Non relativistic field theories and Black Holes

A thesis submitted towards partial fulfilment of
BS-MS Dual Degree Programme

by

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under the guidance of

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Certificate

This is to certify that this thesis entitled Non relativistic field theories and Black Holes submitted towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research Pune represents original research carried out by Ashish Kakkar at Indian Institute of Science, Bangalore, under the supervision of Chethan Krishnan during the academic year 2015-2016.

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Declaration

I hereby declare that the matter embodied in the report entitled Non relativistic field theories and Black Holes are the results of the investigations carried out by me at the Centre for High Energy Physics, Indian Institute of Science, Bangalore, under the supervision of Dr. Chethan Krishnan and the same has not been submitted elsewhere for any other degree.

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Abstract

This thesis is dedicated to studying field theories and has three parts. The first two parts are on aspects of non-relativistic field theories. By taking 2 classical field theories namely, Landau's theory of superfluidity and Yang Mills theories whose relativistic versions are very well known, we study features of non relativistic versions of theories. Importantly, we use 2 different ways to approach the problem of studying the non relativistic field theory of a known relativistic field theory.

In case of fluids, we follow the procedure of [1] by studying fluids in one higher dimension in a space time carrying a null killing vector. The non relativistic physics is arrived at by doing a null reduction along this direction. In case of Yang Mills theories, we arrive at the non relativistic equations of motion by sending the speed of light c to infinity in consistent ways. We find that we can do this in more than one way.

The third part of this thesis is on studying certain aspect of black holes in string theory, particularly on the construction of supergravity solutions which could be microstates of certain black holes. We do this by reviewing the works of [2] and [3] and study topics in string theory from [4] and [5].

Contents

1 Null Fluids and Relativistic Fluids **4**
1.1 Some notation for relativistic fluid dynamics 4
1.2 Equilibrium Partition function for relativistic fluids 5
1.2.1 Setup and ideal order equations 5
1.2.2 1st order corrections 8
1.3 Entropy current for relativistic fluids 10
1.3.1 Ideal order 10
1.3.2 Field redefinition invariance in fluids 11
1.4 Non relativistic fluid dynamics from null fluids 14
1.4.1 Galilean algebra as a subgroup of a one higher dimensional Poincare algebra 14
1.4.2 Building fluids with null isometry 15
1.4.3 Setting up Equilibrium and the equilibrium Partition function 16
1.4.4 Light cone reduction 18
1.5 Null superfluids 22

2 Gallilean Yang Mills Theories **23**
2.1 Galilean Conformal Field Theories 23
2.2 Different non relativistic limits of Yang Mills 24
2.3 Conclusion and Discussion 26

3 Black holes and fundamental strings **27**
3.1 Introduction 27
3.2 Low Energy Effective Theory 28
3.3 Strings wrapping S^1 29
3.4 Fundamental string solution 31
3.4.1 Ansatz 32
3.5 Putting Oscillations On The String 33
3.5.1 Solution generating technique 34
3.5.2 Using solution generation for oscillating strings 35

3.6 Making black holes from fundamental strings 38
3.7 Fundamental string NS5 intersection 39
3.8 Conclusion and Future Directions 41

References **42**

Chapter 1

Null Fluids and Relativistic Fluids

Fluid dynamics is a very well studied field theory. Our aim in Chapter 1 is to re derive the ideal order non relativistic fluid constitutive relations and obtain constraints on the fluid constitutive relations. We would be doing this by treating relativistic fluid dynamics as an effective field theory and demanding the existence and local postivity of an entropy current and by contraining the form of an Equilibrium Partition function following [1] and [6]. We would then review the procedure outlined in [7] of obtaining the constitutive relations of non relativistic fluid dynamics by constructing a non relativistic system on null backgrounds, with the null direction being a killing direction for the fluid variables. We will then pose our problem of writing the constitutive relations for non relativistic superfluids by applying the equilibrium partition function method and positivity of entropy current on null fluids and discuss the approach we are taking.

1.1 Some notation for relativistic fluid dynamics

The fluid variables are the velocity field u^μ, temperature $T(x)$ and chemical potential $\mu(x)$. The fluid velocity is normalised to $u^\mu u_\mu = -1$ The fluid equations of motion are

$$\nabla_\mu T^{\mu\nu} = F^{\nu\mu} J_\mu \tag{1.1}$$

$$\nabla_\mu J^\mu = 0 \tag{1.2}$$

General forms of current are:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} + T^{\mu\nu}_{diss} \tag{1.3}$$

$$J^\mu = qu^\mu + J^\mu_{diss} \tag{1.4}$$

1.2 Equilibrium Partition function for relativistic fluids

In this section, we will look at the constrains put by an equilibrium partition function on fluids based on the discussions in [1]. The existence of an equilibrium partition function for fluids in a curved background with electric sources and the form it takes due to symmetries puts stringent constraints on fluid constitutive relations. The physical requirements we will demanding on our system will be:

1. The fluid background is slowly varying and admits an equilibrium solution.

2. We can write down a partition function for the system

3. The stress energy current and charge current can be derived by varying the partition function

Our aim will be to show how these physical requirements let us write down:

1. The correct ideal order constitutive relations

2. Relations between 1st order correction coefficients to the stress energy and charge current.

1.2.1 Setup and ideal order equations

We begin by writing the most general background metric which has a timelike killing vector (we can physically motivate this by anticipating that there is a time-like direction in which our equilibrium configuration does not change).

We take our $d+1$ dimensional back ground metric with $\vec{x}$ being the d spatial coordinates and ∂_t being the time like killing direction to be:

$$ds^2 = -e^{-2\sigma(\vec{x})}(dt^2 - a_i(\vec{x})dx^i)^2 + g_{ij}dx^i dx^j \tag{1.5}$$

The background gauge field in the system is given by:

$$\mathcal{A} = \mathcal{A}_i dx^i + \mathcal{A}_t dx^t \tag{1.6}$$

We want to constrain the form the partition function on this background. We will do this assuming that we working in a background with weakly varying curvature and a temperature field $T(x)$ such that at any point, we have a well defined local temperature $T(x)$ ie. each point is in local thermal equilibrium. By weakly varying, we mean that at each point $T(x)$ admits derivative corrections.

$$T(\vec{x}) = T_0 e^{-\sigma} + ...$$

The generic form of the partition function of a system with a Hamitonian H at temperature T_0 is

$$Z = Tre^{-\frac{H}{T_0}}$$

For our case, we can write the generic form of the partition function as

$$lnZ = \int dx^d \sqrt{g_d} \frac{P(\vec{x})}{T_0} \tag{1.7}$$

where $P(\vec{x})$ is an unknown function which we will later identify to be the local pressure function of the fluid. This approach relies on the fact this form of the partition can be constrained by symmetries: diffeomorphism invariance, $U(1)$ gauge symmetry of the background gauge field, Kaluza Klein invariance (defined in ((1.8))).

As a result, as we will see, the ideal order fluid constitutive relations are determined and identified with known results using thermodynamic identities. More importantly, this constrains the form of derivative corrections admissible to the partition function.

The metric is invariant under the Kaluza Klein transformations, which represent moving along the:

$$t \longrightarrow t + \phi(\vec{x}), x \longrightarrow x \tag{1.8}$$

Under these transformations, as in usual Kaluza Klein transformations, we find that the $0i$ component of the metric transforms like a $U(1)$ gauge transformation if we look at coordinate transformations of the metric.

$$a_i \longrightarrow a_i - \partial_i \phi(\vec{x}) \tag{1.9}$$

We can construct Kaluza Klein invariant combinations of vectors by figuring out how general vectors transform, for eg. $V_i \longrightarrow V_i - \partial_i \phi V_0$ but V^i is invariant. This tells us that $g_{ij} V^i = V_i - a_i V_0$ is invariant. We use such an invariant made form the $\mathcal{A}$ source, ie. we define $A_i = \mathcal{A}_i - a_i\mathcal{A}_0$ and $\mathcal{A}_0 = A_0$. We will write the Partition function as a function A_i instead of $\mathcal{A}_i$.

We will now talk about how to extract physically meaningful quantities from the Partition function.

To get the currents $T^{\mu\nu}$ and J^{μ} which are defined via

$$\delta\mathcal{S} = \int \sqrt{-g}dx^{d+1}(T^{\mu\nu}\delta g_{\mu\nu} + J^{\mu}\delta\mathcal{A}_{\mu}) \tag{1.10}$$

It is clear that that, diffeomorphism invariance and gauge invariance give the current conservation equations.

For our case,

$$\delta\mathcal{W} = \int dx^{d+1}\left(T^{\mu\nu}\delta g_{\mu\nu} + J^{\mu}\delta\mathcal{A}_{\mu}\right) \tag{1.11}$$

Putting our field theory a temperature T_0 by compactifying the euclidean time direction,

$$\delta\mathcal{W} = \int \sqrt{-g}dx^{d}\frac{1}{T_0}\left(T^{\mu\nu}\delta g_{\mu\nu} + J^{\mu}\delta\mathcal{A}_{\mu}\right) \tag{1.12}$$

ie,

$$T_{\mu\nu} = -2T_0\frac{\delta\mathcal{W}}{\delta g^{\mu\nu}} \tag{1.13}$$

$$J_{\mu} = T_0\frac{\delta\mathcal{W}}{\delta\mathcal{A}_{\mu}} \tag{1.14}$$

Our partition function $\mathcal{W}$ is a generic functional of the background fields $\mathcal{W}[e^{\sigma}, a_i, A_0, A_i, \mu_0, g^{ij}]$

Using the form of the metric and $U(1)$ source and writing these in terms of A_i and A_0 instead of $\mathcal{A}_{\mu}$ using chain rule, we get the above equations in terms of components:

$$T_{00} = -2T_0e^{2\sigma}\frac{\delta\mathcal{W}}{\delta\sigma} \tag{1.15}$$

$$T_0^i = T_0\left(\frac{\delta\mathcal{W}}{\delta a_i} - A_0\frac{\delta\mathcal{W}}{\delta A_i}\right) \tag{1.16}$$

$$T_{ij} = -2T_0\frac{\delta\mathcal{W}}{\delta g^{ij}} \tag{1.17}$$

$$J_0 = -2T_0e^{2\sigma}\frac{\delta\mathcal{W}}{\delta A^0} \tag{1.18}$$

$$J^i = T_0 \frac{\delta \mathcal{W}}{\delta A_i} \tag{1.19}$$

Let's recall the general form of $T^{\mu\nu}_{ideal}$ and J^{μ}_{ideal}

$$T^{\mu\nu} = (\rho + \mathcal{P}) u^\mu u^\nu + \mathcal{P} \eta^{\mu\nu} \tag{1.20}$$

$$J^\mu = q u^\mu \tag{1.21}$$

Here ρ and $\mathcal{P}$ are independent and have not yet been identified as thermodynamic quantities. We will now show that this matching the above ((1.20)) and ((1.21)) to the results for $T^{\mu\nu}$ and J^μ from the partition function, gives us relations between ρ and $\mathcal{P}$ and lets us identify $u^\mu = e^{-\sigma}(1,0,0,0)$ as the equilibrium fluid velocity.

