 2.5 - Coro. 2.7] Following, you can see different versions of Nakayama's lemma, which build up gradually from the last.*

1. *Let A be a ring, M be a finitely generated A-module and $I \subseteq A$ be an ideal of A such that $I \subseteq Jac(A)$. Then there exists an element $A \in I$ such that $(1 - a)M = 0$.*

2. *Let A be a ring, M be a finitely generated A-module and let $I \subseteq Jac(A)$. Then, $IM = M$ implies $M = 0$.*

3. *Let A be a ring, M a finitely generated A-module, N a submodule of M and $I \subseteq Jac(A)$. Then, $M = N + IM$ implies $M = N$.*

Definition 1.1.16 (Prime spectrum of a Ring, Zariski Topology)**.** [1, Exercise 15, 17, Chap. 1] For a ring A, the set of all its prime ideals $\mathfrak{p} \subseteq A$ is called its spectrum, denoted as $Spec(A)$.
For all ideals $I \subseteq A$, we can define sets of the form $V(I) := \{\mathfrak{p} \supseteq I \,|\, \mathfrak{p} \in Spec(A)\}$. These sets have the following properties:

1. $V(0) = A$, $V(I) = \emptyset \iff I = A$.

2. $V(I \cap J) = V(IJ) = V(I) \cup V(J)$ for ideals $I, J \subseteq A$.

3. $\bigcap_{i \in I} V(I_i) = V(\bigcup_{i \in I} I_i)$ for all ideals $I_i \subseteq A$.

4. $V(I) = V(\sqrt{I})$.

5. $V(I) = V(J) \iff \sqrt{I} = \sqrt{J}$ for ideals I, J.

By the first 3 points, the collection of these $V(I)$'s would thus form a topology with these sets being closed sets (Def. 3.1.1). This topology endowed on $Spec(A)$ is called a *Zariski Topology*.
Let $X_a := A - V((a))$ for all $a \in A$. Then we can get the following properties for these X_a's:

1. $X_a \cap X_b = X_a b$ for $a, b \in A$.

2. $X_a = \emptyset \iff a \in Nil(A)$.

3. $X_a = A \iff a \in A^{\times}$.

4. $X_a = X_b \iff \sqrt{(a)} = \sqrt{(b)}$.

These X_as act as basic open sets for the Zariski topology.
Finally, it can be found that $Spec(A)$ under the Zariski topology is compact (Def. 3.1.5). Moreover, each X_a is also compact on its own.

1.2 Free and Projective Modules

Definition 1.2.1 (Free modules)**.** An A-module is said to be *free* if it has a basis, i.e., a generating set whose elements are linearly independent.

Example. 1. All modules over a field (i.e. vector spaces) are free as a consequence of Zorn's Lemma.

2. The affine space $A^{(n)}$ has the basis consisting of all $e_i = (0, \cdots, 1, \cdots, 0)$, $1 \leq i \leq n$, where only the i^{th} place is nonempty and 1, making it free.

3. In $\mathbb{Q}$, any two nonzero elements $\frac{a}{b}$, $\frac{c}{d}$ are $\mathbb{Z}$-linearly dependent, as $cb\frac{a}{b} - ad\frac{c}{d} = 0$. Thus, if $\mathbb{Q}$ were free over $\mathbb{Z}$, its basis would be a singleton, say $\{\frac{p}{q}\}$. But, that would make $\mathbb{Q}$ an infinite cyclic group ($\cong \mathbb{Z}$), which it is not. Thus, $\mathbb{Q}$ is not a free $\mathbb{Z}$-module.

Proposition 1.2.2. *Every short exact sequence of A-modules of the form $0 \longrightarrow M \longrightarrow N \xrightarrow{f} P \longrightarrow 0$ splits if P is a free module.*

Definition 1.2.3 (Projective modules)**.** An A-module P is said to be *projective* if it is a direct summand of a free module, i.e. $\exists\, A$-module M such that $P \oplus M$ is free. Equivalently,

1. Given a homomorphism $f : P \longrightarrow N$ and a surjective homomorphism $g : M \longrightarrow N$, there exists a homomorphism $h : P \longrightarrow M$ making the following diagram commutative:
$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "f"] & \\ M \arrow[r, two heads, "g"'] & N \arrow[r] & 0 \end{tikzcd}.$$

2. Every exact sequence $0 \longrightarrow M \longrightarrow N \xrightarrow{f} P \longrightarrow 0$ splits,
i.e. there exists a homomorphism $g : P \longrightarrow N$ such that $f \circ g = Id_P$.

3. The short exact sequence of A-modules $0 \longrightarrow M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} M_2 \longrightarrow 0$ induces another short exact sequence:
$$0 \longrightarrow Hom_A(P, M_0) \xrightarrow{f_1'} Hom_A(P, M_1) \xrightarrow{f_2'} Hom_A(P, M_2) \longrightarrow 0.$$

Proposition 1.2.4. *A finitely generated projective module over a local ring is free.*

Proof. Let A be a ring with $\mathfrak{m}$ the only maximal ideal, and let P be a f.g. projective A-module. Then, we can have $P/\mathfrak{m}P$ as an f.g. projective module over the field A/m, thus becoming free. Suppose the basis of $P/\mathfrak{m}P$ is $\{\overline{x_1}, \overline{x_2}, \cdots, \overline{x_n}\}$, finite as P itself is finitely generated. Accordingly, we can have $Q := x_1P \oplus x_2P \oplus \cdots \oplus x_nP$ as a submodule of P, which is free by construction. Then we can write $P = Q \oplus \mathfrak{m}P$. As $\mathfrak{m}$ is the sole maximal ideal, $\mathfrak{m} = Jac(A)$, and thus, by 3. of Lemma 1.1.15, $P = \langle x_1, x_2, \cdots, x_n \rangle$. ∎

Definition 1.2.5 (Rank of a module)**.** For a finitely generated free A-module M, we say that M *is of rank* n, if $M \cong A^n$ for some n. For a finitely generated projective A-module P, as per 1.2.4, we can define a map $rkP : Spec(A) \longrightarrow \mathbb{Z}$, $\mathfrak{p} \longmapsto rk(P_{\mathfrak{p}})$. This function is then called the *rank of* P. We say P has a constant rank r, if rkP is constant at r, i.e., if $rk(P_{\mathfrak{p}}) = r$ for all $\mathfrak{p} \in Spec(A)$.

Proposition 1.2.6. *[5, Prop. I.3.7, pp. 17] Let A be a semilocal ring and let P be a finitely generated projective module with constant rank n. Then, P is free.*

Proof. Let $\mathfrak{m}_1, \mathfrak{m}_2, \cdots, \mathfrak{m}_{\mathfrak{r}}$ be the maximal ideals of A. For $1 \leq i \leq r$, choose $x_{i1}, x_{i2}, \cdots, x_{in} \in P$ such that their images in $P_{\mathfrak{m}_i}$ are a free basis for said $P_{\mathfrak{m}_i}$. by Chinese Remainder Theorem, $\exists\, x_j$, $1 \leq j \leq n$ such that $x_j \equiv x_{ij}\,(mod\, P_{\mathfrak{m}_i})$, $1 \leq i \leq r$. Accordingly, we get $x_1, x_2, \cdots, x_n$ as a free basis for all $P_{\mathfrak{m}_i}, 1 \leq i \leq r$, after localization. ☐

Definition 1.2.7 (Stably free modules)**.** An A-module P is said to be *stably free*, if there exists $m \in \mathbb{Z}^+$ such that $P \oplus A^m$ is free.

Remark 1.2.8. In the definition, we stated that $m \in \mathbb{Z}^+$. This is because, if we do not restrict m to be finite, then *all* projective modules would end up satisfying the condition. The proof, also known as the Eilenberg swindle, goes as follows:

Let $M = P \oplus F$ be free and and let $N = M \oplus M \oplus M \oplus \cdots$. Thus N is also free. Now, we can have:

$$\begin{aligned} P \oplus N &\cong P \oplus M \oplus M \oplus M \oplus \cdots \\ &\cong P \oplus (F \oplus P) \oplus (F \oplus P) \oplus \cdots \\ &\cong (P \oplus F) \oplus (P \oplus F) \oplus \cdots \\ &\cong M \oplus M \oplus M \oplus \cdots \\ &\cong N, \text{ which itself is free.} \end{aligned}$$

We can also assume all stably free modules to be finitely generated, as:

Proposition 1.2.9 (Gabel)**.** *A stably free module that is not finitely generated is free.*

Proof. [5, Prop. I.4.2, pp.23] Let $M := N \oplus A^k$ be free, where N is not finitely generated. Then M itself is not finitely generated. Let $\{e_i\}_{\mathcal{I}}$ be a basis for M, $\mathcal{I}$

itself being infinite.
Now, the projection of M onto A^k is a surjective homomorphism $f : M \twoheadrightarrow A^k$, where $ker(f) = N$. By the definition of a generating set, the standard basis of A^k would be in the image of the span of finitely many $e_i's$, say the submodule $M' := Ae_1 + Ae_2 + \cdots Ae_l$ would have $f(M') = A^k$. Thus for each $x \in M$, $f(x) = f(x')$ for some $x' \in M'$. Thus $x - x' \in ker(f) = N$, so $M = M' + N$. Now, the module M' is finite free by construction, and by 1st isomorphism theorem, $A^k \cong M'/(M' \cap N)$. Now, writing $Q = M' \cap N$, we get two exact sequences;

$$0 \longrightarrow Q \longrightarrow N \longrightarrow N/Q \longrightarrow 0,$$
$$0 \longrightarrow Q \longrightarrow M' \longrightarrow A^k \longrightarrow 0.$$

By 2nd isomorphism theorem, $N/Q \cong M/M'$. As $M/M' = \bigoplus_{i \neq 1, \cdots, l} Ae_i$ is free with infinite rank, we can write $M/M' \cong A^k \oplus M''$ with another free A-module M'' of infinite rank. As N/Q and A^k are both free, both sequences split, thus giving:

$$N \cong Q \oplus N/Q \cong Q \oplus A^k \oplus M'' \cong M' \oplus M'',$$

which is free in itself. $\square$

1.3 Unimodular Rows

Definition 1.3.1 (Unimodular Rows)**.** A row $[a_1, a_2, \cdots, a_n] \in A^n$ is said to be *unimodular*, if the ideal with these generators, $\langle a_1, a_2, \cdots, a_n \rangle = A$.

