Quantum Fields in Curved Spacetimes

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled Quantum Fields in Curved Spacetimes submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Nihal S. Rao at Harish Chandra Research Institute, Allahabad, India, under the supervision of Dileep P. Jatkar during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled Quantum Fields in Curved Spacetimes are the results of the investigations carried out by me at the Department of Physics, Harish Chandra Research Institute, Allahabad, India, under the supervision of Dileep P. Jatkar and the same has not been submitted elsewhere for any other degree.

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Abstract

QFT in curved spacetime is a semi classical approximation to any theory of quantum gravity which as of date remains elusive to find. Though the foundations of this subject were firmly established as early as 1970, there is still a lot of debate and discussion which has sometimes culminated in 'wars' on topics such as information paradox, the violation of basic symmetries in black holes, the final stages of evaporation of a black hole and many others even as of today.

We work in a paradigm where gravity is considered as a classical theory and the matter fields are quantum in nature. Also we work at energy scales much less than the Plank scale so that we are justified in neglecting the non perturbative effects of the quantum nature of gravity. As we shall see non trivial effects of gravitation occur in the quantum field modes only then the wavelength of the modes is of the order of characteristic length scales of the background metric which implies we can verify the predicted results only at large energies which were prevalent at early times just after the big bang. As a consequence most of the results remain theoretical and experimental checks are rare.

Hawking's result of thermal radiation from black holes and the connection between quantum black holes and thermodynamics has shed some light for new fundamental physics in which gravity, thermodynamics and QFT pull all the strings. This could in turn lead to looking at physics with new eyes and making advances that have some experimental checks. It is with this motivation that we embark on a journey of understanding these historical and path breaking results.

The motivation to study curved space QFT is multidimensional as a case about it being a first approximation to quantum gravity has been made. The study of black holes and effects like Hawking radiation are consistency checks for many new theories like DBI theory, AdS/CFT duality thus anyone interested exploring these fields must have a firm knowledge of curved space QFT. Also the study of anomalies is central to the understanding of low energy limit of string theories as well as statistical physics. This subject consolidates previous knowledge and brings one to the doorstep of new and exciting physics which is reason enough to pursue it.

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Chapter 1

Quantum Fields in Curved spacetimes

Here we shall discuss the basic formalism of quantum fields in curved spacetimes by extending and generalizing flat space QFT in a straight forward manner. While the formalism is at least mathematically well defined as we shall see we struggle with the interpretation of results in physical terms.
In few cases, mostly those that deal with spacetimes conformal to flat spacetimes and static spacetimes we can make physical sense of particles and vacuum as the notion of absence of particles. But in majority of the cases we do not have a concrete definition of particles and we need approximation schemes at best, to make any sense of particles.
In this chapter we shall start off with the generalization of results in flat spacetime QFT to curved spacetimes QFT. Then we shall solve an explicit example and build upon it by studying quasi static spacetimes and adiabaticity. We discover that the high frequency behaviour of fields is independent of global geometry or the quantum states and depend only on the local structure of spacetime. We shall end the chapter by studying about conformal vacuum and the experiences of a comoving observer in such spacetimes.

1.1 Scalar field Quantization

We shall make some progress by closely imitating the steps involved in the canonical quantization of flat space QFT. Let us first consider the simple case of massive conformally coupled scalar fields. The action for such a field can be written as

$$S[\phi] = \int d^d x \; L \tag{1.1}$$

where the Lagrangian density is

$$L(x) = \frac{1}{2}\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - [m^2 + \xi R(x)]\phi^2\right) \tag{1.2}$$

here m is the mass of the scalar field and ξ is the conformal coupling constant and R is the Ricci scalar which has been included in the action as it is the unique local scalar coupling of the correct dimension which has a mass dimension of 2.

The field equation is generated by setting the variation of the action with respect to ϕ to 0.

$$\left[\bigtriangledown + m^2 + \xi R(x)\right]\phi = 0 \tag{1.3}$$

where the Laplacian in general spacetimes defined in an operational sense is given by

$$\bigtriangledown = \frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\right) \tag{1.4}$$

There are two values of ξ that are of particular interest- the minimally coupled case $\xi = 0$ and the conformally coupled case $\xi = \xi(d) = \frac{d-2}{4(d-1)}$ where d is the dimension of the space. The special property of the conformally coupled case is that the original and the conformally coupled fields satisfy similar equations.

The scalar product is defined as

$$(\phi_1, \phi_2) = -i\int_\Sigma \sqrt{-g_\Sigma}\left[\phi_1\partial_\mu\phi_2^* - \phi_2^*\partial_\mu\phi_1\right] d\Sigma^\mu \tag{1.5}$$

where $d\Sigma^\mu$ is the volume element of the Cauchy hypersurface Σ over which the integration is carried out.

We can find a complete set of mode solutions $u_i(x)$ of (1.3) that are orthogonal with respect to the scalar product defined in (1.5) as

$$(u_i, u_j) = \delta_{ij}, (u_i^*, u_j^*) = -\delta_{ij}, (u_i, u_j^*) = 0 \tag{1.6}$$

We now decompose the field ϕ in terms of these field modes as in the flat space case

$$\phi = \sum_k [a_k u_k + a_k^\dagger (u_k)^*] \tag{1.7}$$

and define a vacuum state with respect to u_k as

$$a_k \left|0_{in}\right\rangle = 0 \tag{1.8}$$

and demand the same canonical relations between the operators a and $a^\dagger$ as

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}, etc... \tag{1.9}$$

The problem in curved spacetime is in general there are no Killing vectors with respect to which we can define 'natural' mode functions. In flat spacetime the Poincare group is a symmetry group and there exists natural coordinates whose associated Killing vectors define mode functions. The non availability of natural mode decomposition means that there can be other set of mode solutions that are equally "physical" for decomposition. Thus considering a second set of orthonormal modes $\bar{u}_k$ exist we can write

$$\phi = \sum_k [b_k \bar{u}_k + b_k^\dagger (\bar{u}_k)^*] \tag{1.10}$$

and define a vacuum state with respect to u_k as

$$b_k \left|\bar{0}\right\rangle = 0 \tag{1.11}$$

Since both the sets of mode functions are complete we can expand one in terms of the other and these are related by what is formally called Bogolubov relations or Bogolubov transformations. We thus have

$$\bar{u}_k = \int_\Sigma dk' [\alpha_{kk'} u_{k'} + \beta_{kk'} (u_{k'})^*] \tag{1.12}$$

$$u_k = \int_\Sigma dk' [\alpha^*_{kk'} \bar{u}_{k'} - \beta_{kk'} (\bar{u}_{k'})^*] \tag{1.13}$$

The α and β are called the Bogolubov coefficient and can be evaluated at least in principle if the mode functions are known in exact form using their orthonormal properties as

$$\alpha_{ij} = (\bar{u}_i, u_j)\,, \beta_{ij} = -\left(\bar{u}_i, u_j^*\right) \tag{1.14}$$

Also one can go further ahead and relate the creation and annihilation operators in terms of these Bogolubov coefficients.

$$a_i = \sum_j \left(\alpha_{ij} b_j + \beta^*_{ij} b_j^\dagger\right) \tag{1.15}$$

$$b_i = \sum_j \left(\alpha^*_{ij} a_j - \beta^*_{ij} a_j^\dagger\right) \tag{1.16}$$

There exists a normalization relation between these coefficients which can be interpreted as conservation of probability in some sense that turns out to be quite useful.

$$\sum_k \left(\alpha_{ik}\alpha^*_{kj} - \beta_{ik}\beta^*_{kj}\right) = \delta_{ij} \tag{1.17}$$

$$\sum_k \left(\alpha_{ik}\beta_{kj} - \beta_{ik}\alpha_{kj}\right) = 0 \tag{1.18}$$

As long as the β coefficient is non zero the Fock spaces based on the two sets of mode functions are different. We shall make the most of these equations in two explicit examples where shall see that the changing gravitational fields lead to creation of particles.

1.2 Particle creation: An example

We shall consider a two dimensional FRW universe with static in and out regions with the metric

$$ds^2 = dt^2 - a^2(t)dx^2 \tag{1.19}$$

where $a(t)$ is the scale factor which is only dependent on time as this universe is spatially homogeneous. To simplify further we introduce the conformal time η which is related to the cosmic time t as $d\eta = \frac{dt}{a(t)}$ and the metric(1.19) can be written in the conformal time as

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2) \tag{1.20}$$

and we choose a particular form of the scale factor as $C(\eta) = a^2(\eta) = A + B\tanh(\rho\eta)$ where ρ can be interpreted as the rate of expansion and at lagre times we asymptotically approach static spacetimes i.e. $C(\eta) \to A \pm B$ =constant.

We shall consider a massive scalar field in two dimension (in two dimensions the conformal coupling constant is 0 and it doesn't matter whether it is minimally coupled or conformally coupled). Since we have spatial homogeneity the mode solutions to the field equation (1.3) with the metric (1.20) decouples as

$$u_k(\eta, x) \propto \exp(ikx)\chi_k(\eta) \tag{1.21}$$

where the functions $\chi_k(\eta)$ satisfy the equation

$$\frac{\partial^2\chi(\eta)}{\partial\eta^2} + [k^2 + C(\eta)m^2]\chi(\eta) = 0 \tag{1.22}$$

whose solutions are in general hypergeometric function. Since we are interested in asymptotic limit where we expand into static spacetimes we take the large time limit of the hypergeometric solutions of (1.22) which indeed coincides with the plane wave solutions in remote past and remote future. As $\eta \to -\infty$

$$u_k^{in}(\eta, x) \to e^{ikx - i\omega_{in}\eta} \tag{1.23}$$

and as $\eta \to +\infty$

$$u_k^{out}(\eta, x) \to e^{ikx - i\omega_{out}\eta} \tag{1.24}$$

where

$$\omega_{in}^2 = k^2 + m^2(A - B) \tag{1.25}$$

$$\omega_{out}^2 = k^2 + m^2(A + B) \tag{1.26}$$

$$\omega_{\pm} = \frac{1}{2}(\omega_{out} \pm \omega_{in}) \tag{1.27}$$

Since we see that u^{in} and u^{out} are not equal and the Bogolubov coefficients are non trivial. These can be computed easily using (1.14) and the transformation properties of hypergeometric functions. We shall only quote the result for the modulo square of these coefficients.

$$u_k^{in}(\eta, x) = \alpha_k u_k^{out}(\eta, x) + \beta_k u_{-k}^{out}(\eta, x) \tag{1.28}$$

$$|\alpha_k|^2 = \frac{\sinh^2(\pi\omega_+/\rho)}{sinh(\pi\omega_{in}/\rho)sinh2(\pi\omega_{out}/\rho)} \tag{1.29}$$

$$|\beta_k|^2 = \frac{\sinh^2(\pi\omega_-/\rho)}{sinh(\pi\omega_{in}/\rho)sinh2(\pi\omega_{out}/\rho)} \tag{1.30}$$

and these satisfy the normalization condition trivially

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{1.31}$$

If we assume that the field started out at time $\eta = -\infty$ in the state $|0_{in}\rangle$ defined by the in modes then an inertial detector would register an absence of particles in this region but in the out region as $\eta = +\infty$ the spacetime is still Minkowskian and as we are in the Heisenberg picture the state of the field continues to be $|0_{in}\rangle$. But this is not regarded as the true vacuum absent of particles as that privilege is reserved for the state $|0_{out}\rangle$ that is defined with respect to the out modes. The expected number of particles in each mode k is given by (1.30) and this is explained by the fact that as the universe expands the changing gravitational fields pump in the energy that excite the vacuum fields and particles are created. In the next section we shall look at another example of particle creation.

1.3 Mass change: Another example

We shall now consider a rather simple case where the scalar field changes its mass as it evolves as $m_{eff}^2(\eta) = m^2$ for $\eta < 0$ and $\eta > \eta_1$ and $m_{eff}^2(\eta) = -m^2$ for $0 < \eta < \eta_1$. We again consider the *in* and *out* modes at early and late times. By again solving (1.3) for the appropriate mass profile and using

(1.14) to calculate the Bogolubov coefficients where the positive frequency modes are gives by

$$u_k^{in} = \frac{1}{\sqrt{\omega_k}} \exp(-i\omega_k \eta) \tag{1.32}$$

$$u_k^{out} = \frac{1}{\sqrt{\omega_k}} \cos(\Omega_k \eta) - i\frac{\sqrt{\omega_k}}{\Omega_k} \sin(\Omega_k \eta) \tag{1.33}$$

This form of the mode functions is derived by demanding continuity of the function and its first derivative at the boundary points $\eta = 0$ and at $\eta = \eta_1$ with $\omega_k^2 = k^2 + m^2$ and $\Omega_k^2 = k^2 - m^2$.
Since the mode functions in the *in* and *out* regions do not match we expect that a state $|0_{in}\rangle$ defined as the vacuum state at early times would register particles at late times. We compute the Bogolubov coefficients to be

$$\alpha_k = \frac{e^{-i\Omega\eta_1}}{4} \left(\sqrt{\frac{\omega_k}{\Omega_k}} - \sqrt{\frac{\Omega_k}{\omega_k}} \right)^2 - c.c. \tag{1.34}$$

$$\beta_k = \frac{1}{2} \left(\frac{\Omega_k}{\omega_k} - \frac{\omega_k}{\Omega_k} \right) \sin(\Omega_k \eta_1) \tag{1.35}$$

We can compute the number density of particles in the mode k at late times in the vacuum state of early times to be

$$n_k = |\beta_k|^2 = \frac{m^4}{|k^4 - m^4|} \sin^2(\Omega_k \eta_1) \tag{1.36}$$

We shall now move on to the discussion of the meaning of particles.

