Boundary behavior of the Carathéodory and Kobayashi-Eisenman volume elements and the Kobayashi–Fuks metric

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Certified that the work incorporated in the thesis entitled "*Boundary behavior of the Carathéodory and Kobayashi-Eisenman volume elements and the Kobayashi–Fuks metric*", submitted by *Debaprasanna Kar* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: December 17, 2021

Dr. Diganta Borah
Thesis Supervisor

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Contents

Abstract

We will compute the boundary asymptotics of the Carathéodory and Kobayashi-Eisenman volume elements on convex finite type domains and Levi corank one domains in $\mathbb{C}^n$ using the standard scaling techniques. We will show that their ratio, the so-called C/K ratio or the quotient invariant, can be used to detect strong pseudoconvexity. Some properties of a Kähler metric called the Kobayashi–Fuks metric will also be observed on planar domains as well as on strongly pseudoconvex domains in $\mathbb{C}^n$. We study the localization of this metric near holomorphic peak points and show that this metric shares several properties with the classical Bergman metric on strongly pseudoconvex domains.

Notations

$\mathbb{R}:$ Set of real numbers

$\mathbb{C}:$ Set of complex numbers

$\Delta:$ The unit disc in $\mathbb{C}$

$\Delta(a,r):$ Disc of radius r centered at $a \in \mathbb{C}$

$\mathbb{B}^n:$ The unit ball in $\mathbb{C}^n$

$B(a,r):$ Ball of radius r centered at $a \in \mathbb{C}^n$

$\Delta^n:$ The unit polydisc $\Delta \times \cdots \times \Delta$ in $\mathbb{C}^n$

$\overline{D}:$ Topological closure of a domain D

$\partial D:$ Topological boundary of a domain D

$\delta_D(z):$ The (shortest) Euclidean distance of $z \in D$ to its boundary ∂D

$\delta_D(z,u):$ The (shortest) Euclidean distance of $z \in D$ to ∂D in the direction of the vector u

$\nabla:$ The gradient operator

$\mathbb{I}:$ The identity matrix or the identity transformation

$M^t:$ Transpose of a matrix M

$\overline{M}:$ Complex conjugate of a matrix M

$M^*:$ Conjugate transpose of M

$\operatorname{ad} M:$ Adjugate of a matrix M

$\det M:$ Determinant of a matrix M

$\operatorname{diag}\{d_1,\ldots,d_n\}:$ A diagonal matrix with entries $d_1,\ldots,d_n$ in the exact order

$\mathcal{O}(D_1,D_2):$ The collection of holomorphic maps from a domain D_1 to another domain D_2

$\det \psi'(p):$ Determinant of the complex Jacobian of a holomorphic map ψ at a point p

$u \in \mathbb{C}^n:$ An $n \times 1$ column vector in $\mathbb{C}^n$

$u^\alpha:$ α-th component of a vector u

$('u, u_n) \in \mathbb{C}^n:$ $'u = (u_1,\ldots,u_{n-1}) \in \mathbb{C}^{n-1}$ and $u_n \in \mathbb{C}$

$(u_1, u') \in \mathbb{C}^n:$ $u_1 \in \mathbb{C}$ and $u' = (u_2,\ldots,u_n) \in \mathbb{C}^{n-1}$

$a_j \to a:$ Sequence a_j converges to a as j goes to ∞

$\square:$ End of a proof

Chapter 1

Introduction

The study of intrinsic or invariant objects has been of fundamental importance in several complex variables for several decades. This played a pivotal role in understanding the geometry of a wide range of domains in the complex Euclidean space, even on complex manifolds. One of the most celebrated results in complex analysis of single variable is the Riemann mapping theorem, which says that, every simply connected domain in $\mathbb{C}$, except for the whole complex plane, is biholomorphically equivalent to the unit disc. Therefore the topological property "simply connectedness" is sufficient to describe a large class of planar domains, up to biholomorphism. On the other hand in 1907, H. Poincaré proved that the group of holomorphic automorphisms of the open ball and the open polydisc in $\mathbb{C}^2$ are not isomorphic. Hence they are, although topologically equivalent, not equivalent biholomorphically. Therefore, it seemed important to associate with domains in $\mathbb{C}^n$ obedient objects those are invariant under biholomorphic mappings. Generalizing the Schwarz-Pick Lemma, C. Carathéodory provided the first example of such an object, which was different from the automorphism group, and later this object was called the *Carathéodory pseudodistance*. Roughly speaking, his principal idea was to use the set of bounded holomorphic functions on a domain D as an invariant. Apart from the class of bounded holomorphic functions, S. Bergman in 1933 considered the Hilbert space of square integrable holomorphic functions on D to obtain a Kähler metric which is now known as the Bergman metric. In 1967, using the set of analytic discs to cook up new biholomorphic invariants, S. Kobayashi introduced a pseudodistance, which is in some sense dual to the Carathéodory pseudodistance, and is called the Kobayashi pseudodistance.

The estimates and study of the limiting behavior of these invariant pseudodistances or metrics play an important role in a wide range of problems in complex analysis like biholomorphic equivalence or non-equivalence of domains, continuation of holomorphic mappings, asymptotic estimates of various classes of holomorphic functions, description of domains with noncompact groups of automorphisms, among many others. Geometry and analysis of bounded domains, especially those of pseudoconvex domains, are broad and rich. In this thesis, we study the limiting behavior of certain invariant objects near the boundary of various classes of pseudoconvex domains and try to convince the readers the importance of studying the boundary behavior by giving a number of applications later on.

Let us now shed some light on the contents of the thesis. The next chapter (Chapter 2) will be on preliminaries, where we list some of the definitions and concepts those will be used in the discourse later. This will serve as a basic prerequisite which provides the necessary groundwork to understand the results in this thesis. Chapters 3 and 4 comprise the study of the Carathéodory and Kobayashi-Eisenman volume elements, most of which can be found in [5]. In 1969, D. A. Eisenman initiated the study of these volume elements as intrinsic measures on complex manifolds (see [18]). The construction of these measures was modelled in analogy with the intrinsic distances of Carathéodory and Kobayashi. One can refer to the exposition [34] and the article [35] of Kobayashi for more survey on these measures. We adopt the following definitions of the volume elements on domains in $\mathbb{C}^n$: For a domain $D \subset \mathbb{C}^n$, the Carathéodory and Kobayashi-Eisenman volume elements on D at a point $p \in D$ are defined respectively by

$$c_D(p) = \sup\Big\{ \big|\det \psi'(p)\big|^2 : \psi \in \mathcal{O}(D, \mathbb{B}^n), \psi(p) = 0 \Big\},$$
$$k_D(p) = \inf\Big\{ \big|\det \psi'(0)\big|^{-2} : \psi \in \mathcal{O}(\mathbb{B}^n, D), \psi(0) = p \Big\}.$$

In Chapter 3, along with some literature survey, we will list some of the properties these volume elements enjoy. A biholomorphic invariant can be defined using these volume elements, called the *quotient invariant*, given by

$$q_D(p) = \frac{c_D(p)}{k_D(p)}.$$

Using the behavior of this invariant near the boundary of bounded convex domains, a criterion is derived that detects strongly pseudoconvex domains in $\mathbb{C}^n$, which says:

Theorem 1.0.1. *For any positive integer n and $\alpha \in (0,1)$, there exists some $\epsilon = \epsilon(n,\alpha) > 0$ with the following property: If $D \subset \mathbb{C}^n$ is a bounded convex domain with $C^{2,\alpha}$ boundary and if*

$$q_D(p) \geq 1 - \epsilon$$

outside a compact subset of D, then D is strongly pseudoconvex.

Chapter 4 is basically the study of boundary behavior of these volume elements on two special kinds of pseudoconvex domains, namely– convex finite type domains and Levi corank one domains. In particular we procure that as we move towards the boundary of these domains, the Kobayashi-Eisenman volume element blows up at a rate reciprocal to the size of certain polydiscs in each of the cases. The method of scaling of domains comes in handy in obtaining these boundary asymptotics, and as we will see, several variants of the scaling technique will be used in this dissertation to achieve our desired objectives. Briefly speaking, scaling is a convenient tool that converts boundary problems into interior problems, and the latter of which is easier to tackle in general. Let us first discuss the convex finite type case.

A smooth boundary point p^0 of a domain $D \subset \mathbb{C}^n$ is said to be a point of finite type (in the sense of D'Angelo) if the maximum order of contact of one-dimensional complex analytic varieties with ∂D at p^0 is bounded. D is called a finite type domain if all the points on the boundary ∂D are finite type points. More on this "finite type-ness" will be explained later in Chapter 2. Let $D = \{\rho < 0\}$ be a smoothly bounded convex finite type domain and $p^0 \in \partial D$. For each point $p \in D$ sufficiently close to p^0, and $\epsilon > 0$ sufficiently small, McNeal's orthogonal coordinate system $z_1^{p,\epsilon}, \ldots, z_n^{p,\epsilon}$ centred at p is constructed as follows (see [40]). Denote by $D_{p,\epsilon}$ the domain

$$D_{p,\epsilon} = \Big\{ z \in \mathbb{C}^n : \rho(z) < \rho(p) + \epsilon \Big\}.$$

Let $\tau_n(p,\epsilon)$ be the distance of p to $\partial D_{p,\epsilon}$ and $\zeta_n(p,\epsilon)$ be a point on $\partial D_{p,\epsilon}$ realising this distance. Let H_n be the complex hyperplane through p and orthogonal to the vector $\zeta_n(p,\epsilon) - p$. Compute the distance from p to $\partial D_{p,\epsilon}$ along each complex line in H_n. Let $\tau_{n-1}(p,\epsilon)$ be the largest such distance and let $\zeta_{n-1}(p,\epsilon)$ be a point on $\partial D_{p,\epsilon}$ such that $|\zeta_{n-1}(p,\epsilon)-p| = \tau_{n-1}(p,\epsilon)$. For the next step, define H_{n-1} as the complex hyperplane through p and orthogonal to the span of the vectors $\zeta_n(p,\epsilon)-p, \zeta_{n-1}(p,\epsilon)-p$ and repeat the above construction. Continuing in this way, we define the numbers $\tau_n(p,\epsilon), \tau_{n-1}(p,\epsilon), \ldots, \tau_1(p,\epsilon)$, and the points $\zeta_n(p,\epsilon)$, $\zeta_{n-1}(p,\epsilon)$, $\ldots, \zeta_1(p,\epsilon)$ on $\partial D_{p,\epsilon}$. Let $T^{p,\epsilon}$ be the translation sending p to the origin and $U^{p,\epsilon}$ be a unitary mapping aligning $\zeta_k(p,\epsilon)-p$ along the z_k-axis and $\zeta_k(p,\epsilon)$ to a point on the positive $\operatorname{Re} z_k$ axis. Set

$$z^{p,\epsilon} = U^{p,\epsilon} \circ T^{p,\epsilon}(z).$$

The polydisc

$$P(p,\epsilon) = \Big\{ z^{p,\epsilon} : |z_1^{p,\epsilon}| < \tau_1(p,\epsilon), \ldots, |z_n^{p,\epsilon}| < \tau_n(p,\epsilon) \Big\}$$

is known as McNeal's polydisc. The scaling method, which will be briefly explained in Chapter 4, shows that every sequence in D that converges to $p^0 \in \partial D$ furnishes limiting domains

$$D_\infty = \left\{ z \in \mathbb{C}^n : -1 + \operatorname{Re} \sum_{\alpha=1}^{n} b_\alpha z_\alpha + P_{2m}('z) < 0 \right\}, \tag{1.1}$$

where b_α are complex numbers and P_{2m} is a real convex polynomial of degree at most $2m$ ($m \geq 1$), where $2m$ is the 1-type of ∂D at p^0. The polynomial P_{2m} is not unique in general and depends on how the given sequence approaches p^0. The exact asymptotic expression for the Kobayashi-Eisenman volume element on convex finite type domains is:

Theorem 1.0.2. *Let $D = \{\rho < 0\}$ be a smoothly bounded convex finite type domain in $\mathbb{C}^n$ and $p^j \in D$ be a sequence converging to $p^0 \in \partial D$. Let $\epsilon_j = -\rho(p^j)$. Then up to a subsequence,*

$$k_D(p^j) \prod_{\alpha=1}^{n} \tau_\alpha(p^j, \epsilon_j)^2 \to k_{D_\infty}(0)$$

as $j \to \infty$, where D_∞ is a limiting domain associated with D at p^0.

Now we consider the Levi corank one case. A boundary point p^0 of a domain $D \subset \mathbb{C}^n$ is said to have Levi corank one if there exists a neighborhood of p^0 where ∂D is smooth, pseudoconvex, of finite type, and the Levi form has at least $(n-2)$ positive eigenvalues. If every boundary point of D has Levi corank one, then D is called a Levi corank one domain. The collection of Levi corank one domains includes the class of all smoothly bounded pseudoconvex finite type domains in $\mathbb{C}^2$. A basic example in higher dimension is the egg

$$E_{2m} = \Big\{ z \in \mathbb{C}^n : |z_1|^{2m} + |z_2|^2 + \ldots + |z_n|^2 < 1 \Big\},$$

where $m \geq 2$ is an integer. Let $D = \{\rho < 0\}$ be a smoothly bounded Levi corank one domain and $p^0 \in \partial D$. It was proved in [10] that for each point p in a sufficiently small neighborhood U of p^0, there are holomorphic coordinates $\zeta = \Phi^p(z)$ such that

$$\begin{aligned} \rho \circ (\Phi^p)^{-1}(\zeta) &= \rho(p) + 2 \operatorname{Re} \zeta_n + \sum_{\substack{j+k \leq 2m \\ j,k>0}} a_{jk}(p) \zeta_1^j \bar{\zeta}_1^k + \sum_{\alpha=2}^{n-1} |\zeta_\alpha|^2 \\ &+ \sum_{\alpha=2}^{n-1} \sum_{\substack{j+k \leq m \\ j,k>0}} \operatorname{Re}\Big(\big(b_{jk}^\alpha(p) \zeta_1^j \bar{\zeta}_1^k\big) \zeta_\alpha \Big) + O\big(|\zeta_n||\zeta| + |\zeta_*|^2|\zeta| + |\zeta_*||\zeta_1|^{m+1} + |\zeta_1|^{2m+1}\big), \end{aligned} \tag{1.2}$$

where $\zeta_* = (0, \zeta_2, \ldots, \zeta_{n-1}, 0)$. To construct the distinguished polydiscs around p, set

$$\begin{aligned} A_l(p) &= \max\Big\{ \big|a_{jk}(p)\big| : j+k=l \Big\}, \quad 2 \le l \le 2m, \\ B_{l'}(p) &= \max\Big\{ \big|b^{\alpha}_{jk}(p)\big| \ : \ j+k=l',\ 2 \le \alpha \le n-1 \Big\},\ 2 \le l' \le m. \end{aligned} \tag{1.3}$$

Now define for each $\delta > 0$, the special-radius

$$\tau(p,\delta) = \min\left\{ \Big(\delta/A_l(p)\Big)^{1/l},\ \Big(\delta^{1/2}/B_{l'}(p)\Big)^{1/l'} \ : \ 2 \le l \le 2m,\ 2 \le l' \le m \right\}. \tag{1.4}$$

It was shown in [10] that the coefficients b^{α}_{jk}'s in the above definition of $\tau(p,\delta)$ are insignificant and may be ignored, so that

$$\tau(p,\delta) = \min\left\{ \Big(\delta/A_l(p)\Big)^{1/l} \ : \ 2 \le l \le 2m \right\}. \tag{1.5}$$

Set

$$\tau_1(p,\delta) = \tau(p,\delta) = \tau, \tau_2(p,\delta) = \cdots = \tau_{n-1}(p,\delta) = \delta^{1/2}, \tau_n(p,\delta) = \delta.$$

The distinguished polydiscs $Q(p,\delta)$ of Catlin are defined by

$$Q(p,\delta) = \Big\{ (\Phi^p)^{-1}(\zeta) : |\zeta_1| < \tau_1(p,\delta), \ldots, |\zeta_n| < \tau_n(p,\delta) \Big\}.$$

The scaling method (which is well known in this case and will be briefly explained in Chapter 4) shows that every sequence in D that converges to $p^0 \in \partial D$ furnishes limiting domains

$$D_\infty = \left\{ z \in \mathbb{C}^n : 2\operatorname{Re} z_n + P_{2m}(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 < 0 \right\}, \tag{1.6}$$

where $P_{2m}(z_1, \bar{z}_1)$ is a subharmonic polynomial of degree at most $2m$ $(m \ge 1)$ without harmonic terms, $2m$ being the 1-type of ∂D at p^0. Observe that the point $b = ('0, -1)$ lies in every such D_∞. The boundary asymptotics of the Kobayashi-Eisenman volume element on Levi corank one domains is as follows:

Theorem 1.0.3. *Let $D = \{\rho < 0\}$ be a smoothly bounded Levi corank one domain in $\mathbb{C}^n$ and $p^j \in D$ be a sequence converging to $p^0 \in \partial D$. Let $\delta_j > 0$ be such that $\tilde{p}^j = (p^j_1, \ldots, p^j_{n-1}, p^j_n + \delta_j)$ is a point on ∂D. Then up to a subsequence,*

$$k_D(p^j) \prod_{\alpha=1}^{n} \tau_\alpha(\tilde{p}^j, \delta_j)^2 \to c(\rho, p^0) k_{D_\infty}(b)$$

as $j \to \infty$, where $c(\rho, p^0)$ is a positive constant that depends only on ρ and p^0, and D_∞ is a limiting domain associated with D at p^0.

Based on [6], Chapters 5-7 consist of the study on the Kobayashi–Fuks metric, a Kähler metric which is closely related to the classical Bergman metric. In these chapters we explore a few similarities of this metric with the Bergman metric by exploring localization of some of its associated invariants and later studying their boundary behavior on strongly pseudoconvex domains. If we denote the Bergman kernel (on the diagonal) for a domain D by K_D, the Bergman metric is given by

$$ds^2_{B,D} := \sum_{\alpha,\beta=1}^{n} g^{B,D}_{\alpha\overline{\beta}}(z)\, dz_\alpha d\overline{z}_\beta,$$

where

$$g^{B,D}_{\alpha\overline{\beta}}(z) = \frac{\partial^2 \log K_D}{\partial z_\alpha \partial \overline{z}_\beta}(z).$$

Denoting by $G_{B,D}(z)$ the matrix $\big(g^{B,D}_{\alpha\overline{\beta}}(z)\big)_{n\times n}$, the components of the Ricci tensor of $ds^2_{B,D}$ are given by

$$\mathrm{Ric}^{B,D}_{\alpha\overline{\beta}}(z) = -\frac{\partial^2 \log \det G_{B,D}}{\partial z_\alpha \partial \overline{z}_\beta}(z),$$

and the Ricci curvature of $ds^2_{B,D}$ is given by

$$\mathrm{Ric}_{B,D}(z,u) = \frac{\sum_{\alpha,\beta=1}^{n} \mathrm{Ric}^{B,D}_{\alpha\overline{\beta}}(z) u^\alpha \overline{u}^\beta}{\sum_{\alpha,\beta=1}^{n} g^{B,D}_{\alpha\overline{\beta}}(z) u^\alpha \overline{u}^\beta}.$$

Kobayashi [33] showed that the Ricci curvature of the Bergman metric on a bounded domain in $\mathbb{C}^n$ is strictly bounded above by $n+1$, and hence the following matrix

$$G_{\tilde{B},D}(z) = \Big(g^{\tilde{B},D}_{\alpha\overline{\beta}}(z)\Big)_{n\times n} = \Big((n+1) g^{B,D}_{\alpha\overline{\beta}}(z) - \mathrm{Ric}^{B,D}_{\alpha\overline{\beta}}(z)\Big)_{n\times n}$$

is positive definite on a bounded domain D (see also Fuks [19]). Using the above result,

$$ds^2_{\tilde{B},D} := \sum_{\alpha,\beta=1}^{n} g^{\tilde{B},D}_{\alpha\overline{\beta}}(z)\, dz_\alpha d\overline{z}_\beta$$

turns out to be an invariant Kähler metric with Kähler potential $\log(K_D^{n+1} \det G_{B,D})$. We call this the Kobayashi–Fuks metric on D. We denote the length of a vector u at a point $z \in D$ in $ds^2_{\tilde{B},D}$ by $\tau_{\tilde{B},D}(z,u)$, i.e.,

$$\tau^2_{\tilde{B},D}(z,u) = \sum_{\alpha,\beta=1}^{n} g^{\tilde{B},D}_{\alpha\overline{\beta}}(z) u^\alpha \overline{u}^\beta.$$

The Riemannian volume element of the Kobayashi–Fuks metric on D will be denoted by $g_{\tilde{B},D}(z)$, i.e., $g_{\tilde{B},D}(z) = \det G_{\tilde{B},D}(z)$. Note that in dimension one, the metric $ds^2_{\tilde{B},D}$ has the form

$$ds^2_{\tilde{B},D} = g_{\tilde{B},D}(z)|dz|^2, \quad \tau_{\tilde{B},D}(z,u) = \sqrt{g_{\tilde{B},D}(z)}|u|$$

and the Gaussian curvature of the Kobayashi–Fuks metric is given by

$$R_{\tilde{B},D}(z) = -\frac{1}{g_{\tilde{B},D}(z)} \frac{\partial^2 \log g_{\tilde{B},D}}{\partial z \partial \overline{z}}(z).$$

Chapter 5, apart from formally defining the Kobayashi–Fuks metric, gives a brief survey of some of the work that has been done on this metric recently. Towards the end of that chapter, some of the monotonicity or the comparison results on the Kobayashi–Fuks metric will be derived which will help us in localising the invariants related to it. Chapter 6 is completely dedicated towards obtaining the localization results associated to the Kobayashi–Fuks metric, and the main result there is the following:

Theorem 1.0.4. *Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain with a holomorphic peak point $p^0 \in \partial D$. If U is a sufficiently small neighborhood of p^0, then*

(i) $\lim_{z \to p^0} \frac{\tau_{\tilde{B},D}(z,u)}{\tau_{\tilde{B},U\cap D}(z,u)} = 1$ *uniformly in unit vectors* $u \in \mathbb{C}^n$.

(ii) $\lim_{z \to p^0} \frac{g_{\tilde{B},D}(z)}{g_{\tilde{B},U\cap D}(z)} = 1$.

(iii) *If* $n = 1$, *then* $\lim_{z \to p^0} \frac{2 - R_{\tilde{B},D}(z)}{2 - R_{\tilde{B},U\cap D}(z)} = 1$.