We also have to evaluate the partition function at the equilibrium configuration. It has to be a scalar, invariant under Kaluza Klein transformations, and as we are looking at ideal order equilibrium configuration, not involve derivatives of background or fluid variables. This constrains it to be of the form:

$$\mathcal{W} = \int \sqrt{-g} \frac{dx^d}{T_0} P(T_0 e^{-\sigma}, A_0 e^{-\sigma}) \tag{1.22}$$

Now we will do the matching. To ease calculations, we relabel variables $e^{-\sigma} T_0 = a$ and $e^{-\sigma} A_0 = b$ From ((1.15))-((1.19)), applied to ((1.20))and ((1.21)), we get

$$T^{ij} = P g^{ij} \tag{1.23}$$

To get this expression, we used the identity $\delta(\sqrt{-g}) = \frac{1}{2}\sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}$

and the rest of the expressions are:

$$T_{00} = e^{2\sigma}(P - a\partial_a P - b\partial_b P) \tag{1.24}$$

$$J^0 = e^{-\sigma} \partial_b P \tag{1.25}$$

The expressions for T^i_0 and J^i are 0 as the equilibrium partition function is not a function of A_i or a_i.

The expression for J^i tells us that $u^\mu = e^{-\sigma}(1,0,0,0)$. The expression for T^{ij} implies $\mathcal{P} = P$ ie. the function P in the partition function is indeed the pressure function. The expression for T_{00} tells us that $\rho = -P + a\partial_a P + b\partial_b P$.

1.2.2 1st order corrections

Here we will outline the strategy to compute first order corrections to constitutive relations by the equilibrium partition function.

1. Find out the admissible terms we can add to the partition function made of 1 derivatives and which are gauge invariant and diffeomorphism invariant. They are the Chern Simons like terms of the form:

$$\frac{C_0}{2}\int AdA + \frac{T_0^2}{2}\int ada + \frac{T_0 C_2}{2}\int Ada \tag{1.26}$$

2. Apply ((1.15)) to ((1.19)) to the first order corrected partition function with above corrected terms to get expressions for $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}.

3. List all terms that can involve fluid or background variables and carry one derivative, such that they do not vanish at the equilibrium value of the variables. Linear combinations of such tensor variables add to $T^{\mu\nu}_{diss}$ and vectors add to J^{μ}_{diss}

4. Add linear combinations from these variables to fluid variables u^μ, $T(x)$ and $\mu(x)$ to get 1 derivative corrections to u^μ, $T(x)$ and $\mu(x)$. Substitute these into expressions for $T^{\mu\nu}$ and J^μ and absorb the corrections coming from here to $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}.

5. Match components of $T^{\mu\nu}_{diss}$ and J^{μ}_{diss} coming from (2) and (4). This constrains the allowed terms which were added arbitrarily in (2) and gives dependence of terms with each.

1.3 Entropy current for relativistic fluids

Now, we would like to approach the problem of determining fluid constitutive relations again, but from a completely different starting point. Here we will follow the discussions in the non-superfluid part of the paper [8]. Superfluids can also be dealt with using entropy current, which is the major question answered in the paper, but we will not venture into those discussions for now.

To constrain fluid dynamics in an arbitrary curved background, we start with physical assumption that there exists a local entropy current J^μ_S. The second law of thermodynamics says that the entropy of any infinitesimal region of the fluid must increase with time, ie.

$$\partial_\mu J^\mu_S \geq 0 \tag{1.27}$$

This requirement not only fixes the form $T^{\mu\nu}$ at ideal order but also constrains the derivative corrections possible.

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} \tag{1.28}$$

The general form of $T^{\mu\nu}$ including derivative corrections is:

$$T^{\mu\nu} = (\rho + P)u^\mu u^\nu + P\eta^{\mu\nu} + T^{\mu\nu}_{diss} \tag{1.29}$$

To find constrains on the form of $T^{\mu\nu}_{diss}$, the general principle is to add all possible derivative corrections, made from fluid data which are consistent with Lorentz invariance and keep only those terms which are consistent with (1.27).

1.3.1 Ideal order

Let us do a first ideal order check on weather imposing 1.27 as discussed above is consistent with what is known at ideal order. The fluid variables are $u^\mu(x)$,$T(x)$ and $\mu_(x)$ At ideal order, the only possibility to make a Lorentz covariant entropy current without derivatives of fluid data is:

$$J^\mu_S = su^\mu \tag{1.30}$$

where s is the local entropy density. Using the thermodynamic relations,

$$\rho + P = sT + \mu q \tag{1.31}$$

and

$$dP = sdT + qd\mu \tag{1.32}$$

we see that ((1.27)) holds. We can consider this as a non trivial consistency check of our principle of ((1.27)) with known thermodynamics relations.

1.3.2 Field redefinition invariance in fluids

We have in section 1.3, outlined our strategy to look at how the positivity of entropy current constrains the form of $T^{\mu\nu}$. We now add a final ingredient.

Our fluid variables u^{μ},$T(x)$ and $\mu(x)$ describe the fluid at some equilibrium configuration, and at each x we can redefine them such that we stay in the same equilibrium configuration. An obvious subset of these transformations would be changing the reference frame of the fluid ie. $u^{\mu} \longrightarrow u^{\mu} + \delta u^{\mu}$. So, we have the freedom to change u^{μ}, $T(x)$ and $\mu(x)$ in such a way that $T^{\mu\nu}$ and J^{μ} don't change.

Lets look at the local transformation:

$$T(x) \longrightarrow T'(x) = T(x) + \delta T(x) \tag{1.33}$$

$$\mu(x) \longrightarrow \mu'(x) = \mu(x) + \delta\mu(x) \tag{1.34}$$

$$u^{\mu}(x) \longrightarrow u^{\mu\prime}(x) = u^{\mu}(x) + \delta u^{\mu}(x) \tag{1.35}$$

These are 6 transformations. Ensuring the normalisation of u^{μ} stays -1 makes them 5. So, we have 5 extra local transformations. These 5 transformations give physically equivalent $T^{\mu\nu}$ and J^{μ}

As we are interested in looking at the terms which contribute to $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}, we can either fix this invariance by imposing 5 conditions on $T^{\mu\nu}_{diss}$ and J^{μ}_{diss}. For example, imposing $T^{\mu\nu}_{diss}u_{\mu} = 0$ fixes 4 and $J^{\mu}_{diss}u_{\mu}$ fixes 1 of these transformations.

Another way we could approach dealing with this redefinition invariance is, we must add only those combinations of terms to $T^{\mu\nu}_{diss}$ an $J^{\mu\nu}_{diss}$ which are field redefinition invariant.

These combinations can be easily found by looking at what the transformations ((1.33)) do to the general form of the $T^{\mu\nu}$.

As $T^{\mu\nu}$ doesn't change, under field redefinitions,

$$\delta T^{\mu\nu}_{diss} = (u^{\mu}\delta u^{\nu} + u^{\nu}\delta u^{\mu})(\rho + P) + u^{\mu}u^{\nu}dP + \eta^{\mu\nu}dP \tag{1.36}$$

$$\delta J^{\mu}_{diss} = (u^{\mu}dq + q\delta u^{\mu}) \tag{1.37}$$

To construct field redefinition invariant quantities, the trick we use is to construct quantities which are manifestly invariant under

$$u^{\mu}(x) \longrightarrow u^{\mu\prime}(x) = u^{\mu}(x) + \delta u^{\mu}(x)$$

by looking at vectors and tensors projected perpendicular to u^{μ}. This is easily done by projecting vectors and tensors perpendicular to u^{μ} using the projection tensor

$$P^{\mu\nu} = \eta^{\mu\nu} + u^{\mu}u^{\nu}$$

For example the invariant tensor is easy to write as in this case projecting perpendicular to u^μ:

$$P^\mu_\alpha P^\nu_\beta T^{\alpha\beta}_{diss} - \frac{1}{3}P^{\mu\nu}P_{\alpha\beta}T^{\alpha\beta}_{diss} \tag{1.38}$$

To construct vectors, we need to first project perpendicular to u^μ and then ensure invariance under the rest of the transformations of ((1.33)), eg:

$$\delta(P^\mu_\alpha T^{\alpha\beta}_{diss} u_\beta = -(P+\rho)\delta u^\mu$$

and

$$\delta(P^\mu_\alpha J^\alpha_{diss}) = q\delta u^\mu$$

lets us write the invariant vector as

$$P^\mu_\alpha J^\alpha_{diss} + \frac{q}{P+\rho}(P^\mu_\alpha T^{\alpha\beta}_{diss} u_\beta) \tag{1.39}$$

And the invariant scalar can be written as a combination of projected vectors.

$$\delta(P_{\alpha\beta}T^{\alpha\beta}_{diss} = 3dP) \tag{1.40}$$

$$\delta(u_\alpha u_\beta T^{\alpha\beta}) = d\rho \tag{1.41}$$

$$\delta(u_\alpha J^\alpha) = -dq \tag{1.42}$$

So the invariant scalar is:

$$\frac{1}{3}(P_{\alpha\beta}T^{\alpha\beta}) - \frac{\partial P}{\partial \rho}(u_\alpha u_\beta T^{\alpha\beta}) + \frac{\partial P}{\partial q}(u_\alpha J^\alpha) \tag{1.43}$$

Now that have these invariants built out of fluid and background data, our job is almost done. The divergence of the entropy current must be frame invariant. A proof of this is in [9]. As a result, the $\partial_\mu J^\mu_S$ must re arrange itself in the form of frame invariant scalars made from $T^{\mu\nu}_{diss}$ and J^μ_{diss} expanded in a basis of one derivative scalar, vector and tensor data.

