

On K-theory of Leavitt Path Algebras

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by

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Certificate

This is to certify that this dissertation entitled On K-theory of Leavitt Path Algebras towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Goutham Dev C R at Indian Institute of Science Education and Research under the supervision of Dr. Rabeya Basu, Professor, Department of Mathematics, during the academic year 2021-2022.



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This thesis is dedicated to my parents.

Declaration

I hereby declare that the matter embodied in the report entitled On K-theory of Leavitt Path Algebras are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rabeya Basu and the same has not been submitted elsewhere for any other degree.



Goutham Dev C R

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Abstract

In 1960's, W. G. Leavitt introduced a type of non-commutative algebras that are now called Leavitt Algebras. In 2005, Gene Abrams and Gonzalo Aranda Pino introduced Leavitt path algebras as a generalization of Leavitt algebras. It was also done by Pere Ara, María Moreno, and Enrique Pardo simultaneously. In this thesis we explain K_0 and K_1 groups of Leavitt path algebras, based on the studies of Gene Abrams, Pere Ara and Mercedes Siles Molina.

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Introduction

in 1960's, W. G. Leavitt introduced an algebraic structure that are now called as Leavitt algebras. Later in 2005, Gene Abrams and Gonzalo Aranda Pino introduced a broader class of algebras called Leavitt path algebras(LPA). LPA are non-commutative graded graph algebras. Grothendieck group(K_0) and Whitehead group(K_1) are structures in K-theory. Here we study K_0 and K_1 groups of Leavitt path algebras and an important relation between them.

The layout of this thesis is as follows:

In chapter 1, we study basic definitions and results related to graphs and directed graphs. This is a precursor to LPA in chapter 3. More details can be found in [1]. In chapter 2, we study graded theory in detail. We study graded rings, graded modules and graded projective modules, for more details refer [2]. In chapter 3, we study Leavitt path algebras that is a precursor to chapter 7. The graded structure of LPA is also discussed here. For more details on this topic, refer [4] and [2].

Let $K_0(L_K(E))$ denote the group completion of the monoid $V(L_K(E))$. The monoid $V(L_K(E))$ has some interesting properties. In chapter 4, we study some properties in classical ring theory. In chapter 7 we check if the monoid $V(L_K(E))$ satisfy these properties. More details on classical ring theory can be referred from [3]. In chapter 5, we cover elementary subgroups in detail. This is a precursor to the K_1 -group that will be discussed in chapter 6. For more details, refer [5]. Chapter 6 is an introduction to K-theory. Here we discuss Grothendieck group and Whitehead group. For more details, refer [6] and [5]. In chapter 7, we prove refinement and separative properties of monoid M_E . We define K_0 and K_1 of LPA. For more details, refer [4].

Let E be a row-finite graph, then we show the relation

$$K_0(L_K(E)) \cong \text{Coker}((I_{ns} - A_{ns})^t) : \bigoplus_{v \in \text{Reg}(E)} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$$

Let E be a finite graph without sources, then we show the relation

$$K_1(L_K(E)) \cong (\text{Coker}((I_m - A_E)^t) \bigoplus (\text{Ker}((I_m - A_E)^t))$$

Let E and F be finite graphs without sinks or sources, and $|E_0| = |F_0|$. If $K_0(L_K(E)) \cong K_0(L_K(F))$, then we show the following relation

$$K_1(L_K(E)) \cong K_1(L_K(F))$$

We also show the Morita equivalence of Leavitt path algebras. Let E and F be finite graphs with $E \equiv_{\det} F$, we show that Morita equivalence of $L_K(E)$ and $L_K(F)$ exists, $L_K(E) \sim_M L_K(F)$.

Chapter 1

Graph theory

In this chapter we study graphs, directed graphs, extended graph, and some properties of graphs like adjacency matrix, incidence matrix, paths, trees, etc. The definitions and major results in this chapter are referred from [1]. All graphs are self drawn and examples are self written.

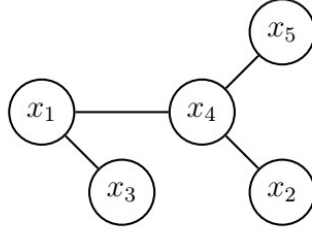
1.1 Graphs

A graph is a representation of n objects and relations between them. In this thesis, we study algebra constructed from graphs.

Definition 1.1.1. *A graph $E = (E^0, E^1, r)$ is a triple. Let E^0 be the set of vertices, and E^1 be the set of edges. Here $r : E^1 \rightarrow E^0$ is a relation that assigns two vertices to each edge. These vertices are called endpoints of the edge. An edge with both vertices same is called a loop, and edges with the same endpoints are called multiple edges.*

Definition 1.1.2. *A graph E without loops or multiple edges is called a simple graph. So an edge $e \in E^1$ in simple graph can be represented by its vertices as $e = uv = vu$, where $u, v \in E^0$.*

Example 1.1.3. A simple graph on 5 vertices :



Definition 1.1.4. A subgraph of a graph $E = (E^0, E^1, r)$ is a graph $F = (F^0, F^1, r')$ such that $F^0 \subseteq E^0$, $F^1 \subseteq E^1$ and $r' = r_F$.

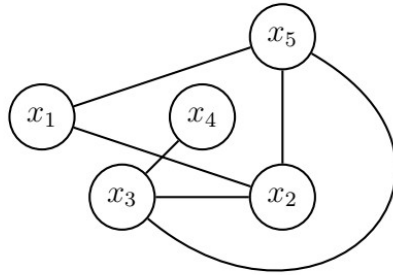
Definition 1.1.5. Let $v \in E^0$. Then the number of edges connected to v is called the degree of v . In the graph in Example 1.1.3, x_4 is of degree 3.

Definition 1.1.6. The complement $\bar{E}$ of a simple graph E is a simple graph with

$$\bar{E}^0 = E^0$$

$$\bar{E}^1 = \{e | e \notin E^1\}$$

Example 1.1.7. The complement of the graph in Example 1.1.3 is the graph



Definition 1.1.8. An independent set is a set of vertices with no edges between them. A clique is a set of vertices with edges between each pair of vertices.

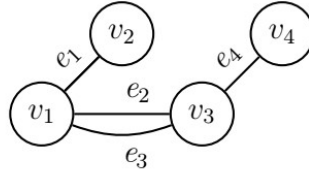
Definition 1.1.9. A bipartite graph is a graph whose vertex set is a union of two independent sets.

Graphs are related to matrices, we can specify a graph using a matrix rather than listing

edges or drawing a graph. Matrix representations are powerful tools in computations related to graphs.

Definition 1.1.10. An adjacency matrix A of a loopless graph E with vertex set $E^0 = \{v_1, v_2, \dots, v_n\}$ is an $n \times n$ matrix with a_{ij} as the number of edges between v_i and v_j . It is a symmetric matrix with diagonal elements 0 for any loopless graph G .

Example 1.1.11. A graph and its adjacency matrix :



$$\begin{bmatrix} 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Definition 1.1.12. Let $E^0 = \{v_1, v_2, \dots, v_m\}$, and $E^1 = \{e_1, e_2, \dots, e_n\}$. Then the incidence matrix M is an $m \times n$ matrix with m_{ij} equal 1 if v_i is an endpoint of e_j , or else 0. The incidence matrix of a graph may not be symmetric.

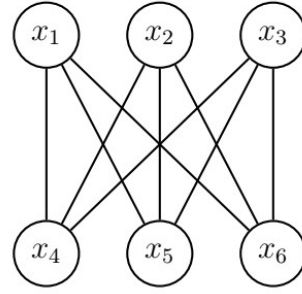
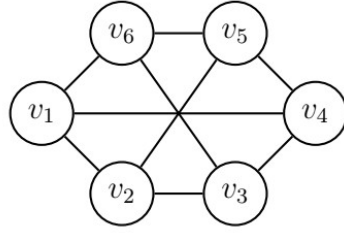
Example 1.1.13. Incidence matrix of graph in Example 1.1.11 :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition 1.1.14. An isomorphism of graphs $E = (E^0, E^1, r)$ and $F = (F^0, F^1, r')$ is a bijection $f : E^0 \rightarrow F^0$ such that $uv \in E^1$ iff $f(u)f(v) \in F^1$ for $u, v \in E^0$. If graph E is isomorphic to graph F , then we write $E \cong F$.

Example 1.1.15. The below two graphs are isomorphic with bijective function :

$$f : (v_1, v_2, v_3, v_4, v_5, v_6) \rightarrow (x_1, x_4, x_2, x_5, x_3, x_6)$$



Remark 1.1.16. If two graphs are isomorphic, then the adjacency matrix of one can be transformed into another by placing the rows in a different order. Isomorphism relation is an equivalence relation on the set of graphs and forms isomorphism classes of graphs.

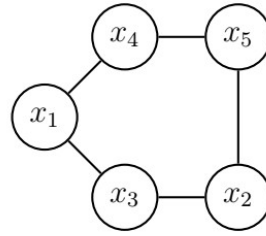
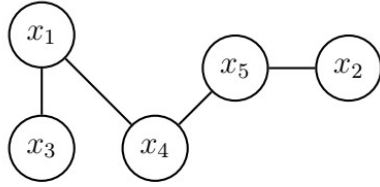
Definition 1.1.17. A graph is called self-complementary if the complement of the graph is isomorphic to the original graph.

Definition 1.1.18. Decomposition of a graph is a set of subgraphs such that each edge of the original graph is present in exactly one subgraph in the set.

Definition 1.1.19. A path is a sequence of edges that joins sequence of distinct vertices.

Definition 1.1.20. A circle is a sequence of edges that forms a cycle, and only the first and last vertex is equal.

Example 1.1.21. A path and cycle on 5 vertices:



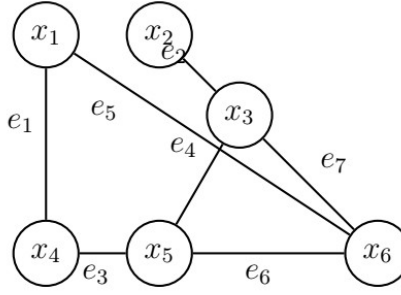
Definition 1.1.22. A walk in a graph $E = (E^0, E^1, r)$ is a sequence

$$v_0, e_0, v_1, e_1, v_2, \dots, v_i, e_i, v_{i+1}$$

such that e_j has endpoints v_j and v_{j+1} , for all j .

Definition 1.1.23. A trail is a walk without a repeated edge. A trail with endpoints u and v is called a (u, v) -trail. If a trail or walk has the same vertex as endpoints, then it's a closed trail or walk.

Example 1.1.24. Consider a graph with six vertices :



1. $x_1, e_1, x_4, e_3, x_5, e_4, x_3, e_7, x_6, e_7, x_3, e_2, x_2$ is a walk in the graph above.
2. $x_1, e_1, x_4, e_3, x_5, e_4, x_3, e_7, x_6, e_6, x_5$ is a x_1, x_5 -trail.
3. x_5, x_3, x_6, x_5 is a cycle in the graph. As it is a simple graph only vertices need to be listed in the sequence.
4. x_1, x_6, x_5, x_3, x_2 is a x_1, x_2 -path.

Definition 1.1.25. The length of the trail, walk, cycle, or path is the number of edges in the sequence.

Remark 1.1.26. The difference between a trail and a path is that edges cannot be repeated in a trail, but in a path, vertices cannot be repeated. Path as a subgraph has 1-degree vertices at the endpoints and all else are 2-degree vertices, this is not true for a trail.

Proposition 1.1.27. Every (u, v) -walk in a graph contains a (u, v) -path.

Proof. Consider a (u, v) -walk in a graph E , where (u, v) are vertices in E .

Case 1 : If the vertices do not repeat in the given walk, then the walk is a path.

Case 2 : If the vertices repeat in the walk, then deleting vertices and edges between the repetition gives a path. This is a (u, v) -path contained in the (u, v) -walk. $\square$

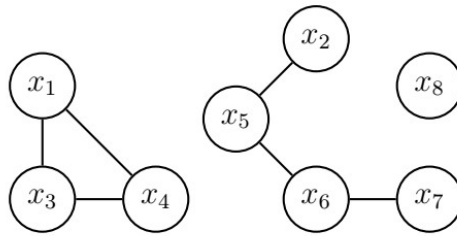
Definition 1.1.28. A graph is connected, if for all pairs of vertices (u, v) in the given graph, there exist a (u, v) -path. The graph is disconnected if (u, v) -path do not exist for at least 1 pair.

Definition 1.1.29. The maximal connected subgraphs in a graph are called its connected components.

Remark 1.1.30. A connected graph only has 1 component but a disconnected graph has more than one. A vertex of degree 0 is called isolated vertex.

Definition 1.1.31. If deleting an edge or a vertex increases the number of components, then such a edge(vertex) is called cut-edge(cut-vertex).

Example 1.1.32. Consider the graph :



The graph has 3 components, x_5x_6 is a cut edge, x_6 is a cut vertex and x_8 is an isolated vertex.

Definition 1.1.33. Let T be a set of vertices, a T induced subgraph in graph E is the graph with all vertices of set $E^0 \cap T$ together with the edges connecting them.

Definition 1.1.34. For graphs H and E , graph E is called H free if E does not have any induced subgraphs isomorphic to H .

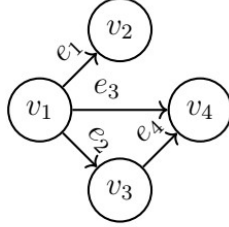
1.2 Directed graphs

We have looked at graphs whose edges have no direction. Graphs with directed edges are called directed graphs.

Definition 1.2.1. A directed graph $E(E^0, E^1, r, s)$ has two sets E^0 and E^1 ; and two relations r and s . Let E^0 be the set of vertices, and E^1 be the set of edges. Here r and s are relations from E^1 to E^0 . All edges in E^1 have direction. Here $s(e_i)$ = source vertex, and $r(e_i)$ = range vertex; for $e_i \in E^1$.

Definition 1.2.2. The source and range together are called the endpoints of the edge. A loop is an edge with same endpoints and multiple edges have same source and range. A graph is called loopless if it does not have any loops.

Example 1.2.3. Directed graph with four vertices :



Here $s(e_1) = v_1, s(e_2) = v_1, r(e_2) = v_3, r(e_4) = v_4$, etc.

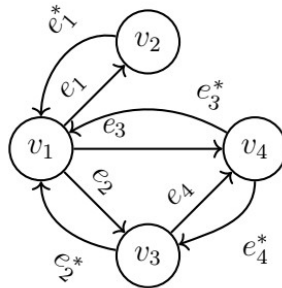
Definition 1.2.4. A path $\mu = e_1 e_2 e_3 \dots e_m$ is a sequence of edges such that $r(e_i) = s(e_{i+1})$. Here $s(e_1)$ is the source of μ , $r(e_m)$ the range, and m is the length. Vertex is considered to be a path of length 0. The set of all paths in E is called $\text{Path}(E)$.

Definition 1.2.5. In a directed graph there are four types of vertices; v with $r^{-1}(v) = \phi$ are called source, v with $s^{-1}(v) = \phi$ is called a sink. If $s^{-1}(v)$ is infinite then it is called infinite emitter and all vertices other than sink or infinite emitter are regular. Here $\text{Source}(E)$, $\text{Sink}(E)$, $\text{Inf}(E)$ and $\text{Reg}(E)$ are the set notations resp.

Definition 1.2.6. Let $(E^1)^* = \{e^* | e \in E^1\}$, then The extended graph is a directed graph $E(E^0, E^1, (E^1)^*, r, s, r', s')$. Apart from the properties of directed graph, maps r' and s' acts on $(E^1)^*$. For $e_i \in E^1$; define $s'(e_i^*) = r(e_i)$, and $r'(e_i^*) = s(e_i)$.

Definition 1.2.7. The edge e^* is called a ghost edge, and μ^* of path $\mu = e_1 e_2 e_3 \dots e_n$ is defined as $\mu^* = e_n^* e_{n-1}^* \dots e_1^*$.

Example 1.2.8. The extended graph of graph in Example 1.2.3 :



Definition 1.2.9. A graph is called row-finite graph if $s^{-1}(v)$ is finite for every $v \in E^0$.

Definition 1.2.10. A directed graph is called simple if at most 1 edge gets mapped to 1 ordered pair. So a single loop is allowed but multiple edges are not allowed.

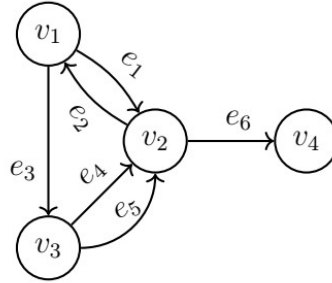
Cycle, trail, walk, subgraphs, complement, isomorphisms, and decomposition in directed graphs are defined in similar manner as in graphs. The difference to be noted is that edge uv and edge vu are different in the case of directed graphs, for $u, v \in V$.

Definition 1.2.11. A graph is called acyclic if there are no cycles at any vertex.

Definition 1.2.12. The underlying graph of a directed graph is a graph obtained by removing the arrows or treating the ordered pairs as unordered pairs. The underlying graph of a simple graph may not be simple.

Definition 1.2.13. The adjacency matrix $A(E)$ of a loopless directed graph E has a_{ij} as the number of edges from v_i to v_j . Adjacency matrix may not be symmetric as in the case of graphs. Incidence matrix $M(E)$ has m_{ij} equal $+1$ if v_i is the source and m_{ij} equal -1 if v_i is the range.

Example 1.2.14. The adjacency matrix and incidence matrix of a loopless directed graph :



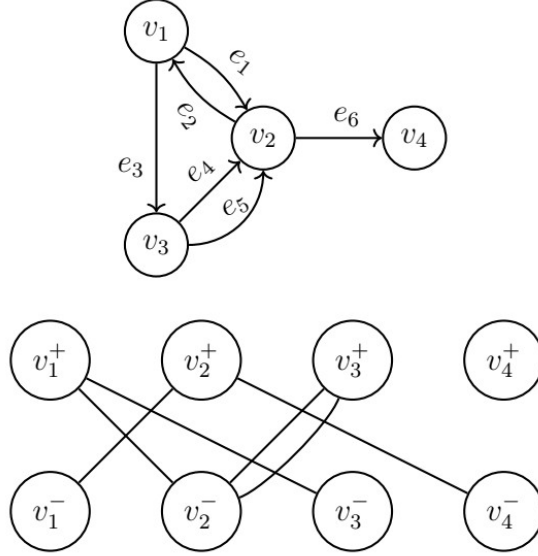
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} +1 & -1 & +1 & 0 & 0 & 0 \\ -1 & +1 & 0 & -1 & -1 & +1 \\ 0 & 0 & -1 & +1 & +1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Definition 1.2.15. A directed graph is called strongly connected if for all ordered pairs (v, v') there is a path from v to v' , for $v, v' \in V$. A directed graph is called weakly connected if its underlying graph is a connected graph.

Definition 1.2.16. Let v be a vertex. The indegree $d^-(v)$ of v is the number of edges with range v . The outdegree $d^+(v)$ is the number of edges with source v .

Definition 1.2.17. The split of a directed graph $E(E^0, E^1, r, s)$ is a non-directed bipartite graph F with vertex sets $V^+ \cup V^-$ where $F^+ = \{v^+ | v \in E^0\}$ and $F^- = \{v^- | v \in E^0\}$. If there is an edge from u to v in E , then F has an edge connecting u^+ and v^- .

Example 1.2.18. A directed graph and its split :



Definition 1.2.19. The preorder $\leq$ on the set E^0 is such that $u \leq v$ if $\exists$ a path μ from $v \rightarrow u$; for $u, v \in E^0$.

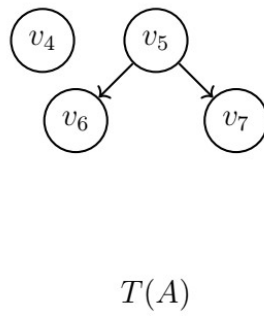
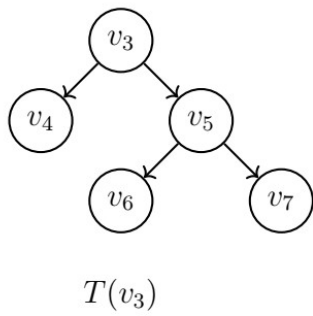
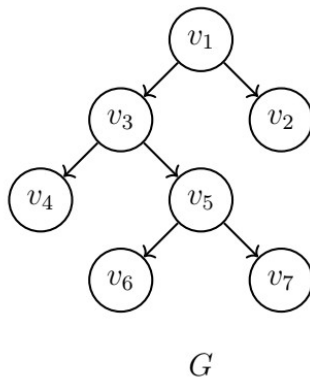
Definition 1.2.20. The tree of a vertex v is an induced subgraph of $E(E^0, E^1, r, s)$:

$$T(v) := \{u | u \leq v; u, v \in E^0\}$$

For a subset $A \subseteq E^0$

$$T(A) := \bigcup_{v \in A} T(v)$$

Example 1.2.21. Let G be a graph, and $A = \{v_4, v_5\}$. Then



Chapter 2

Graded rings and modules

In this chapter we study properties of graded rings, graded ideal, graded algebra, graded modules and graded projective modules. This is a precursor to grading on LPA in Chapter 3. The definitions, major results and some examples in this chapter are referred from [2]. Few examples are self written.