Definition 1.3.2 (Completability)**.** A unimodular row $[a_1, a_2, \cdots, a_n] \in A^n$ is said to be *completable*, if there exists $\alpha \in GL_n(A)$ whose first row is $(a_1, a_2, \cdots, a_n)$.

This can be extended for the case that the given row is instead a first column as, if $\alpha \in GL_n(A)$, then $\alpha^t \in GL_n(A)$ as well.

Definition 1.3.3. Let $(a_1, a_2, \cdots, a_n)$, $(b_1, b_2, \cdots, b_n) \in A^n$. We say that $(a_1, a_2, \cdots, a_n) \overset{GL_n(A)}{\sim} (b_1, b_2, \cdots, b_n)$, if there exists $\sigma \in GL_n(A)$ such that $\sigma(a_1, a_2, \cdots, a_n)^t = (b_1, b_2, \cdots, b_n)^t$.

Remark 1.3.4. The relation defined above is an equivalence relation as, if there exists $\sigma \in GL_n(A)$ such that $\sigma(a_1, a_2, \cdots, a_n)^t = (b_1, b_2, \cdots, b_n)^t$, then we have $\sigma^{-1} \in GL_n(A)$ such that $\sigma^{-1}(b_1, b_2, \cdots, b_n)^t = (a_1, a_2, \cdots, a_n)^t$. Of course, $I_n(a_1, a_2, \cdots, a_n)^t = (a_1, a_2, \cdots, a_n)^t$; and if $\exists\, \sigma, \tau \in GL_n(A)$ such that $\sigma(a_1, a_2, \cdots, a_n)^t = (b_1, b_2, \cdots, b_n)^t$ and $\tau(b_1, b_2 \cdots, b_n)^t = (c_1, c_2, \cdots, c_n)^t$, then $\sigma\tau(a_1, a_2, \cdots, a_n)^t = (c_1, c_2, \cdots, c_n)^t$.

Also, when $(a_1, a_2, \cdots, a_n) \overset{GL_n(A)}{\sim} (b_1, b_2, \cdots, b_n)$, we have $\langle b_1, b_2, \cdots, b_n\rangle \subseteq \langle a_1, a_2, \cdots, a_n\rangle$, and vice versa via symmetry, thus $\langle a_1, a_2, \cdots, a_n\rangle = \langle b_1, b_2, \cdots, b_n\rangle$.

Theorem 1.3.5. *A unimodular row $(a_1, a_2, \cdots, a_n)$ is completable iff $(a_1, a_2, \cdots, a_n) \overset{GL_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Proof. Assuming $(a_1, a_2, \cdots, a_n)$ is completable, there exists $\sigma \in GL_n(A)$ whose first column is $(a_1, a_2, \cdots, a_n)$, say:

$$\sigma = \begin{pmatrix} a_1 & a_{21} & \cdots & a_{n1} \\ a_2 & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{2n} & \cdots & a_{nn} \end{pmatrix}.$$

Now,

$$\sigma \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 & a_{21} & \cdots & a_{n1} \\ a_2 & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

thus making $(a_1, a_2, \cdots, a_n) \overset{GL_n(A)}{\sim} (1, 0, \cdots, 0)$.

Conversely, supposing that we have $(a_1, a_2, \cdots, a_n) \overset{GL_n(A)}{\sim} (1, 0, \cdots, 0)$, there exists $\sigma \in GL_n(A)$ such that:

$$\sigma \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

assuming $\sigma = (b_{ij})$. Then we get:

$$\begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{n1} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

thus:

$$\begin{matrix} \sigma \\ (\in GL_n(A)) \end{matrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix},$$

making $(a_1, a_2, \cdots, a_n)$ completable. □

Remark 1.3.6 (Correspondence between Stably Free Modules and Unimodular Rows)**.** [5, pp. 24-26] If P is a stably free A-module, then $\exists\, n, m \in \mathbb{Z}^+$ such that $P \oplus A^m = A^n$ by 1.2.8 and 1.2.9. Thus P can be placed in the short exact seq. $0 \longrightarrow P \xhookrightarrow{i} A^n \xrightarrow{f} A^m \longrightarrow 0$. Now, if the map f were represented by a matrix $M_{m\times n}$, then as the domain of f is free, the series splits, suppose, $A^m \xrightarrow{g} A^n$, then as $g \circ f = Id_{A^n}$, we can get $N_{n\times m}$, basically the representative of g, such that $MN = I_m$. In this case, any row of M and any column of N is unimodular, as the product of the i^{th} row of M and i^{th} column give the product $(I_m)_{ii}$=1.

On the other hand, if given a unimodular row $a = (a_1, \cdots, a_n)$, we can get the *solution space* for a as follows:

$$P := \left\{ \sigma = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in A^n \,\middle|\, a \cdot \sigma = 0 \right\}.$$

Now, one could define a map $f : A^m \longrightarrow A, x \longmapsto ax$. As per the ranks, this map is surjective, thus we have $P = ker(A^m \xrightarrow{f} A)$. We can thus define a short exact sequence $0 \longrightarrow P \xhookrightarrow{i} A^m \xrightarrow{f} A \longrightarrow 0$. As A is free, the sequence splits, and thus $P \oplus A = A^m$, making P stably free. Now, this is not unique to only unimodular rows, this can certainly be extended for rectangular matrices $M_{m\times n}$, $m \leq n$. This leads to an interesting similarity:

[5, Prop. I.4.3', pp. 26] For any right invertible matrix $M_{m\times n}$, the stably free solution space of M is free iff M can be completed to an invertible matrix by adding a suitable number of new rows.

Definition 1.3.7 (Elementary Group of Matrices)**.** Let A be a ring, and $n \in \mathbb{Z}$.

1. For $i, j \leq n$, let e_{ij} be the matrix whose only non-zero entry is 1 on the ij^{th} place, and let $E_{ij}(\lambda) := I_n + \lambda e_{ij}$, for $i \neq j$, $\lambda \in A$. Such E_{ij}'s are called *elementary matrices.*

2. Let $E_n(A)$ be the subgroup of $SL_n(A)$ generated by all such $E_{ij}(\lambda)$. This $E_n(A)$ is called the *elementary group.*

Remark 1.3.8. Just like we had the relation $\overset{GL_n(A)}{\sim}$, we can also have the relations $\overset{SL_n(A)}{\sim}$, $\overset{E_n(A)}{\sim}$. In fact, all these relations are equivalence relations on $Um_n(A)$. In that sense, we can say that a unimodular row $(a_1, a_2, \cdots, a_n) \in A^n$ is the first row of an elementary matrix $\alpha \in E_n(A)$, iff $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} 1, 0, \cdots, 0$. In this case, we say that the row $(a_1, a_2, \cdots, a_n)$ can be transformed to row $(1, 0, \cdots, 0)$ via elementary matrices, or that it is elementary completable.

Proposition 1.3.9. *[5, Prop. I.5.3, pp.35] Let A be a ring and let $(a_1, a_2, \cdots, a_n)$ be a unimodular row of length n such that a shorter unimodular row is contained in it. Then the row $(a_1, a_2, \cdots, a_n)$ is completable. In fact, it is elementary completable.*

Proof. Suppose $(a_{i_1}, a_{i_2}, \cdots, a_{i_m})$ is the shorter part of the original row that is also unimodular. Let $i \notin \{i_1, i_2, \cdots, i_n\}$. Now, as $(a_{i_1}, a_{i_2}, \cdots, a_{i_m})$ is unimodular, there exist $c_{i_1}, c_{i_2}, \cdots, c_{i_m} \in A$ such that $a_{i_1}c_{i_1} + a_{i_2}c_{i_2} + \cdots + a_{i_m}c_{i_m} = 1$. Then, we can get $a_i - \sum_{j=1}^{m} a_{i_j}c_{i_j}(a_i - 1) = a_i - (a_{i_1}c_{i_1} + a_{i_2}c_{i_2} + \cdots + a_{i_m}c_{i_m})(a_i - 1) = 1$. Now, as one can get, for $1 < i < j < n$:

$$E_{ij}(\lambda)\begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & \lambda & & \\ & & \vdots & \ddots & \vdots & & \\ & & 0 & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_i + \lambda a_j \\ \vdots \\ a_j \\ \vdots \\ a_n \end{pmatrix} \textit{etc.},$$

we can get $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (a_1, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_n)$
$\overset{E_n(A)}{\sim} (0, \cdots, 0, 1, 0, \cdots, 0)$ (1 at i^{th} place) $\overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$. □

Chapter 2

Number of Generators for a Projective Module

2.1 Preliminaries

Theorem 2.1.1. *[3, Prop. 5.3.6] Let $\alpha \in M_{m\times n}(A)$ and let $\mu_\alpha : A^n \longrightarrow A^m, X \longmapsto \alpha X$ be an A-module homomorphism, where $n \geq m$. Then μ_α is surjective iff $I_m(\alpha) = A$, where $I_m(\alpha)$ denotes the ideal generated by all $m \times m$ minors of α.*

Lemma 2.1.2. *Let $\alpha = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \in M_{m\times n}(A)$ and let $\mu_\alpha : A^n \longrightarrow A^m, X \longmapsto \alpha X$ be the A-module homomorphism, where $n \geq m$. Then, μ_α is surjective iff columns of α generate A^m.*

Proof. ($\Rightarrow$) Suppose μ_α is surjective. Suppose $Y \in A^m$. Then, $\exists\, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in A^n$

such that $Y = \alpha X$, *i.e.*,

$$Y = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n \end{pmatrix} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} x_2 + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n,$$

where $x_i \in A, 1 \leq i \leq n$. Thus, columns of α do generate A^m.