1. Independent scalar: $\partial_\mu u^\mu$

2. Independent vectors:$E_\mu = F^{\mu\nu}u_\nu$,$B_\mu = \epsilon_{\mu\nu\alpha\beta}u^\nu F^{\alpha\beta}$ and $V_{1\mu} = \frac{E_\mu}{T} - P^{\mu\nu}\partial_\nu(\frac{\mu}{T})$

3. Independent tensor: $\sigma^{\mu\nu} = \frac{1}{2}P^{\mu\alpha}P^{\nu\beta}(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - P_{\alpha\beta}\nabla_\lambda u^\lambda)$

Divergence of the entropy current gives,

$$\partial_\mu J^\mu_S = -\nabla_\mu \frac{u^\mu}{T}(\frac{1}{3}(P_{\alpha\beta}T^{\alpha\beta}) - \frac{\partial P}{\partial \rho}(u_\alpha u_\beta T^{\alpha\beta}) + \frac{\partial P}{\partial q}(u_\alpha J^\alpha))$$
$$+V_{1\mu}(P^\mu_\alpha J^\alpha_{diss} + \frac{q}{P+\rho}(P^\mu_\alpha T^{\alpha\beta}_{diss} u_\beta)) + \sigma^{\mu\nu}(P^\mu_\alpha P^\nu_\beta T^{\alpha\beta}_{diss} - \frac{1}{3}P^{\mu\nu}P_{\alpha\beta}T^{\alpha\beta}_{diss})$$
$$-T_{\mu\nu\ diss}\ \sigma^{\mu\nu} \tag{1.44}$$

For the expression to always be positive, we must have, the terms in the bracket to make positive squares with the terms outside upto multiplicative constants and these are the 3 sets of constraints on $T^{\mu\nu}$ and J^μ

1.4 Non relativistic fluid dynamics from null fluids

So far, we have reviewed relativistic fluids and techniques which tell us their structure from basic physical principles. We can use these tools to study non relativistic fluids, by setting up a map between the non relativistic fluid theory and a higher dimensional theory "null fluid" theory, by applying these techniques to the "null fluid" theory and then mapping the results back to the non relativistic theory. This section is a review of the paper [7], where these ideas are put forth concretely. We will stress on the process building a map from a relativistic theory to non relativistic theory as this is done differently from a parametric $c \longrightarrow \infty$ limit on the equations of motion as is done in chapter 3 for Yang Mills theories.

There are other approaches to deal with non relativistic fluids in the literature. Newton Cartan formulation of non relativistic curved spaces is used in [10] [11]. We will comment on how these approaches differ from the one taken here but give the same results.

1.4.1 Galilean algebra as a subgroup of a one higher dimensional Poincare algebra

Lets see how we can embed the Galilean group in a one higher dimensional Poincare group.

We will look at a theory, with d spatial directions, a t coordinate and an x^- coordinate, with metric:

$$ds^2 = -2dx^- dt + \vec{dx}.\vec{dx} \tag{1.45}$$

The (d+1) dimensional Galilean algebra sits as a subgroup within this algebra with all its generators commuting with ∂_- The formal way to show this is to write the d+2 dimensional Poincare algebra, perform a change of basis on the generators due to the metric being of the form ((1.45)) instead of the usual Minkowski, and see that ∂_- commutes with all the generators and the rest of the generators form the Galilean algebra.

We will show this in a slightly more intuitive manner, following the discussion in [7]. The Klein Gordon equation in the (d+2) dimensional space is given by:

$$(-2\partial_- \partial_t + (\partial_i)^2)\psi = 0 \tag{1.46}$$

Lets say the ∂_- operator is '-im' where m is the mass. Then the Klein Gordon equation becomes:

$$(2im\partial_t + (\partial_i)^2)\psi = 0 \tag{1.47}$$

which is the free Schrodinger equation. We know that the symmetry algebra of this equation is the Schrodinger algebra which has as a subalgebra the Galilean algebra along with a central extension which is the mass operator.

So, ((1.46)) and ((1.47)) indicate that if we look at a $d+2$ dimensional theory on a background with metric 1.45, the $d+1$ dimensional theory is non relativistic ie. has Schrodinger algebra symmetry. This is true if the mass operator, ∂_- commutes with all other operators in the $d+2$ dimensional theory.

Hence, we can make the following statement: *Theories on this background in $d+2$ dim which have the symmetry:*

$$x^M \longrightarrow x^M + \xi^M(t, \vec{x}) \tag{1.48}$$

where x^M is (x^-, t, x^i), have Galilean symmetry in $d+1$ dim.

So we can study Galilean theories in (d+1) space time dimensions by building them in d+2 dimensions such that they the symmetry ((1.48)) and then compactifying the x^- direction.

We will be building Galilean fluids based on this general principle. We will write the theory for fluids such that is has this symmetry in d+2 dimensions (this is also referred to as a null fluid and the back ground a null background). We will then see that we can apply all the machinery we have reviewed in the previous sections on null fluids and constrain the form of the Partition function to get constrains on the constitutive relations. Dimensional reduction of these relations will let us get constrains on the constitutive relations of Galilean fluids.

1.4.2 Building fluids with null isometry

So, the flat $d+2$ dimensional background on which we should build our theory is

$$ds^2 = -2dx^- dt + \vec{dx}.\vec{dx}$$

Turning on x^- independent fluctuations on this background gives

$$ds^2 = -2e^{-\phi}(dt + a_i dx^i)(dx_- - \mathcal{B}_t dt - \mathcal{B}_i dx^i) + g_{ij}dx^i dx^j \tag{1.49}$$

We can also turn on a background gauge field,

$$\mathcal{A} = \mathcal{A}_t dt + \mathcal{A}_i dx^i \tag{1.50}$$

So, the background fields for a potential Galilean fluid theory which woulld act as sources will be: $[\phi, \mathcal{B}_t, \mathcal{B}_i, g_{ij}, a_i, \mathcal{A}_t, \mathcal{A}_i]$.

We now need to define more precisely what we mean by null curved backgrounds as in the previous section we had only spoken about null flat backgrounds.

For a d+2 dimensional theory, we define a null background as having a null vector $V^M \partial_M$ such that: $\delta_V G^{MN} = \mathcal{L}_V G^{MN} = 0$ and $\delta_V A_M = \mathcal{L}_V \mathcal{A}_M = 0$ and V does not change under covariant transport ie. $\nabla_M V^N = 0$. We also take the component of $\mathcal{A}$ along V to be $-\Lambda_{(V)}$.

Given a null background defined above, we need to define null fluids on this background. To do so we take 3 steps:

1. We first define the fluid theory we are looking at by defining the background fields, the metric G^{MN} and the background $U(1)$ gauge field $\mathcal{A}_M$.

2. We then define the symmetries of the theory, parametrised by the infinitesimal parameters $\xi = \xi^M \partial_M$, $\Lambda_{(\xi)}$ and how they act on the background fields ie. $\delta_\xi G_{MN} = \mathcal{L}_\xi G_{MN}$ (diffeomorphisms) and $\delta_\xi \mathcal{A}_M = \partial_M(\delta_{(\xi)} + \xi^N \mathcal{N}) + \xi^N \partial_{[N} A_{M]}$ (gauge transformations on curved backgrounds).

3. Null fluids are those for which $[\delta_\xi, \delta_V] = 0$. This is the analog of the statement in flat space that ∂_- commutes with all other generators

1.4.3 Setting up Equilibrium and the equilibrium Partition function

To have an equilibrium direction in the theory we should have a timelike $\xi = K$. We can fix a basis $V = \partial_-, \Lambda_{(V)} = 0$ and $K = \partial_t, \Lambda_{(K)} = 0$. In this basis,

$$G_{MN} = \begin{bmatrix} 0 & -e^{-\phi} & -e^{-\phi} a_i \\ -e^{-\phi} & e^{-\phi} B_t & e^{-\phi}(B_i + a_i) \\ -e^{-\phi} a_j & e^{-\phi}(B_i + a_i) & g_{ij} + a_i B_j e^{-\phi} \end{bmatrix} \tag{1.51}$$

Here too, we can get T^{MN} by assuming the existence of an equilibrium partition function and varying it with respect to sources as we did in (1.2), variation of the Partition function has the general form,

$$\delta \mathcal{W} = \int dx^M \sqrt{-G}(\frac{1}{2}T^{MN}\delta G_{MN} + J^M \delta \mathcal{A}_M) \tag{1.52}$$

We now have to compactify the x^- direction whose radius of compactification is called $\tilde{R}$ as well as the euclidean time direction whose radius is $\tilde{\beta}$. Defining $\tilde{\vartheta} = \frac{1}{\tilde{\beta}\tilde{R}}$. The equilibrium temperature is $\vartheta_0 = \tilde{\vartheta}e^{\phi}$ just like in section (1.2) because we are looking at the first term in a derivative expansion of the temperature field. We will not repeat the discussions of section (1.2), except that the partition function now a functional of 3 background scalars instead of 2 and the

Some redefinitions of the scalars are required: $\varpi_0 = \frac{\mathcal{B}_t}{\tilde{\vartheta}}$ and $\nu_0 = \frac{\mathcal{A}_t}{\tilde{\vartheta}}$ and we can write the partition function as:

$$\delta \mathcal{W} = \int dx^i \sqrt{g}\tilde{\vartheta}(\frac{1}{2}T^{MN}\delta G_{MN} + J^M \delta \mathcal{A}_M) \tag{1.53}$$

As before, we work with Kaluza Klein invariant combinations of the gauge fields by defining: $A_i = \mathcal{A}_i - a_i\mathcal{A}_t$ and $B_i = \mathcal{B}_i - a_i\mathcal{B}_t$.

working out variations by using the explicit forms of the metric and gauge field gives us the following equations to determine the T^{MN} and J^M

$$\frac{T^{ij}}{2\vartheta_0} = \frac{\delta W}{\delta g_{ij}} \tag{1.54}$$

$$T_{--} = \frac{\delta W}{\delta \varpi_0} \tag{1.55}$$

$$\frac{T^i_0}{\vartheta_0} = -\frac{\delta W}{\delta B_i} \tag{1.56}$$

$$T_{t-} = e^{-\phi}\vartheta_0^2\frac{\delta W}{\delta \vartheta_0} - \mathcal{B}\frac{\delta W}{\delta \varpi_0} \tag{1.57}$$

$$\frac{T^i_t}{\vartheta_0} = \frac{\delta W}{\delta a_i} - \mathcal{A}_t\frac{\delta W}{\delta A_i} \tag{1.58}$$

$$J_- = \frac{\delta W}{\delta \nu_0} \tag{1.59}$$

$$\frac{J^i}{\vartheta_0} = \frac{\delta W}{\delta A_i} \tag{1.60}$$

These equations will be very useful for us later while calculating the currents for a null superfluid.

