# A Study of Riemannian Geometry

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by

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This is to certify that this dissertation entitled A Study Of Riemannian Geometry towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents original work carried out by Safeer K M at Indian Institute of Science Education and Research under the supervision of Dr. Tejas Kalelkar, assistant professor, Department of Mathematics, during the academic year 2015-2016.

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Dedicated to my parents for their unconditional love and support

# Declaration

I hereby declare that the matter embodied in the report entitled A Study Of Riemannian Geometry are the results of the investigations carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research,Pune, under the supervision of Dr. Tejas Kalelkar and the same has not been submitted elsewhere for any other degree.

Safeer K M

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# Abstract

In this reading project I studied some interesting results in Riemannian geometry. Starting from the definition of Riemannian metric, geodesics and curvature this thesis covers deep results such as Gauss-Bonnet theorem, Cartan-Hadamard theorem, Hopf-Rinow theorem and the Morse index theorem. Along the way it introduces useful tools such as Jacobi fields, variation formulae, cut locus etc. It finally builds up to the proof of the celebrated Sphere theorem using some basic Morse theory.

# Contents

# Chapter 1

# Introduction

The main aim of this chapter is to introduce the essential theory of smooth manifolds and fix the notations used throughout this report. Most of the theorems are presented without proofs. Smooth manifolds are the generalization of curves and surfaces in $\mathbb{R}^2$ and $\mathbb{R}^3$ to arbitrary dimensions.

## 1.1 Some Basic Results From the Theory of Smooth Manifolds

**Definition 1.1.1.** *A smooth manifold of dimension $n$ is a set $M$ and a family of injective maps $x_\alpha$: $U_\alpha \longrightarrow M$ of open subsets $U_\alpha$ of $\mathbb{R}^n$ such that:*

1. $\bigcup_\alpha x_\alpha(U_\alpha) = M$
2. *For any pair $\alpha$, $\beta$ with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mapping $x_\beta^{-1} \circ x_\alpha$ is smooth.*
3. *The family $\{(U_\alpha, x_\alpha)\}$ is maximal with respect to conditions 1 and 2.*

We now define a smooth map between two smooth manifolds.

**Definition 1.1.2.** *Let $M$ and $N$ be manifolds of dimension $m$ and $n$ with coordinate mapping $\{(U_\alpha, x_\alpha)\}$ and $\{(V_\beta, y_\beta)\}$ respectively. A map $F : M \longrightarrow N$ is called smooth at $p \in M$ if given a parametrization $y : V \subset \mathbb{R}^n \longrightarrow N$ at $F(p)$ there exist a parametrization $x : U \subset \mathbb{R}^m \longrightarrow M$ at $p$ such that $F(x(U)) \subset y(V)$ and the map $y^{-1} \circ F \circ x : U \subset \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is smooth at $x^{-1}(p)$. $F$ is smooth on an open set of $M$ if it is smooth at all the points of the open set.*

Smooth manifolds $M$ and $N$ are said to be diffeomorphic if there exists a smooth map $F : M \longrightarrow N$ such that $F$ is bijective and its inverse is also smooth. $F : M \longrightarrow N$ is said to be a local diffeomorphism if for all $p \in M$ there exist open neighborhoods $U$ around $p$ and $V$ around $F(p)$ such that $F : U \longrightarrow V$ is a diffeomorphism.

It can be easily seen that condition *2* is essential in defining smooth maps unambiguously. Following are some of the examples of smooth manifolds.

*Example 1:* Most obvious example of a smooth manifold is Euclidean space $\mathbb{R}^n$ itself with identity map as the coordinate map.

*Example 2:* Consider projective space $\mathbb{RP}^n$ which is the set of all straight lines through origin in $\mathbb{R}^{n+1}$. $\mathbb{RP}^n$ can also be realized as the quotient space of $\mathbb{R}^{n+1} \setminus \{0\}$ by an equivalence relation $(x_1, .., x_{n+1}) \sim (\lambda x_1, .., \lambda x_{n+1})$ with $\lambda \in \mathbb{R}$. Let us denote a point in $\mathbb{RP}^n$ as $[x_1 : ... : x_{n+1}]$. Consider open sets $V_i = \{[x_1 : ... : x_{n+1}] \,|\, x_i \neq 0\}$. It can also be represented as $V_i = \{[\frac{x_1}{x_i} = y_i : ... : 1 : ... : \frac{x_{n+1}}{x_i} = y_n]\}$ with 1 at the $i^{th}$ position. It is easy to see that $\{(V_i, \phi_i)\}$ is a coordinate chart for $\mathbb{RP}^n$ where $\phi_i : \mathbb{R}^n \longrightarrow V_i$ is defined by $\phi_i(y_1, .., y_n) = [y_1 : ...y_{i-1} : 1 : y_i : ... : y_n]$.

There is a notion of directional derivative of a real valued function in Euclidean space which is uniquely determined by the tangent vector in which the directional derivative is calculated. We define tangent vector in abstract manifolds using the properties of tangent vector in a Euclidean space. The following is a working definition of tangent vectors in arbitrary manifolds. We call a smooth function $\gamma : (-\varepsilon, \varepsilon) \subset \mathbb{R} \longrightarrow M$ a smooth curve on $M$. Let $\mathcal{C}^\infty(p)$ denote the set of all real valued functions defined in some neighborhood of $p$ which are smooth at $p$. (To avoid confusion we usually denote real valued functions on manifolds with lower case alphabets while functions between manifolds are denoted by upper case alphabets)

**Definition 1.1.3.** *Let $M$ be a smooth manifold and $\gamma$ be a smooth curve with $\gamma(0) = p$. The tangent vector to the curve at $p$ is a function $\gamma'(0) : \mathcal{C}^\infty(p) \longrightarrow \mathbb{R}$ given by*

$$\gamma'(0)f = \frac{d(f \circ \gamma)}{dt}|_{t=0}, \ \forall f \in \mathcal{C}^\infty(p)$$

A *tangent vector* at $p$ is the tangent vector at $t = 0$ of some smooth curve $\gamma : (-\varepsilon, \varepsilon) \longrightarrow M$ with $\gamma(0) = p$. We denote the set of all tangent vectors at $p$ by $T_pM$. $T_pM$ is an n dimensional vector space. If $x : U \longrightarrow M$ is a given parametrisation around $p$ then $((\frac{\partial}{\partial x_1})_p, ..., (\frac{\partial}{\partial x_n})_p)$ is a basis for $T_pM$, where $(\frac{\partial}{\partial x_i})_p$ is the tangent vector at $p$ of the curve $t \longmapsto x(0, ..0, t, 0, ..0)$ with $t$ at the $i^{th}$ place.

**Proposition 1.1.4.** *Let $M, N$ be smooth manifolds and $F : M \longrightarrow N$ be a smooth map. For every $\nu \in T_pM$ choose $\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\alpha(0) = p$ and $\alpha'(0) = \nu$. Take $\beta = F \circ \alpha$, then the map $dF_p : T_pM \longrightarrow T_{f(p)}N$ defined by $dF_p(\nu) = \beta'(0)$ is a linear map and is independent of the choice of the curve $\alpha$.*

The map $dF_p$ defined in the above proposition is called the differential of $F$ at $p$. It is an easy consequence of the chain rule that if $F$ is diffeomorphism then $dF_p$ is an isomorphism. A weak converse of the above statement can be obtained using inverse function theorem as follows.

**Theorem 1.1.5.** *Let $F : M \longrightarrow N$ be a smooth map such that $\forall p \in M$, $dF_p : T_pM \longrightarrow T_{F(p)}N$ is an isomorphism then $F$ is a local diffeomorphism.*

If $M$ is an $n$ dimensional manifold then $TM = \{(p, \nu) | p \in M, \nu \in T_pM\}$ is a $2n$ dimensional manifold.

**Definition 1.1.6.** *Let $M$ and $N$ be smooth manifolds. We say $F : M \longrightarrow N$ is an immersion at $p \in M$ if $dF_p : T_pM \longrightarrow T_{f(p)}N$ is injective. $F$ is called an immersion if it is an immersion at $p$ for all the $p \in M$. We say $F : M \longrightarrow N$ is an embedding if it is an immersion and $F : M \longrightarrow F(M) \subset N$ is a homeomorphism, where $F(M)$ has the subspace topology. If $M \subset N$ and the inclusion $i : M \hookrightarrow N$ is an embedding then we say that $M$ is a submanifold of $N$.*

**Definition 1.1.7.** *Let $M$ be a smooth manifold. We say it is orientable if it has smooth structure $\{(U_\alpha, x_\alpha)\}$ such that whenever $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the differential of the change of coordinate $d(x_\beta^{-1} \circ x_\alpha)$ has positive determinant.*

Such a choice of smooth structure is called orientation. If such an orientation does not exist we say the manifold is non-orientable. An interesting fact to note here is that tangent bundle of a smooth manifold is orientable even if the manifold is non-orientable. This fact is shown by the following calculation. Let $M^n$ be a smooth manifold with coordinate chart $\{(U_\alpha, x_\alpha)\}$, then it is easy to see that $TM$ is a $2n$ dimensional manifold with a coordinate atlas $\{U_\alpha \times \mathbb{R}^n, y_\alpha\}$, where $y_\alpha : U_\alpha \times \mathbb{R}_n \longrightarrow TM$, $y_\alpha(x_1^\alpha, ..., x_n^\alpha, u_1, ..., u_n) = (x_\alpha(x_1^\alpha, ..., x_n^\alpha), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha})$. But $y_\beta^{-1} \circ y_\alpha(q_\alpha, v_\alpha) = ((x_\beta^{-1} \circ x_\alpha)(q_\alpha), d(x_\beta^{-1} \circ x_\alpha)(v_\alpha))$ when $(q_\alpha, v_\alpha) \in y_\alpha^{-1}(y_\alpha(U_\alpha \times \mathbb{R}^n \cap y_\beta(U_\beta \times \mathbb{R}^n))$. In order to check the orientability look at,

$$d(y_\beta^{-1} \circ y_\alpha) = \begin{bmatrix} d(x_\beta^{-1} \circ x_\alpha) & 0 \\ 0 & d(x_\beta^{-1} \circ x_\alpha) \end{bmatrix}$$

As $\det(d(x_\beta^{-1} \circ x_\alpha)) \neq 0$, $\det(d(y_\beta^{-1} \circ y_\alpha)) > 0$, and thus $TM$ is orientable.

We now define the concept of a vector field which is crucial in the context of Riemannian geometry.

**Definition 1.1.8.** *A vector field $X$ on a smooth manifold $M$ is a map that associates for every point $p \in M$ a vector $X(p) \in T_pM$ i.e a vector field is a map $X : M \longrightarrow TM$ with the property that $\pi \circ X = Id_M$ where $\pi$ is the projection map from $TM$ to $M$. We say a vector field is smooth if the above map is smooth.*

If we consider $f \in \mathcal{C}^\infty(U)$, $U \subseteq M$, then $Xf$ is a real valued function on $M$, defined by $Xf(p) = X(p)f$. It can be shown that $X$ is a smooth vector field if and only if $Xf$ is smooth for all the $f \in \mathcal{C}^\infty(U)$ for every open set $U \subseteq M$. We denote by $\tau(M)$ the set of all smooth vector fields on $M$. $\tau(M)$ is a module over $\mathcal{C}^\infty(M)$. We now define lie brackets.

**Definition 1.1.9.** *Let $X, Y \in \tau(M)$, then the vector field $[X, Y]$, defined by $[X, Y]f = (XY - YX)f$ is called the lie bracket of $X, Y$.*

It can easily be shown that such a vector field is unique by using the coordinate representation of $X$ and $Y$. It is obvious that if $X$ and $Y$ are smooth then $[X, Y]$ is smooth. The following proposition summarizes the properties of lie brackets.

**Proposition 1.1.10.** *If $X, Y, Z \in \tau(M)$, $a, b \in \mathbb{R}$ and $f, g \in \mathcal{C}^\infty(M)$ then:*

1. $[X, Y] = -[Y, X]$
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$
4. $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$

## 1.2 Riemannian Metrics

The concept of inner product on a Euclidean space allows us to define the angle between curves and length of the curves. In order to perform geometry on an arbitrary smooth manifold we need the concept of an inner product.

**Definition 1.2.1.** *A Riemannian metric on a smooth manifold is a 2-tensor field $g \in \tau^2(M)$ which is:*

1. *Symmetric, i.e* $g(X,Y) = g(Y,X)$

2. *Positive definite, i.e* $g(X,X) > 0$ *if* $X \neq 0$

This determines an inner product in each of its tangent spaces $T_pM$. It is usually denoted as $\langle X, Y\rangle := g(X,Y)$ for $X, Y \in T_pM$. A smooth manifold $M$ equipped with a Riemannian metric is called a Riemannian manifold, usually denoted as $(M, g)$ We can define norm of a tangent vector $X \in T_pM$ as $|X| := \langle X, X\rangle^{1/2}$. We also define angle $\theta \in [0, \pi]$ between two non zero tangent vectors $X, Y \in T_pM$, by $cos\theta = \langle X, Y\rangle/|X||Y|$. Let $(E_1, ..., E_n)$ be a local frame and $(\varphi_1, ..., \varphi_n)$ be its dual coframe then a Riemannian metric can be written locally as $g = g_{ij}\varphi_i \otimes \varphi_j$, where $g_{ij} = g(E_i, E_j) = \langle E_i, E_j\rangle$. If we consider coordinate frame it can be written as $g = g_{ij}dx^i \otimes dx^j$. Because of the symmetry of the metric it can be also written as $g = g_{ij}dx^i dx^j$ where $dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + dx^j \otimes dx^i)$.

*Example:* Most obvious example of a Riemannian manifold is $\mathbb{R}^n$ with usual inner product. We can identify the tangent space $T_p\mathbb{R}^n$ with $\mathbb{R}^n$ itself. The Riemannian metric can be represented as $g = \sum_i dx^i dx^i$.

Let $f : M \longrightarrow \tilde{M}$ be an immersion and $\tilde{g}$ be a Riemannian metric on $\tilde{M}$. We can define a metric $g$ on $M$ as $g = f^*\tilde{g}$, i.e $g(X,Y) = f^*\tilde{g}(X,Y) = \tilde{g}(dfX, dfY)$. This gives us several examples of Riemannian manifold.

*Example:* Let $i : S^2 \longrightarrow \mathbb{R}^3$ be the inclusion map and consider the standard metric on $\mathbb{R}^3$, $g_{ij} = \delta_{ij}$. We can define an induced metric on $S^2$ as $\overset{o}{g} = i^*g$. Consider stereographic atlas, $x : \mathbb{R}^2 \longrightarrow S^2 \setminus \{N\}$ given by

$$x(x_1, x_2) = \frac{(2x_1, 2x_2, x_1 + x_2 - 1)}{x_1^2 + x_2^2 + 1}$$

Then the induced metric is given by,

$$\overset{o}{g}_{ij} = \frac{4}{(1 + x_1^2 + x_2^2)^2} g_{ij}$$

This is called the round metric on a sphere.

Now the natural question that arises is if every smooth manifold admits a Riemannian structure. The answer is yes and can be proved using partitions of unity.

**Definition 1.2.2.** *Let* $(M, g)$, $(\overline{M}, \overline{g})$ *be Riemannian manifolds. A diffeomorphism* $f : M \longrightarrow \overline{M}$ *is called an isometry if* $f^*\overline{g} = g$.

## 1.3 Connections on a Riemannian manifold

In order to generalize the concept of a straight line in a Euclidean space onto an arbitrary Riemannian manifold the obvious property of straight line, that it is length minimizing in a small enough neighborhood cannot be used as it is technically difficult to work with. The property that we use instead is that straight lines in Euclidean spaces have zero acceleration. Therefore we need to find a way to take the directional derivative of vector fields. We cannot use the usual directional derivative as vectors at different points lies in different tangent spaces (this problem does not arise in the case of $\mathbb{R}^n$ as different tangent spaces are identified with $\mathbb{R}^n$ itself).

**Definition 1.3.1.** *Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve on a Riemannian manifold $M$ then $V : I \longrightarrow TM$ is called a vector field along the curve $\gamma$ if $V(t) \in T_{\gamma(t)}M$.*

$V$ is smooth if for any smooth function $f$ the real valued function $V(t)f$ is smooth.

**Definition 1.3.2.** *Let $X, Y, Z \in \tau(M)$ and $f, g \in \mathcal{C}^\infty(M)$. An affine connection on $M$ is a map $\nabla : \tau(M) \times \tau(M) \longrightarrow \tau(M)$ denoted by $\nabla(X, Y) := \nabla_X Y$ which satisfies,*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$

The concept of connections provide us a method to carry out the directional derivative of one vector field along another vector field. On intuitive terms it connects the tangent spaces of manifolds. Consider the local coordinate frame $E_i = \frac{\partial}{\partial x_i}$. We can write $\nabla_{E_i} E_j = \sum_k \Gamma^k_{ij} E_k$, where local functions $\Gamma^k_{ij}$ are called the Christoffel symbols of $\nabla$ with respect to the given frame. If $X$ and $Y$ are smooth vector fields and $X = \sum_i X^i E_i$ and $Y = \sum_i Y^i E_i$ when expressed in terms of coordinate local frame then $\nabla_X Y = \sum_k (\sum_{ij} X^i Y^j \Gamma^k_{ij} + X(Y^k))E_k$.

**Proposition 1.3.3.** *Let $M$ be a smooth manifold with an affine connection $\nabla$ and $\gamma$ be a smooth curve and $V$ be a vector field along $\gamma$ then there exist a unique vector field along $\gamma$ associated with $V$ called the covariant derivative of $V$, denoted by $\frac{DV}{dt}$ satisfying:*

1. $\frac{D(V+W)}{dt} = \frac{DV}{dt} + \frac{DW}{dt}$
2. $\frac{D(fV)}{dt} = \frac{df}{dt} V + f\frac{DV}{dt}$

*3. If $V$ is induced by a vector field $Y \in \tau(M)$ i.e, $Y(\gamma(t)) = V(t)$, then* $\frac{DV}{dt} = \nabla_{\frac{d\gamma}{dt}} Y$

*Proof.* We first prove that if such a correspondence exists then it is unique. Consider the coordinate chart $x : U \longrightarrow M$ such that $c(I) \cap x(U) \neq \phi$. We can write $c(t) = (x_1(t), .., x_n(t))$ in terms of local coordinates, and write $V = \sum_i V^i E_i$ where $E_i = \frac{\partial}{\partial x_i}$. Assuming such a correspondence exists and using the properties *1* and *2* we have $\frac{DV}{dt} = \sum_j \frac{dV^j(t)}{dt} E_j + \sum_j V^j \frac{DE_j}{dt}$. We can write $\frac{DE_j}{dt} = \sum_i \frac{dx_i}{dt} \nabla_{E_i} E_j$ by *3* and the definition of affine connections. Hence

$$\frac{DV}{dt} = \sum_j \frac{dV^j}{dt} E_j + \sum_{i,j} \frac{dx_i}{dt} V^j \nabla_{E_i} E_j.$$

Therefore if such a correspondence exists then it is unique by the above expression. Now existence is easy to show. We define such a correspondence in a coordinate chart using the above expression. If there is another coordinate chart we define the correspondence using the same expression in the new coordinate chart. At the intersection of two coordinate chart it coincides because of its uniqueness.

□

**Definition 1.3.4.** *Let $M$ be a smooth manifold with an affine connection $\nabla$. A vector field $V$ along a curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is called parallel if* $\frac{DV}{dt} = 0 \; \forall t \in I$

The following proposition enables us to carry out transportation of vectors through different vector spaces without losing information. (This is always possible in the case of Euclidean space. Surprisingly it is possible also in the case of smooth manifolds.)

**Proposition 1.3.5.** *Let $M$ be a smooth manifold with an affine connection $\nabla$. Let $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ be a smooth curve and $V_0 \in T_{\gamma(t_0)}M$, then there exist a unique parallel vector field along $\gamma$ such that $V(t_0) = V_0$.*

*Proof.* Consider $x : U \longrightarrow M$ a coordinate chart such that $\gamma(I) \cap x(U) \neq \phi$. In such a coordinate chart we can represent $\gamma(t) = (x_1(t), .., x_n(t))$. Let $V_0 = \sum_i V_0^j E_j$ where $E_j$'s are the coordinate frame ($E_j$'s here are $\frac{\partial}{\partial x^i}|_{\gamma(t_0)}$). Suppose there exists a parallel vector field $V$ along the curve in $x(U)$ satisfying $V(t_0) = V_0$. In local coordinate frame express $V = \sum_i V^i E_i$. Then it will satisfy the following condition. $\frac{DV}{dt} = 0 = \sum_j \frac{dV^j}{dt} E_j + \sum_{i,j} \frac{dx_i}{dt} V^j \nabla_{E_i} E_j$. Rewriting this equation in terms of Christoffel symbols we get, $\sum_k (\frac{dV^k}{dt} + \sum_{i,j} V^j \frac{dx_i}{dt} \Gamma_{ij}^k) E_k = 0$. This gives us a system of $n$ first order linear differential equations. Hence it possess a unique solution for the given initial condition $V^k(t_0) = V_0$ by the existence and uniqueness of solutions of linear ODEs. We can define $V$ in different coordinate chart as the solution of the

ODE above in the given coordinate chart and when two of them intersect the solution should be the same because of uniqueness. Hence we have our desired parallel vector field $V$ along the whole curve $\gamma$ with $V(t_0) = V_0$. $\square$

### 1.3.1 Riemannian connections

While talking about affine connections we have not considered the Riemannian structure on the smooth manifold. The natural choice of connection one should introduce in a Riemannian manifold should be the one that preserves angle between two parallel vector fields along the curve. Hence the following definition.

**Definition 1.3.6.** *Let $M$ be a Riemannian manifold, an affine connection is said to be compatible with the Riemannian metric if for any smooth curve $\gamma$ and a pair of parallel vector fields $P, P'$ we have $\langle P, P' \rangle$ =constant.*

**Proposition 1.3.7.** *Let $M$ be a Riemannian manifold. An affine connection $\nabla$ is compatible with the metric if and only if for any two vector fields $V, W$ along a curve $\gamma$ we have $\frac{d\langle V,W\rangle}{dt} = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle$*

**Corollary 1.3.8.** *For a Riemannian manifold $M$ an affine connection $\nabla$ is compatible with the metric if and only if $\forall X, Y, Z \in \tau(M)$, $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$*

**Definition 1.3.9.** *An affine connection $\nabla$ on a Riemannian manifold $M$ is defined to be symmetric if $\nabla_X Y - \nabla_Y X = [X, Y]$*

**Theorem 1.3.10.** *Given a Riemannian manifold $M$ there exists a unique affine connection which is symmetric and compatible with the metric.*

Therefore we can unambiguously consider this which is symmetric and compatible with the metric on a Riemannian manifold. It is called Levi-Civita connection or Riemannian connection.

Connections on a manifold is a way to *connect* different tangent spaces of a manifold and calculate the directional derivative. Historically the idea of connection was derived from the parallel transport formalism. The fact that we can obtain connection from parallelism is shown in the next exercise.