($\Leftarrow$) Suppose that the columns of α generate A^m. Thus, for any $Y \in A^m$, $\exists\, \beta_1, \beta_2, \cdots, \beta_n \in A$ such that:

$$Y = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} \beta_1 + \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} \beta_2 + \cdots + \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \beta_n.$$

Thus there exists $X = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} \in A^n$ such that $\mu_\alpha(X) = \alpha X = Y$. Thus, μ_α is surjective. $\square$

Remark 2.1.3. From Theorem 2.1.1 and Lemma 2.1.2, we can say that:

$$\left\langle \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\rangle = A^m$$

iff the ideal generated by all the $m \times m$ minors is A.

Lemma 2.1.4. *[2, Lemma 4.1.2] Let A be a Noetherian ring of finite dimension and let s be a non-zero divisor of A. Then, $dim(A_{s\langle 1+sA\rangle}) < dim(A)$.*

Proof. First, a claim: *For a maximal ideal $\mathfrak{m} \subseteq A$, $(\mathfrak{m} \cap \langle s\rangle \neq \emptyset) \vee (\mathfrak{m} \cap \langle 1+sA\rangle \neq \emptyset)$.*

If this claim is true, then any maximal ideal from A does not survive in $A_{s\langle 1+sA\rangle}$, thus making $dim(A_{s\langle 1+sA\rangle}) \leq dim(A) - 1$, thus proving the lemma.

Now, if $s \in \mathfrak{m}$ then the claim is satisfied. If not, $\langle s, \mathfrak{m}\rangle = A$, thus $\exists\, a \in A, b \in \mathfrak{m}$ such that $1 = b + sa$. Thus, $b = 1 - sa \in \langle 1 + sA\rangle$, thus $\mathfrak{m} \cap \langle 1 + sA\rangle \neq \emptyset$, thus the claim and the statement is true. □

Lemma 2.1.5. *Let A be a Noetherian ring and let S be the set of all non-zero divisors of A. If P is a f.g. projective module such that $S^{-1}A = \left\langle \frac{c_1}{s_1}, \cdots, \frac{c_k}{s_k} \right\rangle$, $s_i \in S$, $1 \leq i \leq k$, then $\exists\ s \in S$ such that $P_s = \left\langle \frac{c_1}{s_1}, \cdots, \frac{c_k}{s_k} \right\rangle$.*

Proof. Suppose $x_1, x_2, \cdots, x_n$ generate P. Then, we get $\frac{x_i}{1} \in S^{-1}/A$, i.e., $\frac{x_i}{1} = \sum_{j=1}^{k} \frac{r_{ij}c_j}{s_{ij}s_j}$, where $r_{ij} \in A$, $s_{ij} \in S$. Thus, by definition of equality within localisations, $\exists\, \dot{s}_i \in S$ such that $\dot{s}_i(x_i(\prod_{j=1}^{k} s_{ij}s_j) - \sum_{j=1}^{k} r_{ij}c_j(\prod_{l=1,l\neq j}^{k} s_{il}s_l)) = 0$. Now, let $s = \prod i = 1^n \ddot{s}_i = \prod_{i=1}^{n}\prod_{j=1}^{k} \dot{s}_i s_{ij} s_i$, where $\ddot{s}_i \coloneqq \prod_{j=1}^{k} \dot{s}_i s_{ij} s_i$. With this s, you can have $P_s = \left\langle \frac{c_1}{s_1}, \cdots, \frac{c_k}{s_k} \right\rangle$. □

Lemma 2.1.6. *[2, Lemma 3.2.4] Let A be a ring, P be a projective A-module and $s \in A$ be a non-zero divisor in A. Suppose there exist surjections $f : A_s^n \twoheadrightarrow P_s$ and $g : A_{1+sA}^n \twoheadrightarrow P_{1+sA}$. If $\exists\, \sigma \in GL_n(A_{s\langle 1+sA\rangle})$ such that $\hat{f}\sigma = \hat{g}$, where $\hat{f}, \hat{g} : A_{s\langle 1+sA\rangle} \twoheadrightarrow P_{s\langle 1+sA\rangle}$ are induced by f, g respectively; and if $\sigma = (\tau_1)_{1+sA}(\tau_2)_s$, where $(\tau_1)_{1+sA} \in GL_n(A_s), (\tau_2)_s \in GL_n(A_{1+sA})$, then P is generated by n elements.*

Theorem 2.1.7. *(Bass Cancellation Theorem, Matrix-Theoretic Version) [5, Thm. II.7.3', pp. 74] Let A be a Noetherian ring with $dim(A) = d$. Then for $n \geq d + 2$, $E_n(A)$ acts transitively on $Um_n(A)$. In other words, any unimodular row of length n is elementary completable if $n \geq d + 2$.*

Proof. Let $(a_1, a_2, \cdots, a_n) \in Um_n(A)$, where $n \geq d + 2$. Then we can find $b_1, b_2, \cdots, b_n \in A$ such that $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (b_1, b_2, \cdots, b_n)$ and that $ht\langle b_1, b_2, \cdots, b_i\rangle \geq i$, $1 \leq i \leq n$ [2, Lemma 2.1.9]. Thus, we can have $ht\langle b_1, b_2, \cdots, c_{d+1}\rangle \geq d+1 > d = dim(A)$, making $(b_1, b_2, \cdots, b_{d+1})$ unimodular. Now, as $n > d + 1$, we get that the unimodular row $(b_1, b_2, \cdots, b_n)$ contains the shorter unimodular row $(b_1, b_2, \cdots, b_{d+1})$. Therefore, we get $(b_1, b_2, \cdots, b_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$. Thus, $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$. □

Lemma 2.1.8. *[2, Corollary 2.6.5] Let A be a ring, $a, b \in A$ be non-zero divisors in A such that $aA + bA = A$ and $\sigma \in GL_n(A_{ab})$ such that σ can be connected to the identity matrix. Then, $\sigma = (\tau_1)_b(\tau_2)_a$, where $(\tau_1)_b \in GL_n(A_a), (\tau_2)_a \in GL_n(A_b)$. Of course, this directly holds true if $\sigma \in E_n(A_{ab})$.*

2.2 Bounds on the Number of Generators of Projective Modules

Theorem 2.2.1. *Let A be a Noetherian ring with $dim(A) = d$. Suppose that:*

$$\left\langle \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\rangle = A^m.$$

If $n \geq d + m + 1$, then $\exists\, \sigma \in E_n(A)$ such that:

$$\sigma \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} I_m \\ \\ 0 \end{pmatrix},$$

where I_m is the $m \times m$ identity matrix and 0 is the $(n-m) \times m$ zero matrix respectively.

Proof. We prove this by induction on m. The $m = 1$ case is covered by Theorem 2.1.7. Assume the theorem to be true for $m - 1$, $m > 1$. As we have:

$$\left\langle \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix} \right\rangle = A^m,$$

we get that the ideal generated by all the $m \times m$ minors itself is A, particularly $\langle a_{11}, a_{12}, \cdots, a_{1n} \rangle = A$. As $n \geq d + m + 1 \geq d + 2$, by Theorem 2.1.7, there exists

$\alpha \in E_n(A)$ such that:

$$\alpha \begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1n} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, taking:

$$\alpha \begin{pmatrix} a_{21} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{2n} & \cdots & a_{mn} \end{pmatrix} =: \begin{pmatrix} c_{21} & \cdots & c_{m1} \\ \vdots & \ddots & \vdots \\ c_{2n} & \cdots & c_{mn} \end{pmatrix},$$

We get:

$$\alpha \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} 1 & c_{21} & \cdots & c_{m1} \\ 0 & c_{22} & \cdots & c_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{2n} & \cdots & c_{mn} \end{pmatrix}.$$

Now, as $\alpha \in GL_n(A)$, we also get that:

$$\left\langle \begin{pmatrix} 1 \\ c_{21} \\ \vdots \\ c_{m1} \end{pmatrix}, \begin{pmatrix} 0 \\ c_{22} \\ \vdots \\ c_{m2} \end{pmatrix}, \cdots, \begin{pmatrix} 0 \\ c_{2n} \\ \vdots \\ c_{mn} \end{pmatrix} \right\rangle = A^m.$$

Thus,

$$\left\langle \begin{pmatrix} c_{22} \\ \vdots \\ c_{m2} \end{pmatrix}, \begin{pmatrix} c_{23} \\ \vdots \\ c_{m3} \end{pmatrix}, \cdots, \begin{pmatrix} c_{2n} \\ \vdots \\ c_{mn} \end{pmatrix} \right\rangle = A^{m-1}.$$

By assumption for $m-1$, $\exists\, \beta \in E_{n-1}(A)$ such that

$$\beta \begin{pmatrix} c_{22} & \cdots & c_{m2} \\ \vdots & \ddots & \vdots \\ c_{2n} & \cdots & c_{mn} \end{pmatrix} = \begin{pmatrix} I_{m-1} \\ \\ 0 \end{pmatrix}.$$

Taking $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & \beta \end{pmatrix}$, we get:

$$\gamma \begin{pmatrix} 1 & c_{21} & \cdots & c_{m1} \\ 0 & c_{22} & \cdots & c_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{2n} & \cdots & c_{mn} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & I_{m-1} \\ 0 & 0 \end{pmatrix},$$

where $b = (c_{21}, c_{31}, \cdots, c_{m1})$. Also, we can have:

$$\begin{pmatrix} 1 & -b \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & I_{m-1} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_m \\ \\ 0 \end{pmatrix}.$$

It's easy to see that this $\tau = \begin{pmatrix} 1 & -b \\ 0 & I_{n-1} \end{pmatrix} \in E_n(A)$.

Now just take $\sigma = \tau\gamma\alpha$, then $\sigma \begin{pmatrix} a_{21} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{2n} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} I_m \\ \\ 0 \end{pmatrix}$, thus proving the case for m. □

An easy remark:

Remark 2.2.2. If $\phi, \psi : A^n \longrightarrow A^m$ are two surjective homomorphisms, and if $n \geq d + m + 1$, then $\exists\, \sigma \in E_n(A)$ such that $\phi\sigma = \psi$.