1.4.4 Light cone reduction

Null decomposition All the quantities calculated till now have been quantities on $\mathcal{M}_{d+2}$, the $d+2$ dimensional manifold on which the null fluid is defined. We need to dimensionally reduce the V direction to get physical quantities in terms of coordinates on $\mathcal{M}_{d+1}$. This is a subtle procedure because the decomposition of space with a null direction cannot be done uniquely. A null vector is transverse to itself, so we cannot do the conventional decomposition as a vector $\times$ transverse directions. To do this, we need to introduce a time-like vector (which will correspond to the time direction) $T = T^M \partial_M$. Now, we can define a direction orthogonal to V using this T (by a Gram-Schmidt orthogonalisation) as:

$$\bar{V}^M_{(T)} = \frac{-1}{T^N V_N}(T^M - \frac{T_R T^R V^M}{2T^S V_S})$$

Using this definition, we can see that $\bar{V}^M_{(T)}$ is null and orthogonal to V.

This lets decompose the metric into these 2 directions and the transverse direction.

$$G^{MN} = P_T^{MN} - 2\bar{V}^{(M}_{(T)} V^{N)} \tag{1.61}$$

$\mathcal{M}_d$ is spanned by $\psi_M P^{MN}$ where ψ_M is a vector in $\mathcal{M}_{d+2}$. This is the decomposition $\mathcal{M}_{d+2} = S^1_V \times \mathbb{R}_T \times \mathcal{M}_d$. We will now show how Milne transformations (which arise in Newton Cartan structures) also arise via this procedure of decomposing the metric.

Newton Cartan structures from null decomposition We will now show that Newton Cartan structures arise if we pick a basis for the null decomposition described above. $V = \partial_-$ and the coordinates as $x^M = x^-, x^\mu$.

$$V^M = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$V_M = \begin{bmatrix} 0 \\ -n^\mu \end{bmatrix}$$

$$\bar{V}^M_{(T)} = \begin{bmatrix} v^\mu B_\mu \\ v^\mu \end{bmatrix}$$

$$\bar{V}_{M(T)} = \begin{bmatrix} -1 \\ B_\mu \end{bmatrix}$$

$$P_{(T)MN} = \begin{bmatrix} 0 & 0 \\ 0 & p_{\mu\nu} \end{bmatrix} \tag{1.62}$$

We see that given that $n^\mu v_\mu = 1$, $v^\mu p_{\mu\nu} = 0$, $n_\mu p^{\mu\nu} = 0$, $p^{\mu\rho} p_{\rho\mu} + n^\mu v_\nu = \delta^\mu_\nu$ the definition of $\bar{V}^{(M}_{(T)}$ is satisfied and from the last relation, P^{MN} can be seen to be the transverse metric.

This is what is called a Newton Cartan structure [cite Kristen Jensen]. We have shown that it arises by picking a basis in our Null background. It can also be constructed independently, without reference to null back grounds [cite Kristen Jensen]. This construction consists of splitting a degenerate space time metric $g^{\mu\nu}$ (as the metric generates in the non relativistic limit) into closed one form n_μ which defines the local time direction and the transverse spatial metric $p^{\mu\nu}$ such that $n_\mu p^{\mu\nu} = 0$ and $g^{\mu\nu} = n^\mu n^\nu + p^{\mu\nu}$

We will now show an important advantage of constructing theories on null backgrounds compared to working with Newton Cartan structures. While studying non relativistic theories on Newton Cartan manifolds, Milne transformations are demanded as symmetries of the theory [cite Jenen]. In case of theories on null backgrounds, they arise as properties of the null background itself, and hence theories on this background are automatically Milne invariant.

We will first show how the arise in null background theories. Null theories have a symmetry in the definition of the T field. Under the transformation:

$$T^M \longrightarrow T^M - T^N V_N \psi^M \tag{1.63}$$

where $\psi^M V_M = 0$,

$$\bar{V}_{M(T)} \longrightarrow \bar{V}_{M(T)} + \bar{\psi}^M + \frac{1}{2}\bar{\psi}^2 V^M \tag{1.64}$$

$$P^{MN}_{(T)} \longrightarrow P^{MN}_{(T)} + 2V^{(M}\bar{\psi}^{N)} + \bar{\psi}^2 V^M V^N \tag{1.65}$$

These transformations written in the Newton Cartan basis are given by:

$$v^\mu \longrightarrow v^\mu + \bar{\psi}^\mu \tag{1.66}$$

$$B_\mu \longrightarrow B_\mu + \bar{\psi}_\mu - \frac{1}{2} n_\mu \bar{\psi}^2 \tag{1.67}$$

$$p_{\mu\nu} \longrightarrow p_{\mu\nu} - 2n_{(\mu}\bar{\psi}_{\nu)} + n_\mu n_\nu \bar{\psi}^2 \tag{1.68}$$

which are the Milne boosts in Newton Cartan structures [Jensen].

These Milne boosts can be introduced without reference to null backgrounds as a freedom in adding a vector along the transverse directions to v^μ. This non uniqueness corresponds to the choice of different reference frames we have in Galilean physics.

Reduction of currents on null backgrounds Now that we have the null background metric, we want to look at the reduction of the currents T^{MN} and J^M. This would let study the variation of the partition function with these reduced currents and make contact with non relativistic fluids.

We can decompose ie. look at components along linearly independent directions of the currents T^{MN} and J^M on $\mathcal{M} = S^1_V \times \mathbb{R}^1_T \times \mathcal{M}^T_{(d)}$ as:

$$T^{MN} = \rho \bar{V}^M_{(T)} \bar{V}^N_{(T)} + 2\epsilon_{tot} V^{(M}\bar{V}^{N)}_{(T)} + 2j^{(M}_\rho \bar{V}^{N)}_{(T)} + 2j^{(M}_\epsilon V^{N)}_{(T)} + t^{MN} + \theta V^M V^N$$

$$J^M = q\bar{V}^M_{(T)} + j^M_q + \theta' V^M \tag{1.69}$$

θ and θ' are not physically important terms as the variation of the current will give $V\delta V = 0$ via the null condition.

The scalars ρ, ϵ_{tot} and q will be interpreted as mass density, energy density and charge density, the vectors j^μ_ρ, j^μ_ϵ, j^μ_q as mass current, energy current and charge current and $t^{\mu\nu}$ as the stress energy tensor.

We can now reduce the variation of the partition function:

$$\delta\mathcal{W} = \int \sqrt{-G} dx^{d+2} (T^{MN}\delta G_{MN} + J^M \delta \mathcal{A}_M) \tag{1.70}$$

now becomes (using the decomposition of the currents and the null background):

$$\begin{aligned} \delta\mathcal{W} \;\; = \;\; & \int \sqrt{-G} dx^{d+2} [(\epsilon_{tot}\bar{V}^M_{(T)} + j^M_\epsilon)\delta V_M + (\rho \bar{V}^M_{(T)} + j^M_\rho)\delta \bar{V}_{(T)M} \quad (1.71) \\ & + (j^M_\rho \bar{V}^N_{(T)} + \frac{1}{2} t^{MN})\delta P_{(T)MN} + (q\bar{V}^M_{(T)} + j^M_q)\delta\mathcal{A}_M] \end{aligned}$$

This variation in the Newton Cartan basis is:

$$\begin{aligned} \delta\mathcal{W} &= \int \sqrt{-G}dx^{d+2}[(\epsilon_{tot}v^{\mu}+j^{\mu}_{\epsilon})\delta n_{\mu}+(\rho v^{\mu}+j^{\mu}_{\rho})\delta B_{\mu} \\ &+(p^{\mu}v^{\nu}+\frac{1}{2}t^{\mu\nu})\delta p_{\mu\nu}+(qv^{\mu}+j^{\mu}_{q})\delta\mathcal{A}_{\mu}] \end{aligned} \tag{1.72}$$

Now that we have the variation of the partition function, the ward identities corresponding to the symmetries of the reduced theory can be found [7].

1.5 Null superfluids

Relativstic superfluids

Superfluids we will study will be on the same background as the charged fluids we were studying except that they break the $U(1)$ gauge symmetry. The equilibrium partition function technique can be used to constrain the constitutive relations of relativistic superfluids upto first order in derivative expansion. We will outline some of the notation and setup outlined there.

The condensed scalar ϕ develops a vacuum expectation value which leads to spontaneous symmetry breaking. It gives us a one parameter set of background equilibrium configurations. If ϕ is the phase of the scalar condensate, under a gauge transformation: $\mathcal{A}_i \longrightarrow \mathcal{A}_i + \partial_i \alpha$, $\phi \longrightarrow \phi + \alpha$.

The gauge invariant combination of these is:

$$\xi_i = \partial_i \phi + \mathcal{A}_i \tag{1.73}$$

We also define $\xi_0 = \mathcal{A}_0$ We must also construct Kaluza Klein invariant combinations of ξ to use in the equilibrium partition function, as discussed in the ordinary fluid case. $\varsigma_i = \xi_i - a_i A_0$

$\chi = -\xi^2$

The equilibrium partition function, $\mathcal{W}_{eqb}$ can be written down and its form constrained.

Construction of null superfluids

We would now like to discuss some aspects of the problem we are currently working on using the techniques reviewed in the previous sections. As these calculations are ongoing, we will outline them here for now.

