Asymptotic symmetries and dual theories of (2+1) dimensional (super)gravity theories

A thesis submitted in partial fulfilment of the requirements

for the award of the degree of

DOCTOR OF PHILOSOPHY

by

ARINDAM BHATTACHARJEE

20152035

to the

DEPARTMENT OF PHYSICS

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH PUNE

PUNE - 411 008

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DECLARATION

I hereby declare that this written submission **Asymptotic symmetries and dual theories of (2+1) dimensional (super)gravity theories** is my own work and, to the best of my knowledge, it contains no materials previously published or written by any other person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at IISER Pune or any other educational institution, except where due acknowledgement is made in the thesis. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the Institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Arindam Bhattacharjee

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[August, 2021]
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"I am not sure that I exist, actually. I am all the writers that I have read, all the people that I have met, ... all the cities I have visited." — Jorge Luis Borges

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Dedicated to the spirit of Emmy Noether

ABSTRACT

In this thesis we use the asymptotic symmetries of 3D (super)gravity theories to explore the dual theories.

Using the Chern-Simons formulation of (2+1)D gravity we have constructed a two dimensional theory dual to 3D asymptotically flat supergravity in presence of two supercharges with(out) internal R symmetry. In both cases, the dual theory is a chiral Wess-Zumino-Witten type model. We then explore the symmetries of the dual theory and find the most generic, so far unknown, quantum N = 2 superBMS$_3$ symmetry under which this is invariant. We have also commented on the phase space description of the duals.

Next, we use similar techniques to understand the dual dynamics of 3D asymptotically de-Sitter supergravity. We write down the Chern-Simons description of the bulk theory using $OSp(1|2,\mathbb{C})$ as the gauge group. Next we describe the holographic screen of 3D de-Sitter and impose our boundary conditions. We finally end up with a super-Liouville theory at the boundary as the holographic dual of the bulk supergravity theory.

Finally we use conformal field theory techniques to write a Matrix model partition function with BMS$_3$ constraints. We start from the free field realisation of the algebra in terms of a twisted $\beta-\gamma$ system and solve the constraints through it. We end up with an eigenvalue representation of this partition function. Since BMS$_3$ is the asymptotic symmetry algebra of the pure gravity in 3D flat background, we expect this partition function to illuminate our understanding of 3D holography. We comment on qualitative properties of this partition function.

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Chapter 1

Asymptotic Symmetries and Holography

1.1 Asymptotic symmetries in gauge and gravity theories

Ever since Emily Noether discovered the relation between symmetries and conserved quantities, it has become one of the central themes of modern physics.

It was discovered long back that there exist non-trivial enhanced symmetries at the asymptotic region of spacetime for gravitational theories. For asymptotically flat spacetimes, Bondi, Metzner, Van der Burg and Sachs found that the boundary symmetry algebra was not Poincaré algebra but rather an infinite dimensional algebra (later named BMS algebra) which contained Poincaré algebra as its subgroup. Recently, similar symmetry enhancement has also been found for gauge theories as well. These enhanced symmetries or *asymptotic symmetries* have physical consequences and constrains the scattering matrices of these theories. For instance, the ward identity of these symmetries is related to soft theorems that constrain the S matrices' infrared behavior.

Gauge theories have local degrees of freedom that arise from making the global symmetry of the system a function of spacetime points. To relate this system with its original counterpart, we identify the states that differ by gauge transformations to be the same physical state. Thus, it was expected that the gauged system's final symmetry would be the same as the initial global symmetry algebra. It turns out not to be the case. There are essentially two types of gauge transformations one can perform. Trivial gauge transformations are those that fall off fast enough at large distances. On the other hand, large gauge transformations are those allowed transformations that are non-zero at the boundary of the spacetime. The latter one is not just the redundancy of description; instead, it affects the physical state of the system. Thus the asymptotic symmetry group is described as the quotient of all allowed gauge transformations by trivial gauge symmetries.

Because the asymptotic symmetries are defined at the boundary of spacetime, they are heavily influenced by the boundary conditions imposed on the fields. For example, the original BMS group was recently enhanced to extended BMS group by realising that the original boundary conditions can be relaxed to allow for analytic singularities at the null boundary of the spacetime. Thus it is important to mention the boundary conditions along with the Lagrangian of the theory while describing asymptotic symmetries. From the point of view of quantum field theories this makes sense. Since, the Hilbert Space of a QFT is determined not only by the Lagrangian of the theory but also the boundary conditions imposed on the fields.

In this thesis, we will firstly study asymptotic symmetries of supergravity theories in (2+1) dimensions. Using the Chern-Simons formulation of gravity, we will first specify the boundary conditions for such theories and then study the classical holographic dual theories. These holographic duals will possess the asymptotic symmetry algebra of the supergravity theory as its gauge symmetry. We will study these systems in asymptotically flat and de-Sitter spacetimes. In the next part, we consider AdS space time and study a possible implication of asymptotic symmetry. We analyse classical scattering in 4D AdS-Schwarzschild spacetimes and study the soft gravitational radiation. We comment on the relation between this soft radiation and asymptotic symmetries of AdS4.

1.2 Gravity in (2+1) dimensions

In Einstein's theory of gravity, the dynamical field that governs the nature of gravitational force is the metric. In other theories of gravity, for instance in supergravity, along with metric, we also have other dynamical fields like gravitini etc. In 4D, it can be shown from Einstein's field equations that perturbations in gravitational field propagate as spin-2 massless excitations (gravitational waves). These excitations have two local degrees of freedom encoded in their polarisation. This situation is quite different in the case of 2+1 dimensional theories.

A simple calculation shows, that in a D dimensional theory of gravity, the metric which is a symmetric $D \times D$ matrix has $\frac{D(D+1)}{2}$ independent components. Now, Einstein's field equations are $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and conservation of Stress-tensor puts D number of constraints $\nabla^\mu T_{\mu\nu} = 0 \Rightarrow \nabla^\mu G_{\mu\nu} = 0$. Also, locally, one can change the co-ordinates $x^\mu \to x^\mu + \xi^\mu(x)$, which means another D number of components can be fixed. Hence, the physical degrees of freedom is given by

$$\frac{D(D+1)}{2} - D - D = \frac{D(D-3)}{2}. \tag{1.2.1}$$

Putting $D = 4$ we get that there are 2 local degrees of freedom. Now, if we put $D = 3$, we see that the above formula gives zero. This implies that pure gravity does not possess any local d.o.f. in (2+1) dimensions.

This inference can also be reached more formally [1]. In (2+1) dimensions, the Riemann curvature tensor can be decomposed in terms of the Ricci tensor and the Ricci scalar only as the Weyl tensor vanishes identically. We may write

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R \tag{1.2.2}$$

Putting this expression in Einstein's equation we get

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu} \tag{1.2.3}$$

which implies constant curvature. That means, given the cosmological constant, the local structure of spacetime is fixed. In particular, gravitational waves, which are local excitations of the metric, are disallowed in (2+1) D.

1.3 Supergravity in (2+1) dimensions

In order to consider Supergravity, we add the action of gravitino field with pure Einstein-Hilbert action. The gravitino field ψ^μ_α is spin-3/2 field and it's action reads:

$$I_{sug} = -\frac{1}{2}\int d^3x\sqrt{g}\epsilon^{\mu\nu\rho}\bar{\psi}_\mu D_\nu \psi_\rho \tag{1.3.4}$$

It can be shown that even after adding the gravitino part, the action does not possess any local degrees of freedom. The equation of motion for ψ_μ reads $\epsilon^{\mu\nu\rho}\partial_\nu\psi_\rho = 0$ which implies a general solution $\psi_\mu = \partial_\mu\phi$. But local SUSY suggests that ψ has the symmetry

$\delta\psi_\mu = \partial_\mu \epsilon$, where ϵ is the SUSY parameter. This can be used to completely gauge away the field ϕ. Hence, on shell, the gravitino field contains no local degree of freedom in 2+1 dimensions.

The theory described above is a minimal supergravity theory. But in principle extended supergravity theories can also be discussed in 2+1 dimensions. These theories have more than one gravitino fields and also have auxillary fields in them. A similar counting of degrees of freedom shows that even extended SUGRA theories do not contain any local physical degrees of freedom.

But this in no way means that gravity is trivial in this setting. Even without any local excitations there are topological degrees of freedom. These arises from considering geometries which are locally isometric but have different topological structures associated with them. Another non-triviality arises from considering the boundary conditions. When we have a manifold with boundaries, the behaviour of fields at the boundary is an extra condition that we impose. It can be shown that in (2+1) dimensions, with proper boundary conditions, infinite degrees of freedom may reside at the boundary.
These subtleties make gravity at lower dimensions interesting. Since, all the degrees of freedom (apart from the topological ones) in this system resides at the boundary, the ideas of holography seems natural in this setting. In fact the works of [2] were a precursor to AdS/CFT proposal by Maldacena [3]. Now, we will see that it is related to another non-dynamical theory in 2+1D, the Chern-Simons(CS) theory. We will also briefly discuss how the asymptotic phase space of gravity translates into the CS language and how the gauge symmetries of the theory becomes large diffeomorphism of gravity.

1.4 Vielbein and Spin Connection

In order to connect the theory of gravity with CS theory, we will use the vielbein formalism of gravity. In Einstein's gravity, the dynamical variable is the metric of spacetime $g_{\mu\nu}(x)$. The metric is diffeomorphism covariant. We define a flat metric at every point through the following formula:

$$g_{\mu\nu}(x) = e^a_\mu(x)\eta_{ab}e^b_\nu(x) \tag{1.4.5}$$

Thus the dynamical information of the metric is now encoded in the functions e^μ_a called the vielbeins. The indices $\{\mu, \nu, ...\}$ denote the curved space index whereas the indices

$\{a, b, ..\}$ denote the tangent space index. These new quantitites e^a_μ are invertible everywhere and hence have non-zero determinant. Vielbeins can also be used to interchange curved and tangent indices. This is particularly useful while doing QFT in curved spacetime since fields are usually defined according to their trasnformation properties in flat spacetime. Vielbeins are not unique for a given metric. We can perform a local Lorentz transformation in the tangent space which will transform the vielbeins but not the spacetime metric.

Vielbeins are also used for defining a basis for one forms. We can define the 1-form $e^a = e^a_\mu dx^\mu$ which transforms like a vector under local lorentz transformations. Then a co-vector A can be expanded as $A = A_\mu dx^\mu = A_a e^a$ where $A_a = A_\mu e^\mu_a$.

A related concept is that of spin connections. To motivate it, notice that the 2-form de^a does not transform like a Lorentz vector under local transformations. To make a locally Lorentz covariant quantity we may define $T^a = de^a + \omega^a_b e^b$. Deamnding T^a transforms in the correct way, i.e. $T^a \to \Lambda^a_b T^b$ we get the following transformation property for ω:

$$\omega \to \Lambda^{-1}(d + \omega)\Lambda \tag{1.4.6}$$

where we have suppressed the indices. The components of the quantity ω^a_b are called spin connections. Essentially, in the vielbein formalism, the vielbeins e^a_μ play the role of a metric and ω^{ab}_μ play the role of Christoffel symbols. We can write the Einstein-Hilbert action in vielbein formalism. In 3D, the action looks like:

$$S_{EH} = \int \epsilon_{abc}[e^a \wedge R^{bc} - \frac{\Lambda}{3} e^a \wedge e^b \wedge e^c] \tag{1.4.7}$$

with the curvature tensor defined as $R^{bc} = \frac{1}{2}\left(\partial_\mu \omega^{bc}_\nu - \partial_\nu \omega^{bc}_\mu + [\omega^b_{a\mu}, \omega^{ac}_\nu]\right) dx^\mu \wedge dx^\nu$. A plus point of writing the action in this form is that now it looks like the action of a gauge theory. This fact is true for 3D gravity as we will see below.

1.5 Brief Introduction to Chern-Simons theory

Before discussing the relation between gravity and Chern-Simons(CS) theory let us briefly recall some basic features of Chern-Simons gauge theory. It is a topological field theory and the action is given by:

$$S_{CS}[A] = \frac{\kappa}{2}\int_{\mathcal{M}} Tr(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \tag{1.5.8}$$

Where $A = A_\mu dx^\mu$ is the 1 form gauge field for some gauge group $\mathcal{G}$. The field strength dA is defined in a similar way $dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu$. The trace in the action is over the generators $\{\mathcal{T}^a\}$ of the Lie algebra. Thus the action does not have any metric dependence, so the theory is explicitly metric independent or topological.

To see the gauge invariance of the theory, we can transform the gauge field by $A \to G^{-1}AG + G^{-1}dG$ and observe that the action transforms as

$$\delta S_{CS} = Tr\left[\frac{\kappa}{2}\int_{\mathcal{M}} d[dG\, G^{-1}A]\right] \tag{1.5.9}$$

which certainly vanishes locally. But this also highlights one important aspect. On a manifold with boundary, the gauge transformation of the field A has non-trivial effects at the boundary. As we know, these boundary terms play important role in defining Noether Charges and we will see its implications in asymptotic symmetries.

Another important characteristic of the CS theory is its equations of motion. Varying the action with respect to the field A we get,

$$F \equiv dA + A \wedge A = 0 \tag{1.5.10}$$

This shows that the field strength vanishes at every point and thus the solution is pure gauge $A = G^{-1}dG$. This also means that no local excitations exist for this theory, quite similar to 3D gravity. All these results will be extremely crucial when we relate the CS theory with gravity.

1.6 From Chern-Simons to (super)gravity

The relation between CS theory and pure gravity in 2+1 dimensions was discovered in [4] for negative cosmological constant and later expanded by Witten [5].

To show the equivalence of these actions we start from (1.4.7) and interpret the fields e^a and ω^{ab} as gauge fields. Since we are interpreting gravity as a gauge theory, it is natural to assume that the local isometries of the theory of gravity will construct the gauge group. Hence, for zero cosmological constant case, the gauge group would be $ISO(2,1)$ and similarly for $\Lambda < 0$ the gauge group would be $SO(2,2)$ and for $\Lambda > 0$ it's $SO(3,1)$.

But we need to be careful. If we expand the "kinetic" term of the CS action with $A = A^a T_a$, we get

$$S_{kin} = \int_{\mathcal{M}} A^a \wedge dA^b \, Tr[T_a T_b] \tag{1.6.11}$$

Thus in order to ensure that every component of the gauge field has a kinetic piece, we need the matrix $Tr[T_a T_b]$ to be non-degenerate. This plays the role of *metric* in the Lie Algebra vector space and the existence of such an invertible metric is crucial for this formalism to work. Fortunately, for each of the algebras mentioned above, such a non-degenerate bilinear form exists.

Let us see the example of $\Lambda = 0$ case first. For this the symmetry generators are the translations P_a and the Lorentz generators J_{ab}. But in 3D, we can write dual generators for Lorentz transformations $J^a = \frac{1}{2}\epsilon^{abc} J_{bc}$, which puts them in the equal footing as P_a. This allows us to consider the most generic bilinear of the form,

$$W = aP_a P^a + bJ_a J^b + cP_a J^b \tag{1.6.12}$$

Demanding it to be a quadratic casimir gives the solution $a = c = 0$; $b \neq 0$ or $a = b = 0$; $c \neq 0$. The second one is non-degenerate and hence can be used to construct our theory of gravity. From their structure it is quite natural to interpret the vielbeins as the generators associated with translation and spin connections as the generators of Lorentz transformation. Hence we choose our gauge field as

$$A = e^a P_a + \omega^a J_a \tag{1.6.13}$$

where $\omega^a = \epsilon^{abc}\omega_{bc}$ and see that we get (1.4.7) from (1.5.8). To reach the final step we need to take $Tr[P_a J_b] = \delta_{ab}$, this normalisation factor is fixed by demanding the resulting action be equivalent to Einstein-Hilbert action.

For non-zero Λ, there are two ways of approaching. The isometry algebra can be written as

$$[P_a, P_b] = \Lambda \epsilon_{abc} J^c \qquad [J_a, J_b] = \epsilon_{abc} J^c \qquad [P_a, J_b] = \epsilon_{abc} P^c \tag{1.6.14}$$

and proceeding as above, with a similar bilinear form, we get EH action with the cosmological constant. The extra term is generated by the last piece of CS action because the term $A \wedge A \wedge A \sim A \wedge [A, A]$ and the commutator now contains an extra piece due to non-zero Λ.

Interestingly, there is one other option for the case of non-zero Λ. The first Casimir that we discarded for flat case actually becomes non-degenerate in the presence of a cosmological constant. In this case, this becomes $Tr[P_a P_b] = \lambda \delta_{ab}$, $Tr[J_a J_b] = \delta_{ab}$ and gives rise to an action, that is different from 1.4.7. We will not persue this further here.

To extend the above discussion to supergravity is natural. We must now take the corresponding supergroup as the gauge group of the CS theory. The Chern-Simons field now will also contain a component along the supercharges. The co-efficient of this component will be gravitini field. For extended supergravity theories additional characteristics will occur, we mention them in the relevant chapters later.

1.7 Diffeomorphism and Gauge symmetries in CS formalism

Since we can write the Einstein-Hilbert action (1.4.7) in Chern-Simons language, we expect that the diffeomorphism invariance of the gravity action manifests as gauge invariance in this formalism. As we will see below, this is indeed true [5].

Indeed one can show that a generic ISO$(2, 1)$ gauge transformation parametrized by the element $\lambda = E^a P_a + \Omega^a J_a$, act on the gauge field as

$$\delta A_\mu = -D_\mu \lambda = -(\partial_\mu \lambda + [A_\mu, \lambda]). \tag{1.7.15}$$

In terms of the gravity fields (e^a_μ, ω^a_μ) the gauge transformation reads:

$$\delta e^a_\mu = -\partial_\mu E^a - \epsilon^{abc} e_{\mu b} \Omega_c - \epsilon^{abc} \omega_{\mu b} E_c \tag{1.7.16}$$

$$\delta \omega^a_\mu = -\partial_\mu \Omega^a - \epsilon^{abc} \omega_{\mu b} \Omega_c \tag{1.7.17}$$

which are the expected local Lorentz transformations generated by Ω^a and local diffeomorphism transformations generated by E^a. Recall that under a generic diffeomorphism transformation $x^\mu \to x^\mu + V^\mu$, the fields (e^a_μ, ω^a_μ) transforms as

$$\tilde{\delta} e^a_\mu = V^\nu(\partial_\nu e^a_\mu - \partial_\mu e^a_\nu) + \partial_\mu(V^\nu e^a_\nu), \quad \tilde{\delta}\omega^a_\mu = V^\nu(\partial_\nu \omega^a_\mu - \partial_\mu \omega^a_\nu) + \partial_\mu(V^\nu \omega^a_\nu). \tag{1.7.18}$$

Thus for $E^a = e^a_\mu V^\mu$ and turning off the local Lorentz transformation, we can show that the difference between (1.7.16) and (1.7.18) is:

$$\tilde{\delta} e^a_\mu - \delta e^a_\mu = V^\nu(D_\nu e^a_\mu - D_\mu e^a_\nu) - \epsilon^{abc} V^\nu \omega_{\nu b} e_{\mu c} \,. \tag{1.7.19}$$

The 1st term of the RHS of the above equation, the torsion, vanishes on-shell, while the 2nd term can be identified with a local Lorentz transformation with parameter $\Omega^a = \omega^a_\mu V^\mu$ [6]. Thus we see that, on-shell, gauge transformation of Chern Simons theory is identical to local Lorentz and diffeomorphism transformation of 3D Gravity.

1.8 Asymptotic Phase space and symmetries

Now that we have a relation between the action of CS theory and (super)gravity theories, we can ask how to study the asymptotic symmetries of gravity theories in CS formalism. Since, there are no local excitations in 3D gravity, all the dynamics are confined in the boundary. This makes the study of asymptotic phase space of this theory extremely crucial.

Generally, the asymptotic phase space is defined by putting boundary conditions on the components of the metric. These conditions are not unique but they should be chosen in a way that they allow for all physically interesting metrics satify them. But on the other hand, they should not be too loose to allow for systems with unphysical behaviours. We now review a particular set of conditions in asymptotically flat spacetime and explain how to study asymptotic symmetries using CS theory.

In 3D, the oldest set of boundary conditions on the metric was given by Brown and Henneaux [2] for asymptotically AdS_3 spacetimes. The flat limit of these conditions were studied by Barnich et al [7, 8]. They take the 3D analogue of BMS ansatz

$$ds^2 = e^{2\beta}\frac{V}{r}du^2 - 2e^{2\beta}dudr + r^2(d\phi - U du)^2 \tag{1.8.20}$$

which was originally put forward in 4 dimensions [9]. The fall off conditions are then specified as the large u behaviour of the functions V, β, U all of which are at this point arbitrary function of the co-ordinates $\{u, r, \phi\}$. The chosen fall-off are

$$\frac{V}{r} = O(1); \qquad \beta = O(1/r); \qquad U = O(1/r^2) \tag{1.8.21}$$

Deamnding that these also satisfy Einstein's equation gives the form of the asymptotic metric for our case

$$ds^2 = \mathcal{M}du^2 - 2dudr + 2\mathcal{N}dud\phi + r^2 d\phi^2 \tag{1.8.22}$$

Where $\mathcal{M}, \mathcal{N}$ are two arbitrary functions of $\{u, \phi\}$ that span the asymptotic phase space. Then the asymptotic symmetries would be the infinitesimal transformations that keep the above form of the metric unchanged. Thus we solve for the asymptotic killing vector field ξ that satisfies

$$\begin{aligned} &\mathcal{L}_\xi g_{rr} = 0 \qquad \mathcal{L}_\xi g_{r\phi} = 0 \qquad \mathcal{L}_\xi g_{\phi\phi} = 0 \\ &\mathcal{L}_\xi g_{ur} = o(1/r) \qquad \mathcal{L}_\xi g_{uu} = o(1) \qquad \mathcal{L}_\xi g_{u\phi} = o(1) \end{aligned} \tag{1.8.23}$$

This vector field equipped with a modified Lie bracket gives the BMS_3 algebra.

The above computation can be equivalently done in CS formalism. In this case the conditions on the metric components need to be translated into boundary conditions for the field A. For this we need to find out the vielbeins and spin connections associated with the asymptotic metric (1.8.22). But Chern-Simons theory is a gauge theory and hence we need to impose an extra condition on A to fix the gauge. Inspired by WZW model, [10] chose the gauge $\partial_\phi A_r = 0$. This implies that the field A can be expanded as $A = h^{-1}dh + h^{-1}ah$ where h contains all the r dependence and a is just a function of u, ϕ co-ordinates. Thus we can write the boundary conditions just in terms of a which looks like:

$$a = (\frac{\mathcal{M}}{2}du + \frac{\mathcal{N}}{2}d\phi)P_0 + duP_1 + \frac{\mathcal{M}}{2}d\phi J_0 + d\phi J_1 \tag{1.8.24}$$

To find the asymptotic symmetries, we need to find the gauge transformations that leave the form of the field A at the boundary as above. Under a gauge transformation the field A transforms like

$$\delta A = d\lambda + [A, \lambda] \tag{1.8.25}$$

putting the above form into this equation yields the correct form of λ. The conserved charge corresponding to these transformations can be found following

$$\delta Q[\lambda] = \int Tr[\lambda, \delta A_\phi] \tag{1.8.26}$$

The Poisson bracket associated with this charge $\delta_{\lambda_1} Q[\lambda_2] = \{Q[\lambda_1], Q[\lambda_2]\}$ gives again the BMS_3 algebra. This equivanlence is more than just a formal one. It helps us use the machinery of gauge theories [11] to explore the gravity systems. As we will see in this thesis, this also helps us find classical holographic duals for (super)gravity theories.

Although, the above example shows the asymptotic symmetry analysis of pure gravity theory, for supergravity, its very similar. Here along with the metric, we also need to specify the boundary conditions for the gravitini fields and other fields appearing in extended supergravity theories. One consistent way of doing it is to start with the bosonic configuration (which must definitely be included in asymptotic phase space) and then act on it by the exact symmetries of the full supergravity theory. This in principle produces the most generic field A consistent with the boundary conditions of the bosonic sector.

1.9 A brief Review of literature

In this section we present a lightning review of the works done in (super-)gravity in (2+1)D. This will help put this work in the broader context and also hint at the possible roads that lay open for future research in lower dimensional gravity.
In [12, 13] the authors discussed solutions of Einstein's equations with particle source in (2+1)D with zero and non-zero cosmological constants, respectively. They found that the presence of particles imply conical singularities at their position and there's a limit on the total mass supported by the spacetime. Following these [14] wrote down exact non-perturbative scattering amplitudes which were shown to be analogous with Aharonov-Bohm effect in gauge theories [15].

A new direction of research started with the seminal work of Brown and Henneaux [2]. They studied the asymptotic symmetries of pure gravity system with negative cosmological constant and found that the symmetry algebra enhances to Virasoro algebra at the boundary. Since Virasoro algebra is also the symmetry algebra of 2D CFTs, this work indicated a close connection between gravity in 3D with a gauge theory at lower dimension.

In this sense, [2] was precursor to AdS/CFT and holography in general. More interestingly, they studied only the classical charges and their Poisson brackets but the resulting algebra was centrally extended and the central charge was related to AdS radius.

Almost at the same time a new formalism for studying gravity at (2+1)D was developed in [4,5]. These works showed that Einstein's gravity action in 2+1D can be rewritten as a Chern-Simons theory with a suitable gauge group. This was the first realisation of gravity as a gauge theory where the vielbeins and spin connections acted as gauge fields. This gave a new handle on gravitational problems as they can now be recast as a problem in CS gauge theory and studied. In particular, the asymptotic symmetries of gravity can now be studied by studying the large gauge trasformations in gauge theory [16].

One of the most fascinating discoveries in these context was [17, 18] which showed that Black holes exist in (2+1)D with negative cosmological constant. Since we have already mentioned that all solutions of pure gravity in (2+1)D has constant curvature everywhere, it was unexpected that a solution analogous to black holes would be found there. In fact these BTZ black holes are smooth manifolds and at r=0 they possess a 'causal singularity' rather than a curvature singularity. Geometrically these solutions are quotients of global AdS_3 spacetime with discrete subgroups of it's isometry group. This discovery showed that the phase space of gravity is extremely rich and non-trivial in 2+1 D. Later [19] calculated the microscopic entropy for these Black holes and showed that it follows Bekenstein-Hawking area law.

The main focus of this thesis is asymptotic dynamics in 2+1 dimensional gravity which started from the works of [16]. Starting from the boundary conditions of [2] they used the Chern-Simons formalism and showed that the dynamics is described by a Liouville theory. This connection uses the reduction of Chern-Simons theory to Wess-Zumino-Witten (WZW) theory under suitable boundary conditions [20,21]. Then further imposing constraints it reduced to Liouville action. This connection was further persued and quickly generalised to the case of supergravity [22,23]. In [10], the author showed the connection between global charges in Chern-Simons theory and the asymptotic symmetry group of AdS_3 using the methods of [24].

The Brown-Henneaux boundary conditions are by no means unique, many different sets of boundary conditions were proposed subsequently, for example [25, 26]. All these were shown to be special cases of a most generic set of boundary conditions [27]. The interpretation of these conditions as asymptotic charges are of extreme physical interest.

Most of the above work are early development in the area of (2+1) dimensional gravity and are naturally restricted to negative cosmological constant case. It was already known that an infinite dimensional algebra sits at the boundary of 3+1 dimensional flat spaces [9, 28]. The problem of asymptotic symmetry of 2+1 D flat space was first considered in [29]. They started with the asymptotic algebra of [30] and chalked out the phase space of allowed metrics. They also found the central charge of the extended charge algebra and analogues of BTZ black holes were cosmological horizons [7]. These ideas extended the realm of AdS/CFT to a newer field of holography, namely Flat space holography (or BMS/CFT correspondence) [31].