1.4 Particle concept

We have seen in the previous two examples that even a vacuum state defined with respect to a certain set of mode functions contains particles as defined by another set of mode functions and this ambiguity begs the question which set of mode functions is the "correct" one. As we shall show in this section that this question does not have a clear answer. It turns out that we require the state of motion of the detector too to define particles since an inertial and non inertial detector have different experiences in trying to measure particles. Thus the particle concept is essentially observer dependent and does not have any universal significance.
Flat space QFT is spared of this ambiguity and one can clearly define vacuum states and what is meant by "particles" since the vacuum state defined is invariant under the Poincare group thus all sets of inertial observers will

agree upon the same experiences. We, in most cases, will be considering spacetimes that are asymptotically static or flat as we can define in these regions the "natural" vacuum state and record the experiences of an inertial or comoving observer (as may be the case). As we shall be working in the Heisenberg picture the vacuum state defined in the *in* region may not be truly devoid of particles in the *out* region which may be interpreted by thinking of "particle creation" by time dependent gravitational fields.
We now intend to illustrate these claims by considering a model for a particle detector with internal energy levels E and coupled to the scalar field ϕ in four dimensions via a monopole coupling. We shall closely follow the work of Dewitt and Unruh(1979).
We shall consider a particle detector moving along the world line $x^\mu(\tau)$ where τ is its proper time. The detector is coupled to the field ϕ by the interaction term $cm(\tau)\phi(x)$ where $m(\tau)$ is the monopole moment and c is the coupling constant which we assume to be small to use perturbation theory. The field ϕ is assumed to be in the vacuum state and the detector in its ground state E_0. When the detector moves on a general world line it excites to a higher energy state with $E > E_0$ and the field too excites from the vacuum state $|0_M\rangle$ to $|\psi\rangle$. Using first order perturbation theory the amplitude for the transition is given as

$$c\left\langle E_0, 0_M \left| \int d\tau m(\tau)\phi[x(\tau)] \right| E, \psi \right\rangle \tag{1.37}$$

If we now expand the field ϕ in terms of the mode functions we realize that transition can only happen to 1 particle state $|\psi\rangle = |1_k\rangle$ and further we assume that the detector follows an inertial trajectory $x = x + vt$. Then (1.37) can be computed to be

$$\delta\left(E - E_0 + (\omega_k - k.v)(1 - v^2)^{-\frac{1}{2}}\right) \tag{1.38}$$

where $\omega_k^2 = k^2 + m^2$. Since the argument of the δ function is positive we get the expected result that the transition amplitude vanishes as a consequence of energy conservation for an inertial detector.
If we had chosen a general trajectory then we would have to consider all E and ψ to which transition is possible. The result is

$$P \propto c^2 \sum_E |\left\langle E \left| m(0) \right| E_0 \right\rangle|^2 F(E - E_0) \tag{1.39}$$

and

$$F(E) = \int d\tau' \int d\tau G^+(x(\tau), x(\tau'))e^{-iE(\tau-\tau')} \tag{1.40}$$

where $F(E)$ represents the response function which is independent of the details of the detector and represents a bath of particles the detector experiences due to its motion entirely determined by the positive frequency Wightman functions G^+. The prefactor known as the *Selectivity* factor clearly depends upon the details of the detector.

We now shall discuss the Unruh effect and Rindler particles briefly to which we shall return later. The problem has got to do with the experiences of an uniformly accelerating observer in Minkowski vacuum whom we consider to be moving along in a hyperbolic trajectory in the (t, x) plane.

$$y = z = 0, x^2 = t^2 + \alpha^{-2} \tag{1.41}$$

where α is the detector's acceleration whose Wightman Green function can be computed as

$$G^+(\triangle\tau) = -\left[\frac{16\pi^2}{\alpha^2}\sinh^2\left(\frac{\triangle\tau\alpha}{2}\right)\right]^{-1} \tag{1.42}$$

where $\triangle\tau = \tau - \tau'$. (1.39) can be computed to be

$$\frac{c^2}{2\pi}\sum_E \frac{(E-E_0)|\left\langle E\left|m(0)\right|E_0\right\rangle|^2}{\exp(\frac{2\pi}{\alpha}(E-E_0))-1} \tag{1.43}$$

The Plank factor in the response function indicates that an accelerated observer perceives the Minkowski vacuum as a thermal bath at a temperature

$$k_B T = \frac{\alpha}{2\pi} \tag{1.44}$$

which is the famous Unruh temperature. This result shall be discussed in more detail later on but now we shall try to understand what does this physically imply.

The inertial particle detector register no quanta while the accelerated particle detector will 'see' a flux of thermal radiation. Certainly both will agree upon locally defined tensor quantities which can be related to the others frame via the usual tensor transformations. In particular in the inertial x^μ and accelerated frames y^μ the stress tensor is related as

$$T'_{\mu\nu}(y) = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}T_{\alpha\beta}(x) \tag{1.45}$$

and since $\left\langle 0_M\left|T_{\mu\nu}(x)\right|0_M\right\rangle = 0$ this implies $\left\langle 0_M\left|T'_{\mu\nu}(y)\right|0_M\right\rangle = 0$. This leads to the interpretation of the particles detected by the accelerated observer as 'fictitious' but this only illustrates that the particle concept is applicable in a very restrictive sense. The agency which causes the acceleration of the detector is responsible for supplying energy to the fields as well as the detector. The agent of acceleration needs to do work against the resistance offered by the production of field quanta against the accelerating detector. The vacuum field acts as a "mediator" borrowing energy from the agent and exciting the detector as if it were in a thermal bath. The acceleration required to produce a measurable temperature is extremely large and therefore the Unruh effect seems impossible to measure at least in the near future.

1.5 Adiabatic Vacuum

Referring to the examples discussed previously we see that from (1.30) and (1.36) in the $m = 0$ case we have no particle production which are examples of a much wider class called conformally trivial situations that refer to a conformally invariant field propagating in a conformally flat spacetime. The production of particles is a consequence of breaking the conformal symmetry by the introduction of a mass scale in the problem.
In curved spacetimes there is no "natural" definition of particles available unless there is some degree of symmetry. But in a FRW universe as a consequence of its symmetries we do have a privileged class of observers-the comoving observers who see the universe expand isotropically. We then expect to be able to define particles as the excitations of the comoving detectors.
Even if we are able to define some notion of particles their number would be ill defined and its subsequent measurement makes it uncertain. If $|A|$ is the average rate of particle production over an interval $\triangle t$ then to get the precise number of particles we need to choose $\triangle t$ such that $|A|.\triangle t << 1$. Because of the time energy uncertainty choosing a small time interval inherently means using large energy measurements which would cause excitation of field modes and particle production. On the other hand choosing a large time interval allows $|A|.\triangle t$ to be large enough to cause uncertainty in measurement of the precise number of particles. Thus the total uncertainty over time $\triangle t$ is

$$\triangle N > |A|.\triangle t + (m.\triangle t)^{-1}$$

which has a minimum value of $2(\frac{|A|}{m})^{\frac{1}{2}}$ which is non vanishing as long as $|A| \neq 0$ or $m \neq \infty$. This suggests that if we either have a very low average production rate or high mass or high momentum particles we can hope for a well defined notion of particle number.
As the creation rate and particle production density are controlled by the dynamics of spacetime expansion we expect that in the limit of weak expansion we recover Minkowski space QFT. For very weak expansion $\rho \to 0$ we

have

$$|\beta_k|^2 \to \exp(-2\pi\frac{\omega_{in}}{\rho}) \to 0 \tag{1.46}$$

Based on intuition one expects the expansion to excite field modes for which $\omega \approx$ expansion rate. For high k or large m modes $\omega >>$ expansion rate and such modes are inefficiently excited as large energy from the changing gravitational fields during expansion needs to be pumped into the particles rest energy which is unfavourable. This implies that if we started out in the *in* vacuum $|0_{in}\rangle$ which is vacuous in all modes then in the *out* region the detection of particles in high k or m modes will be extremely infrequent. The slower the rate of expansion the higher the probability that a given mode will be devoid of quanta. Thus the approximation to neglect quanta during expansion becomes more and more accurate with increasing energy. However for lower energy modes the detector would register quanta during expansion and a useful approximation of a vacuum state would break down.

We shall put all of this intuitive notions on a firm footing by giving a precise mathematical description. We here shall consider conformally coupled scalar fields in spatially flat FRW universe with the line element

$$ds^2 = C(\eta)\left[d\eta^2 - (dx^i)^2\right] \tag{1.47}$$

and field modes as

$$u_k(\eta, x) \propto \exp(ik.x)\chi_k(\eta) \tag{1.48}$$

where χ_k satisfies

$$\frac{\partial^2\chi(\eta)}{\partial\eta^2} + [k^2 + C(\eta)m^2]\chi(\eta) = 0 \tag{1.49}$$

This equation can be solved using WKB method and possesses solutions of the WKB type

$$\chi_k(\eta) = \frac{1}{\sqrt{2W_k}}\exp\left[-i\int W_k(\eta)d\eta\right] \tag{1.50}$$

where W_k satisfies the non linear equation

$$W_k^2 = \omega_k^2 - \frac{1}{2}\left(\frac{W_k''}{W_k} - \frac{3}{2}\frac{W_k'^2}{W_k^2}\right) \tag{1.51}$$

where the prime denotes differentiation with respect to η.If the spacetime is slowly varying all higher order derivatives of W can be neglected and in the $C(\eta) \to$ constant case get back the Minkowski modes.

We shall quantify the slowness of the expansion by the introduction of the adiabatic parameter T. We define $\eta_1 = \eta/T$ and study expansion in the limit

$T \to \infty$ which reproduces the effects of a slowly varying spacetime and take $T = 1$ at the end of the calculations. The equation of motion now becomes

$$\frac{\partial^2 \chi(\eta_1)}{\partial \eta_1^2} + T^2[k^2 + C(\eta_1)m^2]\chi(\eta_1) = 0 \tag{1.52}$$

An expansion of quantities in the inverse powers of T is called an adiabatic expansion and a term of order T^{-n} is said to be of the nth adiabatic order. We can solve (1.51) iteratively to any adiabatic order and using (1.50) denote this as χ_k^A where A is the adiabatic order.
If one uses adiabatic approximate solution of order A then this would differ from the exact solution only by higher adiabatic orders i.e. by terms of order $A + 1$. Then one can proceed to define *exact* mode solutions to the field equation as

$$u_k = \alpha_k^A(\eta)u_k^A + \beta_k^A(\eta)(u_k^A)^* \tag{1.53}$$

where α and β are constants upto order A which we can choose at some time η_0 as

$$\alpha_k^A(\eta_0) = 1 + O(T^{-(A+1)}) \tag{1.54}$$

$$\beta_k^A(\eta_0) = 0 + O(T^{-(A+1)}) \tag{1.55}$$

Here even though u_k is called a positive frequency mode of adiabatic order A it is an exact solution of the field equation and not some approximate solution.
Now if instead of using exact solutions that reduce to the standard Minkowskian *in* modes in the far past we choose to match an exact solution to an *Ath* order adiabatic approximation at some finite time η_0 then these exact solutions will not reduce to the standard *in* modes in the far past and will in general be a combination of positive and negative frequency modes. A vacuum state defined with respect to these modes is called the "Adiabatic Vacuum" and in general an inertial particle detector will register particles in this state. Nevertheless the number spectra will fall off at high energies as $k^{-(A+1)}$ or $m^{-(A+1)}$ emphasising the fact that these modes differ from the standard *in* modes which are vacuous in all modes k only by terms of adiabatic order higher than $A + 1$. Thus a comoving detector will register no quanta in the high m modes in the adiabatic vacuum with high probability.
Let us now summarize this rather important section before moving ahead.
1. As mentioned above the adiabatic vacuum is an exact state of the field ϕ and not some approximate state based on approximate field modes. These states are defined as vacuum states with respect to exact solutions of the field equation which are matched at some finite time to an approximate solution of order A. These states are perfectly valid quantum states as far as the

theory is concerned.
2. There is no unique *Ath* order adiabatic vacuum. The matching of the exact modes with the approximate modes may make take place at different times. Thus these associated exact modes will differ from each other only by terms of order greater than A. All these modes can be quantized to give equally respectable adiabatic vacuum states which will have identical high energy behaviour but will differ in the structure for low energy modes.
The high frequency modes of a massive scalar field are insensitive to the time dependence of a slowly expanding FRW universe.
This is a consequence of the fact that the high frequency modes or short wavelength modes only probe local behaviour as in an adiabatically expanding universe the metric changes slowly with time. But the long wavelength modes or the low frequency modes can probe the entire manifold and thus are sensitive to the global structure as well as the quantum states used in the adiabatic construction.