This just follows from the fact that from Theorem 2.1, $\exists\, \beta_1, \beta_2 \in E_n(A)$ such that $\phi\beta_1 = \psi\beta_2 = \begin{pmatrix} I_m & 0 \end{pmatrix}_{m \times n}$. Taking $\sigma = \beta_1\beta_2^{-1}$ gives the result.

Definition 2.2.3 (Reduced ring)**.** A ring A is said to be reduced, if it has no nontrivial nilpotent elements.

Not all reduced rings are domains.

Example. $\mathbb{Z}^2$ has $(1, 0)$, $(0, 1)$ as zero divisors, but it is still reduced as per the definition.

Theorem 2.2.4. *[8, Thm. 3.3.5] Let A be a reduced Noetherian ring of dimension d and let P be a f.g projective A-module of constant rank n. Then, P is generated by $n+d$ elements over A.*

Theorem 2.2.5. *[8, Thm. 3.3.6] Let A be a Noetherian ring of dimension d and let P be a finitely generated projective A-module of constant rank n. Then P is generated by $n+d$ elements over A.*

Proof. The case where A is reduced is covered in Thm. 2.2.4.

Now let $B \coloneqq A/Nil(A)$. Then, B is reduced and $dim(A) = dim(B) = d$. As P is a finitely generated A-module of constant rank n, $P/Nil(A)P$ is a finitely generated B-module of constant rank n. By Thm. 2.2.4, $P/Nil(A)P$ is generated by $n+d$ elements, say $\overline{x_1}, \overline{x_2}, \cdots, \overline{x_{n+d}}$, thus $P = \langle x_1, x_2, \cdots, x_n\rangle + Nil(A)P$. As $Nil(A) \subseteq Jac(A)$, by Nakayama's Lemma, $P = \langle x_1, x_2, \cdots, x_n\rangle$. □

Theorem 2.2.6. *Let A be a Noetherian ring of dimension d such that all f.g. projective A-modules of constant rank d have a unimodular element. Let P be a f.g. projective A-module of constant rank $n \geq 2$. Then P is generated over $n+d-1$ elements over A.*

Proof. By Thm. 2.2.5, the given module P is generated by $n+d$ elements, giving us a surjective $A^{n+d} \overset{h}{\twoheadrightarrow} P$, leading to the short exact sequence
$0 \longrightarrow ker(h) \xrightarrow{i} A^{n+d} \xrightarrow{h} P \longrightarrow 0$ which splits as P is projective, thus $P \oplus ker(h) = A^{n+d}$. Now let
$Q = ker(h)$. For all $\mathfrak{p} \in Spec(A)$, we get $P_\mathfrak{p} \oplus Q_\mathfrak{p} \cong A_\mathfrak{p}^{n+d} \implies A_\mathfrak{p}^n \oplus Q_\mathfrak{p} \cong A_\mathfrak{p}^{n+d} \implies Q_\mathfrak{p} \cong A_\mathfrak{p}^d$ as per 1.2.4. This makes Q a projective module of constant rank d, thus Q has a unimodular element(say x), i.e., $f(x) = 1$ for some homomorphism $Q \xrightarrow{f} A$. As $P \oplus Q \cong A_\mathfrak{p}^{n+d}$, this f can be extended to an $f' : A^{n+d} \longrightarrow A$ such that $f'(x) = 1$, making x unimodular in A^{n+d} as well. As $x \in Q = ker(h)$, $h(x) = 0$ and thus we can have another surjection $h' : A^{n+d}/\langle x\rangle \longrightarrow P$. $n \geq 2$ means $n+d \geq d+2$ and thus by 2.1.7, x is elementary completable, making $A^{n+d}/\langle x\rangle \cong A^{n+d-1}$. Thus we can have a surjection from A^{n+d-1} to P, which means P can be generated by $n+d-1$ elements. □

Corollary 2.2.7. *Let A be an affine algebra of dimension d over an algebraically closed field (or over a finite field) such that all f.g. projective A-modules of constant* *rank d have a unimodular element. Let P be a f.g. projective A-module of constant rank n. Then, P is generated by $n+d-1$ elements.*

Proof. If $n \geq 2$, then the result is just the same as 2.2.6 as an affine algebra is Noetherian. For n=1 case, we can follow the proof of Thm. 2.2.6 up until the point that we get a surjection $h' : A^{n+d}/\langle x \rangle \longrightarrow P$, in this case $h' : A^{d+1}/\langle x \rangle \longrightarrow P$, as the value of n is irrelevant until then. Finally, pertaining the stably free A-modules of rank $dim(A)$ over an affine algebra A over a field, their freeness has been proven by Suslin for algebraically closed fields [7], and by Mohan Kumar et al. for finite fields [6]. Thus, for both cases, the unimodular rows of length $\geq d+1$ are completable, making the unimodular element x completable. Thus $A^{d+1}/\langle x \rangle \cong A^d$, allowing us to have a surjection from A^d to P. □

Corollary 2.2.8. *[8, Coro. 3.3.9] Let A be an affine algebra of dimension d over $\overline{\mathbb{F}_p}$, the algebraic closure of the finite field with p elements, such that one of the following conditions hold:*

1. *$d = 2$ and A is regular*
2. *$d \geq 3$.*

Let P be a f.g. projective A-module of constant rank n. Then P is generated by $n+d-1$ elements over A.

Chapter 3

Unimodular Rows From a Topological Perspective

In this chapter, we try to explain the topological interpretation of some algebraic results regarding unimodular rows. The main reference for this is [9].

3.1 Preliminaries

3.1.1 Naïve set Topology and Basic Algebraic Topology

Definition 3.1.1 (Topology)**.** Let $\mathcal{U}$ be a set and let τ be a collection of its subsets. Then, we say that τ is a topology for $\mathcal{U}$, if $\emptyset, \mathcal{U} \in \tau$, and if *arbitrary unions and finite intersections* of elements of τ are also in τ, in which case these elements of τ are called open sets. The complements of these elements are called closed sets. With this, you could define a topology with the closed sets, with *arbitrary intersections and finite unions* instead being in τ. The pair (X, τ) is called a *topological space*, sometimes referred just by the base set, if the given topology is clear from the context.

For this chapter, we will be mainly concerned with the Zariski Topology, defined below.

Definition 3.1.2 (Neighbourhoods)**.** In a topological space $(\mathcal{U}, \tau)$,

1. A neighbourhood for a point $x \in \mathcal{U}$ is a set N_x that contains an open set U_x containing x, i.e., $x \in U_x \subseteq N_x$.

2. A neighbourhood for a subset $S \subseteq \mathcal{U}$ is a set N_S that contains an open set U_S containing S itself, i.e., $S \subseteq U_S \subseteq N_S$.

Definition 3.1.3 (Continuous maps)**.** Let $(X_1, \tau_1), (X_2, \tau_2)$ be two topological spaces, and let $f : X_1 \longrightarrow X_2$ be a function. f is said to be *continuous*, if for all open sets $U \subseteq X_2$, the preimage $f^{-1}(U) = \{x \in X_1 \,|\, f(x) \in U\}$ is also open in X_1.

As the preimages preserve intersections and unions, the same definition can be extended to the case of closed sets as well, i.e., $f : X_1 \longrightarrow X_2$ is said to be continuous, if, for all closed sets $V \in X_2$, $f^{-1}(V)$ is also closed in X_1.

Unless specified otherwise, all the maps between topological spaces will be considered continuous.

Definition 3.1.4 (Homeomorphism)**.** Given two topological spaces X_1 and X_2, we say that a function $f : X_1 \longrightarrow X_2$ is a *homeomorphism*, if f is continuous, f is bijective, and if the inverse $f^{-1} : X_2 \longrightarrow X_1$ is also continuous. This is denoted as $X_1 \simeq X_2$.

Definition 3.1.5 (Compact Space)**.** A topological space $(\mathcal{U}, \tau)$ is said to be *compact*, if for every open cover $\{U_i\}_{i \in \mathcal{I}}$ of $\mathcal{U}$, there exists a finite subcover $\{U_i\}_{i \in \mathcal{F}}$ of $\mathcal{U}$, where $\mathcal{F} \subseteq \mathcal{I}$ and $|\mathcal{F}| < \infty$.

Definition 3.1.6 (Hausdorff Space)**.** A topological space $\mathcal{U}$ is called *Hausdorff* or T_2, if for any two points $x, y \in \mathcal{U}$, there exist neightbourhoods N_x, N_y such that $x \in N_x$, $y \in N_y$ and $N_x \cap N_y = \emptyset$.

Definition 3.1.7 (Regular and Normal spaces)**.** Let $\mathcal{U}$ be a topological space. $\mathcal{U}$ is said to be *regular*, if given a point x and a closed set V, there exist disjoint neighbourhoods $N_x \ni x$ and $N_V \supseteq V$; and *normal*, if given two closed sets V, W, there exist disjoint neighbourhoods $N_V \supseteq V$, $N_W \supseteq W$. Regular Hausdorff spaces and normal Hausdorff spaces are called T_3 and T_4 spaces respectively.

Proposition 3.1.8. *[10, Thm. 17.10] A compact Hausdorff space is regular and normal.*

Definition 3.1.9 (Homotopy)**.** A homotopy between two continuous functions $f_1, f_2 : X_1 \longrightarrow X_2$ between two topological spaces X_1, X_2 is a continuous map $H(x,t) : X_1 \times \mathcal{I}_{[0,1]} \longrightarrow X_2$ such that $H(x,0) = f_1$ and $H(x,1) = f_2$; where $\mathcal{I}_{[0,1]}$ is the closed interval $[0,1] \subseteq \mathbb{R}$.