The dual theory of pure gravity in the case for flat spaces turned out to be a flat limit of Liouville theory [32]. The dynamics for minimal supergravity was discussed in [33]. It was also realised that the cosmological horizons can also be endowed with 'soft hairs' by means of boundary conditions there [34].

One of the most exciting direction of research in three dimensional gravity is to compute the exact partition function of pure gravity including quantum corrections. Taking into account all the known contributions [35], such a computation [36,37] revealed caveats. There are negative densities of states and the spectrum seems to be continuous but finite. The resolution of these problems is yet to be determined and some newer proposals are being actively persued [38, 39].

We end our short and lightning review of the literature here. Of course this review mainly focused on the works directly related to the content of this thesis and mentioned only a few other avenues in the vast field of 2+1 dimensional gravity. There are lots and lots of interesting work that has been going on and that sheds new lights into the properties of gravity in lower dimensions. This review was just an attempt to motivate the works of this thesis.

Chapter 2

Some Properties of constrained WZW models

2.1 Wess-Zumino-Novikov-Witten Model

The gauge theory that we are going to relate our 3D (super)gravity theory to is a particular type of non-linear sigma model (NLSM) called Wess-Zumino-Novikov-Witten Model (or WZW model).

A general NLSM generically has fields that map the spacetime to a target manifold. The fields of the theory act as co-ordinates of the target manifold. In particular we can choose a 2D flat space (with co-ordinates x^{μ}) and choose the target space to be the group manifold of some (semisimple) Lie group G. The action of NLSM then takes the form,

$$S_{\sigma} = \int dx^{\mu} Tr[\partial_{\mu} g \partial^{\mu} g^{-1}] \tag{2.1.1}$$

where it is evident that the Tr of the generators plays the role of the target space metric. For the above theory to be a 2D CFT, we expect that there will be a holomorphic and anti-holomorphic conserved current, in line with the holomorphic factorization of 2D CFTs. It turns out not to be the case. The current $j_{\mu} = g^{-1}\partial_{\mu} g$, can be factorized but those are not separately conserved.

To remedy the situation, it is required to add a 3D term to the above action. This new action is the WZW action

$$S_{WZW} = S_{\sigma} + \frac{\kappa}{3}\int_{V} Tr[G^{-1}dG, (G^{-1}dG)^2] \tag{2.1.2}$$

where G is the extension of the field g to the 3D manifold V, which is bounded by the compactification of the 2D space where the NLSM was defined. But we immediately see a potential ambiguity, since a compact 2D surface can be the boundary of two different 3D bulks. It can be shown, that if we choose κ above to be integer, then this ambiguity can be eliminated at the level of partition function of the theory.

This new theory has two conserved currents which are given by

$$J(z) = \partial_z g g^{-1} \qquad \bar{J}(\bar{z}) = g^{-1}\partial_{\bar{z}} g \tag{2.1.3}$$

The existence of these separate currents also implies that the general solution of the field g can be written as

$$g(z, \bar{z}) = f(z)\bar{f}(\bar{z}). \tag{2.1.4}$$

2.2 Sugawara Construction

The currents above (2.1.3) can be expanded in the lie algebra basis as

$$J(z) = \sum_a J^a(z) t_a \tag{2.2.5}$$

where $\{t_a\}$ are the generators. On the other hand, the fields are also spin 1 primaries and hence admits a Laurent series expansion. Thus,

$$J^a(z) = \sum_n z^{-n-1} J^a_n. \tag{2.2.6}$$

By studying the transformation properties of $J(z)$ it can be shown that the above modes J^a_n satisfy the commutation relation of the sort

$$[J^a_n, J^b_m] = f^{ab}_c J^c_{n+m} + \kappa n g^{ab} \delta_{n+m}. \tag{2.2.7}$$

This is an affine Lie algebra of level k and it's called the current algebra of the theory. g^{ab} appearing in the central term is a bilinear form related to the metric of the gauge group. The structure constant appearing in the above commutators are the ones of the lie algebra G.

Provided that the theory has currents of conformal weight 1, we can construct the stress-tensor of the theory through *Sugawara Construction*. A generic Lie algebra can support a Sugawara construction, provided that the algebra has an invariant, invertible metric [40]. Since, this is also the requirement for the CS action to have (super)gravity correspondence, our cases of interest falls under this category.

If we assume that the Lie algebra $[T^a, T^b] = f_c^{ab} T^c$ has a invariant bilinear Ω^{ab} which is symmetric and invertible, it must satisfy

$$f_c^{ab}\Omega^{cd} + f_c^{ad}\Omega^{cb} = 0; \qquad \Omega^{ab}\Omega_{bc} = \delta_c^a. \tag{2.2.8}$$

The associativity properties of (2.2.7) requires that the metric g^{ab} also satisfies such a property, thus $g^{ab} = a\Omega^{ab}$, a being an arbitrary constant.

Once we have these, the prescription of Sugawara construction suggests that we write our stress tensor as a quadratic sum

$$T(z) = \kappa_{ab} : J^a(z) J^b(z) : . \tag{2.2.9}$$

To determine the matrix κ, we demand that the currents $J^a(z)$ are conformal primaries of weight 1. This fixes the OPE of the current with the above stress tensor as

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{(z-w)}. \tag{2.2.10}$$

This OPE leads to two constraint equations

$$\begin{aligned} f^{aeb}\kappa_{bc} + f^{acb}\kappa_{be} &= 0 \\ 2k\kappa_{cb}g^{ba} + \kappa_{bd} f_e^{ab} f_c^{ed} &= 0. \end{aligned} \tag{2.2.11}$$

Solving these yields $\kappa^{ab} = l\Omega^{ab}$, l being an arbitrary number. The constant a now is fixed in terms of l as $a = 1/(2kl)$.

Thus we see that the bilinears of current formed by a contraction with the metric, gives the stress tensor of the theory. This idea will be crucial in recovering the stress tensor of the holographic dual of our theory and proving that the theory is invariant under the asymptotic symmetry group of bulk (super)gravity theory.

2.3 Constrained dynamical systems

The main feature of a gauge theory is the existence of constraints. In a gauge theory formulation, by definition, there are more degrees of freedom than the physical system. Hence, some of the degrees of freedom of the theory are related to others by the means of constraint relations.

To understand how constraints work, let's start with the Lagrangian formulation of classical mechanics. Here the equation of motion takes the following form:

$$\frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{q}_n}) - \frac{\partial \mathcal{L}}{\partial q_n} = 0 \tag{2.3.12}$$

Where $\mathcal{L}(q_n, \dot{q}_n)$ is the Lagrangian and $\{q_n, \dot{q}_n\}$ are the generalised positions and velocities respectively, which depend on the time parameter t. Writing the above equation in terms of derivatives of q_ns and $\dot{q}_n$s only we get [11]:

$$\ddot{q}_m(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_m \partial \dot{q}_n}) + \dot{q}_m(\frac{\partial^2 \mathcal{L}}{\partial q_m \partial \dot{q}_n}) - \frac{\partial \mathcal{L}}{\partial q_n} = 0 \tag{2.3.13}$$

Thus it is obvious from the above equation that if we want to find the acceleration of the particle uniquely in terms of the (generalised) position and velocity of the particle, then the matrix $(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_m \partial \dot{q}_n}) \equiv M$ must be invertible. The cases where it fails to do so, the accelerations are no longer uniquely determined.

The matrix M also appears in the Hamiltonian formulation of classical mechanics. We define the momentum of a particle by the relation $p_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n}$. Thus if we want to write the velocities uniquely as a function of position and momentum of the particle, we end up demanding again that the matrix M is invertible.

When M is non-invertible, we understand that all the the velocities (or momentum) of the system are not independent. There exists some relations between them and thus the actual degrees of freedom are less than the ones we started with. These are the systems that have *gauge degrees of freedom.*

We will be working in Hamiltonian formalism where the phase space is spanned by $\{q_n, p_n\}$ variables ($n \in 1, ...N$). Since in a gauge system all these are not independent, there exists some relations between them of the form:

$$\phi_a(q,p) = 0 \tag{2.3.14}$$

where $a \in 1,...M$ and $M < N$. These equations then define a $(2N - M)$ dimensional submanifold in the phase space, known as the primary constraint surface. Since, all physical quantities must respect the constraints, they must be defined only in this surface. Thus, for example, the Hamiltonian can be extended arbitrarily outside of this surface by a transformation $H \to H + c^a(p,q)\phi_a$. Using this property, we can write an improve our Hamiltonian so that the Legendre transformation matrix M defined above is invertible. The new Hamiltonian is defined as

$$H_{new} = H_{can} + u^a\phi_a \tag{2.3.15}$$

where H_{can} is the canonical Hamiltonian defined by the usual Legendre transformation of the Lagrangian. The u^a variables are Lagrange multipliers that impose the constraints. With this Hamiltonian the equations of motion now looks like:

$$\dot{A} = \{A, H_{new}\}_{PB} = \{A, H\}_{PB} + u^a\{A, \phi_a\}_{PB} \tag{2.3.16}$$

where A(q,p) is some physical variable and the Poisson bracket is defined as usual.
Now an obvious requirement of consistency is that the constraints should be preserved in the time evolution. So, if we choose one of the constraints ϕ_a as the variable A in (2.3.16) we must get:

$$\dot{\phi}_a = 0 \implies \{\phi_a, H\}_{PB} + u^b\{\phi_a, \phi_b\}_{PB} = 0 \tag{2.3.17}$$

In principle this demand can put additional constraints between the variables (q,p). These are called Secondary Constraints as they are consequences of equations of motion unlike the original (primary) constraints. We must keep on checking whether all the constraints(both primary and secondary) in our theory satisfy $\dot{\phi}_a = 0$ and if not, we must add new constraints that come from the resulting e.o.m. Doing so, we'll finally end up with a full set of constraints.

2.3.1 First and Second class Constraints

The set of constraints are fundamentally classified into two categories: First and Second class constraints. As we'll see below, first class constraints 'generate' gauge transformations while the second class constraints 'arise' from gauge fixation.
In general the first class functions are those that commute with all the constraints. By commute, we mean that the Poisson bracket with the constraints are zero on the constraint surface. We consider primary constraints that are First class[1]. Now for a given function A, the dynamics is governed by (2.3.16) where ϕ^a are now first class constraints. But the new Hamiltonian depends on the Lagrange multipliers $\{u^a\}$ and choice of different $\{u^a\}$ should not alter the final physical state of the system. Using (2.3.15 and 2.3.16), the difference can be written as

$$\delta A = \{A, u^a_{diff}\phi_a\} \tag{2.3.18}$$

where $\{u_{diff}\}$ are the difference between two sets of transformation parameters. The above equation clearly indicates that the first class constraints generate a transformation δA that's not physical, i.e. a gauge transformation. It can be shown that the Poisson bracket of two such primary first class constraint also gives rise to gauge transformation.

On the other hand second class constraints are different and they have non-vanishing Poisson brackets with other constraints

$$\{\phi_a, \phi_b\} = M_{ab} \tag{2.3.19}$$

. It can be shown that transformations generated by them can take the state of the system outside of the constraint surface. They have no physical significance. It is often thought of as arising from the gauge fixation of some underlying first class constraint. Dirac [42] realised that in order to accommodate the second class constraints, the Hamiltonian formalism needed some modifications. He introduced Dirac Brackets which are generalisations of Poisson Brackets as

$$\{f, g\}_{DB} = \{f, g\}_{PB} - \{f, \phi_a\}M^{ab}\{g, \phi_b\} \tag{2.3.20}$$

[1]Secondary First class constraints are little subtle. See for instance, [41]

where M^{ab} is the inverse of the matrix in (2.3.19). It can be shown that Dirac Brackets satisfy all the properties of Poisson Bracket such as anti-symmetry and Jacobi identity. Additionally, the Dirac bracket of any physical quantity with constraints is zero.

In the next chapter we will see how these concepts play important role in analysing the holographic dual WZW theory. We will need to define the operators of the theory in a way that respects the constraints imposed on them.

Chapter 3

$\mathcal{N} = 2$ Super-BMS_3 invariant holographic dual theory

3.1 Introduction and Summary

Our discussions in the previous chapter concludes that in the absence of a cosmological constant, all solutions of pure gravity are locally isomorphic to Minkowski spacetime $\eta_{\mu\nu}$. This feature does not make 3D gravity trivial as a large variety of gravitational solutions exists whenever global topological structures are considered. If the global topology consists of non-contractible cycles, the global solution differs from $\eta_{\mu\nu}$ ([1] and references there in). It is known that 3D gravity solutions with non-trivial topology correspond to stress-energy tensors of a two dimensional theory. These two dimensional theories are usually referred as dual theory. The existence of a dual is more evident in the Chern-Simons formulation of 3D gravity [4, 6]. The dual theory, in general a (chiral) Wess-Zumino-Witten model [43](that we shall introduce in the next paragraph), is defined on a closed spatial section and is obtained by solving the constraints in the Chern-Simons theory [21, 44, 45]. In particular ordinary asymptotically flat 3D gravity can be understood as a $ISO(2,1)$ Chern-Simons gauge theory with flat boundary condition at null infinity where the Chern-Simons level k is identified with Newton's constant. Here the spatial section is a plane and the choice of boundary conditions is crucial in determining the dual theory.

It is well known that a generic Chern-Simons theory (with a compact gauge group G) in presence of a boundary reduces to a Wess-Zumino-Witten (WZW) model [43] at the boundary. The WZW model is constructed by adding a non-linear sigma model (of matrix valued field g) in two dimensions Σ with a three-dimensional WZW term $\Gamma[G]$ that lives in V, such that Σ is the boundary of V and G is the extension of the element g to V [46]:

$$I_{WZW} = \frac{1}{4a^2}\int_\Sigma \langle \partial_\mu g, \partial^\mu(g^{-1})\rangle + \kappa\Gamma[G], \quad \Gamma[G] = \frac{1}{3}\int_V \langle G^{-1}dG, (G^{-1}dG)^2\rangle, \tag{3.1.1}$$

where a and κ are two constants. Although the model contains an explicit three dimensional part, its variation is two dimensional. Thus WZW model describes the dynamics of two dimensional fields g. Such reductions have been mostly performed for asymptotically AdS 3D gravity [16, 23, 47–53]. Reduction of $ISO(2,1)$ Chern-Simons to WZW model was first studied in [54]. But we shall follow the route taken in [55], where the dual WZW model has been constructed for flat ordinary 3D gravity. In this chapter other than $ISO(2,1)$ gauge algebra, the boundary conditions suitable for flat asymptotics at null infinity have been applied for the gauge field. As a result, the dual WZW model, after gauge fixing , shows invariance under infinite dimensional quantum BMS_3 algebra, the asymptotic symmetry of flat 3D gravity.

In this chapter we shall use this construction for finding the dual of 3D asymptotically flat Supergravity theories with two supercharges. We reported these results in [56]. Similar analysis has been done earlier for minimal supersymmetric extension of gravity in [57]. The two supercharges may rotate among themselves if an internal $R-$symmetry is present. In our study both the scenarios, absence and presence of the internal $R-$symmetry, are considered. The resultant dual for both cases corresponds to a richer chiral WZW model at the boundary. We further study the symmetries of these duals. Imposing the constraints coming from appropriate boundary conditions at null infinity, we find that the dual theory is invariant under most generic quantum $\mathcal{N} = 2$ $\mathrm{SuperBMS}_3$ symmetry. In presence of an R symmetry, the $\mathcal{N} = 2$ $\mathrm{SuperBMS}_3$ algebra has three different kinds of central extensions and is so far not reported in the literature. The phase space description can be found by a Hamiltonian reduction of the models and are expected to be a generalised Liouville type theory [58].

The motivation behind our construction goes as follows : the dual theory for 3D asymptotically flat (super)gravity at null infinity is important to establish its connection with the corresponding AdS_3 results [59]. The presence of internal $R-$ charges gives a wide handle on the system. They are also crucial for the study of flat space holography in three dimensions. Most importantly these dual theories can be treated as a toy model for cosmological scenarios [60] due to the existence of time-dependent cosmological solutions that were found in [61].

Throughout the chapter, we are concerned with 3D supergravity. The chapter is organised as follows: in section 3.2 we present the two different kinds of $\mathcal{N} = 2$ SuperPoincaré algebras and their invariant bilinears. We briefly mention the 3D $\mathcal{N} = 2$ Supergravity theory and its asymptotic symmetry in section 3.3. Section 3.4 contains essential details about construction of a 2D dual theory of 3D flat gravity. In section 3.5 we present the dual theory, i.e. $\mathcal{N} = 2$ SuperPoincaré chiral WZW model. Later in sections 3.6 and 3.7 we study symmetries of this model. In section 3.8 we present a new $\mathcal{N} = 2$ SuperBMS$_3$ algebra. The results require some heavy computations and to maintain a correct flow we have presented only the important steps in the chapter. The details have been presented in six appendices that are referred at the relevant junctions in the main text.

3.2 $\mathcal{N} = 2$ SuperPoincaré algebra and Invariant Bilinears

In this chapter, we are interested in finding a two dimensional theory dual to $\mathcal{N} = 2$ Supergravity. As we shall see in details in later sections, to reach to our goal, we need to begin with $\mathcal{N} = 2$ SuperPoincaré algebras, i.e. supersymmetric extension of Poincaré algebra with two supercharges. In this section, we shall present two distinct versions of this algebra and the invariant bilinears associated with them. These will be the building blocks of our construction.

3.2.1 Two distinct $\mathcal{N} = 2$ SuperPoincaré algebras

There are two different versions of $\mathcal{N} = 2$ SuperPoincaré algebras known in the literature [59]. First one given as ,

$$\begin{aligned}
[J_a, J_b] &= \epsilon_{abc} J^c, \qquad [J_a, \mathcal{Q}^{1,2}_\alpha] = \frac{1}{2}(\Gamma_a)_\alpha^{\beta}\mathcal{Q}^{1,2}_\beta, \\
[J_a, P_b] &= \epsilon_{abc} P^c, \qquad [P_a, \mathcal{Q}^{1,2}_\alpha] = 0 \\
[P_a, P_b] &= 0, \qquad \{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} = -\frac{1}{2}(C\Gamma)^a_{\alpha\beta} P_a \delta^{ij},
\end{aligned} \tag{3.2.2}$$

Here $J_a, P_a(a=0,1,2)$ are the Poincare generators and $\mathcal{Q}^i_\alpha$ are two distinct $i=1,2$ two component $\alpha=+1,-1$ spinors which play the role of the two fermionic generators of the algebra. The above algebra (3.2.2) is known as $\mathcal{N}=(1,1)$ SuperPoincaré algebra. The other algebra is richer and it looks as ,

$$\begin{aligned} [J_a,J_b] &= \epsilon_{abc}J^c \qquad [J_a,P_b]=\epsilon_{abc}P^c \\ [J_a,Q^i_\alpha] &= \tfrac{1}{2}(\Gamma^a)^\beta_\alpha Q^i_\beta \qquad [Q^i_\alpha,T]=\epsilon^{ij}Q^j_\alpha \\ \{Q^i_\alpha,Q^j_\beta\} &= -\tfrac{1}{2}\delta^{ij}(C\Gamma^a)_{\alpha\beta}P_a + C_{\alpha\beta}\epsilon^{ij}Z. \end{aligned} \tag{3.2.3}$$

As in the previous case, J_a, P_a are Poincare generators and Q^i_α are two fermionic generators and various indices are running over same values. The important difference compared to the last case is that the two fermionic generators transform under a spinor representation of an internal R-symmetry generator T. As shown in [59], the above algebra is interesting due to the presence of a central term Z. This is known as $\mathcal{N}=(2,0)$ SuperPoincaré algebra. Our conventions are presented in A. In this chapter, we shall work with both these algebras. For the first one (3.2.2), our results are a trivial extension of [33], whereas for the second one (3.2.3), we get new physics , as we shall present in next sections.

3.2.2 Most Generic Non-degenerate Invariant Bilinears

In the context of the present chapter, an algebra is physically interesting when one can define a non-degenerate invariant bilinear or the quadratic Casimir for it. In the context of both the $\mathcal{N}=2$ SuperPoincaré algebras that we have written in the last section, the bilinears exist. Below we present the detailed computation for $\mathcal{N}=(2,0)$ case.

For computing the bilinear, we begin with the most general quadratic combination of the generators as,

$$C^2 = a\eta^{ab}P_aP_b + b\eta^{ab}J_aJ_b + c\eta^{ab}P_aJ_b + d_iC^{\alpha\beta}Q^i_\alpha Q^i_\beta + eC^{\alpha\beta}\epsilon^{ij}Q^i_\alpha Q^j_\beta + fTZ + gTT + hZZ,$$

where a,b,c,d_i,e,f,g,h are constants that we need to determine. For it to be a Casimir, it must commute with every generators of the algebra. An explicit computation shows that commutators of C^2 with Q^i, J_c, P_c do not vanish while others are identically zero.

Equating the four non vanishing ones to zero we get ,

$$b = e = g = 0, \quad c = d_1 = d_2.$$

This shows that the coefficients are fixed up to an overall factor and we fix it[1] by choosing $c = 1$. This procedure does not put any constraint on the coefficients a and h. Thus their values can be taken to be arbitrary.
In this chapter, we wish to write down Supergravity theories invariant under $\mathcal{N} = (2, 0)$ and $\mathcal{N} = (1, 1)$ SuperPoincaré algebra. For that purpose, we need to compute the supertrace elements between various generators. The supertrace elements can be thought of as the elements of the inverse metric in the space of algebra. Thus, taking inverse we get the supertrace elements as,

$$< J_a, P_b >= \eta_{ab} \quad < J_a, J_b >= \mu\eta_{ab} \quad < Q^I_\alpha, Q^J_\beta >= \delta^{IJ} C_{\alpha\beta} \quad < T, Z >= -1 \quad < T, T >= \bar{\mu}.$$

The arbitrariness in coefficients a and h manifests itself in arbitrariness of supertraces in $\langle J_a, J_b\rangle$ and $\langle T, T\rangle$ which are related by $a = \frac{1}{\mu}$, $h = \frac{1}{\bar{\mu}}$. One point to notice that, even for either or both of $\mu = \bar{\mu} = 0$, the supertrace matrix is non degenerate and hence will give us a valid theory, as the one considered in [61][2] On the contrary we can not set the off diagonal elements in the first and last two blocks to zero as that will make the determinant of this matrix vanishing and hence it will be degenerate.

For the $\mathcal{N} = (1, 1)$ case, we do not have the internal generators T, Z and thus redoing the whole calculation for only the remaining generators we get ,

$$< J_a, P_b >= \eta_{ab} \quad < J_a, J_b >= \mu\eta_{ab} \quad < Q^I_\alpha, Q^J_\beta >= \delta^{IJ} C_{\alpha\beta}.$$

We shall use these supertraces in the next section.

[1] It can be fixed by demanding that the bosonic Chern-Simons action reduces rightly to Einstein-Hilbert action, as we shall see in the next section.

[2] In [62], both of $\mu = \bar{\mu}$ were considered to be identical, but as it is clear from above analysis they are independent.

3.3 3-dimensional $\mathcal{N}=2$ Supergravity and its asymptotic symmetry

In this section, we shall study some aspects of 3-dimensional supergravity theories invariant under the above two symmetry algebras (3.2.2) and (3.2.3). As described in introduction (Sec. 1.6) 3-dimensional (super)gravity theories can be formulated as Chern-Simons theories with suitable gauge groups.

For our purpose, we shall consider the gauge groups to be $\mathcal{N}=(1,1)$ and $\mathcal{N}=(2,0)$ SuperPoincaré groups. The 3-manifold will be flat with *null boundary* and we shall identify the level k with Newton's constant as $k=\frac{1}{4G}$. For $\mathcal{N}=(1,1)$, the basis elements $\{T_a\}$ are $J_a, P_a, \mathcal{Q}^i_\alpha$, satisfying algebra (3.2.2) and for $\mathcal{N}=(2,0)$, the basis elements $\{T_a\}$ are $J_a, P_a, Q^i_\alpha, T, Z$ satisfying algebra (3.2.3). The Chern-Simons field A in each case is expanded in the basis as follows:

$$A_{(1,1)} = e^a P_a + \hat{\omega}^a J_a + \psi^\alpha_i \mathcal{Q}^i_\alpha \tag{3.3.4}$$

$$A_{(2,0)} = e^a P_a + \hat{\omega}^a J_a + \psi^\alpha_i Q^i_\alpha + BT + CZ \tag{3.3.5}$$

Using the supertrace elements as obtained in the last section we get the corresponding supergravity actions and they are respectively given as,

$$I^{(1,1)}_{\mu,\gamma} = \frac{k}{4\pi}\int [2e^a \hat{R}_a + \mu L(\hat{\omega}_a) - \bar{\Psi}^i_\beta \nabla \Psi^\beta_i] \tag{3.3.6}$$

and

$$I^{(2,0)}_{\mu,,\bar{\mu},\gamma} = \frac{k}{4\pi}\int [2e^a \hat{R}_a + \mu L(\hat{\omega}_a) - \bar{\Psi}^i_\beta \nabla \Psi^\beta_i - 2BdC + \bar{\mu} BdB] \tag{3.3.7}$$

where $\hat{\omega}^a = \omega^a + \gamma e^a$, for some constant γ and $\bar{\Psi}^i_\beta$ is the Majorana conjugate gravitino . The curvature two form $\hat{R}_a$, Lorentz Chern-Simons three form L_a and the covariant derivative of the gravitino appearing in (3.3.7) are respectively defined as,

$$\begin{aligned}
\hat{R}_a &= d\hat{\omega}_a + \frac{1}{2}\epsilon_{abc}\hat{\omega^b}\hat{\omega}^c \\
L_a &= \hat{\omega^a} d\hat{\omega}_a + \frac{1}{3}\epsilon^{abc}\hat{\omega}_a\hat{\omega}_b\hat{\omega}_c \\
\nabla\Psi^\beta_i &= d\Psi^\beta_i + \frac{1}{2}\hat{\omega}^a \Psi^\delta_i (\Gamma^a)^\beta_\delta + B\Psi^\beta_j \epsilon^{ij}.
\end{aligned} \tag{3.3.8}$$

It was first noticed in [63] that the shift in the spin connection is needed in order to formulate a general class of theories which are invariant under local lorentz transformations and gives linear equations of motion for the fields e^a and ω^a. It is worth mentioning that $\mathcal{N}=2$ supergravity as discussed in [61] is recovered by setting $\mu=\bar{\mu}=\gamma=0$. Action (3.3.6) is recovered from action (3.3.7) when we set the internal symmetry field parameters B, C to zero. This aspect holds true for all computations and final results of this chapter. Thus for the rest of the chapter, to describe our results, we shall work in details for $\mathcal{N}=(2,0)$ group and the corresponding supergravity action (3.3.7). For completion, we shall also present the results for $\mathcal{N}=(1,1)$ case side by side. Appendix A(Part 3) contains computational details for this case.