*Exercise:* Let $X$ and $Y$ be smooth vector fields on a Riemannian manifold with Levi-Civita connection $\nabla$. Let $p \in M$ and $\alpha : I \longrightarrow M$ be an integral curve of $X$ through $p$. Let

$P_{t_0,t} : T_{\alpha(t_0)}M \longrightarrow T_{\alpha(t)}M$ denote the parallel transport along the curve $\alpha$. Prove that

$$(\nabla_X Y)(p) = \frac{d}{dt}(P_{t_0,t}^{-1})(Y(\alpha(t)))\Big|_{t=t_0}$$

*Proof.* Consider an orthonormal frame $\{E_i\}$ at $p$ and parallel transport it along $\alpha$. We can write $Y(\alpha(t)) = \sum_{i=1}^n g_i(t)E_i(t)$. Note that $\langle E_i, E_j\rangle = \delta_{ij}$.

$$\begin{aligned}
\frac{d}{dt}(P_{t_0,t}^{-1})(Y(\alpha(t)))\Big|_{t=t_0} &= \frac{d}{dt}(P_{t_0,t}^{-1})(\sum_{i=1}^n g_i(t)E_i(t))\Big|_{t=t_0} \\
&= \frac{d}{dt}(\sum_{i=1}^n g_i(t)E_i)\Big|_{t=t_0} \\
&= \frac{d}{dt}(\sum_{i=1}^n g_i(t))\Big|_{t=t_0} E_i \\
&= (\nabla_{\alpha'(t)}Y(\alpha(t)))(p) \\
&= (\nabla_X Y)(p)
\end{aligned}$$

Second last step follows from the fact that $E_i(t)$'s are orthonormal. $\square$

It is not very hard to obtain the following formula for Christoffel symbols.

$$\Gamma_{ij}^m = \frac{1}{2}\sum_{k=1}^n \Big(\frac{\partial}{\partial x_i}g_{jk} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_k}g_{ij}\Big)g^{km}$$

where $(g^{km})$ denotes the inverse of the symmetric matrix $(g_{km})$. Using this we will now calculate the Christoffel symbols for a 2-sphere of radius $R$ with round metric. For ease of calculation we will consider spherical coordinates $(\theta, \phi)$ on $S_R^2 \setminus \{(x, y, z) : x \leq 0, y = 0\}$ given by $(x, y, z) = (R \sin\phi \cos\theta, R \sin\phi \sin\theta, R \cos\phi)$, $\theta \in (-\pi, \pi)$. The round metric is given by $\overset{o}{g} = i^*g = (d(R \sin\phi \cos\theta))^2 + (d(R \sin\phi \sin\theta))^2 + (d(R \cos\phi))^2$. After simplification the expression becomes $\overset{o}{g} = R^2 \sin^2(\phi)d\theta^2 + R^2 d\phi^2$. Using the equation for Christoffel symbols we calculate them for a 2-sphere. $\Gamma_{11}^1 = \Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{21}^1 = 0$, $\Gamma_{12}^2 = \Gamma_{21}^2 = \cot\phi$, $\Gamma_{22}^1 = \sin\phi\cos\phi$ and $\Gamma_{22}^2 = 0$.

# Chapter 2

# Geodesics and Curvature

## 2.1 Geodesics

Now we have enough machinery to introduce the concept of *straight lines* onto arbitrary Riemannian manifolds. Remember that velocity vectors of a curve is in particular a vector field along a curve and we can differentiate vector field along curves using covariant derivative. Hence defining geodesics are curves of zero acceleration we get:

**Definition 2.1.1.** *A curve $\gamma : I \subseteq \mathbb{R} \longrightarrow M$ is called a geodesic at $t_0$ if $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ at $t = t_0$. If $\gamma$ is a geodesic at $t$, $\forall t \in I$, then we say $\gamma$ is a geodesic.*

Henceforth we shall assume that the connection on $M$ is Levi-Civita connection. By definition if $\gamma$ is a geodesic then $\frac{d}{dt}\langle \frac{d\gamma}{dt} \frac{d\gamma}{dt} \rangle = 2\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle = 0$, which implies geodesics are curves with velocity vectors of constant length. The arc length $L(\gamma(t)) = \int_{t_0}^{t} |\frac{d\gamma}{dt}| dt$ is proportional to the parameter in the case of a geodesic.

If we consider a geodesic $\gamma$ in a system of local coordinates $(U, x)$ around a point $\gamma(t_0)$, then $\gamma(t) = (x_1(t), .., x_n(t))$ in $U$. This will be a geodesic if and only if $0 = \frac{D}{dt}(\frac{d\gamma}{dt}) = \sum_k (\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt})$. Thus we obtain a second order ODE:

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma^k_{i,j} \frac{dx_i}{dt}\frac{dx_j}{dt} = 0, \quad k = 1, .., n \tag{2.1}$$

**Theorem 2.1.2** (Fundamental theorem on flows)**.** *Let $X \in \tau(M)$ and $V \subseteq M$ open $p \in V$. Then there exist open $V_0 \subseteq V$, $p \in V_0$, $\delta > 0$ such that $\forall q \in V_0$ there exist a smooth map $\varphi : (-\delta, \delta) \times V_0 \longrightarrow V$ such that $t \mapsto \varphi(t, q)$ is the unique integral curve of $X$ at $t = 0$ passes through $q \in V_0$.*

Every curve on a manifold $M$ determines a unique curve in $TM$. If $t \longrightarrow \gamma(t)$ is a smooth curve in $M$ then $t \longrightarrow (\gamma(t), \frac{d\gamma}{dt}(t))$ is a unique curve in $TM$. Taking coordinate for $(q, v)$ as $(x_1, ..x_n, y_1, .., y_n)$ in a coordinate neighborhood, we obtain the following system of first order differential equations.

$$\frac{dx_k}{dt} = y_k,$$
$$\frac{dy_k}{dt} = -\sum_{i,j} \Gamma^k_{ij} y_i y_j$$

Existence and uniqueness of ODE affirms the fact that geodesics exist on an arbitrary Riemannian manifold.

**Lemma 2.1.3.** *There exists a unique vector field $G$ on $TM$ whose integral curves are $t \mapsto (\gamma(t), \frac{d\gamma}{dt})$, where $\gamma$ is a geodesic.*

We call $G$ the geodesic field on the tangent bundle and the flow of it the geodesic flow. We now apply the fundamental theorem of flows onto $G$ around a point $(p, 0) \in TM$. By the theorem there exist $TU_0 \subseteq TU$, a number $\delta > 0$ and a smooth map $\varphi : (-\delta, \delta) \times TU_0 \longrightarrow TU$ such that $t \to \varphi(t, q, v)$ is the unique trajectory of $G$ which has the initial condition $\varphi(0, q, v) = (q, v)$ for each $(q, v) \in TU_0$. In order to obtain this result in a useful form, consider $TU_0$ of the following form. $TU_0 = \{(q, v) \in TU \mid q \in V \textit{ and } v \in T_qM \textit{ with } |v| < \varepsilon\}$ where $V \subseteq U$ is a neighborhood of $p \in M$. If we define $\gamma = \pi \circ \varphi$, where $\pi : TM \longrightarrow M$ is the canonical projection, then the above result becomes:

**Proposition 2.1.4.** *Given a point $p \in M$, there exist an open set $V \subset M$, numbers $\delta > 0$ $\varepsilon > 0$ and a smooth mapping $\gamma : (-\delta, \delta) \times TU_0 \longrightarrow M$ ($TU_0 = \{(q, v) \in TU \mid q \in V \textit{ and } v \in T_qM \textit{ with } |v| < \varepsilon\}$) such that the curve $t \mapsto \gamma(t, q, v)$ is the unique geodesic of $M$ which at $t = 0$ passes through $q$ with a velocity $v$ for all $q \in V$ and for all $v \in T_qM$ where $|v| < \varepsilon$.*

This proposition allows us to talk about geodesic starting from any point in a Riemannian manifold in any given direction. The following lemma says that it is possible to control the velocity by controlling the size of the interval in which the geodesic is defined.

**Lemma 2.1.5.** *If the geodesic $\gamma(t, q, v)$ is defined on the interval $(-\delta, \delta)$ then the geodesic $\gamma(t, q, av)$ is defined on the interval $(-\frac{\delta}{a}, \frac{\delta}{a})$ where $a$ is a positive real number and $\gamma(t, q, av) = \gamma(at, q, v)$*

By the use of this lemma we can make the interval in which a geodesic is defined uniformly large. Using this we can introduce exponential map which allows us to study geodesics in a more elegant manner.

**Definition 2.1.6.** *Let $p \in M$ and $TU_0$ as before with $\epsilon < \delta$ then the map* $\exp : TU_0 \longrightarrow M$ *defined by* $\exp(q, v) = \gamma(1, q, v) = \gamma(|v|, q, \frac{v}{|v|})$ *is called the exponential map on $TU_0$.*

We often consider the map $\exp_q : B_\epsilon(0) \subset T_qM \longrightarrow M$ defined by $\exp_q(v) = \exp(q, v)$, where $B_\epsilon(0)$ is an open ball around $0 \in T_qM$ with radius $\epsilon$. If we analyze the definition $\exp_q(v)$ is the point obtained by moving a distance $|v|$ along the geodesic starting from $q$ in the direction of $v$ with unit speed.

**Proposition 2.1.7.** *Given $q \in M$ there exists an open neighborhood $V$ of $0 \in T_qM$ such that $\exp_q : V \longrightarrow M$ is a diffeomorphism onto its image.*

**Definition 2.1.8.** *We call the image of $V \subset T_qM$ in which the $\exp_q$ map is a diffeomorphism, a normal neighborhood of $q \in M$. In particular image of $B_\epsilon(0) \subset V \subset T_qM$ is called a normal ball of radius $\epsilon$ (or sometimes geodesic ball) around $q \in M$.*

Till now we have not studied the minimizing property of geodesic which we expected it to satisfy. We say a curve joining two points is minimizing if it has length less than or equal to length of all the piecewise smooth curve joining the two points. The following lemma due to Gauss is crucial in the proof of length minimizing property of geodesics.

**Lemma 2.1.9.** *Let $p \in M$ and $v \in T_pM$ such that $\exp_p v$ is defined. Let $w \in T_v(T_pM) \equiv T_pM$, then $\langle (d\exp_p)_v(v), (d\exp_p)_v(w) \rangle = \langle v, w \rangle$*

If we consider the case of a 2-sphere in which geodesics are great circles, they are not length minimizing as soon as it passes the antipodal point of its starting point. Hence it is clear that geodesics are not globally length minimizing. Next proposition clears the point.

**Proposition 2.1.10.** *Let $p \in M$, $U$ be a normal neighborhood of $p$ and $\gamma : [0, a] \longrightarrow U$ be a geodesic segment which lies entirely inside $U$. If any other piecewise smooth curve $\alpha : [0, a] \longrightarrow M$ joining $\gamma(0)$ and $\gamma(a)$ then $L(\gamma) \leq L(\alpha)$ and if they are equal then $\gamma([0, a]) = \alpha([0, a])$*

In order to prove the converse of this we need to introduce a slightly stronger condition than a normal neighborhood has.

**Theorem 2.1.11.** *For every $p \in M$ there exist a neighborhood $W$ of $p$ such that for every $q \in W$, $W$ is a normal neighborhood of $q$ also.*

**Corollary 2.1.12.** *If a piecewise smooth curve $\gamma : I \longrightarrow M$ with parameter proportional to its arc length has length less than or equal to any other piecewise smooth curve joining the end points of $\gamma$ then $\gamma$ is a geodesic.*

## 2.2 Curvature

A vector in $\mathbb{R}^2$ can be parallely transported throughout the whole space by moving it parallely along the coordinate axises. But if we try to do it for any arbitrary surfaces, it turns out that such a transport is not possible. If at all such a transport is possible then the surface must be isometric to $\mathbb{R}^2$. Our intuitive perception of such a behavior is due to the *curvature* of the surface. A little analysis shows that this arises due to the non-commutativity of covariant derivative. Thus we define the curvature as:

**Definition 2.2.1.** *Let $X, Y, Z \in \tau(M)$, then the Riemann curvature endomorphism is a map $R : \tau(M) \times \tau(M) \times \tau(M) \longrightarrow \tau(M)$ defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$*

Riemann curvature endomorphism is a $\binom{3}{1}$ tensor field. We define a Riemann curvature tensor to be a covariant 4-tensor defined by $Rm(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle$

**Theorem 2.2.2.** *A Riemannian manifold is locally isometric to a Euclidean space if and only if its Riemann curvature tensor vanishes identically.*

**Proposition 2.2.3.** *The Riemannian curvature tensor has the following properties. If $X, Y, Z, W \in \tau(M)$ then:*

1. $Rm(W, X, Y, Z) = -Rm(X, W, Y, Z)$
2. $Rm(W, X, Y, Z) = -Rm(W, X, Z, Y)$
3. $Rm(W, X, Y, Z) = Rm(Y, Z, W, X)$
4. $Rm(W, X, Y, Z) + Rm(X, Y, W, Z) + Rm(Y, W, X, Z) = 0$

We now define the sectional curvature.

**Definition 2.2.4.** *Let $\sigma$ be a two dimensional subspace of $T_pM$ where $M$ is a Riemannian manifold and $X, Y$ be two linearly independent vectors in $\sigma$, then the sectional curvature is defined to be $K(\sigma) := K(X, Y) = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2 - \langle X,Y \rangle^2}$*

It is easy to see that the definition of $K(\sigma)$ does not depend on the choice of the basis. Using the properties of curvature tensor one can do the elementary transformations by keeping the sectional curvature the same. i.e $K(X, Y) = K(Y, X) = K(\lambda X, Y) = K(X + \lambda Y, Y)$. We can go from one basis to any other basis using these elementary transformations.

We will mention some special combination of sectional curvature, Ricci curvature and scalar curvature.

**Definition 2.2.5.** *Let $M$ be a Riemannian manifold and $p \in M$ and $X \in T_pM$ such that $|X| = 1$. Extend $X$ to a set of orthonormal basis $\{E_i\}_{i=1}^{n-1}$. Then*

$$Ric_p(X) = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(X, E_i)E_i, X\rangle$$

*And scalar curvature,*

$$K(p) = \frac{1}{n}\sum_{j=1}^{n} Ric_p(E_j) = \frac{1}{n(n-1)}\sum_{ij}\langle R(E_i, E_j)E_j, E_i\rangle$$

Sectional curvature is worth investigating because it determines the Riemannian curvature of a manifold completely.

**Lemma 2.2.6.** *Let $V$ be an $n(\geq 2)$ dimensional vector space with an inner product. $R : V \times V \times V \longrightarrow V$ and $R' : V \times V \times V \longrightarrow V$ be tri-linear maps and $Rm(X, Y, Z, W) := \langle R(X, Y)Z, W\rangle$, $Rm'(X, Y, Z, W) := \langle R'(X, Y)Z, W\rangle$ satisfies the four symmetries of Riemannian metric. Define $K(\sigma) = \frac{Rm(X,Y,Y,X)}{|X|^2|Y|^2-\langle X,Y\rangle^2}$ and $K'(\sigma) = \frac{Rm'(X,Y,Y,X)}{|X|^2|Y|^2-\langle X,Y\rangle^2}$ where $X, Y$ are two linearly independent vectors in the two dimensional subspace $\sigma \subset V$. If $K(\sigma) = K'(\sigma)$ for all $\sigma \subset V$ then $R = R'$*

From the above lemma it is clear that sectional curvature captures all the information of Riemannian curvature. A geometric explanation of sectional curvature will be given later in the section where we explain isometrically immersed manifolds. We define two other curvature appear during our study.

# Chapter 3

# Gauss Bonnet Theorem

Gauss-Bonnet theorem which applies to a compact orientable 2 dimensional Riemannian manifolds links integral of gaussian curvature which is a local property to Euler characteristic of the manifold which is a global property. A curve $\gamma : [0, a] \longrightarrow \mathbb{R}^2$ is called an admissible curve if it is piecewise smooth and regular, i.e $\frac{d\gamma}{dt} \neq 0$ whenever it is defined. We call such a curve simple if it is injective in $[a, b)$ and closed if $\gamma(0) = \gamma(a)$. We can define the tangent angle $\theta : [0, a] \longrightarrow (-\pi, \pi]$ such that $\frac{d\gamma(t)}{dt} = (cos\theta(t), sin\theta(t))$. Existence of such a map can be proved using theory of covering maps. For smooth closed curves we can define rotation angle $Rot(\gamma) := \theta(a) - \theta(0)$. This is clearly a multiple of $2\pi$. We can extend this concept to piecewise smooth curves also. If $t = t_i$ is a point where the jump occurs we define exterior angle $\varepsilon_i$ to be the angle from $\lim_{t \to a_i^-} \frac{\gamma(t)-\gamma(t_i)}{t-t_i}$ to $\lim_{t \to t_i^+} \frac{\gamma(t)-\gamma(t_i)}{t-t_i}$ which are left hand tangent vectors and right hand tangent vectors respectively. By a curved polygon we mean a piecewise smooth simple closed curve which has exterior angle $\varepsilon_i' s \neq \pm\pi$. In the case of curved polygon we define tangent angle as follows. Let $0 = t_0 < t_1 .. < t_k = a$ be the points where the jumps occur. We define $\theta : [0, t_1) \longrightarrow \mathbb{R}$ as earlier and at $t_1$, $\theta(t_1) = lim_{t \to t_1^-} \theta(t) + \varepsilon_1$. We continue this inductively at each jump and finally $\theta(a) = lim_{t \to t_k^-} \theta(t) + \varepsilon_k$

**Theorem 3.0.1.** *If $\gamma$ is a positively oriented curved polygon in a plane the rotation angle is exactly $2\pi$.*

## 3.1 Gauss Bonnet Formula

Consider an oriented Riemannian 2-manifold $M$. A piecewise smooth curve is called a curved polygon in $M$ if is the boundary of an open set with compact closure and there exists a coordinate chart which contains the curve and the image of the curve under the coordinate map is a curved polygon in $\mathbb{R}^2$

**Lemma 3.1.1.** *If $\gamma$ is a positively oriented curved polygon then the rotation angle of $\gamma$ is $2\pi$.*

If $\gamma$ is a positively oriented curved polygon then we can talk about a unit normal vector $N(t)$ such that $(\frac{d\gamma}{dt}, N(t))$ forms an orthonormal basis for $T_{\gamma(t)}M$ whenever $\frac{d\gamma}{dt}$ is defined. We can insist it to be inward pointing if we assume the condition that $(\frac{d\gamma}{dt}, N(t))$ has the same orientation as that of the open set enclosed in the curve $\gamma$. We define signed curvature of $\gamma$ at its smooth points to be $\kappa_N(t) = \langle \frac{D}{dt}(\frac{d\gamma}{dt}), N(t)\rangle$. Since $\frac{D}{dt}\frac{d\gamma}{dt}$ is orthogonal to $\frac{d\gamma}{dt}$ $(\because |\frac{d\gamma}{dt}|^2 \equiv 1)$, we have $\frac{D}{dt}\frac{d\gamma}{dt} = \kappa_N(t)N(t)$.

**Theorem 3.1.2** (Gauss Bonnet Formula)**.** *Let $\gamma$ be a curved polygon on a two dimensional Riemannian manifold and $\Omega$ be an open set of which $\gamma$ is the boundary of then:*

$$\int_\Omega K dA + \int_\gamma \kappa_N(s)ds + \sum_i \varepsilon_i = 2\pi \tag{3.1}$$

*where $K$ is the gaussian curvature and $dA$ is the Riemannian volume form.*

*Proof.* Consider $0 = t_0 < t_1 < ... < t_k = a$ subdivision of $[0, a]$. From Lemma 3.1.1 and fundamental theorem of calculus we get

$$2\pi = \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \theta'(t)dt \tag{3.2}$$

Let $\mathcal{U}$ be the coordinate neighborhood containing $\Omega$ (hence $\gamma$). Take an orthonormal frame $E_1, E_2$ such that $E_1 = c\frac{\partial}{\partial x}$, where $c > 0$. (This can be done by Gram-Schmidt process on $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})$) As $\theta(t)$ is the angle between $E_1$ and $\gamma'(t)$ we can write

$$\gamma(t) = \cos\theta(t)E_1 + \sin\theta(t)E_2 N(t) = -\sin\theta(t)E_1 + \cos\theta(t)E_2$$

Taking the covariant derivative of $\gamma'(t)$ we get

$$D_t\gamma' = \cos\theta\nabla_{\gamma'}E_1 - \theta'(\sin\theta)E_1 + \sin\theta\nabla_{\gamma'}E_2 \tag{3.3}$$
$$= \theta' N + \cos\theta\nabla_{\gamma'}E_1 + sin\theta\nabla_{\gamma'}E_2 \tag{3.4}$$

Since $E_1$ and $E_2$ are orthonormal for any vector $X$ we have,

$$\nabla_X |E_1|^2 = 0 = 2\langle \nabla_X E_1, E_1 \rangle \tag{3.5}$$
$$\nabla_X |E_2|^2 = 0 = 2\langle \nabla_X E_2, E_2 \rangle \tag{3.6}$$
$$\nabla_X \langle E_1, E_2 \rangle = 0 = \langle \nabla_X E_1, E_2 \rangle + \langle E_1, \nabla_X E_2 \rangle \tag{3.7}$$

From (3.5) and (3.6) it follows that $\nabla_X E_2 = c_1 E_1$ and $\nabla_X E_1 = c_2 E_2$ for some $c_1, c_2 \in \mathbb{R}$. Thus we define a $1-$form $\eta(X) = \langle \nabla_X E_1, E_2 \rangle = -\langle E_1, \nabla_X E_2 \rangle$. Therefore $\nabla_X E_1 = -\eta(X) E_2$ and $\nabla_X E_2 = \eta(X) E_1$. Using this information we calculate $\kappa_N$.

$$\begin{aligned}
\kappa_N &= \langle D_t \gamma', N \rangle \\
&= \langle \theta' N, N \rangle + \sin\theta \langle \nabla_{\gamma'} E_2, N \rangle + \cos\theta \langle \nabla_{\gamma'} E_1, N \rangle \\
&= \theta' + \sin\theta \langle \eta(\gamma') E_1, N \rangle - \cos\theta \langle \eta(\gamma') E_2, N \rangle \\
&= \theta' - \cos^2\theta \eta(\gamma') - \sin^2\theta \eta(\gamma') \\
&= \theta' - \eta(\gamma')
\end{aligned}$$

Therefore we can write (3.2) as

$$\begin{aligned}
2\pi &= \sum_{i=1}^{k} \epsilon_i + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \kappa_N(t) dt + \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \eta(\gamma'(t)) dt \\
&= \sum_{i=1}^{k} \epsilon_i + \int_\gamma \kappa_N + \int_\gamma \eta
\end{aligned}$$

It remains to show that $\int_\gamma \eta = \int_\Omega K dA$. By using Stoke's theorem we get $\int_\gamma \eta = \int_\Omega d\eta$. (Even if Stoke's theorem is for smooth curves we can derive the same result for piecewise smooth curves also). Thus $2\pi = \sum_{i=1}^{k} \epsilon_i + \int_\gamma \kappa_N + \int_\Omega d\eta$. As $(E_1, E_2)$ are orthonormal we have $dA(E_1, E_2) = 1$. As $K dA(E_1, E_2) = K = R(E_1, E_2, E_2, E_1)$, using the definition and properties of curvature we obtain, $K dA(E_1, E_2) = d\eta(E_1, E_2)$ which concludes the proof.