2.1 Graded rings

In this section we study properties of graded rings.

Definition 2.1.1. *Let Γ be an abelian group, and R be a ring. Then R is called Γ graded ring if we can write $R = \oplus_{i \in \Gamma} R_i$ where R_i are subgroups of R with operation addition, and $R_i R_j \subseteq R_{i+j}$; for all $i, j \in \Gamma$.*

Example 2.1.2. *The polynomial ring $R = K[X_1, X_2, \dots, X_n]$ over a field K is graded with*

$$R_i = \begin{cases} \langle X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} \mid a_1 + a_2 + \cdots + a_n = i \rangle, & \text{if } i \geq 0 \\ 0, & \text{if } i < 0 \end{cases}$$

Definition 2.1.3. *The abelian group used for the grading is called grading group.*

Definition 2.1.4. *When the grading group is $\mathbb{Z}$, then the ring is called $\mathbb{Z}$ -graded ring.*

Example 2.1.5. The set of all Laurent polynomials over $\mathbb{R}$ with grading given by $\mathbb{R}_{i \in \mathbb{Z}} = \mathbb{R}X^i$ is a $\mathbb{Z}$ -graded ring.

Definition 2.1.6. Let R, S be rings. Let M be an abelian group. If M is a left R -module, and a right S -module, then M is called an $R - S$ bimodule.

Definition 2.1.7. If the grading group is $\mathbb{Z}_2 = \{0, 1\}$ the ring is called $\mathbb{Z}_2$ -graded ring.

Example 2.1.8. $\mathbb{R}(X)$ with grading :

$$\mathbb{R}_0 = \mathbb{R}X^i, i \text{ is even}$$

$$\mathbb{R}_1 = \mathbb{R}X^i, i \text{ is odd}$$

Definition 2.1.9. An algebra over a ring R is a module M equipped with a bilinear product, i.e.,

$$m, m' \in M \implies mm' \in M$$

Example 2.1.10. The Laurent polynomials $\mathbb{Z}[X, X^{-1}]$ over $\mathbb{Z}$, and $\mathbb{C}$ (complex numbers) over $\mathbb{C}$ are both algebras.

Definition 2.1.11 (Graded algebra). Let A be an algebra over R . Then A is called graded algebra if the underlying additive group of A can be written as a direct sum of abelian groups. That is $A = \oplus A_i$ and $A_i A_j \subseteq A_{i+j}$; for $i, j \in \Gamma$.

Definition 2.1.12. Let $R = \oplus_{i \in \Gamma} R_i$ be a Γ -graded ring. Then the set of homogeneous elements of R denoted by R^h

$$R^h = \bigcup_{i \in \Gamma} R_i$$

Definition 2.1.13. Let $R = \oplus_{i \in \Gamma} R_i$ be a Γ -graded ring. Degree of an element a in R is $\deg(a) = i$ if $a \in R_i$.

Definition 2.1.14. Let $R = \oplus_{i \in \Gamma} R_i$. Then $\Gamma_R = \{\gamma \in \Gamma \mid R_\gamma \neq 0\}$ is called support of R .

Definition 2.1.15. Let f be a ring homomorphism $f : A \rightarrow B$ with the homomorphism relations $f(a)f(b) = f(ab)$ and $f(a) + f(b) = f(a + b)$, for $a, b \in A$. Then f is called a Γ -graded ring homomorphism of degree zero if $f(A_\gamma) \subseteq B_\gamma$; $\forall \gamma$.

Definition 2.1.16. Let R be a Γ -graded ring, and S be a Ω -graded ring. Let $f : R \rightarrow S$ be a ring homomorphism, and $g : \Gamma \rightarrow \Omega$ be a group homomorphism. If $f(R_\gamma) \subseteq S_{g(\gamma)}$, then f is called $(\Gamma - \Omega)$ -graded homomorphism.

Proposition 2.1.17. *Let R be Γ -graded ring and $R = \bigoplus_i R_i$, then*

1. $1_R \in R_0$.
2. The abelian group R_0 is a subring of R .
3. R_γ is a R_0 -bimodule for all $\gamma \in \Gamma$.
4. If $r \in R_\gamma$, then $r^{-1} \in R_{-\gamma}$.

Proof. 1. Assume $1_R \notin R_0$, and let $1 = \bigoplus r_i$ where $r_i \in R_i$. Take b that is homogeneous of $\deg \gamma$. Then $1 \times b = b$, i.e. $\bigoplus r_i \times b = b$. If any element of $\deg \neq 0$ gets multiplies with b its $\deg$ will change and the equality fails, so $r_i \in R_0$ for all $i \neq 0$. Hence 1_R is a homogeneous element of R_0 .

2. We know that $1, 0 \in R_0$, and $R_0 R_0 \subseteq R_0$. Then R_0 is an abelian group under addition, Hence R_0 is a subring of R .
3. We know that $R_\gamma R_0 \subseteq R_\gamma$, and $R_0 R_\gamma \subseteq R_\gamma$. We have already shown R_0 is a ring, hence R_γ is a bimodule.
4. Let $b = \bigoplus b_i$ be the inverse of $r \in R_\gamma$. We know $r \times \bigoplus b_i = 1$ and $\deg(1) = 0$. Then $b_i = 0$ for all $i \neq -\gamma$, if not the $\deg$ on both sides of equation will be different. Hence $r^{-1} \in R_{-\gamma}$.

□

We have discussed gradings on rings. We next consider the graded rings that can be constructed from given graded ring. These include grading formed by considering subgroups, quotients, homomorphisms, and localizations.

Definition 2.1.18. *Let Δ be a subgroup of Γ , and R is a Γ -graded ring. Then R is also a Δ -graded ring. Here $A_\delta A_{\delta'} = A_{\delta+\delta'}$ property is satisfied as $\delta, \delta' \in \Delta \subseteq \Gamma$.*

Definition 2.1.19. *Let f be a group homomorphism from $\Gamma \rightarrow \Delta$. We know that homomorphism preserve the operation $f(\gamma + \gamma') = f(\gamma)f(\gamma')$, for $\gamma, \gamma' \in \Gamma$. Let $\delta \in \Delta$. Then the Δ -grading on a Γ -graded ring R is given by*

$$R_\delta = \bigoplus_{\gamma \in f^{-1}(\delta)} R_\gamma$$

If $f^{-1}(\delta) = \phi$, then $R_\delta = 0$.

Definition 2.1.20. Let Ω be a normal subgroup of Γ . For a given Γ -graded ring R , the quotient grading is given by

$$R = \bigoplus_{\gamma \in \Gamma} R_{\Omega+\gamma}$$

where

$$R_{\Omega+\gamma} = \bigoplus_{\omega \in \Omega} R_{\omega+\gamma}; \omega \in \Omega$$

The product rule is directly derived from Γ -grading. Together with the sum given above, A is Γ/Ω -graded ring.

Example 2.1.21. Consider $\mathbb{Z} \rightarrow \mathbb{Z}$ given by $n \mapsto 3n$ map. Normal subgroup of this homomorphism is $3\mathbb{Z}$. We know that Laurent polynomial $R[X]$ is a $\mathbb{Z}$ -graded ring, then it is a $\mathbb{Z}/3\mathbb{Z}$ -graded ring by

$$R'_i = \bigoplus R_{3n+i}$$

where $i = 0, 1, 2; n \in \mathbb{Z}$, and $R_i = RX^i$.

Definition 2.1.22. The localization of ring R by a multiplicative set S is the ring

$$S^{-1}R = \{r/s | r \in R, s \in S\}$$

Definition 2.1.23. If R is a Γ -graded ring, and S be graded subgroup of R . Then localization $S^{-1}R$ is a Γ -graded ring. We take all the homogeneous elements and divide it by elements of S and then redistribute the new elements according to their degree. Here

$$(S^{-1}R)_\gamma = \{r/s | r \in R^h, s \in S, \deg(r/s) = \gamma\}$$

2.1.1 Strongly graded rings

In this section we discuss properties of strongly graded rings.

Definition 2.1.24. Let $\gamma, \delta \in \Gamma$, and R be Γ -graded ring. If $R_\delta R_\gamma = R_{\delta+\gamma}$, then R is called a strongly graded ring.

Example 2.1.25. The Laurent polynomial over $\mathbb{Z}$ is strongly graded ring with the usual grading.

Definition 2.1.26. For a graded ring, if there is an invertible element in every homogeneous component, then it is called a crossed product ring.

Proposition 2.1.27. If $R = \oplus R_\gamma$ is a γ -graded ring, then

1. R is strongly graded iff $1 \in R_\gamma R_{-\gamma}$, for $\gamma \in \Gamma$.
2. If R is strongly graded, then support of R is Γ .
3. $f : R \rightarrow S$ is a graded homomorphism, and R is strongly graded then S is strongly graded.
4. All crossed product rings are strongly graded.

Proof. 1. As R is strongly, then $R_\gamma R_{-\gamma} = R_0 \ni 1$. Conversely, As $1 \in R_\gamma R_{-\gamma}$, then $R_0 = R_\gamma R_{-\gamma} R_0 = R_\gamma R_{-\gamma}$, and

$$R_{\gamma+\delta} = R_0 R_{\gamma+\delta} = R_\gamma R_{-\gamma} R_{\gamma+\delta} \subseteq R_\gamma R_\delta$$

From the graded ring definition we get that $R_\gamma R_\delta \subseteq R_{\gamma+\delta}$. This proves that $R_\gamma R_\delta = R_{\gamma+\delta}$ and hence R is strongly graded ring.

2. From (1), we get that $1 \in R_\gamma R_{-\gamma}$. This implies R_γ has some element other than 0 for all $\gamma \in \Gamma$. So the support of $R = \Gamma$.
3. Let f be a graded homomorphism $A \rightarrow B$ which means $f(A_\gamma) \subseteq B_\gamma$. From (1) we get $1_A \in A_\gamma A_{-\gamma}$, therefore $1_B \in f(A_\gamma)f(A_{-\gamma})$. Thus $1_B \in B_\gamma B_{-\gamma}$, and again from (1), ring B is a strongly graded ring.
4. Crossed product rings contain at least one invertible element in R_γ for all $\gamma \in \Gamma$. We have proved that if an invertible element exist in R_γ , then it is in $R_{-\gamma}$. Hence $1 \in R_\gamma R_{-\gamma}$ and from (1), we can conclude that it is a strongly graded ring.

□

2.2 Graded ideals

Ideals are subgroups of rings such that the product of an element of ideal and ring is contained in the ideal. There are many theorems associated with ideals, and they help in a better understanding of the rings. Graded ideals are important in the study of the properties of graded rings.

Definition 2.2.1. Let R be a Γ -graded ring, and I be its ideal. An ideal $I \subseteq R$ is called a Γ -graded ideal if

$$I = \bigoplus_{\gamma \in \Gamma} (I \cap R_\gamma)$$

The definition can also be stated as follows : if $x \in I$ and $x = \sum x^i$ for $x^i \in R^h$, then $x^i \in I$. Here $I_\gamma I_\delta \subseteq I_{\gamma+\delta}$ is satisfied as $I_\gamma \subseteq R_\gamma$.

Example 2.2.2. The ideal generated by polynomials of even degree in graded ring of polynomials is a graded ideal as

$$I = \bigoplus X^{2i}; \quad i \in \mathbb{Z}$$

Definition 2.2.3. For a graded ring R and ideal I , the quotient ring R/I is a graded ring if I is a graded ideal. Here,

$$R/I = \bigoplus_{\gamma \in \Gamma} R_\gamma / (I \cap R_\gamma)$$

Example 2.2.4. Ring $K[X]/[X^2]$ is graded ring, where K is a field and $[X^2]$ is ideal generated by X^2 .

Remark 2.2.5. If R is strongly graded (resp. crossed product), then R/I is also strongly graded (resp. crossed product).

Remark 2.2.6. Let I be a two sided ideal. If R/I is a graded ring, then I is a graded ideal.

Definition 2.2.7. Let f be a Γ -graded ring homomorphism from $A \rightarrow B$, and $\text{Img}(f)$ is a graded subring of B . Then $\text{Ker}(f)$ is graded by Remark 2.2.6.

Definition 2.2.8. Let I, J be two graded ideals. An ideal P is called graded prime ideal if $IJ \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$. When the given ring is commutative then $xy \in P$ implies that $x \in P$ or $y \in P$, for $x, y \in R^h$.

Example 2.2.9. The ideal generated by X denoted by $[X]$ in polynomial ring $K[X]$ is a graded prime ideal. Any element of $[X]$ is of the form $X(k + k'X + \cdots)$. As $X \in [X]$, we get $[X]$ is a graded prime ideal.

Definition 2.2.10. Let I be a graded ideal. Ideal P is called graded semi prime if $I^2 \subseteq P$ implies $I \subseteq P$.

Definition 2.2.11. An ideal I is called a graded maximal ideal if among the set of proper graded ideals of A , I is maximal. If there exists an ideal I such that $M \subseteq I$ then $I = M$, for proper graded ideal I , and graded maximal ideal M .

Example 2.2.12. The ideal generated by X and Y denoted by (X, Y) in the ring of polynomials $K[X, Y]$ is a graded maximal ideal. As

$$K[X, Y]/(X, Y) = K$$

(X, Y) is the proper maximal ideal.

Remark 2.2.13. A graded maximal ideal is a graded prime ideal, and the quotient of a graded maximal ideal is a graded field.

Definition 2.2.14. the graded Jacobson radical, $J^{gr}(R)$ is the intersection of all graded maximal right ideals. It can also be defined as the intersection of all graded maximal left ideals. For a $\mathbb{Z}$ -graded ring, ideal $J(R) \subseteq J^{gr}(R)$.

Definition 2.2.15. If R and $\{0\}$ are the only graded ideal in a ring R , then R is graded simple ring.

Example 2.2.16. The ring of complex numbers $\mathbb{C}$ is graded simple ring.

2.3 Graded modules

In this section we study properties of graded modules.

Definition 2.3.1. Let R be a Γ -graded ring, and M be a R -module. Then M is called Γ -graded module if $M = \oplus M_\gamma$, where M_γ is an additive subgroup of M , and

$$R_\gamma M_\delta \subseteq M_{\gamma+\delta}$$

for $\delta, \gamma \in \Gamma$.

Example 2.3.2. 1. Let K be a field. Let the grading of the ring $K[X]$ be the as in Example 2.1.2. Let the grading on $X^2K[X]$ be

$$X^2K[X]_{i \in \mathbb{Z}} = \begin{cases} X^2KX^i, & \text{if } i \geq 0 \\ 0, & \text{if } i < 0 \end{cases}$$

Then $X^2K[X]$ is a $\mathbb{Z}$ -graded $K[X]$ -module.

2. For any homogeneous element a of a graded ring A , the module aA is a graded module.

Definition 2.3.3. Let M, N be Γ -graded R -modules. A homomorphism $f : M \rightarrow N$ is called a graded homomorphism if

$$f(M_\gamma) \subseteq N_\gamma$$

For $\gamma \in \Gamma$.

Definition 2.3.4. Let M be Γ -graded R -module, and $\delta \in \Gamma$. Then δ -shifted graded module $M(\delta)$ is the module M with the grading

$$M(\delta) = \bigoplus_{\gamma} M(\delta)_\gamma$$

$$M(\delta)_\gamma = M_{\delta+\gamma}$$

Example 2.3.5. Let K be a field. Let $X^2K[X]$ be a 3-shifted $\mathbb{Z}$ -graded $K[X]$ -module. Then X^2K is of deg -3 , X^2kX is of deg -2 and so on.

Definition 2.3.6. A submodule N of Γ -graded R -module M is graded if

$$N = \bigoplus_{\gamma} (N \cap M)_\gamma,$$

for $\gamma \in \Gamma$.

Example 2.3.7. Let K be a field, and $K[X]$ be $\mathbb{Z}$ -graded K -module. Then $\langle X^3 \rangle$ graded submodule of $K[X]$.

Definition 2.3.8. If N is a graded submodule of M , then M/N is a graded factor module with grading

$$M/N = \bigoplus (M/N)_\gamma$$

$$(M/N)_\gamma = (M_\gamma + N)/N$$

Example 2.3.9. $\langle 1, X^1, X^2, X^4, \dots \rangle$ is graded factor module for the submodule in Example 2.3.7.

In the case of graded modules, degree can be associated with homomorphisms. The homomorphism that we already defined is of degree zero.

Definition 2.3.10. For Γ -graded R -modules M and N , a graded homomorphism of degree δ is a map $f : M \rightarrow N$ such that

$$f(M_\gamma) \subseteq N_{\gamma+\delta}$$

for $\delta, \gamma \in \Gamma$.

Definition 2.3.11. Let $\text{Hom}_R(M, N)$ be the group of homomorphisms from M to N . Let the group operation be addition of functions, with identity being $M \mapsto 0 \in N$. Let $\text{Hom}_R(M, N)_\delta$ be the subgroup of graded homomorphisms of degree δ .

Theorem 2.3.12. For Γ -graded R -modules M and N , if M is a finitely generated module then the group $\text{Hom}_R(M, N)$ can be written as

$$\text{Hom}_R(M, N) = \bigoplus_{\gamma \in \Gamma} \text{Hom}_R(M, N)_\gamma$$

Moreover $\text{Hom}_R(M, M)$ ring is a Γ -graded ring.

Proof. Refer Theorem 1.2.6 of [2] for detailed proof. □

Definition 2.3.13. The direct sum of Γ -graded R -modules M_i , for $i \in I$ is a graded module with

$$\left(\bigoplus M_i\right)_\gamma = \bigoplus_i (M_i)_\gamma$$

Example 2.3.14. $M = \mathbb{Z}[X] \oplus \mathbb{Z}[X]$ has grading given by $M_i = (\mathbb{Z}X^i \oplus 0) \oplus (0 \oplus \mathbb{Z}X^i)$

Definition 2.3.15. An R -module M is called free if M can be written as finite direct sum of R , i.e.,

$$M = \bigoplus_n R$$

Definition 2.3.16. Let M be a free R -module, it is called a Γ -graded free module if there exists a basis of the module consisting of homogeneous elements.

Example 2.3.17. Consider the Γ -graded R -module $\bigoplus_{i \in I} R(\delta_i)$ for $\delta_i \in \Gamma$. We take all shifts of this module, and then direct sum of shifts. This module has a basis $\{e_i\}$ with 1 in the i th position, and 0 elsewhere. This basis is homogeneous having degree $-\delta_i$ for e_i . Thus $\bigoplus_{i \in I} R(\delta_i)$ is a graded free module.

Definition 2.3.18. If $\Omega \subset \Gamma$ as groups, then Γ -graded R -module M can be also a Γ/Ω -graded R -module.

$$M = \bigoplus_{\Omega + \gamma \in \Gamma/\Omega} M_{\Omega + \gamma}$$

$$M_{\Omega + \gamma} = \bigoplus_{\omega} M_{\omega + \gamma}$$

For $\omega \in \Omega$ and $\gamma \in \Gamma$.

2.4 Graded projective modules

In this section we study graded projective modules as they are an important part of graded K-theory.

Definition 2.4.1. A module M is called stably free R -module if

$$M \oplus R^n = R^m$$

Example 2.4.2. The kernel of any surjection $g : R^n \rightarrow R^m$, where $m < n$ is stably free as $\text{Ker} \oplus R^m = R^n$.

Remark 2.4.3. All free modules are stably free, and all stably free modules are projective modules.

Definition 2.4.4. A sequence of R -modules of the form

$$0 \xrightarrow{f_1} M \xrightarrow{f_2} N \xrightarrow{f_3} M' \xrightarrow{f_4} 0$$

where $\text{Ker}(f_{i+1}) = \text{Im}(f_i)$ for all i is called a short exact sequence.

Definition 2.4.5. If for the sequence in Definition 2.4.4 there exists a map $g : M' \rightarrow N$ such that $f_3 g = \text{Id}_{M'}$, then the sequence is called split exact sequence.

Example 2.4.6. Consider a short exact sequence :

$$0 \rightarrow M \rightarrow M \oplus M' \rightarrow M' \rightarrow 0$$

Let $g : M' \rightarrow M \oplus M'$ be the injection map. As $M \oplus M' \rightarrow M'$ is surjective, the composition gives identity on M' . Hence it is a split exact sequence.

Definition 2.4.7 (Projective module). A module over a ring R is called projective if it satisfies any of the following five equivalent definitions.