1.6 Adiabatic expansion of Green Functions

In this section following the works of Dewitt(1965) and Schwinger(1951) and motivation from the previous section we explore the high frequency or short wavelength behaviour of Green functions such as $G_F(x,x')$ in the limit $x \to x'$.
We introduce normal coordinates y^μ for the point x and assume the origin to be at x' i.e. $y^\mu = 0$. We expand around the origin for small y and solve the Green function equation by iteration to arbitrary adiabatic order.

$$\left[\bigtriangledown + m^2 + \xi R(x)\right] G_F(x,x') = -(\sqrt{-g})^{-\frac{1}{2}}\delta(x-x') \tag{1.56}$$

The result by Petrov(1969) to adiabatic order four is

$$G_F(x,x') \approx \int \frac{d^n k}{(2\pi)^n} e^{-iky}\left[a_0 + a_1\left(\frac{-\partial}{\partial m^2}\right) + a_2\left(\frac{-\partial}{\partial m^2}\right)^2\right](k^2-m^2)^{-1} \tag{1.57}$$

where

$$a_0(x,x') = 1 \tag{1.58}$$

$$a_1(x,x') = (\frac{1}{6}-\xi)R - \frac{1}{2}(\frac{1}{6}-\xi)R_{;\alpha}y^\alpha - \frac{1}{3}a_{\alpha\beta}y^\alpha y^\beta \tag{1.59}$$

$$a_2(x,x') = \frac{1}{2}(\frac{1}{6}-\xi)^2R^2 + \frac{1}{3}a^\alpha_\alpha \tag{1.60}$$

and

$$a_{\alpha\beta} = \frac{1}{2}(\xi - \frac{1}{6})R_{;\alpha\beta} + \frac{1}{120}R_{;\alpha\beta} - \frac{1}{40}R_{\alpha\beta;\gamma}y^{\gamma} + \frac{1}{60}R^{\gamma\delta}_{\alpha\beta}R_{\gamma\delta} + \frac{1}{60}R^{\gamma\delta\nu}_{\alpha}R_{\gamma\delta\nu\beta} \quad (1.61)$$

Defining a half of the square of proper distance between x and x' by

$$\sigma(x, x') = \frac{1}{2}y^{\alpha}y_{\alpha} \quad (1.62)$$

We can write the integral representation of $(k^2 - m^2)^{-1}$ and carry out the k integral in (1.57) explicitly one then can write the final result as called the Dewitt Schwinger representation of the Green function as

$$G_F^{DS}(x, x') = -i\frac{1}{(4\pi)^{n/2}}\triangle^{\frac{1}{2}}(x, x')\int d(is)(is)^{-\frac{n}{2}}F(x, x'; is)e^{-im^2 s + \frac{\sigma}{is}} \quad (1.63)$$

where $\triangle$ is the Van Vleck determinant defined as

$$\triangle(x, x') = -\det[\partial_{\mu}\partial_{\nu}\sigma(x, x')][g(x)g(x')]^{-\frac{1}{2}} \quad (1.64)$$

and the asymptotic adiabatic expansion of $F(x, x'; is)$ is given as

$$F(x, x'; is) \approx \sum_{j=0}^{\infty} a_j(x, x')(is)^j \quad (1.65)$$

This is the Dewitt Schwinger representation of the Green function and is intended to be an exact representation if the exact form of $F(x, x'; is)$ is known. But if (1.65) is used then we only get an asymptotic expansion in the limit of large adiabatic parameter T.

1.7 Conformal Vacuum

In curved spacetimes,in general,the existence of particular mode solutions is of little physical significance as the vacuum state defined by these modes are not devoid of quanta and do not concur with the inertial observer's experience of vacuum state as absence of particles. A special case of interest is that of conformal triviality i.e. a conformally invariant field propagating in a conformally flat spacetime. There is some symmetry defined through the existence of conformal Killing vectors and thus modes defined with respect to these in some sense appear "natural".
The metric tensor of such a spacetime can be cast into the form

$$g_{\mu\nu} = \Omega^2(x)\eta_{\mu\nu} \quad (1.66)$$

where $\eta_{\mu\nu}$ is the metric of the flat space.
The conformally invariant field equation

$$[\bigtriangledown + \xi R(x)]\,\phi(x) = 0 \tag{1.67}$$

with $m = 0$ and ξ chosen as the conformal coupling under the conformal transformation

$$g_{\mu\nu} \to \Omega^{-2}(x) g_{\mu\nu} = \eta_{\mu\nu} \tag{1.68}$$

becomes

$$\bigtriangledown\bar{\phi}(x) = 0 \tag{1.69}$$

where $\bar{\phi}$ is the conformally weighted field with a conformal weight of $\frac{d-2}{2}$ where d is the dimension of space and

$$\bar{\phi} = \Omega^{\frac{d-2}{2}}\phi \tag{1.70}$$

Equation (1.69) is a simple plane wave equation with mode solutions

$$\bar{u}_k(\eta, n) = \frac{1}{\sqrt{2\omega}}\exp(-i\omega\eta + ikx) \tag{1.71}$$

Thus we can define a mode function decomposition of ϕ using (1.70) as

$$\phi = \Omega^{\frac{2-d}{2}}\sum_k [a_k\bar{u}_k + a_k^\dagger(\bar{u}_k)^*] \tag{1.72}$$

and define a vacuum state known as the conformal vacuum state as

$$a_k\,|\bar{0}\rangle = 0 \tag{1.73}$$

Using the definition of Green functions and (1.70) we can related the Green functions in the conformal space to those in the flat space as

$$D_F(x, x') = \Omega^{\frac{2-d}{2}}(x)\bar{D}_F(x, x')\Omega^{\frac{2-d}{2}}(x') \tag{1.74}$$

We shall now explore the experiences of a comoving particle detector in four dimensional FRW universe in the conformal vacuum state whose line element is given by (1.47). The Wightman Green function in the two spacetimes for a comoving observer whose proper time in Minkowski space is the same as cosmic time $\tau = t$ and in the conformal spacetime is related to the conformal time as $\tau = C^{\frac{1}{2}}(\eta)\eta$ are related by an equation similar to (1.74) and thus the Green function in the conformally flat space time is

$$D^+(\eta, \eta') = -\frac{C^{-\frac{1}{2}}(\eta)C^{-\frac{1}{2}}(\eta')}{4\pi^2(\eta - \eta')^2} \tag{1.75}$$

This when substituted in (1.40) gives

$$F(E) = -\frac{1}{4\pi^2} \int d\eta' \int d\eta \frac{\exp[-iE \int d\eta C^{\frac{1}{2}}(\eta)]}{(\eta - \eta')^2} \tag{1.76}$$

which in general does not vanish for an arbitrary conformal factor $C(\eta)$. There are thus important lessons learnt in this simple analysis that a co-moving particle detector will in general register particles in the conformal vacuum. But since the spacetime does have conformal Killing vectors a positive frequency mode defined with respect to the conformal timelike Killing vector ∂_η will remain a positive frequency mode for all time. Thus there is no particle production and the expansion of FRW universe does not create any new massless particles.

Chapter 2

Examples in Flat Spacetime

In this chapter we will review three main topics where the techniques of curved space QFT has been used to make sense of non trivial geometrical effects even though the geometry is flat.
As the first example we shall discuss the Casimir effect which appears as the force of attraction between two parallel neutral conducting plates as a result of the disturbance of electromagnetic vacuum energy. We shall show that $\langle T_{\mu\nu} \rangle$ is non zero even for vacuum.
In the second section we shall treat boundary surfaces which constrain the quantum field as moving or non stationary. This simple case of 'moving mirror' which results in particle production is a heuristic model for more complicated systems and is geometrically equivalent to the case of particle production from black holes.
As a final example we shall study the Rindler observer who is an uniformly accelerating observer in Minkowski vacuum. We shall find that the observer perceives the vacuum states as a thermal bath of radiation at a temperature that is proportional to his acceleration.

2.1 Cylindrical Two-dimensional Spacetimes

We generalize of Minkowski space QFT by introducing non trivial topological structures. We shall introduce compactification of spatial sections where we change from $R^1 \times R^1$ to $R^1 \times S^1$ in 2 dimensions and we identify the points x and $x + L$ where L is the periodicity length.
If we impose periodic boundary conditions on the field modes then we get a discrete set of mode functions

$$u_k = (2L\omega)^{-1/2} \exp(ikx - i\omega t) \tag{2.1}$$

where $k = \frac{2\pi n}{L}$ and $n = 0, \pm1, \pm2, \pm3...$
If we consider the massless case $m = 0$, we see that $\omega = |k|$ and thus for modes with positive n have $\exp(-ik(t-x)) = \exp(-iku)$ with $u = t - x$ representing right moving or retarded waves and for modes with negative n have $\exp(-ik(t+x)) = \exp(-ikv)$ with $v = t + x$ representing left moving ar advanced waves.
In two dimensions the Cartesian components of the stress tensor is given by

$$T_{tt} = T_{xx} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial x}\right)^2 \tag{2.2}$$

$$T_{tx} = T_{xt} = \frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial t} \tag{2.3}$$

If we denote the vacuum associated with the mode (1.1) as $|0_L\rangle$ with the property that as $L \to \infty$ we have $|0_L\rangle \to |0\rangle$ being the usual Minkowski vacuum we observe that

$$< T_{tt} >_L = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n \tag{2.4}$$

which is clearly divergent. This is expected as only the long wavelength modes of the field ϕ are sensitive to changes in geometry while the short wavelength modes still show the same UV divergence as in the Minkowski space.
The UV divergence can be removed by normal ordering which for a general state ψ can be written as

$$\langle\psi\,|:T_{\mu\nu}:|\,\psi\rangle = \langle\psi\,|T_{\mu\nu}|\,\psi\rangle - \langle 0\,|T_{\mu\nu}|\,0\rangle \tag{2.5}$$

which guarantees that $\langle 0\,|:T_{\mu\nu}:|\,0\rangle = 0$. Regarding $|0_L\rangle$ as a general state in the Fock space of the Minkowski operators we have

$$\langle 0_L\,|:T_{tt}:|\,0_L\rangle = \langle 0_L\,|T_{tt}|\,0_L\rangle - \lim_{L'\to\infty}\langle 0_{L'}\,|T_{tt}|\,0_{L'}\rangle \tag{2.6}$$

Since both terms on the RHS are divergent we regularize them using a cut-off factor of $\exp(-\alpha|k|)$ and let $\alpha \to 0$ at the end of the calculations. This yields

$$\langle 0_L\,|T_{tt}|\,0_L\rangle = \frac{1}{2\pi\alpha^2} - \frac{\pi}{6L^2} + O(\alpha^3) \tag{2.7}$$

Thus

$$\lim_{L'\to\infty}\langle 0_{L'}\,|T_{tt}|\,0_{L'}\rangle = \frac{1}{2\pi\alpha^2} \tag{2.8}$$

and

$$\langle 0_L |: T_{tt} :| 0_L \rangle = \langle 0_L |: T_{xx} :| 0_L \rangle = -\frac{\pi}{6L^2} \tag{2.9}$$

Further it can be seen using the same procedure that $\langle 0_L |: T_{tx} :| 0_L \rangle = 0$.
In conclusion we see that $|0_L\rangle$ is a state that contains a finite negative energy density and pressure $\rho = p = -\frac{\pi}{6L^2}$ with a negative vacuum energy of $E = -\frac{\pi}{6L}$.

2.1.1 Casimir Effect

In this section we shall briefly discuss one of the few results of curved space QFT that have actually been verified experimentally- The Casimir effect. The finite shift in the zero point energy of the vacuum fluctuations of the quantized electromagnetic field is measurable and the force of attraction is given by

$$F(L) = -\frac{\mathrm{d}\triangle E(L)}{\mathrm{d}L} \tag{2.10}$$

where L is the distance between the plates and $\triangle E(L) = L \langle 0_L |: T_{tt} :| 0_L \rangle$ which is the shift in zero point energy.
We shall consider a simplified version of the original calculation considering a two dimensional massless scalar field ϕ between the plates at $x = 0$ and $x = L$ demanding that the fields vanish at these plates $\phi(t, x = 0) = \phi(t, x = L) = 0$. Following equations from (2.4) through (2.9) we get the regularized value as $-\frac{\pi}{6L^2}$ and calculating $F(L)$ we get

$$F(L) = -\frac{\pi}{6L^2} \tag{2.11}$$

The negative sign implies that there is a force of attraction between the two electrically neutral conducting plates and a similar calculation in four dimensions reveals

$$F(L) = -\frac{\pi^2}{240L^4} \tag{2.12}$$

which is the celebrated result of Casimir.

2.2 Use of Green Functions

In this section we shall derive the results stated in the previous section by the use of Green functions.
As a matter of convenience we shall work with the null coordinates (u, v)

which lead to the definition of the stress tensor $T_{\mu\nu}$ in two dimensional scalar field theory as follows

$$T_{uu} = \left(\frac{\partial\phi}{\partial u}\right)^2 T_{vv} = \left(\frac{\partial\phi}{\partial v}\right)^2 \tag{2.13}$$

$$T_{uv} = T_{vu} = \frac{1}{2}\left(\frac{\partial\phi}{\partial u}\right)\left(\frac{\partial\phi}{\partial v}\right) \tag{2.14}$$

These are related to the Cartesian components (t, x) by

$$T_{tt} = T_{uu} + T_{vv} + 2T_{vu} \tag{2.15}$$

$$T_{xx} = T_{uu} + T_{vv} - 2T_{vu} \tag{2.16}$$

$$T_{tx} = T_{tx} = T_{vv} - T_{uu} \tag{2.17}$$

Using the general definition of the stress tensor in terms of the Green functions we have

$$\langle 0_L |T_{uu}| 0_L\rangle = \lim_{v'',v'\to v} \lim_{u'',u'\to u} \frac{\partial}{\partial u''}\frac{\partial}{\partial u'}\frac{1}{2}D_L^{(1)}(u', v'; u'', v'') \tag{2.18}$$

We can evaluate the Green function $D^{(1)}$ as

$$D_L^{(1)}(u', v'; u'', v'') = -\frac{1}{4\pi}\ln\left[16\sin^2(\pi\frac{u''-u'}{L})\sin^2(\pi\frac{v''-v'}{L})\right] \tag{2.19}$$

Now the renormalized stress tensor with the appropriate infinite vacuum energy removed can be written as

$$\langle 0_L |: T_{uu} :| 0_L\rangle = \langle 0_L |T_{uu}| 0_L\rangle - \lim_{L'\to\infty}\langle 0_{L'} |T_{uu}| 0_{L'}\rangle \tag{2.20}$$

We then arrive at the finite results that as expected match with the results of the previous sections. We shall anyways state them here for the sake of brevity.

$$\langle 0_L |: T_{uu} :| 0_L\rangle = \langle 0_L |: T_{vv} :| 0_L\rangle = -\frac{\pi}{12L^2} \tag{2.21}$$

$$\langle 0_L |: T_{uv} :| 0_L\rangle = \langle 0_L |: T_{uv} :| 0_L\rangle = 0 \tag{2.22}$$

These results are easy to understand as $D_L^{(1)}$ can be written as a sum of v independent and u independent functions and from its symmetric properties under the exchange of u and v.