In Chapter 7, we first examine the boundary behavior of the Kobayashi–Fuks metric on planar domains by using the Riemann mapping theorem. In particular we find that the Gaussian curvature of the Kobayashi–Fuks metric approaches a fixed negative integer as we move towards the boundary of a smoothly bounded domain.

Theorem 1.0.5. *For a C^2-smoothly bounded domain D in $\mathbb{C}$ with $p^0 \in \partial D$, there exists a constant $C = C(D) > 0$ such that*

(i) $\delta_D^2(z)\tau_{\tilde{B},D}^2(z,u) \to C|u|^2$,

(ii) $R_{\tilde{B},D}(z) \to -\frac{1}{3}$,

as $z \to p^0$. Here $\delta_D(z)$ denotes the Euclidean distance of $z \in D$ to the boundary ∂D.

Theorem 1.0.5 (ii) combined with the arguments in the proof of Theorem 1.17 of [24] immediately yields:

Corollary 1.0.6. *Let $D, D' \subset \mathbb{C}$ be C^2-smoothly bounded domains equipped with the metrics $ds^2_{\tilde{B},D}$ and $ds^2_{\tilde{B},D'}$ respectively. Then every isometry $f : (D, ds^2_{\tilde{B},D}) \to (D', ds^2_{\tilde{B},D'})$ is either holomorphic or conjugate holomorphic.*

The arguments used in the proof of Theorem 1.0.5 are based on the Riemann mapping theorem and thus restricted to dimension one only. To compensate for this we make use of the Pinchuk's scaling method to study the boundary behavior of the Kobayashi–Fuks metric on smoothly bounded strongly pseudoconvex domains in higher dimensions. Let us denote by $\delta_D(z)$ the Euclidean distance from the point $z \in D$ to the boundary ∂D. For z close to ∂D, let $\pi(z) \in \partial D$ be the nearest point to z, i.e., $\delta_D(z) = |z - \pi(z)|$, and for a tangent vector $u \in \mathbb{C}^n$ based at z, let $u = u_H(z) + u_N(z)$ be the decomposition of u along the tangential and normal directions respectively at $\pi(z)$.

Theorem 1.0.7. *Let $D \subset \mathbb{C}^n$ be a C^2-smoothly bounded strongly pseudoconvex domain and $p^0 \in \partial D$. Then there are holomorphic coordinates z near p^0 in which*

(i) $\delta_D(z)\, \tau_{\tilde{B},D}(z,u) \to \frac{1}{2}\sqrt{(n+1)(n+2)}\, |u_N(p^0)|$,

(ii) $\sqrt{\delta_D(z)}\, \tau_{\tilde{B},D}\big(z, u_H(z)\big) \to \sqrt{\frac{1}{2}(n+1)(n+2)\mathcal{L}_{\partial D}\big(p^0, u_H(p^0)\big)}$,

(iii) $\delta_D(z)^{n+1} g_{\tilde{B},D}(z) \to \frac{(n+1)^n(n+2)^n}{2^{n+1}}$,

as $z \to p^0$. Here, $\mathcal{L}_{\partial D}$ is the Levi form of ∂D with respect to some defining function for D.

Note that Theorem 1.0.7 (i), (ii) are analogs of Graham's result [22] for the Kobayashi and Carathéodory metrics. Next, simulating a theorem of Herbort on the Bergman metric, we derive a result that ascertains the existence of closed geodesics in the Kobayashi–Fuks metric with prescribed homotopy class.

Theorem 1.0.8. *Let $D \subset \mathbb{C}^n$ be a smoothly bounded strongly pseudoconvex domain which is not simply connected. Then every nontrivial homotopy class in $\pi_1(D)$ contains a closed geodesic for $ds^2_{\tilde{B},D}$.*

Finally, we end the thesis by posing some relevant open questions on the Kobayashi–Fuks metric as a part of future research plans. Answering these questions will give a comprehensive understanding of this metric in accordance with some of the classical results in the literature on various well-studied metrics such as the Carathéodory metric, the Bergman metric, the Kobayashi metric, etc.

Chapter 2

Preliminaries

2.1 Pseudoconvexity

A domain $D \subset \mathbb{C}^n$ with boundary ∂D is said to have C^k-smooth boundary near $p^0 \in \partial D$, for $k \geq 1$, if there is a real-valued C^k differentiable function ρ defined on a neighborhood U of p^0 such that

(a) $D \cap U = \{z \in U : \rho(z) < 0\}$,

(b) $\partial D \cap U = \{z \in U : \rho(z) = 0\}$,

(c) $\nabla \rho \neq 0 \quad \text{on } \partial D \cap U$.

We call the function ρ a C^k *local defining function* for D near p^0.

If D is a bounded domain, the boundary ∂D is said to be C^k-smooth if there exists a real-valued C^k differentiable function ρ defined on a neighborhood U of $\overline{D}$ such that ρ satisfies the above three conditions (a), (b) and (c). We call ρ a C^k *defining function* for D.

Definition 2.1.1. Let $D \subset \mathbb{C}^n$ be an open set, ∂D be C^1-smooth near p^0, and ρ a local defining function for D near p^0. Under the natural identification $\mathbb{C}^n \approx \mathbb{R}^{2n}$ by $(z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \approx (x_1, y_1, \ldots, x_n, y_n)$, an n-tuple $(w_1, \ldots, w_n) = (u_1 + iv_1, \ldots, u_n + iv_n)$ of complex numbers is called a *tangent vector* to ∂D at p^0 if

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial x_j}(p^0) u_j + \sum_{j=1}^{n} \frac{\partial \rho}{\partial y_j}(p^0) v_j = 0. \tag{2.1}$$

We write $w = (w_1, \ldots, w_n) \in T_{p^0}(\partial D)$. Note that in the complex notation (2.1) can be rewritten as

$$2 \operatorname{Re} \left(\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p^0) w_j \right) = 0.$$

The collection of vectors $w \in \mathbb{C}^n$ that satisfy the above equation is not closed under multiplication by i, and hence is not a natural subject of study in complex analysis. Instead, we restrict our attention to the space of vectors $w \in \mathbb{C}^n$ that satisfy

$$\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p^0) w_j = 0.$$

The collection of all such vectors w is termed as the *complex tangent space* to ∂D at p^0, and is denoted by $H_{p^0}(\partial D)$. Clearly $H_{p^0}(\partial D) \subset T_{p^0}(\partial D)$, and one can check that $H_{p^0}(\partial D)$ is the largest complex subspace of $T_{p^0}(\partial D)$ in the following sense: If S is a real linear subspace of $T_{p^0}(\partial D)$ that is closed under multiplication by i, then $S \subset H_{p^0}(\partial D)$.

Definition 2.1.2. Let D be a domain in $\mathbb{C}^n$ with C^2-smooth boundary near $p^0 \in \partial D$ and ρ be a local defining function for D near p^0. Then D is said to be *pseudoconvex* at $p^0 \in \partial D$ if the Levi form of ρ is positive semidefinite on the complex tangent space at p^0, i.e.,

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \rho}{\partial z_\alpha \partial \overline{z}_\beta}(p^0) w_\alpha \overline{w}_\beta \geq 0 \quad \text{for all } \; w \in H_{p^0}(\partial D). \tag{2.2}$$

Moreover, D is called *strongly pseudoconvex* at p^0, if the Levi form is positive definite on $H_{p^0}(\partial D)$, i.e., the inequality in (2.2) becomes strict for all $w \neq 0 \in H_{p^0}(\partial D)$. A domain $D \subset \mathbb{C}^n$ is called a pseudoconvex (or, strongly pseudoconvex) domain, if D is pseudoconvex (or, strongly pseudoconvex) at every boundary point. It can be checked that the definition of pseudoconvexity is independent of the defining function chosen.

2.2 Concept of finite type

2.2.1 D'Angelo finite 1-type

Given a scalar-valued function $f : \mathbb{C} \to \mathbb{R}$ with $f(0) = 0$, let $\nu(f)$ denote the order of vanishing of f at 0. Hence $\nu(f)$ is the least positive integer such that $\nu(f)$-th derivative of f doesn't vanish at 0. Similarly, for a vector valued function of a complex variable $\phi : \mathbb{C} \to \mathbb{R}^n$ with $\phi(0) = (0, \ldots, 0)$, the order of vanishing $\nu(\phi)$ is defined to be the minimum of the order of vanishing of its components at 0.

Definition 2.2.1. For a domain $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, where ρ is a C^∞ defining function for D, we say that a point $p^0 \in \partial D$ has finite type (or, finite 1-type) in the sense of D'Angelo if

$$\sup \left\{ \frac{\nu(\rho \circ l)}{\nu(l - p^0)} \,\middle|\, l : \mathbb{C} \to \mathbb{C}^n \text{ is a non-constant, one dimensional holomorphic curve such that } l(0) = p^0 \right\} < \infty.$$

The above supremum is called the *1-type* (in the sense of D'Angelo) of ∂D at p^0. If all the points on ∂D has finite 1-type, then D is called a *finite type domain*. It can be shown that the definition of 1-type is independent of the choice of defining function.

Example 2.2.2. Every point on the boundary of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ has 1-type 2. In fact, every smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$ is a finite type domain with every boundary point of 1-type 2.

Example 2.2.3. Consider the following egg domain in $\mathbb{C}^2$,

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$$

with the boundary point $p^0 = (1, 0)$. It can be shown that p^0 has 1-type $2m$ on the boundary.

2.2.2 Catlin's multitype

Corresponding to each point z on the boundary of a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$, D. Catlin [8] introduced an invariant $\mathcal{M}(z)$ generally called as *Catlin's multitype* of z. This invariant arises from the study of the boundary regularity properties of solutions of the $\overline{\partial}$-Neumann problem on finite type domains (in the sense of D'Angelo). Before formally defining the multitype $\mathcal{M}(z)$, we introduce some notations.

Let D be a domain in $\mathbb{C}^n$ and ρ a smooth local defining function for D near $p^0 \in \partial D$. Let Γ_n denote the set of all n-tuples of real numbers $\Lambda = (\lambda_1, \ldots, \lambda_n)$ with $1 \leq \lambda_i \leq +\infty$ such that

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

and for each k, either $\lambda_k = +\infty$ or there is a set of nonnegative integers $a_1, \ldots, a_k$ with $a_k > 0$ such that

$$\sum_{j=1}^{k} \frac{a_j}{\lambda_j} = 1.$$

An element of Γ_n will be referred to as a weight. The set of weights can be ordered lexicographically as follows: If $\Lambda' = (\lambda'_1, \ldots, \lambda'_n)$ and $\Lambda'' = (\lambda''_1, \ldots, \lambda''_n)$, then $\Lambda' < \Lambda''$ if for some k, $\lambda'_j = \lambda''_j$ for all $j < k$, but $\lambda'_k < \lambda''_k$. A weight $\Lambda \in \Gamma_n$ is said to be *distinguished* if there exist holomorphic coordinates $(z_1, \ldots, z_n)$ about p^0 with p^0 mapped to the origin such that

$$\text{If } \sum_{i=1}^{n} \frac{\alpha_i + \beta_i}{\lambda_i} < 1, \text{ then } D^\alpha \overline{D}^\beta \rho(p^0) = 0.$$

Here D^α and $\overline{D}^\beta$ denote the partial derivative operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \quad \text{and} \quad \frac{\partial^{|\beta|}}{\partial \overline{z}_1^{\beta_1} \cdots \partial \overline{z}_n^{\beta_n}}$$

respectively. The Catlin's multitype $\mathcal{M}(\partial D, p^0)$ is defined to be the smallest weight (in the lexicographic sense) $\mathcal{M} = (m_1, \ldots, m_n)$ in Γ_n such that $\mathcal{M} \geq \Lambda$ for every distinguished weight Λ.

Example 2.2.4. Let D be a domain in $\mathbb{C}^n$ and $p^0 \in \partial D$ be a smooth boundary point. If p^0 is a strongly pseudoconvex boundary point, then $\mathcal{M}(\partial D, p^0) = (1, 2, \ldots, 2)$. More generally, if the Levi form of ∂D has rank k at p^0, then $\mathcal{M}(\partial D, p^0) = (m_1, \ldots, m_n)$ where $m_1 = 1$, $m_j = 2$ for $2 \leq j \leq k+1$, and $m_j > 2$ for $j > k+1$.

Example 2.2.5. Consider the following domain $D \subset \mathbb{C}^3$ defined by

$$\left\{ \rho(z_1, z_2, z_3) = 2 \operatorname{Re} z_3 + \left| z_1^2 - z_2^3 \right|^2 < 0 \right\}.$$

Clearly $0 \in \partial D$, and the multitype $\mathcal{M}(\partial D, 0) = (1, 4, 6)$.

2.3 Complete hyperbolicity and tautness

The infinitesimal Kobayashi pseudometric on a domain $D \subset \mathbb{C}^n$ is defined by

$$F_D^K(z, \zeta) = \inf \left\{ \alpha > 0 : \text{there exists } f \in \mathcal{O}(\Delta, D) \text{ with } f(0) = z, \alpha f'(0) = \zeta \right\}.$$

The Kobayashi pseudodistance is the integrated form of the Kobayashi pseudometric described as follows. Let $\gamma : [0,1] \to D$ be a piecewise C^1 curve. The *Kobayashi length* of γ is defined to be

$$L_D^K(\gamma) = \int_0^1 F_D^K\big(\gamma(t), \gamma'(t)\big)\, dt.$$

Let w_1 and w_2 be two points in D. The *Kobayashi pseudodistance* between w_1 and w_2 is defined to be

$$d_D^K(w_1, w_2) = \inf\left\{L_D^K(\gamma) : \gamma \text{ is a piecewise } C^1 \text{ curve such that } \gamma(0) = w_1 \text{ and } \gamma(1) = w_2\right\}.$$

Definition 2.3.1. A domain $D \subset \mathbb{C}^n$ is said to be *hyperbolic*, if the Kobayashi pseudodistance is an actual distance on D. In addition to that, if D is complete with respect to the Kobayashi distance, then D is called a *complete hyperbolic domain.*

Definition 2.3.2. A complex manifold X is said to be *taut* if $\mathcal{O}(\Delta, X)$ is a normal family, i.e., given any sequence $(f_j)_j \subset \mathcal{O}(\Delta, X)$, there exists a subsequence $(f_{j_\nu})_\nu$ which either

- converges uniformly on compact subsets of Δ to a function $f \in \mathcal{O}(\Delta, X)$, or
- diverges uniformly on compact sets, i.e., given any two compact sets $L \subset \Delta$ and $M \subset X$, there exists $k = k(L, M) \in \mathbb{N}$ such that $f_{j_\nu}(L) \cap M = \emptyset$ for all $\nu \geq k$.

It can be shown that any complete hyperbolic domain is a taut domain (see [21]).

2.4 Local Hausdorff convergence

A sequence of domains $D_j \subset \mathbb{C}^n$ is said to converge in the local Hausdorff topology to a domain $D_\infty \subset \mathbb{C}^n$ if the following conditions are satisfied:

(i) For any compact set K contained in the interior of $\cap_{j>m} D_j$ for some positive integer m, $K \subset D_\infty$.

(ii) For any compact subset L of D_∞, there exists a positive integer $n > 0$ such that $L \subset D_j$ for every $j > n$.

2.5 Holomorphic peak points

A boundary point p^0 of a domain $D \subset \mathbb{C}^n$ is called a *global holomorphic peak point*, or simply a *global peak point* of D, if there exists a holomorphic function f such that for any neighborhood W of p^0 with $D \setminus W \neq \emptyset$ one has

$$\sup_{z \in D \setminus W} |f(z)| < 1 = \lim_{z \to p^0} f(z).$$

We say that $p^0 \in \partial D$ is a *local holomorphic peak point*, or simply a *local peak point* of D, if there is a neighborhood U of p^0 such that p^0 is a global peak point of $U \cap D$.

2.6 Ricci and holomorphic sectional curvatures

Let (X, J) be an n-dimensional complex manifold with Kähler metric g. If R is the Riemannian curvature tensor of (X, g), then the *holomorphic sectional curvature* $H_g(v)$ of a non-zero vector v is defined to be the sectional curvature of the 2-plane spanned by v and Jv. That is,

$$H_{g,X}(v) := \frac{R(v, Jv, Jv, v)}{\|v\|_g^4}.$$

In the local coordinates, suppose the Kähler metric g is given in the form

$$ds^2_{g,X}(z) = \sum_{\alpha,\beta=1}^{n} g^X_{\alpha\overline{\beta}}(z)\, dz_\alpha d\overline{z}_\beta,$$

and let

$$G_{g,X}(z) = \left(g^X_{\alpha\overline{\beta}}(z)\right)_{n\times n}.$$

It can be shown that the holomorphic sectional curvature of $ds^2_{g,X}$ is given by

$$H_{g,X}(z, v) = \frac{\sum_{\alpha,\beta,\gamma,\delta=1}^{n} R^{g,X}_{\overline{\alpha}\beta\gamma\overline{\delta}}(z)\overline{v}^\alpha v^\beta v^\gamma \overline{v}^\delta}{\left(\sum_{\alpha,\beta=1}^{n} g^X_{\alpha\overline{\beta}}(z) v^\alpha \overline{v}^\beta\right)^2},$$

where

$$R^{g,X}_{\overline{\alpha}\beta\gamma\overline{\delta}}(z) = -\frac{\partial^2 g^X_{\beta\overline{\alpha}}}{\partial z_\gamma \partial \overline{z}_\delta}(z) + \sum_{\mu,\nu} g_X^{\nu\overline{\mu}}(z)\frac{\partial g^X_{\beta\overline{\mu}}}{\partial z_\gamma}(z)\frac{\partial g^X_{\nu\overline{\alpha}}}{\partial \overline{z}_\delta}(z), \tag{2.3}$$

$g_X^{\nu\overline{\mu}}(z)$ being the (ν, μ)th entry of the inverse of the matrix $G_{g,X}(z)$.

The *Ricci tensors* of $ds^2_{g,X}$ are defined by

$$\mathrm{Ric}^{g,X}_{\alpha\overline{\beta}}(z) = -\frac{\partial^2 \log\det G_{g,X}}{\partial z_\alpha \partial \overline{z}_\beta}(z),$$

and the *Ricci curvature* of $ds^2_{g,X}$ in the direction of v is given by

$$\mathrm{Ric}_{g,X}(z, v) = \frac{\sum_{\alpha,\beta=1}^{n} \mathrm{Ric}^{g,X}_{\alpha\overline{\beta}}(z) v^\alpha \overline{v}^\beta}{\sum_{\alpha,\beta=1}^{n} g^X_{\alpha\overline{\beta}}(z) v^\alpha \overline{v}^\beta}.$$

Chapter 3

The Carathéodory and Kobayashi-Eisenman volume elements

For a domain $D \subset \mathbb{C}^n$, recall that the Carathéodory and Kobayashi-Eisenman volume elements on D at a point $p \in D$ are defined respectively by

$$c_D(p) = \sup\Big\{\big|\det \psi'(p)\big|^2 : \psi \in \mathcal{O}(D, \mathbb{B}^n), \psi(p) = 0\Big\},$$
$$k_D(p) = \inf\Big\{\big|\det \psi'(0)\big|^{-2} : \psi \in \mathcal{O}(\mathbb{B}^n, D), \psi(0) = p\Big\}.$$

By Montel's theorem we will see that $c_D(p)$ is always attained and if D is taut then $k_D(p)$ is also attained. Under a holomorphic map $F : D \to \Omega$, they satisfy the rule

$$v_D(p) \geq \big|\det F'(p)\big|^2 v_\Omega\big(F(p)\big)$$

where $v = c, k$. In particular, equality holds if F is a biholomorphism. Accordingly, if k_D is nonvanishing (which is the case if D is bounded or taut), then

$$q_D(p) = \frac{c_D(p)}{k_D(p)}$$

is a biholomorphic invariant and is called the quotient invariant. If $D = \mathbb{B}^n$, then

$$c_{\mathbb{B}^n}(p) = k_{\mathbb{B}^n}(p) = \big(1 - |p|^2\big)^{-n-1},$$

and thus $q_{\mathbb{B}^n}$ is identically equal to 1. In general, an application of the Schwarz lemma shows that $q_D \leq 1$. It is a remarkable fact that if D is any domain in $\mathbb{C}^n$ and $q_D(p) = 1$ for some point $p \in D$, then $q_D(z) = 1$ for all $z \in D$ and D is biholomorphic to $\mathbb{B}^n$. This was first proved by Wong [48] with the hypothesis that D is bounded and complete hyperbolic, which was relaxed by Rosay [45] to D being any bounded domain. Dektyarev [11] further relaxed this condition to D being only hyperbolic and later Graham and Wu [23] showed that no assumption on D is required for the result to be true, in fact, it is true for any complex manifold. Thus q_D measures the extent to which the Riemann mapping theorem fails for D. This fact is a fundamental step in the proof of the Wong-Rosay theorem and several other applications can be found in [25, 26, 37].

In this chapter, we study a few properties of the Carathéodory and Kobayashi-Eisenman volume elements on various domains. After establishing certain regularity properties of the volume elements in the next section using the normal family arguments, one can remark that these volume elements are well behaved on taut domains. To showcase one of the importances of studying the boundary behavior of biholomorphic invariants, towards the end of this chapter, we will give a proof of Theorem 1.0.1. This result shows an efficacy of the quotient invariant in determining strong pseudoconvexity if its boundary behavior is a priori known—a property enjoyed by the squeezing function and its dual the Fridman invariant as well. We refer the reader to the recent articles [39, 43] and the references therein for the definition and other relevant materials related to these two invariants. Let us denote the squeezing function for a domain D by s_D and the Fridman invariant by h_D. It was proved in [51] that if D is a bounded convex domain with $C^{2,\alpha}$ boundary for some $\alpha \in (0,1)$, then D is strongly pseudoconvex if $s_D(z) \to 1$ as $z \to \partial D$. Mahajan and Verma [39] showed that if D is a smoothly bounded convex domain or if D is a smoothly bounded h-extendible domain (i.e., D is a smoothly bounded pseudoconvex finite type domain for which the Catlin and D'Angelo multitypes coincide at every boundary point), then D is strongly pseudoconvex if either $h_D(z) \to 0$ or $s_D(z) \to 1$ as $z \to \partial D$.