1. An R -module P is called projective if $\exists$ another R -module Q such that $P \oplus Q$ is a free module.
2. A module P is called projective if for every surjective module homomorphism $g : N \rightarrow M$ and module homomorphism $f : P \rightarrow M$, there exist a lifting map $h : P \rightarrow N$ such that $gh = f$.
3. A module P is projective iff every short exact sequence of the form $0 \rightarrow A \rightarrow B \rightarrow P \rightarrow 0$ splits. i.e., for surjective homomorphism $g : B \rightarrow P$, there exist homomorphism $h : P \rightarrow B$ such that $hg = Id_P$.
4. If there are two sets $\{f_i \in \text{Hom}(P, R)\}$, and $\{a_i \in P\}$ for $i \in I$ such that for any $x \in P$, only finite $f_i(x)$ are non-zero. And $x = \sum f_i(x)a_i$, then P is projective R -module.
5. Let $M \rightarrow M'$ be surjective homomorphism of modules. A module P is projective iff $\text{Hom}(P, M) \rightarrow \text{Hom}(P, M')$ is surjective.

Example 2.4.8. 1. Module R^n as a R -module is projective.

2. As $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/6\mathbb{Z}$, the module $\mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/6\mathbb{Z}$ -module is projective. This is a projective module which is not a free module.

Definition 2.4.9. Let M, N and P be graded modules. Module P is called graded projective if in the diagram

$$\begin{array}{ccc} P & & \\ \downarrow h & \searrow f & \\ N & \xrightarrow{g} & M \end{array}$$

f, g and h are graded module homomorphisms and $gh = f$.

Definition 2.4.10. A category is a set of objects and maps. Objects can be groups, rings, graphs, etc; and maps be homomorphisms, functions, etc.

Example 2.4.11. Category with groups as objects, and group homomorphisms as maps.

Definition 2.4.12. A subcategory $\mathcal{C}'$ of category $\mathcal{C}$ is a category whose objects and maps are from $\mathcal{C}$. A full subcategory is a subcategory with all maps in $\mathcal{C}$ for objects in $\mathcal{C}'$.

Definition 2.4.13. A covariant functor is map between categories that sends objects to objects and morphisms to morphisms. Let $f : X \rightarrow Y$ be a morphism in A for objects $X, Y \in A$. Then for functor $F : A \rightarrow B$,

1. $F(f) : F(X) \rightarrow F(Y)$.
2. $F(\text{Id}_X) = \text{Id}_{F(X)}$.
3. $F(f.g) = F(f).F(g)$.

Definition 2.4.14. An exact functor is a functor that preserves short exact sequences. If $F : A \rightarrow B$ is a functor, let

$$0 \xrightarrow{f_1} M \xrightarrow{f_2} N \xrightarrow{f_3} M' \xrightarrow{f_4} 0$$

be an exact sequence for $M, N, M' \in A$. Then

$$0 \xrightarrow{F(f_1)} F(M) \xrightarrow{F(f_2)} F(N) \xrightarrow{F(f_3)} F(M') \xrightarrow{F(f_4)} 0$$

is an exact sequence in B .

Example 2.4.15. Let $\text{Hom}_{Gr-R}(P, -)$ be an exact functor from category of graded modules $Gr-R$ to the category of abelian groups $\text{Hom}_{Gr-R}(P, -)$. Here P is a given graded projective module and $-$ represents any graded module. Exactness of the functor will be proved in Proposition 2.4.18.

Lemma 2.4.16. Let M, N , and P be graded R -modules with module homomorphisms f, g , and h as :

$$\begin{array}{ccc} N & & \\ \downarrow h & \searrow f & \\ P & \xrightarrow{g} & M \end{array}$$

Here $f = gh$ and f is a graded homomorphism. If g (resp. h) is a graded homomorphism, then there exists graded homomorphism h' (resp. g') such that $f = gh'$ (resp. $f = g'h$).

Proof. Let $g : P \rightarrow M$ be a Γ -graded homomorphism. Define a map $h' : N \rightarrow P$ such that for a homogeneous element n of degree γ ,

$$h'(n) = h(n)_\gamma$$

Here $h(n)_\gamma$ is the component of the image of h with degree γ . Linear extension of map to a general element $n \in N$ is

$$h'(n) = \sum_{\gamma} h'(n)_\gamma$$

Here h' is a graded homomorphism and we have to show $f = gh'$. For $n \in N_\gamma$

$$gh'(n) = g(h(n)_\gamma) = (gh(n))_\gamma = f(n)_\gamma = f(n)$$

By comparing degree of elements on both sides, g and f are graded homomorphisms and $n \in N_\gamma$. For general element $n \in N$, let

$$gh'(n) = g\left(\sum_{\gamma} h(n)_\gamma\right) = \sum_{\gamma} f(n)_\gamma = f(n)$$

Hence $\exists$ graded homomorphism h' such that $f = gh'$. Similarly we can prove the case when h is given to be graded. $\square$

Lemma 2.4.17 (Splitting lemma). *A short exact sequence*

$$0 \xrightarrow{f_1} M \xrightarrow{f_2} N \xrightarrow{f_3} M' \xrightarrow{f_4} 0$$

is split if any of the following equivalent conditions are satisfied

1. (Right split) $\exists$ a map $g : N \rightarrow M$ such that $f_3g = Id_{M'}$.
2. (Left split) $\exists$ a map $h : M' \rightarrow N$ such that $f_2h = Id_{M'}$.
3. (Direct sum) We have isomorphism of modules $N \cong M \oplus M'$.

Proposition 2.4.18. *Let P be a Γ -graded R -module. Then the following conditions are equivalent.*

1. P is graded and projective.
2. P is graded projective.
3. The functor $\text{Hom}_{Gr-R}(P, -)$ is an exact functor in $Gr - R$.
4. The exact sequence of graded homomorphisms

$$0 \longrightarrow M \xrightarrow{f_1} N \xrightarrow{f_2} P \longrightarrow 0$$

splits via a graded homomorphism.

5. The module P is graded isomorphic to direct summand of a graded free R -modules.

Proof. (1) $\implies$ (2). Given P is graded and projective, consider the diagram :

$$\begin{array}{ccc} P & & \\ \downarrow h & \searrow f & \\ N & \xrightarrow{g} & M \end{array}$$

Where N, M are graded modules. Let f and g be graded homomorphisms. Here P is projective module, and h is a module homomorphism such that $gh = f$. From Lemma 2.4.16, $\exists$ graded homomorphism h' such that $gh' = f$. Hence P is graded projective.

(2) $\implies$ (3). Let

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'' \longrightarrow 0$$

be an exact sequence of graded modules N', N and N'' ; and graded homomorphisms f, g . To show that the below sequence is exact.

$$0 \longrightarrow \text{Hom}_{Gr-R}(P, N') \xrightarrow{f^*} \text{Hom}_{Gr-R}(P, N) \xrightarrow{g^*} \text{Hom}_{Gr-R}(P, N'') \longrightarrow 0$$

Let $\alpha \in \text{Hom}_{Gr-R}(P, N')$, and

$$f^*(\alpha) = f.\alpha$$

$f^*(\alpha) = 0 \implies f.\alpha = 0$ and as $f \neq 0$, $\alpha = 0$. Hence f^* is injective. For $\beta \in \text{Hom}_{Gr-R}(P, N'')$, consider the diagram

$$\begin{array}{ccc} & N & \\ h \uparrow & \searrow g & \\ P & \xrightarrow{\beta} & N'' \end{array}$$

As P is projective, $\exists h$ such that $gh = \beta$. From Lemma 2.4.16, $\exists$ graded homomorphism h' such that $gh' = \beta$. So for all $\beta \in \text{Hom}_{Gr-R}(P, N'')$, $\exists h' \in \text{Hom}_{Gr-R}(P, N)$. Hence g^* is surjective. Therefore the sequence is exact and $\text{Hom}_{Gr-R}(P, -)$ is an exact functor in $Gr-A$.

(3) $\implies$ (4). Let the exact sequence of graded homomorphisms be

$$0 \longrightarrow M \xrightarrow{f_1} N \xrightarrow{f_2} P \longrightarrow 0$$

As $\text{Hom}_{Gr-R}(P, -)$ is an exact functor in $Gr-R$, the map $\text{Hom}_{Gr-R}(P, N) \rightarrow \text{Hom}_{Gr-R}(P, P)$ is surjective. For $Id_P \in \text{Hom}_{Gr-R}(P, P)$, $\exists g \in \text{Hom}_{Gr-R}(P, N)$ such that

$$f_2.g = Id_P$$

g is a graded homomorphism, and hence the exact sequence splits via a graded homomorphism.

(4) $\implies$ (5). Consider the sequence

$$0 \longrightarrow \text{Ker}(g) \longrightarrow \bigoplus R(-\delta_i) \xrightarrow{g} P \longrightarrow 0$$

Where $\bigoplus R(\delta_i)$ is a graded free R -module with basis $\{e_i\}$. Let the homogeneous generating set of graded projective module P be $\{p_i\}$, with $\deg(p_i) = (\delta_i)$. Let

$$g : \bigoplus R(-\delta_i) \rightarrow P$$

$$e_i \mapsto p_i$$

g is a surjective and $Id : \text{Ker}(g) \rightarrow \bigoplus R(-\delta_i)$ is injective map. Therefore the sequence is exact. The exact sequence splits. From Lemma 2.4.17, $\exists h : P \rightarrow \bigoplus R(-\delta_i)$ such that

$gh = Id_P$, and module isomorphism

$$f : P \bigoplus Ker(g) \rightarrow \bigoplus R(\delta_i)$$

$$(p, k) \rightarrow h(p) + k$$

As h is a graded homomorphism, f is also a graded isomorphism. Therefore

$$P \bigoplus Ker(g) \cong_{Gr} \bigoplus R(-\delta_i)$$

Hence P is graded isomorphic to direct summand of a graded free R -modules.

(5) $\implies$ (1). Graded free modules are free modules. Then P is direct summand of free module. From Definition 2.4.7, P is projective. P is given to be graded. Hence P is graded and projective. $\square$

Theorem 2.4.19 (The dual basis lemma). *Let P be a Γ -graded R -module. P is graded projective iff $\exists p_i \in P^h$ with $\deg(p_i) = \delta_i$ and $f_i \in Hom_{Gr-R}(P, R(-\delta_i))$ such that*

1. *For any $p \in P$, $f_i(p) = 0$ for all but finite i .*

2. *For any $p \in P$, $\sum_i f_i(p)p_i = p$.*

Proof. Module P is graded projective. Then consider the split exact sequence

$$0 \longrightarrow Q \longrightarrow \bigoplus R(-\delta_i) \xrightarrow{\pi} P \longrightarrow 0$$

And the splitting map $\phi : P \rightarrow \bigoplus R(-\delta_i)$, such that

$$\pi\phi = Id_P$$

Let π_i be the projection map :

$$\pi_i : \bigoplus R(-\delta_i) \rightarrow R(-\delta_i)$$

Let $\{e_i\}$ be the standard homogeneous basis of $\bigoplus R(-\delta_i)$. Let $p_i = \pi(e_i)$, and $f_i = \pi_i\phi$. Then,

$$\sum_i \pi_i(a)e_i = a; a \in \bigoplus R(-\delta_i)$$

Here $\deg(p_i) = \delta_i$ as π is graded homomorphism, and $\deg(e_i) = \delta_i$. Then π_i and ϕ are graded homomorphisms. Hence $f_i \in \text{Hom}_{Gr-R}(P, R(-\delta_i))$. Here,

$$f_i(p) = \pi_i \phi(p) = 0$$

For all but finite i ; $p \in P$.

$$\sum_i f_i(p)p_i = \sum \pi_i \phi(p)p_i = \sum \pi(e_i)\pi.\phi(p) = \sum \pi(e_i\pi_i(\phi(p))) = \pi(\sum_i e_i\pi_i(\phi(p))) = \pi\phi(p) = p$$

For converse, consider the exact sequence

$$0 \longrightarrow \text{Ker}(\pi) \longrightarrow \bigoplus R(-\delta_i) \xrightarrow{\pi} P \longrightarrow 0$$

Let $\{p_i, f_i\}$ be the dual basis. Let

$$\pi : \bigoplus R(-\delta_i) \rightarrow P$$

$$\pi : \{a_i\} \mapsto \sum p_i a_i$$

$$\phi : P \rightarrow \bigoplus R(-\delta_i)$$

$$\phi : p \mapsto \{f_i(p)\}$$

Here $\pi\phi = \text{Id}_P$, and the exact sequence splits via a graded homomorphism. Hence P is a graded projective module. $\square$

Chapter 3

Leavitt path algebras

The purpose of this chapter is to look into recent work done on Leavitt path algebras(LPA) and Cohn path algebras(CPA). The definitions and major results in this chapter are referred from [4]. For vector spaces, the number of elements in the different bases of the same structure is the same. We call this property the invariant basis number(IBN) property.

Definition 3.0.1. *IBN property states that if R^m and R^n are two free R -modules that are isomorphic, then $m = n$. This is true for R -modules over fields, $\mathbb{Z}$, polynomial rings, etc.*

We see that for commutative rings, the IBN property is always present. However, many structures do not have this property. Leavitt path algebras are one of them.

Theorem 3.0.2. *Let V be infinite-dimensional vector space over R , and S be the set of linear transformations from $V \rightarrow V$. Then $S^m \cong S^n$.*

Proof. Linear transformations are represented as matrices, so the elements of S are matrices. As V is infinite-dimensional, matrices are also infinite-dimensional. The first step of the proof is to show that $S \cong S^2$. Let

$$f : S \rightarrow S \times S$$

Let the forward map be $f : M \mapsto (MY_1, MY_2)$, and the inverse map be $g : (M_1, M_2) \mapsto$

$M_1X_1 + M_2X_2$. Here,

$$Y_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad Y_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Here M, M_1 and M_2 are matrices in S . From calculation, we see that $fg = I_S$ and $gf = I_{S \times S}$. This proves that $S \cong S^2$. Assume $m > n$,

$$S^m \cong S^{m-2}S^2 \cong S^{m-2}S \cong S^{m-3}S^2 \cong \cdots \cong S^n$$

Hence $S^m \cong S^n$. □

3.1 Leavitt Path Algebras(LPA)

There are rings such that R is isomorphic to R^m but not isomorphic to R^n if $n < m$. A common example is Leavitt algebras.

Definition 3.1.1. *A ring R has module type $(1, n)$ if $n \in \mathbb{N}$ is the lowest positive integer such that R^1 is isomorphic to R^n .*

Example 3.1.2. *The ring $\text{end}(V)$ is of module type $(1, 2)$.*

Definition 3.1.3. *Let K be a field. Then Leavitt algebra of type $(1, n)$ denoted by $L_K(1, n)$*

is the K -algebra

$$K[X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n] / [\sum X_i Y_i = 1, Y_i X_j = \delta_{ij} 1, 1 \leq i, j \leq n]$$

Proposition 3.1.4. *Algebra $L_K(1, n)$ is isomorphic to $L_K(1, n)^n$.*

Proof. Consider the row matrix $X = (X_1, X_2, \dots, X_n)$ and column matrix $Y = (Y_1, Y_2, \dots, Y_n)$. Let

$$X : L_K(1, n) \rightarrow L_K(1, n)^n$$

$$Y : L_K(1, n)^n \rightarrow L_K(1, n)$$

By definition of $L_K(1, n)$, we get $XY = [1]$ and $YX = Id$ matrix. As both are identity matrices, the isomorphism is proved. For $m < n$, $\sum_m X_i Y_i \neq 1$. Hence n is the least positive integer for which the relation is true. $\square$

Remark 3.1.5. *The algebra $L_K(1, n)$ satisfies the universal property. If there is another algebra R that has the property $R^1 \cong R^n$, then there exists an isomorphism between $L_K(1, n)$ and R .*

Definition 3.1.6. *The path algebra KE of directed graph $E = (E^0, E^1, r, s)$ over field K is the free algebra generated by $E^0 \cup E^1$. It has the following relation between the generators :*

1. $vv' = \delta_{v,v'}v$, for all $v, v' \in E^0$.
2. $s(e)e = e = er(e)$, for all $e \in E^1$.
3. $r(e) = v$ and $s(e) = v'$; where v is range of edge e , and v' is source of edge e .

Definition 3.1.7 (Basis). *The paths (includes vertices also) in E form the basis of KE . Multiplication operation is the product of paths :*

$$\mu.\lambda = \begin{cases} \mu\lambda, & \text{if } r(\mu) = s(\lambda) \\ 0, & \text{otherwise} \end{cases}$$

Definition 3.1.8 (LPA). *The Leavitt path algebras, $L_K(E)$ of extended directed graph $E = (E^0, E^1, (E^1)^*, r, s)$ is a free algebra on generators $E^0 \cup E^1 \cup (E^1)^*$. It has the following relations on the set of generators :*

1. $vv' = \delta_{v,v'}v$ for $v, v' \in E^0$.
2. $s(e)e = e = er(e)$ for $e \in E^1$.
3. $r(e)e^* = e^* = e^*s(e)$ for $e^* \in (E^1)^*$.
4. $e^*e' = \delta_{e,e'}r(e)$ for $e, e' \in E^1$.
5. $v = \sum_{e|s(e)=v} ee^*$ for $v \in \text{Reg}(E)$.

Remark 3.1.9. *The paths do not form the basis as in the case of KE .*

Definition 3.1.10. *The spanning set of $L_K(E)$:*

$$\{\mu\sigma^* | r(\mu) = r(\sigma); \mu, \sigma \in \text{path}(E)\}$$

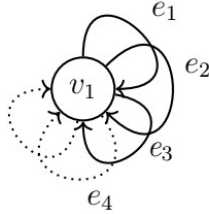
Path product between elements :

$$(\mu\sigma^*)(\psi\rho^*) = \begin{cases} \mu\lambda\rho^*, & \text{if } \psi = \sigma\lambda \\ \mu\kappa^*\rho^*, & \text{if } \sigma = \psi\kappa \\ 0, & \text{otherwise} \end{cases}$$

For $\mu, \lambda, \sigma, \psi, \rho, \kappa \in \text{Path}(E)$. The given set spans the space. But the elements are not linearly independent, and hence do not form a basis.

Few examples of commonly known algebras that are isomorphic to LPA.

Example 3.1.11. *For Leavitt algebra $L_K(1, n)$, consider the rose with n -petals R_n . It is a graph with single vertex v , and n self-loops :*



Proposition 3.1.12. *We have isomorphism of algebras $L_K(1, n) \cong L_K(R_n)$.*

Proof.

$$L_K(1, n) = K[X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n] / [\sum X_i Y_i = 1, Y_i X_j = \delta_{ij} 1 \text{ for } 1 \leq i, j \leq n]$$

$$L_K(R_n) = K[e_1, e_2, \dots, e_n, e_1^*, e_2^*, \dots, e_n^*] / [e^* e' = \delta_{e^*, e'} v, \sum_1^n e_i e_i^* = v; \forall e, e' \in E^1]$$

Let $X_i \mapsto e_i$ and $Y_i \mapsto e_i^*$. As v is the unit in $L_K(R_n)$, it is obvious from the map that it is an isomorphism. $\square$

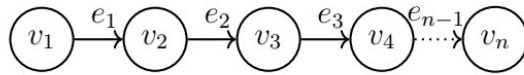
Example 3.1.13. For the Laurent polynomial ring $K[X, X^{-1}]$, consider the rose with 1-petal. The K -algebra form :

$$K[e, e^*] / [ee^* = v, e^*e = v]$$

Proposition 3.1.14. We have isomorphism of algebras $K[X, X^{-1}] \cong K[e, e^*] / [ee^* = v, e^*e = v] \cong L_K(R_1)$

Proof. Let $X \mapsto e$ and $X^{-1} \mapsto e^*$. Since v is the identity in this case also, the isomorphism is established easily. $\square$

Example 3.1.15. For $M_n(K)$ ($n \times n$ matrix over K), consider n -line graph A_n . This is a graph with n vertices and $n - 1$ edges; and they all together form a straight line :



Proposition 3.1.16. We have isomorphism of algebras $M_n(K) \cong L_K(A_n)$.

Proof. Let

$$\sim = [v_i v_j = \delta_{v_i, v_j} v_i, e_i^* e_j = \delta_{e_i^*, e_j} r(e_j), e_i e_i^* = v_i]$$

$$L_K(A_n) = K[v_1, v_2, \dots, v_n, e_1, e_2, \dots, e_{n-1}, e_1^*, e_2^*, \dots, e_{n-1}^*] / \sim$$

Let the matrix with 1 in the ij^{th} position and rest all 0 be denoted by $f_{i,j}$ in $M_n(K)$. Let

$$\phi : L_K(A_n) \rightarrow M_n(K)$$

$$v_i \mapsto f_{i,i} \quad e_i \mapsto f_{i,i+1} \quad e_i^* \mapsto f_{i+1,i}$$

Then

1. $\phi(v_i v_j) = f_{i,i} f_{j,j} = \delta_{f_{i,i} f_{j,j}} f_{i,i} = \phi(\delta_{v_i, v_j} v_i)$.
2. $\phi(e_i^* e_i) = f_{i+1,i} f_{i,i+1} = f_{i+1,i+1} = \phi(r(e_i)) = \phi(v_i + 1) f_{i+1,i} f_{j,j+1} = 0$.
3. $f_{i,i+1} f_{i+1,i} = f_{i,i} = \phi(v_i)$.