□

This theorem has some obvious corollaries

**Corollary 3.1.3.** *In a Euclidean plan the sum of the interior angles of a triangle is* $\pi$

**Corollary 3.1.4.** *In a Euclidean plan circumference of a circle of radius* $r$ *is* $2\pi r$

### 3.1.1 Gauss-Bonnet Theorem

For a smooth compact 2 dimensional manifold a smooth triangulation is a finite collection of curved triangle such that if we consider the closure of the area enclosed by the triangles, it is the whole manifold and the intersection of any of the two curved triangles is either empty, single vertex or single edge. A theorem from algebraic topology asserts that every compact smooth two dimensional manifold has a smooth triangulation also for a given two dimensional manifold the value $\chi(M) = N_v - N_e + N_f$ (where $N_v$ is the number of vertices of a given traingulation, $N_e$ is the number of edges and $N_f$ is the number of faces) does not depend on the triangulation. We call $\chi(M)$ the Euler characteristic of the manifold.

**Theorem 3.1.5** (Gauss-Bonnet Theorem)**.** *Let $M$ be a two dimensional compact orientable Riemannian manifold then $\int_M KdA = 2\pi\chi(M)$.*

*Proof.* Consider any smooth triangulation of $M$. Let $\{\phi_i\}_{i=1}^{N_f}$ denote the faces. For each face $\phi_i$ let $\eta_{ij}$ and $\theta_{ij}$ denote the edges and interior angles respectively ($j = 1, 2, 3$). Applying Gauss-Bonnet formula to each triangle we obtain,

$$\sum_{i=1}^{N_f} \int_{\phi_i} KdA + \sum_{i=1}^{N_f} \sum_{j=1}^{3} \int_{\eta_{ij}} \kappa_N ds + \sum_{i=1}^{N_f} \sum_{j=1}^{3} (\pi - \theta_{ij}) = \sum_{i=1}^{N_f} 2\pi$$

For a smooth triangulation of a compact surface each edge is shared by exactly two triangles. Therefore in the above integral each side appears twice with opposite orientation. Hence,

$$\int_A KdA + 3\pi N_f - \sum_{i=1}^{N_f} \sum_{j=1}^{3} \theta_{ij} = 2\pi N_f$$

At each vertex the sum of angles add upto $2\pi$. Therefore $\sum_{i=1}^{N_f} \sum_{j=1}^{3} \theta_{ij} = 2\pi N_v$. Thus $\int_A KdA + \pi N_f = 2\pi N_v$. From the properties of triangulation we can see that $2N_e = 3N_f$, as each triangle has three edges and each edge is shared by exactly two triangles. Therefore $2\pi N_v - \pi N_f = 2\pi N_v - 2\pi N_e + 2\pi N_f = 2\pi\chi(M)$. Thus we obtain the Gauss-Bonnet theorem.

$$\int_A KdA = 2\pi\chi(M)$$

□

Gauss-Bonnet theorem along with classification theorem for compact surfaces gives a complete picture of curvature on compact surfaces.

# Chapter 4

# Jacobi Fields

## 4.1 Jacobi Fields

Guass-Bonnet theorem was an example of a local-global theorem. Our aim now is to generalize this theorem to higher dimensions. We used Stokes' theorem and differential forms to prove the G-B theorem. We now change our approach with the following observation. When a simply connected manifold has positive curvature geodesics tend to come closer with time while for a simply connected manifold with negative curvature geodesics tend to move away from each other. This link between curvature and the behavior of geodesics are further investigated in this section.

**Definition 4.1.1.** *A smooth family of curves on a Riemannian manifold $M$, $\Gamma : (-\delta, \delta) \times [a, b] \longrightarrow M$, is called a variation through geodesic $\gamma$ if $\Gamma_0(t) = \gamma(t)$ and $\Gamma_s(t)$ is a geodesic for each $s \in (-\delta, \delta)$. A vector field $V$ along $\gamma$ is called the variation field of the variation $\Gamma$ if $V(t) = \frac{\partial \Gamma(0,t)}{\partial s}$. We denote $T(s,t) = \frac{\partial \Gamma(s,t)}{\partial t}$ and $S(s,t) = \frac{\partial \Gamma(s,t)}{\partial s}$.*

We now look at the equation satisfied by the variation field of the variation through geodesic. First consider the following lemma.

**Lemma 4.1.2.** *Let $V$ be any smooth vector field along a smooth family of curves $\Gamma$ then $D_s D_t V - D_t D_s V = R(S,T)V$.*

*Proof.* As covariant derivative is defined locally it is enough to consider this problem in a coordinate neighborhood. Let $V(s,t) = \sum_i V_i(s,t) \frac{\partial}{\partial x_i}$. By definition $D_t V = \sum_i (\frac{\partial V_i}{\partial t} \frac{\partial}{\partial x_i} + V_i D_t \frac{\partial}{\partial x_i})$ and $D_s D_t V = \sum_i (\frac{\partial^2 V_i}{\partial s \partial t} \frac{\partial}{\partial x_i} + \frac{\partial V_i}{\partial t} D_s \frac{\partial}{\partial x_i} + \frac{\partial V_i}{\partial s} D_t \frac{\partial}{\partial x_i} + V_i D_s D_t \frac{\partial}{\partial x_i})$. Similarly we obtain an expression for $D_t D_s V$ and subtracting one from the other we get, $D_s D_t V - D_t D_s V =$

$\sum_i V_i(D_sD_t\frac{\partial}{\partial x_i} - D_tD_s\frac{\partial}{\partial x_i})$. Using the properties of covariant derivative one can see after some manipulations that $D_sD_t\frac{\partial}{\partial x_i} - D_tD_s\frac{\partial}{\partial x_i} = R(S,T)\frac{\partial}{\partial x_i}$ Hence we obtain $D_sD_tV - D_tD_sV = R(S,T)V$. □

**Theorem 4.1.3.** *Let $\gamma$ be a geodesic and $V$ be a variation field of variation of $\gamma$ through geodesics then*

$$D_t^2V + R(V,\gamma')\gamma' = 0 \tag{4.1}$$

*Proof.* Consider vector fields $T$ and $S$ defined before. As the variation we are considering is a variation through geodesic we have, $D_tT \equiv 0$ and therefore $D_sD_tT = 0$. From the previous lemma and the fact that $D_s\frac{\partial\Gamma}{\partial t} = D_t\frac{\partial\Gamma}{\partial s}$ (due to the symmetry of the connection) we get that $D_tD_sT + R(S,T)T = 0 = D_tD_tS + R(S,T)T$. Substituting $s = 0$ we get $D_t^2V + R(V,\gamma')\gamma' = 0$. □

**Definition 4.1.4.** *A smooth vector field $J$ along a geodesic $\gamma$ is called a Jacobi field if $J$ satisfies equation* (4.1)*.*

It can be easily shown that every Jacobi field is a variation field of some variation through geodesics. The existence and uniqueness of Jacobi fields are guaranteed by the following proposition.

**Proposition 4.1.5.** *Let $\gamma$ be a geodesic and $\gamma(a) = p$. For any tangent vectors $X, Y \in T_pM$ there exists a unique Jacobi field $V$ along $\gamma$ satisfying the initial condition $V(a) = X$ and $D_tV(a) = Y$.*

*Proof.* Choose an orthonormal frame $\{E_i\}$ along $\gamma$. (We can obtain this by parallel transporting an orthonormal basis of $T_pM$.) We can write $J(t) = \sum_i J_i(t)E_i(t)$ and $R(E_j,E_k)E_l = \sum_i R^i_{jkl}E_i$. Then the Jacobi equation becomes $\frac{d^2}{dt^2}J_i + \sum_{j,k,l} R^i_{jkl}J_j\gamma'_k\gamma'_l = 0$ which gives $n$ second order linear ODEs. If we make a substitution $\frac{dJ_i}{dt} = V_i$ then we will get $2n$ first order linear equation. Then the existence and uniqueness of solution for linear ODEs we get the desired result. □

There are trivial Jacobi fields along any geodesic $\gamma$. $V(t) = \gamma'(t)$ is a Jacobi field with initial conditions $V(0) = \gamma'(0)$ and $D_tV(0) = 0$. There is another one $V(t) = t\gamma'(0)$ with the initial condition $V(0) = 0$ and $D_tV(0) = \gamma'(0)$. These Jacobi fields does not provide any useful information about the behavior of geodesics. Hence we make a distinction as following; Jacobi fields which are a multiple of the velocity vector of the geodesic as tangential Jacobi field and Jacobi fields which are perpendicular to the velocity vector fields as normal Jacobi fields.

Consider normal neighborhood $\mathcal{U}$ of p and an isomorphism $A : T_pM \longrightarrow \mathbb{R}^n$ which takes orthonormal basis vectors to orthonormal basis vectors. We can find $U \subset \mathbb{R}^n$ such that $\exp_p \circ A^{-1}(U) = \mathcal{U}$. In this manner we can define a coordinate chart for the manifold $M$. We call such a chart the *normal coordinate chart.*

**Lemma 4.1.6.** *Let $(x,\mathcal{U})$ be a normal coordinate chart for $p \in M$ and $\gamma$ be a geodesic starting at p. For any vector $W = \sum_i W_i \frac{\partial}{\partial x_i} \in T_pM$, the Jacobi field $J$ along the geodesic with $J(0) = 0$ and $D_tJ(0) = W$ is given by $J(t) = t\sum_i W_i \frac{\partial}{\partial x_i}$*

*Proof.* We know that in normal coordinates a geodesic $\gamma$ starting from $p$ and $\gamma'(0) = V = \sum_i V_i \frac{\partial}{\partial x_i}$ is given by $\gamma(t) = (tV_1, ..., tV_n)$. Then variation given by $\Gamma(s,t) = (t(V_1 + sW_1), ..., t(V_n + sW_n))$ is easily seen to be a variation of $\gamma$ through geodesics. Thus $\frac{\partial \Gamma(s,t)}{\partial s} = J(t)$ is a Jacobi field ☐

We can see from the above lemma that in such a case a Jacobi field cannot have more than one zeros in a normal neighborhood. This is in fact true for any Jacobi field in a normal neighborhood as we shall prove in the next section. Riemannian manifolds with constant sectional curvature are of special interest. The following lemma gives us an expression of normal Jacobi fields in three different cases.

**Lemma 4.1.7.** *Let $M$ be a Riemannian maifold with constant sectional curvature $K$. Then the normal Jacobi fields along a unit speed geodesic $\gamma$ vanishing at $t = 0$ are $V(t) = u(t)E(t)$ where $E(t)$ is any parallel normal vector field along $\gamma$ and $u(t) = t$ if $C = 0$, $u(t) = Rsin\frac{t}{R}$ if $C = \frac{1}{R^2} > 0$ and $u(t) = Rsinh\frac{t}{R}$ if $C = -\frac{1}{R^2} < 0$.*

These two lemmas give an interesting application of the Jacobi fields. We can have a characterization of metrics of Riemannian manifolds with constant sectional curvature.

**Proposition 4.1.8.** *Let (M,g) be a Riemannian manifold with constant sectional curvature K. Let $(x,\mathcal{U})$ be a normal coordinate chart around $p \in \mathcal{U} \subset M$. Let $r(x) = \sqrt{\sum_i (x^i)^2}$ and $|.|_{\overline{g}}$ be the Euclidean norm in these coordinates. Consider $q \in \mathcal{U} \setminus \{p\}$ and $V \in T_qM$. Write $V = V^{\perp} + V^T$ where $V^T$ is the tangential component of $V$ along the geodesic sphere through $q$ and $V^{\perp}$ is the radial component. The metric $g$ can be written as:*

$$g(V,V) = |V^{\perp}|^2_{\overline{g}} + |V^T|^2_{\overline{g}},\ if\ K = 0$$
$$g(V,V) = |V^{\perp}|^2_{\overline{g}} + \frac{R^2}{r^2}(sin^2\frac{r}{R})|V^T|^2_{\overline{g}},\ if\ K = \frac{1}{R^2} > 0$$
$$g(V,V) = |V^{\perp}|^2_{\overline{g}} + \frac{R^2}{r^2}(sinh^2\frac{r}{R})|V^T|^2_{\overline{g}},\ if\ K = -\frac{1}{R^2} > 0$$

With this complete characterization of Riemannian metric on a manifold with constant sectional curvature one has the following interesting result.

**Proposition 4.1.9.** *Consider two Riemannian manifolds (M,g) and* $(\overline{M}, \overline{g})$ *with constant sectional curvature* $K$*. For any two points* $p \in M$ *and* $\overline{p} \in \overline{M}$ *there exists a neighborhoods* $\mathcal{U}$ *of* $p$ *and* $\overline{\mathcal{U}}$ *of* $\overline{p}$ *which are isometric. In other words any two Riemannian manifold with constant sectional curvature* $K$ *is locally isometric.*

## 4.2 Conjugate Points

Jacobi fields can also answer the question of when an exponential map is a local diffeomorphism. We have seen earlier in the case of 2-sphere that its geodesics are not minimizing past its antipodal point. Jacobi fields shed light on how to determine the normal neighborhood of a point (as we have seen previously in lemma 4.1.4).

**Definition 4.2.1.** *For* $p, q \in M$ *let* $\gamma$ *be a geodesic joining* $p$ *and* $q$*, then* $q$ *is called a conjugate to* $p$ *if there exists a Jacobi field along the geodesic which vanishes at* $p$ *and* $q$ *but not on all of* $\gamma$ *and multiplicity of the conjugate point is the dimension of the space of such Jacobi fields.*

**Proposition 4.2.2.** *Let* $\gamma : [0, a] \longrightarrow M$ *be a geodesic with* $\gamma(0) = p$ *and* $\gamma'(0) = X$*. Consider a Jacobi field with* $J(0) = 0$ *and* $D_t J(0) = V$*. Consider* $\Gamma(s, t) = \exp_p(\frac{t}{a}\alpha(s))$ *where* $\alpha$ *is a curve in* $T_pM$ *with* $\alpha(0) = aX$ *and* $\alpha'(0) = V$*. If we define* $\overline{J} = \frac{\partial \Gamma(t,0)}{\partial s}$ *then* $\overline{J} = J$ *on* $[0, a]$*.*

*Proof.*

$$\begin{aligned} D_t \frac{\partial \Gamma(t,0)}{\partial s} &= D_t(((d\exp_p)_{tX}(tV)) \\ &= D_t(t(d\exp_p)_{tX}(V)) \\ &= (d\exp_p)_{tX}(V) + tD_t(d\exp_p)_{tX}V \end{aligned}$$

Thus for $t = 0$

$$D_t\overline{J}(0) = D_t\frac{\partial}{\partial s}\Gamma(0,0) = (d\exp_p)_0(V) = V$$

Hence by the uniqueness of Jacobi field with same initial condition we get $J = \overline{J}$ □

**Corollary 4.2.3.** *For a geodesic* $\gamma : [0, a] \longrightarrow M$*, Jacobi field* $J$ *along* $\gamma$ *with* $J(0) = 0$ *and* $D_t J = X$ *is given by* $J(t) = (d\exp_p)_{t\gamma'(0)} tX$

**Proposition 4.2.4.** *Let $\gamma : [0, a] \longrightarrow M$ be a geodesic joining $p$ and $q$ and let $q$ be a conjugate point along $\gamma$ if and only if exponential map is not a diffeomorphism around $a\gamma'(0)$.*

*Proof.* By definition $q$ is a conjugate point of $p$ along $\gamma$ if and only if there exist a nontrivial Jacobi field which vanish at both $p$ and $q$. But by the previous corollary Jacobi field $J$ with $D_t J(0) = X$ along $\gamma$ satisfies $J(a) = (d \exp_p)_{a\gamma'(0)} aX = 0$ which implies $(d \exp_p)_{a\gamma'(0)}$ is not injective as $X \neq 0$. Since $(d \exp_p)_{a\gamma'(0)}$ is a linear map between spaces of same dimension by inverse function theorem we conclude that $\exp_p$ is not a local diffeomorphism at $a\gamma'(0)$ and the assertion is proved. □

Jacobi fields are useful tools to study the behavior of geodesics and their relation with curvature. As Jacobi fields can be thought of as the perturbation given to a geodesic such that the perturbed curve remains geodesic. The ideas of conjugate points and Jacobi fields show up almost everywhere in our study from now on.

# Chapter 5

# Isometric Immersions

The manifolds that we often encounter are those that are immersed in the Euclidean space. So it is important to study isometrically immersed manifolds. Consider an immersion $F : M \longrightarrow \overline{M}$ where $M$ is $m$ dimensional and $\overline{M}$ is $n$ dimensional and $n = m+k$. A metric on $\overline{M}$ will naturally induce a metric on $M$. Hence we can consider this as an isometric immersion with the induced Riemannian metric.

## 5.1 Second Fundamental Form

Let $F : M \longrightarrow \overline{M}$ be an immersion. Then for each $p \in M$ there exists an open set $U \subset M$ containing $p$ such that $F(U)$ is a submanifold of $\overline{M}$. Using the metric on $\overline{M}$ $T_p\overline{M} = T_pM \oplus T_pM^{\perp}$. Hence every tangent vector $v \in T_p\overline{M}$ can be written as $v = v^T + v^{\perp}$ where $v^T \in T_pM$ and $v^{\perp} \in T_pM^{\perp}$. If $\overline{\nabla}$ denotes the Riemannian connection on $\overline{M}$, $X, Y$ are local vector fields on $M$ and $\overline{X}, \overline{Y}$ are extensions of $X, Y$ then we define $\nabla_X Y = (\overline{\nabla}_{\overline{X}}\overline{Y})^T$. This is the induced Riemannian connection on $M$. It is easy to see that Riemannian connection coming from induced metric via $F$ is same as the connection defined above. As Riemannian connection on a manifold is unique, in order to see this fact, it is enough to show that the new connection defined will satisfy the compatibility condition and symmetry.

In order to define the second fundamental form we define a map $B : \tau(U) \times \tau(U) \longrightarrow \tau(U)^{\perp}$ such that $B(X, Y) := \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y$ (This equation is called the Gauss formula). This definition does not depend on the extension of the local vector fields $X$ and $Y$. If $\overline{X}$ and $\overline{X_1}$ are two different extension of $X$ then $(\overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y) - (\overline{\nabla}_{\overline{X}_1}\overline{Y} - \nabla_X Y) = \overline{\nabla}_{\overline{X}-\overline{X}_1}\overline{Y} = 0$ as $\overline{X} - \overline{X_1}$ vanish on $M$. Similarly on can see for the second coordinate as well. Hence this map is well defined.

**Proposition 5.1.1.** *The map $B : \tau(U) \times \tau(U) \longrightarrow \tau(U)^{\perp}$ defined above is a symmetric bilinear map.*

*Proof.* Clearly $B$ is additive in both $X$ and $Y$ and $B(fX, Y) = fB(X, Y)$ for all $f \in C^{\infty}(U)$. All of this follows from the definition of the connection. We need to show that $B(X, fY) = fB(X, Y)\ \forall f \in C^{\infty}(U)$. Let $\overline{f}$ be a smooth extension of $f$ to $\overline{U}$. Then,

$$\begin{aligned} B(X, fY) &= \overline{\nabla}_{\overline{X}}(\overline{fY}) - \nabla_X(fY) \\ &= \overline{f}\overline{\nabla}_{\overline{X}}\overline{Y} + \overline{X}(\overline{fY}) - (f\nabla_X Y + X(f)Y) \end{aligned}$$

On $M$, $\overline{X}(\overline{f}) = X(f)$ and $\overline{Y} = Y$. Thus $B(X, fY) = fB(X, Y)$. To show that $B$ is symmetric we write, using the symmetry of connection, $B(X, Y) = (\overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y) = \overline{\nabla}_{\overline{Y}}\overline{X} + [\overline{X}, \overline{Y}] - (\nabla_Y X + [X, Y])$. As $[\overline{X}, \overline{Y}] = [X, Y]$ we get $B(X, Y) = B(Y, X)$.

□

Since it is bilinear $B(X, Y)(p)$ depends only on the values $X(p), Y(p)$. Now let $\eta \in T_pM^{\perp}$ we define $H_\eta : T_pM \times T_pM \longrightarrow \mathbb{R}$ by $H_\eta(X, Y) = \langle B(X, Y), \eta\rangle$. This is a symmetric bilinear form since $B$ is symmetric and bilinear.

**Definition 5.1.2.** *The quadratic form $II_\eta : T_pM \longrightarrow \mathbb{R}$ defined by $II_\eta(X) = H_\eta(X, X)$ is called the second fundamental form of $F$(isometric immersion) at $p$ along the vector $\eta$.*

In literature the term second fundamental form is sometimes used to denote the map $B$ itself or $H_\eta$. There is a self adjoint operator $S_\eta : T_pM \longrightarrow T_pM$ associated with the symmetric bilinear form $H_\eta$ and is given by $\langle S_\eta(X), Y\rangle = H_\eta(X, Y) = \langle B(X, Y), \eta\rangle$.

**Proposition 5.1.3.** *Let $p \in M$ and $X \in T_pM$ and $\eta \in T_pM^{\perp}$. Let $N$ be the local extension of $\eta$, then $S_\eta(X) = -(\overline{\nabla}_X N)^{\perp}$.*

*Proof.* As $X, Y \in T_pM$ extended locally, we denote the local extension which are tangent to $M$ with the same notation. As $\langle N, Y\rangle = 0$ we get

$$\begin{aligned} \langle S_\eta(X), Y\rangle &= \langle B(X, Y)(p), N\rangle = \langle \overline{\nabla}_X Y - \nabla_X Y, N\rangle(p) \\ &= \langle \overline{\nabla}_X Y, N\rangle(p) = -\langle Y, \overline{\nabla}_X N\rangle(p) \\ &= \langle -\overline{\nabla}_X N, Y\rangle \end{aligned}$$

This is true for all $Y \in T_pM$. Hence our proposition is proved. □

The case where immersions with codimension 1 is of particular interest. Let $\eta \in T_pM^{\perp}$ and $|\eta| = 1$. As $S_\eta : T_pM \longrightarrow T_pM$ is a symmetric linear transformation there exists a basis of eigenvectors of $T_pM$ $\{e_1, ..., e_n\}$ with eigenvalues $\lambda_1, .., \lambda_n$. If $M, \overline{M}$ are orientable manifolds then $\eta$ can be uniquely determined. If we demand both $\{e_1, ..., e_n\}$ and $\{e_1, ..., e_n, \eta\}$ to be a basis for the given orientation of $M$ and $N$ respectively then we call $e_i$'s the principle directions and $\lambda_i$'s the principle curvatures of $F$. Product $\lambda_1 ... \lambda_n = det(S_\eta)$ is called the Gauss-Kronecker curvature and $\frac{1}{n}(\lambda_1 + ... + \lambda_n)$ is called the mean curvature of $F$.

Next proposition shows the relation between second fundamental form and the sectional curvature. Let $K(X, Y)$ and $\overline{K}(X, Y)$ be the sectional curvature of $M$ and $\overline{M}$ respectively(of the subspace formed by linearly independent vectors $X, Y$).