3.1 Regularity of the volume elements

In this section, we prove continuity of the volume elements that will be required in computing the boundary asymptotics. The arguments are similar to the case of the Carathéodory-Reiffen and Kobayashi-Royden pseudometrics and we present them only for convenience. First, a few remarks. If $D \subset \mathbb{C}^n$ is any domain and $p \in D$, then $c_D(p)$ is attained. Indeed, choose a sequence $\psi^j \in \mathcal{O}(D, \mathbb{B}^n)$ such that $\psi^j(p) = 0$ and $|\det(\psi^j)'(p)|^2 \to c_D(p)$. By Montel's theorem, passing to a subsequence if necessary, ψ^j converges uniformly on compact subsets of D to a map $\psi \in \mathcal{O}(D, \overline{\mathbb{B}^n})$. Since $\psi(p) = 0$, by the maximum principle $\psi \in \mathcal{O}(D, \mathbb{B}^n)$, and it follows that $c_D(p) = |\det \psi'(p)|^2$. In particular, this implies that $c_D(p)$ is always finite. Note that $c_D(p)$ can vanish (for example if $D = \mathbb{C}$), but is strictly positive if D is not a Liouville domain. Likewise, if D is taut then similar arguments as above shows that $k_D(p)$ is attained. Observe that $k_D(p)$ is finite for any domain D because we can put a ball $B(p, r)$ inside D and consequently $\phi(t) = rt + p$ is a competitor for $k_D(p)$, giving us $k_D(p) \le r^{-2n}$. It is possible that $k_D(p)$ can also vanish but if D is bounded, then by invoking Cauchy's estimates we see that $k_D(p) > 0$. Similarly, if D is taut, then also $k_D(p) > 0$ as it is attained. We will call a map $\psi \in \mathcal{O}(D, \mathbb{B}^n)$ satisfying $\psi(p) = 0$ and $|\det \psi'(p)|^2 = c_D(p)$ a Carathéodory extremal map for D at p. Similarly, a Kobayashi extremal map for D at p is a map $\psi \in \mathcal{O}(\mathbb{B}^n, D)$ with $\psi(0) = p$ and $|\det \psi'(0)|^{-2} = k_D(p)$.

Proposition 3.1.1. *Let $D \subset \mathbb{C}^n$ be a domain. Then c_D is continuous. If D is taut, then k_D is also continuous.*

Proof. We will show that c_D is locally Lipschitz which of course implies that c_D is continuous. Let $B(a, 2r) \subset\subset D$ and fix $p, q \in B(a, r)$. Choose a Carathéodory extremal map ψ for D at p. Then

$$\begin{aligned} c_D(p) - c_D(q) &\le \big|\det \psi'(p)\big|^2 - \big|\det \psi'(q)\big|^2 c_{\mathbb{B}^n}\big(\psi(q)\big) \\ &= \big|\det \psi'(p)\big|^2 - \frac{\big|\det \psi'(q)\big|^2}{\big(1 - |\psi(q)|^2\big)^{n+1}} \le \big|\det \psi'(p)\big|^2 - \big|\det \psi'(q)\big|^2. \end{aligned}$$

Since the distances of p and q to ∂D is at least r, by Cauchy's estimates the right hand side is bounded above by $C_r|p-q|$ where C_r is a constant that depends only on r. Thus we can interchange the role of p and q to have $|c_D(p)-c_D(q)| \leq C_r|p-q|$ that establishes local Lipschitz property of c_D.

For k_D, first we show that it is upper semicontinuous for any domain D. Let $p \in D$ and $\epsilon > 0$. Then there exists $\phi \in \mathcal{O}(\mathbb{B}^n, D)$ with $\phi(0) = p$ such that

$$|\det \phi'(0)|^{-2} < k_D(p) + \epsilon. \tag{3.1}$$

Let $0 < r < 1$ and set for $z \in D$,

$$f^z(t) = \phi\big((1-r)t\big) + (z-p), \quad t \in \mathbb{B}^n.$$

Since $\phi\big(B(0, 1-r)\big)$ is a relatively compact subset of D, there exists $\delta > 0$ such that if $z \in B(p,\delta)$, then $f^z \in \mathcal{O}(\mathbb{B}^n, D)$. Also $f^z(0) = z$ and so f^z is a competitor for $k_D(z)$. Therefore,

$$k_D(z) \leq \big|\det(f^z)'(0)\big|^{-2} = (1-r)^{-2n}\big|\det \phi'(0)\big|^{-2}.$$

Letting $r \to 0^+$ and using (3.1), we obtain that

$$k_D(z) < k_D(p) + \epsilon$$

for all $z \in B(p,\delta)$ which proves the upper semicontinuity of k_D.

Next we assume that D is taut and show that k_D is lower semicontinuous. Let $p \in D$. If possible, assume that k_D is not lower semicontinuous at p. Then $k_D(p) > 0$ and there exist $\epsilon > 0$, a sequence $p^j \to p$, such that

$$k_D(p^j) < k_D(p) - \epsilon.$$

Since D is taut, there are Kobayashi extremal maps g^j for D at p^j. Again by tautness and the fact that $g^j(0) = p^j \to p \in D$, passing to a subsequence, g^j converges uniformly on compact subsets of $\mathbb{B}^n$ to a map $g \in \mathcal{O}(\mathbb{B}^n, D)$. Therefore,

$$k_D(p^j) = \big|\det(g^j)'(0)\big|^{-2} \to \big|\det g'(0)\big|^{-2}.$$

But g is a competitor for $k_D(p)$ and so $k_D(p) \leq |\det g'(0)|^{-2}$. Thus we have

$$k_D(p) \leq k_D(p) - \epsilon$$

which is a contradiction. This proves the lower semicontinuity of k_D and thus k_D is continuous if D is taut. □

3.2 Localization of the Kobayashi-Eisenman volume element

In the process of studying the boundary behavior of any object, its localization plays an important role. In this section, we list one such result for the Kobayashi-Eisenman volume element, which says: the Kobayashi-Eisenman volume element can be localized near a holomorphic peak point of a bounded domain in $\mathbb{C}^n$. We can actually replace the hypothesis "bounded domain" by "hyperbolic domain" and still get the desired localization.

Definition 3.2.1. Let z, w be two points in a domain D, and d_K denote the Kobayashi distance on $\mathbb{B}^n$. Then

$$l_D(z,w) := \inf\big\{d_K(a,b) : \text{there exists } f \in \mathcal{O}(\mathbb{B}^n, D) \text{ such that } f(a) = z, f(b) = w\big\}.$$

Using the homogeneity of $\mathbb{B}^n$, l_D can be rewritten as

$$l_D(z,w) = \inf\big\{d_K(0, r\mathbb{1}) : r \geq 0, \text{ and there exists } f \in \mathcal{O}(\mathbb{B}^n, D) \text{ such that } f(0) = z, f(r\mathbb{1}) = w\big\},$$

where $r\mathbb{1}$ denotes the point $(r, 0') \in \mathbb{B}^n$.

Definition 3.2.2. Let $D_1 \subset D$ and $z \in D_1$. Then

$$l_{D\setminus D_1}(z) := \inf\big\{l_D(z,w) : w \in D \setminus D_1\big\}.$$

Lemma 3.2.3. *Let D be any bounded domain in $\mathbb{C}^n$ and $D_1 \subset D$ be any non-empty subdomain. Then,*

$$k_{D_1}(z_0) \leq \big(\coth l_{D\setminus D_1}(z_0)\big)^{2n} k_D(z_0)$$

for any $z_0 \in D_1$.

Proof. First, observe that $l_{D\setminus D_1}(z_0) > 0$ since D is bounded and hence a hyperbolic domain. Secondly, since $\coth d_K(0, r\mathbb{1}) = 1/r$ for $0 < r < 1$, and coth is a decreasing function, we may write

$$\coth l_D(z,w) = \sup\big\{1/r > 1 : \text{there exists } f \in \mathcal{O}(\mathbb{B}^n, D) \text{ such that } f(0) = z, f(r\mathbb{1}) = w\big\},$$

for $z, w \in D$, $z \neq w$. Similarly,

$$\coth l_{D\setminus D_1}(z_0) = \sup\big\{1/r : \text{there exists } f \in \mathcal{O}(\mathbb{B}^n, D) \text{ such that } f(0) = z_0, f(r\mathbb{1}) = w \in D\setminus D_1\big\}.$$

Now, let $\psi \in \mathcal{O}(\mathbb{B}^n, D)$ be an arbitrary map with $\psi(0) = z_0$. Moreover, choose an $s > 0$ with $1/s > \coth l_{D\setminus D_1}(z_0)$. Then clearly $0 < s < 1$ and we claim that every $f \in \mathcal{O}(\mathbb{B}^n, D)$ with $f(0) = z_0$ maps $B(0, s)$ into D_1. If possible, assume this is not true. Then there would exist $g \in \mathcal{O}(\mathbb{B}^n, D)$, $w \in D \setminus D_1$ such that $g(0) = z_0$ and $g(s\mathbb{1}) = w$. This immediately implies that $\coth l_{D\setminus D_1}(z_0) > 1/s$, which is a contradiction and hence our claim is proved. Now if we put $\tilde{\psi}(z) := \psi(sz)$, $z \in \mathbb{B}^n$, then $\tilde{\psi} \in \mathcal{O}(\mathbb{B}^n, D_1)$ with $\tilde{\psi}(0) = z_0$ and

$$\frac{1}{|\det \tilde{\psi}'(0)|^2} = \left(\frac{1}{s}\right)^{2n} \frac{1}{|\det \psi'(0)|^2}.$$

Since s is arbitrary, by letting $s \to 1/\coth l_{D\setminus D_1}(z_0)$ in the above equation and by the property of infimum, the conclusion of the lemma follows. $\square$

Theorem 3.2.4. *Let $D \subset \mathbb{C}^n$ be a bounded domain with a holomorphic peak point $p^0 \in \partial D$. Then for any neighborhood U of p^0,*

$$\lim_{z\to p^0} \frac{k_D(z)}{k_{U\cap D}(z)} = 1.$$

Proof. By Lemma 3.2.3, we have

$$k_{U\cap D}(z) \leq \left(\coth l_{D\setminus(U\cap D)}(z)\right)^{2n} k_D(z) \tag{3.2}$$

for every $z \in U \cap D$. Similar arguments in the proof of Theorem 19.3.2 in [31], using the fact that p^0 is a holomorphic peak point for D, implies

$$l_{D\setminus(U\cap D)}(z) \to +\infty \quad \text{as } z \to p^0.$$

Therefore letting $z \to p^0$ in (3.2) we obtain

$$\limsup_{z\to p^0} \frac{k_{U\cap D}(z)}{k_D(z)} \leq 1.$$

Again, the monotonicity property of the volume elements imply

$$k_D(z) \leq k_{U\cap D}(z).$$

Therefore, our result follows from the last two inequalities. □

3.3 Detecting strong pseudoconvexity

A convex domain $D \subset \mathbb{C}^n$ is called $\mathbb{C}$-properly convex if it does not contain any affine complex line. Let $\mathbb{X}_n$ denote the set of all $\mathbb{C}$-properly convex domains endowed with the local Hausdorff topology. Consider the space

$$\mathbb{X}_{n,0} = \left\{(D,p) : D \in \mathbb{X}_n, p \in D\right\} \subset \mathbb{X}_n \times \mathbb{C}^n$$

endowed with the subspace topology. It was shown in [2] that a convex domain in $\mathbb{C}^n$ is complete hyperbolic if and only if it is $\mathbb{C}$-properly convex. In particular, $\mathbb{C}$-properly convex domains are taut and hence the quotient invariant on such domains are well-defined. Thus we have a function $q : \mathbb{X}_{n,0} \to \mathbb{R}$ defined by

$$q(D,p) = q_D(p).$$

Recall that a function $f : \mathbb{X}_{n,0} \to \mathbb{R}$ is called intrinsic (see [50]) if $f(D,p) = f(D',p')$ whenever there exits a biholomorphism $F : D \to D'$ with $F(p) = p'$. Thus the function q is intrinsic. The following theorem was proved by Zimmer:

Theorem 3.3.1 ([52])**.** *Let $f : \mathbb{X}_{n,0} \to \mathbb{R}$ be an upper semicontinuous intrinsic function with the following property: if $D \in \mathbb{X}_n$ and $f(D,p) \geq f(\mathbb{B}^n, 0)$ for all $p \in D$, then D is biholomorphic to $\mathbb{B}^n$. Then for any $\alpha > 0$, there exists some $\epsilon = \epsilon(n, f, \alpha) > 0$ such that: if $D \subset \mathbb{C}^n$ is a bounded convex domain with $C^{2,\alpha}$ boundary and*

$$f(D,p) \geq f(\mathbb{B}^n, 0) - \epsilon$$

outside some compact subset of D, then D is strongly pseudoconvex.

Observe that if $D \subset \mathbb{C}^n$ is any domain and if $q_D(p) \geq 1$ for some point $p \in D$, then $q_D(p) = 1$ and so D must be biholomorphic to $\mathbb{B}^n$. Thus, to prove Theorem 1.0.1, we only need to show that the function $q : \mathbb{X}_{n,0} \to \mathbb{R}$ is upper semicontinuous.

Lemma 3.3.2. *Suppose $(D^j, p^j) \to (D_\infty, p)$. If $f^j : \mathbb{B}^n \to D^j$, $f^j(0) = p^j$, then passing to a subsequence, f^j converges uniformly on compact subsets of $\mathbb{B}^n$ to a holomorphic function f on $\mathbb{B}^n$.*

This is precisely Lemma 4.2 of [50] with Δ replaced by $\mathbb{B}^n$ and since the proof is exactly the same we do not repeat it.

Proposition 3.3.3. *The function $q : \mathbb{X}_{n,0} \to \mathbb{R}$ is upper semicontinuous.*

Proof. Let $(D^j, a^j) \to (D_\infty, a)$ in $\mathbb{X}_{n,0}$. We prove the upper semicontinuity of q in two steps, first showing

$$k_{D_\infty}(a) \le \liminf_{j\to\infty} k_{D^j}(a^j), \tag{3.3}$$

and then showing

$$\limsup_{j\to\infty} c_{D^j}(a^j) \le c_{D_\infty}(a) \tag{3.4}$$

in the next steps.

Step 1. Fix $\epsilon > 0$ arbitrarily small. Then there exist $\phi^j \in \mathcal{O}(\mathbb{B}^n, D^j)$ such that $\phi^j(0) = a^j$ and

$$\left|\det(\phi^j)'(0)\right|^{-2} < k_{D^j}(a^j) + \epsilon. \tag{3.5}$$

By Lemma 3.3.2, ϕ^j admits a subsequence which we denote by ϕ^j itself, and which converges uniformly on compact subsets of $\mathbb{B}^n$ to a map $\phi \in \mathcal{O}(\mathbb{B}^n, D_\infty)$. Then from (3.5),

$$\left|\det \phi'(0)\right|^{-2} \le \liminf_{j\to\infty} k_{D^j}(a^j) + \epsilon.$$

But ϕ is a competitor for $k_{D_\infty}(a)$ and ϵ is arbitrary, hence (3.3) follows.

Step 2. Since the Carathéodory volume element is always attained, let us consider ψ^j as Carathéodory extremal maps for D^j at a^j, i.e., $\psi^j \in \mathcal{O}(D^j, \mathbb{B}^n), \psi^j(a^j) = 0$, and

$$c_{D^j}(a^j) = \left|\det(\psi^j)'(a^j)\right|^2. \tag{3.6}$$

Since the target of these extremal maps is $\mathbb{B}^n$, passing to a subsequence if necessary, ψ^j converges uniformly on compact subsets of D_∞ to a holomorphic map $\psi : D_\infty \to \overline{\mathbb{B}^n}$, and since $\psi(a) = 0$, we must have $\psi \in \mathcal{O}(D_\infty, \mathbb{B}^n)$. Now, taking limit in equation (3.6), one obtains

$$\limsup_{j\to\infty} c_{D^j}(a^j) = \left|\det \psi'(a)\right|^2.$$

Since ψ is a candidate function for $c_{D_\infty}(a)$, (3.4) follows from the above identity by the property of supremum.

Combining Step 1 and Step 2, the upper semicontinuity of q is established. $\square$

Thus, we have shown that q satisfies the hypothesis of Theorem 3.3.1 and this completes the proof of Theorem 1.0.1.

Chapter 4

Boundary behavior of the volume elements

Our main purpose here is to study the boundary asymptotics of the Kobayashi volume element on smoothly bounded convex finite type domains and Levi corank one domains in $\mathbb{C}^n$. Boundary behavior of the quotient invariant on strongly pseudoconvex domains had been studied by several authors, see for example [9, 25, 38], and in particular it is known that $q_D(z) \to 1$ if $z \to \partial D$ for a strongly pseudoconvex domain D. Recently in [41], *nontangential* boundary asymptotics of the volume elements near h-extendible boundary points were obtained. Finally, we also note that in [42], a relation between the Carathéodory volume element and the Bergman kernel was observed in light of the multidimensional Suita conjecture. Our goal is to compute the boundary asymptotics of the Kobayashi volume element in terms of the distinguished polydiscs of McNeal and Catlin devised to capture the geometry of a domain near a convex finite type and Levi corank one boundary point respectively.

One of the most convenient tools to compute boundary asymptotics of holomorphic invariants is the scaling method. Roughly speaking, the process of scaling magnifies a small neighborhood of that boundary point via a sequence of biholomorphisms and obtains a sequence of domains which are the images of the biholomorphisms of that small neighborhood. These sequence of domains are generally called the scaled domains. Now, studying the behavior of the volume elements near the specific boundary point of our given domain amounts to studying those objects in the interior of the obtained scaled domains. Since interior problems are much more easier to handle than the boundary problems in general, scaling becomes a handy tool. We will present two different scaling methods, one on convex finite type domains and another on Levi corank one domains, in subsequent sections of this chapter.

4.1 Convex finite type case

In the hypothesis of Theorem 1.0.2, we are given a smoothly bounded convex finite type domain $D = \{\rho < 0\}$ with a sequence $p^j \in D$ converging to $p^0 \in \partial D$. Without loss of generality assume that $p^0 = 0$. The numbers ϵ_j are defined by $\epsilon_j = -\rho(p^j)$. Recall the construction of the translation T^{p^j,ϵ_j} and the unitary transformation U^{p^j,ϵ_j} from Chapter 1. The maps $U^{p^j,\epsilon_j} \circ T^{p^j,\epsilon_j}$ satisfy

$$U^{p^j,\epsilon_j} \circ T^{p^j,\epsilon_j}(p^j) = 0.$$

4.1.1 Scaling

Consider the dilations

$$\Lambda^{p^j,\epsilon_j}(z) = \left(\frac{z_1}{\tau_1(p^j,\epsilon_j)}, \ldots, \frac{z_n}{\tau_n(p^j,\epsilon_j)} \right).$$

The scaling maps are the compositions $S^j = \Lambda^{p^j,\epsilon_j} \circ U^{p^j,\epsilon_j} \circ T^{p^j,\epsilon_j}$ and the scaled domains are $D^j = S^j(D)$. Note that D^j is convex and $S^j(p^j) = 0 \in D^j$, for each j. It was shown in [20] that the defining functions $\rho^j = \frac{1}{\epsilon_j}\rho \circ (S^j)^{-1}$ for D^j, after possibly passing to a subsequence, converge uniformly on compact subsets of $\mathbb{C}^n$ to

$$\rho_\infty(z) = -1 + \operatorname{Re} \sum_{\alpha=1}^{n} b_\alpha z_\alpha + P_{2m}('z),$$

where b_α are complex numbers and P_{2m} is a real convex polynomial of degree less than or equal to $2m$. This implies that after passing to a subsequence if necessary, the domains D^j converge in the local Hausdorff sense to the limiting domain $D_\infty = \{\rho_\infty < 0\}$. It is known that D_∞ possesses a local holomorphic peak function at every boundary point including the point at infinity and hence is complete hyperbolic (see [21]).

4.1.2 Stability of the volume elements

Lemma 4.1.1. *Let $\phi^j \in \mathcal{O}(\mathbb{B}^n, D^j)$ and $\phi^j(0) = a^j \to a \in D_\infty$. Then ϕ^j admits a subsequence that converges uniformly on compact subsets of $\mathbb{B}^n$ to a map $\phi \in \mathcal{O}(\mathbb{B}^n, D_\infty)$.*

Proof. By the arguments in the proof of Lemma 3.1 in [20], observe that the family ϕ^j is normal. Also, $\phi^j(0) = a^j \to a$. Hence, the sequence ϕ^j admits a subsequence, which we denote by ϕ^j itself, and which converges uniformly on compact subsets of $\mathbb{B}^n$ to a holomorphic map $\phi : \mathbb{B}^n \to \mathbb{C}^n$. We will now show that $\phi \in \mathcal{O}(\mathbb{B}^n, D_\infty)$.

Let $0 < r < 1$. Then ϕ^j converges uniformly on $B(0,r)$ to ϕ, and so the sets $\phi^j(B(0,r)) \subset K$ for some fixed compact set K and for all large j. Since $\rho^j(\phi^j(t)) < 0$ for $t \in B(0,r)$ and for all j, we have $\rho_\infty(\phi(t)) \leq 0$, or equivalently $\phi(B(0,r)) \subset \overline{D}_\infty$. Since $r \in (0,1)$ is arbitrary, we have $\phi(\mathbb{B}^n) \subset \overline{D}_\infty$. Since $\phi(0) = a \in D_\infty$, and D_∞ possesses a local holomorphic peak function at every boundary point, the maximum principle implies that $\phi(\mathbb{B}^n) \subset D_\infty$. □

Proposition 4.1.2. *For any $a \in D_\infty$,*

$$\lim_{j\to\infty} k_{D^j}(a) = k_{D_\infty}(a).$$

Moreover, this convergence is uniform on compact subsets of D_∞.