3.3.1 $\mathcal{N}=2$ Super-BMS$_3$ Algebra

It is well known by now that both pure gravity and supergravity theories enjoy an infinite dimensional symmetry enhancement at null infinity (see [61,64,65] for $\mathcal{N}=2$ case). The asymptotic symmetry group is $\mathcal{N}=2$ Super-BMS$_3$ group, which is an extension of BMS$_3$ with supercharges. To get to this symmetry algebra in the Chern-Simons formulation of gravity, we need to find out a proper fall off (at null infinity) condition on the Chern-Simons gauge field. The equation of motion(1.5.10) implies that locally the solutions of a Chern-Simons field are pure gauge $A=G^{-1}dG$, where G is a local group element. Writing the equation of motions in terms of the field parameters of (3.3.7), we get

$$d\hat{\omega}+\hat{\omega}^2=0, \quad (de)^{\gamma}_{\sigma}+[\hat{\omega},e]^{\gamma}_{\sigma}+\frac{1}{4}[\Psi^{\gamma}_{i}\bar{\Psi}^{i}_{\sigma}-\frac{1}{2}\bar{\Psi}^{i}_{\beta}\Psi^{\beta}_{i}\delta^{\gamma}_{\sigma}]=0 \tag{3.3.9}$$

$$d\Psi^{\beta}_{i}+(\hat{\omega}\Psi_i)^{\beta}+B\Psi^{\beta}_{j}\epsilon^{ji}=0, \quad dC=\frac{1}{2}\epsilon^{ij}\Psi^{\alpha}_{i}\Psi^{\beta}_{j}C_{\alpha\beta}, \quad dB=0 \tag{3.3.10}$$

Where the first two equations were written after contracting the original equations with $\frac{1}{2}(\Gamma^a)$ and defining $\hat{\omega}=\frac{1}{2}\hat{\omega}^a\Gamma_a$ and $e=\frac{1}{2}e^a\Gamma_a$. The solution to these equations can be found with a bit of algebra. The $\hat{\omega}$ and B equations easily solve as,

$$\hat{\omega}=\Lambda^{-1}d\Lambda \qquad B=d\tilde{B}. \tag{3.3.11}$$

Coming to the spinor equations, as they are coupled, we use Jordan Decomposition method to decouple them. Defining new variables as $\mathcal{G}^1=\frac{1}{2}(\Psi_1-i\Psi_2)$ and $\mathcal{G}^2=\frac{1}{2}(\Psi_1+$

$i\Psi_2$) we get the new equations to be:

$$d\mathcal{G}_1 + iB\mathcal{G}_1 + \hat{\omega}\mathcal{G}_1 = 0, \quad d\mathcal{G}_2 - iB\mathcal{G}_2 + \hat{\omega}\mathcal{G}_2 = 0, \tag{3.3.12}$$

whose solutions are given as,

$$\mathcal{G}_1 = e^{-i\tilde{B}}\Lambda^{-1}d\eta_1, \quad \mathcal{G}_2 = e^{i\tilde{B}}\Lambda^{-1}d\eta_2. \tag{3.3.13}$$

Thus the $R-$symmetry parameter field acts like a phase to the fermions. Using above results the rest of the equations of motion can be solved to give,

$$C = -\,i(\bar{\eta}_{1\alpha}d\eta_2^\alpha - \bar{\eta}_{2\alpha}d\eta_1^\alpha + d\tilde{C}) \tag{3.3.14}$$
$$e = -\,\Lambda^{-1}[\frac{1}{2}(\eta_1 d\bar{\eta}_2 + \frac{1}{2}d\bar{\eta}_2\eta_1\mathbf{I}) + \frac{1}{2}(\eta_2 d\bar{\eta}_1 + \frac{1}{2}d\bar{\eta}_1\eta_2\mathbf{I}) + db]\Lambda. \tag{3.3.15}$$

Notice that in both of the above expressions of C and e, the phase factors cancel among themselves. Here Λ is an arbitrary $SL(2,R)$ group element of unit determinant. B, C are $SL(2,R)$ scalars, $\eta_i, i=1,2$ are Grassmann-valued $SL(2,R)$ spinors and b is a traceless 2×2 matrix. All these are local functions of three space time coordinates u, ϕ, r. Implementing the radial gauge condition, the above solutions of various field parameters can be further decomposed as[3],

$$\begin{aligned}
\Lambda &= \lambda(u,\phi)\zeta(u,r) \\
\tilde{B} &= a(u,\phi) + \tilde{a}(u,r), \quad \tilde{C} = c(u,\phi) + \tilde{c}(u,r) + \bar{d}_2\lambda\tilde{d}_1 - \bar{d}_1\lambda\tilde{d}_2 \\
\eta_1 &= e^{ia}(\lambda\tilde{d}_1(u,r) + d_1(u,\phi)), \quad \eta_2 = e^{-ia}(\lambda\tilde{d}_2(u,r) + d_2(u,\phi)) \\
b &= \lambda E(u,r)\lambda^{-1} - \frac{1}{2}(d_1\bar{\tilde{d}}_2\lambda^{-1} + \overline{\lambda\tilde{d}_2}d_1\mathbf{I}) - \frac{1}{2}(d_2\bar{\tilde{d}}_1\lambda^{-1} + \overline{\lambda\tilde{d}_1}d_2\mathbf{I}) + F(u,\phi),
\end{aligned} \tag{3.3.16}$$

where $\dot{\zeta}(u,r_0) = \dot{\tilde{a}}(u,r_0) = \dot{\tilde{c}}(u,r_0) = \dot{\tilde{d}}_1(u,r_0) = \dot{\tilde{d}}_2(u,r_0) = \dot{F}(u,r_0) = 0$. At the boundary, these are neither functions of r nor of u and must not have any dynamics. Here we see that, even onshell, the system contains arbitrary local functions $\lambda, F, a, c, d_1, d_2$ of time u (and ϕ). This is a common feature of a gauge theory (like for example Chern-

[3]the decomposition can be obtained as, $\partial_\phi B_r = 0 \Rightarrow \tilde{B} = a(u,\phi) + \tilde{a}(u,r)$ and for $\partial_\phi\omega_r = 0 \Rightarrow \Lambda = \lambda(u,\phi)\zeta(u,r)$. Similarly, for the fermionic fields, demanding $\partial_\phi\mathcal{G}_r^1 = 0$ we find:
$\partial_\phi[e^{-i\tilde{B}}\Lambda^{-1}\partial_r\eta_1] = 0 \Rightarrow e^{-ia}\partial_r(\lambda^{-1}\eta_1) = \tilde{d}_1(u,r)\,(where\ r-dependence\ of\Lambda\ is\ captured\ in\ \tilde{d}_1)$
$\Rightarrow \eta_1 = e^{ia}(\lambda\tilde{d}_1(u,r) + d_1(u,\phi))$. Similarly we can find for other fields.

Simons theory) that the boundary conditions and equations of motion do not uniquely fix the time (u) evolution of all dynamical variable. Rather a general solution of equations of motion contains arbitrary functions of time as residual degrees of freedom of the gauge system. We are looking for the theory that determines the dynamics of these residual degrees $\lambda, F, a, c, d_1, d_2$.

Finally for $\mathcal{N}=2$ supergravity, as proposed in [61], the asymptotic fall of condition on the $r-$independent part of the gauge field gauge field looks like

$$\begin{aligned} a =& \sqrt{2}[J_1 + \frac{\pi}{k}(\mathcal{P} - \frac{4\pi}{k}\mathcal{Z}^2)J_0 + \frac{\pi}{k}(\mathcal{J} + \frac{2\pi}{k}\tau\mathcal{Z})P_0 - \frac{\pi}{k}\psi_i Q^i_+ - \frac{2\pi}{k}\mathcal{Z}T - \frac{2\pi}{k}\tau Z]d\phi \\ &+ [\sqrt{2}P_1 + \frac{8\pi}{k}\mathcal{Z}Z + \frac{\pi}{k}(\mathcal{P} - \frac{4\pi}{k}\mathcal{Z}^2)P_0]du, \end{aligned} \tag{3.3.17}$$

where various fields $\mathcal{P}, \mathcal{J}, \mathcal{Z}, \tau, \psi_i$ are functions of u, ϕ only. These are the residual degree of freedoms and will be in correspondence to $\lambda, F, a, c, d_1, d_2$ as introduced above in (3.3.16). A technical point to note is , although 3D spacetimes can have a non trivial boundary we will not consider the holonomy terms in the following. Consequently the resulting action principle at the boundary only captures the asymptotic symmetries of the original gravitational theory. Computing the conserved charges [66], it can be shown that the asymptotic symmetry of this system is given as,

$$[M_n, J_m] = (n-m)M_{n+m} + n^3 k\,\delta_{n+m,0}, \quad [J_n, J_m] = (n-m)J_{n+m} \tag{3.3.18}$$

$$[M_n, R_m] = -4mS_{n+m}, \quad [J_n, R_m] = -mR_{n+m}, \quad [J_n, S_m] = -mS_{n+m} \tag{3.3.19}$$

$$[R_n, S_m] = n\,k\,\delta_{n+m,0}, \tag{3.3.20}$$

$$[J_n, \mathcal{G}^i_m] = \left(\frac{n}{2} - m\right)\mathcal{G}^i_{n+m}, \quad (i = 1, 2) \tag{3.3.21}$$

$$[R_n, \mathcal{G}^1_m] = \mathcal{G}^1_{n+m}, \quad [R_n, \mathcal{G}^2_m] = -\mathcal{G}^2_{n+m} \tag{3.3.22}$$

$$\{\mathcal{G}^1_n, \mathcal{G}^2_m\} = M_{n+m} + 2kn^2\delta_{n+m,0} + (n-m)S_{n+m} \tag{3.3.23}$$

This is the quantum symmetry algebra of $\mathcal{N}=(2,0)$ theory as given in [61] presented in a diagonal basis for fermionic generators.

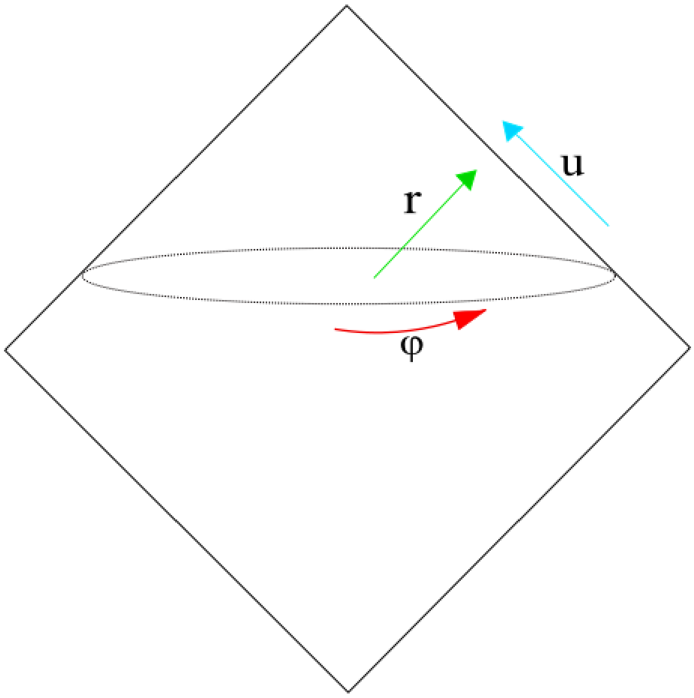

Figure 3.1: 3D Flat Space co-ordinates on a penrose diagram. The null boundary is spanned by $\{u, \phi\}$ co-ordinates.

3.4 The Boundary Theory

We are interested in constructing the two dimensional field theory that governs the dynamics of the 3D residual gauge degrees of freedom. We shall regard this as the dual theory to 3D asymptotically flat $\mathcal{N} = 2$ supergravity and in this section, we shall briefly sketch this construction. Since we are interested in supergravity theories on a 3D manifold with a boundary, we need to add suitable boundary terms to the supergravity action to ensure validity of variational principle. An alternate way to look at the scenario comes from the Chern-Simons formulation of gravity. Presence of a boundary implies a non trivial fall-off conditions on the gauge fields as given in (3.3.17). Hence a boundary term is required to add to the action in order to make solutions with the prescribed asymptotic to be a true extrema of the action under the variational principle. For this purpose, we split the constraints coming from the boundary gauge field into two parts : (a) constraints that relate the u and ϕ components of the gauge field and (b) constraints that various fields of the u component of the gauge field have to satisfy. It has been shown long back (in the context of asymptotically AdS theories) in [21, 44, 45] that pure Chern-Simons theory on a manifold with a boundary is equivalent to a 2-dimensional chiral Wess-Zumino-Witten theory living on that boundary under conditions analogous to (a). In general, decomposing the gauge field $A(u,\phi,r)$ in time and space components as $A = duA_u + \tilde{A}$, the Hamiltonian form of the Chern-Simons action(1.5.8) can be written as,

$$I_H[A] \;=\; -\frac{k}{4\pi}\int \langle \tilde{A}, \dot{\tilde{A}}du\rangle + 2\langle duA_u, \tilde{d}\tilde{A} + \tilde{A}^2\rangle, \tag{3.4.24}$$

upto total derivatives[4]. Since the fields and their derivatives do not go to zero at the boundary, for a well defined variational principle to work, we need to add $-\frac{k}{2\pi}du\tilde{d}\langle A_u, \delta\tilde{A}\rangle$ to the Hamiltonian action. Thus the complete 2D dual theory that contains all dynamical d.o.fs of 3D gravity is governed by

$$I = I_H[A] - \frac{k}{2\pi}\int du\tilde{d}\langle A_u, \delta\tilde{A}\rangle_{r=r_0}. \tag{3.4.25}$$

Furthermore expressing $\tilde{A} = G^{-1}\tilde{d}G$ for some group element $G(u,r,\phi)$, the above action can be written as,

$$I_{WZW} = \frac{k}{4\pi}\int_{\partial M} dud\phi\langle G^{-1}\partial_\phi G, G^{-1}\partial_u G\rangle - \frac{k}{2\pi}\int_{\partial M} du\tilde{d}\langle G^{-1}\partial_u G, \delta G^{-1}\tilde{d}G\rangle + \frac{k}{4\pi}\Gamma[G] \tag{3.4.26}$$

where $\Gamma[G]$ is the three dimensional Wess-Zumino term introduced in (3.1.1).

The above action has an explicit 2D part and a 3D part $\Gamma(G)$. But the variation of this action purely lives in 2 dimensions spanned by u,ϕ (See **Figure 3.1**). The action (3.4.26) reduces to the so called chiral Wess-Zumino-Witten model that is dual to a 3D Chern-Simons theory with a boundary. In the subsequent sections, we shall construct such a Wess-Zumino-Witten model and study its symmetry properties. As we shall see, after incorporating the radial gauge fixing conditions, the dynamics will only depend on two dimensional fields.

3.5 $\mathcal{N}=2$ SuperPoincaré Wess-Zumino-Witten model

Following the prescription outlined in the last section, we first write down (a) type of constraints on the asymptotic gauge field (3.3.17), relating its u and ϕ components as ,

$$e^a_u = \omega^a_\phi, \qquad \omega^a_u = 0, \qquad \psi^{\pm}_{Iu} = 0, \qquad B_u = 0, \qquad -4B_\phi = C_u. \tag{3.5.27}$$

[4]look at appendix A (Part 2) for details.

The u component of the gauge field (3.3.17) is further constrained and we shall come back to this point later. Under these constraints (3.5.27) the surface term at the boundary looks like:

$$I_{surf} = -\frac{k}{2\pi}\int du\tilde{d}\langle A_u, \delta\tilde{A}\rangle_{r_0\to\infty} = -\frac{k}{4\pi}\int_{\partial M} dud\phi[\omega^a_\phi\omega_{a\phi} + 4B^2_\phi]_{r_0\to\infty}, \tag{3.5.28}$$

where the $\phi-$ total derivative has been set to zero. Using the field parameters as defined in (3.3.7) and the supertrace elements the total action (3.4.26) can be expressed as:

$$\begin{aligned} I_{(2,0)} =& \frac{k}{4\pi}\Bigg[\int dud\phi\,[\,e^a_\phi\omega_{au} + \omega^a_\phi e_{au} + \mu\omega^a_\phi\omega_{au} + \bar{\psi}^u_{i\alpha}\psi^\alpha_{i\phi} \\ & - B_\phi C_u - C_\phi B_u - \omega^a_\phi\omega_{a\phi} - 4B^\phi B_\phi + \bar{\mu}B_\phi B_u\,]_{r_0\to\infty} \\ & + \frac{1}{6}\int[3\epsilon_{abc}e^a\omega^b\omega^c + \mu\epsilon_{abc}\omega^a\omega^b\omega^c + \frac{3}{2}\omega^a(C\Gamma_a)_{\alpha\beta}\Psi^\alpha_i\psi^\beta_i + 3B\psi^\alpha_i\psi^\beta_j C_{\alpha\beta}\epsilon^{ij}]\Bigg] \end{aligned} \tag{3.5.29}$$

As has been discussed in section 3.3, in an onshell gauge systems, there are left over residual degrees of freedom. To get the theory (action) that describes the dynamics of these degrees of freedom, we first evaluate the above action on the solutions of equations of motions obtained in section 3.3 as:

$$\begin{aligned} I_{(2,0)} =& \frac{k}{4\pi}\Bigg[\int dud\phi Tr[2\mu\Lambda^{-1}\Lambda'\Lambda^{-1}\dot{\Lambda} - 2(\bar{\eta_1}'\dot{\eta_2} + \bar{\eta_2}'\dot{\eta_1}) + 2i\dot{\tilde{B}}(\bar{\eta}_1\eta_2 - \bar{\eta}_2\eta_1) - 2(\Lambda^{-1}\Lambda')^2 - 4(\tilde{B}')^2 \\ & - 4\dot{\Lambda}\Lambda^{-1}(\frac{1}{2}(\eta_1\bar{\eta}'_2 + \bar{\eta}_2\eta'_1\mathbf{I}) + \frac{1}{2}(\eta_2\bar{\eta}'_1 + \bar{\eta}_1\eta'_2\mathbf{I}) + b') + 2i\dot{\tilde{B}}\tilde{C}' + \bar{\mu}\tilde{B}'\dot{\tilde{B}}] + \frac{2\mu}{3}\int Tr[(d\Lambda\Lambda^{-1})^3]\Bigg] \end{aligned} \tag{3.5.30}$$

Let us briefly mention the origin of various terms appearing in (3.5.30). The terms in (3.5.29) proportional to $\mu, \bar{\mu}$ directly reduces to their counterpart in (3.5.30) onshell whereas the term $3B\psi^\alpha_i\psi^\beta_j C_{\alpha\beta}\epsilon^{ij}$ gives rise to a 2D piece which added with the three other boundary pieces $-B_\phi C_u - C_\phi B_u - 4B^\phi B_\phi$ gives the terms $2i\dot{\tilde{B}}(\bar{\eta}_{1\alpha}\eta^{\alpha'}_2 - \bar{\eta}_{2\alpha}\eta^{\alpha'}_1) - 4(\tilde{B}')^2 + 2i\dot{\tilde{B}}\tilde{C}'$. In a similar way, the bulk terms $3\epsilon_{abc}e^a\omega^b\omega^c$ onshell gives a 2D piece which clubbed with boundary terms $e^a_\phi\omega_{au} + \omega^a_\phi e_{au}$ gives $-4\dot{\Lambda}\Lambda^{-1}(\frac{1}{2}(\eta_1\bar{\eta}'_2 + \bar{\eta}_2\eta'_1\mathbf{I}) + \frac{1}{2}(\eta_2\bar{\eta}'_1 + \bar{\eta}_1\eta'_2\mathbf{I}) + b')$. Finally $\frac{3}{2}\omega^a(C\Gamma_a)_{\alpha\beta}\Psi^\alpha_i\psi^\beta_i$ just vanishes onshell up to total derivatives. The terms proportional to I in (3.5.30) actually give zero contributions.

Further using the gauge decomposed forms of the solutions as in (3.3.16) and neglecting total derivatives in u, ϕ, the above action rightly simplifies to,

$$\begin{aligned}
I_{(2,0)} =& \frac{k}{4\pi}\Big\{ \int du d\phi Tr[2\mu\lambda^{-1}\lambda'\lambda^{-1}\dot{\lambda} - 2(\bar{d_1}'\dot{d}_2 + \bar{d_2}'\dot{d}_1) - 2ia'(\bar{d}_1\dot{d}_2 - \bar{d}_2\dot{d}_1 + \dot{\lambda}\lambda^{-1}(d_2\bar{d}_1 - d_1\bar{d}_2)) \\
& - 4(a')^2 - 2\dot{\lambda}\lambda^{-1}(d_1\bar{d}_2' + d_2\bar{d}_1' + 2F') - 2(\lambda^{-1}\lambda')^2 + 2i\dot{a}c' + \bar{\mu}a'\dot{a}] \\
& + \frac{2\mu}{3}\int Tr[(d\Lambda\Lambda^{-1})^3]\Big\},
\end{aligned} \tag{3.5.31}$$

It is interesting to note that the dependence on $\zeta, \tilde{a}, \tilde{c}, \tilde{d}_1, \tilde{d}_2, E$ drops from the two dimensional part of the last expression. One can easily check that the variation of action (3.5.31) is purely two dimensional [5]. This is the chiral Wess-Zumino-Witten (WZW) model dual to 3D asymptotically flat $\mathcal{N}=(2,0)$ Supergravity and is one of the main results of this chapter. Similarly the chiral WZW model dual to 3D asymptotically flat $\mathcal{N}=(1,1)$ Supergravity takes the following form,

$$\begin{aligned}
I_{(1,1)} = \frac{k}{2\pi}\Big\{ & \int du d\phi Tr[2\dot{\lambda}\lambda^{-1}\alpha' + \frac{1}{2}\sum_{i=1}^{2}\dot{\lambda}\lambda^{-1}\nu^i\overline{\nu}^{i'} - (\lambda'\lambda^{-1})^2 + \mu\lambda'\lambda^{-1}\dot{\lambda}\lambda^{-1} - \frac{1}{2}\sum_{i=1}^{2}\dot{\overline{\nu}}^i\nu^{i'}] \\
& + \frac{\mu}{3}\int Tr(d\Lambda\Lambda^{-1})^3\Big\},
\end{aligned} \tag{3.5.32}$$

where various fields are defined in the appendix A (Part 3).

To further analyse the dynamics of the above two dimensional theory (3.5.31), let us first write down the equations of motion of various fields. They are given as,

$$\begin{aligned}
&eom\ of F: && (\dot{\lambda}\lambda^{-1})' = 0 && (3.5.33)\\
&eom\ of c: && (\dot{a})' = 0 && (3.5.34)\\
&eom\ of d_1: && \dot{\bar{d}}_2' + \bar{d}_2'(\dot{\lambda}\lambda^{-1}) - ia'(\dot{\bar{d}}_2 + \bar{d}_2\dot{\lambda}\lambda^{-1}) = 0 && (3.5.35)\\
&eom\ of d_2: && \dot{\bar{d}}_1' + \bar{d}_1'(\dot{\lambda}\lambda^{-1}) + ia'(\dot{\bar{d}}_1 + \bar{d}_1\dot{\lambda}\lambda^{-1}) = 0 && (3.5.36)\\
&eom\ of a: && \dot{c}' - (\bar{d}_1\dot{d}_2 - \bar{d}_2\dot{d}_1 + \dot{\lambda}\lambda^{-1}(d_2\bar{d}_1 - d_1\bar{d}_2))' + 4ia'' = 0 && (3.5.37)\\
&eom\ of \lambda: && \dot{\hat{\alpha}} - (\dot{\lambda}\lambda^{-1})\hat{\alpha} + \hat{\alpha}(\dot{\lambda}\lambda^{-1}) + 2(\lambda'\lambda^{-1})' = 0 && (3.5.38)
\end{aligned}$$

where we have defined $\hat{\alpha} = 2F' + d_2(\bar{d_1}' + ia'\bar{d}_1) + d_1(\bar{d_2}' - ia'\bar{d}_2)$. The above equations

[5]In particular the variation of the last 3D term is given as

$$\frac{1}{3}\frac{\delta\int Tr[(d\Lambda\Lambda^{-1})^3]}{\delta\lambda^\alpha_\beta} = [(\lambda^{-1})'\dot{\lambda}\lambda^{-1} - (\lambda^{-1})^{\cdot}\lambda'\lambda^{-1}]^\beta_\alpha$$

are simplified versions of original equations of motion, where for simplifying one equation we have iteratively used other ones. For example, in the last equation the μ term drops off with a careful calculation and use of the first equation Similarly,we have used the c eom in deriving the final eom of a. This eliminates the $\bar{\mu}$ dependent piece . Thus we see that the final forms of eom do not contain any of the unfixed supertrace elements μ or $\bar{\mu}$. Hence the solutions of these fields will also be independent of these parameters. The generic solutions of these equations are given as,

$$\lambda = \tau(u)\kappa(\phi) \tag{3.5.39}$$

$$a = a_1(u) + a_2(\phi) \tag{3.5.40}$$

$$d_1 = e^{-ia_2}\tau(\zeta_1^{(1)}(u) + \zeta_2^{(1)}(\phi)) \tag{3.5.41}$$

$$d_2 = e^{ia_2}\tau(\zeta_1^{(2)}(u) + \zeta_2^{(2)}(\phi)) \tag{3.5.42}$$

$$c = \bar{\zeta}_2^{(1)}\zeta_1^{(2)} - \bar{\zeta}_2^{(2)}\zeta_1^{(1)} + c_1(u) + c_2(\phi) - 4iua_2' \tag{3.5.43}$$

$$F = \tau[a_F(\phi) + \delta_F(u) - u\kappa'\kappa^{-1} - \frac{1}{2}(\zeta_1^{(2)}\bar{\zeta}_2^{(1)} + \zeta_1^{(1)}\bar{\zeta}_2^{(2)})]\tau^{-1} \tag{3.5.44}$$

where they further decompose into individual functions of u and ϕ. As it turns out, this chiral WZW model is invariant under rich symmetries. In the next subsection, we shall study these symmetries and their consequences.

3.6 Symmetries of The Chiral WZW Model

3.6.1 Global Symmetry

The Chiral WZW model (3.5.30) that we derived in the last section in invariant under a set of global symmetries. Various fields enjoy a coordinate u, ϕ dependent transformation under these symmetries and they are given as,

$$
\begin{aligned}
&a \to a + A(\phi); \quad c \to c - 4iuA'; \quad d_1 \to e^{-iA}d_1; \quad d_2 \to e^{iA}d_2 \\
&c \to c + \mathcal{C}(\phi) \\
&\lambda \to \lambda\theta^{-1}(\phi); \quad F \to F + u\lambda(\theta^{-1}\theta')\lambda^{-1} \\
&F \to F + \lambda N(\phi)\lambda^{-1} \\
&d_1 \to d_1 + \lambda D_1(\phi); \quad c \to c + \bar{D}_1(\phi)\lambda^{-1}d_2; \quad F \to F - \frac{1}{2}d_2\bar{D}_1(\phi)\lambda^{-1} \\
&d_2 \to d_2 + \lambda D_2(\phi); \quad c \to c - \bar{D}_2(\phi)\lambda^{-1}d_1; \quad F \to F - \frac{1}{2}d_1\bar{D}_2(\phi)\lambda^{-1}
\end{aligned}
$$

In each of the above expressions, the fields that are not written remain unchanged under that corresponding symmetry transformation. Thus, we find that there are six finite symmetry transformations, generated by scalar parameters $A(\phi), \mathcal{C}(\phi)$, matrix valued parameters $\theta(\phi), N(\phi)$ and spinor parameters $D_1(\phi), D_2(\phi)$.

One possible way to get to these symmetry transformations is to look for the symmetries of the solutions given in (3.5.39)[6]. We have presented relevant details of the computations in appendix A (Part 4). Once we obtain these transformations, they can be proved to be symmetries of the action as well.