We follow the following steps:

- We assume the existence of a partition function on the null background and let it depend on the super-fluid $\chi = \xi_i^2$ where x_i is the superfluid velocity.
- We vary the partition function as is done in 1.54 evaluating it at ideal order and get expressions for values of T^{MN} and J^M.
- We match this expression with the most general T^{MN} and J^M we can construct at ideal order which is compatible with positivity of entropy current.

Chapter 2

Gallilean Yang Mills Theories

Having looked at fluid dynamics and a method of studying non relativistic fluid dynamics, we will in this chapter look at a different classical relativistic field theory again, and study it's non relativistic dynamics. Yang Mills is a very important theory to study because it governs the Standard Model. In this chapter, we will look at the non relativistic limit of free Yang Mills theory, and study the symmetries of its equations of motion. We will also look at some general properties of Galilean Conformal Field Theories and study how they arise as limits of relativistic theories. We will be presenting some of the results and discussions of [12].

2.1 Galilean Conformal Field Theories

The Galilean Conformal Algebra (GCA) is the Inounu Wigner contraction of the conformal algebra. The Inounu Wigner method parametrically contracts the generators of an algebra, to give a new algebra. The most common example of an Inounu Wigner contraction is the contraction of the Poincare algebra to give the Galilean algebra. The GCA arises as a contraction of the conformal group [13], as the contraction parameter here becomes the speed of light c, and we get the GCA as $c \longrightarrow \infty$. This is implemented by working in units in which c is 1 and implenting the contraction on space time as $x \longrightarrow \epsilon x$ and $t \longrightarrow t$ with $\epsilon = \frac{x}{t} \longrightarrow 0$. The upside of this approach is, we can write down the representation of the algebra on fields, and take the space-time contraction immediately.

The conformal group generators represented on fields are :

$$P_\mu = \partial_\mu \ , K_\mu = -(2x_\mu x_\nu \partial^\nu - x^\nu x_\nu \partial_\mu), M_{\mu\nu} = -(x_\mu \partial_\nu - x_\nu \partial_\mu) \ , D = -x^\mu \partial_\mu \quad (2.1)$$

where P_μ is the momentum generator, K_μ is the special conformal transformation generator, $M_{\mu\nu}$ is the boost and rotation generator and D is the Dilatation generator.

Contracting the algebra by taking $\epsilon = \frac{x}{t} \longrightarrow 0$ gives us the GCA. The interesting observation [13] is that about the finite-GCA algebra, can be written as:

$$[L^{(n)}, L^{(m)}] = (n-m)\,L^{(n+m)}, \quad [L^{(n)}, M_i^{(m)}] = (n-m)\;M_i^{(n+m)}, \quad [M_i^{(n)}, M_j^{(m)}] = 0$$
$$[J_{ij}, J_{kl}] = \delta_{k[i}J_{j]l} - \delta_{l[i}J_{j]k}, \quad [L^{(n)}, J_{ij}] = 0, \quad [M_i^{(n)}, J_{jk}] = M_{[k}^{(n)}\delta_{j]i}. \qquad (2.2)$$

where $L^{(-1,0,1)} = H, D, K$ and $M_i^{(-1,0,1)} = P_i, B_i, K_i$ .H, D and K are the Galilean hamiltonian, Dilatation and scalar special conformal transformation. P_i, B_i and K_i represent momentum, Galilean boost and vector special conformal transformation. J_{ij} is the rotation generator. And here the algebra continues to hold for $-1 > n$ and $n > 1$. So we get an infinite dimensional algebra even though the algebra we contracted is infinite dimensional only in $d = 2$.

The interesting result we have in [12] is that Galilean Yang Mills Theory is invariant under (2.2) and moreover arises as a representation of the algebra when we label it with D and J_{ij} as Casimirs.

2.2 Different non relativistic limits of Yang Mills

Let us look at the transformation law for 4-vectors:

$$u'_0 = u_0 - \frac{v_i}{c}u_i \qquad (2.3)$$

$$u'_i = u_i - \frac{v_i}{c}u_0 \qquad (2.4)$$

Where we have put γ to 1 (ie. ignored terms of order (v/c^2))

We take non relativistic limits by sending $c \to \infty$ or equivalently calling $1/c = \epsilon$ and $\epsilon \to 0$ So this is just,

$$u'_0 = u_0 - \epsilon v_i u_i \qquad (2.5)$$

$$u'_i = u_i - \epsilon v_i u_0 \qquad (2.6)$$

It is clear that unless u_0 or u_i scale as $u_0 \to \epsilon u_0$ or $u_i \to \epsilon u_i$, taking ϵ to 0 gives us a trivial transformation in (2.5) and (2.6) ie. take non relativistic limit of the Lorentz transformations.

As as example, lets look at $u_0 = ct$ and $u_i = x_i$, Taking the non relativistic limit forces us to take $u_i = \epsilon x_i$ gives us the usual Galilean transformation of coordinates.

$$t'_0 = t_0 \tag{2.7}$$

$$\epsilon x'_i = \epsilon x_i - \epsilon v_i t_0 \tag{2.8}$$

If we are looking at Yang Mills with an $SU(N)$ gauge group, we will have $N^2 - 1$ gauge fields A^a_μ. Each of these is a 4-vector, so according to our discussion above, A^a_0 and A^a_i components of each of the $N^2 - 1$ gauge fields will scale in 1 of 2 ways described above to have consistent Galilean limit.

For concreteness, we look at the possible non relativistic limits we can have with the gauge group being $SU(2)$, ie, 3 gauge fields:

Magnetic limit

$$A^1_0 \longrightarrow \epsilon A^1_0 \qquad A^1_i \longrightarrow A^1_i$$
$$A^2_0 \longrightarrow \epsilon A^2_0 \qquad A^2_i \longrightarrow A^2_i$$
$$A^3_0 \longrightarrow \epsilon A^3_0 \qquad A^3_i \longrightarrow A^3_i$$

Skewed limit 1

$$A^1_0 \longrightarrow \epsilon A^1_0 \qquad A^1_i \longrightarrow A^1_i$$
$$A^2_0 \longrightarrow \epsilon A^2_0 \qquad A^2_i \longrightarrow A^2_i$$
$$A^3_0 \longrightarrow A^3_0 \qquad A^3_i \longrightarrow \epsilon A^3_i$$

Skewed limit 2

$$A^1_0 \longrightarrow \epsilon A^1_0 \qquad A^1_i \longrightarrow A^1_i$$
$$A^2_0 \longrightarrow A^2_0 \qquad A^2_i \longrightarrow \epsilon A^2_i$$
$$A^3_0 \longrightarrow A^3_0 \qquad A^3_i \longrightarrow \epsilon A^3_i$$

Electric limit

$$A^1_0 \longrightarrow A^1_0 \qquad A^1_i \longrightarrow \epsilon A^1_i$$
$$A^2_0 \longrightarrow A^2_0 \qquad A^2_i \longrightarrow \epsilon A^2_i$$
$$A^3_0 \longrightarrow A^3_0 \qquad A^3_i \longrightarrow \epsilon A^3_i$$

2.3 Conclusion and Discussion

We demonstrated how there is more than one consistent way of looking at the non relativistic limit of Yang Mills theory woth gauge group $SU(2)$. In the paper [12], the authors show how to generalise this to $SU(N)$. Furthermore, we can look at the equations of motion in each of these possible limits. It turns out that the equations of motion in all possible limits for general $SU(N)$ gauge fields, are invariant under the infinite dimensional GCA. This is an interesting result because more symmetry constrains the theory much more. The expectation is that this large amount of symmetry in $\mathcal{N} = 4$ supersymmetric Yang Mills will give us an integrable subsector of the gauge-gravity correspondence. Studying non relativistic Yang Mills is small step in this direction.

Chapter 3

Black holes and fundamental strings

3.1 Introduction

The work of Bekenstein and Hawking showed that black holes have entropy and that this entropy is proportional to the area of the horizon. Since then, there has been a programme to explain the microscopic origin of this entropy, ie. to reproduce this entropy from an explicit counting of microstates of the black hole.

Recently, the fuzzball programme has included an attempt to construct smooth horizonless solutions of supergravity to account for the microscopic structure of black holes, see [14] and references within. A programme [15] [16] and references, has been used to construct all possible horizonless solutions of supergravity with the same mass and charges as certain black holes and attempt to calculate the entropy coming from these microstates to see if they can account for the complete entropy of these black holes.

We will in this chapter, try to understand, via a specific example, how these microstate solutions are constructed. We also have another motivation to pick the particular example of solution we want to construct.

States carrying momentum and winding in a system of k NS5-branes wrapping $T^4 \times S^1$ in Type II string theory have a black hole description when the branes are coincident. But when the branes are separated, it has recently been argued in [17] that the system undergoes a black hole/string transition and the elliptic genus and the entropy change discontinuously. In light of this, we construct smooth, horizonless supergravity solutions corresponding to fundamental strings carrying momentum and winding in the background of the separated NS5-branes. These are *not* microstates of the black hole by the above argument, but they have the same global charges as the black hole. This would demonstrate that identifying microstates purely via supergravity constructions can be misleading.

The plan of the chapter is to first review how to couple strings to low energy effective actions in string theory. This is followed by reviewing Dabholkar and Harvey's [2] construction of the fundamental string solution and their subsequent paper on oscillating fundamental strings [3] and some discussions on Vashaspati and Garfinkle's solution generating mechanism of putting oscillations on static fundamental string solutions [18]. Following the methods in these discussions, we will write try to write the metric for an oscillating fundamental string in $NS5$ background.

3.2 Low Energy Effective Theory

We know that the perturbative spectrum of bosonic strings at level one consists of a tachyon and at level 2 consists of massless representations of $SO(D-2)$. The massless fields are a scalar field (dilaton), a symmetric rank 2 tensor ($G_{\mu\nu}$) and an antisymmetric rank 2 tensor ($B_{\mu\nu}$).

A natural question to ask is what would be the motion of strings in a background given by these fields. While answering this question, we are looking at an effective theory because we are only looking at how strings propagate and back-react with a background made of massless excitations.

Our notion of back-reacting is that we're looking at motion of the string in a curved background instead of a flat background and we let the string source these background fields.

$$L = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{mn}G_{\mu\nu}(X)\partial_m X^\mu\partial_n X^\nu$$

where X_μ are target space coordinates and g_{mn} is the world sheet metric. A theory governed by this Lagrangian is called a non-linear sigma model.