2.3 Moving Mirrors

Here we shall review a dynamic effect i.e. creation of particles due to the motion of the boundary that constrains the quantum fields. As we shall see the acceleration of the boundary is is responsible for the excitation of field modes that appear to an inertial detector as a flux of thermal radiation. To simplify calculations we shall treat the problem in two dimensions following closely the treatment of Fulling and Davies(1976).
Let us assume that the mirror remains at rest until $t = 0$ and then follows a trajectory $x = z(t)$ The trajectory is illustrated in the picture below

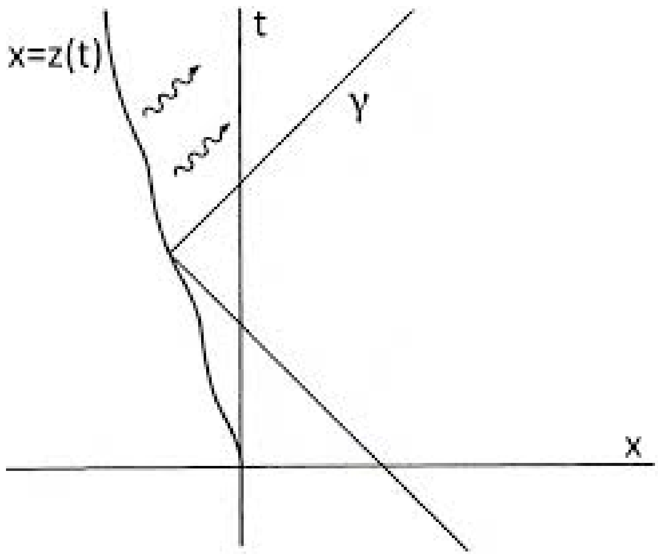

Figure 2.1: Radiation from a moving mirror

In the conformally trivial case we have $m = 0$ and $\xi = \xi(2) = 0$. The field equation becomes

$$\nabla\phi = \frac{\partial}{\partial u}\frac{\partial\phi}{\partial v} = 0 \tag{2.23}$$

with the reflection boundary condition

$$\phi(t, z(t)) = 0 \tag{2.24}$$

This equation has the mode solution

$$u_k^{in}(u, v) = \frac{i}{\sqrt{4\pi\omega}}(\exp(-i\omega v)) - exp(-i\omega(2\tau - u)) \tag{2.25}$$

where τ is determined implicitly by the trajectory as $\tau - z(\tau) = u$.
The left moving or incident waves come in from $\Im^-$ as standard exponential modes $\exp(-i\omega v)$ and reflect off the moving mirror suffering a Doppler shift in the outgoing or right moving modes that reach $\Im^+$ as $\exp(-i\omega(2\tau - u))$. We may now do the field mode expansion in terms of u_k^{in} and define the *in* vacuum in the standard way.

$$\phi = \sum_k [a_k u_k^{in} + a_k^\dagger (u_k^{in})^*] \tag{2.26}$$

$$a_k \left|0_{in}\right\rangle = 0 \tag{2.27}$$

We observe that at early times i.e. $t < 0$ the in modes reduce to the standard positive frequency Minkowski space modes

$$u_k^{in}(u,v) = \frac{i}{\sqrt{4\pi\omega}}(\exp(-i\omega v)) - \exp(-i\omega u)) = (\pi\omega)^{-\frac{1}{2}} \sin(\omega x) \exp(-i\omega t) \tag{2.28}$$

Thus the state $\left|0_{in}\right\rangle$ represents the true physical vacuum and an inertial particle detector would fail to detect any particle.
As the mirror undergoes an acceleration at $t = 0$ the modes suffer a sudden change which is encapsulated by the Doppler shift in these modes. The right moving wave in this region for $t > 0$ is represented by the piece $exp(-i\omega(2\tau - u))$ and thus the state $\left|0_{in}\right\rangle$ no longer represents the true vacuum and an inertial detector would register quanta now. The physical reason for this is easy to understand. As the mirror accelerates from rest to motion at $t = 0$ the agent responsible for acceleration of the mirror supplies the energy that excites the field modes that an inertial detector would perceive as a flux of quanta.
We shall now emphasis on the point made in the above paragraph using a toy model but it is powerful enough to understand the situation of a collapsing star to form a black hole which shall be dealt later.
We assume that the mirror follows the trajectory

$$z(t) \to B - t - Ae^{-2kt}, t \to \infty \tag{2.29}$$

where $A, B, k > 0$ and are all constants. We see that as $t \to \infty$

$$z(t) + t = v \to B \tag{2.30}$$

Thus the trajectory asymptotes to the null ray $v = B$ and only null rays with $v < B$ can be reflected off the mirror, rays for which $v > B$ continue undisturbed to $\Im^+$ not contributing to the thermal radiation.

The evolution of null rays of constant u when traced back from $\Im^+$ and reflected off the mirror all the way to $\Im^-$ seem to crowd around the asymptote $v = B$.

With this insight we wish to study the nature of flux from the Bogolubov transformation between u_k^{in} and u_k^{out} where the modes u_k^{in} have a simple exponential form on $\Im^-$ and a complicated form on $\Im^+$ and vice-versa for u_k^{out}.

The modes u_k^{out} have a simple form $e^{-i\omega u}$ on $\Im^+$ and a complicated form $e^{-i\omega f(v)}$ on $\Im^-$ where the form of f(v) depends upon the trajectory used and for our particular choice it turns out to be

$$f(v) \to k^{-1} \ln \left[\frac{(B-v)}{A} \right] - B, v < B \tag{2.31}$$

and for $v > B$ from the preceding argument we set $u_k^{out} = 0$. We now have the modes u_k^{out} and u_k^{in} expressed as

$$u_k^{out} = \int_{t=0} dk' [\alpha_{kk'} u_{k'}^{in} + \beta_{kk'} (u_{k'}^{in})^*] \tag{2.32}$$

$$u_k^{in} = \int_{t=0} dk' [\alpha_{kk'}^* u_{k'}^{out} - \beta_{kk'} (u_{k'}^{out})^*] \tag{2.33}$$

Performing the required integrations we may express the Bogolubov coefficients in terms of ordinary Γ functions as

$$\alpha_{\omega\omega'} \propto \Gamma(1 - i\omega/k) e^{+\pi\omega/2k - i\omega' B} \omega'^{(-i\omega/k)} \tag{2.34}$$

$$\beta_{\omega\omega'} \propto \Gamma(1 + i\omega/k) e^{-\pi\omega/2k + i\omega' B} \omega'^{(+i\omega/k)} \tag{2.35}$$

from which we deduce that

$$|\beta_{\omega\omega'}|^2 = \frac{1}{2\pi k\omega'} \left(\frac{1}{e^{\omega/k_B T} - 1} \right) \tag{2.36}$$

with

$$k_B T = \frac{k}{2\pi} \tag{2.37}$$

The average number of particles in the mode ω diverges logarithmically and this is to be expected since if the mirror accelerates for all time it would accumulate infinite number of quanta in each mode.

2.4 Unruh Effect

We now look at some detail into one of the most celebrated results in curved space QFT- the Unruh effect. It represents the experiences of an uniformly accelerating detector in Minkowski vacuum who sees a particle flux at a temperature proportional to his acceleration. We shall derive this result in two dimensional spacetime for a massless scalar field where we can exploit conformal triviality to simplify calculations
Consider a two dimensional Minkowski space with the metric

$$ds^2 = d\bar{u}d\bar{v} = dt^2 - dx^2 \tag{2.38}$$

Under the following coordinate transformation from (t, x) to (η, ξ)

$$t = \frac{1}{a}\exp(a\xi)\sinh(a\eta) \tag{2.39}$$

$$x = \frac{1}{a}\exp(a\xi)\cosh(a\eta) \tag{2.40}$$

or equivalently

$$\bar{u} = -\frac{1}{a}\exp(-au) \tag{2.41}$$

$$\bar{v} = \frac{1}{a}\exp(av) \tag{2.42}$$

where $a > 0$ is the constant acceleration of the detector and $-\infty < \xi, \eta < \infty$ and (u, v) is the light cone coordinates in the (η, ξ) frame we can write the metric as

$$ds^2 = e^{2a\xi}dudv = e^{2a\xi}(d\eta^2 - d\xi^2) \tag{2.43}$$

From the relations

$$\eta = \frac{1}{a}\tanh^{-1}\left(\frac{t}{x}\right) \tag{2.44}$$

$$\xi = \frac{1}{a}\ln[a^2(x^2 - t^2)] \tag{2.45}$$

We can see that with the above coordinate transformations the (η, ξ) also known as the Rindler coordinates cover only a quadrant also called the Rindler wedge of the entire Minkowski space that for which $|t| < x$ since the arguments of ln needs to be positive. In the (t, x) space lines of constant η are straight lines with slope $\tanh(a\eta)$ and lines of constant ξ are hyperbola with $x^2 - t^2 = a^-2e^{2a\xi}$ =constant. The observer's proper time(τ) and acceleration(α) defined with respect to this time is related to a, ξ, η by

$$\tau = \exp(a\xi)\eta \tag{2.46}$$

$$\alpha = a^{-1} \exp(a\xi) \tag{2.47}$$

The higher the proper acceleration of the observer the closer he approaches $(t = 0, x = 0)$ and as $\eta \to \pm\infty$ the observer approaches the speed of light. This is illustrated in the diagram below

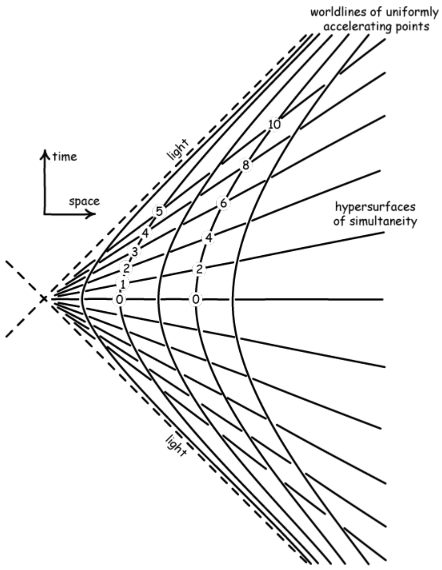

Figure 2.2: Trajectory on a Rindler observer

A second Rindler wedge can be constructed for $|t| > x$ by reversing the direction of time. The incompleteness of coordinates (η, ξ) implies that $\bar{u} = 0, \bar{v} = 0$ act as event horizons and also there are events in the whole Minkowski spacetime that cannot be causally influenced or seen by accelerated observers. We shall now treat the problem of Unruh effect using the approach of Bogolubov transforms.

2.4.1 Bogolubov Transformation Method

Particle detectors register particles as excitations of the vacuum field which are defined with respect to a particular set of mode functions. Further these mode functions are defined to be positive or negative frequency modes with respect to the proper time of the comoving observer. In the case of the inertial observer in Minkowski space these modes are defined with respect to the cosmic time t but for an accelerated observer they are defined with respect to his proper time τ defined above. Since there is a non trivial relation between t and τ we expect that a positive frequency mode with respect to t must comprise of positive and negative frequency modes with respect to

τ. In other words the Bogolubov transformations are non trivial and the Minkowski vacuum appears to an accelerated observer as containing Rindler particles.
The action for a two dimensional massless scalar field theory is conformally trivial i.e. it is the same under a conformal transformation$g_{\mu\nu} \to \Omega^2 g_{\mu\nu}$.

$$S[\phi] = \frac{1}{2} \int d^2x \sqrt{-g} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \tag{2.48}$$

Since the Rindler space is conformally related to the Minkowski space we expect the action to look to same in both spacetimes.

$$S[\phi] = \frac{1}{2} \int dt dx \left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{1}{2} \int d\eta d\xi \left(\frac{\partial \phi}{\partial \eta}\right)^2 - \left(\frac{\partial \phi}{\partial \xi}\right)^2 \tag{2.49}$$

Rewriting this action in lightcone coordinates we have

$$S[\phi] = \int du dv \left(\frac{\partial \phi}{\partial v}\right) \left(\frac{\partial \phi}{\partial u}\right) = \int d\bar{u} d\bar{v} \left(\frac{\partial \phi}{\partial \bar{v}}\right) \left(\frac{\partial \phi}{\partial \bar{u}}\right) \tag{2.50}$$

The field equations $\phi_{,uv} = 0$ and $\phi_{,\bar{u}\bar{v}} = 0$ now have simple solutions

$$\phi(u, v) = A(u) + B(v) \tag{2.51}$$

$$\phi(\bar{u}, \bar{v}) = \bar{A}(\bar{u}) + \bar{B}(\bar{v}) \tag{2.52}$$

where A, B... are arbitrary smooth functions. We can decompose the field into left moving and right moving modes since they do not affect each other and can be treated separately. Writing the field ϕ in the domain of overlap in the first quadrant

$$\phi = \sum_k [a_k \bar{u}_k + a_k^\dagger (\bar{u}_k)^*] \tag{2.53}$$

$$\phi = \sum_k [b_k u_k + b_k^\dagger (u_k)^*] \tag{2.54}$$

Where a_k is the annihilation operator on Minkowski vacuum $|0_M\rangle$ and b_k is the annihilation operator on Rindler vacuum $|0_R\rangle$.

$$a_k |0_M\rangle = 0 \tag{2.55}$$

$$b_k |0_R\rangle = 0 \tag{2.56}$$

Using the standard mode relations using Bogolubov transformations between u and $\bar{u}$ as given by equations (1.21) and (1.22) we can compute the coefficients in terms of Gamma functions as

$$\alpha_{\omega\omega'} = +\frac{1}{2\pi a} \sqrt{\frac{\omega}{\omega'}} \exp(+\frac{\pi\omega}{2a}) \exp(\frac{i\omega}{a} \ln \frac{\omega'}{a}) \Gamma\left(\frac{-i\omega}{a}\right) \tag{2.57}$$

$$\beta_{\omega\omega'} = -\frac{1}{2\pi a}\sqrt{\frac{\omega}{\omega'}}\exp(-\frac{\pi\omega}{2a})\exp(\frac{i\omega}{a}\ln\frac{\omega'}{a})\Gamma\left(\frac{-i\omega}{a}\right) \tag{2.58}$$

and the useful relation between them is

$$|\alpha_{\omega\omega'}|^2 = \exp\frac{2\pi\omega}{a}|\beta_{\omega\omega'}|^2 \tag{2.59}$$

The average density of Rindler particles in the Minkowski vacuum is given by the expectation value of the number density operator of Rindler particles in Minkowski vacuum.

$$\langle n_\omega \rangle = \frac{1}{V}\langle N_\omega \rangle = \frac{1}{V}\left\langle 0_M \left| b_\omega^\dagger b_\omega \right| 0_M \right\rangle \tag{2.60}$$

$$\langle n_\omega \rangle = \left[\exp\left(\frac{2\pi\omega}{a}\right) - 1\right]^{-1} \tag{2.61}$$

Thus the Rindler particles as detected by an accelerating observer in the Minkowski vacuum is seen to have a Bose Einstein spectra from which we can read off the Unruh temperature as

$$k_B T = \frac{a}{2\pi} \tag{2.62}$$

Hence we have shown that an accelerating observer sees particles with a thermal spectrum at a temperature directly proportional to his acceleration.