Hence all the equality's are established and it is an isomorphism. $\square$

3.2 Cohn Path Algebras(CPA)

Definition 3.2.1. *The Cohn path algebra, $C_K(E)$ of directed graph $E = (E^0, E^1, (E^1)^*, r, s)$ is a free algebra on generators $E^0 \cup E^1 \cup (E^1)^*$ with the following relations on the set of generators :*

1. $vv' = \delta_{v,v'}v$, for all $v, v' \in E^0$.
2. $s(e)e = e = er(e)$, for all $e \in E^1$.
3. $r(e)e^* = e^* = e^*s(e)$, for all $e^* \in (E^1)^*$.
4. $e^*e' = \delta_{e,e'}r(e)$, for all $e, e' \in E^1$.

Definition 3.2.2 (Basis). *A basis of $C_K(E)$:*

$$B = \{\mu\sigma^* | r(\mu) = r(\sigma); \mu, \sigma \in \text{path}(E)\}$$

The set forms the basis of $C_K(E)$. The product of paths is as defined in Definition 3.1.10. This is a linearly independent spanning set, and hence a basis of $C_K(E)$.

Proposition 3.2.3. *Let E be a directed graph and I be the ideal generated by the elements of the form $\{v - \sum_{e|s(e)=v} ee^* | v \in \text{Reg}(E)\}$. Then $L_K(E) \cong C_K(E)/I$ as K -algebras.*

Proof. Quotient of $C_K(E)$ with relation $v - \sum_{e|s(e)=v} ee^*$ gives the fifth condition of $L_K(E)$

$$v = \sum_{e|s(e)=v} ee^* | v \in \text{Reg}(E)$$

Hence all condition of LPA is satisfied and $L_K(E) \cong C_K(E)/I$. $\square$

Rather than generating ideal by taking all $v \in \text{Reg}(E)$, consider $X \subseteq \text{Reg}(E)$. Let I^X be ideal of $C_K(E)$ generated by $\{v - \sum_{e|s(e)=v} ee^* | v \in X\}$ where $X \subseteq \text{Reg}(E)$.

Definition 3.2.4. *Relative Cohn path algebra is Cohn path algebra, $C_K^X(E)$ of directed graph E relative to X , where $X \subseteq \text{Reg}(E)$. It is the quotient K -algebra $C_K(E)/I^X$.*

Definition 3.2.5. *The basis of relative Cohn path algebra, $C_K^X(E)$:*

$$B' = B \setminus \{\lambda e_i^v (e_i^v)^* \mu^* | r(\lambda) = r(\mu) = v \in X\}$$

Where $e_i^v \in \{e_1^v, e_2^v, \dots, e_n^v\} = s^{-1}(v)$ and $B = \{\lambda \mu^* | r(\lambda) = r(\mu); \mu, \lambda \in \text{path}(E)\}$

From Proposition 3.2.3 we can get the basis of LPA of graph.

Definition 3.2.6 (Basis). *Let $X = \text{Reg}(E)$, then basis of $L_K(E)$:*

$$B' = B \setminus \{\lambda e_i^v (e_i^v)^* \mu^* | r(\lambda) = r(\mu) = v \in \text{Reg}(E)\}$$

3.3 Grading on LPA

Every path algebra has a natural $\mathbb{Z}$ -graded structure, which assign's $\deg(\text{path}) = \text{length of path}$. The $\mathbb{Z}$ -grading on Leavitt path algebra can also be defined in similar way.

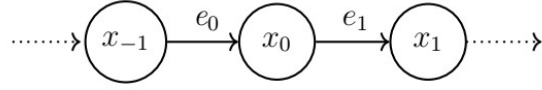
Definition 3.3.1. *Let $L_K(E)$ of a graph E with generators $\{v, e, e^* | v \in E^0, e \in E^1\}$ be Leavitt path algebra over field K . Let $\deg(v) = 0, \deg(e) = 1$, and $\deg(e^*) = -1$. Here e^* is ghost edge of the edge e . For a path $\mu = e_1 e_2 e_3 \dots e_n$, the $\deg(\mu) = \deg(e_1) + \deg(e_2) + \dots + \deg(e_n)$. Then*

$$L_K(E) = \bigoplus_{n \in \mathbb{Z}} L_K(E)_n$$

$$L_K(E)_n = \left\{ \sum_i k_i \mu_i \lambda_i^* | \deg(\mu_i) - \deg(\lambda_i) = n; \mu_i, \lambda_i \in \text{Path}(E), k_i \in K \right\}$$

This gives $L_K(E)$ a $\mathbb{Z}$ -graded structure.

Example 3.3.2. Consider the infinite line graph $E = (E_0, E_1, E_1^*)$



For $n \in \mathbb{Z}$, the $\mathbb{Z}$ -grading on $L_K(E)$:

$$L_K(E)_n = \left\{ \sum_i k_i e_{i1} e_{i2} \cdots e_{ip} e_{ip}^* e_{i(p-1)}^* \cdots e_{i(p-q)}^* \mid p - q = n; k_i \in K \right\}$$

Definition 3.3.3. The $L_K(E)$ is also $\mathbb{Z}_2$ -graded module :

$$L_K(E)_0 = \left\{ \sum_i k_i \mu_i \lambda_i^* \mid \deg(\mu_i) - \deg(\lambda_i) = \text{even}; \mu_i, \lambda_i \in \text{Path}(E), k_i \in K \right\}$$

$$L_K(E)_1 = \left\{ \sum_i k_i \mu_i \lambda_i^* \mid \deg(\mu_i) - \deg(\lambda_i) = \text{odd}; \mu_i, \lambda_i \in \text{Path}(E), k_i \in K \right\}$$

Example 3.3.4. For the graph in Example 3.3.2, the $\mathbb{Z}_2$ -grading :

$$L_K(E)_0 = \left\{ \sum_i k_i e_{i1} e_{i2} \cdots e_{ip} e_{ip}^* e_{i(p-1)}^* \cdots e_{i(p-q)}^* \mid p - q = \text{even}; k_i \in K \right\}$$

$$L_K(E)_1 = \left\{ \sum_i k_i e_{i1} e_{i2} \cdots e_{ip} e_{ip}^* e_{i(p-1)}^* \cdots e_{i(p-q)}^* \mid p - q = \text{odd}; k_i \in K \right\}$$

Definition 3.3.5. Let R be a ring, and $a \in R$. If there is an isomorphism from R to the corner ring aRa , then the isomorphism is called corner isomorphism.

Definition 3.3.6. Let p be an idempotent in R . Let $\phi : R \rightarrow pRp$ be a corner isomorphism. Then the corner skew Laurent polynomial ring is $R[t_+, t_-, \phi]$ where

$$R[t_+, t_-, \phi] := \{t_-^j r_{-j} + t_-^{j-1} r_{-j+1} + \cdots + t_- r_{-1} + r_0 + r_1 t_+ \cdots + r_i t_+^i\}$$

Here $r_{-n} \in p_n R$, $r_n \in R p_n$, $p_0 = 1$, and $p_n = \phi^n(p_0)$. Multiplication in the ring has the following identities,

$$t_- t_+ = 1 \quad t_+ t_- = p \quad r t_- = t_- \phi(r) \quad t_+ r = \phi(r) t_+$$

Lemma 3.3.7. Let $R = \bigoplus_i R_i$ be a $\mathbb{Z}$ -graded ring. Let $t_+ \in R_1$, and $t_- \in R_{-1}$ such that $t_- t_+ = 1$. Then there exists a corner isomorphism $\alpha : R_0 \rightarrow t_+ t_- R_0 t_+ t_-$ given by $\alpha : r \mapsto t_+ r t_-$ and $R = R_0[t_+, t_-; \alpha]$.

Proof. For detailed proof refer Lemma 2.4 of [8]. $\square$

Proposition 3.3.8. *The algebra $L_K(E) = L_K(E)_0[t_+, t_-; \phi]$ for a finite graph E with no source vertex.*

Proof. Let $E^0 = \{v_1, v_2, \dots, v_n\}$ be the vertex set of the graph E . Then for each i choose an edge e_i with $r(e_i) = v_i$. Let $t_+ = e_1 + e_2 + \dots + e_n$, and $t_- = e_1^* + e_2^* + \dots + e_n^*$. Then $t_+ \in L_K(E)_1$, $t_- \in L_K(E)_{-1}$ and $t_- t_+ = 1$. From Lemma 3.3.7, we get $L_K(E) = L_K(E)_0[t_+, t_-; \phi]$. With $\phi : L_K(E)_0 \rightarrow t_+ t_- L_K(E)_0 t_+ t_-$ given by $\phi : a \mapsto t_+ a t_-$. $\square$

Definition 3.3.9. *An idempotent $e \in R$ is called full idempotent if $ReR = R$.*

Proposition 3.3.10. *Let R_0 be a ring with identity, and $R = R_0[t_+, t_-; \phi]$ be corner skew Laurent polynomial ring. Then R is strongly graded iff $\phi(1)$ is full idempotent.*

Proof. Refer Proposition 1 of [7] for detailed proof. $\square$

Lemma 3.3.11. *Element $t_+ t_-$ in Proposition 3.3.8 is a full idempotent iff E does not have sinks.*

Proof. Refer Theorem 2 of [7] for detailed proof. $\square$

Theorem 3.3.12. *Let E be a finite graph. Then $L_K(E)$ is strongly graded iff E does not have sinks.*

Proof. We first show for the case when E has no source vertex. From Proposition 3.3.8, $L_K(E)$ can be written as

$$L_K(E) = L_K(E)_0[t_+, t_-; \phi]$$

Here $\phi(1) = t_+ t_-$, and $t_+ t_-$ is a full idempotent from Lemma 3.3.11. $L_K(E)_0$ is a ring with identity, and by Proposition 3.3.10 $L_K(E)$ is strongly graded iff E does not have source or sink.

We now show for the case when E has source vertices. Let $E' = E - u$ where u is a source vertex. We prove that $L_K(E')$ is strongly graded iff $L_K(E)$ is strongly graded. Graph homomorphism $E' \rightarrow E$ induces a natural inclusion algebra homomorphism

$$L_K(E') \rightarrow L_K(E)$$

$$1_{L_K(E)} = \sum v_i \in E^0 v_i = u + \sum_{v_i \in E'^0} v_i = u + 1_{L_K(E')}$$

Here $1_{L_K(E')} \in L_K(E')_n L_K(E')_{-n} \subseteq L_K(E)_n L_K(E)_{-n}$.

$$u = \sum_{s(\alpha)=u} \alpha \alpha^*$$

As α is an edge whose range is contained in E' , then

$$r(\alpha) \in L_K(E')_{n-1} L_K(E')_{-(n-1)}$$

Since $\alpha \in L_K(E)_1$, $\alpha^* \in L_K(E)_{-1}$, $L_K(E')_n \subseteq L_K(E)_n$, we get

$$u = \sum_{s(\alpha)=u} \alpha r(\alpha) \alpha^* \in L_K(E)_1 L_K(E')_{n-1} L_K(E')_{-(n-1)} L_K(E)_{-1} \subseteq L_K(E)_n L_K(E)_{-n}$$

Therefore $1_{L_K(E)} = u + 1_{L_K(E')} \in L_K(E)_n L_K(E)_{-n}$, and $L_K(E)$ is strongly graded. For converse, as $L_K(E)$ is strongly graded, and $L_K(E')$ is a subalgebra of $L_K(E)$. Then $L_K(E')$ is strongly graded. Hence $L_K(E')$ is strongly graded iff $L_K(E)$ is strongly graded. By induction we can bring the case of E with n sources to the first case of E without any sources. Hence $L_K(E)$ is strongly graded iff E does not have sinks. $\square$

Chapter 4

Classical ring theory

Here we discuss some important properties of general rings like exchange property, refinement property, separative rings, cancellation property and uniqueness of n -th root which will be used for the main result in Chapter 7. The definitions and major results in this chapter are referred from [3].

4.1 Exchange rings

Definition 4.1.1. An R -module M is said to have exchange property, if there are decompositions

$$A = M' \oplus N = \bigoplus_{i \in I} A_i$$

Here N, A and A_i are R -modules. Let $M' \cong M$, and $i \in I$ is the indexing set. Then A can be written as

$$A = M' \bigoplus (\bigoplus_{i \in I} A'_i)$$

Here $A'_i \subseteq A_i$ is a submodule.

Definition 4.1.2. An R -module M is said to have finite exchange property if the indexing set I is finite.

Definition 4.1.3. An R -module M is said to have m -exchange property if the cardinality of the indexing set I is $\leq m$.

Definition 4.1.4. A ring R is called an exchange ring if R satisfies the exchange property as an R -module.

Example 4.1.5. The set of real numbers $\mathbb{R}$ over $\mathbb{R}$ is an exchange ring. Let

$$A = \mathbb{R} \oplus N = \bigoplus_i A_i$$

Here $\mathbb{R}$ is a field. Let $A = \mathbb{R}^m$, and $A_i = \mathbb{R}$. Then

$$\mathbb{R}^m = \mathbb{R} \oplus N = \bigoplus_m \mathbb{R}$$

Here A can be written as $A = \mathbb{R} \bigoplus (\bigoplus_{m-1} \mathbb{R})$ and $\mathbb{R}, 0 \subseteq \mathbb{R}$. Hence $\mathbb{R}$ is an exchange ring.

Lemma 4.1.6. Let $n \in \mathbb{N}$. Let $B_0, B_1, \dots, B_n$ be subalgebra of algebra B , and $B = B_0 \times B_1 \times \dots \times B_n$. The algebra B has m -exchange property iff $B_i (1 \leq i \leq n)$ has m -exchange property.

Proof. Refer Lemma 3.10 of [9] for detailed proof. □

Lemma 4.1.7. If R is exchange ring then all finitely generated modules of R have exchange property.

Proof. Let M be finitely generated module of R . Let $\{x_1, x_2, \dots, x_n\}$ be the generating set, then $M = x_1R + x_2R + \dots + x_nR$. As M has finite generating set, consider component wise multiplication in M . Hence M is an R -algebra. Algebra $x_1R \cong R$. Here R has exchange property. Then from Lemma 4.1.6, algebra M has exchange property. □

Definition 4.1.8. Let R be a ring with identity. Then R is regular ring if for any element $a \in R$, there exists b such that $aba = a$.

Example 4.1.9. Any field F is regular, take $b = a^{-1}$.

Definition 4.1.10. The modules P and Q over R are said to be stably isomorphic to each other if there exists $n, m \in \mathbb{N}$ such that

$$P \oplus R^n \cong Q \oplus R^m$$

Definition 4.1.11. For a ring R , let $FP(R)$ denote the set of finitely generated modules over R .

Definition 4.1.12. For a ring R , let $V(R)$ be the monoid of stably isomorphism classes of finitely generated projective modules. Let $[P] + [Q] = [P \oplus Q]$ be the operation in the monoid $V(R)$, for $P, Q \in FP(R)$.

Lemma 4.1.13. A ring R is an exchange ring iff for any $x \in R$ there exists idempotent $e \in R$ such that $e \in Rx$ and $1 - e \in R(1 - x)$.

Proof. Refer Theorem 2.1 of [10] for detailed proof. □

4.2 Refinement property

Definition 4.2.1. Let $A_1, A_2, B_1, B_2 \in FP(R)$. The set $FP(R)$ has refinement property if for $A_1 \oplus A_2 \cong B_1 \oplus B_2$, then there exists $A_{ij}, 1 \leq i, j \leq 2$ such that

$$A_i \cong A_{i1} \oplus A_{i2}$$

$$B_j \cong A_{1j} \oplus A_{2j}$$

Definition 4.2.2. Let $a, b, c, d \in$ monoid M . Then M is called refinement monoid if for $a + b = c + d$, there exists decompositions $c = x + x'$ and $d = y + y'$, such that $a = x + y$ and $b = x' + y'$.

Example 4.2.3. Every abelian group G is refinement monoid. For $a + b = c + d$ in G , assume $c > a$. Let

$$x = a, x' = c - a, y = 0, y' = d$$

Then we get $c = x + x', d = y + y', a = x + y$, and $b = x' + y'$.

Definition 4.2.4. A ring R is called a refinement ring if $FP(R)$ has refinement property.

Example 4.2.5. Real numbers $\mathbb{R}$ is a refinement ring as it is a field. All finitely generated projective modules are free for a field. Then the monoid is isomorphic to $\mathbb{Z}$ which is an abelian group. From Example 4.2.3, module $FP(\mathbb{R})$ has refinement property.

Definition 4.2.6. A refinement matrix is a matrix for writing down refinements in a monoid. The elements in the border are sum of each row or column.

Example 4.2.7. The refinement matrix for the case in Definition 4.2.2 :

$$\begin{array}{cc} & a & b \\ c & \begin{pmatrix} x & x' \end{pmatrix} \\ d & \begin{pmatrix} y & y' \end{pmatrix} \end{array}$$

Theorem 4.2.8. Let R be an exchange ring and $A_1, A_2, B_1, B_2, A_{ij} (1 \leq i, j \leq 2) \in FP(R)$. If $A_1 \oplus A_2 \cong B_1 \oplus B_2$, then there exists decompositions $A_1 \cong A_{11} \oplus A_{12}$, and $A_2 \cong A_{21} \oplus A_{22}$. Such that $B_1 \cong A_{11} \oplus A_{21}$, and $B_2 \cong A_{12} \oplus A_{22}$.

Proof. Let $P \cong A_1 \oplus A_2 \cong B_1 \oplus B_2$. From Lemma 4.1.7, we know that if R is an exchange ring, then all elements of $FP(R)$ have exchange property. Let B_1 have exchange property, then

$$P \cong B_1 \oplus B_2 \cong (A_{11} \oplus A_{21}) \oplus B_2 \cong A_1 \oplus A_2$$

Here $A_{11} \subseteq A_1$, and $A_{21} \subseteq A_2$. Then

$$B_1 \oplus B_2 \cong (A_{11} \oplus A_{21}) \oplus B_2$$

By cancellation, we get $B_1 \cong (A_{11} \oplus A_{21})$. As $A_{11} \subseteq A_1$, and $A_1 \in FP(R)$; we have $A_{12} \subseteq A_1$ such that $A_{11} \oplus A_{12} \cong A_1$. Similarly there exists $A_{22} \subseteq A_2$ such that $A_{21} \oplus A_{22} \cong A_2$. Then

$$P \cong A_1 \oplus A_2 \cong (A_{11} \oplus A_{12}) \oplus (A_{21} \oplus A_{22}) \cong (A_{11} \oplus A_{21}) \oplus (A_{12} \oplus A_{22})$$

We already shown $B_1 \cong (A_{11} \oplus A_{21})$. Hence $P \cong B_1 \oplus (A_{12} \oplus A_{22}) \cong B_1 \oplus B_2$. And by cancellation we get $B_2 \cong (A_{12} \oplus A_{22})$. Hence the theorem is proved. $\square$

Corollary 4.2.9. For a exchange ring R , monoid $V(R)$ is a refinement monoid.

Proof. In the proof of Theorem 4.2.8, use isomorphism classes of modules $[A_1], [A_2], [B_1], [B_2], [A_{ij}] (1 \leq i, j \leq 2) \in V(R)$. And as $[P] \oplus [Q] \cong [P \oplus Q] \in V(R)$ for $P, Q \in FP(R)$, the proof follows. Hence monoid $V(R)$ is a refinement monoid. $\square$

Proposition 4.2.10. Let M be refinement monoid, and $a + c = b + c$ for $a, b, c \in M$. Then

1. There exists decompositions $a = a_1 + a_2$, $b = b_1 + b_2$, and $c = c_1 + c_2$; such that $a_1 = b_1$, and $a_2 + c_2 = b_2 + c_2 = c$.

2. Let $c \leq a$, and $c \leq b$. Then $\exists$ decompositions as in (1); such that $c_2 \leq a_2$, and $c_2 \leq b_2$.

Proof. 1. Let $b = b_1 + b_2$, and $c = c_1 + c_2$. let $b_1 = a_1$ and $c_1 = a_2$. Then the refinement matrix :

$$\begin{matrix} & b & c \\ a & \left(\begin{matrix} a_1 & a_2 \end{matrix} \right) \\ c & \left(\begin{matrix} b_2 & c_2 \end{matrix} \right) \end{matrix}$$

Hence $a_1 = b_1$, and $a_2 + c_2 = b_2 + c_2 = c$.