Lemma 3.1.10. *Let T, T_1, T_2 be topological spaces, and suppose $f, f' : T \longrightarrow T_1$, $g, g' : T_1 \longrightarrow T_2$ are all homotopic. Then, $g \circ f \sim g' \circ f'$.*

Proof. Suppose there exist homotopies $F : f \sim f'$ and $G : g \sim g'$. Then we can have the function $H(x,t) = G(F(x,t),t)$. H is continuous as a composition of continuous functions. Moreover, $H(x,0) = G(F(x,0),0) = g(F(x,0)) = g(f(x)) = g \circ f$ and similarly $H(x,1) = g' \circ f'$. □

Definition 3.1.11 (Homotopic Equivalence)**.** Let $f : T \longrightarrow T'$ be continuous. We say that f is a homotopic equivalence between T and T', if $\exists$ continuous $g : T' \longrightarrow T$ such that $g \circ f \sim Id_T$, $f \circ g \sim Id_{T'}$.

Theorem 3.1.12. *If two spaces T, T' are homotopically equivalent, and if any continuous map from T to S^{n-1} is homotopic to a constant map, then any continuous map from T' to S^{n-1} is also homotopic to a constant map.*

Proof. Let $h' : T' \longrightarrow S^{n-1}$ be continuous. Now, $T' \sim h' \circ Id_{T'} \sim (h' \circ f) \circ g$, where $h' \circ f : T \longrightarrow S^{n-1}$ is also continuous and is thus homotopic to some constant map, say, $\overline{h} : T \longrightarrow \{x_0\} \subseteq S^{n-1}$. Thus we get $(h' \circ f) \circ g \sim \overline{h} \circ g$, where $\overline{h} \circ g : T' \to T \to \{x_0\}$. □

Definition 3.1.13 (Contractible Space)**.** A topological space X is called *contractible*, if the identity map Id_X is homotopic to a constant map on X.

Example. Any Euclidean space $\mathbb{R}^n$ is contractible, as the identity map $Id_{\mathbb{R}^n}$ is homotopic to $f : \mathbb{R}^n \longrightarrow \{0\}$ via the homotopy $H((x_1, x_2, \cdots, x_n), t) = ((1-t)x_1, (1-t)x_2, \cdots, (1-t)x_n)$. In the same vein, as $S^{n+1} \backslash \{x_0\} \simeq \mathbb{R}^n$ for all $n \in \mathbb{Z}^+$, the set $S^n \backslash \{x_0\}$ is also contractible.

Definition 3.1.14 (Separable Space)**.**

1. A subset X of a topological space $(\mathcal{U}, \tau)$ is said to be *dense*, if $X \cap U \neq \emptyset$ for all $U \in \tau$; or equivalently if $\overline{X} = \mathcal{U}$.

2. A topological space is said to be *separable* if it has a countable dense set.

3.1.2 Simplicial Complexes

Before the next fact, we define the concepts of simplexes, simplicial complexes and some of their properties.

Definition 3.1.15 (Simplex)**.**

1. A set of points $\{x_0, x_1, \cdots, x_k\} \in \mathbb{R}^k$ is said to be *geometrically independent*, if, given $\sum_{i=0}^k a_i x_i = 0$ and $\sum_{i=0}^k a_i = 0$, the only solution is $x_i = 0$, $1 \leq i \leq k$, or, equivalently, if a set of the form $\{x_1 - x_i, \cdots, x_{i-1} - x_i, x_{i+1} - x_i, \cdots, x_k - x_i\}$, $1 \leq i \leq k$ is linearly independent.

2. A *k-simplex* in $\mathbb{R}^n$, $n \geq k$, is a set of the form $\{\sum_{i=0}^k a_i x_i \mid \sum_{i=0}^k a_i = 1\}$ where $x_0, x_1, \cdots, x_k$ are geometrically independent. These x_i are called the *vertices* of the k-simplex, and the number of vertices k is called as the *dimension* of the k-simplex.

For example, a 0-simplex, 1-simplex, 2-simplex and 3-simplex would be a point, a line, a solid triangle and a solid tetrahedron respectively.

Definition 3.1.16 (Simplicial Complex)**.**

1. For two simplices $\sigma, \tau \subseteq \mathbb{R}^k$, we say that σ is a *face* of τ, if the set of vertices of σ is a subset of the set of vertices of τ.

2. A finite collection K of simplices in $\mathbb{R}^k$ is called a *simplicial complex*, if the following conditions are satisfied:

   (a) If σ is in K, then so are its faces,

   (b) If $\sigma_1, \sigma_2 \in K$, then either $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2$ is a common face for σ_1 and σ_2.

   The *dimension* of a simplicial complex K is the greatest integer n such that K contains an n-simplex.

Definition 3.1.17 (Subcomplex)**.** Let L, K be simplicial complexes. We say that L is a subcomplex of K if $L \subseteq K$.

Definition 3.1.18 (Geometric Carrier of a Simplicial Complex)**.** For a simplicial complex K, its geometric carrier, denoted by $|K|$, is the union of all its elements, i.e., $\bigcup_{\sigma \in K} \sigma$.

Definition 3.1.19 (Simplicial Map)**.** Let K, L be two simplicial complexes. Then, a map $f : K \longrightarrow L$ is called a *simplicial map*, if f sends vertices of K to vertices of L, and if the vertices $\{x_1, x_2, \cdots, x_k\}$ generate a simplex in K, then
$\{f(x_1), f(x_2), \cdots, f(x_k)\}$ also generate a simplex in L.

Now, a few relevant results about simplicial complexes [4]:

Theorem 3.1.20. *Let T be a simplicial complex of dimension $\leq r$ and let T' be a closed subcomplex of T. Suppose $f : T' \longrightarrow S^r$ is a continuous map. Then, f can be extended to $f' : T \longrightarrow S^r$ such that $f'|_{T'} = f$.*

Theorem 3.1.21. *Let T be a simplicial complex of dimension n and let T' be a closed subcomplex of T. Then, any continuous map $f : T' \longrightarrow \mathbb{R}^n - 0$ can be extended to a continuous map $f' : T \longrightarrow \mathbb{R}^n$ such that $f'(\{0\})$ is finite.*

Theorem 3.1.22. *Let T be a separable metric space and T' be a closed subspace of T. Suppose two maps $f, g : T' \longrightarrow \mathbb{R}^n$ are homotopic. If there exists an extension $f' : T \longrightarrow \mathbb{R}^n$ of f, then there also exists an extension $g' : T \longrightarrow \mathbb{R}^n$ such that $f' \sim g'$.*

Finally, we get to the simplicial approximation theorem.

Definition 3.1.23. Let K, L be two simplicial complexes, and let $f : |K| \longrightarrow |L|$ be a continuous map.

1. The *open star* of a vertex v in K, denoted as $ost(v)$, is the set $\bigcup\{Int(\sigma) \mid \sigma \in K,$ v vertex of $\sigma\}$.
2. The map $g : K \longrightarrow L$ is said to be a *simplicial approximation* of f, if, for each vertex v of K, $h(ost(v)) \subseteq ost(g(v))$.
3. K is said to be *star-related* to L with respect to f, if for each vertex v of K, there exists a vertex v' of L such that $f(ost(v)) \subseteq ost(v')$.

Theorem 3.1.24 (Simplicial Approximation Theorem)**.** *[4, Thm. 3.3.8] Let K and L be simplicial complexes, and let $f : |K| \longrightarrow |L|$ be a continuous map. Suppose K is star-related to L with respect to f. Then there exists a simplicial map $g : K \longrightarrow L$ such that g is a simplicial approximation of f. In particular, the induced map $|g| : |K| \longrightarrow |L|$ is homotopic to f.*

Definition 3.1.25 (Variety of an Ideal)**.** Let I be an ideal of $k[x_1, x_2, \cdots, x_m]$, k being a field. Then we can define the variety of the ideal I, denoted as $V(I)$, as the set of all tuples in k^m, on which all the elements of I vanish.

$$V(I) := \{(x_1, x_2, \cdots, x_n) \in k^m \,|\, \forall f \in I,\ f(x_1, x_2, \cdots, x_m) = 0\}.$$

Definition 3.1.26 (Ideal of a Variety)**.** The ideal of a variety V of $k[X_1, X_2, \cdots, X_m]$, k being a field, is the set of polynomials that are zero on the variety.

$$I(V) := \{\alpha \in k[X_1, X_2, \cdots, X_m] \,|\, \forall x \in V,\ \alpha(x) = 0\}.$$

Unless specified otherwise, for the ring $\mathbb{R}[X_1, X_2, \cdots, X_m]/I$, I will be the ideal of a real algebraic variety on $\mathbb{R}[X_1, X_2, \cdots, X_m]$, thus making $V(I) \subseteq \mathbb{R}^m$. This also lets us assume V(I) as triangulated Euclidean subspaces, as geometric carriers of certain simplicial complexes.

As $k[x_1, x_2, \cdots, x_m]$ is Noetherian by Hilbert's Basis Theorem, all such I are finitely generated. Thus $V(I)$ is the set of common zeroes of a finite number of polynomials, and thus closed under the Euclidean topology. In general, for any field k, there exists a (coarser) topology in which all such $V(I)$ are closed, called the Zariski topology. This is, in fact, the basic form of the general Zariski Topology with prime ideals containing a certain ideal being the closed sets instead of the $V(I)$s.

Also, by the Tietze Extension Theorem [10, Thm. 15.8], *any continuous map from a closed subset X of a normal topological space T to $\mathbb{R}$ can be extended to a continuous map from T itself to $\mathbb{R}$.* The algebraic analogue to this can be that *any polynomial function on V(I), I an ideal of $\mathbb{R}[X_1, X_2, \cdots, X_n]$ can be treated as a restriction of a polynomial function on $\mathbb{R}^m$.*

3.2 Approaching Results for Unimodular Rows from the Topological Point of View

The results in this section are taken from [9]. First, we look at some generic functions useful for this section.