**Theorem 5.1.4** (Gauss Equation)**.** *Let $p \in M$ and $X, Y$ be orthonormal vectors in $T_pM$, then*

$$K(X, Y) - \overline{K}(X, Y) = \langle B(X, X), B(Y, Y)\rangle - |B(X, Y)|^2.$$

*Proof.* Let us denote the local extension of $X, Y$ tangent to $M$ as $X, Y$ itself and extension to $\overline{M}$ as $\overline{X}, \overline{Y}$.

$$\begin{aligned} K(X, Y) - \overline{K}(X, Y) &= \langle \nabla_X \nabla_Y X - \nabla_Y \nabla_X X - (\overline{\nabla}_{\overline{X}} \overline{\nabla}_{\overline{Y}} \overline{X} - \overline{Y} \overline{\nabla}_{\overline{X}} \overline{X}), Y\rangle(p) \\ &= \langle \nabla_{[X,Y]} X - \overline{\nabla}_{[\overline{X},\overline{Y}]} \overline{X}, Y\rangle(p) \end{aligned}$$

But $\langle \nabla_{[X,Y]} X - \overline{\nabla}_{[\overline{X},\overline{Y}]} \overline{X}, Y\rangle(p) = \langle -(\overline{\nabla}_{[\overline{X},\overline{Y}]} \overline{X})^N, Y\rangle(p) = 0$. Now choose an orthonormal fields which are normal to $M$, $\{E_i\}_{i=1}^m$, where $m$ is the codimension of $M$ in $\overline{M}$. Then we can write $B(X, Y) = \sum_{i=1}^m H_{E_i}(X, Y)E_i$, where $H_{E_i}$ is the symmetric bilinear form we defined earlier. Thu we can write,

$$\begin{aligned} \overline{\nabla}_{\overline{X}} \overline{\nabla}_{\overline{Y}} \overline{X} &= \overline{\nabla}_{\overline{X}} (\sum_{i=1}^m H_{E_i}(X, Y)E_i + \nabla_Y X) \\ &= \sum_{i=1}^m H_{E_i}(X, Y) \overline{\nabla}_{\overline{X}} E_i + \overline{X} H_{E_i}(X, Y)E_i + \overline{\nabla}_{\overline{X}} \nabla_Y X \end{aligned}$$

Thus at $p$ we get $\langle \overline{\nabla}_{\overline{X}} \overline{\nabla}_{\overline{Y}} \overline{X}, Y\rangle = -\sum_i H_{E_i}(X, Y)H_{E_i}(X, Y) + \langle \nabla_X \nabla_Y X, Y\rangle$. Similarly we get an expression for $\langle \overline{\nabla}_{\overline{Y}} \overline{\nabla}_{\overline{X}} \overline{X}, Y\rangle = -\sum_i H_{E_i}(X, X)H_{E_i}(Y, Y) + \langle \nabla_Y \nabla_X X, Y\rangle$. Combining both the expressions we get the desired result.

□

An immersion is called geodesic at $p \in M$ if for every $\eta \in T_pM^\perp$ the second fundamental form $H_\eta$ is identically zero. An immersion is called totally geodesic if it is geodesic for all $p \in M$.

**Proposition 5.1.5.** *An immersion $F : M \longrightarrow \overline{M}$ is geodesic at $p \in M$ if and only if every geodesic of $M$ starting at $p$ is a geodesic of $\overline{M}$ at $p$.*

*Proof.* Let $\gamma$ be a geodesic starting at $p$ with initial velocity vector $X$ (we denote the normal extension of $X$ as $X$ itself). Let $\eta$ and $N$ be as before, we have $\langle X, N\rangle = 0$

$$\begin{aligned} H_\eta(X, X) &= \langle S_\eta(X), X\rangle = -\langle \overline{\nabla}_X N, X\rangle \\ &= X\langle N, X\rangle + \langle N, \overline{\nabla}_X X\rangle = \langle N, \overline{\nabla}_X X\rangle \end{aligned}$$

Thus immersion $F$ is geodesic at $p$ if and only if for all $X \in T_pM$ geodesic with initial velocity vector $X$ has the following property, $\overline{\nabla}_X X(p)$ does not have a normal component. Hence the proposition. □

We can give a geometric interpretation of sectional curvature using the above result. Consider an open neighborhood $U$ of $0 \in T_pM$ in which $\exp_p$ is a diffeomorphism. Let $\sigma$ be a two dimensional subspace of $T_pM$. Then $\exp_p(\sigma \cap U) = S$ is a submanifold of $M$ of dimension 2. In other words $S$ is a surface formed by the geodesics starting at $p$ and has initial velocity vector lying in $\sigma$. Therefore by the previous proposition $S$ is geodesic at $p$. It follows from the Gauss formula that $\kappa_S(p) = K(\sigma)$, where $\kappa_S$ is the Gaussian curvature of the surface $S$. Hence the sectional curvature $K(\sigma)$ is the Gaussian curvature of the surface formed by geodesics starting from $p$ and whose initial tangent vector lies in $\sigma$.

Totally geodesic immersions are rare. There is a weaker condition than being totally geodesic, *minimal*. We say an immersion is minimal for if every $p \in M$ and $\eta \in T_pM^\perp$ the trace of $S_\eta = 0$. Such immersions are important as it minimizes the volume of the immersed manifold and it is an active field of study.

## 5.2 The Fundamental Equations

These set of equations provide the geometric relationship between immersed submanifolds with its ambient manifold. Let $F : M \longrightarrow \overline{M}$ be an isometric immersion. Using the inner product we can write $T_p\overline{M} = T_pM \oplus T_pM^\perp$. As it smoothly depends on $p \in M$ we can write the tangent bundle as $T\overline{M} = TM \oplus TM^\perp$. Now we can define the *normal connection* of immersion $F$, $\nabla^\perp : TM \times TM^\perp \longrightarrow TM^\perp$ as

$$\nabla^{\perp}_{X}\eta = \overline{\nabla}_X\eta - (\overline{\nabla}_X\eta)^T = \overline{\nabla}_X\eta + S_\eta(X)$$

. One can verify that the above defined normal connection will satisfy all the properties of a connection, i.e. it is $C^\infty M^\perp$ linear in the first coordinate, $\mathbb{R}$ linear in the second coordinate and also it satisfies $\nabla^{\perp}_{X} f\eta = f\nabla^{\perp}_{X}\eta + X(f)\eta$ where $f \in C^\infty M$. Using normal connection we define *normal curvature* on $M^\perp$ in a similar manner as we define curvature.

$$R^\perp(X,Y)\eta = \nabla^{\perp}_{X}\nabla^{\perp}_{Y}\eta - \nabla^{\perp}_{Y}\nabla^{\perp}_{X}\eta + \nabla^{\perp}_{[X,Y]}\eta.$$

**Theorem 5.2.1.** *Let $X, Y, Z, T \in \tau(M)$ and $\zeta, \eta \in \tau(M^\perp)$, then*

*(i)* $\langle \overline{R}(X,Y)Z, T\rangle = \langle R(X,Y)Z,T\rangle - \langle B(Y,T), B(X,Z)\rangle + \langle B(X,T), B(Y,Z)\rangle.$

*(ii)* $\langle \overline{R}(X,Y)\eta, \zeta\rangle - \langle R^\perp(X,Y)\eta,\zeta\rangle = \langle [S_\eta, S_\zeta]X, Y\rangle$, *where* $[S_\eta, S_\zeta] = S_\eta \circ S_\zeta - S_\zeta \circ S_\eta$.

*Proof.* Note that $\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)$. Since

$$\begin{aligned}
\overline{R}(X,Y)Z &= \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_X\overline{\nabla}_Y Z + \overline{\nabla}_{[X,Y]}Z \\
&= \overline{\nabla}_Y(\nabla_X Z + B(X,Z)) - \overline{\nabla}_X(\nabla_Y Z + B(Y,Z)) \\
&+ \nabla_{[X,Y]}Z + B([X,Y],Z),
\end{aligned}$$

we have

$$\begin{aligned}
\overline{R}(X,Y)Z &= R(X,Y)Z + B(Y,\nabla_X Z) + \nabla^{\perp}{}_Y B(X,Z) \\
&- S_{B(X,Z)}Y - B(X,\nabla_Y Z) - \nabla^{\perp}{}_X B(Y,Z) \\
&+ S_{B(Y,Z)}X + B([X,Y],Z).
\end{aligned}$$

Taking the inner product of the above expression with $T$, since the normal terms vanish, we will get

$$\begin{aligned}
\langle \overline{R}(X,Y)Z,T\rangle &= \langle R(X,Y)Z,T\rangle - \langle S_{B(X,Z)}Y,T\rangle + \langle S_{B(Y,Z)}X,T\rangle \\
&= \langle R(X,Y)Z,T\rangle - \langle B(Y,T),B(X,Z)\rangle + \langle B(X,T),B(Y,Z)\rangle
\end{aligned}$$

which is the Gauss equation. To get the Ricci equation, we calculate

$$\begin{aligned}
\overline{R}(X,Y)\eta &= \overline{\nabla}_Y\overline{\nabla}_X\eta - \overline{\nabla}_X\overline{\nabla}_Y\eta + \overline{\nabla}_{[X,Y]}\eta \\
&= \overline{\nabla}_Y(\nabla^{\perp}{}_X\eta - S_\eta X) - \overline{\nabla}_X(\nabla^{\perp}{}_Y\eta - S_\eta Y) + \nabla^{\perp}{}_{[X,Y]}\eta - S_\eta[X,Y] \\
&= R^\perp(X,Y)\eta - S_{\nabla^{\perp}{}_X\eta}Y - \nabla_Y(S_\eta X) - B(S_\eta X, Y) + S_{\nabla^{\perp}{}_Y\eta}X \\
&+ \nabla_X(S_\eta Y) + B(X, S_\eta Y) - S_\eta[X,Y].
\end{aligned}$$

Multiplying the expression by $\zeta$ and observing that $\langle B(X,Y),\eta\rangle = \langle S_\eta X, Y\rangle$, we get

$$\begin{aligned}\langle \overline{R}(X,Y)\eta,\zeta\rangle &= \langle R^\perp(X,Y)\eta,\zeta\rangle - \langle B(S_\eta X,Y),\zeta\rangle + \langle B(X,S_\eta Y),\zeta\rangle \\ &= \langle R^\perp(X,Y)\eta,\zeta\rangle + \langle (S_\eta S_\zeta - S_\zeta S_\eta)X,Y\rangle \\ &= \langle R^\perp(X,Y)\eta,\zeta\rangle + \langle [S_\eta,S_\zeta]X,Y\rangle,\end{aligned}$$

which is Ricci's equation. □

Thus we obtain a set of algebraic equations which relates second fundamental form of the immersion with curvature of tangent and normal bundle. From this one can see that for an immersion its geometry decomposes to geometries of normal and tangent bundle.

# Chapter 6

# Complete Manifolds

Many of the manifolds that we deal with have the property that geodesics are defined for all $t \in \mathbb{R}$. So we study them as a separate class of manifolds.

## 6.1 Hopf-Rinow Theorem

**Definition 6.1.1.** *A Riemannian manifold $M$ is said to be geodesically complete if for all $p \in M$, any geodesic starting from $p$ is defined for all $t \in \mathbb{R}$.*

The above definition is same as saying $\forall p \in M$, $\exp_p$ is defined for all $X \in T_pM$. A Riemannian manifold $M$ is said to be *extendible*, if $M$ is isometric to a proper open subset of another Riemannian manifold, otherwise we say $M$ is non-extendible. It is easy to see that if a manifold is complete then it is non-extendible. We can define a metric on Riemannian manifold using the length of curves.

**Proposition 6.1.2.** *Let $p, q \in M$, $d : M \times M \longrightarrow \mathbb{R}$ defined by $d(p,q)$ equals the infimum of length of all piecewise smooth curves joining $p$ and $q$, then $d$ is a metric on $M$.*

The idea of completeness is most useful because of the following proposition which says we can join any two points in complete manifold with a minimizing geodesic.

**Proposition 6.1.3.** *If a Riemannian manifold is complete then for any $p, q \in M$ there exists a geodesic $\gamma$ joining $p$ and $q$ such that $d(p,q) = L(\gamma)$.*

*Proof.* Let $d(p,q) = r$ and $B_\epsilon(p)$ be a normal ball centered at $p$. Our aim is to find a minimizing geodesic joining $p$ and $q$. Denote the boundary of $B_\epsilon(p)$ as $\partial B_\epsilon(p)$. Let $x_0$ be the point such that the $d(q,x)$, $x \in \partial B_\epsilon(p)$ attains its minimum (This occurs as $\partial B_\epsilon(p)$ is compact). As $B_\epsilon(p)$ is a normal ball there exist $X \in T_pM$ such that $\exp_p \epsilon X = x_0$. We claim that $\gamma(t) = \exp_p tX$ is the desired geodesic. Such a curve is defined by our assumption of geodesic completeness. Therefore it remains to show that $\gamma(r) = q$. For that consider the set $A = \{t \in [0,r] \,|\, d(\gamma(t), q) = r - t\}$. $A \neq \phi$ as $0 \in A$. Clearly $A$ is closed in $[0,r]$. If we show that $\sup A = r$ it implies $r \in A$ and $\gamma(r) = q$. For that we will show that if $t_0 \in A$ then $t_0 + \epsilon' \in A$ for some small enough $\epsilon'$

Consider $B_{\epsilon'}(\gamma(t_0))$ and $x_0'$ be the point where $d(q,x)$ attains its minimum for $x \in \partial B_{\epsilon'}(\gamma(t_0))$. If we assume that $\gamma(t_0 + \epsilon') = x_0$ then

$$\begin{aligned} r - t_0 = d(\gamma(t_0), q) &= \epsilon' + d(x_0', q) \\ &= \epsilon' + d(\gamma(t_0 + \epsilon), q) \end{aligned}$$

Which implies $d(\gamma(t_0+\epsilon'), q) = r-(t_0+\epsilon')$ which says $t_0+\epsilon' \in A$. Hence it suffices to show that $\gamma(t_0+\epsilon') = x_0'$. To show this observe $d(p, x_0') \geq d(p,q) - d(q, x_0') = r - (r - (t_0 + \epsilon')) = t_0 + \epsilon$. But the curve obtained by concatenating $\gamma$ from $p$ to $\gamma(t_0)$ and geodesic joining $\gamma(t_0)$ to $x_0$ has length exactly $t_0 + \epsilon'$. From the Corollary 2.1.12 $\gamma$ is a geodesic joining $p$ and $x_0$ as $d(p, x_0') = t_0 + \epsilon'$. □

However the converse is not true, as one can see from the example of an open interval in $\mathbb{R}$. Observe that in the proof we only used the fact that $\exp_p$ is defined for all $X \in T_pM$ for atleast one $p$. We do not demand this for all $p \in M$. Precisely this condition is used in the proof of the following Hopf-Rinow theorem. Hence it can be regarded as an equivalent definition of completeness. The following theorem will also explain the motivation behind why such a property is called completeness.

**Theorem 6.1.4.** *Let $M$ be a Riemannian manifold and $p \in M$, then the following are equivanlent.*

1. *$M$ is geodesically complete*
2. *If $K \subset M$, where $K$ is closed and bounded then $K$ is compact*
3. *$M$ is complete as a metric space*

*Proof.* $1 \Rightarrow 2$. Consider a closed and bounded set $S \subset M$. As it is bounded from previous proposition we find $r$ such that $B_r(0) \subset T_pM$ such that $S \subset \exp_p(\overline{B_r(0)})$. As $S$ is closed set contained in a compact set $S$ is compact.

$2 \Rightarrow 3$. Any Cauchy sequence in bounded and hence has a compact closure by our assumption. Thus it has a converging subsequence and being Cauchy makes the whole sequence convergent.

$3 \Rightarrow 1$. Assume on the contrary that there exist some unit speed geodesic $\gamma$ which defined only for $t < t_0$. Consider a sequence $t_n < t_0$ which converges to $t_0$. Then it is easy to see that $\gamma(t_n)$ is Cauchy hence it converges to some point $p_0$. Consider a totally normal neighborhood $W$ of $p_0$. We can find $\gamma(t_n)$ and $\gamma(t_m)$ such that both the points are contained in $W$. We can find a unique minimizing geodesic joining both the points. As it is unique it coincides $\gamma$. This geodesic can be extended as $\exp_{p_0}$ is a diffeomorphism. □

This theorem provide us some obvious corollaries which are useful. Such as every compact Riemannian manifold is complete and every closed submanifold of a complete manifold is complete. Complete manifolds are ideal to study global properties as it gives us the freedom to join any two points in the manifolds with a minimizing geodesic.

## 6.2 Hadamard Theorem

The main theorem presented in this section due to Hadamard is one of the important theorem concerning complete manifolds which relates a local property, sectional curvature to a global diffeomorphism of the manifold.

**Lemma 6.2.1.** *Let $M$ be a complete manifold whose sectional curvature $K(p, \sigma) \leq 0$ for all $p \in M$ and for all two dimensional $\sigma \subset T_pM$. Then $p$ does not have conjugate points along any geodesic $\gamma$ from $p$ for all $p \in M$. i.e $\exp_p$ is a local diffeomorphism for all $p \in M$.*

*Proof.* Let $\gamma : [0, \infty) \longrightarrow M$ be a geodesic starting at $p$. Consider a $J$ along $\gamma$ such that $J(0) = 0$ and $J$ is non trivial. Then $\frac{d^2}{dt^2}\langle J, J\rangle = 2\langle D_tJ, D_tJ\rangle + 2\langle D_t^2 J, J\rangle$. From the Jacobi equation $\frac{d^2}{dt^2}\langle J, J\rangle = 2|D_tJ|^2 - 2\langle R(J, \gamma')\gamma', J\rangle$. By our assumption on curvature $\frac{d^2}{dt^2}\langle J, J\rangle = 2|D_tJ|^2 - 2K(J, \gamma')(|J|^2|\gamma'|^2 - \langle J, \gamma'\rangle^2) > 0$. Therefore $\frac{d}{dt}\langle J, J\rangle$ is increasing, i.e if $t_2 > t_1$ then $\frac{d}{dt}\langle J, J\rangle(t_2) \geq \frac{d}{dt}\langle J, J\rangle(t_1)$. But $D_tJ(0) \neq 0$ and $\frac{d}{dt}\langle J, J\rangle = 0$. Thus for $t > 0$ small enough $\langle J, J\rangle(t) > \langle J, J\rangle(0)$. Hence for all $t > 0$, $\langle J, J\rangle(t) > 0$. Thus we proved the lemma as $\gamma$ and $p$ were arbitrary. □

**Lemma 6.2.2.** *Let $M, N$ be a Riemannian manifolds and $M$ be complete. Let $F : M \longrightarrow N$ be a local diffeomorphism which satisfies $|dF_p(X)| \geq |X|$ for all $p$ and for all $X \in T_pM$ then $F$ is a covering map.*

*Proof.* It is enough to show that $F$ has path lifting property for curves in $N$. In other words we have to show that given a smooth curve $\gamma : [0,1] \longrightarrow N$ and $q \in M$ such that $F(q) = \gamma(0)$ then there exist a unique curve $\overline{\gamma} : [0,1] \longrightarrow M$ with $\overline{\gamma}(0) = q$ and $F \circ \overline{\gamma} = \gamma$. As $F$ is a local diffeomorphism around an open neighborhood of $q$ we can uniquely define for some small enough $\epsilon > 0$, $\overline{\gamma} : [0,\epsilon] \longrightarrow M$ such that $\overline{\gamma}(0) = q$ and $F \circ \overline{\gamma} = \gamma$. Since $F$ is a local diffeomorphism the set $A \subset [0,1]$ such that $\gamma$ can be lifted to $M$ is open in $[0,1]$. Therefore $A = [0,t_0)$ for some $t_0 \in [0,1]$. If we show that $t_0 \in A$ then $A = [0,1]$.

For that consider an increasing sequence $\{t_n\}$ in $A$ converging to $t_0$. Suppose $\{t_n\}$ is not contained in a compact set then as $M$ is complete $d(\overline{\gamma}(t_n), \overline{\gamma}(0)) \to \infty$. (This follows from a result in topology stating every closed and bounded set in a manifold is compact if and only if $M$ can be covered by compact $\{K_n\}$ such that $K_n \subset$ interior of $K_{n+1}$ and if $q_n \notin K_n$ then $d(p,q_n) \to \infty$. This is sometimes stated as an equivalence condition for completeness of manifold). Then by our assumption,

$$\begin{aligned} L(\gamma[0,t_n]) = \int_0^{t_n} |\gamma'(t)|dt &= \int_0^{t_n} |dF_{\overline{\gamma}(t)}(\overline{\gamma}'(t))|dt \\ &\geq \int_0^{t_n} |\overline{\gamma}(t)|dt \\ &> d(\overline{\gamma}(t_n), \overline{\gamma}(0)) \end{aligned}$$

which says $L(\gamma[0,t_n]) \to \infty$, hence a contradiction.

Therefore assume that $\{\overline{\gamma}(t_n)\} \in K \subset M$, a compact set. Hence there exist a limit point of $\{\overline{\gamma}(t_n)\}$, $r \in M$. Consider an open neighborhood $\mathcal{U} \subset M$ of $r$ such that $F$ is a diffeomorphism. Then $\gamma(t_0) \in F(\mathcal{U})$ and there exist $I \subset [0,1]$ with $t_0 \in I$ such that $\gamma(I) \subset F(\mathcal{U})$ by the continuity of $F$. Choose $n$ such that $\overline{\gamma}(t_n \in \mathcal{U})$. Consider the lift $\alpha$ of $\gamma$ on $I$ which passes through $r \in M$. Since $F|_{\mathcal{U}}$ is a diffeomorphism both $\alpha$ and $\overline{\gamma}$ coincides in their common domain. Thus $\alpha$ is an extension of $\overline{\gamma}$ to $I$. Hence $\gamma$ can be defined for $t_0 \in I$ which concludes the proof of the lemma.

□

**Theorem 6.2.3** (Hadamard)**.** *Let $M^n$ be a complete, simply connected Riemannian manifold with sectional curvature $K(p,\sigma) \leq 0$ then $M$ is diffeomorphic to $\mathbb{R}^n$. The diffeomorphism is given by* $\exp_p : T_pM \longrightarrow M$

*Proof.* By Lemma 6.2.1 $\exp_p : T_pM \longrightarrow M$ is a local diffeomorphism. Thus we can define a metric on $T_pM$ in which $\exp_p$ is a local isometry. Such a metric is complete as geodesics passing through $0 \in T_pM$ are straight lines (Condition 1 of Hopf-Rinow theorem is satisfies). Therefore conditions for Lemma 6.2.2 are satisfied and we get that $\exp_p$ is a covering map. As $M$ is simply connected it is a diffeomorphsim. $\square$

# Chapter 7

# Spaces with Constant Sectional Curvature

Riemannian geometry emerged as a result of the study of non-Euclidean geometries such as spherical geometry and hyperbolic geometry. These are spaces with constant sectional curvature. In fact, we will prove in this chapter that these are the only complete, simply connected manifolds with constant sectional curvature. As we can multiply a Riemannian metric with a positive constant $c$ and scale the sectional curvature by $\frac{1}{c}$, we can assume, without loss of generality, that the constant sectional curvature is either $+1,\ 0$ or $-1$.

## 7.1 Theorem by Cartan

The following is essentially a comparison theorem, in which we derive the relation between metrics of two manifolds in terms of their curvature. In order to state the theorem we need the following set up. Let $M,\ \overline{M}$ be two Riemannian manifolds with same dimension and curvature $R, \overline{R}$ respectively. $p \in M,\ \overline{p} \in \overline{M}$ and $i : T_pM \longrightarrow T_{\overline{p}}\overline{M}$ be a linear isometry. We can always find a normal neighborhood $U \subset M$ of $p$ such that $\exp_{\overline{p}}$ is defined on $i \circ \exp_p^{-1}(U)$. Define $F : U \longrightarrow \overline{M}$ by $F(q) = \exp_{\overline{p}} \circ i \circ \exp_p^{-1}(q)$. Since $U$ is a normal neighborhood of $p$ there exist a unit speed geodesic $\gamma : [0, a] \longrightarrow M$ joining $p$ and $q$. Also consider the geodesic $\overline{\gamma} : [0, a] \longrightarrow \overline{M}$ with $\overline{\gamma}(0) = p$ and $\overline{\gamma}'(0) = i(\gamma'(0))$. Denote the parallel transport along $\gamma$ from $\gamma(0) = p$ to $\gamma(t)$, $t \in [0, a]$ as $P_t$ and similarly parallel transport along $\overline{\gamma}(0)$ to $\overline{\gamma}(t)$ as $\overline{P}_t$. With all these machinery we define $\phi_t : T_qM \longrightarrow T_{F(\gamma(t))}\overline{M}$, as $\phi_t(X) = \overline{P}_t^{-1} \circ P_t(X)$. To avoid notational clutter we denote $\phi_t(X) = \overline{X}$.