2. Let $a = a_1 + a_2$, $c \leq a$, and assume $a_2 \leq b_2$. Consider $c_2 = c' + c''$ such that $c' \leq a_1$, and $c'' \leq a_2$. Let $a_1 = d + c'$ for some d . Then from the decomposition in (1), we get

$$a = d + (c' + a_2) \quad b = d + (c' + b_2) \quad c = c_1 + c' + c''$$

From (1), we get $c = a_2 + c_2$, and $c = b_2 + c_2$. Then

$$c = (a_2 + c') + c'' \quad c = (b_2 + c') + c''$$

By the new decomposition of a, b and c ; we get a refinement matrix :

$$\begin{matrix} & b & c \\ a & \left(\begin{matrix} d & a_2 + c' \end{matrix} \right) \\ c & \left(\begin{matrix} b_2 + c' & c'' \end{matrix} \right) \end{matrix}$$

Then

$$\begin{aligned} c'' &\leq a_2 \leq a_2 + c' \\ c'' &\leq a_2 \leq b_2 \leq b_2 + c' \end{aligned}$$

Hence the proposition is proved. □

4.3 Separative property

Definition 4.3.1. A class $\mathbb{S}$ of modules is said to have separative property if for $A, B \in \mathbb{S}$,

$$A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$$

Definition 4.3.2. Ring R is said to be separative ring if the class of modules $FP(R)$ has separative property.

Definition 4.3.3. A monoid M is separative if for $a, b \in M$,

$$a + a = a + b = b + b \implies a = b$$

Definition 4.3.4. A class $\mathbb{S}$ of modules is said to have cancellation property if for $A, B \in \mathbb{S}$,

$$A \oplus C \cong B \oplus C \implies A \cong B$$

Definition 4.3.5. A class $\mathbb{S}$ of modules satisfies uniqueness of n – th root if

$$\bigoplus_n A \cong \bigoplus_n B \implies A \cong B$$

Proposition 4.3.6. For a ring R , and $A, B, C \in FP(R)$, the following conditions are equivalent :

1. R is a separative ring if $A \oplus A \cong A \oplus B \cong B \oplus B \implies A \cong B$.
2. If $2A \cong 2B$, and $3A \cong 3B \implies A \cong B$.
3. Let $n \in \mathbb{N}$. If $nA \cong nB$, and $(n+1)A \cong (n+1)B \implies A \cong B$.
4. (Separative cancellation) Let $C \oplus D \cong kA$, and $C \oplus D' \cong lB$; for $k, l \in \mathbb{N}$. If $A \oplus C \cong B \oplus C$, then $A \cong B$.

Proof. $1 \implies 2$. Given $2A \cong 2B$ and $3A \cong 3B$. Then

$$4A \cong 2A \oplus 2A \cong 2A \oplus 2B \cong 2(A \oplus B)$$

$$4B \cong 3B \oplus B \cong 3A \oplus B \cong 2A \oplus (A \oplus B)$$

And $4A \cong 2(2A) \cong 2(2B) \cong 4B$

$$\implies 2(2A) \cong 2A \oplus (A \oplus B) \cong 2(A \oplus B)$$

From (1), we get

$$2A \cong A \oplus B$$

And its given that $2A \cong 2B$. Hence by (1)

$$A \cong B$$

2 $\implies$ 3 Given $nA \cong nB$ and $(n+1)A \cong (n+1)B$. Then

$$(n+1)A \cong nA \oplus A \cong nB \oplus A \cong nB \oplus B \cong (n+1)B$$

For $k \geq 1$

$$\begin{aligned} (n+1)A \oplus kA &\cong (n+1)B \oplus kA \cong B \oplus (nB \oplus A) \oplus (k-1)A \\ &\cong B \oplus (nB \oplus B) \oplus (k-1)A \cong B \oplus B \oplus (nB \oplus A) \oplus (k-2)A \\ &\cong B \oplus (nB \oplus kB) \cong (n+1)B \oplus kB \\ &\implies (n+1)A \oplus kA \cong (n+1)B \oplus kB \end{aligned}$$

So for all $n' \geq n$, we have $n'A \cong n'B$. Let $2n-2, 3n-3 \geq n$, for $n > 1$. Then

$$2(n-1)A \cong 2(n-1)B$$

$$3(n-1)A \cong 3(n-1)B$$

From (2), we get

$$(n-1)A \cong (n-1)B$$

Therefore by induction we can prove that $2A \cong 2B$, and $3A \cong 3B$. Hence by (2), we get

$$A \cong B$$

3 $\implies$ 4. Given $A \oplus C \cong B \oplus C$, and $C \oplus D \cong kA$; $k \in \mathbb{N}$. For $r \in \mathbb{N}$, let

$$\begin{aligned}
(k+r)A &\cong rA \oplus kA \cong (r-1)A \oplus (A \oplus C) \oplus D \\
&\cong (r-1)A \oplus B \oplus C \oplus D \cong (r-2)A \oplus B \oplus (A \oplus C) \oplus D \cong rB \oplus kA \\
&\implies rA \oplus kA \cong rB \oplus kA
\end{aligned} \tag{4.1}$$

Similarly by taking $C \oplus D' \cong kB$, we get

$$rB \oplus kB \cong rA \oplus kB$$

Let $r = k$, then

$$2kA \cong 2kB \tag{4.2}$$

$$(2k+1)A \cong kA \oplus (k+1)A$$

Let $r = 1$ in Equation 4.1, then $(k+1)A \cong B \oplus kA$

$$kA \oplus (k+1)A \cong (kA \oplus kA) \oplus B \cong 2kB \oplus B \cong (2k+1)B$$

$$\implies (2k+1)A \cong (2k+1)B$$

Combine this result with Equation 4.2. Then from (3), we get

$$A \cong B$$

4 $\implies$ 1. Given $A \oplus A \cong A \oplus B \cong B \oplus B$. Let $C \cong A \cong B$, then $A \oplus C \cong C \oplus B$. From (4), we get

$$A \cong B$$

□

Definition 4.3.7. An ideal I is called separative ideal if the monoid $V(I)$ has separative property.

Definition 4.3.8. A class $\mathbb{S}$ of modules is called strongly separative if any of the following conditions hold for $A, B, C \in \mathbb{S}$

1. Let $C \cong nA$ for $n \in \mathbb{N}$. If $A \oplus C \cong B \oplus C \implies A \cong B$.

$$2. \ 2A \cong A \oplus B \implies A \cong B.$$

$$3. \ A \oplus 2C \cong B \oplus C \implies A \oplus C \cong B.$$

Lemma 4.3.9. *The class of separative exchange rings are closed under arbitrary direct product, and direct limit.*

Proof. Refer Proposition 2.2 of [3] for detailed proof. □

Theorem 4.3.10 (Extension theorem). *Let R be an exchange ring and I be an ideal of R . Then R is separative ring iff I and R/I are separative.*

Proof. Refer Theorem 4.2 of [3] for detailed proof. □

Chapter 5

Elementary subgroup of $GL_n(R)$

Here in this chapter we study elementary matrices, and properties associated with them. The group K_1 of a ring R , studied in the following chapter is constructed using elementary matrices. Let R be a ring, and $M_n(R)$ be ring of all $n \times n$ matrices with entries in R . The definitions and major results in this chapter are referred from [5].

Definition 5.0.1. *Let $GL_n(R)$ be the group of all $n \times n$ invertible matrices. The group operation is matrix multiplication and identity element is identity matrix Id_n .*

Definition 5.0.2. *Let $SL_n(R)$ be the group of all $n \times n$ invertible matrices with determinant 1.*

Definition 5.0.3. *Let $E_n(R)$ be the group of all elementary matrices :*

$$e_{ij}(\lambda) = I + \lambda E_{ij}; \quad i \neq j, \quad \lambda \in R$$

Where E_{ij} is a matrix with 1 in the ij^{th} position and rest 0.

Proposition 5.0.4. *We have the relation of groups $E_n(R) \subseteq SL_n(R) \subseteq GL_n(R)$.*

Proof. All elementary matrices are invertible, and of $\det = 1$. All elements of $SL_n(R)$ are invertible matrices. □

Definition 5.0.5. *The group $GL_n(R)$ can be embedded within $GL_{n+m}(R)$:*

$$\alpha \mapsto \begin{bmatrix} \alpha & 0 \\ 0 & I_m \end{bmatrix}$$

Where $\alpha \in GL_n(R)$.

Definition 5.0.6. With the embedding defined in Definition 5.0.5, we can define groups :

$$GL(R) = \bigcup_{n=1}^{\infty} GL_n(R)$$

$$SL(R) = \bigcup_{n=1}^{\infty} SL_n(R)$$

$$E(R) = \bigcup_{n=1}^{\infty} E_n(R)$$

Definition 5.0.7. The commutator $[E_n(R), E_n(R)]$ of $E_n(R)$ is the subgroup with elements :

$$\{e_{ij}(\lambda)e_{ij}(\mu)e_{ij}(\lambda)^{-1}e_{ij}(\mu)^{-1}; \lambda, \mu \in R\}$$

5.1 Properties of $E_n(R)$

Definition 5.1.1. The inverse element in $E_n(R)$:

$$e_{ij}(\lambda)^{-1} = e_{ij}(-\lambda)$$

Example 5.1.2. Let $e_{12}(4)^{-1} = e_{12}(-4)$ in $E_3(\mathbb{R})$

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Id$$

Definition 5.1.3. The splitting property in $E_n(R)$ states

$$e_{ij}(\lambda + \mu) = e_{ij}(\lambda)e_{ij}(\mu)$$

Example 5.1.4. Let $e_{23}(6)e_{23}(3) = e_{23}(9)$ in $E_3(\mathbb{R})$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition 5.1.5. *Some properties of commutator are*

1. $[e_{ij}(\lambda), e_{kl}(\mu)] = Id; j \neq k, i \neq l.$
2. $[e_{ij}(\lambda), e_{jk}(\mu)] = e_{ik}(\lambda\mu); j \neq k.$
3. $[e_{ij}(\lambda), e_{ki}(\mu)] = e_{kj}(-\mu\lambda); j \neq k.$

Lemma 5.1.6. *If $\alpha, \beta \in GL_n(R)$, then*

$$\begin{bmatrix} \alpha\beta & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \beta\alpha & 0 \\ 0 & I_n \end{bmatrix} \text{ mod } E_{2n}(R)$$

Proof. Consider a ring R , and $u, v \in R^*$. Here R^* is the set of invertible elements of R . Then

$$\delta = \begin{bmatrix} 1 & v^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & v^{-1} - 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \in E_2(R)$$

Multiply $[u, v]$ diagonal matrix with δ , then

$$\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \delta = \begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix} \tag{5.1}$$

This gives the first equality in lemma. Let $v = u^{-1}$ in Equation 5.1, then

$$\begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix} \delta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Here $I, \delta \in E_2(R)$, then $[u, u^{-1}] \in E_2(R)$. Therefore $[v^{-1}u, u^{-1}v] \in E_2(R)$.

$$\begin{bmatrix} v^{-1}u & 0 \\ 0 & u^{-1}v \end{bmatrix} \in E_2(R) \implies \begin{bmatrix} v^{-1} & 0 \\ 0 & u^{-1} \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \in E_2(R)$$

$$\begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix}^{-1} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \in E_2(R)$$

If $[u, v]$ is contained in some coset, then $[v, u]$ is also contained in the same coset. From

Equation 5.1,

$$\begin{bmatrix} uv & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} v & 0 \\ 0 & u \end{bmatrix} = \begin{bmatrix} vu & 0 \\ 0 & 1 \end{bmatrix} \text{ mod } E_2(R) \quad (5.2)$$

Here we took some random ring R , and $u, v \in R^*$. Let $R = M_n(R)$, and $R^* = GL_n(R)$. Then the equation is true for $\text{mod } E_2(M_n(R))$. But $E_2(M_n(R)) \subseteq E_{2n}(R)$. For $\alpha, \beta \in GL_n(R)$, Equation 5.2 becomes

$$\begin{bmatrix} \alpha\beta & 0 \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \beta\alpha & 0 \\ 0 & I_n \end{bmatrix} \text{ mod } E_{2n}(R)$$

Hence the lemma is proved. □

Lemma 5.1.7. *For any ring R ,*

$$[GL(R), GL(R)] = E(R) = [E(R), E(R)]$$

Proof. First we show that $E_n(R) = [E_n(R), E_n(R)]$. Write $e_{ij}(\lambda)$ in the form $aba^{-1}b^{-1}$:

$$e_{ij}(\lambda\mu) = e_{ik}(\lambda)e_{kj}(\mu)e_{ik}(\lambda)^{-1}e_{kj}(\lambda)^{-1}$$

Here $i \neq j$, $i \neq k$, and $k \neq j$. Then any element of $E_n(R)$ can be written as an element of $[E_n(R), E_n(R)]$ for $n \geq 3$. For $n = 2$ take the embedding in $n = 3$, and the equation holds. Then

$$E_n(R) \subseteq [E_n(R), E_n(R)]$$

It is obvious that $[E_n(R), E_n(R)] \subseteq E_n(R)$ for all n . Therefore

$$\begin{aligned} E_n(R) &= [E_n(R), E_n(R)] \\ \bigcup_{n=1}^{\infty} E_n(R) &= \bigcup_{n=1}^{\infty} [E_n(R), E_n(R)] \\ E(R) &= [E(R), E(R)] \end{aligned} \quad (5.3)$$

Next we show that $[GL_n(R), GL_n(R)] \subseteq E_{2n}(R) \subseteq E(R)$. Let $\alpha, \beta \in GL_n(R)$. From Lemma 5.1.6, we get

$$\begin{bmatrix} \alpha\beta & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \beta\alpha & 0 \\ 0 & I \end{bmatrix} \text{ mod } E_{2n}(R)$$

$$\begin{bmatrix} \alpha\beta & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \beta\alpha & 0 \\ 0 & I \end{bmatrix}^{-1} \in E_{2n}(R)$$

$$\begin{bmatrix} \alpha\beta\alpha^{-1}\beta^{-1} & 0 \\ 0 & I \end{bmatrix} \in E_{2n}(R)$$

Therefore

$$[GL_n(R), GL_n(R)] \subseteq E_{2n}(R) \subseteq E(R)$$

$$[GL(R), GL(R)] \subseteq E(R)$$

And

$$E(R) = [E(R), E(R)] \subseteq [GL(R), GL(R)]$$

$$[GL(R), GL(R)] = E(R)$$

Combining Equation 5.3, we get

$$[GL(R), GL(R)] = E(R) = [E(R), E(R)]$$

Hence the lemma is proved. □

5.2 $SL_n(R) - E_n(R)$ relations

Proposition 5.2.1. *If R is a euclidean ring, then $SL_n(R) = E_n(R)$ for all $n \geq 1$.*

Proof. We have to show that identity matrix can be transformed into any element in $SL_n(R)$ by elementary transformation. Let $\alpha \in SL_n(R)$:

$$\alpha = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ r_{2,1} & r_{2,2} & \cdots & r_{2,n} \\ \vdots & & & \vdots \\ r_{n,1} & r_{n,2} & \cdots & r_{n,n} \end{bmatrix}$$

Elementary column transformation in $E_n(R)$ sends

$$C_n \rightarrow C_n + aC_m$$

Here $m \neq n$, $a \in R$, and $C_k = k^{th}$ column. Similar transformation are defined in terms of rows also. Assume $r_{1,1}$ is the least in α . Here R is euclidean. By the euclidean algorithm, and the above elementary transformation we can make the first column and row 0 except for $r_{1,1}$. If $r_{1,2} > r_{1,1}$, let $r_{1,2} = xr_{1,1} + b$ by euclidean algorithm; $x \in R$. Then by elementary transformation $C_2 \rightarrow C_2 - xC_1$, we get

$$\begin{bmatrix} r_{1,1} & b & \cdots & r_{1,n} \\ r_{2,1} & & \cdots & r_{2,n} \\ \vdots & & & \vdots \\ r_{n,1} & & \cdots & r_{n,n} \end{bmatrix}$$

Here $r_{1,1} > b$. Let $r_{1,1} = x'b + b'$, $x' \in R$. Repeating the same process, till we get 0 in place of $r_{1,2}$. Then

$$\begin{bmatrix} r'_{1,1} & 0 & \cdots & r_{1,n} \\ r'_{2,1} & r'_{2,2} & \cdots & r_{2,n} \\ \vdots & & & \vdots \\ r'_{n,1} & r'_{n,2} & \cdots & r_{n,n} \end{bmatrix}$$

Continuing this manner for all elements except diagonal, by column and row transformation. Then

$$\alpha' = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

By Lemma 5.1.6, there exist $e \in E_n(R)$ such that

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} e = \begin{bmatrix} \lambda_1 \lambda_2 \cdots \lambda_n & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

Here $\alpha \in SL_n(R)$, then $\lambda_1 \lambda_2 \cdots \lambda_n = 1$. Hence resulting matrix is Id_n . So by applying the above multiplications in reverse, we can get any element of $SL_n(R)$ from I . Hence

$$SL_n(R) = E_n(R)$$

□

5.2.1 Cohn's example

In this subsection we show that $SL_n(R) \neq E_n(R)$ for non euclidean rings. Consider $R = K[X, Y]$, where K is a field. We prove that

$$\begin{bmatrix} 1 + XY & X^2 \\ -Y^2 & 1 - XY \end{bmatrix} \notin E_2(R)$$

Lemma 5.2.2. *Group $E_2(R)$ is generated by*

$$E(r) = \begin{bmatrix} r & 1 \\ -1 & 0 \end{bmatrix}; r \in R$$

Proof. Let $e_{12}(r)$ and $e_{21}(r)$ generate $E_2(R)$. We can prove that these two elements can be generated by $E(r)$. Here $E(r) \in E_2(R)$ as

$$E(r) = \begin{bmatrix} 1 & 1-r \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

For $e_{12}(r)$ and $e_{21}(r)$:

$$e_{12}(r) = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -r & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^3$$

$$e_{21}(r) = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^3 \begin{bmatrix} r & 1 \\ -1 & 0 \end{bmatrix}$$

Hence the lemma is proved. □

Definition 5.2.3. *Inverse of $E(r)$:*

$$\begin{bmatrix} 0 & -1 \\ 1 & r \end{bmatrix}$$

Definition 5.2.4. *Let $D_n(R)$ be the set of $n \times n$ invertible diagonal matrices with entries in R . Its elements are denoted by $[r_1, r_2, \dots, r_n]$.*

Definition 5.2.5. Group $E_n(R)$ is normalized by $D_n(R)$:

$$[r_1, r_2, \dots, r_n]^{-1} e_{ij}(\lambda) [r_1, r_2, \dots, r_n] = [r_1^{-1}, r_2^{-1}, \dots, r_n^{-1}] e_{ij}(\lambda) [r_1, r_2, \dots, r_n] = e_{ij}(r_i^{-1} \lambda r_j)$$

Lemma 5.2.6. Let $\alpha \in D_2(R)E_2(R)$, which is a subgroup of $GL_2(R)$. Then α is of the form

$$\alpha = [a, b]E(r_1)E(r_2) \cdots E(r_n); \quad r_i \notin R^* \cup \{0\} \text{ for } 1 < i < n$$

Proof. We check the below three identities.

$$1. \quad E(x)E(0)E(y) = -E(x+y)$$

$$\begin{aligned} \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} -1 & x \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y & 1 \\ -1 & 0 \end{bmatrix} = \\ &= \begin{bmatrix} -x-y & -1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} x+y & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

$$2. \quad E(x)[a, b] = [b, a]E(b^{-1}xa)$$

$$\begin{aligned} \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} &= \begin{bmatrix} xa & b \\ -a & 0 \end{bmatrix} \\ \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} b^{-1}xa & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} xa & b \\ -a & 0 \end{bmatrix} \\ \implies \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} &= \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix} \begin{bmatrix} b^{-1}xa & 1 \\ -1 & 0 \end{bmatrix} \end{aligned}$$

$$3. \quad \text{If } a \in A^*, \text{ then } E(x)E(a)E(y) = E(x-a^{-1})[a, a^{-1}]E(y-a^{-1})$$

$$\begin{aligned} \begin{bmatrix} x & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} xa & b \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} xa-1 & x \\ -a & -1 \end{bmatrix} \begin{bmatrix} y & 1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} xay-y-x & xa-1 \\ 1-ay & -a \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} x - a^{-1} & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \begin{bmatrix} y - a^{-1} & 1 \\ -1 & 0 \end{bmatrix} &= \begin{bmatrix} xa - 1 & a^{-1} \\ -a & 0 \end{bmatrix} \begin{bmatrix} y - a^{-1} & 1 \\ -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} xay - y - x & xa - 1 \\ 1 - ay & -a \end{bmatrix} \\
\implies E(x)E(a)E(y) &= E(x - a^{-1})[a, a^{-1}]E(y - a^{-1})
\end{aligned}$$

Consider $\alpha \in D_2(R)E_2(R)$:

$$\alpha = [a', b']E(r'_1)E(r'_2) \cdots E(r'_n); r_i \in R$$

When $r_i \in R^* \cup \{0\}$ (for $1 < i < n$), we can use the above identities to get a smaller expression for α . If $r_i = 0$, then use first identity. If $r_i \in R^*$, then use third identity. If two elements of subgroup are multiplied, use second identity to bring diagonal matrix to the left end. Hence

$$\alpha = [a, b]E(r_1)E(r_2) \cdots E(r_n); r_i \notin R^* \cup \{0\} \forall 1 < i < n$$

□

Lemma 5.2.7. Let $R = K[X, Y]$, where K is a field. Let $\beta \in D_2(R)E_2(R)$. Then

$$\beta = \begin{bmatrix} f & g \\ * & * \end{bmatrix} = [u, v]E(r_1)E(r_2) \cdots E(r_n)$$

Here $u, v \in R^* = K^*$ and $r_n \notin K^* \cup \{0\} \forall 1 < i < n$. Then $\deg(f) > \deg(g)$.