Next we look for the Noether currents corresponding to the above symmetries. For this purpose, we need the infinitesimal versions of the above symmetry transformations that are as follows,

$$\delta_A a = A(\phi); \quad \delta_A c = -4iuA'; \quad \delta_A d_1 = -iAd_1; \quad \delta_A d_2 = iAd_2 \tag{3.6.45}$$

$$\delta_{\mathcal{C}} c = \mathcal{C}(\phi) \tag{3.6.46}$$

$$\delta_\theta \lambda = -\lambda\Theta(\phi); \quad \delta_\theta F = +u\lambda\Theta'\lambda^{-1} \tag{3.6.47}$$

$$\delta_N F = \lambda N(\phi)\lambda^{-1} \tag{3.6.48}$$

$$\delta_{D1} d_1 = \lambda D_1(\phi); \quad \delta_{D1} c = \bar{D}_1(\phi)\lambda^{-1}d_2; \quad \delta_{D1} F = -\frac{1}{2}d_2\bar{D}_1(\phi)\lambda^{-1} \tag{3.6.49}$$

$$\delta_{D2} d_2 = \lambda D_2(\phi); \quad \delta_{D2} c = -\bar{D}_2(\phi)\lambda^{-1}d_1; \quad \delta_{D2} F = -\frac{1}{2}d_1\bar{D}_2(\phi)\lambda^{-1}. \tag{3.6.50}$$

Here we have used same $A(\phi), \mathcal{C}(\phi), N(\phi), D_1(\phi), D_2(\phi)$ as infinitesimal transformation parameters and $\Theta(\phi)$ is the infinitesimal parameter for θ transformation as $\theta(\phi) = I +$

[6]We are thankful to Prof. Glenn Barnich for clarifying this point to us.

$\Theta(\phi)$. For a theory $S[\phi_i] = \int \mathcal{L}(\phi_i, \partial_\mu\phi_i)$, the Noether current associated to a global symmetry generated by parameter ϵ is given as,

$$\mathcal{J}^\mu{}_\epsilon = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\delta_\epsilon\phi_i - K^\mu_\epsilon, \qquad \partial_\mu K^\mu_\epsilon = \delta_\epsilon\mathcal{L}.$$

The current is conserved i.e. $\partial_\mu\mathcal{J}^\mu{}_\epsilon = 0$ onshell. From this definition there is a clear ambiguity in the identification of current, as

$$J^\mu_\epsilon \sim \mathcal{J}^\mu{}_\epsilon + \partial_\nu S^{[\mu\nu]}_\epsilon + T^\mu_\epsilon, \tag{3.6.51}$$

where $S^{[\mu\nu]}_\epsilon$ is any antisymmetric tensor in its indices and T^μ_ϵ is a possible vector that is onshell divergenceless. Both the currents will generate the same symmetry for the system. Below we present the currents corresponding to above symmetries (3.6.45),

$$\begin{aligned}
J^\mu_A &= \delta^\mu_0\frac{k}{4\pi}Tr[2\bar{\mu}a' + 2ic' - 8ua'' + 2i(\bar{d_2}'d_1 - \bar{d_1}'d_2 - ia'(\bar{d}_2d_1 + \bar{d}_1d_2))]A = \delta^\mu_0[(-Q^A)(-A)]\\
J^\mu_C &= \delta^\mu_0\frac{k}{4\pi}Tr[2ia'\mathcal{C}] = \delta^\mu_0[Q_C(-i\mathcal{C})], \quad Q_C = -\frac{ka'}{2\pi}\\
J^\mu_\Theta &= \delta^\mu_0\frac{k}{2\pi}Tr[\{\lambda^{-1}\hat{\alpha}\lambda + 2u(\lambda^{-1}\lambda')' - \mu\lambda^{-1}\lambda'\}\theta] = \delta^\mu_0 2Tr[Q^J_a\Theta^a]\\
J^\mu_N &= \delta^\mu_0\frac{k}{4\pi}Tr[-4\lambda^{-1}\lambda' N] = \delta^\mu_0 2Tr[Q^P_a(-N^a)]\\
J^\mu_{D_2} &= \delta^\mu_0(-\frac{k}{\pi})Tr[(\bar{d_1}'\lambda + ia'\bar{d}_1\lambda)D_2] = Tr[Q^{G_2}_\alpha D^\alpha_2]\\
J^\mu_{D_1} &= \delta^\mu_0(-\frac{k}{\pi})Tr[(\bar{d_2}'\lambda - ia'\bar{d}_2\lambda)D_1] = Tr[Q^{G_1}_\alpha D^\alpha_1],
\end{aligned} \tag{3.6.52}$$

where $N(\phi)$ and $\Theta(\phi)$ are infinitesimal $SL(2,\mathbb{R})$ matrices which can be further expanded in the basis of Γ matrices as $N(\phi) = N^a(\phi)\Gamma_a$ and $\Theta(\phi) = \Theta^a(\phi)\Gamma_a$. The details of the above computations can be found in appendix A (Part 4). Here we have chosen $S^{[\mu\nu]}_\epsilon, T^\mu_\epsilon$ for certain transformations, as we want the current to be non zero only along the time u component. This way we directly get the canonical generators of the corresponding transformation. From these currents, one can find the corresponding current algebra using the usual procedure [66]. Alternate way to get to the same algebra is, in Hamiltonian formalism, the computation of the Dirac bracket algebra of the canonical generators of the symmetries using the relation below,

$$\delta_{\epsilon_2}J^0_{\epsilon_1} = \{J^0_{\epsilon_1}, J^0_{\epsilon_2}\}_{DB}$$

The Dirac brackets calculated are given below:

$$
\begin{aligned}
\left\{Q^P_a(\phi), Q^P_b(\phi')\right\}_{DB} &= \left\{Q^P_a(\phi), Q^A(\phi')\right\}_{DB} = \left\{Q^P_a(\phi), Q^C(\phi')\right\}_{DB} = 0 \\
\left\{Q^P_a(\phi), Q^{G_1}_\alpha(\phi')\right\}_{DB} &= \left\{Q^P_a(\phi), Q^{G_2}_\alpha(\phi')\right\}_{DB} = 0 \\
\left\{Q^P_a(\phi), Q^J_b(\phi')\right\}_{DB} &= \left\{Q^J_a(\phi), Q^P_b(\phi')\right\}_{DB} = \epsilon_{abc} Q^P_c(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}\eta_{ab}\partial_\phi\delta(\phi-\phi') \\
\left\{Q^J_a(\phi), Q^J_b(\phi')\right\}_{DB} &= \epsilon_{abc} Q^J_c(\phi)\delta(\phi-\phi') + \mu\frac{k}{2\pi}\eta_{ab}\partial_\phi\delta(\phi-\phi') \\
\left\{Q^J_a(\phi), Q^A(\phi')\right\}_{DB} &= \left\{Q^J_a(\phi), Q^C(\phi')\right\}_{DB} = 0 \\
\left\{Q^{G_1}_\alpha(\phi), Q^J_a(\phi')\right\}_{DB} &= -\frac{1}{2}(\Gamma_a)^\beta_\alpha Q^{G_1}_\beta(\phi)\delta(\phi-\phi') \\
\left\{Q^{G_2}_\alpha(\phi), Q^J_a(\phi')\right\}_{DB} &= -\frac{1}{2}(\Gamma_a)^\beta_\alpha Q^{G_2}_\beta(\phi)\delta(\phi-\phi') \\
\left\{Q^C(\phi), Q^C(\phi')\right\}_{DB} &= \left\{Q^C(\phi), Q^{G_1}_\alpha(\phi')\right\}_{DB} = \left\{Q^C(\phi), Q^{G_2}_\alpha(\phi')\right\}_{DB} = 0 \\
\left\{Q^C(\phi), Q^A(\phi')\right\}_{DB} &= \frac{k}{2\pi}\partial_\phi\delta(\phi-\phi'), \quad \left\{Q^A(\phi), Q^A(\phi')\right\}_{DB} = \frac{k}{2\pi}\bar{\mu}\partial_\phi\delta(\phi-\phi') \\
\left\{Q^{G_1}_\alpha(\phi), Q^A(\phi')\right\}_{DB} &= -iQ^{G_1}_\alpha(\phi)\delta(\phi-\phi'), \quad \left\{Q^{G_2}_\alpha(\phi), Q^A(\phi')\right\}_{DB} = iQ^{G_2}_\alpha(\phi)\delta(\phi-\phi') \\
\left\{Q^{G_1}_\alpha(\phi), Q^{G_2}_\beta(\phi')\right\}_{DB} &= -(C\Gamma^a)_{\alpha\beta}Q^P_a\delta(\phi-\phi') - \frac{k}{\pi}C_{\alpha\beta}\partial_\phi\delta(\phi-\phi') + ia'\frac{k}{\pi}C_{\alpha\beta}\delta(\phi-\phi'),
\end{aligned}
\tag{3.6.53}
$$

here we have used $\frac{k}{\pi}C_{\alpha\beta}(\lambda^{-1}\lambda')^\beta_\gamma = (C\Gamma^a)_{\alpha\gamma}Tr[\Gamma_a(\lambda^{-1}\lambda')]$. This is the same affine extended $\mathcal{N}=(2,0)$ SuperPoincaré algebra after a change of basis for the fermionic generators as,

$$
Q^1_\alpha(\phi) = \frac{1}{2}(Q^{G_1}_\alpha(\phi) + Q^{G_2}_\alpha(\phi)), \quad Q^2_\alpha(\phi) = \frac{1}{2i}(Q^{G_1}_\alpha(\phi) - Q^{G_2}_\alpha(\phi)).
$$

In this new basis the fermionic Dirac Brackets take the form:

$$
\begin{aligned}
\{Q^1_\alpha(\phi), Q^2_\beta(\phi')\}_{DB} &= -\frac{k}{2\pi}a'(\phi)C_{\alpha\beta}\delta'(\phi-\phi') \\
\{Q^I_\alpha(\phi), Q^I_\beta(\phi')\}_{DB} &= -\frac{1}{2}(C\Gamma^a)_{\alpha\beta}Q^P_a(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}C_{\alpha\beta}\delta'(\phi-\phi') \\
\{Q^1_\alpha(\phi), Q^A(\phi')\}_{DB} &= Q^2_\alpha(\phi)\delta(\phi-\phi'), \quad \{Q^2_\alpha(\phi), Q^A(\phi')\}_{DB} = -Q^1_\alpha(\phi)\delta(\phi-\phi') \\
\left\{Q^J_a(\phi), Q^1_\alpha(\phi')\right\}_{DB} &= \frac{1}{2}(\Gamma_a)^\beta_\alpha Q^1_\beta(\phi)\delta(\phi-\phi'), \quad \left\{Q^J_a(\phi), Q^2_\alpha(\phi')\right\}_{DB} = \frac{1}{2}(\Gamma_a)^\beta_\alpha Q^2_\beta(\phi)\delta(\phi-\phi').
\end{aligned}
$$

The above modified Dirac brackets along with bosonic ones in (3.6.53) reproduce the exact affine extended $\mathcal{N}=(2,0)$ SuperPoincaré algebra that we started with in (3.2.3). Thus, we see that the global symmetry of the chiral WZW theory is exactly same as that of the dual 3D Supergravity. Similarly for $\mathcal{N}=(1,1)$ case we get following affine extension,

$$\begin{aligned}
\{Q^P_a(\phi), Q^P_a(\phi')\}_{DB} &= 0 \\
\{Q^P_a(\phi), Q^J_b(\phi')\}_{DB} &= \epsilon_{abc}Q^P_c(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}\eta_{ab}\delta'(\phi-\phi') \\
\{Q^P_a(\phi), Q^i_\alpha(\phi')\}_{DB} &= 0 \quad (i=1,2) \\
\{Q^J_a(\phi), J_b(\phi')\}_{DB} &= \epsilon_{abc}Q^J_c\delta(\phi-\phi') + \frac{\mu k}{2\pi}\eta_{ab}\delta'(\phi-\phi') \\
\{Q^J_a(\phi), Q^i_\alpha(\phi')\}_{DB} &= \frac{1}{2}(Q^i\Gamma_a)_\alpha(\phi)\delta(\phi-\phi') \quad (i=1,2) \\
\{Q^i_\alpha(\phi), Q^j_\beta(\phi')\}_{DB} &= \delta^{ij}[-\frac{1}{2}(C\Gamma^a)_{\alpha\beta}Q^P_a(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}C_{\alpha\beta}\delta'(\phi-\phi')]
\end{aligned}$$

The explicit derivation of various canonical generators for $\mathcal{N}=(1,1)$ case has been provided in appendix A (Part 3).

3.6.2 Gauge Symmetry

Other than the above global symmetry, the chairal WZW model (3.5.30) is also invariant under a gauge symmetry. The gauge transformations of various fields can be obtained from the Polyakov-Wiegmann identities and for an arbitrary gauge transformation parameter $\Sigma(u)$, the transformations are given as follows :

$$\lambda(u) \to \Sigma(u)\lambda, \quad d_i(u) \to \Sigma(u)d_i(u) \quad F(u) \to \Sigma(u)F\Sigma(u)^{-1}, \tag{3.6.54}$$

while $a(u), c(u)$ remains invariant. This makes the dynamics of this system constrained and we need to take into account its implications in defining the conserved charges of the theory. We shall come back to this issue in the next section.

3.7 Enhanced Symmetries of $\mathcal{N}=2$ SuperPoincaré Wess-Zumino-Witten theory

In order to get an infinite dimensional mode algebra from the above current algebra usual conformal field theory techniques of [67] can be used after a slight modification. We implement the modified Sugawara construction following [57] to get the stress-tensor. In this case, we are looking for four bosonic generators and two fermionic generators.

These can be achieved by defining the followings:

$$\begin{aligned}
H &= \frac{\pi}{k}Q^P_aQ^P_a + 4\frac{\pi}{k}Q^CQ^C \\
J &= -\mu\frac{\pi}{k}Q^P_aQ^P_a - 2\frac{\pi}{k}Q^J_aQ^P_a + \frac{\pi}{k}C_{\alpha\beta}Q^{G_1}_\alpha Q^{G_2}_\beta + 2\frac{\pi}{k}Q^AQ^C - \bar{\mu}\frac{\pi}{k}Q^CQ^C \\
\mathcal{G}^1 &= \frac{\pi}{k}(Q^P_2Q^{G_1}_+ + \sqrt{2}Q^P_0Q^{G_1}_-) + 2i\frac{\pi}{k}Q^{G_1}_+Q^C, \\
\mathcal{G}^2 &= \frac{\pi}{k}(Q^P_2Q^{G_2}_+ + \sqrt{2}Q^P_0Q^{G_2}_-) - 2i\frac{\pi}{k}Q^{G_1}_+Q^C
\end{aligned} \tag{3.7.55}$$

along with Q^A and Q^C defined in the last section. Here H, J are both weight two bosonic generators and J corresponds to the stress-tensor. Q^A, Q^C are two weight one bosonic generators and $\mathcal{G}^1, \mathcal{G}^2$ are two weight 3/2 fermionic generators. The values of the relative coefficients in the bilinear of currents are fixed by demanding that the Dirac brackets of stress mode J with other bosonic generators should be proportional to them i.e. $\{J(\phi), Q(\phi')\} \sim Q(\phi)$ for each current mode $Q(\phi)$. We refer the readers to appendix A (Part 5) for computational details.

There is a subtle point to note here. The chiral WZW model that we are studying is a gauge theory. Thus, in the usual Hamiltonian formulation, we must study it as a constrained system. The constraints arise from the imposed boundary (asymptotic) value of the radial gauge fixed Chern-Simons field, $a = g^{-1}dg$, as given in (3.3.17). We have already taken into account part of the constraints (type (a)) for constructing the corresponding WZW model. The remaining constraints(type (b)) on the gauge field parameters are

$$\hat{\omega}^1_\phi = \sqrt{2}; \quad \omega^2_\phi = 0; \quad \psi^{1+}_\phi = \psi^{2+}_\phi = 0; \quad e^1_\phi = e^2_\phi = 0.$$

These conditions, that need to be imposed only at the boundary, manifest themselves through constraints on the fields of the WZW model (3.5.31). This is because, at the boundary we can as well identify the onshell CS gauge field parameters $\hat{\omega}, \psi^1, \psi^2, e$ of (3.3.11),(3.3.13),(3.3.14) with the WZW fields $\lambda, d_1, d_2, \hat{\alpha}$ of (3.3.16)[7]. Thus the con-

[7] at the boundary, the non dynamical functions (of u, r) can get absorbed into the WZW fields.

straints on fields are:

$$(\lambda^{-1}\lambda')^1 = \sqrt{2} \qquad (\lambda^{-1}\lambda')^2 = 0$$
$$(\lambda^{-1}d_1' + ia'\lambda^{-1}d_1)^- = (\lambda^{-1}d_2' - ia'\lambda^{-1}d_2)^- = 0$$
$$(\lambda^{-1}\frac{\hat{\alpha}}{2}\lambda)^1 = (\lambda^{-1}\frac{\hat{\alpha}}{2}\lambda)^2 = 0.$$

The above constraints can as well be expressed in terms of the canonical generators of (3.6.52)as,

$$\begin{aligned} Q_0^P &= \sqrt{2}\frac{k}{2\pi}, \quad Q_2^P = 0 \\ Q_0^J &= -\sqrt{2}\frac{\mu k}{2\pi}, \quad Q_2^J = 0 \\ Q_{1+}^1 &= 0, \qquad Q_+^2 = 0 \end{aligned} \tag{3.7.56}$$

Let us denote the above constraints as $\{\Phi_l\}, l = 1, \cdots 6$ respectively as presented above. They collectively define the constrained hypersurface. It can be easily verified that, four out of these six constraints, denoted as $\{\gamma_p\} = \{\Phi_1, \Phi_3, \Phi_4, \Phi_6\}, p = 1, \cdots 4$ have a vanishing Dirac brackets with all of $\{\Phi_l\}$ on the constrained hypersurface [8]. Thus $\{\gamma_p\}$ are the firstclass constraints and they generate the gauge symmetry that we presented in subsection 3.6.2. We know that in a constrained system, one needs to usually modify the canonical generators (conserved charges) such that they commute with the first class constraints on the constraint hypersurface (defined by first class ones) as otherwise they are not gauge invariant (and hence are not physical observable). The charges defined in(3.7.55) fail to satisfy this property, as we have shown in appendix A (Part 5). Thus we need to further modify them using (3.6.51), such that the resultant charges are true observables of the theory. The required minimal shifts in generators that achieve the above requirements are given by:

$$\mathcal{H} = H + \partial_\phi Q_2^P; \quad \mathcal{J} = J - \partial_\phi Q_2^J; \quad \hat{\mathcal{G}}^I = \mathcal{G}^I + \partial_\phi Q_+^I$$

Finally we compute the Dirac brackets of these new gauge invariant canonical generators

[8] $\{\phi_2, \phi_4\} = \{Q_2^P, Q_2^J\} \neq 0$, as there is a central term.

and they are given as[9],

$$
\begin{aligned}
&\{\mathcal{J}(\phi),\mathcal{J}(\phi')\}_{DB}=(\mathcal{J}(\phi)+\mathcal{J}(\phi'))\partial_\phi\delta(\phi-\phi')-\mu\frac{k}{2\pi}\partial_\phi^3\delta(\phi-\phi')\\
&\{\mathcal{H}(\phi),\mathcal{J}(\phi')\}_{DB}=(\mathcal{H}(\phi)+\mathcal{H}(\phi'))\partial_\phi\delta(\phi-\phi')-\frac{k}{2\pi}\partial_\phi^3\delta(\phi-\phi')\\
&\{\tilde{\mathcal{H}}(\phi),\tilde{\mathcal{H}}(\phi')\}_{DB}=0,\quad \{\mathcal{H}(\phi),Q^A(\phi')\}_{DB}=4Q^C(\phi)\partial_\phi\delta(\phi-\phi')\\
&\{\mathcal{J}(\phi),Q^A(\phi')\}_{DB}=Q^A(\phi)\partial_\phi\delta(\phi-\phi'),\quad \{\mathcal{J}(\phi),Q^C(\phi')\}_{DB}=Q^C(\phi)\partial_\phi\delta(\phi-\phi')\\
&\{\mathcal{J}(\phi),Q^A(\phi')\}_{DB}=Q^A(\phi)\partial_\phi\delta(\phi-\phi') \qquad (3.7.57)\\
&\left\{Q^C(\phi),Q_A(\phi')\right\}_{DB}=\frac{k}{2\pi}\partial_\phi\delta(\phi-\phi'),\quad \left\{Q^A(\phi),Q^A(\phi')\right\}_{DB}=\frac{k}{2\pi}\bar{\mu}\partial_\phi\delta(\phi-\phi')\\
&\{\mathcal{J}(\phi),\hat{\mathcal{G}}^i(\phi')\}_{DB}=(\hat{\mathcal{G}}^i(\phi)+\frac{1}{2}\hat{\mathcal{G}}^i(\phi'))\partial_\phi\delta(\phi-\phi'),\quad (i=1,2)\\
&\{\mathcal{H}(\phi),\hat{\mathcal{G}}^i(\phi')\}_{DB}=0,\quad (i=1,2)\\
&\{\hat{\mathcal{G}}^1(\phi),Q^A(\phi')\}_{DB}=-i\hat{\mathcal{G}}^1(\phi)\delta(\phi-\phi'),\quad \{\hat{\mathcal{G}}^2(\phi),Q^A(\phi')\}_{DB}=i\hat{\mathcal{G}}^2(\phi)\delta(\phi-\phi')\\
&\{\hat{\mathcal{G}}^1(\phi),\hat{\mathcal{G}}^2(\phi')\}_{DB}=\mathcal{H}(\phi)\delta(\phi-\phi')-\frac{k}{\pi}\partial_\phi^2\delta(\phi-\phi')-2i(Q^C(\phi)+Q^C(\phi'))\delta'(\phi-\phi')
\end{aligned}
$$

The above Dirac brackets are expected to be the ones of the physical observables $\mathcal{H},\mathcal{J},\hat{\mathcal{G}}^i,Q^A,Q^C$ of the reduced two dimensional Super Liouville theory that is classically equivalent to the 3D Supergravity. Such a structure was found in a subsequent paper [58].

Let us also present the enhanced symmetry for the $\mathcal{N}=(1,1)$ case here :

$$
\begin{aligned}
\{\mathcal{H}(\phi),\mathcal{H}(\phi')\}_{DB} &= 0\\
\{\mathcal{H}(\phi),\mathcal{J}(\phi')\}_{DB} &= (\mathcal{H}(\phi)+\mathcal{H}(\phi'))\delta'(\phi-\phi')-\frac{k}{2\pi}\partial^3\delta(\phi-\phi')\\
\{\mathcal{H}(\phi),\mathcal{G}^i(\phi')\}_{DB} &= 0 \qquad (3.7.58)\\
\{\mathcal{J}(\phi),\tilde{\mathcal{J}}(\phi')\}_{DB} &= (\mathcal{J}(\phi)+\mathcal{J}(\phi'))\delta'(\phi-\phi')-\frac{\mu k}{2\pi}\partial^3\delta(\phi-\phi')\\
\{\mathcal{J}(\phi),\tilde{\mathcal{G}}^i(\phi')\}_{DB} &= (\tilde{\mathcal{G}}^i(\phi)+\frac{1}{2}\tilde{\mathcal{G}}^i(\phi'))\delta'(\phi-\phi')\\
\{\tilde{\mathcal{G}}^i(\phi),\tilde{\mathcal{G}}^i(\phi')\}_{DB} &= \delta^{ij}(\tilde{\mathcal{H}}(\phi)\delta(\phi-\phi')-\frac{k}{2\pi}\partial^2\delta(\phi-\phi')).
\end{aligned}
$$

Here the physical observables $\mathcal{H},\mathcal{J},\tilde{\mathcal{G}}^i$ of the reduced two dimensional Super Liouville theory is dynamically equivalent to a 3D Supergravity with two supercharges but without any internal R-symmetry.

[9] look at appendix A (Part 6)

3.8 A new $\mathcal{N}=2$ SuperBMS$_3$ algebra

Finally we write the quantum algebra that corresponds to the above Dirac brackets (3.7.58) and (3.7.57). For this purpose, we define The modes of the above fields as.

$$\begin{aligned} M_n &= \frac{1}{2\pi}\int d\phi e^{in\phi}\mathcal{H}(\phi), \quad J_n = \frac{1}{2\pi}\int d\phi e^{in\phi}\mathcal{J}(\phi), \\ \mathcal{G}_n^{1,2} &= \frac{1}{2\pi}\int d\phi e^{in\phi}\hat{\mathcal{G}}^{1,2}(\phi), \quad \tilde{\mathcal{G}}_n^{1,2} = \frac{1}{2\pi}\int d\phi e^{in\phi}\tilde{\mathcal{G}}^{1,2}(\phi), \\ R_n &= \frac{1}{2\pi}\int d\phi e^{in\phi}Q^A(\phi), \quad S_n = \frac{1}{2\pi}\int d\phi e^{in\phi}Q^C(\phi). \end{aligned}$$

We further use the identification for bosonic and fermionic commutator brackets respectively as

$$i\{,\}_{DB} \to [,] \qquad \{,\}_{DB} \to \{,\}.$$

The non zero brackets of the algebra corresponding to (3.7.57) looks as,

$$\begin{aligned} [M_n, J_m] &= (n-m)M_{n+m} + n^3 k\ \delta_{n+m,0}, \quad [J_n, J_m] = (n-m)J_{n+m} + n^3\mu\ k\ \delta_{n+m,0} \\ [M_n, R_m] &= -4mS_{n+m}, \quad [J_n, R_m] = -mR_{n+m}, \quad [J_n, S_m] = -mS_{n+m} \\ [R_n, S_m] &= n\ k\ \delta_{n+m,0}, \quad [R_n, R_m] = n\ \bar{\mu}\ k\ \delta_{n+m,0} \\ [J_n, \mathcal{G}^i_m] &= \left(\frac{n}{2}-m\right)\ \mathcal{G}^i_{n+m}, \quad (i=1,2) \\ [R_n, \mathcal{G}^1_m] &= \mathcal{G}^1_{n+m}, \quad [R_n, \mathcal{G}^2_m] = -\mathcal{G}^2_{n+m} \\ \{\mathcal{G}^1_n, \mathcal{G}^2_m\} &= M_{n+m} + 2kn^2\delta_{n+m,0} + (n-m)S_{n+m} \end{aligned} \tag{3.8.59}$$

This is a a new SuperBMS$_3$ algebra, so far not identified in the literature. Here the central term for $[J_n, J_m]$ and $[R_n, R_m]$ are independent of each other. The closest one that was formulated in [62, 65] has both these central terms identical and the other one obtained in [61] has zero central extension for both commutators. It is just that the most generic boundary conditions for $\mathcal{N}=2$ theory was not considered before. In our work, we used the most general supertrace elements for the isometry algebra of the bulk and also used relaxed boundary condition consistent with flat space. Hence, the algebra we get is bigger than the ones previously discussed in the literature. When we say that the new algebra cannot be arrived from contraction, we mean the following: Starting from $\mathcal{N}=2$ supergravity theory in AdS$_3$ we cannot contract and end up at the $\mathcal{N}=(2,0)$ super-BMS$_3$

algebra. Of course, it is possible to start from a theory with higher supersymmetry (say $\mathcal{N} = 4$ supergravity in AdS$_3$) and use contraction to reach this algebra. We see that the 2D dual theory constructed in (3.5.31) has a richer quantum symmetry.

A similar analysis form (3.7.58) reproduces the $\mathcal{N} = 2$ SuperBMS$_3$ algebra that was introduced in [68].