A natural question to ask at this point, is how is $G_{\mu\nu}$ which we seemingly put in by hand, the same as the $G_{\mu\nu}$ coming from the perturbative spectrum of the string we referred to earlier. To see this, we follow the discussion in Tong [4]. We look at the weak field expansion of $G_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$.

Let's compute the partition function Z for a string in this background.

$$Z = \int DXDge^{-S_p - V}$$

$$Z = \int DXDge^{-S_p}(1 - V - \frac{V^2}{2!} + ...)$$

where

$$S_p = \int d^2\sigma \sqrt{g} g^{mn} \partial_m X^\mu \partial_n X^\nu$$

is the flat space Polyakov action and where V is given by

$$V = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{mn} h_{\mu\nu}(X) \partial_m X^\mu \partial_n X^\nu$$

Now as a gravitational perturbation, $h_{\mu\nu}$ has a propagating wave solution of the form $\xi_{\mu\nu} e^{iP.X}$ where $\xi_{\mu\nu}$ is the polarisation tensor. So in effect for weak fields we're doing a path integral

$$\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} \xi_{\mu\nu} e^{iP.X} \partial_m X^\mu \partial_n X^\nu$$

which lets us identify it as a vertex operator insertion for gravitons with the polarization given by $\xi_{\mu\nu}$. We can make sense of this by realising that we can make the identification:

$$\int d^2\sigma \sqrt{g} g^{mn} : \partial_m X \partial_n \bar{X} e^{(iP.X)} := (\alpha^\mu_{-1} \bar{\alpha}^\nu_{-1} + \alpha^\nu_{-1} \bar{\alpha}^\mu_{-1})|0>$$

Here, we had only coupled the string to the background metric. We can couple the string to the antisymmetric 2 tensor $B_{\mu\nu}$ which arises as a part of the massless perturbative spectrum. We do so in a way which keeps reparametrisation invariance and Weyl invariance on the world sheet.

$$S = \frac{1}{4\pi\alpha'} \int (\partial_m X^\mu \partial^m X^\nu G_{\mu\nu}(X) + \epsilon^{mn} \partial_m X^\mu \partial^m X^\nu B_{\mu\nu}(X))$$

The 2 form $B_{\mu\nu}$ is gauge invariant under $B_{\mu\nu} \longrightarrow B_{\mu\nu} + \partial_\mu C_\nu - \partial_\nu C_\mu$ and it is useful to introduce the 3 form $H = dB$.

3.3 Strings wrapping S^1

We will now describe the motivation given by [2] to study a low energy macroscopic string solution to these equations. We consider a string on $\mathbb{R}^9 \times S^1$ where the radius R of S^1 is large compared to the string scale. We will take this as an opportunity to study some features of the perturbative spectrum of closed bosonic strings wrapping around S^1 following the discussion in [5].

The perturbative spectrum of such a string consists of a tower of states labelled by the winding number n and the quantized momentum in the compact direction m/R. The quantization of the momentum is a frequently encountered effect in field theory upon compactification of a dimension. Here we argue that $e^{2\pi iRp}$ (where p is the momentum operator) must leave the state invariant which implies $p = m/R$.

The winding number is a stringy effect which arises because there may exist an integer n such that

$$X(\sigma + 2\pi) = X(\sigma) + 2\pi Rn$$

which means the string comes back to itself after winding the compact direction n times. World sheet momentum p comes from the energy momentum tensor by Noether's procedure,

$$p = \frac{1}{2\pi\alpha'}\int dz\partial X - \bar{z}\bar{\partial}X$$

Change in x going around the string comes from,

$$2\pi Rn = \int dz\partial X + \bar{z}\bar{\partial}X$$

On expanding in terms of oscillator modes,

$$\partial X(z) = -i\sqrt{\frac{\alpha'}{2}}\sum_{-\infty}^{\infty}\alpha_m z^{-m-1}$$

$$\bar{\partial}X(\bar{z}) = -i\sqrt{\alpha'2}\sum_{-\infty}^{\infty}\bar{\alpha}_m\bar{z}^{-m-1}$$

only the 0 mode contribution survive,

$$2\pi Rn = 2\pi\sqrt{\frac{\alpha'}{2}}(\alpha_0 - \bar{\alpha_0})$$

$$p = \sqrt{\frac{1}{2\alpha'}}(\alpha_0 + \bar{\alpha_0})$$

Inverting these linear equations gives,

$$\sqrt{\frac{2}{\alpha'}}\alpha_0 = p_L = \sqrt{\frac{2}{\alpha'}}\left(\frac{m}{R} + \frac{nR}{\alpha'}\right)$$

$$\sqrt{\frac{2}{\alpha'}}\bar{\alpha}_0 = p_R = \sqrt{\frac{2}{\alpha'}}\left(\frac{m}{R} - \frac{nR}{\alpha'}\right)$$

We see that $\bar{\alpha}_0 \neq \alpha_0$ in the compact directions due to the winding n. For the non compact directions, $\bar{\alpha}_0 = \alpha_0$ So, the mass shell condition becomes:

$$m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{(\alpha')^2} + \frac{2}{\alpha'}(N + \bar{N} - 2)$$

This gives us the spectrum of arbitrary left moving and right moving oscillations in the bosonic case.

[2] argue that when right moving oscillations are in the ground state but there are an arbitrary number of left moving oscillations subject to the mass shell condition, half the supersymmetry is broken and we get a $\frac{1}{2}$BPS state.

The number of supersymmetries broken by a state if it is a BPS state, is a quantity that would not change if we went to the low energy regime. So, it makes sense to look for half BPS objects in the low energy effective theory which would correspond to the fundamental string solution we are looking for. Also, BPS objects are usually solitonic objects, for example BPS monopoles in gauge theories [ref].

We also find that the fundamental string has an ADM mass $\frac{n}{2\pi\alpha'}$, where n corresponds to the winding number of the corresponding perturbative state. We will perform the ADM momentum calculation (which is similar in spirit) for the oscillating string in Section [ref]. They are localized objects whose mass varies inversely with the string coupling. So we are probing a non-perturbative regime of our theory.

3.4 Fundamental string solution

We will now closely follow the discussion in the paper [2]. The aim of the section is to study a macroscopic string wound around S^1 on $\mathcal{R}^9 \times S^1$ in the presence fields $G_{\mu\nu}, B_{\mu\nu}, \phi$. The Lagrangian for the massless fields is

$$L_B = R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{12}e^{-2\alpha\phi}H^2$$

We motivated how strings couple to the background in the previous section. Following this, the Lagrangian for the string is:

$$L_\sigma = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{mn}G_{\mu\nu}(X)\partial_m X^\mu \partial_n X^\nu + \epsilon^{mn}\partial_m X^\mu \partial_n X^\nu B_{\mu\nu}$$

The low energy effective theory of strings propagating in this background L_B is given by:

$$L = L_B + L_\sigma$$

We do not show how the Lagrangian L_B for the fields in the low energy limit is arrived at. The method is similar to the Kaluza Klein reduction of pure Einstein gravity to give the electromagnetic Lagrangian coupled to gravity and a dilaton field [5].

The equations of motion from varying the fields are:

$$\nabla_\mu(e^{-2\phi}H^{\mu\nu\rho}) = \int d^2\sigma\epsilon^{mn}\partial_m X^\nu \partial_n X^\rho \delta(x - X(\sigma)) \quad (3.1)$$

$$R^{\mu\nu} + 2\nabla^\mu\nabla^\nu\phi - \frac{1}{4}H^{\mu\nu\rho}H^\nu_{\rho\sigma} = \int d^2\sigma\sqrt{g}g^{mn}\partial_m X^\nu \partial_n X^\rho \delta(x - X(\sigma)) \quad (3.2)$$

$$4\nabla^2\phi - 4(\nabla\phi)^2 + R - \frac{1}{12}H^2 = 0 \quad (3.3)$$

The equations of motion form varying the string coordinates are:

$$\nabla_m(g^{mn}\nabla_n X^\mu) = -\Gamma^\mu_{\nu\rho}\partial_m X^\nu \partial_n X^\rho \gamma^{mn} + H^\mu_{\nu\rho}\partial_m X^\nu \partial_n X^\rho \epsilon^{mn} \quad (3.4)$$

We will use this string equation of motion to good effect while finding the net force on a test string in Sec [ref].

3.4.1 Ansatz

In [], a very simple ansatz dependent on only one function was proposed for these equations of motion. We have already seen why we should expect such a solitonic solution in Sec [ref]. The ansatz is

$$ds^2 = -e^{2\phi}(dt^2 - (dx^1)^2) + d\vec{x}.d\vec{x} \quad (3.5)$$

$$B_{uv} = \frac{1}{2}(e^{2\phi} - 1) \quad (3.6)$$

The Einstein equation for this ansatz gives,

$$\partial^2(e^{-2\phi}) = -\frac{\kappa^2}{\pi\alpha'}\delta^D(x - X(\sigma)) \quad (3.7)$$

which gives(for $D > 4$),

$$e^{-2\phi} = 1 + \frac{Q}{r^{D-4}} \quad (3.8)$$

where $Q = \frac{\kappa^2\mu}{\Omega_{D-3}}$.

No force condition

We are claiming that the fundamental string solution is a $\frac{1}{2}$ BPS object. A characteristic feature of BPS objects is that there is no net force between them. For example, in case of magnetic monopoles, the electric and magnetic forces exactly balance each other when two monopoles are placed near each other. We will now show that the net force on a static test string is 0 following the discussion of [2]. Let us look at a stationary test fundamental string placed at a fixed distance from the origin parallel to the fundamental string located at the origin. The field due to the fundamental string will be given by the metric in Eq. (3.5) and working in the string frame ensures that the gravitational force and the force due to the dilaton are exerted. Fixing conformal gauge for the test string, gives us $X^0 = \tau$, $X^1 = \sigma$ and $g_{mn} = (-1, +1)$. Force in the transverse direction experienced by the string is given by the string equation of motion [2],

$$\partial_\tau^2 X^i = 2\Gamma^i_{01} + H^i_{01} \tag{3.9}$$

The $\partial_\sigma X^i$ term vanishes due to the position of the string. That the equality in Eq. (3.9) holds is easy to see on plugging in our ansatz Eq. (3.5).