Chapter 3

Stress Tensor Renormalization

We have time and again seen the futility of trying to define a concrete particle concept in general spacetimes as it is a global concept which is influenced by the structure of the entire manifold. As already argued it would be worthwhile to study physical quantities that are locally defined as most of the physical detectors probe only small regions of spacetime and are intrinsically local in nature. With this motivation and the fact that we require a self consistent model involving gravitational fields coupled to quantum matter fields we embark upon the study of the stress energy tensor $T_{\mu\nu}$.
As such as seen in examples before $\langle T_{\mu\nu} \rangle$ turns out to be a divergent quantity and to make any physical sense we need to regularize and renormalize these divergences which happen to be the central issue to which this chapter is dedicated.
We shall study various regularization techniques such as dimensional regularization and Zeta function regularization. We shall work with the action rather than $T_{\mu\nu}$ itself as many calculations are simplified at least formally. We shall not worry much as to whether these formal manipulations make any mathematical sense at all. We shall end this chapter by briefly discussing the physical significance of a renormalized stress tensor.

3.1 The fundamental problem

We have earlier encountered the fact that the expectation value of T_{tt} or H even in flat space quantum field theory is infinite and this turns out to be the trend whenever we wish to compute quadratic expectation values of field operators which diverge in the coincidence limit or in other words show UV divergences.
In flat space theory this problem is rather dealt ingeniously be defining nor-

mal ordering or if the topology is non trivial a UV cut-off regulator was used such as the function $\exp(-\alpha|k|)$ where the cut off α was set to 0 at the end of the calculations. As is seen in the previous chapter finite quantities were calculated by subtracting the value of the quantity in Minkowski spacetime form its value in the spacetime of interest. Also in non gravitational physics the quantity of interest which is the physical observable is the difference in energies and so it wouldn't hurt to renormalize the zero point by an infinite amount as long as these differences return finite values.

These techniques do not work in curved spacetimes for two simple reasons. The first one is concerned with the role of $T_{\mu\nu}$ in gravitational physics. This term occupies the right hand side of the Einstein equation and is the source of spacetime curvature whose effects we are trying to study thus we cannot rescale terms at our whim and to study the effects of $T_{\mu\nu}$ on curvature we require a more subtle and elaborate renormalization scheme. Secondly, there are examples where the above simple scheme of subtracting the value of the divergent quantity in Minkowski space from its value in the spacetime of interest fails and divergences in T_{tt} cannot be removed by simply discarding the Minkowski terms.

To get a finite result from an otherwise infinite quantity a subtraction of such terms needs to be carried out and since there are many ways to do it we shall seek general coordinate invariance as a guiding principle. There must be other physically reasonable constraints to define a unique renormalized $T_{\mu\nu}$ which we shall consider later.

Drawing analogy with the semi-classical theory of electromagnetism we consider a semi-classical theory of quantum gravity where the gravitational fields are treated classically and the matter fields are treated quantum mechanically. We shall consider a semi classical theory based on the Einstein equation as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda_B g_{\mu\nu} = -8\pi G_B \langle T_{\mu\nu}\rangle \tag{3.1}$$

where the subscript B stands for bare coefficients whose nomenclature will be justified shortly.

The equation (3.1) can be derived from the action

$$S = S_{grav} + W \tag{3.2}$$

using the condition that the variation with respect to the metric yields the equations of motion i.e

$$\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = 0 \tag{3.3}$$

The term on the LHS is generated by the classical gravitational Einstein Hilbert action

$$S_{grav} = \int d^n x \sqrt{-g} \frac{1}{16\pi G_B}(R - 2\Lambda_B) \tag{3.4}$$

and the term on the RHS is generated by the functional differentiation of the effective action$W$ of the matter fields defined as

$$\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} = \langle T_{\mu\nu} \rangle \tag{3.5}$$

We shall use the path integral quantization of QFT to discuss the structure of W. Defining the generating functional as the transition amplitude from *in* vacuum to *out* vacuum in the presence of an external current $J(x)$ we have

$$Z[J] = \int D[\phi] \exp[iS_m[\phi] + i \int J(x)\phi(x) d^n x] = \langle 0_{out} | 0_{in} \rangle \tag{3.6}$$

and noting the variation of Z[0] as

$$\delta Z[0] = i \int D[\phi] \delta S_m \left\{ i \exp S_m[\phi] \right\} = i \left\langle 0_{out} \left| \delta S_m \right| 0_{in} \right\rangle \tag{3.7}$$

and defining

$$W = -i \log Z[0] = -i \log \langle 0_{out} | 0_{in} \rangle = -\frac{1}{2} itr[\log(-G_F)] \tag{3.8}$$

where G_F is the Feynman Green function defined as an operator the acts on a vector space $|x\rangle$ which is normalized as

$$\langle x | x' \rangle = \frac{1}{\sqrt{-g(x)}} \delta(x - x') \tag{3.9}$$

and

$$G_F(x, x') = \langle x \left| G_F \right| x' \rangle \tag{3.10}$$

with the trace of any operator A is defined as

$$trA = \int d^n x \sqrt{-g(x)} A(x, x) = \int d^n x \sqrt{-g(x)} \langle x \left| A \right| x \rangle \tag{3.11}$$

finally we have

$$\frac{2}{\sqrt{-g}} \frac{\delta W}{\delta g^{\mu\nu}} = \frac{\langle 0_{out} \left| T_{\mu\nu} \right| 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle} \tag{3.12}$$

Using the Dewitt Schwinger (DS) representation of Feynman Green function we can simplify the formal expressions further and write W the effective action as

$$W = \frac{1}{2} i \int dm^2 \int d^n x \sqrt{-g} G_F^{DS}(x, x) \tag{3.13}$$

where G_F^{DS} is discussed in section 1.6. From this we may also define the effective Lagrangian L_{eff} as

$$W = \int d^n x \sqrt{-g} L_{eff}(x) \tag{3.14}$$

hence

$$L_{eff}(x) = \lim_{x \to x'} \frac{1}{2} i \int dm^2 G_F^{DS}(x, x') \tag{3.15}$$

Using the Dewitt Schwinger expansion (1.63) and (1.65) we find that the potentially divergent terms in four dimensions is given by

$$L_{div} = - \lim_{x \to x'} \frac{\triangle^{\frac{1}{2}}(x, x')}{32\pi^2} \int \frac{ds}{s^3} e^{-im^2 s + i\frac{\sigma}{2s}} [a_0 + a_1(is) + a_2(is)^2] \tag{3.16}$$

The other terms in the expansion are finite in the limit as $x \to x'$. We observe from (1.58) to (1.60) that all divergent terms are completely local and geometrical and this feature is expected as we are interested in the short wavelength modes that are only affected by local geometry and are insensitive to the quantum state used or large scale structure of the manifold. Thus though L_{div} is generated from the UV behaviour of the quantum fields due to its entirely geometrical nature we can regard it as a contribution to the gravitational piece and use it to renormalize the gravitational coupling constants and this will be done in the next section.

3.2 Renormalization in the Effective action

In this section we shall spend considerable time with dimensional regularization and renormalization and in the passing look at other methods of regularization.

We shall start with dimensional regularization. Our aim is to set up (3.16) in the form of ∞x geometrical terms such that we can compare them with the classical gravitational Lagrangian.

3.2.1 Dimensional Regularization

We shall treat the dimensionality of space as a complex variable and perform an adiabatic expansion of (3.15) using (1.65) and take the limit $x \to x'$ which in d dimensions we observe results in divergence of the first $\frac{d}{2}+1$ terms. We can express L_{eff} as

$$L_{eff} = \frac{1}{2}(4\pi)^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(x)(m^2)^{d/2-j}\Gamma(j-d/2) \tag{3.17}$$

since we intended to retain the dimensionality of L_{eff} as $[length]^{-4}$ it becomes necessary to introduce a mass scale μ and (3.17) can be rewritten as

$$L_{eff} = \frac{1}{2}(4\pi)^{-\frac{d}{2}} \left(\frac{m}{\mu}\right)^{d-4} \sum_{j=0}^{\infty} a_j(x)m^{4-2j}\Gamma(j-d/2) \tag{3.18}$$

and we observe that as $d \to 4$ the first three terms $j = 0, 1, 2$ diverge with poles in the Γ function expressed as

$$\Gamma\left(-\frac{d}{2}\right) = \frac{4}{d(d-2)}\left(\frac{2}{4-d} - \gamma\right) + O(d-4) \tag{3.19}$$

$$\Gamma\left(1-\frac{d}{2}\right) = \frac{2}{(d-2)}\left(\frac{2}{4-d} - \gamma\right) + O(d-4) \tag{3.20}$$

$$\Gamma\left(2-\frac{d}{2}\right) = \left(\frac{2}{4-d} - \gamma\right) + O(d-4) \tag{3.21}$$

We then can collect all terms that diverge near $d = 4$ and upto order $O(d-4)$ we can express them as L_{div} as

$$L_{div} = -(4\pi)^{-2}\left(\frac{1}{4-d} + \frac{1}{2}\left[\gamma + \log\left(\frac{m^2}{\mu^2}\right)\right]\right)\left\{\frac{m^4 a_0}{2} - m^2 a_1 + a_2\right\} \tag{3.22}$$

where a_0, a_1, a_2 are given in section 1.6 which have $y = 0$ in the coincidence limit and γ is the Euler constant. We thus make a very important observation that was motivated in the previous section that L_{div} contains purely local geometrical terms and hence can be absorbed into the gravitational Lagrangian

Thus (3.4) gets modified for L_{grav} as

$$-\left(A + \frac{\Lambda_B}{8\pi G_B}\right) + \left(B + \frac{1}{16\pi G_B}\right)R - \frac{a_2}{(4\pi)^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma + \log\left(\frac{m^2}{\mu^2}\right)\right]\right\} \tag{3.23}$$

where

$$A = \frac{m^4}{32\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma + \log\left(\frac{m^2}{\mu^2}\right)\right]\right\} \tag{3.24}$$

and

$$B = \frac{m^2(\frac{1}{6}-\xi)}{16\pi^2}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma + \log\left(\frac{m^2}{\mu^2}\right)\right]\right\} \tag{3.25}$$

Since the contribution form A is physically indistinguishable from Λ_B and we cannot physically through any experiment measure the bare couplings alone we simply renormalize or change this coupling constant to the physically observed Λ which contributes to the cosmological constant. Similarly the term involving B simply contributes to the renormalization of the Newton's constant G. Thus we have

$$\Lambda = \Lambda_B + \frac{m^4 G_B}{4\pi}\left\{\frac{1}{d-4} + \frac{1}{2}\left[\gamma + \log\left(\frac{m^2}{\mu^2}\right)\right]\right\} \tag{3.26}$$

and

$$G = \frac{G_B}{1 + 16\pi B G_B} \tag{3.27}$$

As for the final term in (3.23) it can be seen from (1.60) that a_2 involves fourth order derivatives of the metric thus it represents higher order corrections to the classical gravitational action which contains terms upto second order derivatives of the metric. When we consider higher order corrections to the gravitational Lagrangian i.e. terms of the type R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ the LHS of (3.1) becomes

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} + \alpha H^{(1)}_{\mu\nu} + \beta H^{(2)}_{\mu\nu} + \gamma H_{\mu\nu} \tag{3.28}$$

where these new tensors involve fourth order derivatives of the metric and the coefficients α , β , γ are all renormalized constants that have the $d-4$ divergences and bare couplings massaged together.
In four dimensions due to the Gauss Bonnet theorem a topological invariant called the Euler number can be constructed

$$\chi = \int d^4x\sqrt{-g(x)}\left[R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right] \tag{3.29}$$

from which we get the relation

$$-H^{(1)}_{\mu\nu} + 4H^{(2)}_{\mu\nu} = H_{\mu\nu} \tag{3.30}$$

This implies that in four dimensions as only two of the three are independent we can set $\gamma = 0$ and claim that α and β need to be experimentally

determined but in a weak gravitational field where the original two derivative action holds good at least as far as experimental sensitivity is concerned we can postulate that the numerical values of these couplings are very small.
Now having removed the divergent pieces from regularization and renormalization of couplings what remains is finite and is symbolically denoted as L_{ren} where $L_{ren} = L_{eff} - L_{div}$ and from (3.18) and (3.22) we have

$$L_{ren} = \frac{1}{32\pi^2} \int \sum_{j=3}^{\infty} a_j(x)(is)^{j-3} \exp(-im^2 s) d(is) \tag{3.31}$$

which may be written after integration by parts as

$$L_{ren} = -\frac{1}{64\pi^2} \int d(is) \log(is) \frac{\partial^3}{\partial(is)^3} [F(x, x; is) e^{-im^2 s}] \tag{3.32}$$

which is ambiguous upto finite renormalization terms. There is a subtle point here,(3.32) can be realized as the renormalized Lagrangian associated with the physical stress tensor $\langle T_{\mu\nu} \rangle$ only if the complete analytical form of $F(x, x'; is)$ is known but if we know only a asymptotical adiabatic expansion for F we cannot make any connection with physical quantities. In the next section we shall take a look at zeta function regularization.