Chapter 4

Supergravity in dS_3 and holographic dual

4.1 Structure of de-Sitter$_3$ spacetime

4.2 dS/CFT and holographic dual

Starting from the works of Brown and Henneaux [2] and taking a flat limit of the AdS_3 spacetime, the boundary symmetry algebra of asymptotically flat spaces was found [7,29]. It was the (2+1)D analogue of BMS algebra originally discovered in [9,28]. Subsequent work showed a flat limit of Liouville theory describes the dynamics of these spacetimes [32]. The work in both AdS_3 and Flat spacetimes were extended to supergravity theories [23,56,57,61,64,69–72]. Because of its strong resemblance to the negative cosmological constant case, a BMS/CFT conjecture has recently been put forward in the same spirit [31,73–75].

Strominger proposed the initial dS/CFT correspondence [76] as a natural generalisation of AdS/CFT to positive cosmological constant case. Even in the original work, a big question centred on where the holographic dual should reside. A direct analytic extension of AdS/CFT ideas would suggest that the dual theory lives both in the past($\mathcal{I}^-$) and future boundary($\mathcal{I}^+$) of dS [77,78]. The reason for the worry is that de-Sitter space has cosmological horizons, which means any static observer, for example, is not able to access the whole space-time (**See Figure 4.1**). Another related problem with de-Sitter spacetime is that with enough matter present, a spacetime which is de-Sitter in the far past, may collapse in finite time and have no future dS-like structure at all. Thus if the

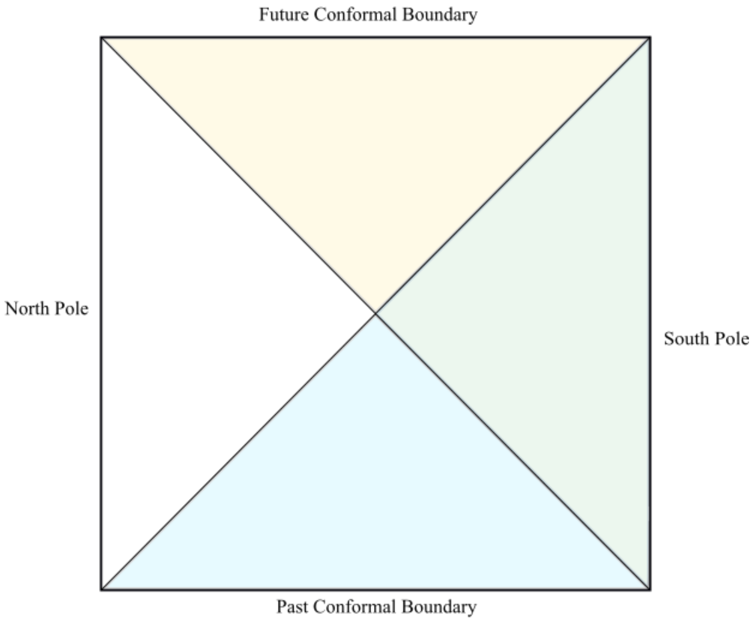

Figure 4.1: Penrose diagram for de Sitter space. For a static observer at the south pole, the blue region is the causal past and the yellow region is the causal future. The overlapping region is the causal diamond for the observer.

holographic dual theory has part of it described on the future boundary, that theory will become meaningless.

By an antipodal matching condition between $\mathcal{I}^+$ and $\mathcal{I}^-$ Strominger was able to show that the holographic description only at one of these is enough. Many other "screens" for the dual were also proposed. But the situation, like many other problems, greatly simplifies in (2+1)D as was shown in [79]. They showed that due to the property of radial gauge choice, it is possible to bring the holographic dual theory in the static patch of the observer itself. Although in principle this enables us to write a consistent boundary theory at any radial slice, it is more natural to impose the boundary conditions to a slice close to the horizon.

In this chapter, we generalize the above construction to minimal supergravity in a dS_3 background while working in Eddington-Finkelstein coordinate. We extend the idea of "asymptotically dS spacetime" to supergravity by giving consistent falloffs for gravitinis. This extends the asymptotic symmetry group, which now lies in the radial slice discussed above. Then, using hamiltonian reduction, we write the classical holographic dual for this supergravity theory which turns out to be a Euclidean super-Liouville theory at the boundary.

This chapter is organized as follows: In section 2 we briefly review the minimal superalgebra that has de-Sitter symmetry group as the bosonic subgroup. We write this algebra in a suitable basis and discuss it's supertraces which is essential to write the CS theory corresponding to supergravity. In section 3, we find out the fall-off of metric and gravitini fields and calculate the asymptotic symmetry algebra. Finally in section 4, we use the constraints coming from the boundary conditions to write a 2D dual for the theory.

4.3 de Sitter algebra and it's supersymmetrization

In a generic dimension d, the symmetry algebra of de Sitter space is given by $\mathfrak{so}(d,1)$. In order to have a supersymmetric extension of this algebra, we must find a superalgebra whose bosonic subalgebra is of the form $\mathfrak{so}(d,1)\oplus\mathfrak{g}$. For $d=3$, we can use the homomorphism $\mathfrak{so}(3,1)\sim\mathfrak{sl}_2(\mathbb{C})$ in order to find the superalgebra. In his work Nahm [80] classified all such algebras. In particular the superalgebra $OSp(N|2,\mathbb{C})$ has the bosonic subgroup $\mathfrak{sl}_2(\mathbb{C})\times\mathfrak{so}(N)$. Hence using the algebra $OSp(1|2,\mathbb{C})$ we can construct minimal de Sitter supergravity theory in (2+1)D.
The algebra $OSp(1|2,\mathbb{C})$ in a particular basis looks like [81]:

$$
\begin{aligned}
&[J^a,J^b]=\epsilon^{abc}J_c; \qquad [P^a,J^b]=\epsilon^{abc}P_c; \qquad [P^a,P^b]=-\lambda\epsilon^{abc}J_c\\
&[J^a,U_\alpha]=-(\sigma^a)_\alpha^\beta U_\beta \qquad [J^a,V_\alpha]=-(\sigma^a)_\alpha^\beta V_\beta \qquad [P^a,U_\alpha]=\sqrt{\lambda}(\sigma^a)_\alpha^\beta V_\beta \qquad [P^a,V_\alpha]=-\sqrt{\lambda}(\sigma^a)_\alpha^\beta U_\beta\\
&\{U_\alpha,U_\beta\}=2(\sigma^a\epsilon_{\alpha\beta})P_a \qquad \{V_\alpha,V_\beta\}=-2(\sigma^a\epsilon_{\alpha\beta})P_a \qquad \{U_\alpha,V_\beta\}=2\sqrt{\lambda}(\sigma^a\epsilon_{\alpha\beta})J_a
\end{aligned}
$$

It has six bosonic generators $\{P_a,J_a\}$ with $a=\{0,1,2\}$ and four fermionic generators $\{U_\alpha,V_\alpha\}$ where $\alpha=\{-,+\}$. The λ parameter appearing in the algebra is inversely proportional to radius of de Sitter space and $\lambda\to0$ limit gives $\mathcal{N}=(1,1)$ super-Poincaré algebra as discussed, for example, in [56].

But in the above basis, the implementation of boundary conditions is difficult. It will be much more convenient for us to go to a new basis which will make the $\mathfrak{sl}_2(\mathbb{C})$ structure of the algebra apparent. To do so, we define the following new combinations:

$$
J_a^\pm=\frac{1}{2}(J_a\pm\frac{i}{\sqrt{\lambda}}P_a), \qquad Q_\alpha=\frac{1}{2\lambda^{1/4}}(U_\alpha-iV_\alpha), \qquad \bar{Q}_\alpha=\frac{1}{2\lambda^{1/4}}(U_\alpha+iV_\alpha).
$$

In this new basis the algebra looks like:

$$
\begin{aligned}
&[J_a^+, J_b^+] = \epsilon_{ab}^c J_c^+, \qquad [J_a^+, Q_\alpha] = -(\sigma^a)_\alpha^\beta Q_\beta, \qquad \{Q_\alpha, Q_\beta\} = -i(\sigma^a \epsilon_{\alpha\beta}) J_a^+, \\
&[J_a^-, J_b^-] = \epsilon_{ab}^c J_c^-, \qquad [J_a^-, \bar{Q}_\alpha] = -(\sigma^a)_\alpha^\beta \bar{Q}_\beta^-, \qquad \{\bar{Q}_\alpha, \bar{Q}_\beta\} = i(\sigma^a \epsilon_{\alpha\beta}) J_a^-, \\
&[J_a^+, J_b^-] = 0, \qquad [J_a^+, \bar{Q}_\alpha] = [J_a^-, Q_\alpha] = 0, \qquad \{Q_\alpha, \bar{Q}_\beta\} = 0.
\end{aligned} \tag{4.3.1}
$$

In our conventions $\varepsilon_{012} = 1$ and the tangent space metric is given by $\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The σ^a matrices satisfy $[\sigma^a, \sigma^b] = \epsilon^{abc}\sigma_c$.

In the basis (4.3.1), we can define invariant bilinears of the algebra. In general, this will be a one-parameter family but the parameter will be fixed by demanding that the bosonic part of the CS action reduces to Einstein gravity action. The nonzero supertraces are given by:

$$
\langle J_a^+, J_b^+ \rangle = \eta_{ab}, \quad \langle Q_\alpha, Q_\beta \rangle = iC_{\alpha\beta}, \quad \langle J_a^-, J_b^- \rangle = \eta_{\alpha\beta}, \quad \langle \bar{Q}_\alpha, \bar{Q}_\beta \rangle = -iC_{\alpha\beta}. \tag{4.3.2}
$$

It must be mentioned that the algebra above actually admits one more invariant bilinear which corresponds to an independent set of supertraces. But this choice does not correspond to a nondegenerate metric as $\lambda \to 0$.

4.4 Writing the supergravity action

In the above basis we can expand the CS gauge field as:

$$
\begin{aligned}
A &= (\omega^a + i\sqrt{\lambda} e^a) J_a^+ + \lambda^{1/4}(\psi^\alpha + i\chi^\alpha) Q_\alpha \\
\bar{A} &= (\omega^a - i\sqrt{\lambda} e^a) J_a^- + \lambda^{1/4}(\psi^\alpha - i\chi^\alpha) \bar{Q}_\alpha
\end{aligned} \tag{4.4.3}
$$

Then the Einstein action can be written as a sum of CS actions of A and $\bar{A}$.

$$S_{EH} = -iS_\kappa[A] + iS_\kappa[\bar{A}] \tag{4.4.4}$$

where $S_\kappa[A]$ is the Chern-Simons action given by:

$$S_\kappa[A] = \frac{\kappa}{2}\int_{\mathcal{M}} \Big\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \Big\rangle$$

In terms of the component fields the equations of motion become

$$\begin{aligned}
&de^c + \varepsilon_{ab}{}^{c} e^a \omega^b + (\sigma^c \epsilon)_{\alpha\beta}(\psi^\alpha \psi^\beta - \chi^\alpha \chi^\beta) = 0 \\
&d\omega^c + \frac{1}{2}\varepsilon_{ab}{}^{c}(\omega^a \omega^b - \lambda e^a e^b) + 2\sqrt{\lambda}\psi^\alpha \chi^\beta (\sigma^c \epsilon)_{\alpha\beta} = 0 \\
&d\psi^\beta - \omega^a \psi^\alpha (\sigma^a)^\beta_\alpha - \sqrt{\lambda} e^a \chi^\alpha (\sigma^a)^\beta_\alpha = 0 \\
&d\chi^\beta - \omega^a \chi^\alpha (\sigma^a)^\beta_\alpha + \sqrt{\lambda} e^a \psi^\alpha (\sigma^a)^\beta_\alpha = 0
\end{aligned} \tag{4.4.5}$$

this we present here just for completeness. For $\lambda \to 0$ limit these equations get decoupled and a generic solution is possible [56].

4.5 Asymptotic symmetry of dS Supergravity

Since gravity is topological in (2+1)D, the boundary conditions play a pivotal role in determining it's behaviour. These conditions give the action a proper variation and also introduce boundary degrees of freedom. So our main goal is to define consistent boundary conditions for dS supergravity and then analyze the asymptotic symmetry algebra that accompanies it.

4.5.1 Falloff for field A and $\bar{A}$

We want to translate the boundary condition from the language of the metric and gravitini field to the gauge field A and $\bar{A}$. Let us briefly review (following [79]) the phase space of asymptotically dS spacetimes and then extend it to supergravity.

In Eddington-Finkelstein coordinates, the fall-off for metric is given by

$$ds^2 = \left(\frac{r^2}{l^2} + 8G\mathcal{M}(u,\phi)\right) du^2 - 2du\, dr + 8G\mathcal{J}(u,\phi)du\, d\phi + r^2 d\phi^2. \tag{4.5.6}$$

Putting this into Einstein's equations yields $\partial_u \mathcal{J} = \partial_\phi \mathcal{M}$ and $\partial_u \mathcal{M} = -\frac{1}{l^2}\partial_\phi \mathcal{J}$. Now we do a coordinate transformation $t_\pm = u \pm il\phi$ and in this new coordinate, the equations take the form: $\partial_+(\mathcal{M} + \frac{i}{l}\mathcal{J}) = 0$ and $\partial_-(\mathcal{M} - \frac{i}{l}\mathcal{J}) = 0$. Hence the fields above can be expanded as

$$\mathcal{M} = \mathcal{L}_+(t_+) + \mathcal{L}_-(t_-) \qquad \mathcal{J} = il(\mathcal{L}_+(t_+) - \mathcal{L}_-(t_-)) \tag{4.5.7}$$

Now we may calculate the vielbeins and spin connections from the above metric. Our tangent space metric is given under (4.3.1) and in that basis the gauge fields take the form:

$$\begin{aligned} a_{bos} &= (\frac{i}{l}J_1^+ + \frac{i8G}{l}\mathcal{L}_-(t_-)J_0^+)dt_- \\ \bar{a}_{bos} &= (-\frac{i}{l}J_1^- - \frac{i}{l}8G\mathcal{L}_+(t_+)J_0^-)dt_+ \end{aligned} \tag{4.5.8}$$

where we have already taken out the radial dependence from both A and $\bar{A}$ using the relation $a_{bos} = k^{-1}(r)(d + A_{bos})k(r)$. Since we'll work in a constant r slice, this reduced form a_{bos} and $\bar{a}_{bos}$ are going to be our dynamical inputs. Also notice that our form is a bit different from that of [79] because of the difference in conventions.

Having found out the fall off for the bosonic part, we want to extend it to the full supergravity field [22]. The bosonic part contains the information of the physical metrics that should be included in the theory. In order to find the appropriate boundary conditions for the fermionic fields as well, we take a_{bos} and $\bar{a}_{bos}$ and act the whole dS supergroup on it. This generates new terms in the boundary and the full field looks like:

$$\begin{aligned} a &= (\frac{i}{l}J_1^+ + \frac{i8G}{l}\mathcal{L}_-(t_-)J_0^+ + \frac{8G}{l}r^-(t_-)Q_-)dt_- \\ \bar{a} &= (-\frac{i}{l}J_1^- - \frac{i}{l}8G\mathcal{L}_+(t_+)J_0^- + \frac{8G}{l}\bar{r}^-(t_+)\bar{Q}_-)dt_+ \end{aligned} \tag{4.5.9}$$

The functions appearing in the expression 4.5.9 are assumed to have a smooth behaviour at the boundary. Of these, the functions r^- and $\bar{r}^-$ are Grassmann valued. The rest are scalar.

The boundary condition on these fields can be divided into two categories. Namely,

(1) $A_{t_+} = 0$ and $\bar{A}_{t_-} = 0$

(2a) $a_{J_2} = a_{Q_+} = 0; \quad a_{J_1} = \frac{i}{l}$ (2b) $\bar{a}_{J_2} = \bar{a}_{Q_+} = 0; \quad \bar{a}_{J_1} = -\frac{i}{l}$

We'll shortly see that these boundary conditions will reduce the boundary dual theory first to a WZW action and then finally to a Liouville type theory.

But before that we'll analyse the asymptotic symmetry algebra that leaves these conditions invariant.

4.5.2 Asymptotic symmetry algebra

In order to obtain the asymptotic algebra we find the variation of the fields that keeps the above structure intact. The generic variation of the field is given by:

$$\delta a_\phi = d\,\lambda + [a_\phi, \lambda] \tag{4.5.10}$$

where λ is the gauge transformation parameter. We expand it in terms of our generators as:

$$\lambda = \xi^a J_a^+ + \theta^\alpha Q_\alpha$$

and then use our asymptotic field expression (4.5.9) to find the variations in (4.5.10). Matching the coefficients on both sides, we see that not all parameters are independent and are be related by:

$$\begin{aligned}
\xi^2 &= -(\xi^1)' \\
\xi^0 &= -(\xi^1)'' + 8G\mathcal{L}_-\xi^1 + 4Gr^-\theta_+ \\
\theta^- &= \sqrt{2}(\theta^+)' - i\,8Gr^-\xi^1
\end{aligned} \tag{4.5.11}$$

Thus the only independent parameters are ξ^1 and θ^+. Next, we will write the variations of the fields appearing in the expression (4.5.9). In terms of independent parameters these are given by:

$$\delta\mathcal{L}_- = -\frac{1}{8G}(\xi^1)''' + 2\mathcal{L}_-(\xi^1)' + \mathcal{L}_-'(\xi^1) + \frac{3}{2}r^-(\theta^+)' + \frac{1}{2}(r^-)'\theta^+ \tag{4.5.12}$$

$$\delta r^- = i\frac{\sqrt{2}}{8G}(\theta^+)'' + \frac{3}{2}r^-(\xi^1)' + (r^-)'\xi^1 - i\frac{1}{\sqrt{2}}\mathcal{L}_-\theta^+ \tag{4.5.13}$$

Now we can construct the conserved charge associated with the variations that preserve the boundary condition [24]. In CS language this charge is given by:

$$\delta Q = \frac{\kappa}{2\pi}\int\langle\lambda, \delta a_\phi\rangle \tag{4.5.14}$$

where the above integration is performed over a constant r slice. With our boundary condition the expression reduces to

$$Q = \int(\xi^1\mathcal{L}_- + \theta^+ r^-) \tag{4.5.15}$$

where the equivalence between the CS level and Newton's constant was used. With this charge the asymptotic symmetry algebra can be written as:

$$\{Q[\lambda_1], Q[\lambda_2]\}_{PB} = \delta_{\lambda_2}Q[\lambda_1]$$

With this and the variations of fields given above, the algebra becomes:

$$\begin{aligned}
\{\mathcal{L}(\theta), \mathcal{L}(\theta')\} &= \frac{1}{8G}\delta'''(\theta-\theta') - (\mathcal{L}(\theta) + \mathcal{L}(\theta'))\delta'(\theta-\theta') \\
\{\mathcal{L}(\theta), r^-(\theta')\} &= -(r^-(\theta) + \frac{1}{2}r^-(\theta'))\delta'(\theta-\theta') \\
\{r^-(\theta), r^-(\theta')\} &= -\frac{i}{\sqrt{2}}\mathcal{L}(\theta)\delta(\theta-\theta') + \frac{\sqrt{2}i}{8G}\delta''(\theta-\theta')
\end{aligned} \tag{4.5.16}$$

In the barred sector, the story runs in parallel. The parameter relations are given by:

$$\begin{aligned}\bar{\xi}^2 &= -(\bar{\xi}^1)' \\ \xi^0 &= -(\bar{\xi}^1)'' + 8G\mathcal{L}_+\xi^1 + 4G\bar{r}^-\bar{\theta}^+ \\ \bar{\theta}^- &= \sqrt{2}(\bar{\theta}^+)' + i\,8G\bar{r}^-\bar{\xi}^1\end{aligned}$$

and with these, the variation of fields become:

$$\delta\mathcal{L}_+ = -\frac{1}{8G}(\bar{\xi}^1)''' + 2\mathcal{L}_+(\bar{\xi}^1)' + \mathcal{L}_+'(\bar{\xi}^1) + \frac{3}{2}\bar{r}^-(\bar{\theta}^+)' + \frac{1}{2}(\bar{r}^-)'\bar{\theta}^+ \tag{4.5.17}$$

$$\delta\bar{r}^- = -i\frac{\sqrt{2}}{8G}(\bar{\theta}^+)'' + \frac{3}{2}\bar{r}^-(\bar{\xi}^1)' + (\bar{r}^-)'\bar{\xi}^1 + i\frac{1}{\sqrt{2}}\mathcal{L}_+\theta^+ \tag{4.5.18}$$

These then produce the asymptotic algebra of the barred sector. The classical poisson brackets are given by:

$$\{\bar{\mathcal{L}}(\theta), \bar{\mathcal{L}}(\theta')\} = -\frac{1}{8G}\delta'''(\theta-\theta') - (\bar{\mathcal{L}}(\theta) + \bar{\mathcal{L}}(\theta'))\delta'(\theta-\theta') \tag{4.5.19}$$

$$\{\bar{\mathcal{L}}(\theta), \bar{r}^-(\theta')\} = -(\bar{r}^-(\theta) + \frac{1}{2}\bar{r}^-(\theta'))\delta'(\theta-\theta') \tag{4.5.20}$$

$$\{\bar{r}^-(\theta), \bar{r}^-(\theta')\} = +\frac{i}{\sqrt{2}}\mathcal{L}(\theta)\delta(\theta-\theta') - \frac{\sqrt{2}i}{8G}\delta''(\theta-\theta') \tag{4.5.21}$$

The results presented above are in terms of Poisson Brackets. To write the quantum algebra we first express the fields in terms of their modes. The left and right movers are identified differently

$$\mathcal{L}(\theta) = \sum_{-\infty}^{\infty} e^{in\theta}L_n \text{ and } \bar{\mathcal{L}}(\theta) = \sum_{-\infty}^{\infty} e^{-in\theta}\bar{L}_n.$$

This identification of modes takes care of the relative minus signs between the bosonic part of the barred and unbarred algebra.

Next, we need to quantise. While converting the Poisson brackets to commutators in the dS case. This factor ensures that the central term in the Virasoro commutator matches with the bosonic result in [79].

4.6 Dual theory at the boundary

Now that we have analysed the asymptotic algebra, we want to write a classical dual of the supergravity theory we are considering at the boundary of the spacetime. Now, since we are working on the static patch of the dS spacetime, the boundary in this case will be an $r = const.$ hypersurface as $r \to \infty$ in Eddington-Finkelstein coordinates.
Using a hamiltonian reduction of the CS theory at the bulk, we will first write a dual WZW type theory at the boundary. This procedure uses the constraint (1) mentioned earlier. Since this constraint essentially is same for pure gravity and supergravity, this procedure goes through in exactly the same way. But we present it here anyway.

4.6.1 Dual WZW model at boundary

To see how the Chern-Simons action gives rise to super-WZW theory at the boundary we employ the techniques discussed in [79]. In fact since the first constraint takes the same form in pure gravity and supergravity, the reduction is exactly the same. Generically, the relations between these two theories were discussed in [21].
To illustrate this let us expand the CS theory explicitly in terms of our coordinates.

$$S_\kappa[A] = \frac{\kappa}{2}\int_{\mathcal{M}} dr\, du\, d\phi \langle A_r(\dot{A}_\phi - A'_u) - A_u(\partial_r A_\phi - \partial_\phi A_r) + A_\phi(\partial_r A_u - \dot{A}_r) + 2A_u[A_\phi, A_r]\rangle \tag{4.6.22}$$

Where the dots are derivatives w.r.t u and dashes are derivatives w.r.t ϕ variables. Now integrating by parts and taking the ϕ boundary terms to be zero because of periodicity the above action upto trivial boundary terms reduces to

$$S_\kappa[A] = \frac{\kappa}{2}\int_{\mathcal{M}} dr\, du\, d\phi \langle A_r \dot{A}_\phi - A_\phi \dot{A}_r + 2A_u F_{\phi r}\rangle \tag{4.6.23}$$

The field strength above is defined as $F_{\phi r} = \partial_\phi A_r - \partial_r A_\phi + [A_\phi, A_r]$. Now we can compute the variation of the above action. We see that apart from the terms proportional to equations of motion ($F = 0$) we also get boundary terms proportional to fields. These come from the commutator in the last term.

$$S_\kappa[A] = \int_{\mathcal{M}} \delta(A)(E.O.M) + \kappa \int_{\partial\mathcal{M}} du\, d\phi \langle A_u \delta A_\phi\rangle \tag{4.6.24}$$

At this point we can use our first set of boundary conditions $A_{t_+} = 0$ at $r \to \infty$. This implies that at the boundary $A_u = A_\phi$. Thus the correct action must be supplemented with an additional boundary term to have a well defined variational principle. The action takes the form:

$$S_\kappa[A] = \frac{\kappa}{2}\int_{\mathcal{M}} \left\langle A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right\rangle + \frac{\kappa}{2}\int_{\partial\mathcal{M}} du\, d\phi \langle A_\phi^2 \rangle$$

Now similarly the barred Chern-Simons theory will also be supplemented with a boundary term. The final action takes the form:

$$S_{IEH} = S_{EH} + i\frac{\kappa}{2}\int_{\partial\mathcal{M}} du\, d\phi \langle A_\phi^2 + \bar{A}_\phi^2 \rangle \tag{4.6.25}$$

where S_{IEH} is our notation for improved Einstein-Hilbert action, whose variation gives us the correct equations of motion with our boundary conditions. S_{EH} is given by [4.4.4].

Since the field strength of the CS action is trivial the fields A and $\bar{A}$ can be expanded as pure gauge

$$A = H^{-1} dH \qquad\qquad \bar{A} = \bar{H}^{-1} d\bar{H} \tag{4.6.26}$$

where $H, \bar{H} \in SL(2, \mathbb{C})$. It's important to understand that the boundary mentioned in the above action isn't the conformal boundary of de-Sitter but rather the boundary of the static patch where we want to see the dual boundary theory. The authors of [79] have shown that the idea of asymptotic symmetry can be extended to anywhere inside the bulk in the case of (2+1)D pure gravity. It seems natural to put the boundary at the boundary of causal diamond of the static observer.

We impose the radial gauge on $H, \bar{H}$ as we've done for the field A earlier. This decomposes the field into $H = h(u, \phi)\, k(r)$ where all the r dependence is now encapsulated in k. We do a similar gauge choice for the barred sector $\bar{H} = \bar{h}\,\bar{k}$.

With this decomposition the unbarred part of the action (alongwith it's corresponding boundary term) reduces to,

$$S_\kappa[A] = \frac{\kappa}{2}\left[\int_{\partial\mathcal{M}} \langle h^{-1}\partial_\phi h\, h^{-1}\partial_+ h \rangle + \frac{1}{3}\int_{\mathcal{M}} \langle (H^{-1} dH)^3 \rangle\right] \tag{4.6.27}$$

where $\partial_+ = \partial_u - \frac{i}{l}\partial_\phi$. This is a chiral WZW model. The barred part reduces similarly to a WZW model of opposite chirality (with $\partial_- = \partial_u + \frac{i}{l}\partial_\phi$)

$$S_\kappa[\bar{A}] = \frac{\kappa}{2}\left[\int_{\partial\mathcal{M}} \langle \bar{h}^{-1}\partial_\phi \bar{h}\, \bar{h}^{-1}\partial_- \bar{h}\rangle + \frac{1}{3}\int_{\mathcal{M}} \langle (\bar{H}^{-1}d\bar{H})^3\rangle\right] \tag{4.6.28}$$

To finally reduce these into a non-chiral WZW model, we define

$$G = H^{-1}\bar{H} \qquad \textit{and} \qquad g = h^{-1}\bar{h} \tag{4.6.29}$$

With this identification, the combination of the barred and unbarred CS theory boils down to a non-chiral WZW model:

$$S_\kappa[A] = \frac{\kappa}{2}\left[\int_{\partial\mathcal{M}} \langle g^{-1}\partial_- g\, g^{-1}\partial_+ g\rangle + \frac{1}{3}\int_{\mathcal{M}} \langle (G^{-1}dG)^3\rangle\right] \tag{4.6.30}$$

In the next section, we will impose the rest of the constraints on this action and reduce this boundary dual to a Liouville theory.