[2] goes on to give an explicit calculation to show that this fundamental string solution is a $\frac{1}{2}BPS$ object by looking at the supersymmetry variation of fermionic fields.

3.5 Putting Oscillations On The String

We followed the discussion in [2], that the perturbative spectrum had states which preserved $\frac{1}{2}$ of the supersymmetry, and these could be mapped to the fundamental string solution which preserved the same fraction of SUSY (we did not prove this, but [2] show that the fundamental string solution is $\frac{1}{2}BPS$), we merely looked at the no force condition between 2 static fundamental strings).

We will now review discussions in [3], where they look for solutions in the low energy theory which preserve a smaller fraction of SUSY ($\frac{1}{4}$) because these states are known to exist in the perturbative regime of the string [3]. They expect to get a solution of this kind by breaking just the right amount of SUSY, which the propose to do by examining what happens to the fundamental string solution if we put oscillations on it.

3.5.1 Solution generating technique

We now discuss [18] where the authors describe a solution generating method for the fields due to an oscillating string, given the fields due to a static string. Though the authors developed the technique to apply to gravitating cosmic strings, it was efficiently used by DGHW [2] to construct the oscillating fundamental string solution.

To give an overview of their technique, we first describe the flat space case discussed in [18]. Here they start with the fields produced by a static string $\phi(x)$ and $A_\mu(x)$, which obey the following equations of motion.

$$D_\mu D^\mu \phi = \frac{1}{2}\lambda(|\phi|^2 - \eta^2)\phi \tag{3.10}$$

$$\nabla_\mu F^{\mu\nu} = \frac{ie}{2}(\phi^\star \nabla^\nu \phi - \phi \nabla^\nu \phi^\star) \tag{3.11}$$

We now define the coordinates,

$$u = t + z$$

$$v = t - z$$

And the new coordinates,

$$x \longrightarrow x' = x - f(u)$$

$$y \longrightarrow y' = y - g(u)$$

We see that $x' = y' = 0$ is the world sheet of a travelling string wave. Let $\phi = \varphi(x, y)$ and $A_\mu = \mathcal{A}_\mu dx^\mu$. The claim is, the fields due to a string with oscillations propagating in the z direction, are given by

$$\phi' = \varphi(x', y') \quad A'_\mu = \mathcal{A}_\mu(x', y') dx'_\mu \tag{3.12}$$

where the functional forms of φ and $\mathcal{A}_\mu$ are the same as the functional form of ϕ and A_μ. (WHAT?) is demonstrated by checking that the equations of motion are satisfied by these transformed fields.

If we want to solve the problem ie. Eqs [ref], in curved space, we define (following the discussion in [18]):

$$g'_{\mu\nu} = g_{\mu\nu} + F k_\mu k_\nu \tag{3.13}$$

The claim is $g'(x')$, $\phi(x')$ and $A_\mu(x')$ satisfy the equations of motion given that the functional forms of g', ϕ' and A'_μ are same as g, ϕ and A_μ and g. k_μ and F satisfy the following conditions:

$$k^\mu k_\mu = 0 \tag{3.14}$$

$$\nabla_{(\mu}k_{\nu)} = 0 \tag{3.15}$$

$$k_{[\mu}\nabla_{\nu}k_{\rho)} = 0 \tag{3.16}$$

$$k^{\mu}A_{\mu} = 0 \tag{3.17}$$

$$\mathcal{L}_k A_{\mu} = 0 \tag{3.18}$$

$$\mathcal{L}_k \phi = 0 \tag{3.19}$$

$$\nabla^{\mu}\nabla_{\mu}(e^{\alpha}F) = 0 \tag{3.20}$$

where α is any scalar.

The authors show that the equations of motion hold for the set $g'(x')$, $\phi(x')$ and $A_\mu(x')$ by writing the equations of motion explicitly in terms of the new metric and using the above properties.

3.5.2 Using solution generation for oscillating strings

We now follow the construction of DHGW [3] using the technology developed in the previous section to put oscillations on the fundamental string. We choose $K = \partial_u$ because $K^2 = 0$ and $\mathcal{L}_K\psi = 0$ where ψ can be any of the fields in the equations of motion. We could have as well chosen $K = \partial_v$ and this would have given us right moving oscillations on the string.

Using the recipe in the previous section, the static solutions now become:

$$ds^2 = -e^{2\phi}(dudv - T(v,\vec{x})dv^2) + d\vec{x}.d\vec{x} \tag{3.21}$$

$$B_{uv} = \frac{1}{2}(e^{2\phi} - 1) \tag{3.22}$$

$$e^{-2\phi} = 1 + \frac{Q}{r^{D-4}} \tag{3.23}$$

With T satisfying,

$$\partial^2 T(v,\vec{x}) = 0 \tag{3.24}$$

where the partial derivatives are over the transverse space.

We can solve for the admissible form for $T(v,\vec{x})$. To do so we have to impose some conditions on our solutions:

1. The metric should be asymptotically flat.

2. The solutions should map onto string sources ie. they should satisfy the equations of motion with the fundamental string as the source.

We will now solve for

$$\partial^2 T(v,x) = 0$$

by decomposing it into $(D-2)$ dimensional spherical harmonics:

$$T(v,x) = \Sigma_{l\geq 0}[(a_l(v)r^l + b_l(v)r^{-D+4-l}]Y_l \tag{3.25}$$

Let us look at the terms carefully. r^0 terms are just additive constants and can be set to 0 by shifting coordinates. r^2 and higher order terms are not asymptotically flat. r^{-D+3} and lower order terms are asymptotically flat but do not contribute to the ADM momentum. This is because in the $D-2$ dimensional transverse space, for metric coefficient $g_{ty} = q$, terms of order r^{-D+4} contribute. These r^{-D+4} give the ADM momentum $\frac{(D-4)\Omega q}{16\pi G}$. So we keep the terms of order r and r^{-D+4}.

The term which goes as r^{-D+4} carries momentum but does not match to a string source (we shall see this in the next section). So it is a momentum wave without oscillations.

$$T(v,x) = f(\vec{v}).\vec{x} + \frac{p(v)}{|\vec{x}|^{D-4}} \tag{3.26}$$

Written in these coordinates with this form for T the metric isn't asymptotically flat due to the r^1 order term in T. We can make it flat by doing the following co-ordinate transformation:

$$v = v' \tag{3.27}$$

$$u = u' - 2\dot{\vec{F}}.\vec{x}\prime + 2\dot{\vec{F}}.\vec{F} - \int^{v'} \dot{F}^2 dv \tag{3.28}$$

$$\vec{x} = \vec{x'} - \vec{F} \tag{3.29}$$

where the derivatives are w.r.t. to v and $\vec{f} = -2\ddot{\vec{F}}$ The metric and fields in these coordinates are:

$$ds^2 = -e^{2\phi}dudv + [e^{2\phi}p(v)r^{-D+4} - (e^{2\phi}-1)\dot{F}^2]dv^2 + 2(e^{2\phi}-1)\dot{\vec{F}}.\vec{x}dv + dx.dx \tag{3.30}$$

$$B_{uv} = \frac{1}{2}(e^{2\phi-1} - 1) \tag{3.31}$$

$$B_{vi} = \dot{F}_i(e^{2\phi} - 1) \tag{3.32}$$

where

$$e^{-2\phi} = 1 + \frac{Q}{|\vec{x} - \vec{F}|^{D-4}} \tag{3.33}$$

A very important feature of these solutions is that it is very easy to construct multi-string solutions due to fundamental strings localized at different points. The importance of this is underlined in [3], where they describe a method of making solitons in lower dimensions by superimposing a periodic array of fundamental strings on a compactified dimension.

Given a solution, we can write down a multi-centre solution by linear superposition, because for the static case the linear equation $\partial^2 e^{-2\phi} = 0$ is true even for solutions of the form:

$$e^{-2\phi} = 1 + \Sigma_i \frac{Q}{|\vec{x} - \vec{x_i}|^{D-4}} \tag{3.34}$$

This is analogous to electrodynamics where linearly superimposing delta function sources (at particle positions) in $\partial^2 V(\vec{x}) = \frac{\rho}{\epsilon_0}$ gives the net V. The linear nature of the equation $\partial^2 T(v, \vec{x})$ lets us superimpose solutions corresponding to many oscillating solutions.

$$T(v, x) = f(\vec{v}).\vec{x} + \sum_i \frac{p_i(v)}{|\vec{x} - \vec{x_i}|^{D-4}} \tag{3.35}$$

Making the same coordinate transformation to asymptotically flat coordinates,

$$e^{-2\phi} = 1 + \Sigma_i \frac{Q}{|\vec{x} - \vec{x_i} - \vec{F}(v)|^{D-4}} \tag{3.36}$$

It is important to note that because $T(v, \vec{x})$ carries a piece which looks like $\vec{f}(v).\vec{x}$, the solutions which can be superimposed all carry the same oscillation profile $\vec{F}$ as it is determined by $\vec{f}$. They can carry different momenta given by different functions $p_i(v)$.

Matching onto string sources

We now want to check if the solutions constructed actually do satisfy the equations of motion ie. whether or not there exist string sources which can source these field configurations. We also want to impose the Virasoro constraints on the string. Taking the radius of the macroscopic string to be much larger than the string scale, we can ignore normal ordering effects of the CFT (this is evident from the mass shell formula). This is further explained in [3].