3.2.2 Zeta function regularization

In this method of regularization we seek an eigenfunction expansion of the Feynman Green function and express certain quantities as generalized ζ functions. The advantage of this approach is that there is no need to explicitly renormalize coupling constants as in the previous case. Somehow the formal procedure of analytical continuation is sufficient to get rid of all divergent terms. Here we shall briefly make our case.
We consider a formal eigenfunction expansion of G_F as

$$G_F = -\sum_m \frac{|m\rangle \langle m|}{\lambda_m} \tag{3.33}$$

where the eigenfunctions are orthonormal and form a complete set and

$$(-G_F)^\nu = \sum_m \frac{|m\rangle \langle m|}{\lambda_m^\nu} \tag{3.34}$$

hence we may compute the trace of G_F using (3.11) and the completeness relation of the vectors $|x\rangle$ to be

$$tr(-G_F)^\nu = \sum_m \lambda_m^{-\nu} = \zeta(\nu) \tag{3.35}$$

where $\zeta(\nu)$ is the generalized zeta function which is a close cousin of the Riemann ζ function defined as $\sum_{m=1}^{\infty} m^{-\nu}$.
We now wish to compute the effective action W in (3.8) in terms of ζ functions and to make the argument of the log term dimensionless we introduce a mass scale μ and write

$$W = -i \log Z[0] = -\frac{1}{2} itr[\log(-\mu^2 G_F)] \tag{3.36}$$

using the expansion of a^x for x near 0 we can define the log function as

$$\log(a) = \lim_{x \to 0} \frac{\mathrm{d}(a^x)}{\mathrm{d}x} \tag{3.37}$$

thus we have

$$W = \lim_{\nu \to 0} \left\{ -\frac{1}{2} i\mu^{2\nu} \left[\zeta'(\nu) + \zeta(\nu) \log(\mu^2) \right] \right\} \tag{3.38}$$

In general we cannot guarantee the convergence of $\zeta'(\nu)$ and $\zeta(\nu)$ thus we take the limit of $\nu = 0$ and find that $\zeta'(0)$ and $\zeta(0)$ are finite thus assuming analytical continuation from regions where it converges to the point $\nu = 0$ does get rid of divergences. Thus we have a finite regularized effective action W written as

$$W = -\frac{1}{2} i[\zeta'(0) + \zeta(0) \log(\mu^2)] \tag{3.39}$$

We can show that $\zeta'(0)$ and $\zeta(0)$ are finite by explicitly evaluating the integrals in the Dewitt Schwinger representation of G_F. They turn out to be

$$\zeta(0) = i(4\pi)^{-2} \int d^4x \sqrt{-g} \left[\frac{m^4}{2} a_0 - m^2 a_1 + a_2 \right] \tag{3.40}$$

and

$$\zeta'(0) = -\frac{1}{2} i(4\pi)^{-2} \int d^4x \sqrt{-g} \int d(is) \log(is) \frac{\partial^3}{\partial(is)^3} [F(x, x; is) e^{-im^2 s}] \tag{3.41}$$

hence again using (3.41) and (3.40) in (3.39) and using (3.14) we rederive the result (3.32) that

$$L_{ren} = -\frac{1}{64\pi^2} \int d(is) \log(is) \frac{\partial^3}{\partial(is)^3} [F(x, x; is) e^{-im^2 s}] \tag{3.42}$$

We shall now point out some technical details of renormalization. From (3.5) and (3.12) it is evident that for the RHS of the Einstein equation we get

$$\langle T_{\mu\nu} \rangle = \frac{\langle 0_{out} | T_{\mu\nu} | 0_{in} \rangle}{\langle 0_{out} | 0_{in} \rangle} \tag{3.43}$$

but the renormalization schemes do not make use of any particular vacuum state and these states enter expressions in a purely formal manner which is implicitly controlled by the boundary conditions one imposes on the particular Green functions used. Rather one would be interested in expectation values of the kind $\langle 0_{in} |T_{\mu\nu}| 0_{in}\rangle$ or $\langle 0_{out} |T_{\mu\nu}| 0_{out}\rangle$ but since the high energy behaviour is state independent we expect the same kind of divergences to exist in different forms of the expectation values.
This naive expectation turns out to be true as proved by Dewitt(1975) where one expresses the *out* vacuum $|0_{out}\rangle$ in terms of many particle *in* states and simplify using the action of *in* creation and annihilation operators acting on $|0_{in}\rangle$. The result is

$$\langle 0_{in} |T_{\mu\nu}| 0_{in}\rangle = \frac{\langle 0_{out} |T_{\mu\nu}| 0_{in}\rangle}{\langle 0_{out}|0_{in}\rangle} - i\sum_{ijk} \alpha^{-1}_{ik}\beta_{kj}T_{\mu\nu}(u^*_{in,i}, u^*_{in,j}) \tag{3.44}$$

where α and β are the Bogolubov coefficients of transformation and u_{in} are the in modes with respect to which $|0_{in}\rangle$ is defined. The final term in the RHS of (3.44) is finite thus proving our claim that both sets of expectation values have the same nature of UV divergences.

3.3 Conformal anomalies

In this section we shall focus mainly on conformal anomalies or better known as trace anomalies and show that the trace of the renormalized stress tensor is intimately related to a particular coefficient in the adiabatic regularization. We shall also discuss counter terms in the effective action and their significance in renormalization.
Classically it is simple to show that if the action is invariant under conformal transformations then it is always possible to construct a stress tensor(which sometimes includes the addition of Belifant improvement terms) that is traceless. This is easy to see if we consider a conformal transformation

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \tag{3.45}$$

which under a functional differentiation results in

$$\delta S[\bar{g}_{\mu\nu}] = S[\bar{g}_{\mu\nu}] - S[g_{\mu\nu}] = \int d^n x \delta\bar{g}^{\alpha\beta}\frac{\delta S[\bar{g}_{\mu\nu}]}{\delta\bar{g}^{\alpha\beta}} \tag{3.46}$$

from where using $\delta\bar{g}^{\alpha\beta} = -2\bar{g}^{\alpha\beta}\Omega^{-1}(x)\delta\Omega(x)$ one obtains the trace of the stress tensor using (3.5) as

$$T^\mu_\mu[g_{\alpha\beta}] = g^{\mu\nu}T_{\mu\nu} = -\frac{\Omega(x)}{\sqrt{-g}}\frac{\delta S[\bar{g}_{\alpha\beta}]}{\delta\Omega(x)} \tag{3.47}$$

from which it is clear that if the classical action is conformally invariant then the RHS vanishes thus we have a traceless stress tensor. The presence of a mass scale or any length scale breaks conformal invariance and thus we shall now deal with the regularization and renormalization in the massless limit. In four dimensions it is clear from from the DS expansion of L_{eff} in (3.18) that as we need only concentrate on UV divergent terms i.e. terms for $j = 0, 1, 2$ the first two terms are regular in the limit $m \to 0$ hence we need to only focus on the potentially UV divergent term $j = 2$

$$\frac{1}{2}(4\pi)^{-\frac{d}{2}} \left(\frac{m}{\mu}\right)^{d-4} a_2 \Gamma(2 - d/2) \tag{3.48}$$

working with the conformally coupled $\xi(4) = \frac{1}{6}$ massless scalar field the divergent term in the effective action W in (3.14) may be written as

$$W_{div} = \frac{1}{2}(4\pi)^{-\frac{d}{2}} \left(\frac{m}{\mu}\right)^{d-4} \Gamma(2-d/2) \int d^n x \sqrt{-g} \left[\alpha F + \beta G\right] + O(d-4) \tag{3.49}$$

where

$$F = \frac{1}{3}R^2 - 2R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \tag{3.50}$$

and

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \tag{3.51}$$

and the coefficients are $\alpha = \frac{1}{120}$ and $\beta = -\frac{1}{360}$. This particular decomposition of a_2 in terms of F and G was done because F is the Weyl squared tensor $F = C_{\alpha\beta\mu\nu}C^{\alpha\beta\mu\nu}$ and G is the topological invariant Euler number both of which are conformal invariants thus rendering W_{div} to be conformally invariant. One may using (3.5) and certain identities(Duff 1977)

$$\frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g} F = -(d-4)(F - \frac{2}{3} \bigtriangledown R) \tag{3.52}$$

and

$$\frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^n x \sqrt{-g} G = -(d-4)G \tag{3.53}$$

compute the contribution of W_{div} to the trace of the stress tensor to be

$$\left\langle T^{\mu}_{\mu} \right\rangle_{div} = \frac{1}{16\pi^2}[\alpha(F - \frac{2}{3} \bigtriangledown R) + \beta G] \tag{3.54}$$

Now we can take the massless limit and continue with the finite result we got in (3.54) which is local and state independent.

Now from the previous arguments we know that as W_{eff} is conformally invariant in the massless conformally coupled case it must have a traceless stress tensor which from the fact that $W_{ren} = W_{eff} - Wdiv$ implies that the renormalized stress tensor has a trace that is finite and of the same value as the divergent piece except it is of the opposite sign. Thus

$$\left\langle T^{\mu}_{\mu} \right\rangle_{ren} = -\frac{1}{16\pi^2}[\alpha(F - \frac{2}{3} \bigtriangledown R) + \beta G] = -\frac{a_2}{16\pi^2} \tag{3.55}$$

The classical stress tensor is traceless and in four dimensions W and W_{div} are conformally invariant but we still have a trace anomaly because W_{div} is not conformally invariant away form $d = 4$ though W still is and since we analytically continue our results to $d = 4$ the breaking of symmetry leaves an imprint due to the divergent $(d-4)^{-1}$ nature of W_{div}.
These results can be easily extended to spacetimes of other dimensions. There is no anomaly in odd dimensions as (3.17) is finite. For even dimensions say $d = 2n$ only the first $n+1$ terms of (3.17) are UV divergent as a consequence of poles in the Gamma functions and in the massless limit all but 1 term vanish. The term with a_n survives as it has a mass term m^{d-2n} that does not vanish and its contribution to the trace tensor can be easily computed to be

$$\left\langle T^{\mu}_{\mu} \right\rangle_{ren} = -\frac{a_n}{(4\pi)^n} \tag{3.56}$$

We have the trace anomaly in two dimensions as

$$\left\langle T^{\mu}_{\mu} \right\rangle_{ren} = -\frac{a_1}{4\pi} = -\frac{R}{24\pi} \tag{3.57}$$

It is natural to wonder whether trace anomaly is an unwanted artifact of the choice of Lagrangian and whether there exist counter terms whose addition can completely cancel the anomaly. We could consider a dynamical theory with no anomalies but it would come at the expense of considering extremely complicated actions with no physical motivation available to guess their structure as shown by Brown and Dutton(1978).
In four dimensions we can indeed remove the $\bigtriangledown R$ term from the anomaly (3.55) by adding an R^2 term in the effective action whose variation is given by

$$\frac{2}{\sqrt{-g}} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x \sqrt{-g} R^2 = -12 \bigtriangledown R \tag{3.58}$$

The coefficient of this term can be set to exactly cancel the $\bigtriangledown R$ term but adding this term by hand inadvertently breaks conformal invariance of W_{eff}. There is no compelling theoretical reason to add such counter terms breaking the symmetry of the theory by hand unless guided by some experimental evidence. In general if W is not conformally invariant then the anomaly can contain additional terms like R^2 or $\bigtriangledown R$. In addition there may also be a non-anomalous state dependent part which may be non local too.