4.6.2 Super-Liouville action at the boundary

Now that we've used the first set of boundary conditions to reduce the dual holographic theory at the boundary into a WZW model [4.6.30], we want to implement the second set of constraints.

To do so, we closely follow the construction of [82] and expand do gauss decomposition of the elements of the supergroup close to identity. A generic element then can be written as:

$$g = G^+\, G^0\, G^-$$

where,

$$\begin{aligned} G^+ &= \exp(x\Gamma_1 + \psi_+ Q^+) \\ G^- &= \exp(y\Gamma_0 + \psi_- Q^-) \\ G^0 &= \exp(\phi\Gamma_2) \end{aligned} \tag{4.6.31}$$

In general the fields $x, y, \psi_\pm, \phi$ will be functions of (r, u, θ) but due to our gauge choice at the boundary the r dependence is absent. With this form of of the element, we can now calculate the currents of the WZW theory. The theory has 2 independent currents given by:

$$J = g^{-1}dg \qquad \bar{J} = -dg\, g^{-1} \tag{4.6.32}$$

Now since in the previous section we've seen that $g = h^{-1}\bar{h}$ we can substitute this in the above expression and get:

$$J = \bar{a} - g^{-1}\, a\, g \qquad \bar{J} = a - g\, \bar{a}\, g^{-1} \tag{4.6.33}$$

Here we use the first set of constraints once again and we see that

$$\begin{aligned} J_+ &= \bar{a}_+ - g^{-1}\, a_+\, g \\ &= \bar{a}_+ \end{aligned}$$

as $a_+ = 0$. Similarly for the barred sector $\bar{J}_- = a_-$. Thus the constraints on the fields can be directly imposed on the components of currents of the theory as well. Using (4.6.31) we now expand the current J in terms of the unbarred basis. We get

$$\begin{aligned} J_+ =& [e^\phi(\partial_+ x) - \frac{i}{\sqrt{2}} e^\phi(\partial_+\psi_+)\psi_+] J_1^+ \\ &+ [-e^\phi(\partial_+ x)y + \frac{i}{\sqrt{2}} y e^\phi(\partial_+\psi_+)\psi_+ - \frac{i}{2} e^{\phi/2}(\partial_+\psi_+)\psi_- + \partial_+\phi] J_2^+ \\ &+ [(\partial_+ y) - (\partial_+\phi)y - \frac{i}{2} e^{\phi/2} y(\partial_+\psi_+)\psi_- + \frac{i}{\sqrt{2}}(\partial_+\psi_-)\psi_-] J_0^+ \\ &+ [-\frac{1}{\sqrt{2}} e^\phi(\partial_+ x)\psi_- + e^{\phi/2}(\partial_+\psi_+) + \frac{i}{2} e^\phi(\partial_+\psi_+)\psi_+\psi_-] Q_+ \\ &+ [\frac{1}{2} e^\phi(\partial_+ x)\psi_- + \frac{1}{\sqrt{2}} e^{\phi/2} y(\partial_+\psi_+) + \frac{i}{2\sqrt{2}} e^\phi y(\partial_+\psi_+)\psi_+\psi_- + \frac{1}{2}(\partial_+\phi)\psi_- + (\partial_+\psi_-)] Q_- \end{aligned} \tag{4.6.34}$$

In this expression, using the constraints above we find following relations:

$$
\begin{aligned}
&e^{\phi}(\partial_+ x) - \frac{i}{\sqrt{2}} e^{\phi}(\partial_+\psi_+)\psi_+ = \frac{i}{l} \\
&y = -\frac{l}{2} e^{\phi/2}(\partial_+\psi_+)\psi_- - il(\partial_+\phi) \\
&\psi_- = -i\sqrt{2} l e^{\phi/2}(\partial_+\psi_+)
\end{aligned}
$$

Similarly, we expand the $\bar{J}$ current in barred basis and get the expression:

$$
\begin{aligned}
\bar{J}_- = &- [e^{\phi}(\partial_- y)] J_1^- \\
&+ [-e^{\phi} x(\partial_- y) - \frac{i}{2} e^{\phi/2}(\partial_-\psi_-)\psi_+ + \partial_-\phi] J_2^- \\
&- [(\partial_- x) - (\partial_-\phi) x - \frac{i}{2} e^{\phi/2} x(\partial_-\psi_-)\psi_+] J_0^- \\
&- [\frac{1}{\sqrt{2}} e^{\phi}(\partial_- y)\psi_+ + e^{\phi/2}(\partial_-\psi_-)] \bar{Q}_+ \\
&- [(\partial_-\psi_+) - \frac{1}{\sqrt{2}} e^{\phi/2} x(\partial_-\psi_-) - \frac{1}{2} e^{\phi} x(\partial_- y)\psi_+ + \frac{1}{2}(\partial_+\phi)\psi_-] \bar{Q}_-
\end{aligned}
\tag{4.6.35}
$$

Then imposing the constraints here gives:

$$
\begin{aligned}
&e^{\phi}(\partial_- y) = \frac{i}{l} \\
&x = -il(\partial_-\phi) - \frac{l}{2} e^{\phi/2}(\partial_-\psi_-)\psi_+ \\
&\psi_+ = i\sqrt{2} l e^{\phi/2}(\partial_-\psi_-)
\end{aligned}
\tag{4.6.36}
$$

Thus we see that in the theory the fields x and y can be completely substituted by the fields ϕ, ψ_+, ψ_- and the final action of the constrained theory will be written in terms of these.

Now after substituting these [4.6.35,4.6.36] into the original action improved with the boundary term, we get the Super-Liouville action at the boundary:

$$
\begin{aligned}
S_E = \kappa \int_{\partial bulk} [2(\partial_+\phi)(\partial_-\phi) + \frac{1}{l^2}(e^{2\phi} + \frac{1}{\sqrt{2}} e^{\phi}\psi_+\psi_-) \\
+ \psi_+\partial_-\psi_+ + \psi_-\partial_+\psi_-] dx_+\, dx_-
\end{aligned}
\tag{4.6.37}
$$

This can be treated as the classical boundary dual of the bulk supergravity theory. This theory is the supersymmetrized version of the theory found in [79].

Chapter 5

A Matrix Model with BMS_3 Constraints

5.1 Introduction

Although we are far from understanding the complete picture of quantum gravity, matrix models have proven to be very successful in the study of 2D quantum gravity [83, 84]. A one-dimensional Hermitian matrix model in the double-scaling limit describes a two-dimensional string theory which can be interpreted as a Liouville theory coupled to $c = 1$ matter [85]. This connection generated a huge interest in other possible relations between gravity theories and matrix models. The 2D quantum gravity models are usually formulated as conformal field theories, therefore, one natural direction is to look for a CFT formulation of matrix models [86]. The connection between random matrix models and conformal field theory (CFT) is bilateral. While matrix model techniques can be useful for computing certain correlators in a conformal field theory, in some cases, the techniques of CFT might be useful for solving matrix models.

A matrix model possesses an infinite number of symmetries, which gives rise to a recursive relation between correlation functions through Loop equations [87]. The existence of infinite symmetries points to a possible integrability structure, which has been extensively explored in the literature [88–91]. The partition function of matrix models is known to play the role of tau-function of some integrable systems. Their underlying integrability structure makes them exactly solvable and thus, an extremely important tool in the study of lower-dimensional quantum field theories.

The loop equations can be reformulated in terms of linear differential constraints on the partition function, where the differential operators satisfy an infinite dimensional algebra [92–95]. The most famous example is that of Hermitian one matrix model for which the operators are known to satisfy the Virasoro algebra. The matrix model partition function can then be described as a solution to Virasoro-constraints. It is possible to invert this relation and start from the Virasoro algebra (in fact any infinite dimensional algebra) to write a corresponding matrix model partition function. A systematic approach was developed in [86, 96] (see [97] for review). Their approach gives a formulation of matrix models in terms of conformal field theory, where the constraints imposed on the partition function are translated to conditions imposed on the correlators of a CFT. Our case of interest in this chapter, is the BMS_3 algebra that arises as the asymptotic symmetry algebra of (2+1)-dimensional flat space-times [7, 31]. We have already established its extreme importance in understanding the behaviour of these algebras from the context of flat space holography [31]. While this serves as an example of the method developed in [86], our bigger motivation is to look for a possible connection between matrix models and higher-dimensional gravity theories. Given the success of matrix models in the study of 2D gravity, we believe that a framework for higher-dimensional theories in terms of matrix models might be useful. A matrix model possessing BMS_3 invariance in it's partition function may help us explore the integrability structure that underlines it. We start with a set of linear differential constraints, which we refer to as *BMS_3-constraints*. Assuming that those constraints describe the loop equations of a matrix model, we use the free field realisation of BMS_3 to write down a solution for those loop equations. The contents of this paper are organised as follows: In section 5.2, we review the formulation of loop equations for Hermitian one matrix model as Virasoro-constraints. In section 5.3, we discuss the method of [86], to compute a matrix model partition function as a solution to a set of infinite constraints. In section 5.3, we discuss about the BMS_3 algebra and its free field realisation. Finally, in section 5.4, we present our computations of a matrix model partition function which satisfies BMS_3 constraints. These results were reported in [98]

5.2 Loop equations

Loop equations are an infinite set of recursion relations among the correlation functions of a matrix model that follow from the invariance of the matrix integral under a change of integration variables [87].

5.2.1 The Hermitian one matrix model : Virasoro constraints

Consider an ensemble E of $N \times N$ Hermitian matrices, $H \in E$. Invariance of the partition function

$$\mathcal{Z} = \int_E dH e^{-\mathrm{Tr}V(H)}, \tag{5.2.1}$$

under an infinitesimal change of integration variables $H \to H + \epsilon H^n$ leads to the first loop equation,

$$\sum_{k=0}^{n-1} \langle \mathrm{tr}H^k \mathrm{tr}H^{n-k-1} \rangle - \langle \mathrm{tr}H^n V'(H) \rangle = 0. \tag{5.2.2}$$

Higher loop equations can be obtained by considering more general change of variables, or equivalently, from

$$\sum_{i,j} \int dH \frac{\partial}{\partial H_{ij}} \left((H^{\mu_1})_{ij} \mathrm{tr}H^{\mu_2} \cdots \mathrm{tr}H^{\mu_n} e^{-\mathrm{tr}V(H)} \right) = 0. \tag{5.2.3}$$

The case $n = 1$ with $\mu_1 = k$, corresponds to (5.2.2). In general,

$$\begin{aligned} \sum_{l=0}^{\mu_1-1} \langle \mathrm{tr}H^l \mathrm{tr}H^{\mu_1-l-1} \prod_{i=2}^{n} \mathrm{tr}H^{\mu_i} \rangle + \sum_{j=2}^{n} \mu_j \langle \mathrm{tr}H^{\mu_1+\mu_j-1} \prod_{\substack{i=2 \\ i\neq j}}^{n} \mathrm{tr}H^{\mu_i} \rangle \\ = \langle \mathrm{tr}V'(H) H^{\mu_1} \prod_{i=2}^{n} \mathrm{tr}H^{\mu_i} \rangle. \end{aligned} \tag{5.2.4}$$

The exact loop equations are difficult to solve for finite N. However, in the large-N limit, these can be efficiently used to compute correlation functions, order by order in $1/N$ expansion [99]. They also admit a topological expansion and can be formulated as *topological recursion relations* among the correlation functions [100, 101].

The Hermitian matrices can be diagonalized as

$$H \to U \Lambda U^\dagger, \tag{5.2.5}$$

where U are unitary matrices, and $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$ is a diagonal matrix. The eigenvalues λ_i are real. In the eigenvalue basis, the matrix model reduces to an eigenvalue integral,

$$\mathcal{Z} = \int \prod_{i=1}^{N} d\lambda_i \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^2 e^{V(\{\lambda_i\})}, \tag{5.2.6}$$

with a normalization factor which depends on the volume of $U(N)$. The integral measure, dH remains invariant under the change of basis (5.2.5), and the most general form of potential $V(H)$ consistent with this invariance is

$$V(H) = -\sum_{k=0}^{\infty} t_k H^k. \tag{5.2.7}$$

Thus, (5.2.5) acts as gauge symmetry on the matrix model partition function having the potential (5.2.7). The correlators can be obtained as derivatives of the partition function, with respect to the parameters t_k,

$$\begin{aligned}\langle \mathrm{tr}H^{\mu_1}\cdots\mathrm{tr}H^{\mu_n}\rangle &= \int dH\ e^{\sum_{k=0}^{\infty} t_k \mathrm{tr}H^k}\ \mathrm{tr}H^{\mu_1}\cdots\mathrm{tr}H^{\mu_n} \\ &= \frac{\partial^n}{\partial t_{\mu_1}\cdots\partial t_{\mu_n}}\mathcal{Z}.\end{aligned} \tag{5.2.8}$$

This relation can be used to rewrite loop equations as linear differential constraints,

$$L_n \mathcal{Z} = 0 \quad \text{for} \quad n \geq -1, \tag{5.2.9}$$

where

$$L_n = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}} + \sum_{k=0}^{n} \frac{\partial}{\partial t_k}\frac{\partial}{\partial t_{n-k}}. \tag{5.2.10}$$

The operators L_n satisfy a closed algebra,

$$[L_n, L_m] = (n-m)L_{n+m}, \tag{5.2.11}$$

which is similar to the Virasoro algebra, except that $n, m \geq -1$. Therefore, it is referred to as “discrete Virasoro algebra". The constraints (5.2.9) along with

$$\frac{\partial}{\partial t_0}\mathcal{Z} = N\mathcal{Z}, \tag{5.2.12}$$

are called the *Virasoro constraints*. The matrix model partition function is given by a solution of these infinite set of constraint equations.

Virasoro algebra is well-known to be the symmetry algebra of a two-dimensional conformal field theory. Therefore, a formulation of matrix model partition function as a solution to Virasoro constraints points to a connection between Hermitian one matrix model and 2d CFT, which we discuss in the next section.

5.3 From Infinite Dimensional Algebras to Matrix Models

In [86], a systematic approach was developed to construct solutions of Virasoro and W-constraints [92], using the methods of conformal field theory. The approach can be generalised to other set of constraints, provided, there exists a free field realisation of the algebra satisfied by the corresponding differential operators.
The idea is to identify the operators, L_n with the modes of stress tensor, T_n of a conformal field theory. The solution to differential constraints is then obtained as a correlator in the CFT, and the annihilation of that correlator by L_n is translated to the annihilation of vacuum state by T_n. Thus, finding an integral expression of the partition function essentially involves two main steps:

(i) Finding a t-dependent "Hamiltonian" operator that relates L_n with the modes of stress tensor of a CFT. The identification is expressed through

$$L_n\langle N|e^{H(t)} = \langle N|e^{H(t)}T_n. \tag{5.3.13}$$

where $\langle N|$ is a charged vacuum state of the theory.

(ii) Finding states $|G\rangle$ in the CFT, which are annihilated by T_n, $n \geq -1$,

$$T_n|G\rangle = 0. \tag{5.3.14}$$

Once we find $H(t)$ and $|G\rangle$, the solution is given by

$$\mathcal{Z} = \langle N|e^{H(t)}|G\rangle. \tag{5.3.15}$$

The construction of operator $H(t)$ is somewhat ad hoc. This Hamiltonian operator, in general, does not have any relation to the CFT Hamiltonian, since there is no obligation for a CFT Hamiltonian to satisfy such a relation. However, the state $|G\rangle$ is well-known. It is given by the action of an operator G which commutes with the stress tensor, on the uncharged vacuum state,

$$|G\rangle = G|0\rangle, \tag{5.3.16}$$

where

$$[T_n, G] = 0, \quad n \geq -1. \tag{5.3.17}$$

Any function of the screening charges satisfy such a commutation relation. In a CFT, screening charges are operators with non-zero "charge" under the conserved current but zero conformal dimension. Thus, adding them to a correlation function will not change their conformal behaviour, however, it will change their total charge.
To get this operator, we first need a dimension-1 primary field in our theory (say ψ with $h_\psi = 1$). Then define a non-local operator, A as

$$A = \oint dz \psi(z) \tag{5.3.18}$$

One can show $[T_n, A] = \oint dz\, \partial(z^{n+1}\psi(z)) = 0$. Thus, the operator has zero conformal dimension.

5.3.1 CFT formulation of Hermitian one matrix model

A free field realisation of the Virasoro algebra is given in terms of a free bosonic CFT. To find a solution of the Virasoro constraints, we only need to consider the holomorphic part of the theory. The mode expansion of the scalar field is given by

$$\phi(z) = \phi_0 + \pi_0 \log z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^k, \tag{5.3.19}$$

where the modes satisfy

$$[J_n, J_m] = n\delta_{n+m,0}\ , \quad [\phi_0, \pi_0] = 1. \tag{5.3.20}$$

The stress tensor of the theory,

$$T(z) = \frac{1}{2}[\partial\phi(z)]^2 \tag{5.3.21}$$

admits a mode expansion

$$T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n-2}, \quad \text{with} \quad T_n = \sum_{k>0} J_{-k}J_{k+n} + \frac{1}{2}\sum_{\substack{a+b=n \\ a,b\geq 0}} J_a J_b. \tag{5.3.22}$$

In order to relate the above T_n with L_n given in (5.2.10), we first define vacuum states,

$$\begin{aligned} &\pi_0|0\rangle = 0, \quad J_k|0\rangle = 0, \quad k>0 \\ &\langle N|\pi_0 = N\langle N|, \quad \langle N|J_{-k} = 0, \quad k>0. \end{aligned} \tag{5.3.23}$$

It follows from (5.3.22) and (5.3.23) that

$$T_n|0\rangle = 0, \quad n \geq -1. \tag{5.3.24}$$

One can check that (5.3.13) holds for (5.3.22) and (5.2.10) with

$$H(t) = \frac{1}{\sqrt{2}}\sum_{k>0} t_k J_k = \frac{1}{\sqrt{2}}\oint U(z)J(z), \tag{5.3.25}$$

where

$$U(z) = \sum_{k>0} t_k z^k, \quad J(z) = \partial\phi(z). \tag{5.3.26}$$

The next step is to find a state $|G\rangle$, annihilated by the stress tensor. In a scalar field theory, the screening charges are defined as

$$Q_\pm = \oint : e^{\pm\sqrt{2}\phi} : \tag{5.3.27}$$

with charges $+1$ and -1, respectively. Any function of these charges satisfy (5.3.17), but the correlator (5.3.15) is non-zero only when the charge conservation condition is satisfied i.e. $|G\rangle$ carries the same charge as $\langle N|$. Therefore, if we choose Q_+ to construct $|G\rangle$, only a term with Q_+^N will contribute, irrespective of the form of the function $G(Q_+)$. For example, we can take, $G(Q_+) = e^{Q_+}$, a solution to Virasoro constraints is then given by

$$\begin{aligned} \mathcal{Z} &= \frac{1}{N!}\langle N|e^{H(t)}(Q_+)^N|0\rangle, \\ &= \frac{1}{N!}\langle N| : e^{\frac{1}{\sqrt{2}}\oint_{C_0} U(z)J(z)} : \prod_{i=1}^{N}\oint_{C_i} dz_i : e^{\sqrt{2}\phi(z_i)} : |0\rangle \end{aligned} \tag{5.3.28}$$

This correlator can be evaluated using the identity

$$\langle : e^{A_1} :: e^{A_2} : .... : e^{A_n} : \rangle = \exp \sum_{i<j}^{n} \langle A_i A_j \rangle \tag{5.3.29}$$

and the OPE of scalar fields,

$$\phi(z)\phi(z') = \log(z - z'). \tag{5.3.30}$$

Finally, we get

$$\mathcal{Z} = \frac{1}{N!} \prod_{i=}^{N} \oint_{C_i} dz_i \; e^{U(z_i)} \prod_{1\leq i<j\leq N} (z_i - z_j)^2, \tag{5.3.31}$$

which is same as the eigenvalue integral representation (5.2.6) of Hermitian one matrix model (5.2.1).

Another way to look at the connection between Hermitian one matrix model and free bosonic CFT, is to define a field [93]

$$\Phi(z) = \frac{1}{\sqrt{2}} \sum_{k\geq 0} t_k z^k - \sqrt{2} \mathrm{tr} \log \left(\frac{1}{z - H} \right), \tag{5.3.32}$$

such that the following contour integral, where C encloses all the eigenvalues $(\lambda_1, \cdots \lambda_N)$ of H but not the point z,

$$\oint_C \frac{dz'}{2\pi i} \frac{1}{z - z'} \langle \mathbf{T}(z') \rangle = 0, \quad \text{where} \quad \mathbf{T}(z) = \frac{1}{2} [\partial \phi(z)]^2, \tag{5.3.33}$$

gives the loop equation

$$\langle \sum_{i,j} \frac{1}{z - \lambda_i} \frac{1}{z - \lambda_j} + \sum_{i=1}^{N} \frac{1}{z - \lambda_i} \sum_{k\geq 0} k t_k \lambda_i^{k-1} \rangle = 0 \tag{5.3.34}$$

This equation is just a reformulation of (5.2.2) in terms of eigenvalues, with the potential given by (5.2.7). (5.2.8) allows one to represent (5.3.32) as

$$\phi(z) = \frac{1}{\sqrt{2}} \sum_{k>0} t_k z^k + \sqrt{2} N \log z - \sqrt{2} \sum_{k\geq 0} \frac{z^{-k}}{k} \frac{\partial}{\partial t_k}. \tag{5.3.35}$$

Thus, $\{\frac{\partial}{\partial t_k}, t_k\}$ can be thought of as creation and annihilation operators in the mode expansion of a scalar field (5.3.19). It can be checked that the modes of the stress tensor evaluated using (5.3.35) are given by (5.2.10).

5.3.2 Conformal multi-matrix models

Having formulated the loop equations of Hermitian one matrix model as Virasoro constraints, one think of a generalisation to other set of constraints. There exists a class of constraints called the *W-constraints*, whose solutions correspond to multi-matrix models. For example, the solution of W_{r+1}-constraints,

$$W_n^{(a)} Z = 0, \quad n \geq 1 - a, \quad a = 2, \cdots r+1, \tag{5.3.36}$$

where $W_n^{(a)}$ satisfy W_{r+1}-algebra, is given by an r-matrix integral [Mironov,Morozov,...], with r being the rank of the algebra. The conformal field theory techniques applied to solve the Virasoro constraints can be easily generalised to the case of W-constraints. The associated CFT is that of r free scalar fields. For $r = 2$ ($SL(3)$ algebra), the differential operators are given by

$$W_n^2 = \sum_{k=0}^{\infty} \left(k t_k \frac{\partial}{\partial t_{k+n}} + k\bar{t}_k \frac{\partial}{\partial \bar{t}_{k+n}} \right) + \sum_{a+b=n} \left(\frac{\partial^2}{\partial t_a \partial t_b} + \frac{\partial^2}{\partial \bar{t}_a \partial \bar{t}_b} \right) \tag{5.3.37}$$

$$\begin{aligned} W_n^3 = \sum_{k,l>0} & \left(k t_k \; l t_l \frac{\partial}{\partial t_{k+l+n}} - k\bar{t}_k \; l\bar{t}_l \frac{\partial}{\partial t_{k+l+n}} - 2kl t_k \bar{t}_l \frac{\partial}{\partial \bar{t}_{k+l+n}} \right) \\ & + 2 \sum_{k>0} \sum_{a+b=n+k} \left(k t_k \frac{\partial^2}{\partial t_a \partial t_b} - k t_k \frac{\partial^2}{\partial \bar{t}_a \partial \bar{t}_b} - 2k\bar{t}_k \frac{\partial^2}{\partial t_a \partial \bar{t}_b} \right) \\ & + \frac{4}{3} \sum_{a+b+c=n} \left(\frac{\partial^3}{\partial t_a \partial t_b \partial t_c} - \frac{\partial^3}{\partial t_a \partial \bar{t}_b \partial \bar{t}_c} \right), \end{aligned} \tag{5.3.38}$$

and the solution to W_3-constraints is a two-matrix model,

$$Z = \frac{1}{N_1! N_2!} \prod_{i=1}^{N_1} \int dx_i e^{U(x_i)} \prod_{j=1}^{N_2} \int dy_j e^{\tilde{U}(y_j)} \triangle(x) \triangle(y) \triangle(x,y). \tag{5.3.39}$$

Here, $\triangle(x) = \prod_{1 \leq i < j \leq N_1} (x_i - x_j)$ (similarly for y) is the Vandermonde factor and

$$\triangle(x,y) = \triangle(x) \triangle(y) \prod_{i,j} (x_i - y_j). \tag{5.3.40}$$

A solution (5.3.15) can be constructed for any algebra of constraints, provided, the following three conditions are fulfilled:

- The algebra admits a free field realisation.
- One can find a vacuum annihilated by relevant generators in the corresponding field theory.
- One can find a free field representation of the screening charges.

In this paper, we adopt this construction for the case of BMS$_3$ algebra, which is the asymptotic symmetry algebra of 3D flat spacetimes.

5.3.3 Free field realisation of BMS$_3$

BMS$_3$ algebra is spanned by two spin-2 fields, T_n and M_n and their commutators are given by

$$\begin{aligned}
[T_n, T_m] &= (n-m)T_{n+m} + \frac{c_1}{12}n(n^2-1)\delta_{n+m,0}, \\
[T_n, M_m] &= (n-m)M_{n+m} + \frac{c_2}{12}n(n^2-1)\delta_{n+m,0}, \\
[M_n, M_m] &= 0.
\end{aligned} \tag{5.3.41}$$

As it turns out, there exists a free field realisation of this algebra in terms of the $\beta - \gamma$ bosonic ghost CFT. It was shown [102] that a twisted ghost system with spin (2,-1) of the fields, respectively, can realise the above algebra (5.3.41).