**Theorem 7.1.1** (Cartan)**.** *Let $M, \overline{M}$ are Riemannian manifolds as above. If for all $q \in U$*

*and for all $X, Y, Z, W \in T_qM$ we have $\langle R(X,Y)Z,W\rangle = \langle \overline{R}(\overline{X},\overline{Y})\overline{Z},\overline{W}\rangle$, then $F$ is a local isometry.*

*Proof.* As inner product is symmetric and bilinear, to prove $F$ is an isometry in $U$ it is enough to show that for all $q \in U$ and $X \in T_qM$, $|dF_q(X)| = |X|$. Consider any $q \in U$ and $X \in T_qM$. Let $\gamma : [0,a] \longrightarrow M$ be a unit speed geodesic joining $p$ and $q$ and $J : [0,a] \longrightarrow TM$ be a Jacobi field along $\gamma$ such that $J(0) = 0$ and $J(a) = X$. Consider an orthonormal basis of $T_pM$, $E_1, ..E_{n-1}, E_n = \gamma'(0)$ and parallelly transport it along the geodesic $\gamma$. Thus we obtain an orthonormal frame along $\gamma$. Write the Jacobi field $J(t) = \sum_i f_i(t)E_i(t)$. By Jacobi equation we get, $f_j'' + \sum_i \langle R(E_i, E_n)E_n, E_j\rangle = 0$.
As described before we consider $\overline{\gamma} : [0,a] \longrightarrow M$ with $\overline{\gamma}(0) = \overline{p}$ and $\overline{\gamma}'(0) = i(\gamma(0))$. Let $\overline{J}$ be a vector field along $\overline{\gamma}$ defined by $J(t) = \sum_i f_i(t)\overline{E}_i(t)$, where $\overline{E}_i(t) = \phi_t(E_i(t))$. It turns out that $\overline{J}$ is a Jacobi field since it satisfies $f_j''(t) + \sum_i \langle \overline{R}(\overline{E_i}, \overline{E_n})\overline{E_n}, \overline{E_j}\rangle = 0$ which is an easy consequence of the hypothesis of the theorem. Since $J$ and $\overline{J}$ are Jacobi fields with initial vector 0, $J(t) = (d\exp_p)_{t\gamma'(0)}(tD_tJ(0))$ and $\overline{J}(t) = (d\exp_{\overline{p}})_{t\overline{\gamma}'(0)}(tD_t\overline{J}(0))$. As $D_t\overline{J}(0) = i(D_t(J(0)))$,

$$\begin{aligned}
\overline{J}(a) &= a(d\exp_{\overline{p}})_{a\overline{\gamma}'(0)}(i(D_tJ(0))) \\
&= (d\exp_{\overline{p}})_{a\overline{\gamma}'(0)} \circ i \circ ((d\exp_p)a\gamma'(0))^{-1}(J(a)) \\
&= dF_q(J(a)) \\
&= dF_qX
\end{aligned}$$

But $|X| = |J(a)| = |\overline{J}(a)|$ as $\phi_t$ is an isometry. Hence we proved the theorem. □

One can easily see that if $\exp_p$ and $\exp_{\overline{p}}$ are diffeomorphism then $F$ becomes an isometry. As mentioned earlier this theorem gives us a way to figure out the metric if some information about the curvature is given.

**Corollary 7.1.2.** *Let $M$ and $\overline{M}$ be two Riemannian manifolds of same dimension and same constant curvature. Consider $p \in M$ and $\overline{p} \in \overline{M}$, $\{E_i\}$ and $\{\overline{E_i}\}$ be orthonormal basis of $T_pM$ and $T_{\overline{p}}\overline{M}$ respectively. Then there exist a neighborhood of $p$, $U \subset M$ and an isometry $F : U \subset M \longrightarrow F(U) \subset \overline{M}$ such that $dF_p(E_i) = \overline{E_i}$.*

*Proof.* Take $i : T_pM \longrightarrow T_{\overline{p}}\overline{M}$ such that $i(E_i) = \overline{E_i}$. $dF_p = (d\exp_p)_0 \circ i \circ (d\exp_{\overline{p}})_0^{-1}$ and $(d\exp_p)_0$ is identity for all $p$, therefore $dF_p = i$. □

Next corollary makes the spaces of all constant curvature interesting. It gives us freedom to freely move "small" triangles and check whether they are congruent to each other. This is possible because there are a lot of isometries as evident from the following corollary.

**Corollary 7.1.3.** *Let $M$ be a Riemannian manifold with constant sectional curvature. $p, q \in M$ and $\{E_i\}$ and $\{F_i\}$ be orthonormal basis vectors at $T_pM$ and $T_qM$ respectively. Then there exist an open neighborhoods $U \ni p$, $V \ni p$ and an isomorphism $G : U \longrightarrow V$ such that $dG_p(E_i) = F_i$.*

## 7.2 Classification Theorem

With the following theorem we essentially classifies all the manifolds with constant sectional curvature. Before stating the theorem we prove an interesting lemma.

**Lemma 7.2.1.** *Let $F, G$ be local isometry between two connected Riemannian manifolds $M$, $N$. If for some $p \in M$ $F(p) = G(p)$ and $dF_p = dG_p$ then $F = G$.*

*Proof.* As our manifolds are connected we essentially need to prove the above statement for an open set. For that a normal neighborhood of $p$, $U$ st that $F$ and $G$ are diffeomorphisms in that neighborhood. Let $\Phi = G^{-1} \circ F : U \longrightarrow U$.Then by the hypothesis $\Phi(p) = p$ and $d\Phi_p = id$. Choose $q \in U$, then there exist a unique vector $X \in T_pM$ such that $\exp_p X = q$ as $U$ is a normal neighborhood of $p$. As differential of a local isometry and exponential map commute we have,

$$q = \exp_p X = \exp_p d\Phi(X) = \Phi(\exp_p X) = \Phi(q)$$

Hence we proved the lemma. □

**Theorem 7.2.2.** *Universal covering with covering metric of an $n$ dimensional complete Riemannian manifold with constant sectional curvature is isometric to:*

*(i)* $S^n$ *if curvature is* 1

*(ii)* $\mathbb{R}^n$ *if curvature is* 0

*(iii)* $\mathbb{H}^n$ *if curvature is* $-1$

*Proof. Case(i)*: Let the sectional curvature of the Riemannian manifold $M$ is 1. Let $\overline{M}$ be the universal cover of $M$ with covering metric (Note that $\overline{M}$ also has constant sectional curvature 1). Choose $p \in S^n$, $\overline{p} \in \overline{M}$ and $i : T_pS^n \longrightarrow T_{\overline{p}}\overline{M}$ be a linear isometry. Define $F : S^n \backslash \{-p\} \longrightarrow \overline{M}$ as $F(q) = \exp_{\overline{p}} \circ i \circ \exp_p^{-1}(q)$. Then from Theorem 7.1.1 it is clear that $F$ is a local isometry. Choose $p' \in S^n$ other than $p$ or $-p$ and choose $\overline{p'} = F(p')$, linear isometry $i' = dF_{p'} : T_{p'}S^n \longrightarrow T_{\overline{p'}}\overline{M}$. Now define $G : S^n \setminus \{-p'\} \longrightarrow \overline{M}$ as $G(q) = \exp_{\overline{p'}} \circ i' \circ \exp_{p'}^{-1}$. As $F(p') = G(p')$ and $dG_{p'} = dF_{p'}$ from Lemma 7.2.1 $G = F$ on their common domain. Thus we define $H : S^n \longrightarrow \overline{M}$ as,

$$H(q) = \begin{cases} F(q) & \text{if } q \in S^n \setminus \{-p\}, \\ G(q) & \text{if } q \in S^n \setminus \{-p'\} \end{cases}$$

By definition the new map we defined is a local isometry, hence a local diffeomorphism. Any local diffeomorphism from a compact space to a connected space is covering map. As $\overline{M}$ is simply connected this map becomes a diffeomorphism and thus an isometry.

*Case(ii)*: The case when sectional curvature is 0 or $-1$ can be treated simultaneously. Let $\mathbb{k}^n = \mathbb{R}^n$or $\mathbb{H}^n$ for convenience. Now for $p \in \mathbb{k}^n$, $\overline{p} \in \overline{M}$ and a linear isometry $i : T_p\mathbb{k}^n \longrightarrow T_{\overline{p}}\overline{M}$ the map $F : \mathbb{k}^n \longrightarrow \overline{M}$ defined by $F(q) = \exp_p \circ i \circ \exp_p^{-1}(q)$ is well defined by Hadamard's theorem(Theorem 6.0.6). Now from Theorem 7.1.1 this is a local isometry. This map therefore is a diffeomorphism and hence an isometry from Lemma 6.2.2. □

Using this theorem, we can see that any complete manifold with constant sectional curvature is isometric to one of the above manifolds quotiented by a group of isometries which act totally discontinuously. Thereby the problem of determining all the complete manifold with constant sectional curvature is converted to a problem in group theory.

# Chapter 8

# Variation of Energy

Earlier we have defined geodesic as a solution of a certain system of differential equations. In this chapter we attempt to give a different characterization for geodesics using ideas from calculus of variation. We have seen that geodesics are length minimizing in a small enough neighborhood (*normal neighborhoods*). As it is a minima of lengths of curves in normal neighborhoods we can imitate the derivative test to find whether a curve is geodesic or not.

## 8.1 First and Second Variation Formulas

To study the minimizing property of a curve we need the idea of neighboring curves. For that we have the following set up. By an *admissible curve* we mean a regular piecewise smooth curve. Let $\gamma : [0, a] \longrightarrow M$ be an admissible curve with $0 = a_0 < a_1 < ... < a_n = a$ as points where it fails to be smooth. A continuous mapping $\Gamma : (-\epsilon, \epsilon) \times [0, a] \longrightarrow M$ is called a variation of $\gamma$ if $\Gamma(0, t) = \Gamma_0(t) = \gamma(t)$ and for all $s \in (-\epsilon, \epsilon)$, $\Gamma_s(t)$ is an admissible curve with $0 = a_0 < a_1 < ... < a_n = a$ as points where it fails to be smooth. A variation is said to be proper if for all $s \in (-\epsilon, \epsilon)$, $\Gamma_s(0) = \gamma(0)$ and $\Gamma_s(a) = \gamma(s)$. We call $V(t) = \frac{\partial \Gamma(0,t)}{\partial t}$ to be the variation field of $\Gamma$. A variation is said to be a variation through geodesic if all the intermediate curves are geodesics.

A variation can be thought of as a perturbation of the given curve and the way to specify a given curve is by specifying a vector field along the curve. This vector field is precisely the variation vector field. In this regard we have the following lemma.

**Lemma 8.1.1.** *Let $V : [0, a] \longrightarrow M$ be a smooth vector field along an admissible curve $\gamma : [0, a] \longrightarrow M$. Then there exist a variation $\Gamma : (-\epsilon, \epsilon) \times [0, a] \longrightarrow M$ such that $V(t) = \frac{\partial \Gamma(0,t)}{\partial t}$. Moreover if we choose $V(0) = V(a) = 0$ then we can find $\Gamma$ which is a proper variation.*

*Proof.* Define $\Gamma(s,t) = \exp_{\gamma(t)}(sV(t))$. This can be done for $s \in (-\epsilon, \epsilon)$ for some $\epsilon$ as $\gamma[0,a]$ is compact. By definition $\Gamma$ is continuous and it has all the properties needed. $\square$

Earlier we have defined length of a piecewise smooth curve. Similarly we define energy of a piecewise smooth curve $\gamma : [0,a] \longrightarrow M$ as

$$E(\gamma) = \int_0^a \left|\frac{d\gamma}{dt}\right|^2 dt$$

From Cauchy-Schwarz inequality we get,

$$(\int_0^a \left|\frac{d\gamma}{dt}\right| dt)^2 \leq (\int_0^a 1^2 dt)(\int_0^a \left|\frac{d\gamma}{dt}\right|^2 dt)$$

i.e $L(\gamma)^2 \leq aE(\gamma)$ with equality if and only if $\gamma$ is a constant speed curve. Because of the above inequality energy functional has an advantage over length functional while characterizing the geodesic as energy minimizing as made clear from the following lemma.

**Lemma 8.1.2.** *Let $\gamma : [0,a] \longrightarrow M$ be a minimizing geodesic with $\gamma(0) = p$ and $\gamma(a) = q$. Then for any other piecewise smooth curve $c : [0,a] \longrightarrow M$ joining $p$ and $q$ we have $E(\gamma) \leq E(c)$ and $E(\gamma) = E(c)$ if and only if $c$ is a minimizing geodesic.*

*Proof.* As $\gamma$ is a minimizing geodesic we have $L(\gamma)^2 \leq L(c)^2$. Since $\gamma$ is of constant speed we have $aE(\gamma) = L(\gamma)^2 \leq L(c) \leq aE(c)$. If $E(\gamma) = E(c)$ we have $L(\gamma) = aE(c)$ which means $c$ is a constant speed curve and $L(\gamma) = L(c)$. Hence $c$ is a minimizing geodesic. The other implication is easy to see. $\square$

Observe that here we did not have to specify that $c$ is a constant speed curve. It is guaranteed by the above inequality. Now we present a formula for first variation of energy.

**Proposition 8.1.3.** *Let $\gamma : [0,a]$ be an admissible curve and $\Gamma : (-\epsilon, \epsilon) \times [0,a] \longrightarrow M$ be a proper variation of $\gamma$ then*

$$\frac{1}{2}\frac{dE(\Gamma_0)}{ds} = -\int_0^a \langle V(t), D_t\gamma'(t)\rangle - \sum_{i=1}^{n-1} \langle V(a_i), \triangle_i \gamma'(a_i)\rangle$$

*where $V$ is the variation field of the variation $\Gamma$ and $\triangle_i\gamma'(a_i) = \gamma'(a_i^+) - \gamma'(a_i^-)$ where $a_i$'s are the corners of $\gamma$.*

*Proof.* : We know,

$$
\begin{aligned}
E(\Gamma_s) &= \int_0^a \left|\frac{\partial\Gamma(s,t)}{\partial t}\right|^2 dt \\
&= \sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} \left|\frac{\partial\Gamma(s,t)}{\partial t}\right|^2 dt
\end{aligned}
$$

Now we differentiate the expression and obtain,

$$
\begin{aligned}
\frac{dE(\Gamma_s)}{ds} &= \frac{d}{ds}\sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} \left|\frac{\partial\Gamma(s,t)}{\partial t}\right|^2 dt \\
&= \sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} \frac{d}{ds}\Big\langle\frac{\partial\Gamma(s,t)}{\partial t}, \frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt \\
&= \sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} 2\Big\langle D_s\frac{\partial\Gamma(s,t)}{\partial t}, \frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt \\
&= 2\sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} \Big\langle D_t\frac{\partial\Gamma(s,t)}{\partial s}, \frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt \\
\frac{1}{2}\frac{dE(\Gamma_s)}{ds} &= \sum_{i=1}^{n-1}\int_{a_i}^{a_{i+1}} \frac{d}{dt}\Big\langle\frac{\partial\Gamma(s,t)}{\partial s}, \frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt - \int_{a_i}^{a_{i+1}} \Big\langle\frac{\partial\Gamma(s,t)}{\partial s}, D_t\frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle \\
&= \sum_{i=1}^{n-1}\Big\langle\frac{\partial\Gamma(s,t)}{\partial s}, \frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle\Big|_{a_i}^{a_{i+1}} - \int_0^a \Big\langle\frac{\partial\Gamma(s,t)}{\partial s}, D_t\frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt
\end{aligned}
$$

Notice when $s = 0$, $\frac{\partial\Gamma(s,t)}{\partial s} = V(t)$ and $\frac{\partial\Gamma(s,t)}{\partial t} = \gamma'(t)$. Hence we obtain the desired formula. □

The first variation formula has a nice geometric interpretation. Observe that if we choose the variation field in the direction of the acceleration of the curve (covariant derivative of velocity vector field), then the derivative of energy is negative. In other words if we perturb the curve in the direction of the acceleration vector the length of the resultant curve decreases. We use this idea to prove the following proposition.

**Proposition 8.1.4.** *An admissible curve $\gamma : [0,a] \longrightarrow M$ is a geodesic if and only if for all the proper variation of $\gamma$ the derivative of energy functional at 0 vanishes.*

*Proof.* If $\gamma$ is a geodesic it is immediately clear that $\frac{dE(0)}{ds} = 0$ as $D_t\gamma'(t) \equiv 0$. Now to prove the converse assume that $\frac{dE(0)}{ds} = 0$ for all the proper variations. Choose a variation field $V(t) = \phi(t)D_t\gamma'(t)$, where $\phi(t)$ is a smooth real valued function on $\mathbb{R}$ with $\phi > 0$ on $(a_i, a_{i+1})$ and zero outside. The we get $0 = -\int_{a_i}^{a_{i+1}} \phi|D_t\gamma'(t)|^2 dt$ which implies $D_t\gamma'(t) = 0$ on $(a_i, a_{i+1})$. We can do this for any smooth interval and conclude that $\gamma$ is a geodesic on those intervals where it is smooth. Now to prove that $\gamma$ is smooth we take a variation field $V$ such that $V(a_i) = \triangle_i\gamma(a_i)$ and zero on all other corners. As $\gamma$ is a geodesic on the smooth components we get $-|\triangle_i \gamma(a_i)|^2 = 0$. Thus $\gamma$ has no corners and $\gamma$ is a geodesic. □

From first variation formula we are able to obtain a characterization of geodesic as the critical points of energy functional. Notice that we need not to invoke the second derivative test to check whether a critical point is minima or maxima. Using first variation formula we were able to give a global definition of geodesics. Even though from the first variation formula we obtain that a critical point of energy is a geodesic we calculate the second derivative of energy functional. It turns out to be very useful when we study the relationship between geodesic and curvature.

**Proposition 8.1.5.** *Let $\gamma : [0, a] \longrightarrow M$ be a geodesic and $\Gamma : (-\epsilon, \epsilon) \times [0, a] \longrightarrow M$ be a proper variation of $\gamma$, then*

$$\frac{1}{2}\frac{d^2E(0)}{ds^2} = -\int_0^a \langle V(t), D_t^2 V + R(V, \gamma')\gamma'\rangle dt \tag{8.1}$$

*Proof.*

$$\begin{aligned}
\frac{1}{2}\frac{dE(\Gamma_s)}{ds} &= \int_0^a \Big\langle \frac{\partial\Gamma(s,t)}{\partial s}, D_t\frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt \\
\frac{1}{2}\frac{d^2E(s)}{ds^2} &= -\int_0^a \frac{\partial}{\partial s}\Big\langle \frac{\partial\Gamma(s,t)}{\partial s}, D_t\frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt \\
&= \int_0^a \Big\langle D_s\frac{\partial\Gamma(s,t)}{\partial s}, D_t\frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt - \int_0^a \Big\langle \frac{\partial\Gamma(s,t)}{\partial s}, D_sD_t\frac{\partial\Gamma(s,t)}{\partial t}\Big\rangle dt
\end{aligned}$$

As $\gamma$ is a geodesic at $s = 0$ the first term becomes zero and using the definition of Riemannian curvature tensor we obtain

$$\frac{1}{2}\frac{d^2E(s)}{ds^2} = -\int_0^a \langle V(t), D_t^2V(t) + R(V(t), \gamma'(t))\gamma'(t)\rangle dt$$

Since $\frac{d}{dt}\langle V(t), D_tV(t) = \langle V(t), D_t^2V\rangle + \langle D_tV(t), D_tV(t)$, we can write

$$\begin{aligned}\frac{1}{2}\frac{d^2E(s)}{ds^2} &= -\int_0^a \langle V(t), D_t^2V(t) + R(V(t), \gamma'(t))\gamma'(t)\rangle dt \\ &= -(\int_0^a \langle V(t), D_t^2V(t)\rangle + \langle V(t), R(V(t), \gamma'(t))\gamma'(t)\rangle dt) \\ &= \int_0^a \frac{d}{dt}\langle V(t), D_tV(t)\rangle + \langle D_tV(t), D_tV(t) - \langle V(t), R(V(t), \gamma'(t))\gamma'(t)\rangle dt \\ &= \int_0^a \langle D_tV(t), D_tV(t) - \langle V(t), R(V(t), \gamma'(t))\gamma'(t)\rangle dt\end{aligned}$$

We obtain the last expression from fundamental theorem of calculus and the fact that the variation is proper. □

## 8.2 Applications of Variation formulas

From our intuition it is clear that if a manifold has a strictly positive curvature then it tends to curve inwards and eventually form a compact manifold. This intuition is formalized in the following theorem due to Bonnet and Myers.

**Theorem 8.2.1.** *If Ricci curvature of a complete Riemannian manifold satisfies $Ric_p(X) \geq \frac{1}{r^2}$ for all $p \in M$ and for all unit tangent vectors in $T_pM$ then $M$ is compact and $diam(M) = sup\ \{d(p,q)|p,q \in M\} \leq \pi r$.*

*Proof.* Given any two points in the manifold there exists a geodesic which minimizes the length joining them as the manifold is complete. We need to show that given $p, q \in M$, $L(\gamma) \leq \pi r$ where $\gamma$ is the minimizing geodesic joining $p$ and $q$. Assume for a contradiction that there exist $p$ and $q$ such that the minimizing geodesic $\gamma : [0, a] \longrightarrow M$ joining them has $L(\gamma) = l > \pi r$. Consider $\{E_1, .., E_{n-1}, E_n = \frac{\gamma'(0)}{l}\}$ an orthonormal set of basis vectors at $\gamma(0)$. Parallel transport it along the curve $\gamma$. Consider a vector field $V_i(t) = sin\pi tE_i(t)$. By second variation formula

$$\begin{aligned}\frac{1}{2}\frac{d^2E_i(0)}{ds} &= \int_0^l \langle V_i, D_t^2 + R(V_i, \gamma')\gamma'\rangle dt \\ &= \int_0^l sin^2\pi t((\pi^2 - l^2)K(E_n(t), E_i(t)))dt \\ \sum_{i=1}^{n-1}\frac{1}{2}\frac{d^2E_i(0)}{ds} &= \int_0^l (sin^2\pi t((n-1)\pi^2 - (n-1)l^2 Ric_{\gamma(t)}E_n(t)))dt\end{aligned}$$

By our assumption $(n-1)\pi^2 < (n-1)l^2 Ric_{\gamma(t)} E_n(t)$. Thus we obtain $\frac{1}{2}\frac{d^2E_i(0)}{ds} < 0$. This implies for atleast one $i$, $\frac{d^2E_i(0)}{ds} < 0$ which is a contradiction to the fact that $\gamma$ is minimizing. Hence for all $p, q \in M$, $d(p,q) \leq \pi r$. As every subset of $M$ is bounded and manifold is complete, $M$ is compact. □

If we consider $\pi : \overline{M} \longrightarrow M$ cover of a manifold with covering map being local isometry then it can be easily shown that $M$ is complete if and only if $\overline{M}$ is complete. From this observation we can conclude that if a manifold satisfies the hypothesis of Bonnet-Myers theorem then its universal cover is compact and hence the fundamental group is finite.