3.4 Physical significance of renormalization

Having gone through some amount of rigour and a lot of formal mathematical manipulations intended to extract some finite residue from infinite quantities,in this section we shall discuss the physical basis that in some sense guides and motivates these renormalization procedures and justifies the means required to reach the end. If the criteria are too restrictive we may not be able to write down $\langle T_{\mu\nu} \rangle$ at all and if the criteria are too relaxed we may not be able to uniquely specify such an object. Following the works of Wald(1977) and Christensen(1975) we shall impose a set of physically reasonable axioms that $\langle T_{\mu\nu} \rangle$ must possess. They are:
1.Covariant conservation
2.Causality
3.Standard results for 'off-diagonal' elements
4.Standard results in Minkowski space

The first condition is expected as the LHS of the Einstein equation is divergence-less and thus we impose the RHS must also be divergence-less which in equations becomes the vanishing of the covariant derivative of $\langle T_{\mu\nu} \rangle$

$$\left\langle T^{\nu}_{\mu} \right\rangle_{;\nu} = 0 \tag{3.59}$$

The second condition as Wald(1977) states it is:'For a fixed *in* state,$\langle T_{\mu\nu} \rangle$ at a point p in spacetime depends only on the spacetime geometry of the causal past of p'. A similar statement holds true relating the *out* state and the future null cone. Hence as long as the states in the remote past or future are not modified this axiom states that all we need to care is about the structure of the metric in the future and past null cones.
The third condition requires that between any two orthogonal states $|\Phi\rangle$,$|\Psi\rangle$ such that $\langle \Psi | \Phi \rangle = 0$ the matrix elements of $T_{\mu\nu}$ must be finite and if two stress tensors are related upto improvement terms then their off-diagonal values are equal.

$$\langle \Psi \left| T_{\mu\nu} \right| \Phi \rangle = \left\langle \Psi \left| \bar{T}_{\mu\nu} \right| \Phi \right\rangle \tag{3.60}$$

The fourth requires normal ordering of the stress tensor to be valid in Minkowski space QFT so that divergences can be removed without having to go through

any renormalization procedures.
We shall now use the axioms to prove that the stress tensor defined shall be unique upto a locally conserved tensor.
Let us assume that two different stress tensors that describe the same physical situation $T_{\mu\nu}$ and $\bar{T}_{\mu\nu}$ satisfy the above axioms. To show that they are unique upto a locally conserved stress tensor we define

$$U_{\mu\nu} = T_{\mu\nu} - \bar{T}_{\mu\nu} \tag{3.61}$$

Let $|\Phi\rangle$ and $|\Psi\rangle$ be two orthogonal states. We can construct a basis where using these orthonormal states $|\Pi_{\pm}\rangle = \frac{1}{\sqrt{2}}(|\Psi\rangle \pm |\Phi\rangle)$ and using (3.60) we find that such that

$$\langle \Pi_+ | U_{\mu\nu} | \Pi_- \rangle = 0 \tag{3.62}$$

Thus only diagonal elements of $U_{\mu\nu}$ survive and these turn out to be equal i.e.

$$\langle \Psi | U_{\mu\nu} | \Psi \rangle = \langle \Phi | U_{\mu\nu} | \Phi \rangle \tag{3.63}$$

from which we conclude that $U_{\mu\nu}$ is just some multiple of the identity operator.

$$U_{\mu\nu} = u_{\mu\nu} I \tag{3.64}$$

From (3.62) and the causality condition at a particular spacetime point p we have

$$\langle in | U_{\mu\nu}(p) | in \rangle = \langle out | U_{\mu\nu}(p) | out \rangle = u_{\mu\nu}(p) \tag{3.65}$$

This implies that $u_{\mu\nu}(p)$ depends only on the local geometry as it depends on intersection of the past and future null cones i.e it depends on the point p. It is covariantly conserved which is evident from the first condition.
Thus we have proved our claim that $\langle T_{\mu\nu} \rangle$ is unique upto a local conserved tensor which with all respect can be regarded as a part of the LHS of the Einstein equation i.e the gravitational part rather than a part of the dynamics of quantum matter fields.
In conclusion one can show with some amount of work that the stress tensors defined through the regularization and renormalization procedures of the previous sections satisfy the Wald criteria. Since in most cases it is easier to carry out some form of renormalization (at least in a formal sense if not computationally) with a firm belief that they do satisfy some physically reasonable axioms, one could regards removal of divergences through renormalization of physical constants as legitimate.

Chapter 4

Quantum Black holes

Until 1974,black holes as were thought to be massive objects in general relativity from which even light could not escape. But all this changed with the landmark paper of Hawking which showed that black holes emit radiation with a thermal flux due to quantum effects thus showing that probably we were a little bit hasty in naming these beasts after all.
We dedicate this chapter to a detailed analysis of the Hawking effect from a Schwarzschild black hole (SCBH). We shall learn about Penrose diagrams and the causal structure of SCBH before passing onto other serious business. As usual for the sake of mathematical simplicity we shall work in two dimensions as it does permit a complete solution of the for the field equations.
We shall reproduce the result that a collapsing star which implodes to form a black hole produces gravitational perturbations which induces creation of particles with a thermal spectrum. We shall also discuss stability of the black hole in a thermodynamical sense and end this chapter with some questions that still remain unsolved and are a matter of great debate if not discussion.

4.1 Penrose Diagrams

We consider the four dimensional Schwarzschild spacetime in the (t, r, θ, ϕ) coordinates which is described by the line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{4.1}$$

This is the unique spherically symmetric solution of the vacuum Einstein equations that is used to represent the spacetime outside a spherical object of mass$M$. Since these coordinates are singular at $r = 2M$ we shall extended our coordinates beyond this coordinate singularity known as the event horizon all the way upto the true physical singularity $r = 0$ using the Kruskal coordinates(Kruskal 1960). We begin by defining the tortoise coordinate $r^*(r)$ as $r^*(r) = r + 2M \log\left|\frac{r}{2M} - 1\right|$ and the light cone Kruskal coordinates $(\bar{u}, \bar{v})$ become

$$\bar{u} = -4M \exp(-\frac{u}{4M}) \tag{4.2}$$

and

$$\bar{v} = 4M \exp(\frac{v}{4M}) \tag{4.3}$$

where $u = t - r^*$ and $v = t + r^*$. In these light cone coordinates (4.1) can be rewritten as

$$ds^2 = \left(\frac{2M}{r}\right) e^{-r/2M} d\bar{u} d\bar{v} - r^2 \left(d\theta^2 + sin^2\theta d\phi^2\right) \tag{4.4}$$

In this system we see that $r = 2M$ is a regular point. We also see that the $d\bar{u}d\bar{v}$ part of the metric is conformal to the two dimensional Minkowski space thus using the transformations

$$u' = 2\tan^{-1}(\bar{u}) \tag{4.5}$$

$$v' = 2\tan^{-1}(\bar{v}) \tag{4.6}$$

We can compactify our spacetime as conformal to Minkowski spacetime.
The causal structure can be illustrated in the form a conformal Penrose diagram.There are certain features of this diagram. The null rays remain at 45^o and these rays originate and terminate at the past $\Im^-$and future $\Im^+$ null infinities respectively.The spacelike lines will converge on spacelike infinity denoted by i^0 and timelike lines start and end at future i^+ and past i^- timelike infinities.Since these diagrams are drawn on two dimensional surfaces we suppress the (θ, ϕ) coordinates and usually express only the compactified (t, r) coordinates as a result of which each point on the diagram represents a 2-sphere.
Now we consider the Penrose diagram for the Schwarzschild case. Since $\bar{u}, \bar{v}$ are defined only in the quadrant $-\infty < \bar{u} < 0$ and $0 < \bar{v} < \infty$ the left hand edge of the region I represents the points $\bar{u} = 0, \bar{v} = 0$ or $r = 2M, t = \pm\infty$. We analytically extend beyond region I and let $-\infty < \bar{u}, \bar{v} < +\infty$ which represents the maximally extended Kruskal manifold.
The horizontal line $r = 0$ represents a true coordinate singularity. We can construct another asymptotical Minkowskian region III where the time direction is reversed $t \to -t$ and regions I and III are causally disconnected. The null rays $\bar{u} = 0$ or $r = 2M, t = +\infty$ is the latest retarded null ray that reaches $\Im^+$ and thus represents an event horizon for observers in region I. Null rays later than $\bar{u} = 0$ run into the future singularity $r = 0$. Similarly

for observers in region III the null ray $\bar{v} = 0$ acts as the event horizon. Since no event in region II can causally communicate with any event in region I, thus region II becomes a black hole with respect to I. Since time is reversed in region III, region IV appears as a black hole here.
The Penrose Diagram for the Schwarzschild Black hole is given below.

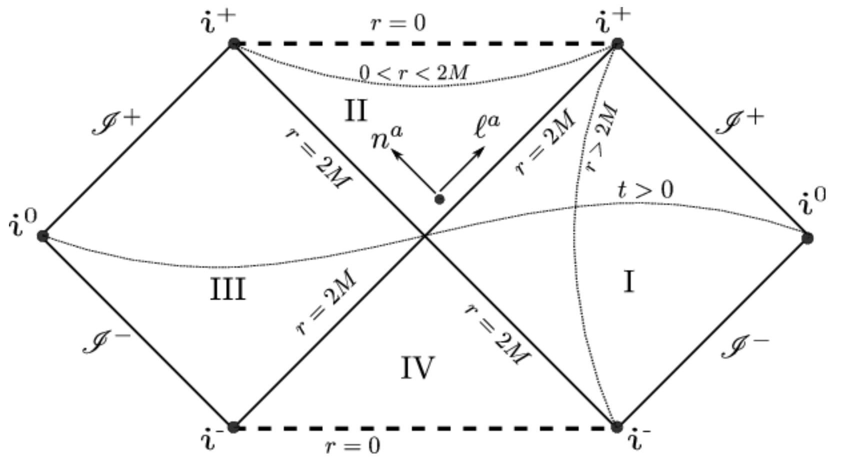

Figure 4.1: Penrose diagram of maximally extended Kruskal manifold

In real physical scenario where a star implodes to form a black hole only a small fragment of the Penrose diagram is relevant.
We shall now continue our discussion of creation of particles by collapsing spherical bodies.

4.2 Particle creation in a Black hole

We consider a spherically symmetric star in empty space. The solution of the vacuum Einstein equations outside the star are the unique Schwarzschild solution given by the line element (4.1). When the star becomes sufficiently compact it implodes forming a black hole in the process. While the exterior is undisturbed we still expect production of particles for any quantum fields propagating through the interior.
We shall here adhere to the treatment given by Hawking(1975) and Parker(1977) where we consider a massless scalar field in four dimensional Schwarzschild spacetime. We shall assume that in the remote past, before the collapse began and the star was static, the spacetime to be Minkowskian and thus we can construct the usual Minkowski vacuum states in the *in* region. As the star collapse in the *out* region we have the Schwarzschild spacetime in the exterior region and shall again construct vacuum states in the *out* region

which is related to the *in* vacuum by the usual Bogolubov transformations. Thus we expect an asymptotic inertial observer to find particles at late times in the *out* vacuum. This scenario is described below by the following Penrose diagram.

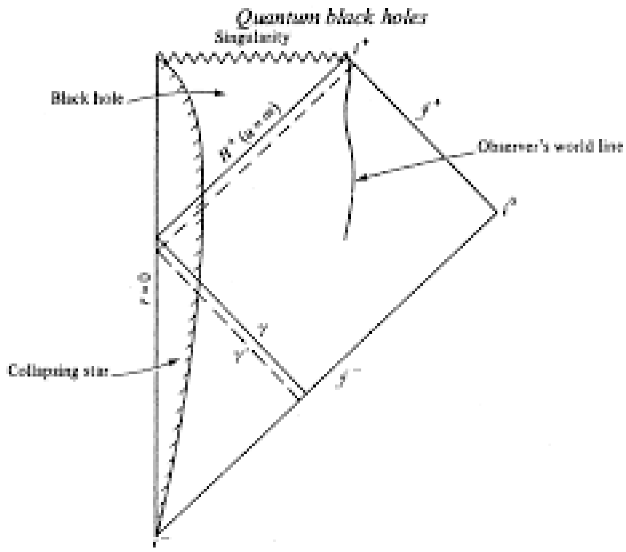

Figure 4.2: Penrose diagram of a collapsing star

As $R = 0$ in Schwarzschild spacetime the wave equation $\bigtriangledown\phi = 0$ have solutions in the form

$$r^{-1}R_{\omega l}(r)Y_{lm}(\theta,\phi)e^{-i\omega t} \tag{4.7}$$

where Y_{lm} are spherical harmonics and the radial function $R(r)$ satisfies the equation

$$\frac{d^2R_{\omega l}}{dr^{*2}} + \left\{\omega^2 - \left[\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\right]\left[1 - \frac{2M}{r}\right]\right\}R_{\omega l} = 0 \tag{4.8}$$

Since we are interested only in the asymptotical limit we can neglect the terms in the square brackets as write the solution as $r \to \infty$ as $e^{\pm i\omega r}$ with this simplification (4.7) reduces when written in terms of null coordinates to

$$r^{-1}Y_{lm}(\theta,\phi)e^{-i\omega u} \tag{4.9}$$

and

$$r^{-1}Y_{lm}(\theta,\phi)e^{-i\omega v} \tag{4.10}$$

The presence of the potential term i.e. the term in the square brackets results in scattering of the incoming modes (4.10) thus resulting in a superposition

of outgoing and incoming modes.
We shall now resort to the general theory of chapter 1 and work out the Bogolubov coefficients. The standard mode decomposition can be written as

$$\phi = \sum_{lm} \int d\omega (a_{\omega lm} f_{\omega lm} + a^{\dagger}_{\omega lm} f^{*}_{\omega lm}) \tag{4.11}$$

where the mode functions $f_{\omega lm}$ are normalized as

$$(f_{\omega_1 lm}, f_{\omega_2 pn}) = \delta(\omega_1 - \omega_2)\delta_{l,p}\delta_{m,n} \tag{4.12}$$

For the purposes of computational ease we shall work in two dimensions suppressing the Y_{lm} part and shall be inserted when required in four dimensions.We are ultimately interested in waves that passes through the collapsing star and reach the asymptotic observer in the *out* region at $\Im^+$.
The story in two dimensions simplifies a bit but we shall repeat it here for the sake of brevity. The field ϕ can be decomposed in terms of the *in* modes as

$$\phi = \int d\omega (a_\omega f_\omega + a^{\dagger}_{\omega} f^{*}_{\omega}) \tag{4.13}$$

where f reduce to the standard incoming spherical modes in the remote past. We define the *in* vacuum with respect to these modes in the usual way

$$a_\omega \left|0_{in}\right\rangle = 0 \tag{4.14}$$

The field ϕ can also be decomposed in terms of the *out* modes i.e in terms of the modes that reach $\Im^+$ after having passes through the center of the collapsing star.

$$\phi = \int d\omega (b_\omega p_\omega + b^{\dagger}_{\omega} p^{*}_{\omega}) \tag{4.15}$$

We define the *out* vacuum with respect to these modes in the usual way

$$b_\omega \left|0_{out}\right\rangle = 0 \tag{4.16}$$

and represent the Bogolubov transformation between the modes as

$$p_\omega = \int d\omega' \alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^{*}_{\omega'} \tag{4.17}$$

4.2.1 Effects of Blueshifts and Redshifts

We first shall consider the static case where as can be seen from (4.10) the incoming modes for different values of v converge to the center of the star

and pass through to become outgoing spherical waves that reach the inertial observer situated at $\Im^+$.
As seen by this asymptotic observer, these waves suffer a blueshift as they approach the surface of the star due to gravitational acceleration. As the waves pass through the center and recede away from the surface of the star they would suffer a redshift again due to decrease in gravitational acceleration as the body moves away to infinity. In the static case these two effects exactly compensate and cancel each other and the waves reaching $\Im^+$ have the form of (4.9) with the same frequency as the incoming modes. Cast in the language of Bogolubov transformations we have the incoming modes as

$$f_\omega \propto \frac{1}{\sqrt{8\pi^2\omega}} r^{-1} \exp(-i\omega v) \tag{4.18}$$

and the outgoing modes can be written as

$$p_{\omega'} \propto \frac{1}{\sqrt{8\pi^2\omega'}} r^{-1} \exp(-i\omega u) \tag{4.19}$$

using (4.17) and (1.14) we find that the Bogolubov coefficients are

$$\alpha_{\omega\omega'} \propto \delta(\omega - \omega'), \beta_{\omega\omega'} = 0 \tag{4.20}$$

Thus we see that there is no particle production in this case and the asymptotic observer 'sees' no thermal flux.
In the case of the collapsing star as the field modes approach the surface they suffer blueshift but as they pass on to the center and emerge out, the star has shrunk and its surface gravity has increased ($\kappa \sim 1/r^2$) and thus the modes suffer a larger redshift that doesn't balance the blueshift. If the star collapses to form a black hole with the transit time of the wave in the order of collapse time, the redshift will be significant actually exponentially greater.
For the same set of ingoing modes as (4.18) the outgoing modes that suffer an exponential redshift can be written down as

$$p_{\omega'} = \frac{1}{\sqrt{8\pi^2\omega'}} r^{-1} \exp\left[4Mi\omega' \log\left(\frac{v_0 - v}{c}\right)\right], v_0 > v \tag{4.21}$$

where v_0 is the constant corresponding to the final null ray γ that forms the event horizon.