Proof. Proof by induction. The diagonal matrix does not change the $\deg$ of terms as $u, v \in R^* = K^*$, so only other terms are considered for the proof. For $r = 2$,

$$\beta = \begin{bmatrix} r_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r_2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} r_1 r_2 - 1 & r_1 \\ -r_2 & -1 \end{bmatrix}$$

The lemma is true for $r = 2$. Assume that the lemma is true for $r = n$. Let

$$\beta = E(r_1)E(r_2) \cdots E(r_n) = \begin{bmatrix} f_1 & g_1 \\ * & * \end{bmatrix}$$

Here $\deg(f_1) > \deg(g_1)$. For $r = n + 1$

$$\beta = \begin{bmatrix} f_1 & g_1 \\ * & * \end{bmatrix} \begin{bmatrix} r_{n+1} & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} f_1 r_{n+1} - g_1 & f_1 \\ * & * \end{bmatrix}$$

Here $r_{n+1} \notin K$, and $\deg(f_1) > \deg(g_1)$. Therefore $\deg(f_1 r_{n+1} - g_1) > \deg(f_1)$. Hence the lemma is proved by mathematical induction. $\square$

Proposition 5.2.8. *For $R = K[X, Y]$, let*

$$\alpha = \begin{bmatrix} f & g \\ * & * \end{bmatrix} \in GL_2(R)$$

Let $\deg(f) = \deg(g)$, and f, g be linearly independent over K . Then $\alpha \notin D_2(R)E_2(R)$.

Proof. Assume $\alpha \in D_2(R)E_2(R)$, then by Lemma 5.2.6

$$\alpha = [a, b]E(r_1)E(r_2) \cdots E(r_n)$$

Here $r_i \notin R^* \cup \{0\} \forall 1 < i < n$. Matrix $[a, b]$ is not considered as $u, v \in K^*$. For $n = 1$,

$$E(r_1) = \begin{bmatrix} r_1 & 1 \\ -1 & 0 \end{bmatrix}$$

Here $r_1 \in K$. As r_1 and 1 are not linearly independent, our assumption fails for this case. For $n = 2$,

$$E(r_1)E(r_2) = \begin{bmatrix} r_1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} r_2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} r_1 r_2 - 1 & r_1 \\ -r_2 & -1 \end{bmatrix}$$

Here $r_1 r_2 - 1$ and r_1 are not linearly independent. Therefore our assumption fails for $n = 2$ case also. For $n > 2$,

$$\alpha = \begin{bmatrix} f & g \\ * & * \end{bmatrix} = E(r_1)E(r_2) \cdots E(r_n)$$

$$\alpha E(r_n)^{-1} = \begin{bmatrix} f & g \\ * & * \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & r_n \end{bmatrix} = \begin{bmatrix} g & -f + gr_n \\ * & * \end{bmatrix}$$

Here f and g were given to be linearly independent. As $r_n \in K$, we get $\deg(g) = \deg(-f + gr_n)$. But $r_{n-1} \notin K$. Then by Lemma 5.2.7, we get $\deg(g) > \deg(-f + gr_n)$. This is a

contradiction. So our assumption is wrong and $\alpha \notin D_2(R)E_2(R)$. $\square$

Theorem 5.2.9. *Let $R = K[X, Y]$, then $SL_2(R) \neq E_2(R)$*

Proof. Let

$$\alpha = \begin{bmatrix} 1 + XY & X^2 \\ -Y^2 & 1 - XY \end{bmatrix}$$

Let $\alpha \in SL_2(R)$, as $\det(\alpha) = 1$. Here $\deg(1 + XY) = \deg(X^2)$. Here $1 + XY$ and X^2 are linearly independent. Therefore by Proposition 5.2.8, we get $\alpha \notin D_2(R)E_2(R)$. As $Id \in D_2(R)$, we can conclude

$$\alpha \notin E_2(R)$$

Hence $SL_2(R) \neq E_2(R)$. $\square$

Chapter 6

K_0 and K_1

In this chapter we study properties of group completion of a monoid, Grothendieck group(K_0) and Whitehead group(K_1). The definitions and major results in this chapter are referred from [6] and [5].

6.1 Group completion

The Grothendieck group is calculated by group completion of a monoid.

Definition 6.1.1. *A groupoid is a set of objects together with invertible maps between them. If $a, b \in G$, then there can be a map from $a \rightarrow b$. If such a map exist then inverse of that map also exist.*

Definition 6.1.2. *A semigroup is set of objects with binary operation between them. The binary operation is associative and closed. But commutativity, existence of inverse and identity element are not necessary for a semigroup.*

Example 6.1.3. *$\{\mathbb{Z}, \min\}$ is a semigroup where $\min$ is binary operation that takes a pair to their minimum value. There is no identity element or inverse.*

Definition 6.1.4. *A monoid is set with a binary operation which is associative and closed. There is an identity element but inverse may not exist.*

Example 6.1.5. *The set of natural numbers with addition operation $\{\mathbb{N} \cup \{0\}, +\}$ is a monoid.*

Definition 6.1.6. *Let $a, b \in M$. A monoid homomorphism $f : M \rightarrow N$, is a set map such that $f(a + b) = f(a) + f(b)$, and $f(0) = 0$.*

Definition 6.1.7. *A group is set together with associativity, closed binary operation, inverse and identity element.*

Example 6.1.8. *The set of integers $\{\mathbb{Z}, +\}$ is a group.*

Definition 6.1.9. *A category is a set of objects and maps between these objects. Map compositions are associative, and there exists an identity map from every object to itself.*

Example 6.1.10. *Consider the set of all groups with groups as objects, and group homomorphisms as maps. Group homomorphisms are associative, and identity map also exist.*

A monoid with all inverse elements is a group. We can construct an abelian group from an abelian monoid by adding the inverse of elements.

Definition 6.1.11. *Let M be an abelian monoid. Then*

$$M^{-1} = \{-m \mid m \in M\}$$

$$M^{-1}M = \{m - n \mid m, n \in M\}$$

Definition 6.1.12. *The group completion of an abelian monoid M is a abelian group $M^{-1}M$ together with a monoid homomorphism $\square : M \rightarrow M^{-1}M$. This homomorphism is universal. That is, if there is a map $f : M \rightarrow A$ where A is any group, then $\exists f' : M^{-1}M \rightarrow A$. Here $f([m]) = f'(m)$ for all $m \in M$.*

Definition 6.1.13 (Construction). *Let $m \in$ monoid M . Consider the free group $F(M)$ generated by $[m]$. Let relation $R(M) = [m + n] - [m] - [n]$. Then*

$$M^{-1}M = F(M)/R(M)$$

Example 6.1.14. *Let $\mathbb{N} = \{0, 1, 2, \dots\}$, then*

$$F(\mathbb{N}) = \{\dots, -[3], -[2] - [1], -[2], -[1], [0], [1], [2], [1] + [2], [3], \dots\}$$

$$R(\mathbb{N}) = \{\dots, [3] - [2] - [1], [1] + [2] - [3], \dots\}$$

$$\mathbb{N}^{-1}\mathbb{N} = F(\mathbb{N})/R(\mathbb{N}) = \mathbb{Z}$$

Remark 6.1.15. *The map $f' : M^{-1}M \rightarrow N^{-1}N$ is unique. Let map composition of $M \rightarrow N$ and $N \rightarrow N^{-1}N$ give $M \rightarrow N^{-1}N$. By universality of monoid homomorphism $[] : M \rightarrow M^{-1}M$, there exist a unique map from $M^{-1}M \rightarrow N^{-1}N$.*

Proposition 6.1.16. *Let M be an abelian monoid, and $M^{-1}M$ be its group completion. Then*

1. *Elements of $M^{-1}M$ are of the form $[m] - [n]$, for $m, n \in M$.*
2. *Element $[m] = [n]$ in $M^{-1}M$ iff $m + p = n + p$, for some $p \in M$.*
3. *Map $M \times M \rightarrow M^{-1}M$ that sends $(m, n) \rightarrow [m] - [n]$ is a surjection.*
4. *The group $M^{-1}M$ is $M \times M$ quotiented by $(m, n) \sim (m + p, n + p)$.*

Proof. 1. Let $F(M)$ have elements of the form $[m_1] + [m_2] + \dots - [m_k] - [m_{k+1}] - \dots$. Then $M^{-1}M$ is this modulo relation $[m + n] - [m] - [n]$. Hence $M^{-1}M$ has elements of the form

$$\begin{aligned} [m_i + m_{i+1} + \dots] - [m_j + m_{j+1} + \dots]; \quad i, j \in I \\ = [m] - [n] \end{aligned}$$

Where $m, n \in M$.

2. Let $[m] = [n]$ in $M^{-1}M$. Then $[m] - [n] = 0$. Let

$$0 = \sum [a_i + b_i] - [a_i] - [b_i] - \sum [c_j + d_j] - [c_j] - [d_j]$$

Separating the positive and negative terms :

$$\sum_i [a_i] + [b_i] + \sum_j [c_j + d_j] = \sum_i [a_i + b_i] + \sum_j [c_j] + [d_j] = p$$

Here $[m] - [n] = 0$. Let

$$[m] - [n] = \sum_i [a_i + b_i] - [a_i] - [b_i] - \sum_j [c_j + d_j] - [c_j] - [d_j]$$

Separating the positive and negative terms :

$$[m] + \sum_i [a_i] + [b_i] + \sum_j [c_j + d_j] = [n] + \sum_i [a_i + b_i] + \sum_j [c_j] + [d_j]$$

$$m + p = n + p$$

Conversely; as $m + p = n + p$, we get

$$[m + p] = [n + p]$$

$$[m] + [p] = [n] + [p]$$

$$\implies [m] = [n]$$

Hence $[m] = [n]$ in $M^{-1}M$ iff $m + p = n + p$.

3. Let $m, n \in M$. From (1), every element of $M^{-1}M$ is of the form $[m] - [n]$. Hence every element has preimage (m, n) for the map $M \times M \rightarrow M^{-1}M$.
4. Let $f : M \times M \rightarrow M^{-1}M$, defined by $(m, n) \mapsto [m] - [n]$. Here $f((m, n)) = [m] - [n]$, and $f((m + p, n + p)) = [m + p] - [n + p] = [m] - [n]$. Therefore $f((m, n)) = f((m + p, n + p))$. Then $(m, n) \sim (m + p, n + p)$ in $M \times M$. We considered for general element $m, n, p \in M$. Then $\text{Img}(f) = M \times M / \sim$, where $\sim$ is the relation $(m, n) \sim (m + p, n + p)$. From (3), we get $\text{Img}(f) = M^{-1}M$. Hence $M^{-1}M = M \times M / \sim$.

□

Definition 6.1.17. Monoid M is called cancellation monoid if for $m, n, p \in M$:

$$m + p = n + p \implies m = n$$

Corollary 6.1.18. Let M be an abelian monoid and $M^{-1}M$ be its group completion. Then the map $f : M \rightarrow M^{-1}M$ is injective iff M is a cancellation monoid.

Proof. Let $m, n, p \in M$. Let $m + p = n + p$, then $[m] = [n]$. That is $f(m) = f(n)$. As f is an injective function, then $m = n$. Hence $m + p = n + p$ implies $m = n$, and the monoid is cancellation monoid.

Conversely, let $f(m) = f(n)$. That is $[m] = [n]$. From Proposition 6.1.16 (2), we get

$[m] = [n]$ in $M^{-1}M$ iff $m + p = n + p$. As the monoid is cancellation, we get $m + p = n + p \implies m = n$. Hence $f(m) = f(n)$ implies $m = n$, and f is an injective map. $\square$

Definition 6.1.19. A submonoid $L \subseteq M$ is cofinal if for $m \in M$, $\exists m' \in M$ such that $m + m' \in L$.

Corollary 6.1.20. If $L \subseteq M$ is cofinal, then

1. Group $L^{-1}L$ is a subgroup of $M^{-1}M$.
2. Every element of $M^{-1}M$ is of the form $[m] - [l]$. Here $m \in M$ and $l \in L$.
3. If $[m] = [m']$ in $M^{-1}M$, then $m + l = m' + l$. Here $l \in L$.

Proof. 1. From Proposition 6.1.16 (1), every element of $L^{-1}L$ is of the form $[l] - [l']$; for some $l, l' \in L$. L is a submonoid of M , then $l, l' \in M$. Therefore $[l] - [l'] \in M^{-1}M$, and $L^{-1}L$ is a subgroup of $M^{-1}M$.

2. Every element of $M^{-1}M$ is of the form $[m] - [n]$, for some $m, n \in M$. Let L be cofinal. Then for n we have $n' \in M$, and $l \in L$ such that $n + n' = l$. Then

$$[n] = [l] - [n']$$

$$[m] - [n] = [m] - [l] + [n'] = [m'] - [l]; \quad m' \in M, \quad l \in L$$

3. Let $[m] = [m']$ in $M^{-1}M$. By Proposition 6.1.16 (2), we get $m + n = m' + n$ for some $n \in M$. Let $m + n + n' = m' + n + n'$, for $n' \in M$. As L is cofinal we have $n' \in M$, and $l \in L$ such that $n + n' = l$. Then

$$m + l = m' + l; \quad l \in L$$

Hence the corollary is proved. $\square$

6.2 K_0 group

Definition 6.2.1. Let $V(R)$ be the monoid of stably isomorphism classes of projective modules over ring R as in Definition 4.1.12. The Grothendieck group of ring R denoted by $K_0(R)$ is group completion of the monoid $V(R)$:

$$K_0(R) = \mathcal{G}(V)$$

Theorem 6.2.2. The monoid map $\mathbb{N} \rightarrow P(R)$ that maps $n \mapsto R^n$ induces a group homomorphism $\mathbb{Z} \rightarrow K_0(R)$.

1. The group homomorphism $\mathbb{Z} \rightarrow K_0(R)$ is injective iff R satisfies invariant basis number property (IBN).
2. Let R satisfies IBN. Then $K_0(R) \cong \mathbb{Z}$ iff every finitely generated projective R -module is stably free.

Proof. 1. Let $f : \mathbb{Z} \rightarrow K_0(R)$ and $n, m \in \mathbb{Z}$. Let $R^m \cong R^n$, then $[R^m] = [R^n]$. As f is injective, we have $f(m) = f(n) \implies m = n$. That is $[R^m] = [R^n] \implies m = n$. Hence R has IBN as $R^m \cong R^n \implies m = n$.

Conversely, Let $[R^m] = [R^n]$. Then from Proposition 6.1.16 (2), $[R^m] = [R^n]$ iff $R^m \oplus P \cong R^n \oplus P$. Here $P \in V(R)$. Let $P = 0$. Then by IBN, we get $R^m \cong R^n \implies m = n$. Hence f is injective as $[R^m] = [R^n] \implies m = n$.

2. Let $g : K_0(R) \rightarrow \mathbb{Z}$ with $g([R^n]) = n$ be an isomorphism. Let P be a finitely generated projective R -module. Then $[P] = [R^n]$ for $n \in \mathbb{Z}$. From Definition 4.1.12, the isomorphism in monoid is stably isomorphism. Therefore $P \oplus R^k \cong R^n \oplus R^k$, for $k \in \mathbb{Z}$. Hence P is stably free as $P \oplus R^k \cong R^{n+k}$.

Conversely, Let P be stably free. Then $P \oplus R^m \cong R^n$. Here $[P] + [R^m] = [R^n]$. Therefore $[P] = [R^n] - [R^m] = [R^{n-m}]$. Hence $K_0(R) \cong \mathbb{Z}$ as any element P is contained in equivalence class $[R^m]$, for $m \in \mathbb{Z}$.

□

6.3 K_1 group

Groups $GL(R)$, $SL(R)$, $E(R)$ were explained in chapter 5. The K_1 group is constructed using these structures.

Definition 6.3.1. *The Whitehead group of ring R denoted by $K_1(R)$ is defined as*

$$K_1(R) = GL(R)/[GL(R), GL(R)]$$

Definition 6.3.2. *The group $SK_1(R)$ of ring R :*

$$SK_1(R) = SL(R)/E(R)$$

Lemma 6.3.3. *The whitehead group $K_1(R) \cong SK_1(R) \oplus R^*$*

Proof. Consider the short exact sequence

$$0 \longrightarrow SK_1(R) \longrightarrow K_1(R) \longrightarrow R^* \longrightarrow 0$$

Where the map $K_1(R) \rightarrow R^*$ is the determinant map, and $SK_1(R) \rightarrow K_1(R)$ is the embedding. To show that the above sequence splits, take

$$f : R^* \rightarrow K_1(R)$$

$$u \rightarrow \begin{bmatrix} u & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$\det \circ f$ gives identity on R^* . Hence the sequence splits. Then by Splitting lemma 2.4.17,

$$K_1(R) \cong SK_1(R) \oplus R^*$$

□

Proposition 6.3.4. *If R is a euclidean ring, then*

$$SK_1(R) = 0$$

Proof. From Proposition 5.2.1; if R is a euclidean ring, then $SL_n(R) = E_n(R)$. Here $n \geq 1$.
Hence

$$SK_1(R) = SL(R)/E(R) = E(R)/E(R) = 0$$

□

Chapter 7

K_0 and K_1 of LPA

LPA of a group was studied in Chapter 3. In this chapter we study properties of the monoid M_E , K_0 and K_1 of LPA. The definitions and major results in this chapter are referred from [4].

Definition 7.0.1. *A graph homomorphism $f : E \rightarrow F$ is set of two compatible homomorphisms*

$$(f^0, f^1) : (E^0, E^1) \rightarrow (F^0, F^1)$$

It satisfies the conditions :

1. $r_F(f^1(e)) = f^0(r_E(e))$, for $e \in E^1$.
2. $s_F(f^1(e)) = f^0(s_E(e))$.

Definition 7.0.2. *A graph homomorphism is called complete homomorphism if*

1. *The homomorphism f^0 is injective.*
2. *The homomorphism $f_{s_E^{-1}(v)}^1 : s_E^{-1}(v) \rightarrow s_F^{-1}(f^0(v))$ is a bijection.*

7.1 Direct limit

Definition 7.1.1. Let $\langle I, \leq \rangle$ be a directed set. Let A_i be objects and f_{ij} be corresponding homomorphisms. Then $\langle A_i, f_{ij}; i \leq j \in I \rangle$ is a directed system of objects if

1. f_{ii} is an isomorphism on A_i .
2. f_{ij} is a homomorphism from $A_i \rightarrow A_j$.
3. $f_{jk} \circ f_{ij} = f_{ik}$ for $i \leq j \leq k \in I$.

Here objects can be group, monoids, rings, etc. Then the corresponding homomorphisms will be group homomorphism, monoid homomorphism, ring homomorphism etc.

Definition 7.1.2. Pair $\langle A, \phi_i \rangle$ is called direct limit of directed system $\langle A_i, f_{ij}; i \leq j \in I \rangle$ if

$$A = \bigsqcup A_i / \sim$$

$$\phi_i : A_i \rightarrow A$$

Let $x_i \in A_i$ and $x_j \in A_j$. Here $x_i \sim x_j$ if $f_{ik}(x_i) = f_{jk}(x_j)$ for some $i, j \leq k \in I$. Homomorphism ϕ_i sends $x_i \mapsto [x_i]$, here $[x_i]$ is the equivalence class of x_i .

Definition 7.1.3. Category $\mathbf{G}$ is a category whose objects are row-finite graphs, and morphisms are complete graph homomorphisms.

Lemma 7.1.4. Every row-finite graph E is a limit in the category $\mathbf{G}$ of directed system of finite graphs.