Proof. Assume that k_{D^j} does not converge to k_{D_∞} uniformly on some compact subset $S \subset D_\infty$. Then there exist $\epsilon_0 > 0$, a subsequence of k_{D^j} which we denote by k_{D^j} itself, and a sequence $a^j \in S$ satisfying

$$\left| k_{D^j}(a^j) - k_{D_\infty}(a^j) \right| > \epsilon_0$$

for all large j. Since S is compact, after passing to a subsequence if necessary, $a^j \to a \in S$. Since D_∞ is complete hyperbolic, and hence taut, k_{D_∞} is continuous by Proposition 3.1.1. Hence for all large j, we have

$$\left| k_{D_\infty}(a^j) - k_{D_\infty}(a) \right| \leq \frac{\epsilon_0}{2}.$$

Combining the above two inequalities we have

$$\big|k_{D^j}(a^j) - k_{D_\infty}(a)\big| > \frac{\epsilon_0}{2} \tag{4.1}$$

for all large j. We will deduce a contradiction in the following two steps:

Step 1. Since D_∞ is taut, we have $0 < k_{D_\infty}(a) < \infty$ and there exists a Kobayashi extremal map ψ for D_∞ at a. Fix $0 < r < 1$ and define the holomorphic maps $\psi^j : \mathbb{B}^n \to \mathbb{C}^n$ by

$$\psi^j(t) = \psi\big((1-r)t\big) + (a^j - a).$$

Since the image $\psi\big(B(0, 1-r)\big)$ is compactly contained in D_∞ and $a^j \to a$ as $j \to \infty$, it follows that $\psi^j \in \mathcal{O}(\mathbb{B}^n, D^j)$ for all large j. Also, $\psi^j(0) = \psi(0) + a^j - a = a^j$ and thus ψ^j is a competitor for $k_{D^j}(a^j)$. Therefore,

$$k_{D^j}(a^j) \le \big|\det(\psi^j)'(0)\big|^{-2} = (1-r)^{-2n}\big|\det\psi'(0)\big|^{-2}.$$

Letting $r \to 0^+$, we get

$$\limsup_{j\to\infty} k_{D^j}(a^j) \le k_{D_\infty}(a).$$

Step 2. Fix $\epsilon > 0$ arbitrarily small. Then there exist $\phi^j \in \mathcal{O}(\mathbb{B}^n, D^j)$ such that $\phi^j(0) = a^j$ and

$$\big|\det(\phi^j)'(0)\big|^{-2} < k_{D^j}(a^j) + \epsilon. \tag{4.2}$$

By Lemma 4.1.1, ϕ^j admits a subsequence which we denote by ϕ^j itself, and which converges uniformly on compact subsets of $\mathbb{B}^n$ to a map $\phi \in \mathcal{O}(\mathbb{B}^n, D_\infty)$. Then from (4.2)

$$\big|\det\phi'(0)\big|^{-2} \le \liminf_{j\to\infty} k_{D^j}(a^j) + \epsilon$$

But ϕ is a competitor for $k_{D_\infty}(a)$ and ϵ is arbitrary. So we obtain

$$k_{D_\infty}(a) \le \liminf_{j\to\infty} k_{D^j}(a^j)$$

as required.

By Step 1 and Step 2, we have $\lim_{j\to\infty} k_{D^j}(a^j) = k_{D_\infty}(a)$ which contradicts (4.1) and thus the proposition is proved. □

We believe that the analog of the above stability result holds for the Carathéodory volume element also, but we do not have a proof. However, we do have the following:

Proposition 4.1.3. *For $a^j \in D^j$ converging to $a \in D_\infty$,*

$$\limsup_{j\to\infty} c_{D^j}(a^j) \le c_{D_\infty}(a).$$

Proof. If possible, assume that this is not true. Then there exists a subsequence of $c_{D^j}(a^j)$ which we denote by $c_{D^j}(a^j)$ itself, and an $\epsilon > 0$, such that

$$c_{D^j}(a^j) > c_{D_\infty}(a) + \epsilon \quad \text{for all } j \ge 1.$$

Let ψ^j be a Carathéodory extremal map for D^j at a^j. Since the target of these maps is $\mathbb{B}^n$, passing to a subsequence if necessary, ψ^j converges uniformly on compact subsets of D_∞ to a holomorphic map $\psi : D_\infty \to \overline{\mathbb{B}^n}$, and since $\psi(a) = 0$ we must have $\psi \in \mathcal{O}(D_\infty, \mathbb{B}^n)$. Now, the above inequality implies that this limit map satisfies

$$\left| \det \psi'(a) \right|^2 \geq c_{D_\infty}(a) + \epsilon.$$

On the other hand, as ψ is a candidate for $c_{D_\infty}(a)$, we also have

$$c_{D_\infty}(a) \geq \left| \det \psi'(a) \right|^2.$$

Combining the last two inequalities, we obtain

$$c_{D_\infty}(a) \geq c_{D_\infty}(a) + \epsilon$$

which is a contradiction. □

4.1.3 Boundary asymptotics on convex finite type domains

Proof of Theorem 1.0.2. By the transformation rule

$$k_D(p^j) = \left| \det(\Lambda^{p^j,\epsilon_j} U^{p^j,\epsilon_j} T^{p^j,\epsilon_j})'(p^j) \right|^2 k_{D^j}(0).$$

Since $|\det(\Lambda^{p^j,\epsilon_j})'(0)|^2 = \prod_{\alpha=1}^n \tau_\alpha(p^j, \epsilon_j)^{-2}$ we get

$$k_D(p^j) \prod_{\alpha=1}^n \tau_\alpha(p^j, \epsilon_j)^2 = k_{D^j}(0).$$

Recall that the domains D^j converge in the local Hausdorff sense to D_∞ up to a subsequence and hence in view of Proposition 4.1.2, a limit of the right hand side is $k_{D_\infty}(0)$. This completes the proof of the theorem. □

Remark 4.1.4. The exact asymptotics as given in Theorem 1.0.2 can be derived near $p^0 \in \partial D$ when D, instead of being a smoothly bounded convex finite type domain, is given to have a smooth convex finite type boundary point p^0 locally. In this case, McNeal's orthogonal coordinate system $z_1^{p,\epsilon}, \ldots, z_n^{p,\epsilon}$ can be defined on $U \cap D$ for a sufficiently small neighborhood U of p^0 (see [20]), and then the scaling is applied to $U \cap D$. Next, the arguments in the above proof provides asymptotics for $U \cap D$, and finally, using the localization result as in Theorem 3.2.4 we obtain the boundary asymptotics for D.

4.2 Levi corank one case

4.2.1 Change of coordinates

Let $D = \{\rho < 0\}$ be a smoothly bounded Levi corank one domain and $p^0 \in \partial D$. We may assume that the Levi form of ρ at p^0 has exactly $n-2$ positive eigenvalues. We recall the definition of the change of coordinates Φ^p that transform ρ into the normal form (1.2). The maps Φ^p are actually holomorphic polynomial automorphisms defined as $\Phi^p = \phi_5 \circ \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ where ϕ_i are described below. Since the volume elements are invariant under unitary rotations, we assume without loss of generality that $\partial\rho/\partial z_n(p^0) \neq 0$. Then there is a neighborhood U of p^0 such that $(\partial\rho/\partial z_n)(p) \neq 0$ for all $p \in U$. Thus,

$$\nu = \left(\frac{\partial\rho}{\partial z_1}, \ldots, \frac{\partial\rho}{\partial z_n}\right)$$

is a nonvanishing vector field on U. Note that the vector fields

$$L_n = \frac{\partial}{\partial z_n}, \quad L_\alpha = \frac{\partial}{\partial z_\alpha} - b_\alpha \frac{\partial}{\partial z_n}, \quad 1 \leq \alpha \leq n-1,$$

where $b_\alpha = \frac{\partial\rho}{\partial z_\alpha} / \frac{\partial\rho}{\partial z_n}$, form a basis of $T^{1,0}(U)$. Moreover, for $1 \leq \alpha \leq n-1$, $L_\alpha\rho \equiv 0$ and so L_α is a complex tangent vector field to $\partial D \cap U$. Shrinking U if necessary, we also assume that

$$\left[\partial\overline{\partial}\rho(L_\alpha, \overline{L}_\beta)\right]_{2\leq\alpha,\beta\leq n-1}$$

has all its eigenvalues positive at each $p \in U$.

(i) The map ϕ_1 is defined by

$$\phi_1(z) = \Big(z_1 - p_1, \ldots, z_{n-1} - p_{n-1}, \langle z - p, \nu(p)\rangle\Big)$$

and it normalises the linear part of the Taylor series expansion of ρ at p. In the new coordinates which we denote by z itself, ρ takes the form

$$\rho \circ \phi_1^{-1}(z) = \rho(p) + 2\,\mathrm{Re}\, z_n + O(|z|^2).$$

(ii) Now

$$A = \left[\tfrac{\partial^2\rho}{\partial z_\alpha \partial \overline{z}_\beta}(p)\right]_{2\leq\alpha,\beta\leq n-1}$$

is a Hermitian matrix and there is a unitary matrix $P = \left[P_{jk}\right]_{2\leq j,k\leq n-1}$ such that $P^*AP = D$, where D is a diagonal matrix whose entries are the positive eigenvalues of A. Writing $\tilde{z} = (z_2, \ldots z_{n-1})$, the map $w = \phi_2(z)$ is defined by

$$w_1 = z_1, \quad w_n = z_n, \quad \tilde{w} = P^T\tilde{z}.$$

Then

$$\sum_{\alpha,\beta=2}^{n-1} \frac{\partial^2\rho}{\partial z_\alpha \partial\overline{z}_\beta}(p) z_\alpha \overline{z}_\beta = \tilde{z}^T A \overline{\tilde{z}} = (\overline{P}\tilde{w})^T A (\overline{\overline{P}\tilde{w}}) = \tilde{w}^T D \overline{\tilde{w}} = \sum_{\alpha=2}^{n-1} \lambda_\alpha |w_\alpha|^2,$$

where $\lambda_\alpha > 0$ is the α-th entry of D. Thus, denoting the new coordinates w by z again,

$$\rho \circ \phi_1^{-1} \circ \phi_2^{-1}(z) = \rho(p) + 2\,\mathrm{Re}\, z_n + \sum_{\alpha=2}^{n-1} \lambda_\alpha |z_\alpha|^2 + O(|z|^2)$$

where $O(|z|^2)$ consists of only the non-Hermitian quadratic terms and all other higher order terms.

(iii) The map $w = \phi_3(z)$ is defined by $w_1 = z_1, w_n = z_n$, and $w_j = \lambda_j^{1/2} z_j$ for $2 \leq j \leq n-1$. In the new coordinates, still denoted by z,

$$\begin{aligned}\rho \circ \phi_1^{-1} \circ \phi_2^{-1} \circ \phi_3^{-1}(z) &= \rho(p) + 2\operatorname{Re} z_n + \sum_{\alpha=2}^{n-1}\sum_{j=1}^{m} 2\operatorname{Re}\big((a_j^\alpha z_1^j + b_j^\alpha \bar{z}_1^j) z_\alpha\big) \\ &+ 2\operatorname{Re}\sum_{\alpha=2}^{n-1} c_\alpha z_\alpha^2 + \sum_{2\leq j+k\leq 2m} a_{jk} z_1^j \bar{z}_1^k + \sum_{\alpha=2}^{n-1} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1}\sum_{\substack{j+k\leq m\\ j,k>0}} 2\operatorname{Re}\big(b_{jk}^\alpha z_1^j \bar{z}_1^k z_\alpha\big) \\ &+ O\big(|z_n||z| + |z_*|^2|z| + |z_*||z_1|^{m+1} + |z_1|^{2m+1}\big) \qquad (4.3)\end{aligned}$$

where $z_* = (0, z_2, \ldots, z_{n-1}, 0)$.

(iv) Next, the pure terms in (4.3), i.e., z_α^2, z_1^k, $\bar{z}_1^k$, as well as $z_1^k z_\alpha$, $\bar{z}_1^k \bar{z}_\alpha$ terms are removed by absorbing them into the normal variable z_n in terms of the change of coordinates $t = \phi_4(z)$ which is defined by

$$\begin{aligned} z_j &= t_j, \quad 1 \leq j \leq n-1, \\ z_n &= t_n - \hat{Q}_1(t_1, \ldots, t_{n-1}), \end{aligned}$$

where

$$\hat{Q}_1(t_1, \ldots, t_{n-1}) = \sum_{k=2}^{2m} a_{k0} t_1^k - \sum_{\alpha=2}^{n-1}\sum_{k=1}^{m} a_k^\alpha t_\alpha t_1^k - \sum_{\alpha=2}^{n-1} c_\alpha t_\alpha^2.$$

(v) In the final step, the terms of the form $\bar{t}_1^j t_\alpha$ are removed by applying the transformation $\zeta = \phi_5(t)$ given by

$$\begin{aligned} t_1 &= \zeta_1, t_n = \zeta_n, \\ t_\alpha &= \zeta_\alpha - Q_2^\alpha(\zeta_1), \quad 2 \leq \alpha \leq n-1, \end{aligned}$$

where $Q_2^\alpha(\zeta_1) = \sum_{k=1}^m \bar{b}_k^\alpha \zeta_1^k$. In these coordinates, ρ takes the normal form (1.2).

It is evident from the definition of Φ^p that $\Phi^p(p) = 0$,

$$\Phi^p(p_1, \ldots, p_{n-1}, p_n - \epsilon) = \left(0, \ldots, 0, -\epsilon \frac{\partial \rho}{\partial \bar{z}_n}(p)\right),$$

and

$$\det(\Phi^p)'(p) = \frac{\partial \rho}{\partial \bar{z}_n}(p)(\lambda_2 \cdots \lambda_{n-1})^{1/2}, \tag{4.4}$$

where $\lambda_2, \ldots, \lambda_{n-1}$ are the positive eigenvalues of

$$\left[\frac{\partial^2 \rho}{\partial z_\alpha \partial \bar{z}_\beta}(p)\right]_{2\leq\alpha,\beta\leq n-1}.$$

4.2.2 Scaling

Suppose $p^0 = 0$ and ρ is in the normal form (1.2) for $p = p^0$; in particular, $\nu(p^0) = ('0, 1)$. Let $p^j \in D$ be a sequence converging to p^0. The points $\tilde{p}^j \in \partial D$ are chosen so that $\tilde{p}^j = p^j + ('0, \delta_j)$ for some $\delta_j > 0$. Then $\delta_j \approx \delta_D(p^j)$, where $\delta_D(p) = d(p, \partial D)$ is the distance of p to the boundary of D. The polynomial automorphisms $\Phi^{\tilde{p}^j}$ of $\mathbb{C}^n$ as described above satisfy $\Phi^{\tilde{p}^j}(\tilde{p}^j) = ('0, 0)$ and

$$\Phi^{\tilde{p}^j}(p^j) = \big('0, -\delta_j d_0(\tilde{p}^j)\big),$$

where $d_0(\tilde{p}^j) = \partial\rho/\partial\bar{z}_n(\tilde{p}^j) \to 1$ as $j \to \infty$.

Define a dilation of coordinates by

$$\Delta^{\tilde{p}^j, \delta_j}(z_1, z_2, \dots, z_n) = \left(\frac{z_1}{\tau(\tilde{p}^j, \delta_j)}, \frac{z_2}{\delta_j^{1/2}}, \dots, \frac{z_{n-1}}{\delta_j^{1/2}}, \frac{z_n}{\delta_j} \right).$$

The scaling maps are $S^j = \Delta^{\tilde{p}^j, \delta_j} \circ \Phi^{\tilde{p}^j}$ and the scaled domains are $D^j = S^j(D)$. Note that D^j contains $S^j(p^j) = \big('0, -d_0(\tilde{p}^j)\big)$ which we will denote by b^j and which converges to $b = ('0, -1)$. From (1.2), the defining function $\rho^j = \frac{1}{\delta_j}\rho \circ (S^j)^{-1}$ for D^j has the form

$$\rho^j(z) = 2\operatorname{Re} z_n + P^j(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n} |z_\alpha|^2 + \sum_{\alpha=2}^{n-1} \operatorname{Re}\big(Q^j_\alpha(z_1, \bar{z}_1) z_\alpha\big) + O(\tau^j_1),$$

where $\tau^j_1 = \tau_1(\tilde{p}^j, \delta_j)$,

$$P^j(z_1, \bar{z}_1) = \sum_{\substack{\mu+\nu \le 2m \\ \mu,\nu > 0}} a_{\mu\nu}(\tilde{p}^j)\delta_j^{-1}(\tau^j_1)^{\mu+\nu} z_1^\mu \bar{z}_1^\nu,$$

and

$$Q^j_\alpha(z_1, \bar{z}_1) = \sum_{\substack{\mu+\nu \le m \\ \mu,\nu > 0}} b^\alpha_{\mu\nu}(\tilde{p}^j)\delta_j^{-1/2}(\tau^j_1)^{\mu+\nu} z_1^\mu \bar{z}_1^\nu.$$

By (1.3) and the definition of τ_1, the coefficients of P^j and Q^j_α are bounded by 1. By Lemma 3.7 in [46], it follows that the defining functions ρ^j, after possibly passing to a subsequence, converge together with all derivatives uniformly on compact subsets of $\mathbb{C}^n$ to

$$\rho_\infty(z) = 2\operatorname{Re} z_n + P_{2m}(z_1, \bar{z}_1) + \sum_{\alpha=2}^{n-1} |z_\alpha|^2,$$

where $P_{2m}(z_1, \bar{z}_1)$ is a polynomial of degree at most $2m$ without harmonic terms. This implies that the corresponding domains D^j converge in the local Hausdorff sense to $D_\infty = \{\rho_\infty < 0\}$. Note that since D_∞ is a smooth limit of pseudoconvex domains, it is pseudoconvex and hence P_{2m} is subharmonic. By Proposition 4.5 of [49] and the remark at the end of page 605 of the same article, D_∞ possesses a local holomorphic peak function at every boundary point. By Lemma 1 of [3], there is a local holomorphic peak function for D_∞ at the point at infinity also. It follows that D_∞ is complete hyperbolic (see [21]).

4.2.3 Stability of the volume elements

Lemma 4.2.1. *Let $\phi^j \in \mathcal{O}(\mathbb{B}^n, D^j)$ and $\phi^j(0) = a^j \to a \in D_\infty$. Then ϕ^j admits a subsequence that converges uniformly on compact subsets of $\mathbb{B}^n$ to a map $\phi \in \mathcal{O}(\mathbb{B}^n, D_\infty)$.*

Proof. We first claim that the sequence $q^j := (S^j)^{-1}(a^j) \in D$ converges to $p^0 \in \partial D$, where $p^0 = 0$ is the base point for scaling. Choose a relatively compact neighborhood K of a in D_∞. Since $a^j \to a \in D_\infty$, $a^j \in K$ for all large j. Now choose a constant $C > 1$ large enough, so that K is compactly contained in the polydisc

$$\Delta(0, C^{1/2m}) \times \Delta(0, C^{1/2}) \cdots \Delta(0, C^{1/2}) \times \Delta(0, C).$$

From (1.5), we have $\tau_1(\tilde{p}^j, C\delta_j) \geq C^{1/2m}\tau_1(\tilde{p}^j, \delta_j)$. Moreover, by definition,

$$\tau_\alpha(\tilde{p}^j, C\delta_j) = (C\delta_j)^{1/2} = C^{1/2}\tau_\alpha(\tilde{p}^j, \delta_j)$$

for $\alpha = 2, \ldots, n-1$, and

$$\tau_n(\tilde{p}^j, C\delta_j) = C\delta_j = C\tau_n(\tilde{p}^j, \delta_j).$$

As a consequence, the above polydisc is contained in

$$\prod_{\alpha=1}^{n} \Delta\left(0, \frac{\tau_\alpha(\tilde{p}^j, C\delta_j)}{\tau_\alpha(\tilde{p}^j, \delta_j)}\right).$$

The pull back of this polydisc by $S^j = \Delta^{\tilde{p}^j, \delta_j} \circ \Phi^{\tilde{p}^j}$ is precisely $Q(\tilde{p}^j, C\delta_j)$. Thus,

$$q^j \in Q(\tilde{p}^j, C\delta_j)$$

for all large j. Since $\tilde{p}^j \to p^0$ and $\delta_j \to 0$ as $j \to \infty$, it follows that $q^j \to p^0$ establishing our claim.

Now we prove that the family ϕ^j is normal. Consider the sequence of maps

$$f^j = (S^j)^{-1} \circ \phi^j : \mathbb{B}^n \to D.$$

Note that $f^j(0) = q^j \to p^0$. It is shown in [46] that (see page 156 in the proof of Theorem 3.11) for every $0 < r < 1$, there exists a constant $C_r \geq 1$ depending only on r such that

$$f^j\big(B(0, r)\big) \subset Q\big(q^j, C_r\epsilon(q^j)\big), \tag{4.5}$$

where $\epsilon(q^j) = |\rho(q^j)|$. Let U be a neighborhood of $p^0 = 0$ in $\mathbb{C}^n$ as defined earlier. One can show that (see page 149 in [46]) there exist constants $0 \leq \alpha \leq 1$ and $C_1, C_2 \geq 1$ such that for $\eta, \eta' \in U$ and $\epsilon \in (0, \alpha]$ the following estimates hold for $\eta \in Q(\eta', \epsilon)$:

$$\rho(\eta) \leq \rho(\eta') + C_1\epsilon, \tag{4.6}$$

$$Q(\eta, \epsilon) \subset Q(\eta', C_2\epsilon) \quad \text{and} \quad Q(\eta', \epsilon) \subset Q(\eta, C_2\epsilon). \tag{4.7}$$

Since $\delta_j \to 0$, passing to a subsequence we may assume

$$Q(\tilde{p}^j, C\delta_j) \subset U \quad \text{and} \quad CC_1C_2C_r\delta_j \leq \alpha$$

for each j. Now, since $q^j \in Q(\tilde{p}^j, C\delta_j)$ and $C\delta_j \in (0, \alpha]$, we have by (4.6)

$$\rho(q^j) \leq \rho(\tilde{p}^j) + CC_1\delta_j = CC_1\delta_j.$$

Using the above inequality in (4.5), one obtains

$$f^j\big(B(0,r)\big) \subset Q\big(q^j, CC_1C_r\delta_j\big). \tag{4.8}$$

Moreover using the fact that $q^j \in Q(\tilde{p}^j, C\delta_j) \subset Q(\tilde{p}^j, CC_1C_r\delta_j)$ and $CC_1C_r\delta_j \in (0, \alpha]$, we obtain from (4.7) that

$$Q\big(q^j, CC_1C_r\delta_j\big) \subset Q\big(\tilde{p}^j, CC_1C_2C_r\delta_j\big). \tag{4.9}$$

Using (4.8) and (4.9), and denoting $K_r := CC_1C_2C_r$, we conclude

$$f^j\big(B(0,r)\big) \subset Q\big(\tilde{p}^j, K_r\delta_j\big)$$

for all large j. This implies that

$$\phi^j\big(B(0,r)\big) \subset \prod_{\alpha=1}^{n} \Delta\left(0, \frac{\tau_\alpha(\tilde{p}^j, K_r\delta_j)}{\tau_\alpha(\tilde{p}^j, \delta_j)}\right)$$

for all large j. Again from (1.5), $\tau_1(\tilde{p}^j, K_r\delta_j) \leq K_r^{1/2}\tau_1(\tilde{p}^j, \delta_j)$. Together with this, using the definition of τ_α for $\alpha = 2, \ldots, n$, we see that the above polydisc is contained in

$$\Delta\big(0, K_r^{1/2}\big) \times \cdots \times \Delta\big(0, K_r^{1/2}\big) \times \Delta\big(0, K_r\big).$$

Using a diagonal argument, it now follows that the family ϕ^j is normal.