In the figure below, the situation of a star collapsing to a black hole is shown. All null rays from remote past pass through the center of the collapsing star and reach far infinity in the remote future only if $v < v_0$. This is similar to the case of moving mirrors. The last ingoing ray that makes it to $\Im^+$ is labelled as γ. This null ray forms the event horizon. All later rays pass through the horizon with $v > v_0$ do not reach $\Im^+$ instead run into the singularity thus do not contribute to the thermal flux of outgoing particles.

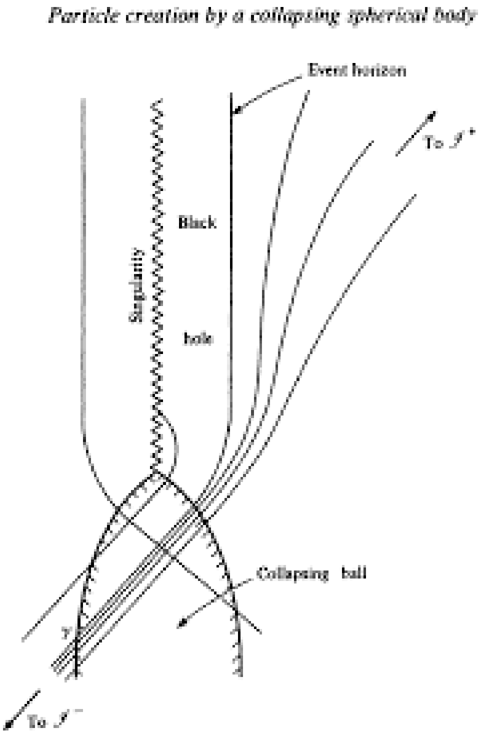

Figure 4.3: Diagram of a rays passing through a collapsing star

There is a similarity between (2.31) in the case of moving mirrors and (4.21) in the case of quantum black holes. However this is expected as the analysis of mode propagation through a black hole is based on geometrical optics with the potential barrier acting as the mirror in this case. Thus the reflection of modes from a moving mirror is similar to the scattering of modes from a collapsing star. The Doppler shift in that case finds an analogue in redshift of outgoing modes in this case. Thus we may exploit the machinery of section 2.3 and just state the results here.
The Bogolubov coefficients connecting the modes (4.21) and (4.18) are given by

$$\alpha_{\omega\omega'}(\beta_{\omega\omega'}) = \pm\frac{1}{2\pi}\sqrt{\frac{\omega'}{\omega}}e^{\pm 2\pi M\omega}\exp\left\{4Mi\omega\log\left[4M\omega\right]\right\}\Gamma\left(-4Mi\omega'\right) \quad (4.22)$$

From which we easily observe that the relation between the coefficients is

$$|\alpha_{\omega\omega'}|^2 = e^{8\pi\omega M}|\beta_{\omega\omega'}|^2 \tag{4.23}$$

from which using the Wronskian relation between the coefficients

$$\sum_{\omega}\left(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega''}|^2\right) = \delta(\omega - \omega'') \tag{4.24}$$

however, from (4.23) and (4.24) we deduce that

$$|\beta_{\omega\omega'}|^2 = \frac{1}{2\pi k\omega'}\left(\frac{1}{e^{\omega/8\pi M} - 1}\right) \tag{4.25}$$

using the fact that for a Schwarzschild black hole $\kappa = (4M)^{-1}$ we have

$$k_B T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M} \tag{4.26}$$

Thus the observer at $\Im^+$ finds a thermal flux of particles with temperature T known as the Hawking temperature. Further the number of particles per unit time passing out of the surface of the collapsing ball in the frequency range of ω to $\omega + d\omega$ is given by a Plank spectrum distribution.

$$n_\omega = \frac{d\omega}{2\pi}\frac{1}{(e^{8\pi M\omega} - 1)} \tag{4.27}$$

There is a slight modification of the above results in four dimensions. Since the wave propagating through a black hole encounters a potential barrier not all of it reaches $\Im^+$ some of the waves are back-scattered. Now only a fraction Γ_ω of the incoming rays reach infinity and the remaining $1-\Gamma_\omega$ gets reflected from the black hole. (4.24) which can be thought of as a probability conservation equation gets modified by a grey body factor Γ_ω and thus (4.25) becomes

$$n_\omega = \frac{d\omega}{2\pi}\frac{\Gamma_\omega}{(e^{8\pi M\omega} - 1)} \tag{4.28}$$

For a real black hole Page (1977) estimated the Hawking temperature in Kelvin(K)(4.26) to be

$$T = \frac{1.2 \times 10^{26}}{M} \tag{4.29}$$

For a body with mass of the order of 1 solar mass i.e $M \sim 10^{30}$ kg we get a temperature $T \sim 10^{-8}$ K. Thus at these energy scales emission of massless quanta is predominant. However if $M \sim 10^{15}$ kg then $T \sim 10^9$ K which is sufficient to create thermal e^+e^- pairs implying that at further lower masses more heavy particles can be created.

4.3 Thermodynamics of black holes

In this section we shall give a brief overview of the thermodynamics of black holes and in the next section we shall try to do some justification with a deeper analysis.
We consider an isolated black hole in an empty space formed as a result of implosion of a spherically symmetrical star with a horizon radius of $r = 2M$ and a surface area of $A = 4\pi r^2 = 16\pi M^2$ and a surface temperature given by (4.26). Since the black hole is emitting radiation in losses mass. Considering classical thermodynamics a spherical body at a temperature T will radiate according to the Stefan-Boltzmann law. The loss of mass as a result of this radiation is

$$\frac{dM}{dt} = -L = -\Gamma_\omega \sigma A T^4 = -\frac{\Gamma_\omega}{15360\pi M^2} \tag{4.30}$$

where L is the flux of radiation and σ is the Stefan constant. From this equation it is evident that an isolated black hole has a finite life time $t \sim M^3$ and as it evaporates it losses mass and its temperature increases rapidly leading to an explosive end whose fate cannot be decided in the context of a semi-classical theory as when the $M \sim 10^{-8}$ kg its size is of the Plank scale(10^{-33} cm) and non perturbative quantum gravity effects can no longer be ignored.
In 1971 Bekenstein conjectured that black holes must possess intrinsic entropy which would be proportional to the area of the event horizon. This famous area law for the Schwarzschild case we are considering has the form

$$S_{BH} = \frac{1}{4}A = 4\pi M^2 \tag{4.31}$$

and the first law of black hole thermodynamics becomes

$$dM = T dS_{BH} \tag{4.32}$$

As the black hole has a negative heat capacity $C_{BH} < 0$ it cannot be in a stable equilibrium with an infinite heat reservoir. Since as the infinite reservoir has a constant temperature it implies that if $T_{BH} \neq T_{rev}$ the black hole would either keep on emitting radiation and becoming hotter or keep on absorbing radiation and becoming colder thus ensuring a stable equilibrium is never reached. It can be further shown using simple thermodynamical considerations that a SCBH can be in stable equilibrium with a finite reservoir only if the heat capacity of the reservoir is in a finite range i.e. $0 < C_{rev} < 8\pi M^2$.
The second law of black hole thermodynamics is similar to the entropy law of classical thermodynamics. It goes by the name of the area theorem (Hawking 1972) which states that the area of the event horizon in all black hole

processes that satisfy the weak energy condition is always non decreasing i.e.

$$dA \geq 0 \tag{4.33}$$

These laws apply not only to adiabatic processes but also non equilibrium cases. It might appear that evaporation violates the area law but this is a false alarm as the negative energy states violate the weak energy condition which in effect states that any observer only observes non negative energy density. However the second law of thermodynamics is not violated as the emission of radiation increases the total entropy i.e.

$$\delta S_{total} = \delta S_{BH} + \delta S_{matter} \geq 0 \tag{4.34}$$

4.4 An explosive end

We shall end this chapter with a bit more detailed analysis of the thermodynamical nature of black holes.

Since the wavelength of the outgoing quanta $\sim M$ i.e. it is of the order of the size of the hole and from basic quantum mechanics we know that it is impossible to specify the location of a particle to within a wavelength the concept of particles produced near the horizon makes little sense.

Nevertheless,the Hawking effect can create virtual particle-antiparticle pairs that have associated wavelength λ. If this $\lambda \sim M$ then gravitational forces prevent the re-annihilation of the pair and the particle escapes to infinity carrying positive energy as a part of the Hawking flux and the antiparticle falls into the black hole carrying negative energy with respect to the observer at infinity. This concept follows on similar lines of the Klein paradox (Klein 1929) where a virtual particle antiparticle pair of charge q and mass m in the Dirac setup separate in the presence of an external electric field E. If the separation is such that $qE\triangle x \approx 2m$ then the energy gained will be sufficient to convert them into a real pair and we have particles with positive energy and antiparticles with negative energy.

As the black hole radiates it is losing mass which results in the area of the event horizon to shrink. The temperature increases and heavier particles antiparticle pairs begin to be created. This is illustrated in the figure below. The problem is that we intuitively expect the horizon to shrink as the hole is losing mass but the horizon evolution equations are non linear and hard to solve also as pointed out earlier $d\log M/dt \sim M^{-1}$. Thus the rate of evaporation is of the order of the frequency of radiation from the hole thus our semi classical theory breaks down and we need a fully quantum theory to study further the final stages of a black hole. Hawking conjectured in 1977

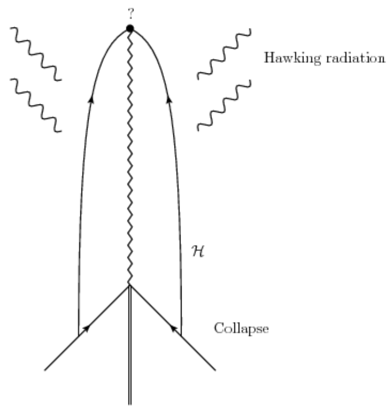

Figure 4.4: Evaporation of a Black hole

that the fate of a black hole is a naked singularity or a Plank mass object or an explosive end.
This of course leads us to the (in)famous information paradox. Since there is a definitive relation between information content and entropy at least in the classical sense due to Shannon and Weaver (1949) we can extended the same idea to black holes as they posses an intrinsic entropy which is related to the large number of internal micro-states that are hidden behind the event horizon. If we assume that some sub atomic particles enter the black hole in a pure quantum state as the hole radiates thermally this pure state would devolve into the mixed state of Hawking radiation thus destroying information.
Hawking further came up with the idea of 'Principle of ignorance' where the boundary conditions at the singularity are responsible for the random nature of thermal radiation form a black hole. As seen using the Feynman picture of an antiparticle travelling backwards in time, a particle can be thought of as originating from the singularity, tunnelling through the horizon and being a part of the thermal flux which is random in nature.
There are many open question that till date generate discussion and debate amongst them are the information paradox, violation of time reversal symmetry in quantum gravity and the explosive end of an evaporating black hole. Thus even though nearly four decades have gone by since the early 1970's this field is far from over. This semi classical approach is surely a first approximation to any theory of quantum gravity and even with its limited perspective it has offered many interesting and surprising results.

Chapter 5

Conclusion

We have certainly come a long way from where we started. We have time and again seen that the notion of particles is obscure and a highly observer dependent concept. Thus we had to resort to some local physical quantity that would help us probe the structure of spacetime and we did this using the stress tensor $\langle T_{\mu\nu} \rangle$. But as we saw this quantity was plagued with divergences and we had to use the theory of regularization and renormalization of coupling constants to extract meaningful physics from them. We further encountered conformal anomalies which arise due to the breaking of the conformal symmetry of the theory.We end the thesis studying about quantum black holes and realized that this was a misnomer since black holes do radiate. We also took notice of the intricate relation between thermodynamics and semi classical quantum gravity.

The techniques and results of curved space QFT which have been consistent from various theoretical view points has bolstered our belief that this might be indeed the right approach to obtaining a full theory of quantum gravity. Using the no-hair theorem (which states that black holes are characterized by just three observables-mass, electric charge and angular momentum) in conjunction with the area law for entropy we see that the black hole entropy is truly a fundamental quantity which could act as a catalyst to study new physics.

In the recent times a lot of advances have taken place in curved space QFT that have branched into the study of instantons, quantum information theory, quantum topology, inflationary models and deSitter QFT to name a few. In conclusion it is fair to say that this subject has brought us to the doorstep of new advances in physics.

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