Proof. Consider a finite subgraph $G_0 = (G_0^0, G_0^1)$ of $E = (E^0, E^1)$. Construct another subgraph F_0 from G_0 :

$$F_0^0 = G_0^0 \cup \{r(e) | s(e) \in G_0^0\}$$

$$F_0^1 = G_0^1 \cup \{e | s(e) \in G_0^0\}$$

Let $f_0 : F_0 \rightarrow E$ be the inclusion map. Map is injective as it is an inclusion map. Then

$$s_{F_0}^{-1}(v) \rightarrow s_E^{-1}(v)$$

is a bijection, as all edges such that $s(e) \in G_0$ is contained in F_0 . Hence f_0 is a complete graph homomorphism. Row-finite graph E is the union of finite subgraphs of E . From each finite subgraph G_i , we can construct F_i such that $f_i : F_i \rightarrow E$ is a complete graph homomorphism. Then $\cup F_i \rightarrow E$ is also a complete graph homomorphism. Then consider the directed system of finite graphs

$$F_0 \longrightarrow F_0 \cup F_2 \longrightarrow \cdots \longrightarrow \cup F_i$$

For $i \in I$. Let

$$E = \bigcup G_i \subseteq \bigcup F_i$$

And $\cup F_i \rightarrow E$ is a complete graph homomorphism. Then E is the direct limit of the directed system. Here

$$E = \lim_i \cup F_i = \lim_i F'_i$$

Where $F'_i = F_1 \cup F_2 \cup \cdots \cup F_i$. Hence the lemma is proved. $\square$

Proposition 7.1.5. *Let $f : E \rightarrow F$ be a complete graph homomorphism. Then $\exists$ canonical algebra homomorphism $L(f) : L_K(E) \rightarrow L_K(F)$*

$$L(f) : (v_1 v_2 \cdots) \mapsto f^0(v_1) f^0(v_2) \cdots$$

$$L(f) : (e_1 e_2 \cdots) \mapsto f^1(e_1) f^1(e_2) \cdots$$

This map preserves the five LPA relations.

Proof. 1. $f(v)f(v') = f(vv') = f(\delta_{v,v'}v) = \delta_{f(v),f(v')}f(v)$; for $v, v' \in E^0$.

2. Let $f(s(e)e) = f(e) = f(er(e))$; for all $e \in E^1$.

$$f^0(s_E(e))f^1(e) = f^1(e) = f^1(e)f^0(r_E(e))$$

From Definition 7.0.1, we get

$$s_F(f^1(e))f^1(e) = f^1(e) = f^1(e)r_F(f^1(e))$$

$$s_F(f(e))f(e) = f(e) = f(e)r_F(f(e))$$

3. $f(r(e)e^*) = f(e^*) = f(e^*s(e))$; for all $e^* \in (E^1)^*$.

$$f^0(r_E(e))f^1(e^*) = f^1(e^*) = f^1(e^*)f^0(s_E(e))$$

From Definition 7.0.1, we get

$$r_F(f^1(e))f^1(e^*) = f^1(e^*) = f^1(e^*)s_F(f^0(e))$$

Here $r_F(f^1(e))f^1(e^*) = f^1(e^*)$, then $f^1(e^*) = f^1(e)^*$. Then

$$r_F(f(e))f(e)^* = f(e)^* = f(e)^*s_F(f(e))$$

4. $f(e)^*f(e') = f(e^*e') = f(\delta_{e^*,e'}r(e)) = \delta_{f(e)^*,f(e')}r_F(f(e))$; for all $e, e' \in E^1$.

5. $f(v) = f(\sum_{e|s(e)=v} ee^*)$; for $v \in \text{Reg}(E)$.

$$f(v) = \sum_{f^1(e)|f^0(s_E(e))=f^0(v)} f^1(e)f^1(e^*)$$

As $s_E^{-1}(v) \rightarrow s_F^{-1}(f^0(v))$ is a bijection for a complete homomorphism, no terms are omitted. Then

$$f(v) = \sum_{f(e)|s_F(f(e))=f(v)} f(e)f(e)^*$$

Therefore all five LPA relations are preserved. Hence the proposition is proved. $\square$

Lemma 7.1.6. *The algebra $L_K(E) = \lim_{i \in I} L_K(F_i)$, where $\{F_i\}$ is directed family of finite graphs with limit E . i.e. L_K commutes with direct limit.*

Proof. Let $\phi_i : F_i \rightarrow E$ be complete graph homomorphisms of the directed system, and $\overline{\phi_i} : L_K(F_i) \rightarrow L_K(E)$ be canonical algebra homomorphism. Let A be a K -algebra. Then consider K -algebra homomorphisms :

$$\gamma_i : L_K(F_i) \rightarrow A$$

$$\gamma : L_K(E) \rightarrow A$$

Here γ_i are compatible homomorphisms that satisfy LPA relations. Let

$$\gamma(\overline{\phi_i}(v)) = \gamma_i(v); \quad \gamma(\overline{\phi_i}(e)) = \gamma_i(e); \quad \gamma(\overline{\phi_i}(e^*)) = \gamma_i(e^*)$$

If γ preserves all relation of LPA, then the diagram below commutes.

$$\begin{array}{ccc} F_i & \xrightarrow{\phi_i} & E \\ \downarrow & & \downarrow \\ L_K(F_i) & \xrightarrow{\overline{\phi_i}} & L_K(E) \\ \downarrow \gamma_i & \swarrow \gamma & \\ A & & \end{array}$$

1. for $v, v' \in E^0$; choose i such that $v, v' \in F_i$. Then

$$\gamma(\overline{\phi_i}(v))\gamma(\overline{\phi_i}(v')) = \gamma_i(v)\gamma_i(v') = \delta_{\gamma_i(v), \gamma_i(v')}\gamma_i(v) = \delta_{\gamma(\overline{\phi_i}(v)), \gamma(\overline{\phi_i}(v'))}\gamma(\overline{\phi_i}(v))$$

Here $\gamma_i(v)\gamma_i(v') = \delta_{\gamma_i(v), \gamma_i(v')}\gamma_i(v)$ as γ_i is compatible homomorphism. Therefore

$$\gamma(f)\gamma(f') = \delta_{\gamma(f), \gamma(f')}\gamma(f)$$

Where $f = \overline{\phi_i}(v)$, $f' = \overline{\phi_i}(v')$.

2. For $e \in E^1$, choose i such that $e \in F_i$. Let $s(\gamma_i(e))\gamma_i(e) = \gamma_i(e) = \gamma_i(e)r(\gamma_i(e))$.

$$\gamma(\overline{\phi_i}(s(e)))\gamma(\overline{\phi_i}(e)) = \gamma_i(s(e))\gamma_i(e) = s(\gamma_i(e))\gamma_i(e) = \gamma_i(e)$$

$$s(\gamma(\overline{\phi_i}(e)))\gamma(\overline{\phi_i}(e)) = \gamma(\overline{\phi_i}(e))$$

$$s(\gamma(f))\gamma(f) = \gamma(f)$$

Where $f = \overline{\phi_i}(e)$. Similarly we can show that

$$\gamma(f) = \gamma(f)r(\gamma(f))$$

Therefore

$$s(\gamma(f))\gamma(f) = \gamma(f) = \gamma(f)r(\gamma(f))$$

3. For all $e^* \in (E^1)^*$, choose i such that $e \in F_i$. Let $r(\gamma_i(e))\gamma_i(e)^* = \gamma_i(e)^* = \gamma_i(e)^*s(\gamma_i(e))$.

$$r(\gamma(f))\gamma(f)^* = r(\gamma_i(e))\gamma_i(e)^* = \gamma_i(e)^* = \gamma(f)^*$$

Where $f = \overline{\phi_i}(e)$. Similarly

$$\gamma(f)^* = \gamma(f)^*s\gamma(f)$$

Therefore

$$r(\gamma(f))\gamma(f)^* = \gamma(f)^* = \gamma(f)^*s\gamma(f)$$

4. For all $e, e' \in E^1$, choose i such that $e, e' \in F_i$.

$$\gamma(f)^*\gamma(f') = \delta_{\gamma(f)^*, \gamma(f')}r(\gamma(f))$$

where $f = \overline{\phi_i}(e)$

5. Let

$$\gamma_i(v) = \sum_{\gamma_i(e)|s(\gamma_i(e))=\gamma_i(v)} \gamma_i(e)\gamma_i(e)^*$$

As γ_i is a complete homomorphism, all such $e \in F_i$ for $v \in Reg(E)$. Here $s_{F_i}^{-1}(v) \rightarrow s_E^{-1}(\overline{\phi_i}(v))$ is a bijection for γ_i . Then $\exists g = \overline{\phi_i}(v) \in L_K(E)$ such that

$$\gamma(g) = \gamma_i(v) = \sum_{\gamma_i(e)|s(\gamma_i(e))=\gamma_i(v)} \gamma_i(e)\gamma_i(e)^* = \sum_{\gamma(f)|s(\gamma(f))=\gamma(g)} \gamma(f)\gamma(f)^*$$

Where $f = \overline{\phi_i}(e)$ for each edge e . The number of terms do not change as it is a bijection.

As γ preserves all relation of LPA, $L_K(E)$ is the direct limit of $L_K(F_i)$. □

Definition 7.1.7. For a graph E let M_E be a monoid on the set of generators

$$\{a_v \mid v \in E^0\}$$

with the following relation on generators

$$a_v = \sum_{e \in E^1 \mid s(e)=v} a_{r(e)}$$

Lemma 7.1.8. *Every complete graph homomorphism $g : E \rightarrow F$ induces a monoid homomorphism $M_g : M_E \rightarrow M_F$.*

Proof. Let $g : E \rightarrow F$

$$g : e \mapsto f$$

Let $M_g : M_E \rightarrow M_F$

$$M_g : a_e \mapsto a_f$$

Then

$$\begin{aligned} M_g(a_v) &= M_g\left(\sum_{e \in E^1 | s(e)=v} a_{r(e)}\right) \\ a_{g(v)} &= \sum_{g(e) \in F^1 | g(s(e))=g(v)} a_{r(g(e))} \end{aligned}$$

As $s_E^{-1}(v) \rightarrow s_F^{-1}(g(v))$ is a bijection, the number of terms remain same in M_F . Let $g(v) = u$, then

$$a_u = \sum_{f \in F^1 | s(f)=u} a_{r(f)}$$

Therefore the relation hold in M_F , and hence the lemma is proved. $\square$

Lemma 7.1.9. *Let E be a graph, and M_E be its monoid. Then the map $\lambda : E \rightarrow M_E$ defined by $\lambda : v \mapsto a_v$ commute with direct limit. Also every monoid M_E of row-finite graphs is a direct limit of monoids $M_{E'}$ of finite graphs.*

Proof. Let $\langle E_i, \psi_{ij}; i \leq j \in I \rangle$ be a directed system of finite graphs and complete homomorphisms with direct limit $\langle E, \psi_i \rangle$. Let $\langle M_{E_i}, \overline{\psi_{ij}}; i \leq j \in I \rangle$ be the corresponding directed system of monoids. Let M be a monoid. Consider

$$\begin{array}{ccc} M_{E_i} & & \\ \downarrow \overline{\psi_i} & \searrow \gamma_i & \\ M_E & \xrightarrow{\gamma} & M \end{array}$$

Here $\overline{\psi_i}$ is the map corresponding to ψ_i . Here γ_i are compatible homomorphisms, that is $\gamma_i(a_{v_i})$ have the monoid relation. Let

$$\gamma(\overline{\psi_i}(a_{v_i})) = \gamma_i(a_{v_i})$$

For a_{v_i} , choose j such that $\{r(v_i)\} \subseteq E_j^0$. As γ_i are compatible homomorphisms

$$\gamma(\overline{\psi_j}(a_{v_i})) = \gamma_j(a_{v_i}) = \sum_{e|s(e)=v_i} \gamma_j(a_{r(e)}) = \sum_{e|s(e)=v_i} \gamma(\overline{\psi_j}(a_{r(e)}))$$

Hence the relation holds for γ , and $\overline{\psi_i}$ are compatible monoid homomorphisms. Therefore $\langle M_E, \overline{\psi_i} \rangle$ is a direct limit and λ commutes with direct limit. From Lemma 7.1.4, every row-finite graph is a direct limit of finite graphs. Hence every monoid M_E of row-finite graphs is a direct limit of monoids $M_{E'}$ of finite graphs. $\square$

7.2 M_E is a refinement monoid

Definition 7.2.1. Let F be a free abelian group on generators :

$$\{x_i \mid x_i \in E^0\}$$

Definition 7.2.2. Let $\rightarrow_1$ be binary relation on F :

$$\sum_{i=1}^n x_i \rightarrow_1 \sum_{i \neq j} x_i + \mathbf{r}(x_j)$$

$$0 \rightarrow_1 0$$

where $\mathbf{r}(x_j) = \sum_{e|s(e)=x_j} r(e)$. Let $\alpha \rightarrow_1 \beta$ if $\alpha = \sum_{i=1}^n x_i$, and $\beta = \sum_{i \neq j} x_i + \mathbf{r}(x_j)$.

Definition 7.2.3. Let $\alpha \rightarrow \beta$ if there exists a finite string

$$\alpha = \alpha_0 \rightarrow_1 \alpha_1 \rightarrow_1 \cdots \rightarrow_1 \alpha_i = \beta$$

Definition 7.2.4. Let $\alpha, \beta \in F$. Then $\alpha \sim \beta$ if there exists a sequence $\alpha_0, \alpha_1, \dots, \alpha_n$ such that $\alpha_0 = \alpha$, $\alpha_n = \beta$. Here for each i , either $\alpha_i \rightarrow_1 \alpha_{i+1}$ or $\alpha_{i+1} \rightarrow_1 \alpha_i$. Let n be called the length of the sequence.

Definition 7.2.5. The monoid M_E can be viewed as quotient of F by congruence relation $\sim$

$$M_E = F / \sim$$

Relation $\sim$ as defined in Definition 7.2.4.

Definition 7.2.6. The support of γ in free abelian group F is the basis of γ in F . Let $\gamma = x_1 + x_2 + \cdots + x_k$, then

$$\text{supp}(\gamma) = \{x_1, x_2, \dots, x_k\} \subseteq E^0$$

Lemma 7.2.7. Assume $\alpha = \alpha_1 + \alpha_2$, and $\alpha \rightarrow \beta$. Then β can be written as $\beta = \beta_1 + \beta_2$; where $\alpha_1 \rightarrow \beta_1$, and $\alpha_2 \rightarrow \beta_2$.

Proof. Let $\alpha \rightarrow_1 \beta$. Then there exists $x \in \text{supp}(\alpha)$, such that $\beta = (\alpha - x) + \mathbf{r}(x)$. Let $\alpha = \alpha_1 + \alpha_2$. Then $x \in \text{supp}(\alpha_1)$, or $x \in \text{supp}(\alpha_2)$. Assume $x \in \text{supp}(\alpha_1)$, then

$$\beta_1 = (\alpha_1 - x) + \mathbf{r}(x), \quad \beta_2 = \alpha_2$$

Then

$$\alpha_1 \rightarrow_1 \beta_1, \quad \alpha_2 \rightarrow \beta_2$$

If $x \in \text{supp}(\alpha_2)$, then proof is similar. Let $\alpha \rightarrow_1 \gamma \rightarrow_1 \beta$. Then there exists x' such that

$$\gamma_1 = (\alpha_1 - x') + \mathbf{r}(x'), \quad \gamma_2 = \alpha_2$$

Here $\alpha_1 \rightarrow_1 \gamma_1$, and $\alpha_2 \rightarrow \gamma_2$. Repeat the process on γ_1 and γ_2 . There exist x'' such that

$$\beta_1 = (\gamma_1 - x'') + \mathbf{r}(x''), \quad \beta_2 = \gamma_2$$

Here $\gamma_1 \rightarrow_1 \beta_1$, and $\gamma_2 \rightarrow \beta_2$. Then $\alpha_1 \rightarrow \beta_1$, and $\alpha_2 \rightarrow \beta_2$. By induction we can prove for any length n . Hence the lemma is proved. $\square$

Lemma 7.2.8 (Confluence lemma). Let $\alpha, \beta \neq 0$ in F . Then $\alpha \sim \beta$ iff there is a $\gamma \in F$ such that $\alpha \rightarrow \gamma$, and $\beta \rightarrow \gamma$.

Proof. Let $\alpha \sim \beta$. Then there exists a sequence $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$. For each i either $\alpha_i \rightarrow \alpha_{i+1}$, or $\alpha_{i+1} \rightarrow \alpha_i$. We prove by induction on n . For $n = 0$, $\alpha = \beta$. Then take $\gamma = \alpha$. Therefore the statement is true for $n = 0$. Assume that the statement is true for $n - 1$. Then there exists λ such that $\alpha_0 \rightarrow \lambda$, and $\alpha_{n-1} \rightarrow \lambda$. For n ; if $\beta \rightarrow \alpha_{n-1}$, then the statement is true. If $\alpha_{n-1} \rightarrow \beta$, let

$$\alpha_{n-1} = x + \alpha'_{n-1}$$

$$\beta = \mathbf{r}(x) + \alpha'_{n-1}$$

Case 1 : If $x \rightarrow \lambda(x)$, let

$$\lambda = \lambda(x) + \lambda'$$

Here $x \rightarrow \lambda(x)$ and $\alpha'_{n-1} \rightarrow \lambda'$. Then there is a sequence such that

$$\mathbf{r}(x) \rightarrow \lambda(x)$$

Then

$$\beta = \mathbf{r}(x) + \alpha'_{n-1} \rightarrow \lambda(x) + \lambda' = \lambda$$

Case 2: If $x = \lambda(x)$, let

$$\lambda = x + \lambda'$$

Let

$$\gamma = \mathbf{r}(x) + \lambda'$$

Then

$$\beta \rightarrow \gamma, \alpha \rightarrow \gamma$$

Therefore the statement is true for all n , by induction. Hence $\alpha \sim \beta$ implies $\exists \gamma \in F$ such that $\alpha \rightarrow \gamma$, and $\beta \rightarrow \gamma$.

Conversely, as $\alpha \rightarrow \gamma \implies \alpha \sim \gamma$. And $\beta \rightarrow \gamma \implies \beta \sim \gamma$. Therefore $\alpha \sim \beta$. Hence the lemma is proved. $\square$

Proposition 7.2.9. *Let M be free abelian monoid. Then M has refinement property.*

Proof. Let $a, b \in M$. Let $a = a_1 + a_2 = b_1 + b_2 = b$. Two elements are equal in a free abelian monoid iff they have same number of terms, and one is permutations of generators of the other. Then $\text{supp}(a) = \text{supp}(b)$, and

$$\text{supp}(a_1, a_2, b_1, b_2) \subseteq \text{supp}(a)$$

Take $c_{11}, c_{12}, c_{21}, c_{22} \in M$ such that $\text{supp}(c_{ij}) = \text{supp}(a_i) \cap \text{supp}(b_j)$; for $1 \leq i, j \leq 2$. The number of terms of each generator are equal, and therefore

$$a_i = c_{i1} + c_{i2}, b_j = c_{1j} + c_{2j}; 1 \leq i, j \leq 2$$

Hence every free abelian monoid has refinement property. $\square$

Theorem 7.2.10. *Monoid M_E is refinement monoid for a row-finite graph E .*

Proof. Let $M_E = F / \sim$. Consider $\alpha, \beta \in M_E$ such that

$$\alpha = \alpha_1 + \alpha_2 \sim \beta_1 + \beta_2 = \beta$$

By Lemma 7.2.8, there exists γ such that $\alpha \rightarrow \gamma$, and $\beta \rightarrow \gamma$. And Lemma 7.2.7, we get

$$\gamma = \alpha'_1 + \alpha'_2 = \beta'_1 + \beta'_2$$

Where

$$\alpha_i \rightarrow \alpha'_i, \beta_i \rightarrow \beta'_i$$

As all elements are contained in the free abelian group F , which has refinement property. Then there exists $\gamma_{ij} (1 \leq i, j \leq 2)$ such that

$$\alpha'_i = \gamma_{i1} + \gamma_{i2}, \beta_j = \gamma_{1j} + \gamma_{2j}; \quad 1 \leq i, j \leq 2$$

Hence M_E is a refinement monoid. □

7.3 M_E is a separative monoid

Definition 7.3.1. *We put an order $\leq$ on monoid M_E :*

$$a \in \text{supp}(b) \implies a \leq b; \quad a, b \in M_E$$

Definition 7.3.2. *Let $p \in$ monoid M . Element p is prime if $p \leq a_1 + a_2$ implies $p \leq a_1$ or $p \leq a_2$. Monoid M is primely generated if all elements are sum of primes.*

Lemma 7.3.3. *Every finitely generated refinement monoid is primely generated*

Proof. For detailed proof refer Corollary 6.8 of [11]. □

Theorem 7.3.4 (Brookfield). *Any primely generated refinement monoid is separative*

Proof. For detailed proof refer Theorem 4.5 of [11]. □

Theorem 7.3.5. *The monoid M_E is a separative monoid for a row-finite graph E .*

Proof. Assume that the graph E is finite. Then M_E is a finitely generated monoid. By Lemma 7.3.3, M_E is a primely generated monoid. From Theorem 7.2.10, M_E is a refinement monoid. Applying the Brookfield result 7.3.4, monoid M_E is a separative monoid. Let E' be a row-finite graph. Any row-finite graph is a direct limit of finite graphs, and $M_{E'}$ is direct limit of M_E . From Lemma 4.3.9, separativity is closed under direct limit. Hence for a row-finite graph E' , monoid $M_{E'}$ is a separative monoid. $\square$

7.4 $K_0(L_K(E))$

Definition 7.4.1. *The group $K_0(L_K(E))$ is the group completion of the monoid $V(L_K(E))$:*

$$K_0(L_K(E)) \cong \mathcal{G}(V(L_K(E)))$$

Definition 7.4.2. *Let $v_1, v_2, \dots, v_n$ be vertices of E . Then module $L_K(E)v_i$ is a finitely generated projective left module of $L_K(E)$.*

$$\bigoplus_{v_i} L_K(E)v_i \cong L_K(E)$$

Lemma 7.4.3. *Let E be a row-finite graph. Then $L_K(E)$ is the directed union of Leavitt path algebras of finite complete subgraphs of E .*

Proof. Refer Corollary 1.6.16 of [4] for detailed proof. $\square$

Theorem 7.4.4. *Let E be a row-finite graph. Then monoid $V(L_K(E)) \cong M_E$.*

Proof. We first prove for the case of finite graph E . Consider the K -algebra B_0

$$B_0 = \prod_{v \in E^0} K$$

There are orthogonal idempotents $p_v = \{0, 0, \dots, 1, \dots, 0\}$ with 1 in v^{th} position in B_0 . Their sum $\sum_{v \in E^0} p_v = 1$. Let $Reg(E) = \{v_1, v_2, \dots, v_n\}$. Let $P_1 = B_0 p_{v_1}$ and $Q_1 = \bigoplus_{s(e)=v_1} B_0 p_{r(e)}$.