# Chapter 9

# Comparison Theorems

Intuitively one can see that on a surface as curvature increases length of geodesic decreases. In this chapter we prove Rauch comparison theorem which generalizes this intuitive idea.

## 9.1 Rauch Comparison Theorem

Consider a Riemannian manifold $M$ and a geodesic $\gamma : [0, a] \longrightarrow M$. For a smooth vector field $V$ along $\gamma$ we define the index $I_a(V, V) = \int_0^a \{\langle D_t V, D_t V\rangle - \langle R(V, \gamma')\gamma', V\rangle\}dt$. (This expression is the second derivative of Energy functional of a variation with variation field $V$). The following theorem (index lemma), regarding the value of index of a Jacobi field, will be used crucially in the proof of Rauch comparison theorem as well as in the the proof of Morse index theorem.

**Theorem 9.1.1** (Index Lemma)**.** *Consider a geodesic $\gamma : [0, a] \longrightarrow M$ on a Riemannian manifold $M$. Assume $\gamma(0)$ does not have conjugate points along $\gamma$ on $(0, a]$. Consider a normal Jacobi field $J$ (i.e $\langle J, \gamma'\rangle = 0$) and a smooth normal vector field $V$ along $\gamma$ (i.e $\langle V, \gamma'\rangle = 0$) with $J(0) = V(0) = 0$ and $J(t_0) = V(t_0)$. Then $I_{t_0}(J, J) \leq I_{t_0}(V, V)$. Equality holds if and only if $J = V$ on $[0, t_0]$.*

*Proof.* We know that normal Jacobi fields with initial condition $J(0) = 0$ form an $n-1$ vector space. Let $\{J_i\}_{i=1}^{n-1}$ form a basis of the vector space. From our assumption, $J_i(t) \neq 0$ for all $t \in (0, a]$, it follows that $J_i(t)$ are linearly independent and lies in $(\gamma'(t))^{\perp}$, the orthogonal complement of $\gamma'(t)$. Thus it forms a basis for $(\gamma'(t))^{\perp}$ for all $t \in (0, a]$. Therefore we can write $V(t) = \sum_{i=1}^{n-1} f_i(t)J_i(t)$ where $f_i : (0, a] \longrightarrow \mathbb{R}$ is a smooth function. We can extend the function $f_i$ to $[0, a]$ in the following way. It is an easy calculus fact that there exists $X_i(t)$

such that $J_i(t) = tX_i(t)$. It follows that $X_i$'s are linearly independent on $[0, a]$, and therefore one can write $V(t) = \sum_{i=1}^{n-1} g_i(t)X_i(t)$ with $g_i(0) = 0$. We can write $g_i(t) = th_i(t)$, where $h_i$'s are defined on $[0, a]$. But $\sum_{i=1}^{n-1} th_i(t)X_i(t) = V(t) = \sum_{i=1}^{n-1} h_i(t)J_i(t)$. Hence $h_i(t) = f_i(t)$ on their common domain and $h_i$'s are smooth thus we obtained a smooth extension of $f_i$'s.

From the Jacobi equation we have $R(V, \gamma')\gamma' = R(\sum_i f_iJ_i, \gamma')\gamma' = \sum_i f_iR(J_i, \gamma')\gamma' - \sum_i f_iD_t^2J_i$.

$$\begin{aligned}
\langle D_tV, D_tV\rangle - \langle R(V, \gamma')\gamma', V\rangle &= \langle \sum_i f_i'J_i + \sum_i f_iD_tJ_i, \sum_j f_j'J_j + \sum_j f_jD_tJ_j\rangle \\
&\quad - \langle R(V, \gamma')\gamma', V\rangle \\
&= \langle \sum_i f_i'J_i, \sum_j f_j'J_j\rangle + \langle \sum_i f_i'J_i, \sum_j f_jD_tJ_j\rangle \\
&\quad + \langle \sum_i f_iD_tJ_i, \sum_j f_j'J_j\rangle + \langle \sum_i f_iD_tJ_i, \sum_j f_jD_tJ_j\rangle \\
&\quad + \langle \sum_i f_iD_t^2J_i, \sum_j f_jJ_j\rangle
\end{aligned}$$

But,

$$\begin{aligned}
\frac{d}{dt}\langle \sum_i f_iJ_i, \sum_j f_jD_tJ_j\rangle &= \langle \sum_i f_i'J_i + \sum_i f_iD_tJ_i, \sum_j D_tJ_j\rangle \\
&\quad + \langle \sum_i f_iJ_i, \sum_j f_j'D_tJ_j + \sum_j f_jD_t^2J_j\rangle \\
&= \langle \sum_i f_i'J_i, \sum_j f_jD_tJ_j\rangle + \langle \sum_i f_iD_tJ_i, \sum_j f_jD_tJ_j\rangle \\
&\quad + \langle \sum_i f_iJ_i, \sum_j f_j'D_tJ_j\rangle + \langle \sum_i f_iJ_i, f_jD_t^2J_j\rangle
\end{aligned}$$

From the symmetry of the curvature and the compatibility of the metric we have,

$$\begin{aligned}
\frac{d}{dt}(\langle D_tJ_i, J_j\rangle - \langle J_i, D_tJ_j\rangle) &= \langle D_t^2J_i, J_j\rangle + \langle D_tJ_i, D_tJ_j\rangle - \langle D_tJ_i, D_tJ_j\rangle - \langle J_i, D_t^2J_j\rangle \\
&= \langle R(J_j, \gamma')\gamma', J_i\rangle - \langle R(J_i, \gamma')\gamma', J_j\rangle \\
&= 0
\end{aligned}$$

As $J_i(0) = 0$ for all $i$ we get $\langle D_tJ_i, J_j\rangle = \langle J_i, D_tJ_j\rangle$. Combining all the expressions above we get

$$
\begin{aligned}
I_{t_0}(V,V) &= \int_0^{t_0} \{\langle D_tV, D_tV\rangle - \langle R(V,\gamma')\gamma', V\rangle\}dt \\
&= \int_0^{t_0} \langle \sum_i f_i' J_i, \sum_j f_j' J_j\rangle + \frac{d}{dt}\langle \sum_i f_i J_i, \sum_j f_j D_t J_j\rangle \\
&= \langle \sum_i f_i(t_0) J_i(t_0), \sum_j fj(t_0) D_t^2 J_j\rangle + \int_0^{t_0} \langle \sum_i f_i' J_i, \sum_j f_j' J_j\rangle dt
\end{aligned}
$$

If we take $J = \sum_i a_i J_i$ where $a_i$'s are constants then $I_{t_0}(J,J) = \langle \sum_i a_i J_i(t_0), \sum_j a_j D_t J_j(t_0)\rangle$. But $J(t_0) = V(t_0)$, therefore $a_i = f_i(t_0)$. Finally therefore $I_{t_0}(V,V) = I_{t_0}(J,J) + \int_0^{t_0} |\sum_i f_i' J_i|^2 dt$. As the integrand is positive we obtain the desired result. $I_{t_0}(J,J) \leq I_{t_0}(V,V)$ and the equality occurs if and only if $f_i(t) = a_i$ for all $t \in [0, t_0]$. Thus we proved the theorem. □

Now we state the Rauch comparison theorem.

**Theorem 9.1.2** (Rauch Comparison theorem)**.** *Consider two Riemannian manifolds $M$ of dimension $n$, $\overline{M}$ of dimension $n+k$, and geodesics $\gamma : [0,a] \longrightarrow M$, $\overline{\gamma} : [0,a] \longrightarrow \overline{M}$ with same speed and $\overline{\gamma}$ does not have any conjugate points in $(0,a]$. Let $J$ and $\overline{J}$ be normal Jacobi fields along $\gamma$ and $\overline{\gamma}$ respectively such that $J(0) = \overline{J}(0) = 0$, $\langle D_t J, \gamma'\rangle = \langle D_t \overline{J}, \overline{\gamma}'\rangle$ and $|D_t J(0)| = |D_t \overline{J}(0)|$. If for all $X \in T_{\gamma(t)}$ and $\overline{X} \in T_{\overline{\gamma}(t)}$ we have $K(X, \gamma'(t)) \leq K(\overline{X}, \overline{\gamma}')$ then $|\overline{J}| \leq |J|$.*

*Proof.* Let $u(t) = |J(t)|^2$ and $\overline{u}(t) = |\overline{J}(t)|^2$. As $\overline{J}$ has no conjugate points in the interval we can consider $\frac{u(t)}{\overline{u}(t)}$. As $lim_{t\to 0} \frac{u(t)}{\overline{u}(t)} = lim_{t\to 0} \frac{u''(t)}{\overline{u}''(t)} = \frac{|D_t J(0)|^2}{|D_t \overline{J}(0)|^2} = 1$ by hypothesis. Hence instead of proving $|\overline{J}| \leq |J|$ we show that $\frac{d}{dt}(\frac{u(t)}{\overline{u}(t)}) > 0$. In other words $\overline{u}(t)u'(t) \geq \overline{u}'(t)u(t)$.

We will prove the above inequality for all $t_0 \in (0,a]$. If $u(t_0) = 0$ for $t_0 \in (0,a]$ then $u'(t_0) = \frac{d}{dt}\langle J(t_0), J(t_0)\rangle = 2\langle D_t J(t_0), J(t_0)\rangle = 0$ then the inequality is satisfied. Now we will consider $t_0 \in (0,a]$ such that $u(t_0) \neq 0$. Define $U(t) = \frac{1}{\sqrt{u(t_0)}} J(t)$ and $\overline{U}(t) = \frac{1}{\sqrt{\overline{u}(t_0)}} \overline{J}(t)$. Then,

$$
\begin{aligned}
\frac{u'(t_0)}{u(t_0)} = \frac{\frac{d}{dt}\langle D_t J(t_0), J(t_0)\rangle}{\langle J(t_0), J(t_0)\rangle} &= \frac{d}{dt}\langle D_t U(t_0), U(t_0)\rangle \\
&= \int_0^{t_0} \frac{d^2}{dt^2}\langle U(t), U(t)\rangle \\
&= 2\int_0^{t_0} \langle D_t U(t), D_t U(t)\rangle - \langle R(U,\gamma')\gamma', U\rangle \qquad = 2I_{t_0}(U,U)
\end{aligned}
$$

Similarly one can deduce that, $\frac{\overline{u}'(t_0)}{\overline{u}(t_0)} = 2I_{t_0}(\overline{U},\overline{U})$. Thus our aim reduces to proving $I_{t_0}(U,U) \geq I_{t_0}(\overline{U},\overline{U})$. For that consider $E_1(t) = \frac{\gamma'(t)}{|\gamma'(t)|}$, $E_2(t_0) = U(t_0)$ and extend it an orthonormal set basis of $\{E_i(t)\}_{i=1}^n$ and similarly $\overline{E_1}(t) = \frac{\overline{\gamma}'(t)}{|\overline{\gamma}'(t)|}$, $E_2(t_0) = \overline{U}(t_0)$ and extend it to $\{\overline{E}_i(t)\}_{i=1}^{n+k}$. Let $U(t) = \sum_{i=1}^n f_i(t)E_i(t)$ and define $\mathcal{U}(t) = \sum_{i=1}^n f_i(t)\overline{E}_i(t)$. It is clear that $\langle U(t), U(t)\rangle = \langle \mathcal{U}(t), \mathcal{U}(t)\rangle$ and $D_tU = D_t\mathcal{U}$. Hence from the restriction on the curvature it follows that $I_{t_0}(\mathcal{U},\mathcal{U}) \leq I_{t_0}(U,U)$. Thus it $\overline{U}$ and $\mathcal{U}$ satisfies all the condition of the index lemma, and it follows that $I_{t_0}(\overline{U},\overline{U}) \leq I_{t_0}(\mathcal{U},\mathcal{U}) \leq I_a(U,U)$ and we proved the theorem. □

Observe that in Rauch comparison theorem only major restriction is on the sectional curvature and we obtain information about the Jacobi field (We do not even need manifolds to be of same dimension), which carries a wealth of information about the behavior of geodesics. Rauch comparison theorem is used in proving many interesting theorems, e.g: Sphere theorem. Following proposition is one application of the theorem where it is used to determine the length between consecutive conjugate points.

**Proposition 9.1.3.** *Let $M$ be a Riemannian manifold whose sectional curvature is strictly positive i.e $0 < K_{inf} \leq K \leq K_{sup}$, then the distance $d$ between any two conjugate points along any unit speed geodesic (length of the geodesic segment joining the conjugate points) satisfies,*

$$\frac{\pi}{\sqrt{K_{sup}}} \leq d \leq \frac{\pi}{\sqrt{K_{inf}}}$$

*Proof.* Let $n$ be the dimension of $M$. Compare $M$ with a sphere $S^n(\frac{1}{\sqrt{K_{sup}}})$, which has constant sectional curvature $K_{sup}$. Let $p \in M$ and $\gamma : [0,a] \longrightarrow M$ be a unit speed geodesic with $\gamma(0) = p$. We only need to show that there does not exist a Jacobi field which vanish at $p$ and vanish before $\gamma(\frac{\pi}{\sqrt{K_{sup}}})$ while moving along $\gamma$. It is enough to show it for normal Jacobi field. Consider a normal Jacobi fields $J$ along $\gamma$, $\overline{J}$ along $\overline{\gamma} : [0,a] \longrightarrow S^n(\frac{1}{\sqrt{K_{sup}}})$ such that $J(0) = \overline{J}(0) = 0$ and $|D_tJ(0)| = |D_t\overline{J}(0)|$. As $\overline{\gamma}$ does not have any conjugate points in the interval $(0, \frac{\pi}{\sqrt{K_{sup}}})$ by Rauch comparison theorem $|\overline{J}| \leq |J|$. Hence the distance $d$ between conjugate point along $\gamma$ satisfies $\frac{\pi}{\sqrt{K_{sup}}} \leq d$. To obtain the other inequality compare $M$ with $S^n(\frac{1}{\sqrt{K_{inf}}})$. □

Following is another application of the Rauch comparison theorem which will be used crucially in the proof of the Sphere theorem.

**Proposition 9.1.4.** *Let $M$ and $\bar{(M)}$ be two Riemannian manifolds of same dimension. Suppose for all $p \in M$ and $\overline{p} \in \overline{M}$ and all the two dimensional subspace $\sigma \subset T_pM$, $\overline{\sigma} \subset T_{\overline{p}}\overline{M}$*

*we have* $\overline{K}_{\overline{p}}(\overline{\sigma}) \geq K_p(\sigma)$. *Let* $i : T_pM \longrightarrow T_{\overline{p}}\overline{M}$ *be a linear isometry. Let* $r > 0$ *be such that the restriction* $\exp_p|B_r(0)$ *is a diffeomorphism and* $\exp_{\overline{p}}|\overline{B}_r(0)$ *is non singular. Consider a smooth curve* $\alpha : [0,a] \longrightarrow \exp_p(B_r(0))$ *and define* $\overline{\alpha} : [0,a] \longrightarrow \exp_{\overline{p}}(\overline{B}_r(0))$ *by* $\alpha(t) = \exp_{\overline{p}} \circ i \circ \exp_p^{-1}(\alpha(t))$ *then* $L(\alpha) \geq L(\overline{\alpha})$

*Proof.* Let $\nu(s) = \exp_p^{-1} t\alpha(s)$. For a fixed $s$ consider the geodesic, $\gamma_s(t) = \exp_p t\nu(s)$. $\Gamma(s,t) = \gamma_s(t)$ forms a parametric surface. Since for each $s$, $\gamma_s(t)$ is a geodesic $J_s(t) = \frac{\partial}{\partial s}\Gamma(s,t) = \frac{\partial}{\partial s}\exp_p t\nu(s) = (d\exp_p)_{t\nu(s)}(t\nu'(s))$ is a Jacobi field. Clearly $J_s(0) = 0$ and $J_s(1) = \frac{\partial}{\partial s}\Gamma(s,1) = \frac{\partial}{\partial s}\exp_p \nu(s) = \alpha'(s)$.

$$\begin{aligned} D_tJ_s(0) =& D_t\{(d\exp_p)_{t\nu(s)}(t\nu'(s))\}|_{t=0} \\ &= D_t\{t(d\exp_p)_{t\nu(s)}(\nu'(s))\}|_{t=0} \\ &= (d\exp_p)_{t\nu(s)}(\nu'(s))|_{t=0} + tD_t(d\exp_p)_{t\nu(s)}(\nu'(s))|_{t=0} \\ &= (d\exp_p)_0(\nu'(s)) = \nu'(s) \end{aligned}$$

Consider $\overline{\Gamma}(s,t)$ on $\overline{M}$ defined by $\overline{\Gamma}(s,t) = \exp_{\overline{p}} ti(\nu(s)) = \overline{\gamma}_s(t)$. As each $\overline{\gamma}_s$ is a geodesic $\frac{\partial}{\partial s}\overline{\Gamma}(s,t) = \overline{J}_s(t)$ is a Jacobi field and $\overline{J}_s(0) = 0$, $\overline{J}_s(1) = \overline{\alpha}'(t)$ and $D_t\overline{J}_s(0) = i\nu'(s)$. As $i$ is an isometry, $|J_s(0)| = |\overline{J}_s(0)|$ and $|D_tJ_s(0)| = |D_t\overline{J}_s(0)|$. Also,

$$\begin{aligned} \langle D_t\overline{J}_s(0), \overline{\gamma}'_s(0)\rangle =& \langle i\nu'(s), i\gamma'_s(0)\rangle \\ &= \langle \nu'(s), \gamma'_s(0)\rangle = \langle D_tJ_s(0), \gamma'_s(0)\rangle \end{aligned}$$

Thus, from the above properties and from the condition on the sectional curvature, we can apply Rauch's Theorem on $\overline{J}_s(t)$ and $J_s(t)$. And it gives us $|\overline{\gamma}'(s) = \overline{J}_s(1)| \leq |J_s(1)| = \alpha'(s)$. Therefore $L(\overline{\alpha}) \leq L(\alpha)$. □

## 9.2 Morse Index Theorem

In this section we will prove the Morse index theorem. As a corollary of this we shall see that no geodesic is minimizing past its conjugate point. Consider a geodesic $\gamma : [0,a] \longrightarrow M$ and define $\mathcal{V}_{\gamma_a}$ be set of all vector fields along $\gamma$ which vanish at the end points. We can define index form on $\mathcal{V}_{\gamma_a}$ as,

$$I_a(V,W) = \int_0^a \langle D_tV, D_tW\rangle - \langle R(V,\gamma')\gamma', W\rangle \tag{9.1}$$

Even though we have calculated $I_a(V,W)$ for smooth vector fields this expression is same for piecewise smooth vector fields. But it becomes

$$I_a(V,W) = -\int_0^a \langle D_t^2 V + R(V,\gamma')\gamma', W\rangle dt - \sum_{i=1}^{k-1} \langle D_t V(a_i^+) - D_t V(a_i^-), W(a_i)\rangle \qquad (9.2)$$

where $\{a_i\}_{i=1}^{k-1}$ are the points where $V$ is not smooth. From the symmetry of Riemannian curvature it follows that $I_a$ is symmetric and bilinear. Given a symmetric bilinear form we define the index of the form as the dimension of subspace where $I_a$ is negative definite. Null space of $I_a$ is the subspace of all vector fields $V$ such that $I_a(V,W) = 0$ for all $W \in \mathcal{V}_\gamma^a$. Following proposition specifies the null space of $I_a$.

**Proposition 9.2.1.** *The null space of $I_a$ is the subspace formed by all Jacobi fields along $\gamma$ which vanish at the end points.*

*Proof.* Let $V \in Null(\mathcal{V}_{\gamma_a})$, then $I_a(V,W) = 0$ for all $W \in \mathcal{V}_\gamma^a$. Let $a_1 < ... < a_{k-1}$ be the points where $V$ fails to be smooth. Consider a the vector field $W \in \mathcal{V}_\gamma^a$ defined by $W(t) = D_t^2 V(t) + R(V(t),\gamma'(t))\gamma'(t)$ for all $t \in (0,a) \setminus \{a_i\}_{i=1}^{k-1}$ and $W(a_i) = D_t V(a_i^+) - D_t V(a_i^-)$. From equation 9.2 it is clear that $V$ is a Jacobi field. Converse is obvious. □

This proposition gives us an easy corollary which says that $I_a$ is degenerate (positive nullity) if and only if $\gamma(a)$ is a conjugate point of $\gamma(0)$ and nullity is precisely the multiplicity of conjugate points.

Now assume $0 = a_0 < a_1 < ... < a_{k-1} < a_k$ be the partition such that $\gamma|_{[a_i,a_{i+1}]}$ has no conjugate points. This can be chosen because $\gamma[0,a]$ is compact and can be covered by finite number of (totally) normal neighborhoods. Let $\mathcal{V}_{\gamma_a}^-$ be the set of all vector fields $V$ such that $V|_{(a_i,a_{i+1})}$ is a Jacobi field. Let $\mathcal{V}_{\gamma_a}^+$ denote the set of all vector fields $V$ such that $V(a_i) = 0$ for all $i$. The notations are suggestive as demonstrated by the following proposition.

**Proposition 9.2.2.** *$\mathcal{V}_{\gamma_a}^+$ and $\mathcal{V}_{\gamma_a}^-$ are orthogonal with respect to $I_a$ and $\mathcal{V}_{\gamma_a} = \mathcal{V}_{\gamma_a}^+ \oplus \mathcal{V}_{\gamma_a}^-$ and restriction of $I_a$ onto $\mathcal{V}_{\gamma_a}^+$ is positive definite.*

*Proof.* Let $V \in \mathcal{V}_{\gamma_a}$. Choose $W$ such that $W(a_i) = V(a_i)$ for all $i$ and $W|[a_i,a_{i+1}]$ is a Jacobi field. Such a Jacobi field exists and is unique as it is the unique solution of boundary condition. Therefore $V - W \in \mathcal{V}_{\gamma_a}^+$ and $W \in \mathcal{V}_{\gamma_a}^-$. Hence $\mathcal{V}_{\gamma_a} = \mathcal{V}_{\gamma_a}^+ \oplus \mathcal{V}_{\gamma_a}^-$ and observe that $I_a(V,W) = 0$ for all $V \in \mathcal{V}_{\gamma_a}^+$ and for all $W \in \mathcal{V}_{\gamma_a}^-$ by equation 9.2. To prove the second part of the proposition choose $V \in \mathcal{V}_{\gamma_a}^+$ then $I_a(V,V) \geq 0$ as $I_a(V,V) = \frac{d^2}{ds^2}E(0)$ where $E$ is the energy of the variation with variation field $V$. Suppose $I_a(V,V) = 0$ for some

$V \in \mathcal{V}^+_{\gamma_a} \setminus \{0\}$. If $W \in \mathcal{V}^-_{\gamma_a}$ then $I_a(V,W) = 0$. If $W \in \mathcal{V}^+_{\gamma_a}$ then $0 \leq I_a(V+xW, V+xW) = 2xI_a(V,W) + x^2 I_a(W,W)$ for any $x \in \mathbb{R}$. As $I_a(W,W) \geq 0$ and the above inequality is true for all $x \in \mathbb{R}$ we conclude that $I_a(V,W) = 0$ which implies $V \in Null(I_a)$. As $Null(I_a)$ is formed by the Jacobi fields and $V(a_i) = 0$, $a_i$ and $a_{i+1}$ are conjugate points contradicting the choice of partition. Hence $V = 0$ which contradicts $V \in \mathcal{V}^+_{\gamma_a} \setminus \{0\}$. □

This proposition tells us that index and nullity of $I_a$ is same as the index and nullity of $I_a$ restricted to $\mathcal{V}^-_{\gamma_a}$ and it is finite. Now we prove the Morse index theorem.