Now, since $\phi^j(0) = a^j \to a \in D_\infty$, ϕ^j admits a subsequence which we denote by ϕ^j itself and which converges uniformly on compact subsets of $\mathbb{B}^n$ to a holomorphic mapping $\phi : \mathbb{B}^n \to \mathbb{C}^n$. Since D_∞ possesses a local holomorphic peak function at every boundary point, arguments similar to Lemma 4.1.1 now implies that $\phi(\mathbb{B}^n) \subset D_\infty$. $\square$

With this lemma, the proof of the following proposition is exactly similar to that of Proposition 4.1.2 and so we do not repeat the arguments.

Proposition 4.2.2. *For any $a \in D_\infty$,*

$$\lim_{j\to\infty} k_{D^j}(a) = k_{D_\infty}(a).$$

Moreover, this convergence is uniform on compact subsets of D_∞.

Similarly, the proof of Proposition 4.1.3 also gives

Proposition 4.2.3. *For $a^j \in D^j$ converging to $a \in D_\infty$,*

$$\limsup_{j\to\infty} c_{D^j}(a^j) \leq c_{D_\infty}(a).$$

4.2.4 Boundary asymptotics on Levi corank one domains

Proof of Theorem 1.0.3. Recall that we are in the case when $p^0 = 0$ and ρ is in the normal form for $p = p^0$. Therefore, $\Phi^{p^0} = \mathbb{I}$, the identity map. Observe that by the transformation rule

$$k_D(p^j) = \left| \det \left(S^j\right)'(p^j)\right|^2 k_{D^j}(b^j),$$

where $S^j = \Delta^{\tilde{p}^j,\delta_j} \circ \Phi^{\tilde{p}^j}$ are the scaling maps. Since

$$\left| \det(\Delta^{\tilde{p}^j,\delta_j})'\left(\Phi^{\tilde{p}^j}(p^j)\right)\right|^2 = \prod_{\alpha=1}^{n} \tau_\alpha(\tilde{p}^j,\delta_j)^{-2},$$

we get

$$k_D(p^j) \prod_{\alpha=1}^{n} \tau_\alpha(\tilde{p}^j,\delta_j)^2 = \left| \det(\Phi^{\tilde{p}^j})'(p^j)\right|^2 k_{D^j}(b^j). \tag{4.10}$$

Now $|\det(\Phi^{\tilde{p}^j})'(p^j)| \to |\det\left(\Phi^{p^0}\right)'(p^0)| = 1$, and recall that after possibly passing to a subsequence, the domains D^j converge in the local Hausdorff sense to D_∞. Hence by Propostion 4.2.2, the right hand side of (4.10) has $k_{D_\infty}(b)$ as a limit, proving the theorem in the current situation.

For the general case, assume that $(\partial\rho/\partial z_n)(p^0) \neq 0$ and make an initial change of coordinates $w = T(z) = \Phi^{p^0}(z)$. Let $\Omega = T(D)$, $q^0 = T(p^0) = 0$, and $q^j = T(p^j)$. Then

$$k_D(p^j) = \left| \det T'(p^j)\right|^2 k_\Omega(q^j). \tag{4.11}$$

To emphasize the dependence of Φ^p, τ, and τ_α on $D = \{\rho < 0\}$, we will write them now as Φ^p_ρ, τ_ρ and $\tau_{\alpha,\rho}$ respectively. Note that the defining function $r = \rho \circ T^{-1}$ for Ω is in the normal form at $q^0 = 0$. Choose η_j such that $\tilde{q}^j = (q^j_1,\dots,q^j_{n-1},q^j_n+\eta_j) \in \partial\Omega$. Then by the previous case

$$k_\Omega(q^j) \prod_{\alpha=1}^{n} \tau_{\alpha,r}(\tilde{q}^j,\eta_j)^2 \to k_{D_\infty}(b) \tag{4.12}$$

up to a subsequence. Since $\delta_\Omega \circ T$ is a defining function for D, we have $\delta_\Omega \circ T \approx \delta_D$ and hence $\delta_j \approx \delta_D(p^j) \approx \delta_\Omega(q^j) \approx \eta_j$. Also, by (3.3) of [46],

$$\rho \circ (\Phi^{p^j}_\rho)^{-1} = r \circ (\Phi^{q^j}_r)^{-1}.$$

It follows from (2.9) of [10] that $\tau_\rho(\tilde{p}^j,\delta_j) \approx \tau_r(\tilde{q}^j,\eta_j)$. Hence, after passing to a subsequence if necessary,

$$\prod_{\alpha=1}^{n} \frac{\tau_{\alpha,\rho}(\tilde{p}^j,\delta_j)}{\tau_{\alpha,r}(\tilde{q}^j,\eta_j)} \to c_0 \tag{4.13}$$

for some $c_0 > 0$ that depends only on ρ. Also,

$$\left|\det T'(p^j)\right| \to \left|\det T'(p^0)\right| = \left|\frac{\partial\rho}{\partial\bar{z}_n}(p^0)\right| \prod_{\alpha=2}^{n-1} \lambda_\alpha^{1/2}, \tag{4.14}$$

by (4.4), where λ_α's are the positive eigenvalues of

$$\left[\frac{\partial^2\rho}{\partial z_\alpha \partial \bar{z}_\beta}(p^0)\right]_{2\le\alpha,\beta\le n-1}.$$

Hence it follows from equations (4.11) to (4.14) that

$$k_D(p^j) \prod_{\alpha=1}^{n} \tau_{\alpha,\rho}(\tilde{p}^j, \delta_j)^2 \to c_0^2 \left| \frac{\partial \rho}{\partial \bar{z}_n}(p^0) \right|^2 \left(\prod_{\alpha=2}^{n-1} \lambda_\alpha \right) k_{D_\infty}(b)$$

up to a subsequence. This completes the proof of the theorem. □

Chapter 5

The Kobayashi–Fuks metric

For a bounded domain $D \subset \mathbb{C}^n$ the space

$$A^2(D) = \left\{ f : D \to \mathbb{C} \text{ holomorphic and } \|f\|_D^2 := \int_D |f|^2 \, dV < \infty \right\}$$

where dV is the Lebesgue measure on $\mathbb{C}^n$ is a closed subspace of $L^2(D)$ and is a reproducing kernel Hilbert space. The associated reproducing kernel denoted by $K_D(z,w)$ is uniquely determined by the following properties: $K_D(\cdot, w) \in A^2(D)$ for each $w \in D$, it is anti-symmetric, i.e., $K_D(z,w) = \overline{K_D(w,z)}$, and it reproduces $A^2(D)$:

$$f(w) = \int_D f(z) \overline{K_D(z,w)} \, dV(z), \quad f \in A^2(D).$$

It also follows that for any complete orthonormal basis $\{\phi_k\}$ of $A^2(D)$,

$$K_D(z,w) = \sum_k \phi_k(z) \overline{\phi_k(w)},$$

where the series converges uniformly on compact subsets of $D \times D$. The reproducing kernel $K_D(z,w)$ is called the Bergman kernel for D. Denote by $K_D(z) = K_D(z,z)$ its restriction to the diagonal. It is known that $\log K_D$ is a strongly plurisubharmonic function and thus is a potential for a Kähler metric which is called the Bergman metric for D and is given by

$$ds^2_{B,D} = \sum_{\alpha,\beta=1}^{n} g^{B,D}_{\alpha\overline{\beta}}(z) \, dz_\alpha d\overline{z}_\beta,$$

where

$$g^{B,D}_{\alpha\overline{\beta}}(z) = \frac{\partial^2 \log K_D}{\partial z_\alpha \partial \overline{z}_\beta}(z).$$

Let

$$G_{B,D}(z) = \left(g^{B,D}_{\alpha\overline{\beta}}(z) \right)_{n \times n} \quad \text{and} \quad g_{B,D}(z) = \det G_{B,D}(z).$$

The components of the Ricci tensor of $ds^2_{B,D}$ are defined by

$$\mathrm{Ric}^{B,D}_{\alpha\overline{\beta}}(z) = -\frac{\partial^2 \log g_{B,D}}{\partial z_\alpha \partial \overline{z}_\beta}(z), \tag{5.1}$$

and the Ricci curvature of $ds^2_{B,D}$ is given by

$$\mathrm{Ric}_{B,D}(z,u) = \frac{\sum_{\alpha,\beta=1}^n \mathrm{Ric}^{B,D}_{\alpha\overline{\beta}}(z)u^\alpha\overline{u}^\beta}{\sum_{\alpha,\beta=1}^n g^{B,D}_{\alpha\overline{\beta}}(z)u^\alpha\overline{u}^\beta}. \tag{5.2}$$

We have already seen (by results of Kobayashi and Fuks mentioned in Chapter 1) that the matrix

$$G_{\tilde{B},D}(z) = \left(g^{\tilde{B},D}_{\alpha\overline{\beta}}(z)\right)_{n\times n}$$

where

$$g^{\tilde{B},D}_{\alpha\overline{\beta}}(z) = (n+1)g^{B,D}_{\alpha\overline{\beta}}(z) - \mathrm{Ric}^{B,D}_{\alpha\overline{\beta}}(z) = \frac{\partial^2 \log(K_D^{n+1}g_{B,D})}{\partial z_\alpha \partial \overline{z}_\beta}(z),$$

is positive definite. Therefore,

$$ds^2_{\tilde{B},D} = \sum_{\alpha,\beta=1}^n g^{\tilde{B},D}_{\alpha\overline{\beta}}(z)\, dz_\alpha d\overline{z}_\beta$$

is a Kähler metric with Kähler potential $\log(K_D^{n+1}g_{B,D})$. Moreover, if $F : D \to D'$ is a biholomorphism, then

$$G_{\tilde{B},D}(z) = F'(z)^t\, G_{\tilde{B},D'}\big(F(z)\big)\overline{F}'(z), \tag{5.3}$$

where $F'(z)$ is the Jacobian matrix of F at z. This implies that $ds^2_{\tilde{B},D}$ is an invariant metric. We will call this metric the Kobayashi–Fuks metric on D.

The boundary asymptotics of the Bergman metric and its Ricci curvature on strongly pseudoconvex domains are known from which it turns out that the Kobayashi–Fuks metric is complete on such domains (to be seen later in Section 7.5). Dinew [15] showed that on any bounded hyperconvex domain the Kobayashi–Fuks metric is complete, and hence in particular, by a result of Demailly [12], this metric is complete on any bounded pseudoconvex domain with Lipschitz boundary. Dinew [13] also observed that the Kobayashi–Fuks metric is useful in the study of the Bergman representative coordinates. Invariant metrics play an important role in understanding the geometry of a domain which makes their study of natural interest in complex analysis and our purpose here is to show that the Kobayashi–Fuks metric shares several properties with the Bergman metric. Let us fix some notations before we state our results. We will denote by h any of B or $\tilde{B}$. We write

$$G_{h,D}(z) = \left(g^{h,D}_{\alpha\overline{\beta}}(z)\right)_{n\times n} \quad \text{and} \quad g_{h,D}(z) = \det G_{h,D}(z).$$

The length of a vector u at a point $z \in D$ in $ds^2_{h,D}$ will be denoted by $\tau_{h,D}(z,u)$, i.e.,

$$\tau^2_{h,D}(z,u) = \sum_{\alpha,\beta=1}^n g^{h,D}_{\alpha\overline{\beta}}(z)u^\alpha\overline{u}^\beta.$$

The holomorphic sectional curvature of $ds^2_{h,D}$ is defined by

$$R_{h,D}(z,u) = \frac{\sum_{\alpha,\beta,\gamma,\delta=1}^n R^{h,D}_{\overline{\alpha}\beta\gamma\overline{\delta}}(z)\overline{u}^\alpha u^\beta u^\gamma \overline{u}^\delta}{\left(\sum_{\alpha,\beta=1}^n g^{h,D}_{\alpha\overline{\beta}}(z)u^\alpha\overline{u}^\beta\right)^2}, \tag{5.4}$$

where

$$R^{h,D}_{\overline{\alpha}\beta\gamma\overline{\delta}}(z) = -\frac{\partial^2 g^{h,D}_{\beta\overline{\alpha}}}{\partial z_\gamma \partial \overline{z}_\delta}(z) + \sum_{\mu,\nu} g^{\nu\overline{\mu}}_{h,D}(z) \frac{\partial g^{h,D}_{\beta\overline{\mu}}}{\partial z_\gamma}(z) \frac{\partial g^{h,D}_{\nu\overline{\alpha}}}{\partial \overline{z}_\delta}(z), \tag{5.5}$$

$g^{\nu\overline{\mu}}_{h,D}(z)$ being the (ν,μ)th entry of the inverse of the matrix $G_{h,D}(z)$. The Ricci curvature of $ds^2_{h,D}$ is defined by (5.2) with B replaced by h. Finally, note that in dimension one, the metric $ds^2_{h,D}$ has the form

$$ds^2_{h,D} = g_{h,D}(z)|dz|^2, \quad \tau_{h,D}(z,u) = \sqrt{g_{h,D}(z)}|u|$$

and both the holomorphic sectional curvature and the Ricci curvature at a point are independent of the tangent vector u and are simply the Gaussian curvature

$$R_{h,D}(z) = -\frac{1}{g_{h,D}(z)} \frac{\partial^2 \log g_{h,D}}{\partial z \partial \overline{z}}(z).$$

5.1 Some examples

Proposition 5.1.1. *For the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$,*

$$ds^2_{\tilde{B},\mathbb{B}^n} = (n+2)ds^2_{B,\mathbb{B}^n} = (n+1)(n+2)\sum_{\alpha,\beta=1}^{n}\left(\frac{\delta_{\alpha\overline{\beta}}}{1-|z|^2} + \frac{\overline{z}_\alpha z_\beta}{(1-|z|^2)^2}\right) dz_\alpha d\overline{z}_\beta.$$

Proof. Recall that for the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$,

$$K_{\mathbb{B}^n}(z) = \frac{n!}{\pi^n}\frac{1}{(1-|z|^2)^{n+1}},$$

and so

$$g^{B,\mathbb{B}^n}_{\alpha\overline{\beta}}(z) = (n+1)\frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta}\log\frac{1}{1-|z|^2} = (n+1)\left(\frac{\delta_{\alpha\overline{\beta}}}{1-|z|^2} + \frac{\overline{z}_\alpha z_\beta}{(1-|z|^2)^2}\right).$$

Denoting the matrix $\overline{z}z^t$ by A_z and using the fact that its characteristic polynomial is $\det(\lambda\mathbb{I} - A_z) = \lambda^n - |z|^2\lambda^{n-1}$, we obtain

$$g_{B,\mathbb{B}^n}(z) = \frac{(n+1)^n}{(1-|z|^2)^{n+1}},$$

and hence

$$\mathrm{Ric}^{B,\mathbb{B}^n}_{\alpha\overline{\beta}}(z) = -(n+1)\frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta}\log\frac{1}{1-|z|^2} = -g^{B,\mathbb{B}^n}_{\alpha\overline{\beta}}(z).$$

It follows that

$$g^{\tilde{B},\mathbb{B}^n}_{\alpha\overline{\beta}}(z) = (n+2)g^{B,\mathbb{B}^n}_{\alpha\overline{\beta}}(z),$$

which completes the proof of the proposition. □

Proposition 5.1.2. *For the unit polydisc $\Delta^n \subset \mathbb{C}^n$,*

$$ds^2_{\tilde{B},\Delta^n} = (n+2)ds^2_{B,\Delta^n} = 2(n+2)\sum_{\alpha=1}^{n}\frac{1}{(1-|z_\alpha|^2)^2}dz_\alpha d\overline{z}_\alpha.$$

Proof. For the unit polydisc $\Delta^n \subset \mathbb{C}^n$, recall that by the product formula for the Bergman kernel,

$$K_{\Delta^n}(z) = \prod_{j=1}^n K_\Delta(z_j) = \frac{1}{\pi^n} \prod_{j=1}^n \frac{1}{(1-|z_j|^2)^2},$$

and therefore

$$g_{\alpha\overline{\beta}}^{B,\Delta^n}(z) = 2\,\frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \sum_{j=1}^n \log \frac{1}{1-|z_j|^2} = \frac{2\delta_{\alpha\beta}}{(1-|z_\alpha|^2)^2}.$$

Thus,

$$g_{B,\Delta^n}(z) = 2^n \prod_{j=1}^n \frac{1}{(1-|z_j|^2)^2},$$

and hence

$$\mathrm{Ric}_{\alpha\overline{\beta}}^{B,\Delta^n}(z) = -2\,\frac{\partial^2}{\partial z_\alpha \partial \overline{z}_\beta} \sum_{j=1}^n \log \frac{1}{1-|z_j|^2} = -g_{\alpha\overline{\beta}}^{B,\Delta^n}(z).$$

It follows that

$$g_{\alpha\overline{\beta}}^{\tilde{B},\Delta^n}(z) = (n+2) g_{\alpha\overline{\beta}}^{B,\Delta^n}(z) = \frac{2(n+2)}{(1-|z_\alpha|^2)^2}\delta_{\alpha\beta},$$

and the proof of the proposition is complete. □

In general, if D is a bounded domain with a transitive group of holomorphic automorphisms, the Bergman metric is Kähler-Einstein, and so $ds^2_{\tilde{B},D}$ is a constant multiple of $ds^2_{B,D}$.

5.2 Some monotonicity results

In this section, we will first express the Kobayashi–Fuks length in terms of a maximal domain function introduced by Krantz and Yu [36]. Using that expression several monotonicity properties of the Kobayashi–Fuks metric will be established. The monotonicity properties are not only interesting results in themselves, they also help us in localising the holomorphic sectional curvature of the Kobayashi–Fuks metric. We begin by recalling the maximal domain function of Krantz and Yu: For a domain $D \subset \mathbb{C}^n$, $z_0 \in D$ and a nonzero vector $u \in \mathbb{C}^n$, let

$$I_D(z_0, u) = \sup\left\{ u^t f''(z_0) \overline{G}_{B,D}^{-1}(z_0) \overline{f''}(z_0) \overline{u} : \|f\|_D = 1, f(z_0) = f'(z_0) = 0 \right\}. \tag{5.6}$$

Here $f''(z_0)$ is the symmetric matrix

$$f''(z_0) = \left(\frac{\partial^2 f}{\partial z_i \partial z_j}(z_0) \right)_{n\times n}.$$

It was shown in Proposition 2.1 (ii) of [36] that whenever $K_D(z_0)$ and $\tau_{B,D}(z_0, u)$ are positive,

$$\mathrm{Ric}_{B,D}(z_0, u) = (n+1) - \frac{1}{\tau^2_{B,D}(z_0, u) K_D(z_0)} I_D(z_0, u). \tag{5.7}$$

Also, from the definition of the Kobayashi–Fuks metric, note that

$$\tau_{\tilde{B},D}(z_0, u) = \tau_{B,D}(z_0, u)\sqrt{n+1-\mathrm{Ric}_{B,D}(z_0, u)}. \tag{5.8}$$

Combining (5.7) and (5.8) we obtain

Proposition 5.2.1. *Let D be a bounded domain in $\mathbb{C}^n$, $z_0 \in D$, and $u \in \mathbb{C}^n$. Then we have*

$$\tau^2_{\tilde{B},D}(z_0,u) = \frac{I_D(z_0,u)}{K_D(z_0)}.$$

For the localization of $g_{\tilde{B},D}$, which will be established in the next chapter, we will need the following lemma. The notation $\mathbb{I}_n$, or simply $\mathbb{I}$, stands for the $n \times n$ identity matrix.

Lemma 5.2.2. *Let D_1, D_2 be two bounded domains in $\mathbb{C}^n$ such that $D_2 \subset D_1$. For any $z_0 \in D_2$, there exist a nonsingular matrix Q and positive real numbers $d_1, \ldots, d_n$ such that*

$$Q^t G_{\tilde{B},D_1}(z_0)\overline{Q} = \operatorname{diag}\{d_1,\ldots,d_n\} \quad \textit{and} \quad Q^t G_{\tilde{B},D_2}(z_0)\overline{Q} = \mathbb{I}.$$

Proof. Note that we have $G_{\tilde{B},D_2}(z_0)$ as a positive definite Hermitian matrix. Hence one can find an invertible matrix A such that

$$A^t G_{\tilde{B},D_2}(z_0)\overline{A} = \mathbb{I}.$$

By the transformation rule (5.3) applied to $A : A^{-1}D_1 \to D_1$ and $A : A^{-1}D_2 \to D_2$,

$$G_{\tilde{B},A^{-1}D_1}(A^{-1}z_0) = A^t G_{\tilde{B},D_1}(z_0)\overline{A} \quad \text{and} \quad G_{\tilde{B},A^{-1}D_2}(A^{-1}z_0) = A^t G_{\tilde{B},D_2}(z_0)\overline{A} = \mathbb{I}.$$

From the first identity above, $A^t G_{\tilde{B},D_1}(z_0)\overline{A}$ is a positive definite Hermitian matrix and hence there exists a unitary matrix B such that

$$B^t(A^t G_{\tilde{B},D_1}(z_0)\overline{A})\overline{B} = \operatorname{diag}\{d_1,\ldots,d_n\} \quad \text{for some} \quad d_1,\ldots,d_n > 0.$$

Now letting $Q = AB$ the lemma follows. □

In general, the Kobayashi–Fuks metric and its associated objects do not satisfy the monotonicity properties with respect to increasing domains. However, we show that they can be compared after taking products with certain invariants. Recall that the Bergman canonical invariant on D is the function defined by

$$J_D(z) = \frac{\det G_{B,D}(z)}{K_D(z)} = \frac{g_{B,D}(z)}{K_D(z)}.$$

From the transformation rule for the Bergman kernel it is evident that J_D is a biholomorphic invariant.

Proposition 5.2.3. *Let D_1, D_2 be two bounded domains in $\mathbb{C}^n$ such that $D_2 \subset D_1$. For any $z_0 \in D_2$ and $u \in \mathbb{C}^n$, we have*

(i) $\tau^2_{\tilde{B},D_1}(z_0,u) \le \left(\frac{K_{D_2}(z_0)}{K_{D_1}(z_0)}\right)^{n+1} \left(\frac{J_{D_2}(z_0)}{J_{D_1}(z_0)}\right) \tau^2_{\tilde{B},D_2}(z_0,u)$,

(ii) $u^t \bar{G}^{-1}_{\tilde{B},D_1}(z_0)\overline{u} \ge \left(\frac{K_{D_1}(z_0)}{K_{D_2}(z_0)}\right)^{n+1} \left(\frac{J_{D_1}(z_0)}{J_{D_2}(z_0)}\right) u^t \bar{G}^{-1}_{\tilde{B},D_2}(z_0)\overline{u}$,

(iii) $u^t\, \overline{\operatorname{ad}} G_{\tilde{B},D_1}(z_0)\overline{u} \le \left(\frac{K_{D_2}(z_0)}{K_{D_1}(z_0)}\right)^{n^2-1} \left(\frac{J_{D_2}(z_0)}{J_{D_1}(z_0)}\right)^{n-1} u^t\, \overline{\operatorname{ad}} G_{\tilde{B},D_2}(z_0)\overline{u}$.