Let $A = \mathbb{R}[X_1, X_2, \cdots, X_m]$, and let $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$. Then we can have a continuous map $\mathbf{F_{\tilde{a}}} : \mathbb{R}^{\mathbf{m}} \longrightarrow \mathbb{R}^{\mathbf{n}}$, defined by
$F_{\vec{a}}(x_1, x_2, \cdots, x_m) = (a_1(x_1, x_2, \cdots, x_m), \cdots, a_n(x_1, x_2, \cdots, x_m))$, for all $(x_1, x_2, \cdots, x_m) \in \mathbb{R}^n$. This can be extended for $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$, I being an ideal of a real algebraic variety in $\mathbb{R}[x_1, x_2, \cdots, x_m]$, to a continuous map $\mathbf{F_{\tilde{a}}} : \mathbf{V(I)} \longrightarrow \mathbb{R}^{\mathbf{m}}$ defined similarly, but with $(x_1, x_2, \cdots, x_m) \in V(I)$.

Now, a form of Hilbert's *Nullstellensatz* is as follows:

Theorem 3.2.1 (Hilbert Nullstellensatz, Weak form)**.** *If $A = k[X_1, X_2, \cdots, X_m]$, where k is an algebraically closed field, and if $a_1, a_2, \cdots, a_n \in A$, then $a_1, a_2, \cdots, a_n$ have a common zero iff $\langle a_1, a_2, \cdots, a_n \rangle \neq A$.*

Similarly, we find that:

Theorem 3.2.2. *For $A = \mathbb{R}[X_1, X_2, \cdots, X_n]/I$, if $\vec{a} = (a_1, a_2, \cdots, a_n)$ is unimodular over A^n, then the a_i $(1 \leq i \leq n)$ do not vanish simultaneously anywhere on $V(I)$.*

Thus, we can have $F_{\vec{a}} : V(I) \longrightarrow \mathbb{R}^n \backslash \{0\}$. Now we define a map $g : \mathbb{R}^n - \{0, 0, \cdots, 0\} \longrightarrow S^{n-1}$, $x \mapsto \frac{x}{||x||}$, $||x||$ being the norm of x. Accordingly we get the map $\mathbf{G_{\tilde{a}}} := \mathbf{g} \circ \mathbf{F_{\tilde{a}}} : \mathbf{V(I)} \longrightarrow \mathbf{S^{n-1}}$. Since $Id_{\mathbb{R}^n \backslash \{0,0,\cdots,0\}} \sim g$, we get that $Id_{\mathbb{R}^n \backslash \{0,0,\cdots,0\}} \circ F_{\vec{a}} = F_{\vec{a}} \sim G_{\vec{a}} = g \circ F_{\vec{a}}$ for all unimodular $\vec{a} = (a_1, a_2, \cdots, a_n)$.

Now we consider $A[X] = (\mathbb{R}[X_1, X_2, \cdots, X_m]/I)[X]$, I again being an ideal for a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_m]$. Then, $a \in I$ vanishes at $(x_1, x_2, \cdots, x_m, x)$ for all $x \in \mathbb{R}$ and $(x_1, x_2, \cdots, x_m) \in V(I)$. Thus, the variety for this $A[X]$ would be $V(I) \times \mathbb{R}$. Thus, for any unimodular row $(a_1(X), a_2(X), \cdots, a_n(X)) \in a[X]^n$ we get two homotopic maps $\mathbf{F_{\tilde{a}}[X]} : \mathbf{V(I)} \times \mathbb{R} \longrightarrow \mathbb{R}^{\mathbf{n}} - \{\mathbf{0}, \cdots, \mathbf{0}\}$ and
$\mathbf{G_{\tilde{a}}[X]} : \mathbf{V(I)} \times \mathbb{R} \longrightarrow \mathbf{S^{n-1}}$.

For the rest of this section, we will maintain the definitions of these four maps $F_{\vec{a}}$, $G_{\vec{a}}$, $F_{\vec{a}}[X]$ and $G_{\vec{a}}[X]$ introduced here.

Proposition 3.2.3. *Let $\vec{a} = (a_1, a_2, \cdots, a_n)$ and $\vec{b} = (b_1, b_2, \cdots, b_n)$ be two unimodular rows over $A = \mathbb{R}[X_1.X_2, \cdots, X_m]/I$, I being the ideal of a real algebraic variety in $\mathbb{R}[X_1.X_2, \cdots, X_m]$, such that $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (b_1, b_2, \cdots, c_n)$. Then the maps $F_{\vec{a}}$ and $F_{\vec{b}}$ are homotopic.*

Proof. As $\vec{a} \overset{E_n(A)}{\sim} \vec{b}$, $\exists\, \sigma = \prod_{i=1}^r E_{ij}(\lambda) \in E_n(A)$ such that $\vec{a}\sigma = \vec{b}$. Now take $\sigma(X) = \prod_{i=1}^r E_{ij}(\lambda X) \in E_n(A[X])$, which gives us $\sigma(0) = I_n$, $\sigma(1) = \sigma$. Thus $\vec{a}\sigma(0) = \vec{a}I_n = \vec{a}$ and $\vec{a}\sigma(1) = \vec{a}\sigma = \vec{b}$. Thus, $(a_1, a_2, \cdots, a_n)\sigma(X)$ is a unimodular row over $A[X]$, which can then let us have $F_{\vec{a}}[X] : V(I) \times \mathbb{R} \longrightarrow \mathbb{R}^n - \{0\}$, such that $F_{\vec{a}}[0] = F_{\vec{a}}$, $F_{\vec{a}}[1] = F_{\vec{b}}$, which makes the maps $F_{\vec{a}}$, $F_{\vec{b}}$ homotopic via $F_{\vec{a}}[X]$. □

Now, for elementary connected $\vec{a}$ and $\vec{b}$, we got that $F_{\vec{a}} \sim F_{\vec{b}}$. We also have $F_{\vec{a}} \sim G_{\vec{a}}$ for unimodular $\vec{a}$. Then, $G_{\vec{a}} \sim F_{\vec{a}} \sim F_{\vec{b}} \sim G_{\vec{b}}$, thus $G_{\vec{a}}$, $G_{\vec{b}}$ are also homotopic.

Now we look the underlying topological ideas for some results on unimodular rows, to illustrate how one can think about these results in a topological way.

Lemma 3.2.4. *Let $\vec{a} = (a_1, a_2, \cdots, a_n)$ be unimodular in A with a_1 being a unit in A. Then, $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Topological Approach. Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_n]/I$, I being an ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_n]$. As $\vec{a}$ is unimodular over A, we have the map $G_{\vec{a}} : V(I) \longrightarrow S^{n-1}$. As a_1 is a unit in A, it does not vanish in $V(I) \subseteq A$. Thus, for an element of p of the form $(0, x_2, \cdots, x_n) \in S^{n-1}$, there does not exist an element $\vec{x} \in V(I)$ such that $G_{\vec{a}}(\vec{x}) = p$. Thus, $Im(G_{\vec{a}}) \subseteq S^{n-1}\backslash\{p\}$. As $S^{n-1}\backslash\{p\}$ is contractible, $G_{\vec{a}}$ homotopic to a constant map by Thm. 3.1.12. □

Lemma 3.2.5. *Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$ be unimodular over A, such that $(a_1, a_2, \cdots, a_i) \in A^i$ is also unimodular for some $i < n$. Then, $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Topological Approach. Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$, I being an ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_m]$. As $(a_1, a_2, \cdots, a_n)$ is unimodular over

A, we have the map $G_{\vec{a}} : V(I) \longrightarrow S^{n-1}$. As $(a_1, a_2, \cdots, a_i)$ is also unimodular over A, $a_1, a_2, \cdots, a_i$ do not vanish simultaneously anywhere on $V(I)$. Thus, for any $q \in S^{n-1} \subseteq \mathbb{R}^n \backslash \{0\}$ whose first i coordinates are zero, there does not exist $\vec{x} \in V(I)$ such that $G_{\vec{a}}(\vec{x}) = q$. Thus, again, $Im(G_{\vec{a}}) \subseteq S^{n-1} \backslash \{q\}$, which is contractible, making $G_{\vec{a}}$ homotopic to a constant map. □

The inclusion map $S^{n-1} \hookrightarrow \mathbb{R}^n \backslash \{0\}$ is not homotopic to a constant map. This algebraically leads us to:

Example 3.2.6. If $\vec{a} = (x_1, x_2) \in A^2$ is unimodular in $A = \mathbb{R}[X_1, X_2]/\langle X_1^2 + X_2^2 - 1\rangle$, then there does not exists a matrix $\sigma \in E_2(A)$ such that $(x_1, x_2)\sigma = (1, 0)$.

Proof. Assume otherwise. As (x_1, x_2) is unimodular in A, and as $V(\langle X_1^2 + X_2^2 - 1\rangle) = S^1$, we can have the map $F_{\vec{a}} : S^1 \longrightarrow \mathbb{R}^2 \backslash \{(0, 0)\}$, which is an inclusion map. If we have $\sigma \in E_2(A)$ such that $\vec{a}\sigma = (1, 0)$, then we get $(x_1, x_2)\sigma(X)$ unimodular over $A[X]$. Thus, $(x_1, x_2)\sigma(X)$ is unimodular in $A[X]$, which gives us the map $F_{\vec{a}} : S^1 \times \mathbb{R} \longrightarrow \mathbb{R}^2 \backslash \{(0, 0)\}$ such that $F_{\vec{a}}[0] = F_{\vec{a}}$ and $F_{\vec{a}}[1]$ is a constant map, which makes $S^{n-1} \hookrightarrow \mathbb{R}^n \backslash \{0\}$ homotopic to a constant function, which is contradictory. □

Theorem 3.2.7. *Let A be a Noetherian ring of dimension d and let $(a_1, a_2, \cdots, a_n) \in Um_n(A)$ with $n \geq d + 2$. Then, $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (1, 0, \cdots, 0)$.*

Topological Approach. Let A be a coordinate ring of V, *i.e.*,
$A = \mathbb{R}[X_1, X_2, \cdots, X_n]/I(V)$, where V is a real algebraic variety of dimension d. Then as $I(V(I)) \supseteq I$ and $I \subseteq J \implies V(I) \supseteq V(J)$, we get $dim\, V(I(V)) \leq d$. As $(a_1, a_2, \cdots, a_n) \in Um_n(A)$, we have the continuous map $G_{\vec{a}} : V(I) \longrightarrow S^{n-1}$. Now, let $V(I)$ be a simplicial complex. By *Simplicial Approximation Theorem* (Thm. 3.1.24), there exists a simplicial map $\varphi : V(I) \longrightarrow S^{n-1}$ such that φ and $G_{\vec{a}}$ are homotopic. As $n \geq d + 2$,, $n - 1 \geq d + 1$, and as a simplicial map cannot raise dimension, $dim\, Im(\varphi) < dim\, S^{n-1}$, which makes φ not surjective. Thus, $Im(\varphi) \subseteq S^{n-1} \backslash \{p\}$ for some p. $S^{n-1} \backslash \{p\}$ is contractible, thus making φ and subsequently $G_{\vec{a}}$ constant maps. □

Theorem 3.2.8. *Let A be a Noetherian ring of dimension d.*
Let $(a_1(X), a_2(X), \cdots, a_n(X)) \in Um_n(A[X])$ such that $n \geq d + 2$. Then,
$(a_1(X), a_2(X), \cdots, a_n(X)) \overset{E_n(A[X])}{\sim} (1, 0, \cdots, 0)$.