The bosonic $\beta - \gamma$ system (see [103]) generically has a field β with spin λ and γ with spin $1 - \lambda$ and they satisfy the following OPE,

$$\gamma(z)\beta(w) \sim \beta(w)\gamma(z) \sim \frac{1}{(z-w)}, \tag{5.3.42}$$

while the OPE among the fields with themselves vanishes. Our interest lies in the case where $\lambda = 2$. For this system, the holomorphic parts of the primary fields can be expanded as

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-2}, \quad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n+1}, \tag{5.3.43}$$

The stress tensor of the theory,

$$T(z) = -\lambda : \beta(z)\partial_z\gamma : +(1-\lambda) : \gamma(z)\partial_z\beta : \tag{5.3.44}$$

$$= -2 : \beta(z)\partial_z\gamma : - : \gamma(z)\partial_z\beta : \quad (\text{for } \lambda = 2), \tag{5.3.45}$$

has the mode expansion

$$T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n-2}, \tag{5.3.46}$$

with

$$T_n = \sum_{m=0}^{\infty}(2n+m)\beta_{-m}\gamma_{m+n} + \sum_{m=0}^{\infty}(n-m)\gamma_{-m}\beta_{m+n} + \frac{1}{2}\sum_{\substack{a+b=n\\ a,b\geq 0}}(2a+b)\beta_a\gamma_b \tag{5.3.47}$$

From the OPE (5.3.42), it is clear that the above stress tensor satisfies the following expansions:

$$T(z)T(w) \sim \frac{1}{2}\frac{26}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{5.3.48}$$

$$T(z)\beta(w) \sim \frac{2\beta(w)}{(z-w)^2} + \frac{\partial\beta(w)}{(z-w)} \tag{5.3.49}$$

This suggests that the modes of $T(z)$ and $\beta(z)$ satisfy an algebra that is almost like BMS$_3$ except that the central charges are different. The central charges of BMS$_3$ (5.3.41) are arbitrary, whereas in this system $c_1 = 26$ and $c_2 = 0$. To get around this problem, [102] twisted the above stress tensor with

$$T(z) \to T(z) - a\partial^3\gamma \tag{5.3.50}$$

This twist introduces an arbitrary central charge of $12a$ in the OPE of $T(z)$ with $\beta(z)$. So now we may say that the modes of the stress tensor gives the T_n generator whereas the modes of the field β acts as M_n. Of course, the c_1 central charge is still fixed to be 26. But we can change that by coupling this system with arbitrary chiral matter whose stress tensor has some non-zero central charge. *We ignore this extra complication for now as whatever we derive with the twisted $\beta - \gamma$ system described above would go through even in that case.*

5.4 A BMS$_3$ invariant Matrix Model

We impose an infinite set of differential constraints

$$B^a_n Z = 0, \quad n \geq -1, a = 1, 2. \tag{5.4.51}$$

such that the operators B^a_n satisfy BMS$_3$ algebra (5.3.41). The explicit form of these operators is given in (5.4.53) and (5.4.54). We call these constraints the BMS$_3$-constraints, and claim that a solution of (5.4.51) gives a BMS$_3$ invariant matrix model partition function. The constraints (5.4.51) should describe the loop equations of that matrix model.

5.4.1 Loop equations

In order to define the differential operators corresponding to Loop equations, we first observe that the OPE (5.3.42) gives the following relation between the modes:

$$[\gamma_n, \beta_m] = \delta_{n+m,0}, \tag{5.4.52}$$

while the rest of the commutators vanish. Thus, the pair $\{\gamma_k, \beta_{-k}\}$ behaves like creation and annihilation operators of a simple harmonic oscillator (SHO) and can be equated with $\{\frac{\partial}{\partial t_k}, t_k\}$ for $k > 0$. A similar set of equivalence can be made for $k < 0$ modes, with another set of SHOs $\{\frac{\partial}{\partial \bar{t}_k}, \bar{t}_k\}$.

Thus, the differential operators of relevance become

$$\begin{aligned} B^1_n \equiv L_n &= \sum_{m=0}^{\infty}(2n+m)t_m\frac{\partial}{\partial t_{m+n}} + \sum_{m=0}^{\infty}(m-n)\bar{t}_m\frac{\partial}{\partial \bar{t}_{m+n}} \\ &+ \frac{1}{2}\sum_{\substack{a+b=n \\ a,b\geq 0}}(2a+b)\frac{\partial}{\partial t_a}\frac{\partial}{\partial \bar{t}_b}, \end{aligned} \tag{5.4.53}$$

$$B^2_n \equiv M_n = \frac{\partial}{\partial \bar{t}_n}, \quad n > 0, \quad M_n = t_{-n}, \quad n < 0, \tag{5.4.54}$$

which satisfy

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m}, \\ [L_n, M_m] &= (n-m)M_{n+m}, \\ [M_n, M_m] &= 0. \end{aligned} \tag{5.4.55}$$

This is the classical version of BMS$_3$ algebra. The matrix model partition function is obtained as a solution to the constraints

$$L_n Z_N = 0, \quad \text{and} \quad M_n Z_N = 0 \quad \text{for} \quad n \geq -1, \tag{5.4.56}$$

where the suffix N of the partition function indicates the charge of the vacuum under the $U(1)$ current of the system,

$$j(z) = - : \gamma\beta : \tag{5.4.57}$$

The rest of this paper is devoted to finding a solution to (5.4.56). To solve the above constraints, we will use our knowledge of the free field realisation of the algebra. Our objective would be to translate the constraints in terms of a CFT correlator in the $\beta - \gamma$ model.

5.4.2 The 'Hamiltonian' function

The procedure to find the partition function that satisfies (5.4.56) involves two vital steps [97]. Firstly, we find a "Hamiltonian" operator that relates the differential operators with the modes of the operators of our CFT.

This relation is expressed in terms of the following expressions:

$$L_n\langle N|e^{H(t,\bar{t})} = \langle N|e^{H(t,\bar{t})}T_n, \quad \text{and} \quad M_n\langle N|e^{H(t,\bar{t})} = \langle N|e^{H(t,\bar{t})}\beta_n. \tag{5.4.58}$$

The state $\langle N|$ is a vacuum of the theory which is charged.

We propose the following operator:

$$H(t,\bar{t}) = \sum_{k>1} t_k\gamma_k + \sum_{k\geq -1} \bar{t}_k\beta_k \tag{5.4.59}$$

Using (5.4.53), we have

$$L_n\langle N|e^{H(t,\bar{t})} = \langle N| \left(\sum_{p=0}^{\infty}(2n+p)t_p\gamma_{n+p} + \sum_{p=0}^{\infty}(p-n)\bar{t}_p\beta_{n+p} + \frac{1}{2}\sum_{a+b=n;a,b\geq 0}\gamma_a\beta_b \right), \tag{5.4.60}$$

whereas to work out the RHS of (5.4.58), we make use of the BCH formula

$$e^X Y = Y e^X + [X,Y], \tag{5.4.61}$$

and also use the fact that the vacuum state $\langle N|$ is annihilated by the modes β_{-k}, γ_{-k} for $k > 0$. It can be easily shown that the first equation of (5.4.58) is satisfied.

We also need to check the second equality in (5.4.58). This is relatively easy to check since the LHS gives

$$M_n\langle N|e^{H(t,\bar{t})} = \langle N|\beta_n e^{H(t,\bar{t})}. \tag{5.4.62}$$

This result is also obtained from RHS quite straightforwardly as β_n commutes with the Hamiltonian for $n > 0$ and hence, our choice of Hamiltonian function is justified.

5.4.3 Screening Charges

To complete our analysis, we now require a ket state $|G\rangle$ such that

$$T_n|G\rangle = 0 \qquad M_n|G\rangle = 0 \tag{5.4.63}$$

If such a state is found, then we may claim that the full partition function of the theory is given by

$$Z_N = \langle N|e^{H(t,\bar{t})}|G\rangle, \tag{5.4.64}$$

which from the properties of (5.4.58) and (5.4.63) satisfies all our constraints.
Generic states, which commute with all positive modes of stress tensor (first equation of (5.4.63)) are generated by Screening Operators. Unfortunately, for a $\beta-\gamma$ system of spin (2,-1) we don't have a spin 1 primary at hand. Hence, the construction of these operators isn't straightforward. For this we need to take an indirect route which we chalk out below.

For a generic bosonic $\beta-\gamma$ system of dimension $(\lambda, 1-\lambda)$ the system has a background charge $(1-2\lambda)$, which implies that our system has a background charge -3. The presence of a background charge makes the $U(1)$ symmetry anomalous and the current (5.4.57) is no longer a primary of the theory.
To get a better handle in the theory, we fermionize the theory [103]. We take a free scalar field, ϕ which satisfies the OPE

$$\phi(z)\phi(w) \sim \ln(z-w), \tag{5.4.65}$$

and two fermionic fields, η and ξ such that $\eta(z)$ and $\partial\xi(z)$ are primary fields of dimension one. Their OPE is given by

$$\eta(z)\xi(w) \sim \frac{1}{(z-w)}. \tag{5.4.66}$$

Then in terms of these fields, we can write

$$\beta(z) = e^{-\phi(z)}\partial\xi, \quad \gamma(z) = e^{\phi(z)}\eta(z). \tag{5.4.67}$$

This map gives us incredible advantage. Since, we know that free scalar fields have Vertex operators

$$\mathcal{V}_\alpha(z,\bar{z}) = e^{i\sqrt{2}\alpha\phi(z,\bar{z})}, \tag{5.4.68}$$

which are primary operators with dimension[1] α^2. Hence, we can construct dimension one primaries now, which in turn gives us our screening charges. Of course we also have fermionic primaries in our theory. It turns out the relevant screening charges in this new theory are

$$Q_1 = \oint e^{-\phi(z)}, \quad Q_2 = \oint e^{-2\phi(z)}, \tag{5.4.69}$$

as T_n and M_n both commute with them[2]. It also implies that we can take any function of these charges acting on vacuum as our state $|G\rangle$.

5.4.4 The matrix model partition function

If we choose G to be an exponential function, then we realise that charge conservation of CFT correlators demand that only the N-th term of the exponential operator would survive. Thus, the final partition function is given by

$$\begin{aligned} Z_N &= \langle N|e^{H(t,\bar{t})}|G\rangle, \\ &= \frac{1}{N_1!N_2!}\langle N|e^{H(t,\bar{t})}(Q_1)^{N_1}(Q_2)^{N_2}|0\rangle, \qquad \text{with } N_1+N_2=N. \end{aligned}$$

To evaluate the above integral, we first write our Hamiltonian in terms of fields,

$$H(t,\bar{t}) = \sum_{k>1} t_k\gamma_k + \sum_{k\geq -1}\bar{t}\beta_k = -\oint V(z)\partial\gamma(z) - \oint U(z)\partial\beta(z), \tag{5.4.70}$$

where

$$V(z) = \sum_{k>1} t_k\frac{z^{k-1}}{k-1}, \quad U(z) = \sum_{k\geq -1}\bar{t}_k\frac{z^{k+2}}{k+2}. \tag{5.4.71}$$

[1] In absence of a background charge. Otherwise $h_{\mathcal{V}_\alpha} = \alpha^2 - 2\alpha_0\alpha$

[2] This is the reason we couldn't use the fermionic screening operators. M_n doesn't commute with them.

Thus, in terms of fields our partition function becomes

$$Z_N = \frac{1}{N_1!N_2!}\langle N| : e^{-\oint V(z)\partial\gamma(z) - \oint U(z)\partial\beta(z)} : \prod_{i=1}^{N_1}\oint_{C_i} dx_i : e^{-\phi(x_i)} : \prod_{j=1}^{N_2}\oint_{C_j} dy_j : e^{-2\phi(y_j)} : |0\rangle$$

We'll also need the OPE relations between the original fields and the new fields, which are

$$\partial\beta(z)\phi(z') \sim \frac{\beta(z')}{z-z'}, \tag{5.4.72}$$

$$\partial\gamma(z)\phi(z') \sim -\frac{\gamma(z')}{z-z'}. \tag{5.4.73}$$

Finally, to evaluate the correlator, we use the identity of exponentiated operators (5.3.29). Using these we get

$$Z_N = \frac{1}{N_1!(N-N_1)!}\prod_{i=1}^{N_1}\oint_{C_i} dx_i \; e^{X(x_i)} \prod_{j=1}^{(N-N_1)}\oint_{C_j} dy_j \; e^{Y(y_j)} \frac{1}{\triangle(x)\triangle^4(y)\triangle^2(x,y)}, \tag{5.4.74}$$

where

$$X(x_i) = V(x_i)\gamma(x_i) - U(x_i)\beta(x_i), \quad Y(y_j) = V(y_j)\gamma(y_j) - U(y_j)\beta(y_j) \tag{5.4.75}$$

$$\triangle(x) = \prod_{i<k}^{N_1}(x_i - x_k), \quad \triangle(y) = \prod_{j<k}^{N_2}(y_j - y_k), \quad \triangle(x,y) = \prod_{i,j}(x_i - y_j) \tag{5.4.76}$$

This is our final result. It shows that the BMS$_3$ constraints give rise to a two-matrix model partition function written in terms of their eigenvalues. It may be interpreted as an orthogonal matrix model and a quaternionic Hermitian model interacting through the measure of the partition function.

Chapter 6

Outlook and Future Directions

In Chapter 3 and 4, we have found the dual field theory to the supergravity theory in 3D bulk. First we looked at asymptotically flat spacetimes in $\mathcal{N} = 2$ Supergravity. We have constructed the duals for both $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ cases. The physical observables dual to 3D graviton (and other supergravity fields) belong to a super Liouville like theory. We found that for $\mathcal{N} = (2, 0)$ case the dual theory enjoys an infinite dimensional most generic $\mathcal{N} = 2$ quantum SuperBMS$_3$ symmetry that so far was not known. The symmetry is a truncated version of $\mathcal{N} = 4$ SuperBMS$_3$ that we developed in [68]. One of the interesting results here was the non-trivial central extensions for $[J_n, J_m]$ and $[R_n, R_m]$ commutators. These can not be derived from the AdS$_3$ case by taking a flat space limit and hence is unique to flat space itself.

Next, we discussed the boundary behaviour of minimal supergravity theory in (2+1)D with $\Lambda > 0$. We found how the asymptotic symmetries can be extended to the static patch for this supergravity model and again ended up in a Liouville type theory. For a Lorentzian signature de-Sitter space, the holographic dual is an Euclidean CFT, which we found here as well. To analyze the symmetries of this dual super-Liouville theory is something we want to persue in the future. Our understanding is that the currents of this theory can be used to do a Sugawara construction which reproduces the asymptotic symmetry algebra of the bulk dS spacetime.

The dual theories that we wrote down are not complete quantum holographic duals of the gravity theory in the bulk. They are to be thought of as effective descriptions of the bulk gravity at asymptotic boundaries. But since symmetries play a major role in constraining the gravitational data, it is expected that analysing these duals may lead to understanding of 3D gravity observables.

Earlier in [68, 102] a free field realisation of SuperBMS$_3$ algebras was presented. It would be interesting to see how the theory constructed in this paper are related to those. Another technically challenging problem would be to extend the above analysis for $\mathcal{N} = 4$ [72, 104] and $\mathcal{N} = 8$ [71, 105] Supergravity theories. With that, we shall have a complete zoo of all 2D duals of all possible 3D Supergravities.

Another interesting problem to study is the quantum corrections to the asymptotic algebras and the dual theory. For instance the work [106] found non-trivial corrections to the central charges of the boundary algebra. It would be interesting to see these corrections for the boundary theory presented here as well. The problem of considering non trivial holonomies of the bulk gravity theory is also interesting.

In Chapter 5, we have found a matrix model partition function with BMS_3 constraints. The BMS_3 constraints are imposed through loop equations which suggests it might also be possible to formulate a topological recursion relation. There has been some recent works trying to understand the possible connections between BMS_3 algebras and integrability [107]. We believe further investigation of this partition function will shed some light over this. BMS_3 algebra is also linked with flat limits of Liouville theory [32] and it would be interesting to understand the connection between this 2D gauge theory and our resulting matrix model. We would also like to understand the links between the correlation functions in the Matrix model and BMS$_3$ invariant correlation functions of the boundary gauge theory.
Another possible direction would be to understand Super-BMS$_3$ algebra in terms of matrix models [33, 56, 61, 71]. They appear as the asymptotic symmetry group of Supergravity theories in 3D flat spacetimes. The non-trivial features resulting from fermionic constraints on a Matrix model partition function would be interesting to study. The free field realisations required for this construction were discussed in [68].

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Appendices

Appendix A

Details of Calculations

Part 1: Conventions and Identities

In this paper, we have mostly followed the conventions of [57]. The tangent space metric $\eta_{ab}, a = 0, 1, 2$ is flat and off-diagonal, given as

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The space time coordinates are u, ϕ, r with positive orientation in the bulk being $dud\phi dr$. Accordingly the Levi-Civita symbol is chosen such that $\epsilon_{012} = 1$.
The three dimensional Dirac matrices satisfy usual commutation relation $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$. They also satisfy following useful identities:

$$\Gamma_a \Gamma_b = \epsilon_{abc}\Gamma^c + \eta_{ab}\mathbb{I}, \qquad (\Gamma^a)^\alpha_\beta (\Gamma_a)^\gamma_\delta = 2\delta^\alpha_\delta \delta^\gamma_\beta - \delta^\alpha_\beta \delta^\gamma_\delta.$$

The explicit form of the Dirac matrices are chosen as,

$$\Gamma_0 = \sqrt{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma_1 = \sqrt{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.0.1}$$

All spinors in this work are Majorana and our convention for the majorana conjugate of the fermions are different from [57] and is given as,

$$\overline{\psi}_{\alpha i} = \psi^\beta_i C_{\beta\alpha}, \qquad C_{\alpha\beta} = \epsilon_{\alpha\beta} = C^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $i = 1, 2$ is the internal index and $C_{\alpha\beta}$ is the charge conjugation matrix that satisfies

$$C^T = -C, \quad C\Gamma_a C^{-1} = -(\Gamma_a)^T, \quad C_{\alpha\beta}C_{\beta\gamma} = -\delta_{\alpha\gamma}$$

For any traceless 2×2 matrix A, it can be shown that $C_{\alpha\beta}A^{\beta}_{\gamma} = (C\Gamma^a)_{\alpha\gamma}Tr[\Gamma_a A]$.
For computing the gauged action the three dimensional Fierz relation is useful and is given as

$$\zeta\bar{\eta} = -\frac{1}{2}\bar{\eta}\,\zeta\,\mathbf{1} - \frac{1}{2}(\bar{\eta}\Gamma^a\zeta)\Gamma_a\ , \tag{A.0.2}$$

Other useful identities are:

$$\bar{\psi}\Gamma_a\,\eta = \bar{\eta}\,\Gamma_a\,\psi\ , \qquad \bar{\psi}\Gamma_a\,\epsilon = -\bar{\epsilon}\,\Gamma_a\,\psi\ , \tag{A.0.3}$$

where ψ, η are Grassmannian one-forms, while ϵ is a Grassmann parameter.

Part 2: Hamiltonian form of the CS action

In this appendix, we shall present the details of the Hamiltonian action and the boundary term corresponding to a Chern-Simons theory on a 3 manifold with boundary. We decompose the gauge field $A(u,\phi,r)$ as $A = duA_u + \tilde{A}$. Thus we give a preference to the time like u direction and other two directions are treated together. The reasoning behind this decomposition is: in variational principle, in general we can not through out the variations of derivatives of gauge fields along the spacelike directions. Next we can decompose the field strength. Using $A = du\ A_u + \tilde{A}$ and $d = du\ \partial_u + \tilde{d}$, we get

$$dA = du\dot{\tilde{A}} + \tilde{d}\ duA_u + \tilde{d}\tilde{A} \tag{A.0.4}$$

Therefore

$$\begin{aligned}
&< A, dA > \\
&=< (du\ A_u + \tilde{A}), (du\dot{\tilde{A}} + \tilde{d}\ duA_u + \tilde{d}\tilde{A}) > \\
&=< du\ A_u, \tilde{d}\tilde{A} > + < \tilde{A}, du\dot{\tilde{A}} > + < \tilde{A}, \tilde{d}duA_u > \\
&=< \tilde{A}, du\dot{\tilde{A}} > +2 < du\ A_u, \tilde{d}\tilde{A} > +\text{total derivative term}
\end{aligned}$$

where we have used

$$\begin{aligned}
\tilde{d} < \tilde{A}, duA_u > &=< \tilde{d}\tilde{A}, duA_u > - < \tilde{A}, \tilde{d}duA_u > \\
&=<, duA_u, \tilde{d}\tilde{A} > - < \tilde{A}, \tilde{d}duA_u >
\end{aligned}$$

using cyclic invariance of trace in the last step.
Also we have, $(A \wedge A) = (duA_u \wedge \tilde{A} + \tilde{A} \wedge duA_u + \tilde{A} \wedge \tilde{A})$. Therefore

$$\begin{aligned}
< A^3 > &=< (du\ A_u + \tilde{A}) \wedge (duA_u \wedge \tilde{A} + \tilde{A} \wedge duA_u + \tilde{A} \wedge \tilde{A}) > \\
&=< duA_u \wedge \tilde{A} \wedge \tilde{A} > + < \tilde{A} \wedge duA_u \wedge \tilde{A} > + < \tilde{A} \wedge \tilde{A} \wedge duA_u > \\
&= 3 < duA_u \wedge \tilde{A} \wedge \tilde{A} >
\end{aligned}$$

Now collecting all the terms and putting in the CS action we get,

$$I_H[A] = \frac{k}{4\pi}\int < \tilde{A}, du\dot{\tilde{A}} > +2 < duA_u, \tilde{d}\tilde{A} + \tilde{A}^2 > \tag{A.0.5}$$

Finally we present construction of the boundary term. Variation of the above Hamiltonian form of the Chern-Simons action (A.0.5)s is given by

$$\delta I_H[A] = \frac{k}{4\pi}\int \delta\langle\tilde{A},\dot{\tilde{A}}du\rangle + 2\langle du\delta A_u, \tilde{d}\tilde{A}+\tilde{A}^2\rangle + 2\langle duA_u, \tilde{d}\delta\tilde{A}+\delta\tilde{A}^2\rangle \tag{A.0.6}$$

$$\begin{aligned}\delta\langle\tilde{A},\dot{\tilde{A}}du\rangle &= dud\phi dr\ Tr[-\delta A_r\partial_u A_\phi + \delta A_\phi\partial_u A_r - A_r\partial_u(\delta A_\phi) + A_\phi\partial_u(\delta A_r)] \\ \tilde{d}\tilde{A} &= d\phi dr(\partial_r A_\phi - \partial_\phi A_r), \quad \tilde{A}^2 = d\phi dr(A_\phi A_r - A_r A_\phi), \quad \tilde{d}\tilde{A}+\tilde{A}^2 = d\phi dr\mathcal{F}_{r\phi}\end{aligned}$$

Substituting all these expressions in (A.0.6), we get

$$\begin{aligned}\delta I_H[A] &= \frac{k}{4\pi}\int dud\phi drTr[-\delta A_r\partial_u A_\phi + \delta A_\phi\partial_u A_r - A_r\partial_u(\delta A_\phi) + A_\phi\partial_u(\delta A_r)] \\ &+\frac{k}{2\pi}\int dud\phi drTr[A_u\partial_r(\delta A_\phi) - A_u\partial_\phi(\delta A_r) + A_u(\delta A_\phi A_r + A_\phi\delta A_r - \delta A_r A_\phi - A_r\delta A_\phi)] \\ &+\frac{k}{2\pi}\int dud\phi drTr[\delta A_u\mathcal{F}_{r\phi}]\end{aligned} \tag{A.0.7}$$

The colored terms can be manipulated to write them as a total derivative plus another term. The total derivative terms from all the blue terms can be integrated out to give zero at the boundary (as the variation of fields are zero at the boundary). The red colored term gives a non-zero term at the boundary $r = r_0$ (marked green in the following expression). All other terms combine to give the following variation of the action:

$$\begin{aligned}\delta I_H[A] &= \frac{k}{2\pi}\int dud\phi drTr[\delta A_u\mathcal{F}_{r\phi}] + \frac{k}{2\pi}\int dud\phi drTr[\delta A_\phi\mathcal{F}_{ur}] + \frac{k}{2\pi}\int dud\phi drTr[\delta A_r\mathcal{F}_{\phi u}] \\ &+\frac{k}{2\pi}\int dud\phi drTr[\partial_r(A_u\delta A_\phi)] - \frac{k}{2\pi}\int dud\phi drTr[\partial_\phi(A_u\delta A_r)]\end{aligned} \tag{A.0.8}$$

The boundary term can be rewritten in form notation as $-\frac{k}{2\pi}\int du\tilde{d}\langle A_u,\delta\tilde{A}\rangle$ as,

$$-\frac{k}{2\pi}\int du\tilde{d}\langle A_u,\delta\tilde{A}\rangle = \frac{k}{2\pi}\int dud\phi drTr[\partial_r(A_u\delta A_\phi) - \partial_\phi(A_u\delta A_r)] \tag{A.0.9}$$

Apart from the boundary term, $\delta I_H[A] = 0 \Longrightarrow \mathcal{F}_{r\phi} = 0$, $\mathcal{F}_{ur} = 0$, $\mathcal{F}_{\phi u} = 0$ which means

$$F = dA + A^2 = 0$$

Part 3: Details of the Computations for Dual WZW theory of $\mathcal{N} = (1,1)$ Supergravity

In this appendix, we shall briefly present an independent computation for the $\mathcal{N} = (1,1)$ case. This is a simpler version of $\mathcal{N} = (2,0)$ case as we do not have any internal symmetry generators T, Z. But in calculations, the exact behaviour of fields (their overall signs) differ from the $\mathcal{N} = (2,0)$ case and also the basis of fermionic generators are different. Thus although the final result is mere a truncation of the $\mathcal{N} = (2,0)$ one. The action is given in (3.3.6). We begin with eoms as, Equations of motion:

$$
\begin{aligned}
de + [\hat{\omega}, e] &= \frac{1}{4}\sum_{i=1}^{2}(\overline{\psi}^i\psi^i - \frac{1}{2}\overline{\psi}^i\psi^i\mathbb{I}), \qquad d\hat{\omega} + \hat{\omega}^2 = 0 && \text{(A.0.10)}\\
D\psi^{\alpha i} &= -\frac{1}{2}\gamma e^a(\Gamma_a)^\alpha_\beta\psi^{\beta i}, \qquad (i = 1, 2) && \text{(A.0.11)}
\end{aligned}
$$

Solutions to equations of motion:

$$\hat{\omega} = \Lambda^{-1}d\Lambda, \qquad \Lambda \in SL(2, \mathbb{R}) \tag{A.0.12}$$

$$\psi^i = \Lambda^{-1}d\eta^i \qquad (i = 1, 2) \tag{A.0.13}$$

$$e = \Lambda^{-1}(\frac{1}{4}\sum_{i=1}^{2}\eta^i d\overline{\eta}^i + \frac{1}{8}\sum_{i=1}^{2}d\overline{\eta}^i\eta^i\mathbb{I} + db)\Lambda \tag{A.0.14}$$

Asymptotic form of the r-independent part of the gauge field in radial gauge:

$$A = (\frac{\mathcal{M}}{2}du + \frac{\mathcal{N}}{2}d\phi)P_0 + duP_1 + \frac{\mathcal{M}}{2}d\phi J_0 + d\phi J_1 + \sum_{i=1}^{2}\frac{\psi^i}{2^{1/4}}\mathcal{Q}^i_+ \tag{A.0.15}$$

Functional form of the solutions in radial gauge:

$$
\begin{aligned}
\Lambda &= \lambda(u, \phi)\xi(u, r) && \text{(A.0.16)}\\
\eta^{i\alpha} &= \nu^{i\alpha}(u, \phi) + \lambda(u, \phi)\rho^{i\alpha}(u, r) && \text{(A.0.17)}\\
b &= \alpha(u, \phi) - \frac{1}{4}\sum_{i=1}^{2}\nu^i(u, \phi)\overline{\rho}^i(u, r)\lambda^{-1}(u, \phi) - \frac{1}{8}\sum_{i=1}^{2}\overline{\rho}^i(u, r)\lambda^{-1}(u, \phi)\nu^i(u, \phi)\mathbb{I} && \text{(A.0.18)}\\
&\quad +\lambda(u, \phi)\beta(u, r)\lambda^{-1}(u, \phi)
\end{aligned}
$$

Constraints on the asymptotic gauge field components:

$$\omega^a_\phi = e^a_u, \quad \omega^a_u = 0, \quad \psi^{i+}_u = 0 = \psi^{i-}_u \tag{A.0.19}$$

Surface term at the boundary:

$$-\frac{k}{2\pi}\int du\tilde{d}\langle A_u, \delta\tilde{A}\rangle = -\frac{k}{4\pi}\int dud\phi\, \omega^a_\phi\omega_{a\phi}|^{r=r_0} \tag{A.0.20}$$