Proof. Fix $z_0 \in D_2$ and $u \in \mathbb{C}^n$. For simplicity of notations, we will write K_i for $K_{D_i}(z_0)$, J_i for $J_{D_i}(z_0)$, G_i for $G_{B,D_i}(z_0)$, and $\tilde{G}_i$ for $G_{\tilde{B},D_i}(z_0)$ for $i = 1, 2$.

(i) In view of Proposition 5.2.1, it is enough to prove that

$$I_{D_1}(z_0, u) \leq \left(\frac{K_2}{K_1}\right)^n \frac{J_2}{J_1} I_{D_2}(z_0, u). \tag{5.9}$$

From the proof of Proposition 2.2 in [36] (see page 236) there exists a nonsingular matrix P (depending on z_0) such that for every $v \in \mathbb{C}^n$,

$$v^t P^{-1} \,\overline{\mathrm{ad}} G_1 (P^*)^{-1} \overline{v} \leq \left(\frac{K_2}{K_1}\right)^{n-1} v^t P^{-1} \,\overline{\mathrm{ad}} G_2 (P^*)^{-1} \overline{v}. \tag{5.10}$$

Now consider $f \in A^2(D_1)$ such that $\|f\|_{D_1} = 1, f(z_0) = f'(z_0) = 0$. We can write $f''(z_0) = (P^t)^{-1} A$ for some matrix A. Putting $v = Au$ in (5.10) and using the fact that $f''(z_0)$ is symmetric, we get

$$u^t f''(z_0) \,\overline{\mathrm{ad}} G_1 \overline{f''}(z_0) \overline{u} \leq \left(\frac{K_2}{K_1}\right)^{n-1} u^t f''(z_0) \overline{\mathrm{ad}} G_2 \overline{f''}(z_0) \overline{u}. \tag{5.11}$$

Define $g : D_2 \to \mathbb{C}$ by

$$g(z) = \frac{f(z)}{\|f\|_{D_2}}.$$

Then $g \in A^2(D_2)$, $\|g\|_{D_2} = 1$, $g(z_0) = g'(z_0) = 0$. Since $f''(z_0) = \|f\|_{D_2} g''(z_0)$, $\|f\|_{D_2} \leq 1$, and $\mathrm{ad}\, G_i = (\det G_i) G_i^{-1}$, we have from (5.11)

$$\begin{aligned} u^t f''(z_0)\, \bar{G}_1^{-1} \overline{f''}(z_0) \overline{u} &\leq \left(\frac{K_2}{K_1}\right)^{n-1} \frac{\det G_2}{\det G_1} \left(u^t g''(z_0)\, \bar{G}_2^{-1} \overline{g''}(z_0) \overline{u}\right) \\ &\leq \left(\frac{K_2}{K_1}\right)^n \frac{J_2}{J_1} I_{D_2}(z_0, u). \end{aligned} \tag{5.12}$$

Taking supremum over f in (5.12) and using Proposition 5.2.1, we obtain (5.9) and hence (i) is proved.

(ii) Let Q be as in Lemma 5.2.2. Let e_j denote the j-th standard unit vector in $\mathbb{C}^n$, i.e., $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$. Taking $u = Qe_j$ in (i), we get

$$d_j \leq \left(\frac{K_2}{K_1}\right)^{n+1} \frac{J_2}{J_1} \quad \text{for} \quad j = 1, \ldots, n. \tag{5.13}$$

From Lemma 5.2.2, it follows that

$$Q^{-1} \big(\overline{\tilde{G}_1}\big)^{-1} (Q^*)^{-1} = \mathrm{diag}\left\{\frac{1}{d_1}, \ldots, \frac{1}{d_n}\right\} \quad \text{and} \quad Q^{-1} \big(\overline{\tilde{G}_2}\big)^{-1} (Q^*)^{-1} = \mathbb{I}.$$

Hence for any $v \in \mathbb{C}^n$, using the inequality (5.13), we get

$$\begin{aligned} v^t Q^{-1} \big(\overline{\tilde{G}_1}\big)^{-1} (Q^*)^{-1} \overline{v} = v^t \,\mathrm{diag}\left\{\frac{1}{d_1}, \ldots, \frac{1}{d_n}\right\} \overline{v} &\geq \left(\frac{K_1}{K_2}\right)^{n+1} \frac{J_1}{J_2} (v^t \mathbb{I} \overline{v}) \\ &= \left(\frac{K_1}{K_2}\right)^{n+1} \frac{J_1}{J_2} \big(v^t Q^{-1} \big(\overline{\tilde{G}_2}\big)^{-1} (Q^*)^{-1} \overline{v}\big). \end{aligned}$$

Putting $u = (Q^t)^{-1}v$ in the above inequality, we get (ii).

(iii) Note that from Lemma 5.2.2, we get

$$\det \tilde{G}_1 = \left(\prod_{j=1}^{n} d_j\right) \det \tilde{G}_2.$$

Therefore using the relation $\mathrm{ad}\,\tilde{G}_i = (\det \tilde{G}_i)\tilde{G}_i^{-1}$, we have

$$Q^{-1}\,\overline{\mathrm{ad}}\tilde{G}_1(Q^*)^{-1} = \left(\det \tilde{G}_1\right) Q^{-1}(\overline{\tilde{G}_1})^{-1}(Q^*)^{-1} = \left(\det \tilde{G}_2\right) \mathrm{diag}\left\{\prod_{j\neq 1} d_j, \ldots, \prod_{j\neq n} d_j\right\}.$$

Hence, for any $v \in \mathbb{C}^n$, using (5.13)

$$\begin{aligned} v^t Q^{-1}\,\overline{\mathrm{ad}}\tilde{G}_1(Q^*)^{-1}\overline{v} &\leq \left(\det \tilde{G}_2\right)\left[\left(\frac{K_2}{K_1}\right)^{n+1}\frac{J_2}{J_1}\right]^{n-1}\left(v^t \mathbb{I}\overline{v}\right) \\ &= \left(\det \tilde{G}_2\right)\left[\left(\frac{K_2}{K_1}\right)^{n+1}\frac{J_2}{J_1}\right]^{n-1} v^t Q^{-1}(\overline{\tilde{G}_2})^{-1}(Q^*)^{-1}\overline{v} \\ &= \left(\frac{K_2}{K_1}\right)^{n^2-1}\left(\frac{J_2}{J_1}\right)^{n-1} v^t Q^{-1}\,\overline{\mathrm{ad}}\tilde{G}_2(Q^*)^{-1}\overline{v}. \end{aligned}$$

Now putting $u = (Q^t)^{-1}v$ in the above equation, we get the desired result. $\square$

Chapter 6

Localizations

The localization of invariant metrics is one of the pivotal components in the study of their boundary behavior. It was Graham (see [22]) who first studied the localization of the Kobayashi metric on strongly pseudoconvex domains with smooth boundary based on the existence of a global peak function at each boundary point of such domains. The Bergman invariants such as – the metric, its Ricci and holomorphic sectional curvatures, etc. are also localized. The localization of the Bergman kernel (on the diagonal) near a local holomorphic peak point of a bounded pseudoconvex domain was proved by Lars Hörmander in [29]. Later several authors (cf. [4, 14, 32, 36]) used his L^2-technique to obtain the localization results for various Bergman invariants. In this chapter, we localize some invariants associated with the Kobayashi–Fuks metric near the holomorphic peak points of bounded pseudoconvex domains in $\mathbb{C}^n$. When we deal with the holomorphic sectional curvature of the Kobayashi–Fuks metric, we will restrict our attention only to the planar case, i.e., $n = 1$.

We start by proving Theorem 1.0.4. A crucial step in the proof of this theorem is to obtain Bergman–Fuks type results for the Kobayashi–Fuks metric and its related invariants, i.e., to express them in terms of certain maximal domain functions. For the holomorphic sectional curvature of the Kobayashi–Fuks metric, we derive such a result only in dimension one, though we believe that in higher dimensions also, an analog of this and hence of Theorem 1.0.4 (iii) should hold.

Proof of Theorem 1.0.4. (i) It was shown in [32] that

$$\lim_{z\to p^0} \frac{K_D(z)}{K_{U\cap D}(z)} = 1, \tag{6.1}$$

and Krantz and Yu [36] showed that

$$\lim_{z\to p^0} \frac{I_D(z,u)}{I_{U\cap D}(z,u)} = 1 \tag{6.2}$$

uniformly in unit vectors u, and hence (i) follows from Proposition 5.2.1.

(ii) By Lemma 5.2.2, there exist an invertible matrix $Q(z)$ and positive real numbers $d_1(z), \ldots, d_n(z)$ such that

$$Q^t(z)G_{\tilde{B},D}(z)\overline{Q}(z) = \operatorname{diag}\{d_1(z), \ldots, d_n(z)\} \quad \text{and} \quad Q^t(z)G_{\tilde{B},U\cap D}(z)\overline{Q}(z) = \mathbb{I}.$$

Taking determinant on both sides of these equations yields

$$\frac{g_{\tilde{B},D}(z)}{g_{\tilde{B},U\cap D}(z)} = \prod_{j=1}^{n} d_j(z).$$

Also, by Proposition 5.2.1,

$$\tau^2_{\tilde{B},D}(z,u) = \frac{K_{U\cap D}(z)}{K_D(z)} \frac{I_D(z,u)}{I_{U\cap D}(z,u)} \tau^2_{\tilde{B},U\cap D}(z,u).$$

Putting $u = Q(z)e_j$ in the above equation, we get

$$d_j(z) = \frac{K_{U\cap D}(z)}{K_D(z)} \frac{I_D\big(z, Q(z)e_j\big)}{I_{U\cap D}\big(z, Q(z)e_j\big)} \quad \text{for} \quad j = 1, \ldots, n.$$

Therefore,

$$\frac{g_{\tilde{B},D}(z)}{g_{\tilde{B},U\cap D}(z)} = \left(\frac{K_{U\cap D}(z)}{K_D(z)}\right)^n \prod_{j=1}^{n} \frac{I_D\big(z, Q(z)e_j\big)}{I_{U\cap D}\big(z, Q(z)e_j\big)} = \left(\frac{K_{U\cap D}(z)}{K_D(z)}\right)^n \prod_{j=1}^{n} \frac{I_D\big(z, v_j(z)\big)}{I_{U\cap D}\big(z, v_j(z)\big)}$$

where $v_j(z) = Q(z)e_j/\|Q(z)e_j\|$. Now (ii) follows immediately from (6.1) and (6.2).

The proof of (iii) will be given later once we express the Gaussian curvature of the Kobayashi–Fuks metric on bounded planar domains in terms of certain maximal domain functions. □

We now introduce two maximal domain functions on planar domains for the purpose of localising the Gaussian curvature of the Kobayashi–Fuks metric. For a bounded domain $D \subset \mathbb{C}$, let

$$I'_D(z_0) = \sup\big\{g^{-1}_{\tilde{B},D}(z_0)|f'(z_0)|^2 : f \in A^2(D), \|f\|_D = 1, f(z_0) = 0\big\},$$
$$I''_D(z_0) = \sup\big\{g^{-3}_{\tilde{B},D}(z_0)|f'''(z_0)|^2 : f \in A^2(D), \|f\|_D = 1, f(z_0) = f'(z_0) = f''(z_0) = 0\big\}.$$

Note that, as D is bounded, the functions I'_D and I''_D are well-defined and strictly positive. It is also evident that the supremums are achieved. Moreover, under biholomorphisms they transform by the same rule as that of the Bergman kernel which we establish in the following:

Proposition 6.0.1. *Let $F : D_1 \to D_2$ be a biholomorphism between two bounded domains in $\mathbb{C}$. Then*

$$I'_{D_1}(z_0) = I'_{D_2}\big(F(z_0)\big)\big|F'(z_0)\big|^2 \quad \textit{and} \quad I''_{D_1}(z_0) = I''_{D_2}\big(F(z_0)\big)\big|F'(z_0)\big|^2.$$

Proof. We will prove the transformation rule only for I''_D, as the case of I'_D is even simpler and follows from similar arguments. Suppose $g \in A^2(D_2)$ is a maximizer for $I''_{D_2}\big(F(z_0)\big)$. Now set

$$f(z) = g\big(F(z)\big)F'(z).$$

It is straightforward to check that $\|f\|_{D_1} = \|g\|_{D_2} = 1$, $f(z_0) = f'(z_0) = f''(z_0) = 0$, and

$$f'''(z_0) = g'''\big(F(z_0)\big)\big(F'(z_0)\big)^4.$$

Therefore,

$$g^{-3}_{\tilde{B},D_1}(z_0)\big|f'''(z_0)\big|^2 = g^{-3}_{\tilde{B},D_1}(z_0)\big|g'''\big(F(z_0)\big)\big|^2\big|F'(z_0)\big|^8. \tag{6.3}$$

Note that from the transformation rule for the Kobayashi–Fuks metric, we have

$$g_{\tilde{B},D_1}^{-1}(z_0)|F'(z_0)|^2 = g_{\tilde{B},D_2}^{-1}\big(F(z_0)\big).$$

Applying this on the right hand side of (6.3), we get

$$g_{\tilde{B},D_1}^{-3}(z_0)|f'''(z_0)|^2 = g_{\tilde{B},D_2}^{-3}\big(F(z_0)\big)\big|g'''\big(F(z_0)\big)\big|^2\big|F'(z_0)\big|^2.$$

As f is a candidate for $I''_{D_1}(z_0)$ and g is a maximizer for $I''_{D_2}(F(z_0))$, we obtain

$$I''_{D_1}(z_0) \geq I''_{D_2}\big(F(z_0)\big)\,|F'(z_0)|^2.$$

Similar arguments when applied to the map $F^{-1} : D_2 \to D_1$ gives the reverse inequality and hence it is an equality. □

The main ingredient for the localization of the Gaussian curvature of the Kobayashi–Fuks metric is the following Bergman–Fuks type result:

Proposition 6.0.2. *Let $D \subset \mathbb{C}$ be a bounded domain and $z_0 \in D$. Then the Gaussian curvature of the Kobayashi–Fuks metric on D satisfies*

$$R_{\tilde{B},D}(z_0) = 2 - \frac{I'_D(z_0)}{K_D(z_0)} - \frac{I''_D(z_0)}{I'_D(z_0)}. \tag{6.4}$$

Observe that both the sides of (6.4) are invariant under biholomorphisms and we will establish their equality by computing them in terms of a suitable orthonormal basis of $A^2(D)$ in some special coordinates. To this end, we fix $z_0 \in D$ and consider the closed subspaces of $A^2(D)$ given by

$$\begin{aligned} A_1(z_0) &= \big\{f \in A^2(D) : f(z_0) = 0\big\},\\ A_2(z_0) &= \big\{f \in A^2(D) : f(z_0) = f'(z_0) = 0\big\},\\ A_3(z_0) &= \big\{f \in A^2(D) : f(z_0) = f'(z_0) = f''(z_0) = 0\big\}. \end{aligned}$$

Observe that the orthogonal complement of $A_1(z_0)$ in $A^2(D)$ has dimension one and let h_0 be a unit vector in this orthogonal complement. It is easy to see that the orthogonal complement of $A_2(z_0)$ inside $A_1(z_0)$ has dimension at most one and the orthogonal complement of $A_3(z_0)$ in $A_2(z_0)$ also has dimension at most one. Without loss of generality, we assume that both these dimensions are exactly one. Let $\{\phi, \psi, h_1, h_2, \ldots\}$ be an orthonormal basis for $A_1(z_0)$ such that ϕ is a unit vector in $A_1(z_0) \setminus A_2(z_0)$, ψ is a unit vector in $A_2(z_0) \setminus A_3(z_0)$, and $\{h_1, \ldots, h_j, \ldots\}$ is an orthonormal basis for $A_3(z_0)$. Note that

$$K_D(z) = \big|h_0(z)\big|^2 + \big|\phi(z)\big|^2 + \big|\psi(z)\big|^2 + \sum_{j=1}^{\infty}\big|h_j(z)\big|^2, \quad z \in D. \tag{6.5}$$

Hence $K_D(z_0) = \big|h_0(z_0)\big|^2$, which in particular implies $h_0(z_0) \neq 0$.

Lemma 6.0.3. *In normal coordinates for the Kobayashi–Fuks metric at z_0,*

(a) $I'_D(z_0) = |\phi'(z_0)|^2$, *and*

(b) $I''_D(z_0) = \sum_{j=1}^{\infty} \left| h'''_j(z_0) \right|^2$.

Proof. (a) Observe that in normal coordinates at z_0, I'_D is reduced to

$$I'_D(z_0) = \sup \left\{ |f'(z_0)|^2 : f \in A_1(z_0), \|f\|_D = 1 \right\}.$$

Since ϕ is a candidate for $I'_D(z_0)$,

$$I'_D(z_0) \geq \left|\phi'(z_0)\right|^2.$$

To see the reverse inequality, consider any $f \in A_1(z_0)$ with $\|f\|_D = 1$. Since f can be represented as

$$f(z) = \langle f, \phi \rangle \, \phi(z) + \langle f, \psi \rangle \, \psi(z) + \sum_{j=1}^{\infty} \langle f, h_j \rangle \, h_j(z),$$

using $\|f\|_D = 1$, we have

$$|f'(z_0)|^2 = \left|\langle f, \phi \rangle\right|^2 \left|\phi'(z_0)\right|^2 \leq \left|\phi'(z_0)\right|^2.$$

Now taking supremum in the left hand side of the above inequality, we get

$$I'_D(z_0) \leq \left|\phi'(z_0)\right|^2.$$

(b) Clearly the right hand side is finite thanks to Cauchy estimates. Observe that in normal coordinates at z_0,

$$I''_D(z_0) = \sup \left\{ |f'''(z_0)|^2 : f \in A_3(z_0), \|f\|_D = 1 \right\}.$$

Note that if $f(z) = \sum_{j=1}^{\infty} a_j h_j(z)$ is an arbitrary member of $A_3(z_0)$, then

$$\left|f'''(z_0)\right|^2 = \left| \sum_{j=1}^{\infty} a_j h'''_j(z_0) \right|^2 = \left| \sum_{j=1}^{\infty} a_j H_j \right|^2,$$

where $H_j = h'''_j(z_0)$. Moreover, as $A_3(z_0)$ is linearly isometric to ℓ_2 via the orthonormal basis $\{h_j\}$, we also have $\|f\|_D = \|a\|_{\ell_2}$, where $a = \{a_j\}_{j \geq 1}$. Hence, we arrive at

$$I''_D(z_0) = \sup \left\{ \left| \sum_{j=1}^{\infty} a_j H_j \right|^2 : a = \{a_j\}_{j \geq 1} \in \ell_2, \|a\|_{\ell_2} = 1 \right\}. \tag{6.6}$$

Now let $H = \{H_j\}_{j=1}^{\infty}$ and define $L_H : \ell_2 \to \mathbb{C}$ by

$$L_H(a) = \sum_{j=1}^{\infty} a_j H_j.$$

Then L_H is a bounded linear operator on ℓ_2 and denoting its operator norm by $\|L_H\|$, observe that

$$\|L_H\|^2 = \sup_{\|a\|=1} \left| L_H(\{a_j\}) \right|^2 = \sup_{\|a\|=1} \left| \sum_{j=1}^{\infty} a_j H_j \right|^2 = I''_D(z_0) \tag{6.7}$$

by (6.6). Also, from the canonical isometry of ℓ'_2 with ℓ_2, we have

$$\|L_H\|^2 = \|H\|^2_{\ell_2} = \sum_{j=1}^{\infty} |H_j|^2 = \sum_{j=1}^{\infty} \left| h'''_j(z_0) \right|^2. \tag{6.8}$$

From (6.7) and (6.8), the lemma follows immediately. □

Proof of Proposition 6.0.2. We work in normal coordinates for the Kobayashi–Fuks metric at z_0. Without loss of generality, we will denote the new coordinates by z, same as the previous ones. Note that in normal coordinates at z_0, we have

$$g_{\tilde{B},D}(z_0) = 1, \quad \frac{\partial g_{\tilde{B},D}}{\partial z}(z_0) = 0, \quad \text{and} \quad R_{\tilde{B},D}(z_0) = -\frac{\partial^2 g_{\tilde{B},D}}{\partial z \partial \overline{z}}(z_0). \tag{6.9}$$

Next, we express the above equations in terms of the basis expansion of K_D. Recall that the Kähler potential of the Kobayashi–Fuks metric in dimension 1 is $\log A(z)$ where

$$A = K_D^2 g_{B,D} = K_D^2 \frac{\partial^2 \log K_D}{\partial z \partial \overline{z}} = K_D \tfrac{\partial^2 K_D}{\partial z \partial \overline{z}} - \tfrac{\partial K_D}{\partial z} \tfrac{\partial K_D}{\partial \overline{z}}. \tag{6.10}$$

Thus,

$$g_{\tilde{B},D} = \frac{\partial^2 \log A}{\partial z \partial \overline{z}} = \frac{1}{A} \frac{\partial^2 A}{\partial z \partial \overline{z}} - \frac{1}{A^2} \frac{\partial A}{\partial z} \frac{\partial A}{\partial \overline{z}}. \tag{6.11}$$