Fixing conformal gauge by using $\sigma^{\pm} = \tau \pm \sigma$ and using light-cone coordinates $U(\sigma^+, \sigma^-)$, $V(\sigma^+, \sigma^-)$ and $\vec{X} = 0$ for the target space string coordinates, the equations of motion of the string become:

$$\partial^2 e^{-2\phi} = \frac{-\kappa^2}{\pi\alpha'} \int d\sigma^+ d\sigma^- [\partial_+ V \partial_- U - \partial_- V \partial_+ U \delta^D(x - X)] \tag{3.37}$$

$$\partial^2 e^{-2\phi} = \frac{-\kappa^2}{\pi\alpha'} \int d\sigma^+ d\sigma^- [\partial_+ V \partial_- U + \partial_- V \partial_+ U \delta^D(x - X)] \tag{3.38}$$

$$0 = \int d\sigma^+ d\sigma^- [\partial_+ V \partial_- V \delta^D(x - X)] \tag{3.39}$$

$$T\partial^2 e^{-2\phi} + e^{-2\phi}\partial^2 T = \int d\sigma^+ d\sigma^- [\partial_+ U \partial_- U \delta^D(x - X)] \tag{3.40}$$

This last equation imposes the condition that T cannot diverge as it approaches $r = 0$ as the RHS is finite. This tells us that terms of order r^{-D+3}, r^{-D+4}... cannot come from delta function string sources. Terms linear in r are allowed as they go to 0 at the source. Also, the Virasoro constraints are:

$$T_{++} = -e^{2\phi}(\partial_+ V \partial_+ U - T\partial_+ V \partial_+ V) = 0 \tag{3.41}$$

$$T_{--} = -e^{2\phi}(\partial_- V \partial_- U - T\partial_- V \partial_- V) = 0 \tag{3.42}$$

Now, as $e^{2\phi} = 0$ at $r = 0$ ie. right at the string source, the equations are trivially satisfied as long as the string lies only at $r = 0$. The constraints can be satisfied if we choose $V = V(\sigma^+)$ and $U = U(\sigma^-)$

3.6 Making black holes from fundamental strings

We have seen that fundamental strings are solitonic objects and we have also seen that we can linearly superpose their solutions. One possibility is to put a periodic array of these strings along a particular transverse direction and compactify along that direction. This would give us a fundamental string in a lower dimension.

A second possibility is that we could get point-like objects whose asymptotic properties match black holes by doing a Kaluza Klein reduction along the length of the fundamental string. We would get point like BPS objects whose asymptotic charges would match those of supersymmetric black holes.

3.7 Fundamental string NS5 intersection

In this section, we will write down a metric ansatz for an oscillating fundamental string in NS5 background. The metric for the fundamental string (static) NS5 brane intersection is given by [19] using an algorithmic procedure for calculating brane intersections:

$$ds^2 = H_1^{-1}(-dudv + \sum_{i=2}^{5} dy_i.dy_i + H_5 \sum_{i=1}^{4} dx_i.dx_i \tag{3.43}$$

where $u = t - y_1$ and $v = t + y_1$ are the coordinates of the fundamental string, y_i are the world volume directions of the NS5 brane, x_i are the transverse directions and r is the radial coordinate in the transverse direction.

$$\begin{aligned} e^{2\phi} &= \frac{H_5}{H_1} \\ H_1 &= 1 + \frac{Q_1}{r^2} \\ H_5 &= 1 + \frac{Q_2}{r^2} \\ B_{uv} &= 2H^{-1} \\ H_{mnk} &= -\epsilon_{mnkl}\partial_l H_5 \end{aligned} \tag{3.44}$$

To put oscillations on the fundamental string we use the solution generating technique of [3] [18]. It lets you transform a static solution produced by a string source to one produced by an oscillating string if the static solution has a null, hyper-surface orthogonal killing vector.

[] admits the killing vector ∂_v which corresponds to a left moving wave. Applying the solution generating technique we have been discussing we get:

$$ds^2 = H_1^{-1}(-dudv + T(v,x)dv^2) + \sum_{i=2}^{5} dy_i.dy_i + H_5 \sum_{i=1}^{4} dx_i.dx_i \tag{3.45}$$

where $\partial^2 T(v, x_i) = 0$ on the transverse directions, and the ϕ and B fields stay the same.

$\partial^2 T(v,x) = 0$ can be solved by expanding in spherical harmonics and only keeping terms which keep the metric *asymptomatically* NS5 and contribute to the ADM mass :

$$\partial^2 T(v,x) = \vec{f(v)}.\vec{x} + p(v)r^{-2} \tag{3.46}$$

The r^{-2} term does not match to a string source. Due to the $\vec{f(v)}.\vec{x}$ term, the transverse metric is not asymptotically $NS5$, but this can be removed by a coordinate transformation,

$$
\begin{aligned}
v &= v' \\
u &= u' - 2\dot{\vec{F}}.\vec{x}' + 2\dot{\vec{F}}.\vec{F} - \int^{v'} \dot{F}^2 dv \\
\vec{x} &= \vec{x}' - \vec{F}
\end{aligned}
\tag{3.47}
$$

where the derivatives are w.r.t. to v and $\vec{f} = -2\ddot{\vec{F}}$. The metric and fields become,

$$
\begin{aligned}
ds^2 &= -H_1^{-1} du dv + 2\dot{F}(H_1^{-1} - H_5) dx dv + ((-H_1^{-1} + H_5)\dot{F}^2 + H_1^{-1} p(v) r^{-2}) dv^2 + \\
&\sum_{i=2}^{5} dy_i.dy_i + H_5 \sum_{i=1}^{4} dx_i.dx_i
\end{aligned}
\tag{3.48}
$$

$$
\begin{aligned}
B_{uv} &= 2H^{-1} \\
B_{vi} &= 4\dot{\vec{F}} H^{-1}
\end{aligned}
$$

where

$$
\begin{aligned}
H_1 &= 1 + \frac{Q_1}{|\vec{x} - \vec{F}|^2} \\
H_5 &= 1 + \frac{Q_2}{|\vec{x} - \vec{F}|^2}
\end{aligned}
$$

and

$$
e^{2\phi} = \frac{H_5}{H_1} \tag{3.49}
$$

The multi string generalization for fundamental strings placed at $\vec{x_i}$ and NS5 branes placed at $\vec{x_j}$ in the transverse direction is given by the above metric with

$$
\begin{aligned}
H_1 &= 1 + \frac{\sum_i Q_1}{|\vec{x} - \vec{x_i} - \vec{F}|^2} \\
H_5 &= 1 + \frac{\sum_j Q_2}{|\vec{x} - \vec{x_j} - \vec{F}|^2}
\end{aligned}
$$

[20] also gives the metric for a fundamental string in an NS5 background by generating the metric for a multiply wound string and performing a set of dualities to reach $NS5$-fundamental string intersection.

3.8 Conclusion and Future Directions

We have reviewed the works of [2], [3] and [18] and following the discussions in these works, we have written an ansatz for the metric and fields produced by an oscillating fundamental string in NS5 background. Along the way, we have covered some topics in string theory from the textbook by Polchinski [5] and lecture notes by D.Tong [4]. Most importantly, we have realised some of the power of string theory in answering fundamental questions on black holes.

We should now explicitly check that this solution is $\frac{1}{4}$ BPS by doing a supersymmetry variation of the fermionic fields and showing that it vanishes. We also wish to understand how and why are solution differs from [20], and how starting from our solution, we might be able to get to their solution.

References

[1] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma, *Constraints on Fluid Dynamics from Equilibrium Partition Functions*, *JHEP* **09** (2012) p. 046, [arXiv:1203.3544].

[2] A. Dabholkar, G. W. Gibbons, J. A. Harvey, and F. Ruiz Ruiz, *Superstrings and Solitons*, *Nucl. Phys.* **B340** (1990) pp. 33–55.

[3] A. Dabholkar, J. P. Gauntlett, J. A. Harvey, and D. Waldram, *Strings as solitons and black holes as strings*, *Nucl. Phys.* **B474** (1996) pp. 85–121, [hep-th/9511053].

[4] D. Tong, *String Theory*, arXiv:0908.0333.

[5] J. Polchinski, *String Theory: Volume 1, An Introduction to the Bosonic String.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1998.

[6] S. Bhattacharyya, S. Jain, S. Minwalla, and T. Sharma, *Constraints on Superfluid Hydrodynamics from Equilibrium Partition Functions*, *JHEP* **01** (2013) p. 040, [arXiv:1206.6106].

[7] N. Banerjee, S. Dutta, and A. Jain, *Null Fluids - A New Viewpoint of Galilean Fluids*, arXiv:1509.0471.

[8] J. Bhattacharya, S. Bhattacharyya, S. Minwalla, and A. Yarom, *A Theory of first order dissipative superfluid dynamics*, *JHEP* **05** (2014) p. 147, [arXiv:1105.3733].

[9] J. Bhattacharya, S. Bhattacharyya, and S. Minwalla, *Dissipative Superfluid dynamics from gravity*, *JHEP* **04** (2011) p. 125, [arXiv:1101.3332].

[10] K. Jensen and A. Karch, *Revisiting non-relativistic limits*, *JHEP* **04** (2015) p. 155, [arXiv:1412.2738].

[11] M. Geracie, K. Prabhu, and M. M. Roberts, *Fields and fluids on curved non-relativistic spacetimes*, *JHEP* **08** (2015) p. 042, [arXiv:1503.0268].

[12] A. Bagchi, R. Basu, A. Kakkar, and A. Mehra, *Galilean Yang-Mills Theory*, arXiv:1512.0837.

[13] A. Bagchi and R. Gopakumar, *Galilean Conformal Algebras and AdS/CFT*, *JHEP* **07** (2009) p. 037, [arXiv:0902.1385].

[14] S. D. Mathur, *The Fuzzball proposal for black holes: An Elementary review*, *Fortsch. Phys.* **53** (2005) pp. 793–827, [hep-th/0502050].

[15] I. Bena and N. P. Warner, *Resolving the Structure of Black Holes: Philosophizing with a Hammer*, arXiv:1311.4538.

[16] I. Bena, S. Giusto, R. Russo, M. Shigemori, and N. P. Warner, *Habemus Superstratum! A constructive proof of the existence of superstrata*, *JHEP* **05** (2015) p. 110, [arXiv:1503.0146].

[17] A. Giveon, J. Harvey, D. Kutasov, and S. Lee, *Three-Charge Black Holes and Quarter BPS States in Little String Theory*, *JHEP* **12** (2015) p. 145, [arXiv:1508.0443].

[18] D. Garfinkle and T. Vachaspati, *Cosmic string traveling waves*, *Phys. Rev.* **D42** (1990) pp. 1960–1963.

[19] A. A. Tseytlin, *Harmonic superpositions of M-branes*, *Nucl. Phys.* **B475** (1996) pp. 149–163, [hep-th/9604035].

[20] O. Lunin and S. D. Mathur, *Metric of the multiply wound rotating string*, *Nucl. Phys.* **B610** (2001) pp. 49–76, [hep-th/0105136].