Topological Approach. Let A be the coordinate ring of a real algebraic variety V of dimension d with the base ring a polynomial ring over $\mathbb{R}$. Then, $dim\, V(I(V)) \leq d$. Assume $V(I)$ is a simplicial complex. For $n \geq d+3$, the theorem follows from Theorem 3.2.7. As $(a_1[x], a_2, [X], \cdots, a_n[X]) \in Um_n(A[X])$, we can have the map $G_{\vec{a}}[X] : V(I) \times \mathbb{R} \longrightarrow S^{n-1}$. Now consider the inclusion and projection maps $i : V(I) \hookrightarrow V(I) \times \mathbb{R}$ and $p : V(I) \times \mathbb{R} \twoheadrightarrow V(I)$. $p \circ i = Id_{V(I)} \sim Id_{V(I)}$, and the map $H(x,t) := t(i \circ p)(x) + (1-t)Id_{V(I)\times\mathbb{R}}$ gives a straight-line homotopy between $i \circ p$ and $Id_{V(I)\times\mathbb{R}}$, thus making the spaces $V(I)$ and $V(I) \times \mathbb{R}$ homotopically equivalent. Also, by Theorem 3.2.7, as any map from $V(I)$ to S^{n-1} is homotopic to a constant map, we get that $G_{\vec{a}}[X]$ is homotopic to a constant map by Theorem 3.1.12. $\square$

Theorem 3.2.9. *Let A be a Noetherian ring of dimension $\leq r$, and J be an ideal of A. Let $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_{r+1}}) \in (A/J)^{r+1}$ be a unimodular row. Then there exist $c_1, c_2, \cdots, c_{r+1} \in A$ such that $\overline{c_i} = \overline{a_i}$, $1 \leq i \leq r+1$, and $(\overline{c_1}, \overline{c_2}, \cdots, \overline{c_{r+1}})$ is unimodular over A.*

Topological Approach. Let $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$. Assume $V(I)$ is a simplicial complex. Now, $V(J)$ is a closed subspace of $V(I)$. As $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_{r+1}}) \in (A/J)^{r+1}$ is unimodular, we can have the map $G_{\vec{a}} : V(J) \longrightarrow S^r$. By Theorem 3.1.12, this $G_{\vec{a}}$ can be extended to $G'_{\vec{a}} : V(I) \longrightarrow S^r$ such that $G'_{\vec{a}}|_{V(J)} = G_{\vec{a}}$. $\square$

Now, the algebraic proof of the theorem can be given from the proof of Theorem 3.1.20, as follows:

Proof. Consider $Spec(A)$ to a simplicial complex, with prime ideals of height k as k-simplexes, *i.e.*, minimal prime ideals as points, prime ideals of height 1 as 1-simplexes etc. . We'll say that an element $a \in A$ vanishes at an edge of $\mathfrak{p}$, if $a \in \mathfrak{p}$. Also assume $J = \langle a_{r+2} \rangle$.

Now, $K := \{\mathfrak{p} \in Spec(A) \,|\, a_{r+2} \in \mathfrak{p}\}$ is a subcomplex of $Spec(A)$, and accordingly the row $(a_1, a_2, \cdots, a_{r+1})$ does not vanish on K.

Now we choose *vertices* of $Spec(A)$ that do not belong to K, *i.e.*, choose minimal prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_s$ of A such that $a_{r+2} \notin \mathfrak{p}_i$, $1 \leq i \leq s$. Thus, $\langle a_1, a_{r+2} \rangle \nsubseteq \mathfrak{p}_i$, $1 \leq i \leq s$. Thus, $\exists\, \lambda_1 \in A$ such that $a'_1 = a_1 + \lambda_1 a_{r+2} \notin \cup_{i=1}^{s} \mathfrak{i}$. We thus get a row $(a'_1, a_2, \cdots, a_{r+1})$ which does not vanish on $K' = K \cup (\cup_{i=1}^{s} \mathfrak{p}_i)$.

Now we choose *edges* of $Spec(A)$ that do not belong to K', *i.e.*, choose prime ideals $\mathfrak{q}_1, \mathfrak{q}_2, \cdots, \mathfrak{q}_l$ of A such that $a_1' \in \mathfrak{q}_j, a_{r+2} \notin \mathfrak{q}_j$, $1 \leq j \leq l$. if there is no such ideal, then $(a_1', a_2, \cdots, a_{r+1})$ will not vanish on $Spec(A)$ and we are done. If not, as $a_{r+2} \notin \mathfrak{q}_j$, $\langle a_2, a_{r+2} \rangle \nsubseteq \mathfrak{q}_j$, $1 \leq j \leq l$, $\exists\, \lambda_2$ such that $a_2' = a_2 + \lambda_2 a_{r+2} \notin \cup_{j=1}^{l} \mathfrak{q}_i$. Continuing this procedure for all simplexes, we get the desired result. $\square$

Theorem 3.2.10. *Let A be a Noetherian ring of dimension n and let J be an ideal of A. Suppose $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \in (A/J)^n$ is a unimodular row. Then $\exists\, b_1, b_2, \cdots, b_n \in A$ such that $\overline{b_i} = \overline{a_i}$ for all i, and the ideal $\langle b_1, b_2, \cdots, b_n \rangle$ has height n.*

Topological Approach. Let $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$ Assume $V(I)$ to be a simplicial complex. Now, $V(J)$ is a closed subspace of $V(I)$. As $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \in (A/J)^n$ is unimodular, we have a continuous map $G_{\vec{a}} : V(J) \longrightarrow S^{n-1}$. By Theorem 3.1.20, we can extend it to continuous $G'_{\vec{a}} : V(I) \longrightarrow S^{n-1}$ where $G'_{\vec{a}}|_{V(J)} = G_{\vec{a}}$. $\square$

Lemma 3.2.11. *The canonical homomorphism of groups from $E_n(A)$ to $E_n(A/I)$ is surjective.*

Proof. This follows from the fact that the generators $E_{ij}(\overline{\lambda})$ of $E_n(A/I)$ can be lifted to generators $E_{ij}(\lambda)$ of $E_n(A)$. $\square$

The same does not hold true for the canonical homomorphism from $SL_n(A)$ to $SL_n(A/I)$, as demonstrated by the following example:

Example 3.2.12. Let $B = \mathbb{R}[X, Y]$, $A = \mathbb{R}[X, Y]/\langle X^2 + Y^2 - 1 \rangle$. Then the unimodular row $(\mathbf{x}, \mathbf{y}) \in A^2$ cannot be lifted to a unimodular row over B^2.

Proof. Let $\alpha = \begin{pmatrix} \mathbf{x} & \mathbf{y} \\ -\mathbf{y} & \mathbf{x} \end{pmatrix} \in SL_2(A)$.

Accordingly, *Claim*: $\nexists\, \beta \in SL_2(B)$ such that $\overline{\beta} = \alpha$.

Since $\alpha \in SL_2(A)$, we have a map $\varphi : S^1 \longrightarrow SL_2(\mathbb{R})$, where $S^1 = V(\langle X^2 + Y^2 - 1 \rangle)$, defined as:

$$\varphi(X, Y) = \alpha(X, Y) = \begin{pmatrix} \mathbf{x}(X, Y) & \mathbf{y}(X, Y) \\ -\mathbf{y}(X, Y) & \mathbf{x}(X, Y) \end{pmatrix} = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}.$$

Now let:

$$\beta = \begin{pmatrix} f_1(X,Y) & f_2(X,Y) \\ f_3(X,Y) & f_4(X,Y) \end{pmatrix} \in SL_2(B)$$

be a lift of α. Then we have another map $\phi : \mathbb{R}^2 \longrightarrow SL_2(\mathbb{R})$ defined as:

$$\phi(X,Y) = \beta(X,Y) = \begin{pmatrix} f_1(X,Y) & f_2(X,Y) \\ f_3(X,Y) & f_4(X,Y) \end{pmatrix}$$

which is an extension of φ. In particular, by looking at the first rows of α and β, we see that the inclusion map $S^1 \hookrightarrow \mathbb{R}^2\backslash\{0\}$ extends to $\mathbb{R}^2 \longrightarrow \mathbb{R}^2\backslash\{0\}$, which is not possible. Thus the claim is proved and thus the unimodular row $(\mathbf{x}, \mathbf{y}) \in A^2$ cannot be lifted to a unimodular row over B. $\square$

Note: The matrix in the last example, $\begin{pmatrix} \mathbf{x} & \mathbf{y} \\ -\mathbf{y} & \mathbf{x} \end{pmatrix} \notin E_2(A)$, as if it were true, then it could be lifted to $E_2(B) \subseteq SL_2(B)$, *i.e.*, the row $(\mathbf{x}, \mathbf{y}) \in A^2$ is not elementary completable.