Using the expansion of the Bergman kernel as in (6.5), we get from (6.10)

$$\begin{aligned}
A(z_0) &= \left|h_0(z_0)\right|^2 \left|\phi'(z_0)\right|^2, \\
\frac{\partial A}{\partial z}(z_0) &= \left|h_0(z_0)\right|^2 \overline{\phi'(z_0)}\, \phi''(z_0), \\
\frac{\partial^2 A}{\partial z \partial \overline{z}}(z_0) &= \left|h_0(z_0)\right|^2 \left(\left|\phi''(z_0)\right|^2 + \left|\psi''(z_0)\right|^2\right), \\
\frac{\partial^2 A}{\partial z^2}(z_0) &= \left|h_0(z_0)\right|^2 \overline{\phi'}(z_0)\phi'''(z_0) + \overline{h_0(z_0)}\, h_0'(z_0)\overline{\phi'}(z_0)\phi''(z_0) \\
&\quad - \overline{h_0(z_0)}\, h_0''(z_0) \left|\phi'(z_0)\right|^2, \\
\frac{\partial^3 A}{\partial z^2 \partial \overline{z}}(z_0) &= \left|h_0(z_0)\right|^2 \left(\overline{\phi''}(z_0)\phi'''(z_0) + \overline{\psi''}(z_0)\psi'''(z_0)\right) - \overline{h_0(z_0)}\, h_0''(z_0)\phi'(z_0)\overline{\phi''}(z_0) \\
&\quad + \overline{h_0(z_0)}\, h_0'(z_0) \left(\left|\phi''(z_0)\right|^2 + \left|\psi''(z_0)\right|^2\right), \text{ and} \\
\frac{\partial^4 A}{\partial z^2 \partial \overline{z}^2}(z_0) &= \left|h_0(z_0)\right|^2 \left(\left|\phi'''(z_0)\right|^2 + \left|\psi'''(z_0)\right|^2 + \sum_{j=1}^{\infty} \left|h_j'''(z_0)\right|^2\right) + \left|h_0'(z_0)\right|^2 \left(\left|\phi''(z_0)\right|^2 + \left|\psi''(z_0)\right|^2\right) \\
&\quad + \left|\phi'(z_0)\right|^2 \left(\left|h_0''(z_0)\right|^2 + \left|\psi''(z_0)\right|^2\right) - 2\operatorname{Re}\left(h_0(z_0)\overline{h_0''}(z_0)\overline{\phi'}(z_0)\phi'''(z_0)\right) \\
&\quad - 2\operatorname{Re}\left(h_0'(z_0)\overline{h_0''}(z_0)\overline{\phi'}(z_0)\phi''(z_0)\right) + 2\operatorname{Re}\left(h_0(z_0)\overline{h_0'}(z_0)\overline{\phi''}(z_0)\phi'''(z_0)\right) \\
&\quad + 2\operatorname{Re}\left(h_0(z_0)\overline{h_0'}(z_0)\overline{\psi''}(z_0)\psi'''(z_0)\right).
\end{aligned} \tag{6.12}$$

Also, differentiating (6.11) one immediately obtains

$$\frac{\partial g_{\tilde{B},D}}{\partial z} = \frac{1}{A} \frac{\partial^3 A}{\partial z^2 \partial \overline{z}} - \frac{2}{A^2} \frac{\partial A}{\partial z} \frac{\partial^2 A}{\partial z \partial \overline{z}} - \frac{1}{A^2} \frac{\partial A}{\partial \overline{z}} \frac{\partial^2 A}{\partial z^2} + \frac{2}{A^3} \frac{\partial A}{\partial \overline{z}} \left(\frac{\partial A}{\partial z}\right)^2, \tag{6.13}$$

and

$$-\frac{\partial^2 g_{\tilde{B},D}}{\partial z\partial\bar{z}}(z) = -\frac{1}{A}\frac{\partial^4 A}{\partial z^2\partial\bar{z}^2} + \frac{2}{A^2}\frac{\partial A}{\partial\bar{z}}\frac{\partial^3 A}{\partial z^2\partial\bar{z}} + \frac{2}{A^2}\frac{\partial A}{\partial z}\frac{\partial^3 A}{\partial z\partial\bar{z}^2} + \frac{2}{A^2}\left(\frac{\partial^2 A}{\partial z\partial\bar{z}}\right)^2 + \frac{1}{A^2}\frac{\partial^2 A}{\partial z^2}\frac{\partial^2 A}{\partial\bar{z}^2} + \frac{6}{A^4}\left(\frac{\partial A}{\partial z}\right)^2\left(\frac{\partial A}{\partial\bar{z}}\right)^2 - \frac{2}{A^3}\left(\frac{\partial A}{\partial\bar{z}}\right)^2\frac{\partial^2 A}{\partial z^2} - \frac{2}{A^3}\left(\frac{\partial A}{\partial z}\right)^2\frac{\partial^2 A}{\partial\bar{z}^2} - \frac{8}{A^3}\frac{\partial A}{\partial z}\frac{\partial A}{\partial\bar{z}}\frac{\partial^2 A}{\partial z\partial\bar{z}}. \tag{6.14}$$

Now, the relation $g_{\tilde{B},D}(z_0) = 1$ using (6.11) and (6.12) gives

$$\left|h_0(z_0)\right|^4\left|\phi'(z_0)\right|^2\left|\psi''(z_0)\right|^2 = \left|h_0(z_0)\right|^4\left|\phi'(z_0)\right|^4.$$

Observe that $h_0(z_0) \neq 0$ and $\phi'(z_0) \neq 0$. Hence the above identity implies

$$\left|\psi''(z_0)\right|^2 = \left|\phi'(z_0)\right|^2.$$

As a consequence, applying a unitary transformation to the basis $\{\psi\}$, we may assume that

$$\psi''(z_0) = \phi'(z_0). \tag{6.15}$$

The relation $\frac{\partial g_{\tilde{B},D}}{\partial z}(z_0) = 0$ using (6.13) and (6.12) gives

$$\left|h_0(z_0)\right|^2\left|\phi'(z_0)\right|^2\overline{\psi''}(z_0)\,\psi'''(z_0) + \overline{h_0(z_0)}\,h_0'(z_0)\,\left|\phi'(z_0)\right|^2\,\left|\psi''(z_0)\right|^2 - 2\left|h_0(z_0)\right|^2\overline{\phi'}(z_0)\,\phi''(z_0)\,\left|\psi''(z_0)\right|^2 = 0.$$

Using (6.15) in the above sum, we obtain the relation

$$h_0(z_0)\,\psi'''(z_0) + h_0'(z_0)\,\phi'(z_0) - 2h_0(z_0)\,\phi''(z_0) = 0. \tag{6.16}$$

Finally, we compute the curvature. It follows from (6.14) and (6.12), by a straightforward but lengthy calculation, that

$$A^4\frac{-\partial^2 g_{\tilde{B},D}}{\partial z\partial\bar{z}} = -|h_0|^8|\phi'|^6\Big(|\psi'''|^2 + \sum_{j=1}^{\infty}|h_j'''|^2\Big) - 2|h_0|^6\,|\phi'|^6\,\mathrm{Re}\,\Big(h_0\overline{h_0'\psi''}\psi'''\Big) - |h_0|^6\,|h_0'|^2\,|\phi'|^6\,|\psi''|^2 - |h_0|^6|\phi'|^8|\psi''|^2 + 4|h_0|^8|\phi'|^4\,\mathrm{Re}\,\big(\phi'\overline{\phi''\psi''}\psi'''\big) + 4|h_0|^6|\phi'|^4|\psi''|^2\,\mathrm{Re}\,\Big(h_0\overline{h_0'\phi'}\phi''\Big) + 2|h_0|^8|\phi'|^4|\psi''|^4 - 4|h_0|^8|\phi'|^4|\phi''|^2|\psi''|^2$$

at the point z_0. In the above equation and in the subsequent steps, if not mentioned, all the terms and partial derivatives are evaluated at the point z_0. Now making use of the relation (6.15), the above equation can be rewritten as

$$|h_0|^8|\phi'|^8\frac{-\partial^2 g_{\tilde{B},D}}{\partial z\partial\bar{z}}(z_0) = 2|h_0|^8|\phi'|^8 - |h_0|^6|\phi'|^{10} - |h_0|^8|\phi'|^6\sum_{j=1}^{\infty}|h_j'''|^2 - |h_0|^6|\phi'|^6\Big\{|h_0|^2|\psi'''|^2 + |h_0'|^2|\phi'|^2 + 4|h_0|^2|\phi''|^2 + 2\,\mathrm{Re}\,\big(h_0\overline{h_0'\phi'}\psi'''\big) - 4|h_0|^2\,\mathrm{Re}\,\big(\overline{\phi''}\psi'''\big) - 4\,\mathrm{Re}\,\big(h_0\overline{h_0'\phi'}\phi''\big)\Big\}.$$

Here one can see that, the terms inside the curly bracket above is exactly equal to

$$|h_0\psi''' + h_0'\phi' - 2h_0\phi''|^2,$$

which vanishes by vitue of the relation (6.16). Hence we finally arrive at

$$R_{\tilde{B},D}(z_0) = -\frac{\partial^2 g_{\tilde{B},D}}{\partial z \partial \overline{z}}(z_0) = 2 - \frac{\left|\phi'(z_0)\right|^2}{\left|h_0(z_0)\right|^2} - \frac{1}{\left|\phi'(z_0)\right|^2} \sum_{j=1}^{\infty} \left|h_j'''(z_0)\right|^2. \tag{6.17}$$

The right hand side of the above identity is finite thanks to the Cauchy estimates. The proposition now follows from Lemma 6.0.3. □

The following result is an immediate consequence of Proposition 6.0.2.

Corollary 6.0.4. *The Gaussian curvature of the Kobayashi–Fuks metric on bounded planar domains is strictly bounded above by* 2*.*

Now we localize the domain funtions I'_Ω and I''_Ω.

Proposition 6.0.5. *The functions I'_D and I''_D can be localized. More precisely, let $D \subset \mathbb{C}$ be a bounded domain, $p^0 \in \partial D$ a local peak point, and U a sufficiently small neighborhood of p^0. Then*

$$\lim_{z \to p^0} \frac{I'_D(z)}{I'_{U\cap D}(z)} = \lim_{z \to p^0} \frac{I''_D(z)}{I''_{U\cap D}(z)} = 1.$$

Proof. We will present the proof only for I''_D here. The proof for I'_D follows in an exact similar manner. Let h be a local holomorphic peak function for p^0 defined on a neighborhood U of p^0. Shrinking U if necessary, we can assume that h is nonvanishing on $U \cap \overline{D}$. Now choose any neighborhood V of p^0 such that $V \subset\subset U$. Then there is a constant $b \in (0,1)$ such that $|h| \leq b$ on $\overline{(U \setminus V) \cap D}$. Let us choose a cut-off function $\chi \in C_0^\infty(U)$ satisfying $\chi = 1$ on V and $0 \leq \chi \leq 1$ on U. Given any $\zeta \in V$, a function $f \in A^2(U \cap D)$, and an integer $k \geq 1$, set

$$\phi(z) = 8 \log|z - \zeta| \quad \text{and} \quad \alpha_k = \overline{\partial}(\chi f h^k).$$

Then ϕ is a subharmonic function on $\mathbb{C}$ and α_k is a $\overline{\partial}$-closed, smooth $(0,1)$-form on D with $\operatorname{supp} \alpha_k \subset (U \setminus V) \cap D$. Now as in [36], applying Theorem 4.2 of [30], we get a solution u to the equation $\overline{\partial} u = \alpha_k$ on D such that

$$\int_D |u(z)|^2 e^{-\phi(z)}(1 + |z|^2)^{-2} dV(z) \leq \int_D |\alpha_k(z)|^2 e^{-\phi(z)} dV(z),$$

where dV denotes the standard Lebesgue measure on $\mathbb{C}$. Then above inequality clearly implies

$$\int_D \frac{|u(z)|^2}{|z - \zeta|^8 (1 + |z|^2)^2} dV(z) \leq \int_{(U\setminus V)\cap D} \frac{|\alpha_k(z)|^2}{|z - \zeta|^8} dV(z). \tag{6.18}$$

Since the right-hand side of (6.18) is bounded, so is the left-hand side. This in particular implies that

$$\frac{\partial^{|A|+|B|} u}{\partial z^A \partial \overline{z}^B}(\zeta) = 0 \quad \text{for all multi-indices } A, B \text{ with } |A| + |B| \leq 3. \tag{6.19}$$

Moreover, since D is bounded, there are positive constants c_1, c_2 independent of k and ζ such that

$$\int_D \frac{|u|^2}{|z-\zeta|^8\,(1+|z|^2)^2}dV(z) \geq c_1 \int_\Omega |u|^2 dV,$$

and

$$\begin{aligned}\int_{(U\setminus V)\cap D} \frac{|\alpha_k|^2}{|z-\zeta|^8}dV(z) &= \int_{(U\setminus V)\cap D} \frac{|\bar\partial\chi|^2|f|^2|h|^{2k}}{|z-\zeta|^8}dV(z) \leq c_2 \int_{(U\setminus V)\cap D} |f|^2|h|^{2k}dV \\ &\leq c_2 b^{2k} \int_{(U\setminus V)\cap D} |f|^2 dV, \quad \text{by the choices of } h, V \\ &\leq c_2 b^{2k} \|f\|^2_{U\cap D}.\end{aligned}$$

For $c = c_2/c_1$, it follows from (6.18) that

$$\|u\|_D \leq c b^k \|f\|_{U\cap D}. \tag{6.20}$$

Here, clearly, c is a constant independent of ζ and k.

Now let $f \in A^2(U\cap D)$ be a maximizing function for $I''_{U\cap D}(\zeta)$, i.e., $\|f\|_{U\cap D} = 1$, $f(\zeta) = f'(\zeta) = f''(\zeta) = 0$, and $g^{-3}_{\tilde B, U\cap D}(\zeta)|f'''(\zeta)|^2 = I''_{U\cap D}(\zeta)$. Choose u as above and set $F_k = \chi f h^k - u$. Then $F_k \in A^2(D)$ and it follows from (6.20) that

$$\|F_k\|_D \leq \|\chi f h^k\|_D + \|u\|_D \leq \|f\|_{U\cap D} + cb^k\|f\|_{U\cap D} = 1 + cb^k. \tag{6.21}$$

Therefore, setting $f_k = F_k/\|F_k\|_D$, we see that $f_k \in A^2(D)$, $\|f_k\|_D = 1$, and $f_k(\zeta) = f'_k(\zeta) = f''_k(\zeta) = 0$. Moreover, by the maximality of $I''_D(\zeta)$, estimate (6.21), and (ii) of Proposition 5.2.3, one obtains

$$\begin{aligned} I''_D(\zeta) &\geq g^{-3}_{\tilde B, D}(\zeta)|f'''_k(\zeta)|^2 \\ &= \|F_k\|_D^{-2} |h^k(\zeta)|^2\, g^{-3}_{\tilde B, D}(\zeta)|f'''(\zeta)|^2 \\ &\geq \|F_k\|_D^{-2}|h(\zeta)|^{2k}\left(\frac{K_D(\zeta)}{K_{U\cap D}(\zeta)}\right)^6 \left(\frac{J_D(\zeta)}{J_{U\cap D}(\zeta)}\right)^3 g^{-3}_{\tilde B, U\cap D}(\zeta)|f'''(\zeta)|^2 \\ &\geq \frac{|h(\zeta)|^{2k}}{(1+cb^k)^2}\left(\frac{K_D(\zeta)}{K_{U\cap D}(\zeta)}\right)^6 \left(\frac{J_D(\zeta)}{J_{U\cap D}(\zeta)}\right)^3 I''_{U\cap D}(\zeta). \end{aligned}$$

This implies that

$$\frac{I''_D(\zeta)}{I''_{U\cap D}(\zeta)} \geq \frac{|h(\zeta)|^{2k}}{(1+cb^k)^2}\left(\frac{K_D(\zeta)}{K_{U\cap D}(\zeta)}\right)^6 \left(\frac{J_D(\zeta)}{J_{U\cap D}(\zeta)}\right)^3.$$

Note that $h(p^0) = 1$. By (6.1), Proposition 2.1, and Proposition 2.4 of [36],

$$\lim_{\zeta\to p^0} \frac{K_D(\zeta)}{K_{U\cap D}(\zeta)} = \lim_{\zeta\to p^0} \frac{J_D(\zeta)}{J_{U\cap D}(\zeta)} = 1.$$

Hence, letting $\zeta \to p^0$ in the above inequality, we get

$$\liminf_{\zeta\to p^0} \frac{I''_D(\zeta)}{I''_{U\cap D}(\zeta)} \geq (1+cb^k)^{-2}.$$

Now letting $k \to \infty$, as $0 \leq b < 1$, and c is independent of k, we obtain

$$\liminf_{\zeta \to p^0} \frac{I''_D(\zeta)}{I''_{U\cap D}(\zeta)} \geq 1. \tag{6.22}$$

On the other hand, consider a candidate function η for $I''_D(\zeta)$, i.e., $\eta \in A^2(D)$, $\|\eta\|_D = 1$, and $\eta(\zeta) = \eta'(\zeta) = \eta''(\zeta) = 0$. Now for $z \in U \cap D$, let us define

$$\gamma(z) = \frac{\eta(z)}{\|\eta\|_{U\cap D}}.$$

Then clearly $\gamma \in A^2(U \cap D)$ with $\|\gamma\|_{U\cap D} = 1$, and $\gamma(\zeta) = \gamma'(\zeta) = \gamma''(\zeta) = 0$. Therefore the maximality of $I''_{U\cap D}(\zeta)$ implies

$$\begin{aligned} I''_{U\cap D}(\zeta) \geq g^{-3}_{\tilde{B},U\cap D}(\zeta)|\gamma'''(\zeta)|^2 &= \|\eta\|^{-2}_{U\cap D}\left(\frac{g_{\tilde{B},U\cap D}(\zeta)}{g_{\tilde{B},D}(\zeta)}\right)^{-3} g^{-3}_{\tilde{B},D}(\zeta)|\eta'''(\zeta)|^2 \\ &\geq \left(\frac{g_{\tilde{B},U\cap D}(\zeta)}{g_{\tilde{B},D}(\zeta)}\right)^{-3} I''_D(\zeta). \end{aligned}$$

The last inequality above follows from $\|\eta\|_{U\cap D} \leq 1$ and the fact that η is an arbitrary candidate function for $I''_D(\zeta)$. Thus we obtain

$$\frac{I''_D(\zeta)}{I''_{U\cap D}(\zeta)} \leq \left(\frac{g_{\tilde{B},U\cap D}(\zeta)}{g_{\tilde{B},D}(\zeta)}\right)^{3}. \tag{6.23}$$

By (ii) of Theorem 1.0.4, the right hand side converges to 1 as $\zeta \to p^0$, and therefore,

$$\limsup_{\zeta \to p^0} \frac{I''_D(\zeta)}{I''_{U\cap D}(\zeta)} \leq 1. \tag{6.24}$$

From (6.22) and (6.24), we conclude that

$$\lim_{\zeta \to p^0} \frac{I''_D(\zeta)}{I''_{U\cap D}(\zeta)} = 1,$$

as required. □

Lemma 6.0.6. *Suppose $\{a_j\}, \{b_j\}, \{c_j\}$ and $\{d_j\}$ are real sequences with $b_j, d_j > 0$ and*

$$\lim_{j\to\infty} \frac{a_j}{b_j} = 1 \quad \text{and} \quad \lim_{j\to\infty} \frac{c_j}{d_j} = 1.$$

Then we have

$$\lim_{j\to\infty} \frac{a_j + c_j}{b_j + d_j} = 1.$$

Proof. To verify this claim let $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\left|\frac{a_j}{b_j} - 1\right| < \frac{\epsilon}{2} \quad \text{and} \quad \left|\frac{c_j}{d_j} - 1\right| < \frac{\epsilon}{2} \quad \text{for } j \geq N.$$

This, in particular implies that

$$\frac{|a_j - b_j|}{b_j + d_j} < \frac{\epsilon}{2} \quad \text{and} \quad \frac{|c_j - d_j|}{b_j + d_j} < \frac{\epsilon}{2} \quad \text{for } j \geq N.$$

Hence we have

$$\left| \frac{a_j + c_j}{b_j + d_j} - 1 \right| \leq \frac{|a_j - b_j|}{b_j + d_j} + \frac{|c_j - d_j|}{b_j + d_j} < \epsilon \quad \text{for } j \geq N,$$

proving the lemma. □

We are now in a state to give a proof of remaining part of Theorem 1.0.4.

Proof of Theorem 1.0.4 (iii). By Proposition 6.0.2, let us write

$$R_{\tilde{B},D}(z) = 2 - \tilde{E}_D(z) - \tilde{F}_D(z),$$

where

$$\tilde{E}_D(z) = \frac{I'_D(z)}{K_D(z)} \quad \text{and} \quad \tilde{F}_D(z) = \frac{I''_D(z)}{I'_D(z)}.$$

From Proposition 6.0.5 and (6.1), we have

$$\lim_{z \to p^0} \frac{\tilde{E}_D(z)}{\tilde{E}_{U \cap D}(z)} = 1 \quad \text{and} \quad \lim_{z \to p^0} \frac{\tilde{F}_D(z)}{\tilde{F}_{U \cap D}(z)} = 1,$$

for a small enough neighborhood U of p^0. Now (iii) follows immediately from Lemma 6.0.6. □

Chapter 7

Boundary behavior of the Kobayashi–Fuks metric

In this chapter, we use Pinchuk's scaling method to find boundary behavior and compute exact asymptotics of the Kobayashi–Fuks metric on bounded strongly pseudoconvex domains. But we will dedicate our first section towards obtaining the boundary asymptotics on domains in $\mathbb{C}$. As we will see, in the planar case, one can bypass the scaling method by virtue of the Riemann mapping theorem to obtain the boundary behavior.

7.1 Boundary behavior on planar domains

Proof of Theorem 1.0.5. Note that because of the localization results in Theorem 1.0.4, it suffices to prove the theorem for $\tau_{\tilde{B},U\cap D}$ and $R_{\tilde{B},U\cap D}$ for a small neighborhood U of p^0. Since the boundary ∂D is smooth near p^0, we can choose a tiny disc U centered at p^0 such that $U \cap D$ is simply connected. Hence, there exists a Riemann map ϕ, which is a biholomorphism between $U \cap D$ and the unit disc Δ. Since $\partial(U \cap D)$ is a Jordan curve and is smooth near the boundary point p^0, by the work of S. E. Warschawski [47], it is known that ϕ can be extended smoothly to p^0 with $\phi'(p^0) \neq 0$.

(i) Consider any sequence of points $\{p^j\} \subset U \cap D$ converging to $p^0 \in \partial D$. Then using the invariance of the Kobayashi–Fuks metric and by Proposition 5.1.1, we have

$$\tau^2_{\tilde{B},U\cap D}(p^j, u) = \tau^2_{\tilde{B},\Delta}\big(\phi(p^j), \phi'(p^j)u\big) = \frac{6\,|\phi'(p^j)|^2}{\big(1-|\phi(p^j)|^2\big)^2}|u|^2.$$

Since $\delta^2_D(p^j) \approx \big(1-|\phi(p^j)|^2\big)^2$, there exists a constant $c > 0$ such that the above identity becomes

$$\lim_{j\to\infty} \delta^2_D(p^j)\tau^2_{\tilde{B},U\cap D}(p^j, u) = c \lim_{j\to\infty} \big(6\,|\phi'(p^j)|^2|u|^2\big) = C(D)\,|u|^2,$$

where $C(D) = 6\,c|\phi'(p^0)|^2$. This proves our claim.