Action in terms of gauge field components:

$$
\begin{aligned}
I &= \frac{k}{4\pi}[\int dud\phi(e^a_\phi\omega_{au} + \omega^a_\phi e_{au} + \mu\omega^a_\phi\omega_{a\phi} - \sum_{i=1}^{2}\overline{\psi}^i_{\alpha u}\psi^{\alpha i}_\phi)|^{r=r_0} && \text{(A.0.21)}\\
&\quad +\frac{1}{6}\int(3\epsilon^{abc}e_a\omega_b\omega_c + \mu\epsilon^{abc}\omega_a\omega_b\omega_c + \sum_{i=1}^{2}\frac{3}{2}\omega^a(C\Gamma_a)_{\alpha\beta}\psi^{\alpha i}\psi^{\beta i})]
\end{aligned}
$$

Action on the solutions of equations of motion:

$$I = \frac{k}{2\pi}(\int dud\phi\, Tr[2\dot{\Lambda}\Lambda^{-1}(-\frac{1}{4}\sum_{i=1}^{2}\eta^i\overline{\eta}^{i'} + b') - (\Lambda'\Lambda^{-1})^2 + \mu\Lambda'\Lambda^{-1}\dot{\Lambda}\Lambda^{-1} - \frac{1}{2}\sum_{i=1}^{2}\eta^{i'}\dot{\overline{\eta}}^i]^{r=r_0} + \frac{\mu}{3}\int Tr(d\Lambda\Lambda^{-1})^3) \tag{A.0.22}$$

Action after using gauge decomposed forms of the solutions:

$$I[\lambda,\alpha,\nu^1,\nu^2] = \frac{k}{2\pi}\Big(\int dud\phi[2\dot{\lambda}\lambda^{-1}\alpha' + \tfrac{1}{2}\textstyle\sum_{i=1}^{2}\dot{\lambda}\lambda^{-1}\nu^i\overline{\nu}^{i'} - (\lambda'\lambda^{-1})^2 + \mu\lambda'\lambda^{-1}\dot{\lambda}\lambda^{-1} - \tfrac{1}{2}\textstyle\sum_{i=1}^{2}\dot{\overline{\nu}}^i\nu^{i'}] + \frac{\mu}{3}\int Tr(d\Lambda\Lambda^{-1})^3\Big) \quad \text{(A.0.23)}$$

Equations of motion:

$$(\dot{\lambda}\lambda^{-1})' = 0 \quad \text{(A.0.24)}$$

$$\dot{\overline{\nu}}^{i'} + \overline{\nu}^{i'}\dot{\lambda}\lambda^{-1} = 0 \quad \text{(A.0.25)}$$

$$\dot{\alpha}' + \alpha'\dot{\lambda}\lambda^{-1} - \dot{\lambda}\lambda^{-1}\alpha' + \frac{1}{4}\sum_{i=1}^{2}\dot{\nu}^i\overline{\nu}^{i'} + \frac{1}{4}\sum_{i=1}^{2}\dot{\lambda}\lambda^{-1}\nu^i\overline{\nu}^{i'} - \partial_\phi(\lambda'\lambda^{-1}) = 0 \quad \text{(A.0.26)}$$

Generic solutions of the equations of motion:

$$\lambda = \tau(u)\kappa(\phi) \quad \text{(A.0.27)}$$

$$\nu^i = \tau(\zeta_1^i(u) + \zeta_2^i(\phi)) \quad \text{(A.0.28)}$$

$$\alpha = \tau(a(\phi) + \delta(u) + u\kappa'\kappa^{-1} - \frac{1}{4}\sum_{i=1}^{2}\zeta_1^i\overline{\zeta}_2^i)\tau^{-1} \quad \text{(A.0.29)}$$

Symmetries of the solutions:

$$\alpha \longrightarrow \alpha + \lambda\Sigma(\phi)\lambda^{-1} \quad \text{(A.0.30)}$$

$$\lambda \longrightarrow \lambda\Theta^{-1}(\phi); \quad \alpha \longrightarrow \alpha - u\lambda\Theta^{-1}\Theta'\lambda^{-1} \quad \text{(A.0.31)}$$

$$\nu^i \longrightarrow \nu^i + \lambda\Upsilon^i(\phi) \quad \text{(i=1 or 2)}; \quad \alpha \longrightarrow \alpha - \frac{1}{4}\nu^i\overline{\Upsilon}^i\lambda^{-1} \quad \text{(A.0.32)}$$

Infinitesimal version of the symmetries:

$$\delta_\sigma\alpha = \lambda\sigma(\phi)\lambda^{-1} \quad \text{(A.0.33)}$$

$$\delta_\theta\lambda = -\lambda\theta(\delta_\theta\lambda^{-1} = \theta\lambda^{-1}); \quad \delta_\theta\alpha = -u\lambda\theta'\lambda^{-1} \quad \text{(A.0.34)}$$

$$\delta_\gamma\nu^1 = \lambda\gamma^1; \quad \delta_\gamma\alpha = -\frac{1}{4}\nu^1\overline{\gamma}^1\lambda^{-1} \quad \text{(A.0.35)}$$

$$\delta_\gamma\nu^2 = \lambda\gamma^2; \quad \delta_\gamma\alpha = -\frac{1}{4}\nu^2\overline{\gamma}^2\lambda^{-1} \quad \text{(A.0.36)}$$

Currents corresponding to the above symmetries:

$$J^\mu_\sigma = \delta^\mu_0(\frac{k}{\pi})Tr[\sigma\lambda^{-1}\lambda'] = 2\delta^\mu_0\sigma^a P_a \quad \text{(A.0.37)}$$

$$J^\mu_\theta = -\frac{k}{\pi}\delta^\mu_0 Tr[\theta(\lambda^{-1}\alpha'\lambda - u(\lambda^{-1}\lambda')' + \frac{1}{4}\sum_{i=1}^{2}\lambda^{-1}\nu^i\overline{\nu}^{i'}\lambda)] = 2\delta^\mu_0\theta^a J_a \quad \text{(A.0.38)}$$

$$J^\mu_{\gamma^i} = \frac{k}{2\pi}\delta^\mu_0 Tr[\gamma^i\overline{\nu}^{i'}\lambda] = \delta^\mu_0 Q^i_\alpha\gamma^{i\alpha} \quad (i = 1,2, \quad i \text{ not summed over}) \quad \text{(A.0.39)}$$

where σ and θ being $SL(2,\mathbb{R})$ matrices are expanded in the basis of Γ matrices.

Dirac brackets:

$$\begin{aligned}
\{P_a(\phi), P_a(\phi')\} &= 0 && \text{(A.0.40)}\\
\{P_a(\phi), J_b(\phi')\} &= \epsilon_{abc}P^c(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}\eta_{ab}\delta'(\phi-\phi') && \text{(A.0.41)}\\
\{P_a(\phi), Q^i_\alpha(\phi')\} &= 0 \quad (i=1,2) && \text{(A.0.42)}\\
\{J_a(\phi), J_b(\phi')\} &= \epsilon_{abc}J^c\delta(\phi-\phi') + \frac{\mu k}{2\pi}\eta_{ab}\delta'(\phi-\phi') && \text{(A.0.43)}\\
\{J_a(\phi), Q^i_\alpha(\phi')\} &= \frac{1}{2}(Q^i\Gamma_a)_\alpha(\phi)\delta(\phi-\phi') \quad (i=1,2) && \text{(A.0.44)}\\
\{Q^i_\alpha(\phi), Q^j_\beta(\phi')\} &= \delta^{ij}[-\frac{1}{2}(C\Gamma^a)_{\alpha\beta}P_a(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}C_{\alpha\beta}\delta'(\phi-\phi')] && \text{(A.0.45)}
\end{aligned}$$

Bilinears for Sugawara construction:

$$\mathcal{H} = \frac{\pi}{k}P^aP_a, \quad \mathcal{P} = -\frac{2\pi}{k}J^aP_a + \mu\mathcal{H} + \frac{\pi}{k}\sum_{i=1}^{2}Q^i_\alpha C^{\alpha\beta}Q^i_\beta, \quad \mathcal{G}^i = 2^{3/4}\frac{\pi}{k}(P_2Q^i_+ + \sqrt{2}P_0Q^i_-) \quad (i=1,2) \tag{A.0.46}$$

Remaining constraints on the gauge field:

$$\omega^1_\phi = 1, \quad e^1_\phi = e^2_\phi = 0, \quad \psi^{1-}_\phi = \psi^{2-}_\phi = 0 \tag{A.0.47}$$

Constraints on the fields:

$$[\lambda^{-1}d\lambda]^1 = 1, \quad [\lambda^{-1}(\frac{1}{4}\sum_{i=1}^{2}\nu^i\overline{\nu}^{i'} + \frac{1}{8}\sum_{i=1}^{2}\overline{\nu}^{i'}\nu^i\mathbb{I} + \alpha')\lambda]^1 = 0, \quad [\lambda^{-1}\nu^{i'}]^- = 0$$

In terms of components of currents,

$$P_0(\phi) = \frac{k}{2\pi}, \quad J_0(\phi) = -\frac{\mu k}{2\pi}, \quad Q^1_+ = Q^2_+ = 0 \tag{A.0.48}$$

Shifted bilinears:

$$\tilde{\mathcal{H}} = \mathcal{H} + \partial_\phi P_2, \quad \tilde{\mathcal{P}} = \mathcal{P} - \partial_\phi J_2, \quad \tilde{\mathcal{G}}^i = \mathcal{G}^i + 2^{3/4}\partial_\phi Q^i_+ \tag{A.0.49}$$

Poisson brackets:

$$\begin{aligned}
\{\tilde{\mathcal{H}}(\phi), \tilde{\mathcal{H}}(\phi')\} &= 0 && \text{(A.0.50)}\\
\{\tilde{\mathcal{H}}(\phi), \tilde{\mathcal{P}}(\phi')\} &= (\tilde{\mathcal{H}}(\phi) + \tilde{\mathcal{H}}(\phi'))\delta'(\phi-\phi') - \frac{k}{2\pi}\partial^3\delta(\phi-\phi') && \text{(A.0.51)}\\
\{\tilde{\mathcal{H}}(\phi), \tilde{\mathcal{G}}^i(\phi')\} &= 0 && \text{(A.0.52)}\\
\{\tilde{\mathcal{P}}(\phi), \tilde{\mathcal{P}}(\phi')\} &= (\tilde{\mathcal{P}}(\phi) + \tilde{\mathcal{P}}(\phi'))\delta'(\phi-\phi') - \frac{\mu k}{2\pi}\partial^3\delta(\phi-\phi') && \text{(A.0.53)}\\
\{\tilde{\mathcal{P}}(\phi), \tilde{\mathcal{G}}^i(\phi')\} &= (\tilde{\mathcal{G}}^i(\phi) + \frac{1}{2}\tilde{\mathcal{G}}^i(\phi'))\delta'(\phi-\phi') && \text{(A.0.54)}\\
\{\tilde{\mathcal{G}}^i(\phi), \tilde{\mathcal{G}}^i(\phi')\} &= \delta^{ij}(\tilde{\mathcal{H}}(\phi)\delta(\phi-\phi') - \frac{k}{2\pi}\partial^2\delta(\phi-\phi')) && \text{(A.0.55)}
\end{aligned}$$

Part 4: Currents corresponding to Global symmetries of WZW theory

Here we present a procedure to get the ϕ dependent symmetries of the solutions with one example. Let us look at the solution of λ:

$$\lambda = \tau(u)\kappa(\phi).$$

Multiplying the solution by an arbitrary ϕ dependent $SL(2,C)$ field θ^{-1} from right is still a symmetry of the solution. In this new solution κ is modified as $\kappa\theta$. Since κ appears in the solution of F, that solution also needs to be transformed accordingly. The κ dependent term in F is $\sim -u\tau\kappa'\kappa^{-1}\tau^{-1}$. This for $\kappa \to \kappa\theta$ this piece transforms as $-u\tau(\kappa\theta^{-1})'(\kappa\theta^{-1})^{-1}\tau^{-1} = -u\tau\kappa'\kappa^{-1}\tau^{-1} + u\lambda(\theta^{-1}\theta')\lambda^{-1}$. Thus the field F changes as $F \to F + u\lambda(\theta^{-1}\theta')\lambda^{-1}$. A similar analysis for all possible symmetries of the solutions yields the transformations presented in the first equation of section 3.6.1. One can then easily derive the infinitesimal versions presented in (3.6.45).Below we present some details of the current computations.

Noether current associated to a global symmetry generated by parameter ϵ is given as,

$$\mathcal{J}^{\mu}{}_{\epsilon} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\delta_\epsilon\phi_i - K^\mu_\epsilon, \qquad \partial_\mu K^\mu_\epsilon = \delta_\epsilon\mathcal{L}. \tag{A.0.56}$$

Another useful way to get the current is to use

$$\partial_\mu\mathcal{J}^{\mu}{}_{\epsilon} = \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)} - \frac{\delta\mathcal{L}}{\delta\phi_i}\right)\delta_\epsilon\phi_i. \tag{A.0.57}$$

We shall write the current such that it only has non-zero component in the $u-$direction. For finding currents corresponding to $\mathcal{C}$ and N transformations, (A.0.57) is useful and that directly gives us J^μ_C, J^μ_N of (3.6.52). For the other four currents we need to use either of (A.0.56), (A.0.57) and improvement terms $S^{[\mu\nu]}_\epsilon, T^\mu_\epsilon$.

First we look at the fermionic currents. For these, we do not require improvement and (A.0.56) directly gives us currents in $u-$direction. In particular the for $J^\mu_{D_1}$, we get

$$K^u = (-\frac{k}{2\pi})Tr[(\bar{d_2}'\lambda - ia'\bar{d_2}\lambda)D_1], \quad K^\phi = \frac{k}{2\pi}Tr[\lambda D_1\dot{\bar{d_2}} + i\dot{a}\overline{\lambda D_1}d_2].$$

Finally for $J^\mu_{D_1}$ we need to take contribution from $\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_i)}\delta_\epsilon\phi_i$ piece, that cancels the K^ϕ part and adds an equal contribution as K^u in the final expression. Similarly we get $J^\mu_{D_2}$.

For the current due to A transformation, direct evaluation with (A.0.56) gives

$$\mathcal{J}^u = \frac{k}{4\pi}[(\bar{\mu}a'+2ic'+2i(\bar{d_2}'d_1-\bar{d_1}'d_2-ia'(\bar{d_2}d_1+\bar{d_1}d_2)))A-\bar{\mu}aA'+8ua'A'], \mathcal{J}^\phi = \frac{k}{4\pi}(\bar{\mu}\dot{a}A-8a'A).$$

Here the fermion terms are traced among themselves. To get the final current in $u-$direction, we need to add

$$S^{[\mu\nu]}_A = -\frac{k}{4\pi}\varepsilon^{\mu\nu}(8ua'A - \bar{\mu}aA), \quad \varepsilon^{u\phi} = 1, \quad T^\mu_A = \delta^\mu_\phi u\frac{k}{4\pi}\dot{a}'.$$

taking these improvement terms into account we finally get J^μ_A as given in (3.6.52).

Finally for Θ transformation, using (A.0.57) we get

$$\mathcal{J}^u = \frac{k}{2\pi}Tr[\{\lambda^{-1}\hat{\alpha}\lambda + \mu\lambda^{-1}\lambda'\}\theta - 2u(\lambda^{-1}\lambda')\theta'], \quad \mathcal{J}^\phi = \frac{k}{2\pi}Tr[\theta\lambda^{-1}\lambda']$$

. Adding the required improvements terms are

$$S_\Theta^{[\mu\nu]} = \frac{k}{2\pi}\varepsilon^{\mu\nu}Tr[u\Theta\lambda^{-1}\lambda'], \quad T_\Theta^\mu = \delta_\phi^\mu u\frac{k}{2\pi}Tr[\Theta(\lambda^{-1}\lambda')^{\cdot}],$$

we finally get the expression for J_Θ^μ as given in (3.6.52).

Part 5: Some important Dirac Brackets

In this appendix, we provide the nontrivial Dirac brackets between various currents and current bilinears that are required for the results presented in the draft.

The non-trivial Dirac brackets of Sugawara modes with currents are given by:

$$\left\{H(\phi), Q_a^J(\phi')\right\} = -Q_a^P(\phi)\delta'(\phi-\phi') \tag{A.0.58}$$
$$\left\{H(\phi), Q^A(\phi')\right\} = 4Q^C(\phi)\delta'(\phi-\phi') \tag{A.0.59}$$
$$\left\{P(\phi), Q_a^J(\phi')\right\} = Q_a^J(\phi)\delta'(\phi-\phi') \tag{A.0.60}$$
$$\left\{P(\phi), Q_a^P(\phi')\right\} = Q_a^P(\phi)\delta'(\phi-\phi') \tag{A.0.61}$$
$$\left\{P(\phi), Q_\gamma^{G_1}(\phi')\right\} = Q_\gamma^{G_1}(\phi)\delta'(\phi-\phi') \tag{A.0.62}$$
$$\left\{P(\phi), Q_\gamma^{G_2}(\phi')\right\} = Q_\gamma^{G_2}(\phi)\delta'(\phi-\phi') \tag{A.0.63}$$
$$\left\{P(\phi), Q^A(\phi')\right\} = Q^A(\phi)\delta'(\phi-\phi') \tag{A.0.64}$$
$$\left\{P(\phi), Q^C(\phi')\right\} = Q^C(\phi)\delta'(\phi-\phi') \tag{A.0.65}$$
$$\left\{\mathcal{G}^1(\phi), Q_+^{G_2}(\phi')\right\} = \sqrt{2}\frac{\pi}{k}Q_0^P\frac{k}{\pi}\delta'(\phi-\phi') \tag{A.0.66}$$
$$\left\{\mathcal{G}^1(\phi), Q_-^{G_2}(\phi')\right\} = H(\phi)\delta(\phi-\phi') - Q_2^P(\phi)\delta'(\phi-\phi') - 2iQ^C(\phi)\delta'(\phi-\phi') \tag{A.0.67}$$
$$\{\mathcal{G}_1(\phi), Q^A(\phi)\} = -i\mathcal{G}_1(\phi)\delta(\phi-\phi') \tag{A.0.68}$$
$$\{\mathcal{G}_2(\phi), Q^A(\phi)\} = i\mathcal{G}_2(\phi)\delta(\phi-\phi') \tag{A.0.69}$$

With above equations we can try to calculate the PBs between different modes of stress tensor. For example:

$$
\begin{aligned}
\{H(\phi), P(\phi')\} &= \left\{H(\phi), (-2\frac{\pi}{k})Q^J_a Q^P_a\right\} \\
&= (-2\frac{\pi}{k})[\left\{H(\phi), Q^J_a(\phi')\right\} Q^P_a(\phi') + Q^J_a(\phi') \left\{H(\phi), Q^P_a(\phi')\right\} Q^P_a(\phi')] \\
&= 2\frac{\pi}{k} Q^P_a(\phi) Q^P_a(\phi') \partial_\phi \delta(\phi - \phi') \\
&= \frac{\pi}{k} Q^P_a(\phi) \partial_\phi [Q^P_a(\phi')\delta(\phi - \phi')] - \frac{\pi}{k}\partial_{\phi'}[Q^P_a(\phi)\delta(\phi - \phi')] Q^P_a(\phi') \\
&= \frac{\pi}{k} Q^P_a(\phi) \partial_\phi [Q^P_a(\phi)\delta(\phi - \phi')] - \frac{\pi}{k}\partial_{\phi'}[Q^P_a(\phi')\delta(\phi - \phi')] Q^P_a(\phi') \\
&= \frac{\pi}{k} Q^P_a(\phi) Q^P_a(\phi) \partial_\phi \delta(\phi - \phi') - \frac{\pi}{k}\partial_{\phi'}\delta(\phi - \phi') Q^P_a(\phi') Q^P_a(\phi') \\
&= (H(\phi) + H(\phi'))\partial_\phi \delta(\phi - \phi')
\end{aligned}
$$

The Dirac Brackets of the above modes among themselves are given by:

$$\{H(\phi), H(\phi')\} = 0 \tag{A.0.70}$$

$$\{H(\phi), P(\phi')\} = (H(\phi) + H(\phi'))\partial_\phi \delta(\phi - \phi') \tag{A.0.71}$$

$$\{P(\phi), P(\phi')\} = (P(\phi) + P(\phi'))\partial_\phi \delta(\phi - \phi') \tag{A.0.72}$$

$$\{P(\phi), \mathcal{G}^I(\phi')\} = (\mathcal{G}^I(\phi) + \mathcal{G}^I(\phi'))\partial_\phi \delta(\phi - \phi') \tag{A.0.73}$$

$$\{H(\phi), Q^A(\phi')\} = 4Q^C(\phi)\partial_\phi \delta(\phi - \phi') \tag{A.0.74}$$

$$\{Q_C(\phi), Q_A(\phi')\} = \frac{k}{2\pi}\partial_\phi \delta(\phi - \phi') \tag{A.0.75}$$

$$\{P(\phi), Q^C(\phi')\} = Q^C(\phi)\partial_\phi \delta(\phi - \phi') \tag{A.0.76}$$

$$\{P(\phi), Q^A(\phi')\} = Q^A(\phi)\partial_\phi \delta(\phi - \phi') \tag{A.0.77}$$

$$\{\mathcal{G}_1(\phi), Q^A(\phi)\} = -i\mathcal{G}_1(\phi)\delta(\phi - \phi') \tag{A.0.78}$$

$$\{\mathcal{G}_2(\phi), Q^A(\phi)\} = i\mathcal{G}_2(\phi)\delta(\phi - \phi') \tag{A.0.79}$$

$$\tag{A.0.80}$$

The modes of Stress tensor as defined by above Sugawara construction do not commute with the First class constraints. In fact,

$$
\begin{aligned}
\{H(\phi), Q_0^P(\phi')\} &= 0\\
\{H(\phi), Q_0^J(\phi')\} &= -Q_0^P(\phi)\delta'(\phi-\phi') = -\sqrt{2}\frac{k}{2\pi}\delta'(\phi-\phi')\\
\{H(\phi), Q_+^{G_1}(\phi')\} &= \{H(\phi), Q_+^{G_2}(\phi')\} = 0\\
\{P(\phi), Q_0^P(\phi')\} &= Q_0^P(\phi)\delta'(\phi-\phi') = \sqrt{2}\frac{k}{2\pi}\delta'(\phi-\phi')\\
\{P(\phi), Q_0^J(\phi')\} &= Q_0^J(\phi)\delta'(\phi-\phi') = \sqrt{2}\frac{\mu k}{4\pi}\delta'(\phi-\phi')\\
\{P(\phi), Q_+^{G_{1,2}}(\phi')\} &= Q_+^{G_{1,2}}(\phi)\delta'(\phi-\phi') = 0\\
\{\mathcal{G}_\alpha^{1,2}(\phi), Q_0^P(\phi')\} &= 0\\
\{\mathcal{G}_\alpha^{1,2}(\phi), Q_0^J(\phi')\} &= Q_0^P(\phi)Q_+^{1,2}(\phi)\delta(\phi-\phi')\\
\left\{\mathcal{G}^1(\phi), Q_+^{G_2}(\phi')\right\} &= \sqrt{2}\frac{\pi}{k}Q_0^P\frac{k}{\pi}\delta'(\phi-\phi') = \frac{k}{\pi}\delta'(\phi-\phi')\\
\{Q^A(\phi), Q_+^{G_1}(\phi')\} &= iQ_+^{G_1}(\phi)\delta(\phi-\phi') = 0\\
\{Q^A(\phi), Q_+^{G_2}(\phi')\} &= -iQ_+^{G_2}(\phi)\delta(\phi-\phi') = 0
\end{aligned}
$$

Part 6: An example of Super BMS$_3$ current commutation

In this appendix we shall present how the shifted fermionic currents $\hat{\mathcal{G}}^1(\phi), \hat{\mathcal{G}}^2(\phi')$ closes to right SuperBMS$_3$ structure under anti-commutation. With these shifts the Dirac bracket becomes:

$$
\begin{aligned}
\{\hat{\mathcal{G}}^1(\phi), \hat{\mathcal{G}}^2(\phi')\} =& \{\mathcal{G}^1(\phi) + \partial_\phi Q_+^{G_1}(\phi), \mathcal{G}^2(\phi') + \partial_{\phi'} Q_+^{G_2}(\phi')\}\\
=& \{\mathcal{G}^1(\phi), \frac{\pi}{k}(Q_2^P Q_+^{G_2} + \sqrt{2}Q_0^P Q_-^{G_2} + 2iQ^C Q_+^{G_2})\} + \partial_\phi\{Q_+^{G_1}(\phi), \mathcal{G}^2(\phi')\}\\
&+ \partial_{\phi'}\{\mathcal{G}^1(\phi), Q_+^{G_2}(\phi')\} + \partial_{\phi'}\partial_\phi\{Q_+^{G_1}(\phi), Q_+^{G_2}(\phi')\}
\end{aligned}
$$

Now we can look at the RHS term by term. The first DB gives:

$$
\begin{aligned}
&\{\mathcal{G}^1(\phi), \frac{\pi}{k}(Q_2^P Q_+^{G_2} + \sqrt{2}Q_0^P Q_-^{G_2} + 2iQ^C Q_+^{G_2})(\phi')\}\\
=&\frac{\pi}{k}Q_2^P(\phi')\{\mathcal{G}^1(\phi), Q_+^{G_2}(\phi')\} + \frac{\pi}{k}\sqrt{2}Q_0^P(\phi')\{\mathcal{G}^1(\phi), Q_-^{G_2}(\phi')\} + 2i\frac{\pi}{k}\{\mathcal{G}^1(\phi), Q_+^{G_2}(\phi')\}
\end{aligned}
$$

Using the fact that we are on the constrained surface defined by (3.7.56), we see that the first term above is 0 since $Q_2^P(\phi) = 0$ on the surface. The last two terms combine to give:

$$
\{\mathcal{G}^1(\phi), \frac{\pi}{k}(Q_2^P Q_+^{G_2} + \sqrt{2}Q_0^P Q_-^{G_2} + 2iQ^C Q_+^{G_2})(\phi')\} = \mathcal{H}(\phi)\delta(\phi-\phi') - 2i(Q^C(\phi) + Q^C(\phi'))\delta'(\phi-\phi')
$$

Where we also needed to use $Q_2^P(\phi)\partial_\phi\delta(\phi-\phi') = -\partial_\phi Q_2^P(\phi)\delta(\phi-\phi')$ on the constrained surface.

Similarly the last rest of the terms of the first DB combine to give:

$$
\partial_\phi\{Q_+^{G_1}(\phi), \mathcal{G}^2(\phi')\} + \partial_{\phi'}\{\mathcal{G}^1(\phi), Q_+^{G_2}(\phi')\} + \partial_{\phi'}\partial_\phi\{Q_+^{G_1}(\phi), Q_+^{G_2}(\phi')\} = -\frac{k}{\pi}\partial_\phi^2\delta(\phi-\phi')
$$

Thus we finally get:

$$\{\hat{\mathcal{G}}^1(\phi), \hat{\mathcal{G}}^2(\phi')\}_{DB} = \mathcal{H}(\phi)\delta(\phi-\phi') - \frac{k}{\pi}\partial_\phi^2\delta(\phi-\phi') - 2i(Q^C(\phi) + Q^C(\phi'))\delta'(\phi-\phi')$$

Similar computations can be performed with other shifted currents to get the final Dirac brackets. [12]

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