The general form of Lemma 3.2.11 is as follows:

Theorem 3.2.13. *Let $\vec{a} = (a_1, a_2, \cdots, a_n) \in A^n$ be a unimodular row. Suppose J is an ideal of A and $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \overset{E_n(\overline{A})}{\sim} (\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$, where bar denotes reduction modulo J. Then $\exists\, (c_1, c_2, \cdots, c_n) \in A^n$ such that $(a_1, a_2, \cdots, a_n) \overset{E_n(A)}{\sim} (c_1, c_2, \cdots, c_n)$ and $(\overline{c_1}, \overline{c_2}, \cdots, \overline{c_n}) = (\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$. In particular, $(\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$ can be lifted to a unimodular row over A.*

Topological Approach. Suppose $A = \mathbb{R}[X_1, X_2, \cdots, X_n]/I$, with I being an ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_n]$. As a subspace of a separable metric space $\mathbb{R}^m$, $V(I)$ itself is a separable metric space. Also, $V(J)$ is a closed subspace of $V(I)$. As $(\overline{a_1}, \overline{a_2}, \cdots, \overline{a_n}) \overset{E_n(\overline{A})}{\sim} (\overline{b_1}, \overline{b_2}, \cdots, \overline{b_n})$, the corresponding continuous maps $G'_{\vec{a}}, G'_{\vec{b}} : V(J) \longrightarrow S^{n-1}$ are homotopic by Prop. 3.2.3, *i.e.*, there exists a continuous map $F : V(J) \times I_{[0,1]} \longrightarrow S^{n-1}$ such that $F(x,0) = G'_{\vec{a}}$ and $F(x,1) = G'_{\vec{b}}$. Now let $\overline{\alpha}$ be a lift of α and let $(c_1, c_2, \cdots, c_n) = (a_1, a_2, \cdots, a_n)\alpha$. Then we have an extension $G_{\vec{a}} : V(I) \longrightarrow S^{n-1}$.

Now, we define $H' : (V(J) \times I_{[0,1]}) \cup (V(I) \times \{0\}) \longrightarrow S^n$, as:

$$H'(x,t) = \begin{cases} F(x,t), & x \in V(J),\, 0 \le t \le 1 \\ G_{\vec{a}}, & x \in V(I),\, t = 0 \end{cases}.$$

which is a continuous map. As $(V(J) \times I_{[0,1]}) \cup (V(I) \times \{0\})$ is a closed subset of $V(I) \times I_{[0,1]}$, by Theorem 3.1.22, H' can be extended to a map $H : V(I) \times I_{[0,1]} \longrightarrow S^n$. Now take $G_{\vec{b}} = H(x,1) : V(I) \longrightarrow S^n$, which gives $G_{\vec{b}}|_{V(J)} = G'_{\vec{b}}$. □

Example 3.2.14. -Let $A = \mathbb{R}[X,Y,Z]/\langle X^2+Y^2+Z^2-1\rangle$. Then $(X,Y,Z) \in A^3$ is a unimodular row. Since S^2 is not contractible, we have, as an algebraic consequence, that $(X,Y,Z) \overset{E_3(A)}{\nsim} (1,0,0)$, i.e., (X,Y,Z) is not elementary completable. In fact, (X,Y,Z) is not even completable to a matrix in $GL_3(A)$.

Proof. The proof that $(X,Y,Z) \overset{E_3(A)}{\nsim} (1,0,0)$ follows the same way as that of example 3.2.6.
Now, assume on the contrary that (X,Y,Z) is completable to some matrix $\sigma \in GL_3(A)$, making (X,Y,Z) completable. Now, the solution space for the corresponding matrix, $P \cong A^3/\langle x,y,z\rangle \cong A^2$. Thus we can get a surjective homomorphism $f : P \longrightarrow A$. Suppose e_1, e_2, e_3 make up the standard basis of A^3 and $\overline{e_i} \overset{f}{\longmapsto} h_i$, $i = 1,2,3$. As $f(x,y,z) = 0$, $xf(\overline{e_1}) + yf(\overline{e_2}) + zf(\overline{e_3}) = Xh_1 + Yh_2 + Zh_3 = 0$, thus it can be classed as a multiple of $X^2+Y^2+Z^2-1$. In particular, for given x_1, x_2, x_3, $x_1^2 + x_2^2 + x_3^2 = 1$ implies $x_1h_1 + x_2h_2 + x_3h_3 = 0$, which gives us that $(x_1,x_2,x_3) \perp (h_1(x_1,x_2,x_3), h_2(x_1,x_2,x_3), h_3(x_1,x_2,x_3))$. Now we define the continuous vector field $\Phi : S^2 \longrightarrow \mathbb{R}^3$, $(x_1,x_2,x_3) \longmapsto (h_1,h_2,h_3)$. As $f(\overline{e_i}) = h_i$ for all i, we get $\langle h_1(x,y,z), h_2(x,y,z), h_3(x,y,z)\rangle = A$, thus Φ gets no zeros on S^2 by Thm. 3.2.1, which is contradictory as per the Hairy Ball Theorem, i.e., *There is no nowhere vanishing vector field on* S^2. Thus (X,Y,Z) is not completable, let alone elementary completable. □

Proposition 3.2.15. *Let* $(a,b,c) \in A^3$ *be a unimodular row. Then, the row* (a^2,b,c) *is completable.*

We first give the algebraic proof for the proposition, to motivate the underlying topological idea.

Proof. As $(a,b,c) \in A^3$ is unimodular, $\exists\, a',b',c' \in A$ such that $aa'+bb'+cc'=1$. Now, consider the matrix:

$$\alpha = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c' & b' \\ -b & -c' & 0 & a' \\ -c & -b' & a' & 0 \end{pmatrix}.$$

Then $det(\alpha) = (aa'+bb'+cc')^2 = 1$. As $\langle 0,-a,-b,-c\rangle = A$, $(0,-a,-b,-c)$ is elementary completable by Lemma 3.2.5, with $\alpha_1\alpha_2$ a completion, where:

$$\alpha_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & 0 & 1 & 0 \\ c & 0 & 0 & 1 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & -a' & -b' & -c' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now take $\alpha' = \alpha_1\alpha_2\alpha$. Then,

$$\alpha' = \begin{pmatrix} 1 & a & b & c \\ 0 & a^2 & ab+c' & ac-b' \\ 0 & ab-c' & b^2 & bc+a' \\ 0 & ac+b' & bc-a' & c^2 \end{pmatrix}$$

and $det(\alpha) = det(\alpha') = 1$. After some replacements, we get:

$$\begin{matrix} a \to a, & b \to b, \\ c \to c, & a' \to a', \\ b' \to b'+ac, & c' \to c'-ab \end{matrix} \quad \Longrightarrow \beta = \begin{pmatrix} 1 & a & b & c \\ 0 & a^2 & c' & -b' \\ 0 & 2ab-c' & b^2 & bc+a' \\ 0 & 2ac+b' & bc-a' & c^2 \end{pmatrix}.$$

Then $det(\beta) = aa' + (-b')(-b+ac') + c(c'-ab) = aa'+bb'+cc' = 1$. By replacing further, we get:

$$\begin{matrix} a \to a, & b \to -b', \\ c \to c', & a' \to a', \\ b' \to -b, & c' \to c \end{matrix} \quad \Longrightarrow \sigma = \begin{pmatrix} 1 & a & -b' & c' \\ 0 & a^2 & c & b \\ 0 & -2ab'-c & b'^2 & -b'c'+a' \\ 0 & 2ac'-b & -b'c'-a' & c'^2 \end{pmatrix}.$$

Then $det(\sigma) = aa + (-b')(-b + ac') + c'(c + ab') = aa' + bb' + cc' = 1$.

Now, as $det(\sigma) = det \begin{pmatrix} a^2 & c & b \\ -2ab' - c & b'^2 & -b'c' + a' \\ 2ac' - b & -b'c' - a' & c'^2 \end{pmatrix}$,

we get that the matrix:

$$\alpha_3 = \begin{pmatrix} a^2 & b & c \\ -2ac' - b' & c'^2 & -b'c' + a' \\ 2ab' - c & -b'c' - a' & b'^2 \end{pmatrix}$$

is a completion of (a^2, b, c). □

Topological Approach. Let $A = \mathbb{R}[X_1, X_2, \cdots, X_m]/I$, I being the ideal of a real algebraic variety in $\mathbb{R}[X_1, X_2, \cdots, X_m]$. As $(a, b, c) \in A^3$ is unimodular, the sequence $(0, -a, -b, -c)$ is elementary completable. Now we consider the exact sequence:

$$0 \longrightarrow SL_3(\mathbb{R}) \xrightarrow{i_1} SL_4(\mathbb{R}) \xrightarrow{i_2} \mathbb{R}^4 \backslash \{0\} \longrightarrow 0 \ ,$$

where i_1 sends $\sigma \in SL_3(\mathbb{R})$ to $\begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix} \in SL_4(\mathbb{R})$, and i_2 sends any matrix $\sigma' \in SL_4(\mathbb{R})$ to its first row.

As $\alpha \in SL_4(\mathbb{R})$, we can have the map $\varphi_1 : V(I) \longrightarrow SL_4(\mathbb{R})$, $\vec{x} \longmapsto \alpha(\vec{x})$, and for $(0, -a, -b, -c) \in Um_4(A)$, we can have the map $F_{(0,-a,-b,-c)} : V(I) \longrightarrow \mathbb{R}^4 \backslash \{0\}$, which ends up equalling $i_2 \circ \varphi$; which is homotopic to the constant map. Also, $\alpha_3 \in SL_3(\mathbb{R})$ gives another map $\varphi_2 : V(I) \longrightarrow SL_3(\mathbb{R})$, such that $i_1 \circ \varphi_2 \sim \varphi_1$, and with the map $i'_2 : \underset{(\in SL_3(\mathbb{R}))}{\sigma'} = \begin{pmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_3 \end{pmatrix} \longmapsto (\sigma'_1)$, we get that $i'_2 \circ \varphi_2 : V(I) \longrightarrow \mathbb{R}^3 \backslash \{0\}$ equals the map $F_{(a^2,b,c)}$. □

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