**Theorem 9.2.3** (Morse Index Theorem)**.** *Let $\gamma : [0,a] \longrightarrow M$ be a geodesic then the index of $I_a$ is equal to the number of conjugate points to $\gamma(0)$ counted with multiplicity. In particular index is finite.*

*Proof.* We denote by $\gamma_t$ the restriction $\gamma|_{[0,t]}$ for $t \in [0,a]$ and index form of $\gamma_t$ by $I_t$. The function $i : [0,a] \longrightarrow \mathbb{N}$ is defined as $i(t) =$ index of $I_t$. Choose $0 = a_0 < a_1 < ... < a_{k-1} < a_k = a$ such that $\gamma|_{[a_i,a_{i+1}]}$ is minimizing. Observe that on a small enough neighborhood of 0, $i(t) = 0$. By definition $i(t) = dim(U)$ such that $U \subset \mathcal{V}_{\gamma_t}$ such that $I_t$ is negative definite. We can extend any $V \in U$ to all of $\gamma$ by defining it to be zero out side the interval $[0,t]$. Thus $i(t') \geq i(t)$ for all $t' > t$, in other words $i$ is increasing function (may not strictly increasing).

We have seen that index of $I_t$ is index of $I_t$ restricted to $\mathcal{V}^-_{\gamma_t}$. (We denote both $I_t$ and its restriction to $\mathcal{V}^-_{\gamma_t}$ by $I_t$ itself.) But elements in $\mathcal{V}^-_{\gamma_t}$ are uniquely determined by values of the vectors at $\gamma(a_i)$s as all its elements are broken Jacobi fields. Therefore we can write,

$$\mathcal{V}^-_{\gamma_t} = T_{\gamma(a_1)} \oplus ... \oplus T_{\gamma(a_{j-1})}$$

We can choose $a_i$s such that $t \in (a_{j-1}, a_j)$ (Remember choice of $a_i$s were upto us). If we vary $t \in (a_{j-1}, a_j)$ we get that all the spaces $\mathcal{V}^-_{\gamma_t}$ are isomorphic. Denote it by $S_j$. $I_t$ on $\mathcal{V}^-_{\gamma_t}$ depend continuously on $t \in (a_{j-1}, a_j)$. If $I_t$ is negative definite on a subspace of $S_j$ then $I_{t-\epsilon}$ is also negative definite in that subspace for some small enough $\epsilon > 0$. Therefore $i(t-\epsilon) \geq i(t)$. As $i$ is increasing it follows that $i(t-\epsilon) = i(t)$. Let the nullity of $I_t$ be $\varphi$.

We claim that for small enough $\epsilon > 0$, $i(t+\epsilon) = i(t) + \phi$. First we will show that $i(t+\epsilon) \leq i(t) + \varphi$. Observe that dimension of $S_j = n(j-1)$, hence $I_t$ is positive definite on a subspace of dimension $n(j-1) - i(t) - \varphi$. By continuity of $I_t$, $I_t + \epsilon$ is positive definite for a small enough $\epsilon > 0$. Hence $i(t+\epsilon) \leq n(j-1) - (n(j-1) - i(t) - \varphi) = i(t) + \varphi$. To prove the other way inequality consider $V \in S_j$ such that $V(a_{j-1}) \neq 0$. $V_{t_0}$ be a broken Jacobi field such that $V_{t_0}(t_0) = 0$ and it is equal to the value of $V$ at all the non-smooth points. Let $W_{t_0}$

be a vector field along $\gamma_{t_0+\epsilon}$ such that $W_{t_0}(t) = V_{t_0}(t)$ for all $t \in [0, t_0]$ and vanishes outside the interval. From index lemma we have $I_{t_0}(V_{t_0}, V_{t_0}) = I_{t_0+\epsilon}(W_{t_0}, W_{t_0}) > I_{t_{0+\epsilon}}(V_{t_0+\epsilon}, V_{t_0+\epsilon})$. If $V(a_{j-1}) = 0$ then either $V$ is identically zero or it is a broken Jacobi field. Hence it does not affect nullity as null space is precisely the space of Jacobi fields (not broken). Hence $\varphi$ remains unchanged. As $I_t(V, V) < I_{t+\epsilon}$ the negative definite space of $I_{t+\epsilon}$ has dimension $i(t+\epsilon) \geq i(t) + \varphi$. Thus $i(t+\epsilon) = i(t) + \varphi$. From this we can conclude that $i(t)$ is a step function which is 0 around a neighborhood of 0 and jumps at conjugate points of $\gamma(0)$ with height same as the multiplicity of conjugate points (This occurs as $\varphi$ precisely measures that). Hence we proved the theorem. □

**Corollary 9.2.4** (Jacobi)**.** *Let $\gamma : [0, a] \longrightarrow M$ be a geodesic such that $\gamma(a)$ is not a conjugate point of $\gamma(0)$. Then $\gamma$ does not have any conjugate points in $(0, a)$ if and only if for all proper variations of $\gamma$ there exist a $\delta > 0$ such that for all $0 < |s| < \delta$, $E(s) < E(\delta)$. In particular if $\gamma$ is minimizing then it does not have any conjugate points on $(0, a)$*

**Corollary 9.2.5.** *The set of conjugate point along a geodesic forms a discrete set.*

We will now look at an interesting exercises which elegantly uses Morse Index Theorem to prove a calculus result.

*Exercise:*(Wirtinger's inequality) Consider the function $f : [0, \pi] \longrightarrow \mathbb{R}$ of class $\mathcal{C}^2$ such that $f(0) = f(\pi) = 0$. Then prove that,

$$\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt$$

and the equality occurs if and only if $f(t) = c \sin t$, where $c$ is a constant.

*Proof.* Let $p \in S^2$, two dimensional sphere with sectional curvature 1. Consider unit speed geodesic $\gamma$ and a smooth parallel vector field $X$ along $\gamma$ such that $\langle X, \gamma' \rangle = 0$ with $|X| = 1$. Define $\overline{X} = fX$. By the second variation formula,

$$\begin{aligned} I_\pi(\overline{X}, \overline{X}) &= \int_0^\pi \langle D_t \overline{X}, D_t \overline{X} \rangle - \langle R(\overline{X}, \gamma')\gamma, \overline{X} \rangle dt \\ &= \int_0^\pi \langle D_t fX, D_t fX \rangle - \langle R(fX, \gamma')\gamma, fX \rangle dt \\ &= \int_0^\pi \langle f'X, f'X \rangle - \langle f^2 R(X, \gamma')\gamma, X \rangle dt \\ &= \int_0^\pi (f')^2 dt - \int_0^\pi f^2 dt \end{aligned}$$

As there are no conjugate points $p$ along $\gamma$ on the interval $(0, \pi)$ from Morse Index Lemma it follows that $I_\pi(\overline{X}, \overline{X}) \geq 0$. Therefore $\int_0^\pi f^2 dt \leq \int_0^\pi (f')^2 dt$. Now $I_\pi(\overline{X}, \overline{X}) = 0$ if and only if $\overline{X}$ is a Jacobi field. Thus from the Jacobi equation, $f'' + f = 0$ and from the initial and final condition it follows that, $f(t) = c \sin t$. □

Both the main theorems presented in this chapter are essential ingredient for the proof of Sphere theorem.

# Chapter 10

# The Sphere Theorem

Sphere theorems are regarded as one of the most beautiful results in global differential geometry. It was first proved by Klingenberg and Berger using Topogonov's theorem. We present here a different proof of Sphere theorem using basic Morse theory. Following is the statement of Sphere theorem:

**Theorem 10.0.1.** *Let $M$ be a compact simply connected Riemannian manifold whose sectional curvature satisfies $0 < \frac{1}{4}K_{max} < K \leq K_{max}$ then $M$ is homeomorphic to a sphere.*

Not that if sectional curvature $K$ is allowed to be equal $\frac{1}{4}K_{max}$ at a point with respect to some 2-plane then the result is not true. A counter example is complex projective space with Fubini-Study metric. The differential version of Sphere theorem was proved in 2007 by Simon Brendle and Richard Schoen using Ricci flow. In dimension two and three this follows from Gauss-Bonnet theorem and Hamilton's theorem. It also follows from the simple connectedness and Poincare conjecture along with theorem of Bonnet-Myers (Theorem 8.2.1).

## 10.1 Cut Locus

The concept of cut locus was introduced by Poincaré. But it was Klingenberg who showed that the idea of cut locus important for proving the Sphere theorem. Throughout this chapter we assume $M$ to be a complete Riemannian manifold and $\gamma$ a unit speed geodesic.

**Definition 10.1.1.** *Let $\gamma : [0, \infty) \longrightarrow M$ be a unit speed geodesic such that $\gamma(0) = p$. Let $t_0 = sup\{t \in [0, \infty) \,|\, d(p, \gamma(t)) = t\}$, then $\gamma(t_0)$ is said to be the cut point of $p$ along $\gamma$. If such a $t_0$ does not exist then we say cut point of $p$ along $\gamma$ does not exist.*

We denote set of all cut points of $p$ along all the possible directions, the cut locus of $p$, as $Cut(p)$

**Proposition 10.1.2.** *Let $q = \gamma(t_0)$ be a cut point of $p$ along $\gamma$ then either $q$ is the first conjugate point of $p$ along $\gamma$ or there exist minimizing geodesic other than $\gamma$ joining $p$ and $q$. Conversely if atleast one of the above condition is valid then $\gamma(t)$ is a cut point of $p$ along $\gamma$ for some $t \in (0, t_0]$*

*Proof.* Assume $q = \gamma(t_0)$ a cut point of $p = \gamma(0)$. Consider the $t_i > t_0$ be a sequence converging to $t_0$ and $\alpha_i$s are unit speed minimizing geodesic joining $p$ and $\gamma(t_i)$. Such a geodesic always exist as the space we are considering is complete. Considering the sequence $\alpha_i'(0)$ we can assume that it converges to $\alpha'(0)$ ($\alpha_i'(0)$ lies in the sphere of radius 1 inside $T_pM$, so without loss of generality we can consider a converging subsequence). By the continuity of exponential map $\alpha$ is a minimizing geodesic joining $p$ and $q$ with initial velocity $\alpha'(0)$. Thus we get $L(\gamma) = L(\alpha)$. If $\gamma \neq \alpha$ then we obtain a minimizing geodesic other than $\gamma$ joining $p$ and $q$. So let us assume that $\gamma = \alpha$. Therefore we have to show that $q$ is the first conjugate point of $p$. It is enough to show that $(d\exp_p)_{t_0\gamma'(0)}$ is singular as $\gamma$ is minimizing upto $t_0$. For a contradiction assume that $(d\exp_p)_{t_0\gamma'(0)}$ is non singular and $\exp_p$ a diffeomorphism around an open neighborhood $U$ of $t_0\gamma'(0)$. Let $t_i = t_0 + \epsilon$ for a small enough $\epsilon > 0$. Then $\alpha_i(t_0 + \epsilon) = \gamma(t_0 + \epsilon')$ where $\epsilon' > \epsilon$ as $\alpha_i$ is minimizing upto $t_i$. By our assumption $\alpha_i'(0) \to \gamma'(0)$ and therefore we can choose $\epsilon > 0$ small enough such that $(t_0 + \epsilon)\alpha_i' \in U$. It follows that $(t_0 + \epsilon')\gamma'(0) \in U$. Since we have assumed that $\exp_p$ is a diffeomorphism in $U$.

$$\begin{aligned}
\exp_p(t_0 + \epsilon')\gamma'(0) &= \gamma(t_0 + \epsilon') \\
&= \alpha_i(t_0 + \epsilon) \\
&= \exp_p(t_0 + \epsilon)\alpha_i'(0)
\end{aligned}$$

Since $\exp_p$ is a diffeomorphsim in $U$, $(t_0+\epsilon')\gamma'(0) = (t_0+\epsilon)\alpha_i'(0)$ which implies $\gamma'(0) = \alpha_i'(0)$. By the definition of $\alpha_i$ this is a contradiction to the fact that $\gamma(t_0)$ is a cut point of $\gamma(0)$.

Conversely assume that $q$ is the first conjugate point of $p$ as no geodesic is minimizing past its conjugate point, the cut point of $\gamma(0)$ occur at $\gamma(t)$ for some $t \in (0, t_0]$. On the other hand assume that there exist a minimizing geodesic $\alpha$ joining $p$ and $q$ other than $\gamma$. Choose an $\epsilon > 0$ small enough such that $\alpha(t_0 - \epsilon)$ and $\gamma(t_0 + \epsilon)$ lies in a totally normal neighborhood. Then there exist a unique minimizing geodesic $\beta$ joining $\alpha(t_0 - \epsilon)$ and $\gamma(t_0 + \epsilon)$. $\beta$ has length strictly less than $2\epsilon$ as $\alpha \neq \gamma$. Consider the curve obtained by concatenating $\alpha$ from $p$ to $\alpha(t_0 - \epsilon)$ and $\beta$. The new curve thus obtained has length less than $t_0 + \epsilon$. Which implies that

$\gamma$ is not minimizing past $\gamma(t_0)$. Therefore cut point can occur at $t$ for some $t \in (0, t_0]$. $\square$

**Corollary 10.1.3.** *$q \in Cut(p)$ if and only if $p \in Cut(q)$*

*Proof.* If $q$ is a cut point of $p$ along $\gamma$ then consider $-\gamma$ joining $q$ and $p$. $L(-\gamma) = d(p, q)$. Now applying the proposition gives us that $p \in Cut(q)$ $\square$

One can see that if $q \notin Cut(p)$ then there exist a unique minimizing geodesic joining $p$ and $q$ (Remember we are always considering complete manifolds). Therefore $M \setminus Cut(p)$ is homeomorphic to an open ball in a Euclidean space. This indicates to us that the cut locus inherit topological information about the manifold. The way cut locus is glued to the open set carries complete topological information of the manifold. This corollary tells us that $\exp_p$ is injective on a open ball of radius $r = d(p, Cut(p))$ around 0. Bearing this fact in mind we define *injectivity radius* of a manifold $M$ as $inj(M) = inf_{p \in M}\{d(p, Cut(p))\}$

We further carry on with the study of cut locus. We have the following theorem which states that cut locus depends continuously on the initial point and initial velocity vector. Let $T_1M = \{(p, v) \in TM \, : \, |v| = 1\}$. Give $\mathbb{R} \cup \{\infty\}$ topology whose basis sets are all open intervals in addition to sets of the type $(a, \infty) \cup \{\infty\}$.

**Proposition 10.1.4.** *Define $F : T_1M \longrightarrow \mathbb{R} \cup \{\infty\}$ as follows,*

$$F(\gamma(0), \gamma'(0)) = \begin{cases} t_0 & \text{if } \gamma(t_0) \text{ is a cut point of } \gamma(0), \\ \infty & \text{if cut point of } \gamma(0) \text{ along } \gamma \text{ does not exist.} \end{cases} \tag{10.1}$$

*is continuous.*

*Proof.* Choose $\gamma_i$s such that $\gamma_i(0) \to \gamma(0)$ and $\gamma_i'(0) \to \gamma'(0)$. Assume that $\gamma_i(t_0^i)$ and $\gamma(t_0)$ are the cut points of $\gamma_i(0)$ and $\gamma(0)$ along their respective curves. In order to prove that $F$ is continuous we need to show that $\lim_{i \to \infty} t_0^i = t_0$.

Choose $\epsilon > 0$ and assume $t_0 < \infty$. There are only infinitely many $i$ such that $t_0 + \epsilon < t_0^i$. Otherwise $d(\gamma_i(0), \gamma_i(t_0 + \epsilon)) = t_0 + \epsilon$ and hence by continuity of the metric $d(\gamma(0), \gamma(t_0 + \epsilon)) = t_0 + \epsilon$ which contradicts the fact that $t_0$ is the cut point of $\gamma(0)$ along $\gamma$. Therefore we have established that $\limsup_i(t_0^i) < t_0$. This inequality is anyway true if $t_0 = \infty$. Denote $t' = \liminf_i(t_0^i)$. In order to prove the proposition it is enough to show that $t' \geq t_0$. If $t' = \infty$ then the claim is proved. So assume $t' < \infty$ Consider any subseqence of $t_0^i$ which converges to $t'$ (denote it by the same). If $\gamma_i(t_0^i)$ is a conjugate point of $\gamma_i(0)$ then $\gamma(t')$ is a conjugate point of $\gamma(0)$. Hence $t' \geq t_0$. The other case where $\gamma_i(t_0^i)$ not conjugate to $\gamma_i(0)$ can be dealt with similar arguments as in the proof of proposition 10.1.2. $\square$

**Corollary 10.1.5.** *For any $p \in M$, $Cut(p)$ is closed, therefore if $M$ is compact then $Cut(p)$ is compact.*

*Proof.* Assume $q$ is a limit point of $Cut(p)$, i.e there exist a sequence $\gamma_i(t_i)$ which converges to $q$ and $t_i = F(p, \gamma_i'(0))$. We can assume that $\gamma_i' \to \gamma'(0)$ (if necessary we can consider a subsequence), where $\gamma$ is a geodesic starting from $p$ with initial velocity vector $\gamma'(0)$. Now we use the continuity of $F$.

$$\begin{aligned} q = lim_i \gamma_i(t_i) &= lim_i(\gamma_i(F(p, \gamma'(0)))) \\ &= lim_i \exp_p(F(p, \gamma'(0))\gamma'(0)) \\ &= \exp_p(F(p, \gamma'(0))\gamma'(0)) \\ &= \gamma(F(p, \gamma'(0))) \in Cut(p) \end{aligned}$$

Hence $Cut(p)$ is closed. □

**Corollary 10.1.6.** *If there exist $p \in M$ such that $p$ has a cut point in all the possible directions then $M$ is compact.*

*Proof.* We $M = \bigcup\{\gamma(t) : t \leq F(p, \gamma'(0))\}$ for some $p \in M$ and $F$ is continuous implies that $M$ is bounded. From Hopf-Rinow theorem it follows that $M$ is compact. □

**Proposition 10.1.7.** *Let $p \in M$ and if there exist $q \in Cut(p)$ such that $d(p, q) = d(p, Cut(p))$ then either there a geodesic $\gamma$ joining $p$ and $q$ such that $L(\gamma) = d(p, q) = l$ and $q$ is a conjugate point of $q$ or there exist exactly two minimizing geodesic $\gamma$ and $\lambda$ joining $p$ and $q$ with $\gamma'(l) = -\lambda'(l)$.*

*Proof.* Let $q \in Cut(p)$ such that $d(p, q) = l = d(p, Cut(p))$ then by proposition 10.1.2 $q$ is a conjugate to $p$ along some minimizing geodesic $\gamma$. This establishes the first part of the proposition. Otherwise, according to the same proposition, there exist a minimizing geodesic $\lambda \neq \gamma$ joining $p$ and $q$ such that $L(\gamma) = L(\lambda)$. To prove the second assertion assume that $q$ is not a conjugate of $p$ and for a contradiction assume that $\gamma'(l) \neq \lambda'(l)$. Thus we can find $X \in T_qM$ such that $\langle X, \gamma'(l)\rangle < 0$ and $\langle X, \lambda'(l)\rangle < 0$. Choose a curve $\sigma : (-\epsilon, \epsilon) \longrightarrow M$ such that $\sigma(0) = q$ and $\sigma'(0) = X$. As we have assumed $q$ is not a conjugate of $p$ we can find $\Sigma : (-\epsilon, \epsilon) \longrightarrow M$ such that $\exp_p \Sigma(s) = \sigma(s)$, as exponential map is a diffeomorphism around $l\gamma'(0)$. Define variation of $\gamma$, $\Gamma_s(t) = \exp_p \frac{t}{l}\Sigma(s)$ for $t \in [0, l]$. From the first variation formula we obtain $\frac{dL(\Gamma_s)}{ds}|_{s=0} = \langle V, \gamma'(l)\rangle < 0$. Similarly we obtain a variation for $\lambda$, $\Lambda_s$ and $\frac{dL(\Lambda_s)}{ds}|_{s=0} = \langle V, \lambda'(l)\rangle$. Hence for small enough $\epsilon > 0$, $L(\Gamma_s) < L(\gamma)$ for $s \in (-\epsilon, \epsilon)$ and

$L(\Gamma_s) < L(\gamma)$. As $d(p, \Gamma'_s) = L(\Gamma_s) < d(p, Cut(p))$ we obtain that if $L(\Gamma_s) = L(\Lambda_s)$ then from proposition 10.1.2 $\Gamma_s(l)$ is a cut point of $p$ which contradicts that $d(p,q) = d(p, Cut(p))$. If $L(\Gamma_s) < L(\Lambda_s)$ then $\Lambda_s$ is not minimizing and hence there exist a cut point $\Lambda_s(t)$ for some $t \in (0, l]$ which again contradicts $d(p,q) = d(p, Cut(p))$. Analogously $L(\Gamma_s) > L(\gamma)$ also gives us a contradiction. □

**Proposition 10.1.8.** *Let $M$ be a complete manifold its sectional curvature $K$ satisfies $0 < K_{min} \le K \le K_{max}$ then $inj(M) \ge \pi/\sqrt{K_{max}}$ or there exist a closed geodesic $\gamma$ in $M$ such that $L(\gamma) < L(\lambda)$ for any other closed geodesic $\lambda$ and $inj(M) = \frac{1}{2}L(\gamma)$*

*Proof.* From the theorem 8.2.1 of Bonnet-Myers we obtain that $M$ is compact. As $T_1M$ is compact and $F$ in proposition 10.1.4 is continuous it follows that there exist $p \in M$ such that $d(p, Cut(p)) = inj(M)$. Since $Cut(p)$ is compact there exist $q \in M$ such that $d(p,q) = d(p, Cut(p))$. If $q$ is a conjugate of $p$ then by the application of Rauch comparison theorem $d(p,q) \ge \frac{\pi}{\sqrt{K_{max}}}$. If $q$ is not a conjugate of $p$ there exist $\gamma$ and $\lambda$ two minimizing geodesic from $p$ to $q$ such that $\lambda'(l) = -\gamma'(l)$. As the relation $q$ cut point of $p$ is symmetric it gives $\lambda'(0) = -\gamma'(0)$ and hence we obtained a closed geodesic concatenating $\lambda$ and $\gamma$ which proves the proposition. □

## 10.2 Theorem of Klingenberg on injectivity radius

The theorem on injectivity radius due to Klingenberg is a crucial step in the proof of Sphere theorem. We state certain Morse theory facts essential for the proof of Klingenberg's theorem.