Both P and Q are finitely generated projective modules over B_0 . From [12] page 38, we get a construction by Bergman

$$B_1 = B_0 \langle i, i^{-1} : \overline{P_1} \cong \overline{Q_1} \rangle$$

Here B_1 is a B_0 algebra adjoint an isomorphism $i : \overline{P_1} \rightarrow \overline{Q_1}$. Here $\overline{P_1} = - \otimes_{B_0} P_1$ and $\overline{Q_1} = - \otimes_{B_0} Q_1$. Let $- = B_1$. Then by examining the construction used, we get

$$B_1 \cong L_K(X_1)$$

Here X_1 is a graph with $X_1^0 = E^0$, and $X_1^1 = \{e | e \in E^1, s(e) = v_1\}$. From definition of B_0 , we see that $B_0 \cong L_K(X_0)$. Here X_0 is a graph with $X_0^0 = E^0$, and $X_0^1 = \phi$. By [12] Theorem 5.2, we get that $V(B_1) = V(B_0)/[P_1] - [Q_1]$. Monoid $V(B_0)$ is generated by $\{a_v\}$, where $a_v = [B_0 p_v]$. Then $V(B_1)$ is generated by $\{a_v\}$ with relation $a_{v_1} = \oplus_{s(e)=v_1} a_{r(e)}$. Repeat the process again with $B_2 = B_1 \langle i, i^{-1} : \overline{P_2} \cong \overline{Q_2} \rangle$. Let $P_2 = B_1 p_{v_2}$ and $Q_2 = \oplus_{s(e)=v_2} B_0 p_{r(e)}$. By induction over all regular vertices, we get

$$B_n \cong L_K(X_n)$$

Here X_n is a graph with $X_n^0 = E^0$, and $X_n^1 = \{e | e \in E^1, s(e) = v_1, v_2, \dots, v_n\} = E^1$. Therefore

$$B_n \cong L_K(E)$$

By [12] Theorem 5.2, we get that $V(B_n) = V(B_{n-1})/[P_n] - [Q_n]$. Then $V(B_n)$ is generated by $\{a_v\}$ with relations $a_{v_1} = \oplus_{s(e)=v_1} a_{r(e)}$, $a_{v_2} = \oplus_{s(e)=v_2} a_{r(e)}$, $\dots$, $a_{v_n} = \oplus_{s(e)=v_n} a_{r(e)}$. But this is the monoid M_E . Hence

$$V(L_K(E)) \cong M_E$$

From Lemma 7.1.9, monoid $M_{E'}$ of row-finite graphs is a direct limit of monoids M_E of finite graphs. Using Lemma 7.4.3, we can show that $V(L_K(E))$ commutes with direct limit. Hence for a row-finite graph E'

$$V(L_K(E')) \cong M'_E$$

□

Definition 7.4.5. Let A_E be the adjacency matrix of graph E . Then A_{ns} is the matrix after removing rows corresponding to sinks (zero rows). Matrix A_{ns} may not be a square matrix.

Definition 7.4.6. Let $I_{|E^0|}$ be the identity matrix with $|E^0| = m$ rows. Then I_{ns} is the matrix after removing rows corresponding to sinks.

Definition 7.4.7. Let $\oplus_{v \in E^0} \mathbb{Z}$ be a group with generators z_v .

Definition 7.4.8. Let T be a subgroup of $\oplus_{v \in E^0} \mathbb{Z}$:

$$T = \{z_v - \sum_{e \in s^{-1}(v)} z_{r(e)} \mid v \in \text{Reg}(E)\}$$

Theorem 7.4.9. We have isomorphism of groups $K_0(L_K(E)) \cong \oplus_{v \in E^0} \mathbb{Z}/T$

Proof. Let $K_0(L_K(E)) \cong \mathcal{G}(V(L_K(E)))$. From Theorem 7.4.4, we get $V(L_K(E)) \cong M_E$. Here M_E is the free monoid on generators $\{a_v\}$ with relation $a_v = \sum_{e \in s^{-1}(v)} a_{r(e)}$; for $v \in \text{Reg}(E)$. Group completion of M_E is a group on generators $\{z_v\}$ with relation $z_v = \sum_{e \in s^{-1}(v)} z_{r(e)}$; for $v \in \text{Reg}(E), z \in \mathbb{Z}$. Hence

$$K_0(L_K(E)) \cong \oplus_{v \in E^0} \mathbb{Z}/T$$

□

Lemma 7.4.10. Let $(I_{ns} - A_{ns})^t$ be the transpose of matrix $I_{ns} - A_{ns}$. Let Z_v denote the column matrix of generators z_v . Then each row of matrix multiplication

$$(I_{ns} - A_{ns})^t Z_v = z_v - \sum_{e \in s^{-1}(v)} z_{r(e)}$$

Proof. Matrix A_{ns} is the adjacency matrix with zero rows removed. Then i^{th} row of A_{ns}^t gives $r(e)$, for $s(e) = v_i$. Hence

$$(I_{ns} - A_{ns})^t Z_v = z_v - \sum_{e \in s^{-1}(v)} z_{r(e)}$$

□

Theorem 7.4.11. Let E be a row-finite graph. Then

$$K_0(L_K(E)) \cong \text{Coker}((I_{ns} - A_{ns})^t)$$

Where

$$(I_{ns} - A_{ns})^t : \bigoplus_{v \in \text{Reg}(E)} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$$

Proof. From Lemma 7.4.10,

$$(I_{ns} - A_{ns})^t Z_v = z_v - \sum_{e \in s^{-1}(v)} z_{r(e)}$$

Therefore the matrix $(I_{ns} - A_{ns})^t$ can be viewed as map from $\bigoplus_{v \in \text{Reg}(E)} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$. The image of generators is the generating set of $\text{Img}((I_{ns} - A_{ns})^t)$. Then $\text{Img}((I_{ns} - A_{ns})^t)$ is generated by

$$\{z_v - \sum_{e \in s^{-1}(v)} z_{r(e)}; v \in \text{Reg}(E)\}$$

Then

$$\text{Coker}((I_{ns} - A_{ns})^t) = \bigoplus_{v \in E^0} \mathbb{Z} / (z_v - \sum_{e \in s^{-1}(v)} z_{r(e)}; v \in \text{Reg}(E))$$

From Theorem 7.4.9, we get $K_0(L_K(E)) \cong \bigoplus_{v \in E^0} \mathbb{Z} / (z_v - \sum_{e \in s^{-1}(v)} z_{r(e)} | v \in \text{Reg}(E))$. Hence

$$K_0(L_K(E)) \cong \text{Coker}((I_{ns} - A_{ns})^t)$$

□

7.5 $K_1(L_K(E))$

Definition 7.5.1. *Whitehead group of $L_K(E)$:*

$$K_1(L_K(E)) = GL(L_K(E)) / [GL(L_K(E)), GL(L_K(E))]$$

Theorem 7.5.2. *Let E be a finite graph without sources. Then $K_1(L_K(E)) \cong$*

$$(\text{Coker}((I_{ns} - A_{ns})^t) : \prod_{v \in \text{Reg}(E)} K^* \rightarrow \prod_{v \in E^0} K^*) \bigoplus (\text{Ker}((I_{ns} - A_{ns})^t) : \bigoplus_{v \in \text{Reg}(E)} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z})$$

Proof. From [4] page 227, we get the exact sequence

$$K_1(R) \xrightarrow{1-\phi^*} K_1(R) \xrightarrow{j} K_1(R[t_+, t_-; \phi]) \xrightarrow{p} K_0(R) \xrightarrow{1-\phi^*} K_0(R)$$

Here $R = L_K(E)_0$. From Proposition 3.3.8, algebra $L_K(E) = L_K(E)_0[t_+, t_-; \phi]$. From the proof of [13] Theorem 5.10, we get that the $Coker(1 - \phi^*) : K_0(R) \rightarrow K_0(R)$, and $Ker(1 - \phi^*) : K_0(R) \rightarrow K_0(R)$ are

$$Coker((I_{ns} - A_{ns})^t) : \prod_{v \in E \setminus Sink(E)} K^* \rightarrow \prod_{v \in E^0} K^*$$

$$Ker((I_{ns} - A_{ns})^t) : \bigoplus_{v \in E \setminus Sink(E)} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$$

As the graph is finite, $E \setminus Sink(E) = Reg(E)$. Here $K_1(L_K(E))$ is an abelian group. From the exact sequence

$$K_1(L_K(E)) = Ker(p) \oplus Img(p)$$

For an exact sequence $Img(f_i) = Ker(f_{i+1})$. Then

$$Ker(p) = Img(j) = K_1(L_K(E)_0)/Ker(1 - \phi^*) = Coker(1 - \phi^*) : K_0(R) \rightarrow K_0(R)$$

$$Img(p) = Ker(1 - \phi^*) : K_0(R) \rightarrow K_0(R)$$

Then

$$K_1(L_K(E)) = (Coker(1 - \phi^*) : K_0(R) \rightarrow K_0(R)) \bigoplus (Ker(1 - \phi^*) : K_0(R) \rightarrow K_0(R))$$

Hence $K_1(L_K(E)) \cong$

$$(Coker((I_{ns} - A_{ns})^t) : \prod_{v \in Reg(E)} K^* \rightarrow \prod_{v \in E^0} K^*) \bigoplus (Ker((I_{ns} - A_{ns})^t) : \bigoplus_{v \in Reg(E)} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z})$$

□

Proposition 7.5.3. *Let E and F be finite graphs without sinks or sources. Let $|E_0| = |F_0|$. If $K_0(L_K(E)) \cong K_0(L_K(F))$, then $K_1(L_K(E)) \cong K_1(L_K(F))$.*

Proof. Let $|E_0| = |F_0| = m$. As the graphs do not have sinks, the matrices I_{ns} and A_{ns} are square matrices. Let A_E be A_{ns} of E , and A_F be A_{ns} of F . Here $I_{ns} = I_m$. From Theorem

7.4.11,

$$\text{Coker}((I_m - A_E)^t) \cong K_0(L_K(E)) \cong K_0(L_K(F)) \cong \text{Coker}((I_m - A_F)^t)$$

Graphs are finite without sinks or sources, hence $E^0 = \text{Reg}(E)$ and $F^0 = \text{Reg}(F)$. Here $(I_m - A_E)^t$ is viewed as a linear transformation from $\bigoplus_{v \in E^0} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$.

$$\text{Coker}((I_m - A_E)^t) : \bigoplus_{v \in E^0} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z} \cong \text{Coker}((I_m - A_F)^t) : \bigoplus_{v \in F^0} \mathbb{Z} \rightarrow \bigoplus_{v \in F^0} \mathbb{Z}$$

The *Coker* are isomorphic and the number of vertices are same. Then the *img* are isomorphic. As the number of vertices are same, the *Ker* are also isomorphic.

$$\text{Ker}((I_m - A_E)^t) : \bigoplus_{v \in E^0} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z} \cong \text{Ker}((I_m - A_F)^t) : \bigoplus_{v \in F^0} \mathbb{Z} \rightarrow \bigoplus_{v \in F^0} \mathbb{Z} \quad (7.1)$$

Here the *img* of $(I_m - A_E)^t$ and $(I_m - A_F)^t$ are isomorphic. Then $\exists$ matrices P and Q such that $P(I_m - A_E)^t Q = (I_m - A_F)^t$. Then

$$\text{Coker}((I_m - A_E)^t) : \prod_{v \in E^0} K^* \rightarrow \prod_{v \in E^0} K^* \cong \text{Coker}((I_m - A_F)^t) : \prod_{v \in F^0} K^* \rightarrow \prod_{v \in F^0} K^* \quad (7.2)$$

By Theorem 7.5.2, Equation 7.1, and Equation 7.2 we get

$$K_1(L_K(E)) \cong K_1(L_K(F))$$

□

Theorem 7.5.4 (Purely infinite simplicity theorem). *$L_K(E)$ is purely infinite simple iff E is simple and E contains a cycle.*

Definition 7.5.5. Let $E^0 = n$, and $F^0 = m$. Then $E \equiv_{\det} F$, if

$$K_0(L_K(E)) \cong K_0(L_K(F))$$

and

$$\det(I_n - A_E^t) = \det(I_m - A_F^t)$$

Definition 7.5.6. Rings R and S are said to be Morita equivalent if the categories of left

R -modules and left S -modules are equivalent. Its denoted by

$$R \sim_M S$$

Lemma 7.5.7. *Let E be a finite graph. Let $L_K(E)$ be purely infinite simple. Then $\exists$ graph E' with no sources such that*

$$L_K(E) \sim_M L_K(E')$$

Proof. Refer Corollary 1.5 of [14] for detailed proof. $\square$

Lemma 7.5.8. *Let E be a finite graph. Let $L_K(E)$ be purely infinite simple. Let v be a source in E^0 . Then*

$$E \equiv_{det} (E \setminus v)$$

Proof. Refer Lemma 1.24 of [14] for detailed proof. $\square$

Definition 7.5.9. *Let $E = (E^0, E^1, r, s)$. Let $v \in E^0$. Let v^*, f be new elements. Then graph $E_v = (E_v^0, E_v^1, r, s)$:*

$$\begin{aligned} E_v^0 &= E^0 \cup \{v^*\} \\ E_v^1 &= E^1 \cup \{f\} \\ s_{E_v}(e) &= \begin{cases} v & \text{if } e = f \\ v^* & \text{if } s_E(e) = v \\ s_E(e) & \text{otherwise} \end{cases} \\ r_{E_v}(e) &= \begin{cases} v^* & \text{if } e = f \\ r_E(e) & \text{otherwise} \end{cases} \end{aligned}$$

Then E_v is called expansion graph of E , and E is called contraction graph of E_v .

Definition 7.5.10. *Let $\mu = e_1 e_2 \cdots e_n$. Then e is called an exit for μ , if $s(e) = s(e_i)$ and $e \neq e_i$, for $1 \leq i \leq n$.*

Theorem 7.5.11 (Simplicity theorem). *$L_K(E)$ is simple iff every cycle has an exit, and there $\exists$ a path from any vertex to any cycle or sink.*

Proposition 7.5.12. *Let E be a finite graph. Let $L_K(E)$ be simple. Let $v \in E^0$. Then*

$$L_K(E) \sim_M L_K(E_v)$$

Proof. Refer Proposition 1.8 of [14] for detailed proof. □

Definition 7.5.13. *Let $E = (E^0, E^1, r, s)$. Consider all v such that $r^{-1}(v)$ is nonempty. Partition the elements of $r^{-1}(v)$ into disjoint nonempty subsets $\Lambda_1^v, \Lambda_2^v, \dots, \Lambda_{m(v)}^v$. Let this partition be called $\mathcal{P}$. Then graph $E_r(\mathcal{P})$:*

$$E_r(\mathcal{P})^0 = \{v_i \mid v \in E^0, 1 \leq i \leq m(v)\} \cup \{v \mid r^{-1}(v) = \phi\}$$

$$E_r(\mathcal{P})^1 = \{e_j \mid v \in E^0, 1 \leq j \leq m(s(e))\} \cup \{e \mid s(e) \in \text{Source}(E)\}$$

For $e \in \Lambda_i^{r(e)}$, let

$$s_{E_r(\mathcal{P})}(e_j) = s(e)_j$$

$$s_{E_r(\mathcal{P})}(e) = s(e)$$

$$r_{E_r(\mathcal{P})}(e_j) = r(e)_i$$

$$r_{E_r(\mathcal{P})}(e) = r(e)_i$$

Graph $E_r(\mathcal{P})$ is called *in-split* of E , and E is called *in-amalgamation* of $E_r(\mathcal{P})$.

Proposition 7.5.14. *Let E be finite graph. Let E have no sinks or sources, and $L_K(E)$ be simple. Let $\mathcal{P}$ be a partition as in Definition 7.5.13, and $E_r(\mathcal{P})$ be in-split graph of E . Then*

$$L_K(E) \sim_M L_K(E_r(\mathcal{P}))$$

Proof. Refer Proposition 1.11 of [14] for detailed proof. □

Definition 7.5.15. *Let $E = (E^0, E^1, r, s)$. Consider all v such that $s^{-1}(v)$ is nonempty. Partition the elements of $s^{-1}(v)$ into disjoint nonempty subsets $\kappa_1^v, \kappa_2^v, \dots, \kappa_{m(v)}^v$. Let this partition be called $\mathcal{P}$. Then graph $E_s(\mathcal{P})$:*

$$E_s(\mathcal{P})^0 = \{v_i \mid v \in E^0, 1 \leq i \leq m(v)\} \cup \{v \mid s^{-1}(v) = \phi\}$$

$$E_s(\mathcal{P})^1 = \{e_j \mid v \in E^0, 1 \leq j \leq m(r(e))\} \cup \{e \mid r(e) \in \text{Sink}(E)\}$$

For $e \in \kappa_i^{s(e)}$, let

$$s_{E_s(\mathcal{P})}(e_j) = s(e)_i$$

$$s_{E_s(\mathcal{P})}(e) = s(e)_i$$

$$r_{E_s(\mathcal{P})}(e_j) = r(e)_j$$

$$r_{E_s(\mathcal{P})}(e) = r(e)$$

Graph $E_s(\mathcal{P})$ is called out-split of E , and E is called out-amalgamation of $E_s(\mathcal{P})$.

Proposition 7.5.16. *Let E be finite graph. Let $\mathcal{P}$ be a partition as in Definition 7.5.15, and $E_s(\mathcal{P})$ be out-split graph of E . Then*

$$L_K(E) \sim_M L_K(E_s(\mathcal{P}))$$

Proof. Refer Proposition 1.14 of [14] for detailed proof. □

Definition 7.5.17. *A graph transformation is called standard if it is expansion, contraction, in-split, in-amalgamation, out-split, or out-amalgamation.*

Lemma 7.5.18. *Let E and F be finite graphs. Let $L_K(E)$ and $L_K(F)$ be purely infinite simple. Let $E \equiv_{\det} F$, $|E^0| = n$, and $|F^0| = m$. Then $\exists$ a sequence of standard graph transformations from E to F .*

Proof. Refer Corollary 1.22 of [14] for detailed proof. □

Theorem 7.5.19. *Let $L_K(E)$ and $L_K(F)$ be purely infinite simple, for finite graphs E and F . Let $E^0 = n$ and $F^0 = m$. Let $E \equiv_{\det} F$. Then $L_K(E)$ and $L_K(F)$ are Morita equivalent to each other.*

$$L_K(E) \sim_M L_K(F)$$

Proof. Let E and F be finite graphs. Let $L_K(E)$ and $L_K(F)$ be purely infinite simple. From Lemma 7.5.7, there $\exists E'$ and F' without sources such that

$$L_K(E) \sim_M L_K(E')$$

$$L_K(F) \sim_M L_K(F')$$

Consider that we get E' from E by removing sources one by one. A source vertex is not part of a cycle. By purely infinite simplicity theorem, $L_K(E')$ is purely infinite simple. Then by Lemma 7.5.8, each graph is $\equiv_{det} E$ in the sequence.

$$E \equiv_{det} E'$$

Similarly

$$F \equiv_{det} F'$$

As $E \equiv_{det} F$, then $E' \equiv_{det} F'$. By Lemma 7.5.18, there exist a sequence of standard graph transformation from E' to F' . By Propositions 7.5.12, 7.5.14 and 7.5.16; each of these transformations preserve Morita equivalence. Therefore

$$L_K(E') \sim_M L_K(F')$$

But $L_K(E) \sim_M L_K(E')$, and $L_K(F) \sim_M L_K(F')$. Hence

$$L_K(E) \sim_M L_K(F)$$

□

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