**Lemma 10.2.1.** *Consider two Riemannian manifolds $M$ and $\overline{M}$ whose sectional curvature $K$ and $\overline{K}$ satisfies $\overline{K}_{sup} \le K_{inf}$. Consider a unit speed geodesic $\gamma : [0, l] \longrightarrow M$ where $\gamma(0) = p$ and a choose a point $\overline{p} \in \overline{M}$. Let $i : T_pM \longrightarrow T_{\overline{p}}\overline{M}$ be a linear isometry and define $\overline{\gamma} : [0, l] \longrightarrow \overline{M}$ as $\exp_{\overline{p}} t(i(\gamma'(0)))$. Then index($\gamma$)$\ge$ index($\overline{\gamma}$)*

*Proof.* Choose a piecewise smooth vector field $\overline{V}$ along the curve $\overline{\gamma}$ and define $V(t) = P_t \circ i^{-1} \circ \overline{P}_t^{-1}(\overline{V}(t))$, where $P_t$ and $\overline{P}_t$ are parallel transport from 0 to $t$ along the curve $\gamma$ and $\overline{\gamma}$ respectively. (As in section 7.1.) Thus from the proof of Rauch comparison theorem we get $\langle V, \gamma' \rangle = \langle \overline{V}, \overline{\gamma}' \rangle$, $|V| = |\overline{V}|$ and $|D_t| = |D_t\overline{V}|$. As $\overline{K}_{sup} \le K_{inf}$ we conclude that $I(V,V) \le I(\overline{V}, \overline{V})$ which proves the theorem. □

Let $f : M \longrightarrow \mathbb{R}$ be a smooth function. We say $p \in M$ is a critical point of $f$ if $df(p) = 0$ and $f(p)$ is called the critical value of $f$. Choose a system of coordinates around $p$, $(x_1, ..., x_n)$ and consider the hessian matrix $\triangle_p f = (\frac{\partial^2 f}{\partial x_i \partial x_j})(p)$. One can show that this does not depend on the choice of coordinate chart. Hessian defines a symmetric bilinear form on $T_pM$. A critical point is said to be non-degenerate if $det(\triangle_p f) \neq 0$. Non-degenerate critical points are isolated. We can define index of the critical point as the dimension of the subspace where hessian is negative definite. An equivalent definition would be to choose a coordinate neighborhood around $p$, $(x_1, ..., x_{n-k}, y_1, ..., y_k)$ such that in that neighborhood $f(x) = f(p) + x_1^2 + ... + x_{n-k}^2 - y_1^2 - ... - y_k^2$, then $k$ is called the index of $p$. We present the following lemma without proof.

**Lemma 10.2.2.** *Let $f : M \longrightarrow \mathbb{R}$ be a smooth map which has only non-degenerate critical points. Given a smooth curve $\gamma : [0, 1] \longrightarrow M$ joining $p$ and $q$ and $a = max\{f(p), f(q)\}$ and denote $M_a = \{x \in M \mid f(x) \leq a$ and $b$ be the maximum value taken by $f$ along the curve $\gamma$. If $f^{-1}([a, b])$ is compact and does not have any critical points of index 0 or 1. Then given any $\delta > 0$, $\gamma$ is path homotopic to $\overline{\gamma}$ such that $\overline{\gamma}([0, 1]) \subset M_{a+\delta}$*

A slight modification of the proof also yield us that if $f^{-1}([0, 1])$ contains critical points of index 0 or 1 then $\gamma$ is homotopic to $\overline{\gamma}$ such that $\overline{\gamma} \subset M_{c+\delta}$ where $c$ is the largest value of such critical points.

We will now look at an interesting construction of a manifold which will be used in the proof of Klingenberg's injectivity radius estimate. Let $\Omega_{p,q}$ be the set of all piecewise smooth curves joining $p$ and $q$. First variation of a proper variation can be seen analogous to the derivative of a function on a smooth manifold. One can see that in this set up tangent vector on a manifold is seen to be the piecewise smooth vector field which vanish at the end points. Energy function, $E$ we defined in chapter 8 is a smooth function on $\Omega_{p,q}$ and $\frac{d}{ds}E(\Gamma_s)$ is derivative of $E$ in the direction of $V$, where $V$ is the variation field of $\Gamma_s$. Difficulty with handling such a set is that we cannot find a diffeomorphism to an open set in Euclidean space of any dimension. But from Morse index theorem we get a way to approximate this space to a finite dimensional manifold under certain restrictions (Morse index theorem originally was introduced for this). Let $\Omega_{p,q}^c$ denote the set of all curves in $\Omega_{p,q}$ whose energy is $\leq c$ ($\mathring{\Omega}_{p,q}$ corresponds to energy $< c$).

We now sketch the way to approximate $\Omega_{p,q}^c(\mathring{\Omega}_{p,q})$ with a finite dimensional manifold using morse index theorem. If endpoints are understood we denote $\Omega_{p,q}$ as $\Omega$. One can see that curves in $\Omega^c$ are contained in a compact $S \subset M$. Choose $\delta > 0$ such that given any two points in $S$ with distance $< \delta$ we can find a unique minimizing geodesic between the two points. We choose a partition of $[0,1]$ such that $|a_i - a_{i-1}| < \frac{\delta^2}{c}$. Consider $B \subset \Omega^c$ such that $B$ consists of curves $\gamma$ such that $\gamma|_{[a_{i-1},a_i]}$ are geodesics. As

$$L(\gamma|_{[a_{i-1},a_i]})^2 = (a_i - a_{i-1})E(\gamma|_{[a_{i-1},a_i]}) < \delta^2$$

we deduce that such curves are determined by their values at $a_i$'s. Therefore we get a map from $B \to M \times M \ldots \times M$ $(k-1$ times) which is bijective. Hence we can define a smooth structure on $B$. By our correspondance, tangent vector of $B$ corresponds to broken Jacobi fields. One can show that $\mathring{B}$ can be a deformation retract of $\mathring{\Omega}$, i.e there is a family $\{h_s : \mathring{\Omega}^c \to \mathring{\Omega}^c : s \in [0,1]\}$ of continuous functions s.t $h_0 = Id_{\mathring{\Omega}}$ and $h_1 : \mathring{\Omega}^c \to \mathring{B}$. Also geodesics on $\mathring{\Omega}^c$ are geodesics on $\mathring{B}$ and are precisely the critical points of $E$. Index and nullity of $I$ restricted to the space of broken Jacobi fields (result similar to prop 9.2.1 and 9.2.2). Thus we can work with $\mathring{B}$ instead of $\mathring{\Omega}$.

**Lemma 10.2.3.** *Let $p, q \in M$ and $\gamma_0$ and $\gamma_1$ be two geodesics joining them with $L(\gamma_0) \leq L(\gamma_1)$. Let $\Gamma_s$, $s \in [0,1]$ be a continuous family of curves such that $\Gamma_0 = \gamma_0$ and $\Gamma_1 = \gamma_1$ i.e $\gamma_0$ and $\gamma_1$ are homotopic. Then there exist $t_0 \in [0,1]$ such that $L(\gamma_0) + L(\Gamma_{t_0}) \geq \frac{2\pi}{\sqrt{K_0}}$*

**Theorem 10.2.4** (Klingenberg)**.** *Let $M$ be a simply connected, compact Riemannian manifold of dimension $\geq 3$ whose sectional curvature $K$ satisfies $1/4 < K \leq 1$ then $inj(M) \geq \pi$*

*Sketch of the proof:* Assume on the contrary that $inj(M) < \pi$. By Proposition 10.1.8 there exist a closed geodesic $\gamma$ in $M$ with $L(\gamma) = l < 2\pi$. Choose $\epsilon > 0$ such that

(i) $\gamma(l - \epsilon)$ is not a conjugate point of $\gamma(0) = p$ (it is possible as set of conjugate points form a discrete set)

(ii) $\exp_p$ is a diffeomorphism on $B_{2\epsilon}(p)$

(iii) $3\epsilon < 2\pi - \frac{\pi}{\sqrt{K_{inf}}}$

(iv) $3\epsilon < 2\pi - l$

(v) $5\epsilon < 2\pi$

By Sard's theorem there exist atleast one regular value of $\exp_p$ $q \in B_\epsilon(\gamma(l-\epsilon))$. By $(i)$ we can choose $q$ such that there is a geodesic $\gamma_1$ joining $p$ and $q$ satisfying $3\epsilon < L(\gamma_1) < l$. Let

$\gamma_0$ be the minimizing geodesic joining $p$ and $q$. Then by triangle inequality and $(ii)$ we get, $L(\gamma_0) \leq d(p, \gamma(l-\epsilon)) + d(\gamma(l-\epsilon), q) < 2\epsilon$. Therefore $\gamma_0$ and $\gamma_1$ are distinct.

Consider $\Omega^c_{p,q}$ and its finite dimensional approximation $B$. Consider the smooth function $\Omega^c_{p,q} : B \longrightarrow \mathbb{R}$. Since $q$ is a regular value of $\exp_p$ all the critical points of $\Omega^c_{p,q}$ are non-degenerate. As $M$ is simply connected $\gamma_1$ and $\gamma_0$ are path homotopic. Let $\Gamma(s,t)$ be the homotopy. Note that $\Gamma^t(s) = \Gamma(s,t)$ is a path in $\Omega^c_{p,q}$. As $B$ is a deformation retract of $\Omega^c_{p,q}$, $\Gamma^t$ can be retracted to a curve in $B$. Apply Lemma 10.2.2(special case after the lemma) to $B$. Then for a given $\delta > 0$, $\Gamma^t$ which is a curve in $B$ is path homotopic to a curve $\overline{\Gamma^t}$ such that all the curves $\overline{\Gamma_s}$ (which are points in $B$) satisfies $L(\overline{\Gamma_s}) < a + \delta$. Here $a = \max\{L(\gamma_0), L(\gamma_1), L(\sigma)\}$ in which $\sigma$ is the geodesic with index$< 2$ and has maximum length among such geodesics with index$< 2$.

Take $\delta = \epsilon$. But $L(\gamma_0) < 2\epsilon$ and $L(\gamma) < l < 2\pi - \epsilon$. Applying Lemma 10.2.1 by taking $\overline{M}$ to be a sphere of curvature $\overline{K} = K_{inf}$ we obtain index$(\overline{\sigma}) \leq$ index$(\sigma) < 2$. From Morse index theorem it follows that if $L(\sigma) > \frac{\pi}{\sqrt{\overline{K}}}$ then Index$(\overline{\sigma}) \geq 2s$. Hence $L(\sigma) \leq \frac{\pi}{\sqrt{\overline{K}}} < 2\pi - 3\epsilon$ by $(iii)$. Let $\overline{\gamma}$ be the curve among the curves $\overline{\Gamma}_s$s which has the maximum length. Then by $(v)$ it follows that $L(\overline{\gamma}) < a + \epsilon < 2\pi - 3\epsilon + \epsilon = 2\pi - 2\epsilon$. But previous lemma says that there exist a curve $\Gamma_{s_0}$ among $\gamma_s$s such that $L(\gamma_0) + L(\Gamma_{s_0})$. Hence $L(\overline{\gamma}) \geq L(\Gamma_{s_0}) \geq 2\pi - L(\gamma_0) > 2\pi - 2\epsilon$. Which is a contradiction and hence we proved the theorem. □

## 10.3 The Sphere Theorem

We now prove the Sphere theorem.

**Lemma 10.3.1.** *Let $M$ be a compact Riemannian manifold and $p, q \in M$ be such that $d(p,q) = diam(M)$. Given any $v \in T_pM, \exists$ a minimizing geodesic $\gamma$ joining $p$ and $q$ and $\langle \gamma'(0), v \rangle \geq 0$.*

*Proof.* Take $\lambda(t) = exp_p(tv)$. Denote $a_t = d(\lambda(t), q)$. Consider a minimizing geodesic, $\gamma_t : [0, a_t] \to M$ such that $\gamma_t(0) = \lambda(t)$ and $\gamma_t(a_t) = q$. Suppose for all integer $n > 0$, there exist a $t_n$ s.t $0 \leq t_n \leq \frac{1}{n}$ and $\langle \gamma'_{t_n}(0), \lambda'(t_n) \rangle \geq 0$ then $\gamma_{t_n}$ converges to $\gamma$ (if needed taking a subsequence) which yields us $\langle \gamma'(0), \lambda'(0) \rangle = \langle \gamma'(0), v \rangle \geq 0$. This proves the lemma under such an assumption.

Now suppose there is an integer $n > 0$ such that $\forall t \in [0, \frac{1}{n}], \langle \gamma'_t(0), \lambda'(t) \rangle < 0$. Consider a totally normal neighbourhood $\mathcal{U}$ of $\lambda(t)$. Choose $q_0 \in \mathcal{U}$ and $q_0 \in \gamma_t([0, a_t])$. Let $\epsilon$ be small enough such that $\forall s \in (t-\epsilon, t+\epsilon)$, $\lambda(s) \in \mathcal{U}$ and $\Gamma_s$ be a minimizing geodesic joining $q_0$

and $\lambda(s)$. Then by the first variation formula and our assumption we get,

$$\frac{1}{2}\frac{d}{ds}E(\Gamma_s)|_{s=t} = -\langle\gamma_t'(0), \lambda'(0)\rangle > 0$$

Hence for $s < t$, $d(q_0, \lambda(s)) < d(q_0, \lambda(t))$ and therefore

$$d(q, \lambda(s)) \leq d(q, q_0) + d(q_0, \lambda(s)) < d(q, q_0) + d(q, \lambda(t)) = d(q, \lambda(t))$$

$$\implies d(q, \lambda(0)) < d(q, \lambda(t)),$$

which is a contradiction to the fact that $d(p, q) = diam(M)$. □

Following lemma is crucial in the proof of Sphere theorem. It is through this lemma Klingenberg's estimation on injectivity radius enters the proof of Sphere theorem. This was first proved by Berger using Topogonov's theorem. We present here a proof by Tsukamoto using Rauch's theorem.

**Lemma 10.3.2.** *Let $M$ be a connected, compact Riemannian manifold whose sectional curvature $K$ satisfies $\frac{1}{4} < \delta \leq K \leq 1$. Let $p,q \in M$ such that $diam(M) = d(p, q)$. Then $M = B_\epsilon(P) \cup B_\epsilon(q)$ such that $\frac{\pi}{2\sqrt{\delta}} < \epsilon < \pi$*

*Proof.* By estimate on injectivity radius, $B_\epsilon(p)$ and $B_\epsilon(q)$ does not contain any cut point of $p$ and $q$ respectively. Thus it is diffeomorphic to a Euclidean ball via exponential map. For a contradiction assume that there exist $r \in M$ such that $r \notin B_\epsilon(p) \cup B_\epsilon(q)$. In other words $d(p, r) \geq \epsilon$ and $d(q, r) \geq \epsilon$. Without loss of generality one can assume that $d(p, r) \geq d(q, r) \geq \epsilon$. Let $q' \in \partial B_q(q)$ be the point of intersection of minimizing geodesic joining q and r with $\partial B_\epsilon(q)$. if $q' \in B_\epsilon(p)$ then $d(q', r) > d(r, B_\epsilon(p)) \geq d(r, B_\epsilon(q)) = d(p, p')$ which is a contradiction. Therefore $q \notin B_\epsilon(P)$.

We have from the theorem of Bonnet - Myers $diam(M) \leq \frac{\pi}{\sqrt{\delta}} < 2\epsilon$. Let $q''$ be the point of intersection of the minimizing geodesic joining p and q with $\partial B_\epsilon(q)$ then $q \in B_\epsilon(P)$ because $d(q'', p) = d(p, q) - d(q, q'') < 2\epsilon = \epsilon$. Therefore $\partial B_\epsilon(p) \cap \partial B_\epsilon(q) \neq \phi$ as boundaries are path connected. Let $r_0 \in \partial B_\epsilon(p) \cap \partial B_\epsilon(q), ied(r_0, q) = \epsilon$. Consider a minimizing geodesic $\lambda$ from $p$ to $r_0$. As $diam(M) = d(p, q)$ from previous lemma there exist $\gamma$ joining p and q such that $\langle\lambda'(0), \gamma'(0)\rangle \geq 0$. Let s $\epsilon$ $\partial B_\epsilon(p)$ be the point of intersection of $\gamma$ with $\partial B_\epsilon(p)$. Then $d(p, s) = \epsilon$. Observe that the angle formed by $\gamma$ and $\lambda$ at 0 is $\leq \frac{\pi}{2}$. By Rauch's theorem by comparing $M$ with a sphere of same dimension whose sectional curvature is $\delta$ will yield.

$$d(r_0, s) \leq \frac{\pi}{2\sqrt{\delta}}$$

There exist atleast one point $s$, whose distance from $r_0 < \epsilon$. Therefore the point at which shortest distance from $\gamma$ to $r_0$ does not occur at its end points $(d(r_0, p) = d(r_0, p) = \epsilon)$. Let $s_0$ be such a point, then $(d(r_0, \gamma) = d(r_0, s_0) = \frac{\pi}{2\sqrt{\delta}})$. As $d(p, q) \leq \frac{\pi}{\sqrt{\delta}}$, either $d(p, r_0)\frac{\pi}{2\delta}$ or $d(q, r_0) \leq \frac{\pi}{2\sqrt{\delta}}$. Let us assume $d(p, s_0) \leq \frac{\pi}{2\sqrt{\delta}}$. As the angle formed by $\gamma$ and geodesic curve joining $r_0$ and $s_0$is $\frac{\pi}{2}$ from Rauch theorem, $d(p, r_0) \leq \frac{\pi}{2\sqrt{\delta}} < \epsilon$ which is a contradiction as $d(p, r_0) = \epsilon$ other case is analogous. □

One can show with some effort that a compact topological manifold covered by two balls is homeomorphic to a sphere. However we will give an explicit homeomorphism in this case.

**Lemma 10.3.3.** *Under the same conditions as the previous lemma, for each geodesic $\gamma$ starting from $p$ of length $\epsilon$, there exist a unique point $r$, on $\gamma$ such that $d(p, r) = d(q, r)$. Similarly for $q$.*

*Proof.* Define a function $f : \mathbb{R} \to \mathbb{R}$ s.t $f(t) = d(q, \gamma(t)) - d(p, \gamma(t))$, which is clearly continuous and $f(0) = d(p, q) > 0$. Let $\gamma(t_0)$ be the cut point of $\gamma$. Then by injectivity radius estimate $d(p, \gamma(t_0)) \geq \pi > \epsilon$. From the previous lemma $d(q, \gamma(t_0)) > \epsilon$. Hence $f(t_0) < 0$. Hence as $f$ is continuous there exist some $t' \in (0, t_0)$ such that $f(t') = 0$. Thus $\gamma(t') = r$. We now need to show that such a point is unique.
Suppose there exist two such points $r_1 \neq r_2$. As $r_1$ and $r_2$ are points on the same geodesic we have

$$d(q, r_2) = d(p, r_2) = d(p, r_1) + d(r_1, r_2) = d(q, r_1) + d(r_1, r_2)$$

From the above equation, unique geodesic joining $q$ and $r_2$ coincides with $\gamma$. As $r_1 \neq r_2$ and $d(p, r_1) = d(q, r_1)$, $d(q, r_1) = d(q, r_2)$, it follows that $p = q$ which is absurd. The other case is similar. Thus the lemma is proved. □

*Proof of Sphere theorem.* Let $p, q \in M$ such that $diam(M) = d(p, q)$. Let $\Gamma_p$ be the set of all geodesics starting from $p$ and $\Gamma_q$ be the set of geodesics starting from $q$. By previous lemma for each $\gamma \in \Gamma_p$ there exist $a(\gamma)$ in the image of $\gamma$ satisfying $d(p, a(\gamma)) = d(q, a(\gamma)) < \epsilon$. Similarly for each $\rho \in \Gamma_q$ there exist an $b(\rho)$ in the image of $\rho$ such that, $d(p, b(\rho)) = d(q, b(\rho)) < \epsilon$. And for each $\gamma \in \Gamma_p$ there is a unique positive real number $\alpha(\gamma)$ such that $\gamma(\alpha(\gamma)) = a(\gamma)$, similarly for each $\rho \in \Gamma_q$ there is a unique $\beta(\gamma)$ such that $\rho(\beta(\rho)) = b(\rho)$. Consider the sets,

$$D_1 = \cup_{\gamma \in \Gamma_p} \{\gamma(t) : t \in [0, \alpha(\gamma)]\}$$

$$D_2 = \cup_{\rho \in \Gamma_q} \{\rho(t) : t \in [0, \beta(\rho)]\}$$

We will first show that $M = D_1 \cup D_2$ and $D_1 \cap D_2 = \partial D_1 = \partial D_2$.
Let $m \in M$ then either $d(p,m) < \epsilon$ or $d(q,m) < \epsilon$. By lemma without loss of generality assume $d(p,m) < \epsilon$. As $d(p, cut(p)) \geq \pi > \epsilon$ we can find a unique minimizing geodesic $\gamma$ joining $p$ and $m$. By lemma there exist a $r_0$ along $\gamma$ such that $d(p,r_0) = d(q,r_0) < \epsilon$. If $d(p,m) < d(q,m)$ then $m$ is not on the endpoints of $\gamma$ ($r_0 \neq m,\ m \in D_1$). If $d(p,m) = d(q,m)$ then by uniqueness of $r_0$, $m = r_0$ and $m \in \partial D_1$. Using analogous arguments one can show that if $d(q,m) < \epsilon$ then $m \in D_2$ or $m \in \partial D_2$. Since choice of $m$ was arbitrary, $M = D_1 \cup D_2$. By uniqueness in lemma if $d(p,m) = d(q,m)$ then $m \in \partial D_1 = \partial D_2 = D_1 \cap D_2$.
Now we provide a homeomorphism from $S^n \to M$. Fix $s \in S^n$ (south pole). Fix a linear isometry $i : T_N S^n \to T_p M$. And let $E$ be the equator of $S^n$ with respect to $N$ and $e \in E$. Let $\gamma : [0, \pi] \to S^n$ be a geodesic such that $\gamma(0) = N$ and $\gamma(\frac{\pi}{2}) = e$. Consider geodesic starting at $p$ with initial velocity vector $i(\gamma'(0))$. Consider $\varphi : S^n \to M$ defined as

$$\varphi(\gamma(s)) = \begin{cases} exp_p(s\frac{2}{\pi}d(p,m)i(\gamma'(0))) & 0 \leq s \leq \frac{\pi}{2} \\ exp_q((2 - \frac{2s}{\pi})d(p,m)x) & \frac{\pi}{2} < s \leq \pi \end{cases}$$

where $m$ is the point given by lemma and $x$ is the initial velocity vector of unique minimizing geodesic joining $q$ and $m$. By definition itself $\varphi$ maps closed northern hemisphere to $D_1$ and closed southern hemisphere to $D_2$ bijectively. As $M = D_1 \cup D_2$, $\varphi$ is surjective. Since $m$ is unique, $\varphi$ is continuous from lemma. $\varphi$ is injective on the set $D_1 \cap D_2 = \partial D_1 = \partial D_2 = \varphi(E)$, hence is injective on all of $S^n$. Thus $\varphi$ is a homeomorphism and we have proved the celebrated Sphere theorem! $\square$

# Summary

We have started the journey from the definition of Riemannian metric, geodesic and curvature. Along the way we have encountered some beautiful results such as Gauss-Bonnet theorem which in dimension two implies the Sphere theorem. Concepts of Jacobi fields and conjugate points are introduced which captured a wealth of geometric and topological information. Jacobi fields were used throughout this study in an extensive manner. As manifolds that we often encounter are the ones immersed in Euclidean space, the study of isometric immersions is essential. Even in dimension two isometric immersions and minimal surfaces are an active field. We saw a proof of the Hopf-Rinow theorem, a theorem which gives us the freedom to join two points in complete manifold with a minimizing geodesic. We also calculated variation formulas which have a variety of applications, in particular to prove the Bonnet-Myers theorem. The Rauch comparison theorem and Morse index theorem presented next are essential ingredients in the proof of Sphere theorem. Finally we presented a proof of the Sphere theorem as given by Klingenberg and Berger. The Sphere theorem is still an active area of research and has ramifications and applications in different areas of geometry.

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