(ii) By invariance of the Gaussian curvature,

$$R_{\tilde{B},U\cap D}(p^j) = R_{\tilde{B},\Delta}\big(\phi(p^j)\big). \tag{7.1}$$

By Proposition 5.1.1, we have for $z \in \Delta$,

$$g_{\tilde{B},\Delta}(z) = \frac{6}{(1-|z|^2)^2},$$

and therefore,

$$R_{\tilde{B},\Delta}(z) = -\frac{1}{g_{\tilde{B},\Delta}(z)} \frac{\partial^2 \log g_{\tilde{B},\Delta}}{\partial z \partial \overline{z}}(z) = -\frac{1}{3}.$$

This, along with (7.1), completes the proof of (ii) and the theorem. □

7.2 Pinchuk's scaling

In this section, we deal with the scaling technique on strongly pseudoconvex domains introduced by S. Pinchuk. As an application of Pinchuk's scaling, we will see how boundary behavior of the Kobayashi–Fuks objects on strongly pseudoconvex domains is dealt with by transferring the problem into computing those objects on the unit ball. The symmetric and geometrically simple nature of the unit ball allows us to handle the above stated problem in a rather perspicuous manner. We begin by recalling the change of coordinates associated with Pinchuk's scaling method. Throughout this section, D is a C^2-smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$ and ρ is a C^2-smooth local defining function for D defined on a neighborhood U of a point $p^0 \in \partial D$. Without loss of generality, we assume that

$$\nabla_{\overline{z}}\rho(p^0) = ('0, 1) \text{ and } \frac{\partial \rho}{\partial z_n}(z) \neq 0 \text{ for all } z \in U. \tag{7.2}$$

Here, $\nabla_z \rho = (\partial\rho/\partial z_1, \ldots, \partial\rho/\partial z_n)$ and we write $\nabla_{\overline{z}}\rho = \overline{\nabla_z \rho}$. Note that the gradient $\nabla\rho = 2\nabla_{\overline{z}}\rho$.

7.2.1 Change of coordinates

The following lemma from [44] illustrates the change of coordinates near strongly pseudoconvex boundary points.

Lemma 7.2.1. *There exist a family of biholomorphic mappings $h^\zeta : \mathbb{C}^n \to \mathbb{C}^n$ depending continuously on $\zeta \in \partial D \cap U$, satisfying the following conditions:*

(a) $h^{p^0} = \mathbb{I}$.

(b) $h^\zeta(\zeta) = 0$.

(c) *The local defining function $\rho^\zeta = \rho \circ (h^\zeta)^{-1}$ of the domain $D^\zeta = h^\zeta(D)$ near the origin has the form*
$$\rho^\zeta(z) = 2\,\mathrm{Re}\left(z_n + K^\zeta(z)\right) + H^\zeta(z) + o(|z^2|)$$
in a neighborhood of the origin, where
$$K^\zeta(z) = \sum_{\mu,\nu=1}^{n} a_{\mu\nu}(\zeta) z_\mu z_\nu \quad \text{and} \quad H^\zeta(z) = \sum_{\mu,\nu=1}^{n} a_{\mu\overline{\nu}}(\zeta) z_\mu \overline{z}_\nu$$
with $K^\zeta('z, 0) \equiv 0$ and $H^\zeta('z, 0) \equiv |'z|^2$.

(d) *The biholomorphism h^ζ takes the real normal $\eta_\zeta = \{z = \zeta + 2t\nabla_{\overline{z}}\rho(\zeta) : t \in \mathbb{R}\}$ to ∂D at ζ into the real normal $\{'z = y_n = 0\}$ to ∂D^ζ at the origin.*

The definition of the map h^ζ and its derivative will play an important role in the computation of the boundary asymptotics and so we quickly recall its construction. We fix $\zeta \in \partial D \cap U$. The map h^ζ is a polynomial automorphism of $\mathbb{C}^n$ defined as the composition $h^\zeta(z) = \phi_3^\zeta \circ \phi_2^\zeta \circ \phi_1^\zeta(z)$, where the maps $\phi_i^\zeta : \mathbb{C}^n \to \mathbb{C}^n$ are biholomorphisms defined as follows: The map $w = \phi_1^\zeta(z)$ is an affine transformation given by

$$\begin{aligned} w_j &= \frac{\partial \rho}{\partial \overline{z}_n}(\zeta)(z_j - \zeta_j) - \frac{\partial \rho}{\partial \overline{z}_j}(\zeta)(z_n - \zeta_n) \quad \text{for} \quad j = 1, \dots, n-1, \\ w_n &= \sum_{\nu=1}^{n} \frac{\partial \rho}{\partial z_\nu}(\zeta)(z_\nu - \zeta_\nu). \end{aligned} \tag{7.3}$$

The map ϕ_1^ζ is nonsingular by (7.2) and it takes the point ζ to the origin. We relabel the new coordinates w as z. Then the Taylor series expansion of the local defining function $\rho \circ (\phi_1^\zeta)^{-1}$ for the domain $\phi_1^\zeta(D)$ near the origin has the form

$$2\,\mathrm{Re}\left(z_n + \sum_{\mu,\nu=1}^{n} a_{\mu\nu}(\zeta) z_\mu z_\nu \right) + H^\zeta(z) + o(|z|^2), \tag{7.4}$$

where $H^\zeta(z)$ is a Hermitian form.

The map $w = \phi_2^\zeta(z)$ is given by

$$w = \left('z, z_n + \sum_{\mu,\nu=1}^{n-1} a_{\mu\nu}(\zeta) z_\mu z_\nu \right) \tag{7.5}$$

and is a polynomial automorphism. Relabelling the new coordinates w as z, the Taylor series expansion of the local defining function $\rho \circ (\phi_1^\zeta)^{-1} \circ (\phi_2^\zeta)^{-1}$ for the domain $\phi_2^\zeta \circ \phi_1^\zeta(D)$ has the form (7.4) with $a_{\mu\nu} = 0$ for $1 \le \mu, \nu \le n-1$.

Finally, the map ϕ_3^ζ is chosen so that the Hermitian form $H^\zeta(z)$ satisfies $H^\zeta('z, 0) = |'z|^2$. Since D is strongly pseudoconvex and the complex tangent space to ∂D at ζ is given by $z_n = 0$ in the current coordinates, the form $H^\zeta('z, 0)$ is strictly positive definite. Hence there exists a unitary map $U^\zeta : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ such that $H^\zeta\big(U^\zeta('z), 0\big)$ is diagonal with diagonal entries $\lambda_1^\zeta, \dots, \lambda_{n-1}^\zeta > 0$. Now consider the stretching map $L^\zeta = \mathrm{diag}\{(\lambda_1^\zeta)^{-1/2}, \dots, (\lambda_{n-1}^\zeta)^{-1/2}\}$. Then the linear map $A^\zeta : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ given by $A^\zeta := L^\zeta \circ U^\zeta$ satisfies $H^\zeta\big(A^\zeta('z), 0\big) = |'z|^2$. Note that U^ζ and L^ζ, and hence A^ζ can be chosen to depend continuously on ζ. Thus, if we define $w = \phi_3^\zeta(z)$ by

$$w = \big(A^\zeta('z), z_n\big),$$

and relabel w as z, then the local defining function $\rho^\zeta = \rho \circ (h^\zeta)^{-1}$ for the domain $D^\zeta = h^\zeta(D)$ near the origin has the Taylor series expansion as in (c). If we compute the defining function explicitly, we get

$$\begin{aligned} K^\zeta(z) &= z_n \left(\sum_{\mu=1}^{n} a_{\mu n}(\zeta) z_\mu + \sum_{\nu=1}^{n-1} a_{n\nu}(\zeta) z_\nu \right) \\ H^\zeta(z) &= |'z|^2 + \overline{z}_n \left(\sum_{\mu=1}^{n} a_{\mu\overline{n}}(\zeta) z_\mu \right) + z_n \left(\sum_{\nu=1}^{n-1} a_{n\overline{\nu}}(\zeta) \overline{z}_\nu \right). \end{aligned} \tag{7.6}$$

Therefore the conditions $K^\zeta('z,0) \equiv 0$ and $H^\zeta('z,0) \equiv |'z|^2$ are satisfied.

It is evident from the construction that $h^{p^0} = \mathbb{I}$ and as $\zeta \to \zeta^0$, $\phi_i^\zeta(z) \to \phi_i^{\zeta^0}(z)$ uniformly on compact subsets of $\mathbb{C}^n$, for $i = 1,2,3$. Hence $h^\zeta(z)$ converges to $h^{\zeta^0}(z)$ uniformly on compact subsets of $\mathbb{C}^n$. Again, note that the linear map ϕ_1^ζ takes the real tangent plane $\mathrm{T}_\zeta(\partial\Omega)$ to $\{z \in \mathbb{C}^n : \operatorname{Re} z_n = 0\}$. Since ϕ_1^ζ takes the point ζ to the origin, now the normal to the boundary at the origin is along $\operatorname{Re} z_n$-axis. Observe that the remaining two maps ϕ_2^ζ and ϕ_3^ζ do not alter the real tangent plane or the normal at the origin. Hence the biholomorphism h^ζ takes the real normal η_ζ to $\partial\Omega$ at p^0 into the real normal line $\{'z = y_n = 0\}$ to ∂D^ζ at the origin.

7.2.2 Scaling of the domain

By strong pseudoconvexity, shrinking U if necessary, there exist local holomorphic coordinates $z_1, \ldots, z_n$ on U in which $p^0 = 0$, and

$$\rho(z) = 2\operatorname{Re} z_n + |'z|^2 + o\big(\operatorname{Im} z_n, |'z|^2\big), \quad z \in U, \tag{7.7}$$

and a constant $0 < r < 1$ such that

$$U \cap D \subset \Omega := \big\{z \in \mathbb{C}^n : 2\operatorname{Re} z_n + r|'z|^2 < 0\big\}. \tag{7.8}$$

Henceforth, we will be working in the above coordinates, and with U, ρ and p^0 as above.

Let us consider a sequence of points p^j in D that converges to $p^0 = 0$ on ∂D. For j sufficiently large, and without loss of generality we assume that for all j, $p^j \in U$ and there exists a unique $\zeta^j \in \partial D \cap U$ that is closest to p^j. Define $\delta_j := \mathrm{d}(p^j, \partial D) = |p^j - \zeta^j|$. Note that $\zeta^j \to p^0 = 0$ and $\delta_j \to 0$ as $j \to \infty$. For each ζ^j, denote by h_j the map h^{ζ^j} given by Lemma 7.2.1. Denoting $\phi_i^{\zeta^j}$ by ϕ_i^j for $j = 1,2,3$, we have $h_j = \phi_1^j \circ \phi_2^j \circ \phi_3^j$. Also set $\rho_j = \rho^{\zeta^j}$. Then by Lemma 7.2.1, near the origin,

$$\rho_j(z) = 2\operatorname{Re}\big(z_n + K_j(z)\big) + H_j(z) + o\big(|z|^2\big),$$

where $K_j = K^{\zeta^j}$ and $H_j = H^{\zeta^j}$. Moreover, thanks to the strong pseudoconvexity of ∂D near $p^0 = 0$, shrinking U if necessary and taking a smaller r in (7.8), we have

$$h_j(U \cap D) \subset \Omega \tag{7.9}$$

for all large j. Note that by Theorem 1.0.4, it is enough to prove Theorem 1.0.7 for the domain $U \cap D$, shrinking U if necessary. Set $D_j = h_j(U \cap D)$, $q^j = h_j(p^j)$, and $\eta_j = \mathrm{d}(q^j, \partial D_j)$.

Now consider the anisotropic dilation map $\Lambda_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Lambda_j(z) = \left(\frac{z_1}{\sqrt{\eta_j}}, \ldots, \frac{z_{n-1}}{\sqrt{\eta_j}}, \frac{z_n}{\eta_j}\right). \tag{7.10}$$

Set $\tilde{D}_j = \Lambda_j(D_j) = \Lambda_j \circ h_j(U \cap D)$. We will call the maps $S_j := \Lambda_j \circ h_j$ the scaling maps and the domains $\tilde{D}_j = S_j(U \cap D)$ the scaled domains. Note that since $S_j(p^j) = \Lambda_j \circ h_j(p^j) = ('0, -1)$, each $\tilde{D}_j$ contains the point $('0, -1)$ and we will denote this point by b^*. A defining function for $\tilde{D}_j$ near the origin, is given by

$$\tilde{\rho}_j(z) = \frac{1}{\eta_j}\rho_j\big(\Lambda_j^{-1}(z)\big) = 2\operatorname{Re}\left(z_n + \frac{1}{\eta_j}K_j\big(\Lambda_j^{-1}(z)\big)\right) + \frac{1}{\eta_j}H_j\big(\Lambda_j^{-1}(z)\big) + o\big(\eta_j^{1/2}|z|^2\big).$$

Since $K_j(z)$ and $H_j(z)$ satisfy condition (c) of Lemma 7.2.1, it follows that

$$\lim_{j\to\infty}\frac{1}{\eta_j}K_j(\Lambda_j^{-1}z)=0 \quad \text{and} \quad \lim_{j\to\infty}\frac{1}{\eta_j}H_j(\Lambda_j^{-1}z)=|'z|^2$$

in C^2-topology on compact subsets of $\mathbb{C}^n$. Evidently, as $\eta_j \to 0$, we have $o(\eta_j^{1/2}|z|^2) \to 0$ as $j \to \infty$ in C^2-topology on any compact set of $\mathbb{C}^n$. Thus, the defining functions $\tilde{\rho}_j$ converge in C^2 topology on compact subsets of $\mathbb{C}^n$ to

$$\rho_\infty(z) = 2\operatorname{Re} z_n + |'z|^2.$$

Hence our scaled domains $\tilde{D}_j$ converge in the local Hausdorff sense to the Siegel upper half-space

$$D_\infty = \big\{z \in \mathbb{C}^n : 2\operatorname{Re} z_n + |'z|^2 < 0\big\}.$$

7.3 A Ramanadov type theorem and stability results

In this section, we will establish the stability results for the quantities associated to the Kobayashi–Fuks metric whose boundary asymptotics will be computed in the later sections. First we state a stability result for the Bergman kernel along with its partial derivatives under a specific perturbation of domains, generally known as a Ramanadov type result. The majority of content of this section can be found in [7] with some more details and we include it here for reader's convenience.

Let G be a domain in $\mathbb{C}^n$. For $q \in \mathbb{C}^n$, $G - q$ will denote the domain that is the the image of G under the translation $v \mapsto v - q$. Similarly, for $r > 0$, rG will denote the image of G under the map $v \mapsto rv$.

Proposition 7.3.1. *Let G_j be a sequence of domains in $\mathbb{C}^n$ converging to a domain G in $\mathbb{C}^n$ in the following manner:*
(i) any compact set of G is eventually contained in each G_j,
(ii) there exists a common interior point q of G and all G_j, such that for every $\epsilon > 0$ there exists $j_\epsilon \in \mathbb{N}$ satisfying

$$G_j - q \subset (1+\epsilon)(G-q), \quad \text{for all } j \geq j_\epsilon.$$

Assume further that G is star-convex with respect to q and K_G is non-vanishing on the diagonal. Then $K_{G_j} \to K_G$ uniformly on compact subsets of G together with all the partial derivatives.

A proof of the above proposition can be found in [1]. We now derive a stability result for the Bergman kernel and its derivatives under Pinchuk's scaling. First note that the Cayley transform Φ defined by

$$\Phi(z_1,\ldots,z_n) = \left(\frac{\sqrt{2}z_1}{z_n-1},\ldots,\frac{\sqrt{2}z_{n-1}}{z_n-1},\frac{z_n+1}{z_n-1}\right) \tag{7.11}$$

is a biholomorphism that maps D_∞ onto $\mathbb{B}^n$. The following properties of the map Φ can also be checked by a routine calculation: Φ is a biholomorphism of its domain

$$\mathcal{D}_\Phi := \mathbb{C}^n \setminus \{z : z_n = 1\}$$

onto itself with $\Phi^{-1} = \Phi$. The domain Ω defined in (7.8) is mapped by Φ onto the bounded domain

$$\Phi(\Omega) = \{z \in \mathbb{C}^n : r|'z|^2 + |z_n|^2 < 1\}. \tag{7.12}$$

Lemma 7.3.2. *For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$,*

$$\frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} K_{\tilde{D}_j}(z) \to \frac{\partial^{|\alpha|+|\beta|}}{\partial z^\alpha \partial \overline{z}^\beta} K_{D_\infty}(z)$$

uniformly on compact subsets of D_∞, as $j \to \infty$.

Proof. We have seen, by (7.9), that Ω contains D_j for all sufficiently large j. The domain Ω can also be checked to be invariant by the dialation maps Λ_j. Therefore, Ω contains $\tilde{D}_j$ for all sufficiently large j. Again it can be easily observed that, since $c < 1$, Ω contains D_∞. We claim that $\Phi(\tilde{D}_j)$ converge to $\Phi(D_\infty) = \mathbb{B}^n$ in the way required by the hypothesis of Proposition 7.3.1 with $q = 0$.

Indeed, first note that if S is a compact subset of $\Phi(D_\infty)$, then $\Phi^{-1}(S)$ is a compact subset of D_∞. Since $\tilde{D}_j \to D_\infty$ in the local Hausdorff sense, $\Phi^{-1}(S)$ is contained in $\tilde{D}_j$ for all sufficiently large j, which in turn implies that S is contained in $\Phi(\tilde{D}_j)$ for all sufficiently large j. Next, if possible, assume that the second condition in the hypothesis of Proposition 7.3.1 is not satisfied with $q = 0$. Then there exists an $\epsilon > 0$, a subsequence of $\Phi(\tilde{D}_j)$ (which we relabel as $\Phi(\tilde{D}_j)$ itself) and $\xi^j \in \Phi(\tilde{D}_j)$ such that ξ^j lies outside $(1+\epsilon)\Phi(D_\infty) = B(0, 1+\epsilon)$. Therefore

$$|\xi^j| \geq 1 + \epsilon. \tag{7.13}$$

Since $\Phi(\tilde{D}_j) \subset \Phi(\Omega)$ for all large j, the sequence $\{\xi^j\}$ is bounded and hence after passing to a subsequence, $\xi^j \to \xi$ for some $\xi \in \overline{\Phi(\Omega)}$. This, in particular, implies that

$$|\xi_n|^2 + r|'\xi|^2 \leq 1 \quad \text{and} \quad |\xi| \geq 1 + \epsilon. \tag{7.14}$$

The first inequality above follows from (7.12) and the second one is obtained taking limit in (7.13). The inequalities in (7.14) together ensure that $\xi_n \neq 1$, i.e., $\xi \in \mathcal{D}_\Phi$. Therefore it follows $\Phi^{-1}(\xi^j) \to \Phi^{-1}(\xi)$, as $j \to \infty$. Since $\Phi^{-1}(\xi^j) \in \tilde{D}_j$, we have $\tilde{\rho}_j\big(\Phi^{-1}(\xi^j)\big) < 0$ for all large j, and hence $\rho_\infty\big(\Phi^{-1}(\xi)\big) \leq 0$. This implies $\Phi^{-1}(\xi) \in \overline{D}_\infty$ and hence $\xi \in \overline{\Phi(D_\infty)} = \overline{\mathbb{B}^n}$, which contradicts the fact that $|\xi| \geq 1 + \epsilon$. This proves our claim.

Therefore, by Proposition 7.3.1, $K_{\Phi(\tilde{D}_j)}(z)$ converges to $K_{\Phi(D_\infty)}(z)$ uniformly on compact subsets of $\Phi(D_\infty)$, together with all partial derivatives. Now, applying the transformation rule of the Bergman kernel, our result follows immediately. $\square$

Proposition 7.3.3. *For $z \in D_\infty$ and $u \in \mathbb{C}^n \setminus \{0\}$, we have*

$$g_{\tilde{B},\tilde{D}_j}(z) \to g_{\tilde{B},D_\infty}(z), \quad \tau_{\tilde{B},\tilde{D}_j}(z,u) \to \tau_{\tilde{B},D_\infty}(z,u) \quad \textit{and} \quad \mathrm{Ric}_{\tilde{B},\tilde{D}_j}(z,u) \to \mathrm{Ric}_{\tilde{B},D_\infty}(z,u)$$

as $j \to \infty$. Moreover, the first convergence is uniform on compact subsets of D_∞ and the second and third convergences are uniform on compact subsets of $D_\infty \times \mathbb{C}^n$.

Proof. Since the Kobayashi–Fuks metric on the domain D has Kähler potential $\log(K_D^{n+1} g_{B,D})$, i.e.,

$$g^{\tilde{B},D}_{\alpha\overline{\beta}} = \frac{\partial^2 \log(K_D^{n+1} g_{B,D})}{\partial z_\alpha \partial \overline{z}_\beta},$$

all that is required is to show that

$$K^{n+1}_{\tilde{D}_j} g_{B,\tilde{D}_j} \to K^{n+1}_{D_\infty} g_{B,D_\infty}$$

uniformly on compact subsets of D_∞, together with all derivatives. But this is an immediate consequence of the fact that $K_{\tilde{D}_j} \to K_{D_\infty}$ together will all derivatives on compact subsets of D_∞, which is precisely Lemma 7.3.2. $\square$

7.4 Boundary asymptotics in higher dimensions

We are now ready to compute the boundary asymptotics of the Kobayashi–Fuks metric on smoothly bounded strongly pseudoconvex domains. Recall that $S_j = \Lambda_j \circ h_j$, $S_j(U \cap D) = \tilde{D}_j$, and $S_j(p^j) = b^* = ('0, -1)$. Denoting the matrix of a linear map by itself, we have

$$S_j'(p^j) = \Lambda_j h_j'(p^j) = \Lambda_j \cdot \phi_3^j \cdot (\phi_2^j)'\big(\phi_1^j(p^j)\big) \cdot (\phi_1^j)'(p^j). \tag{7.15}$$

Note that from the definition of ϕ_1^j,

$$(\phi_1^j)'(p^j) = \begin{pmatrix} \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & 0 & \cdots & 0 & -\dfrac{\partial \rho}{\partial \overline{z}_1}(\zeta^j) \\ 0 & \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & \cdots & 0 & -\dfrac{\partial \rho}{\partial \overline{z}_2}(\zeta^j) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & -\dfrac{\partial \rho}{\partial \overline{z}_{n-1}}(\zeta^j) \\ \dfrac{\partial \rho}{\partial z_1}(\zeta^j) & \dfrac{\partial \rho}{\partial z_2}(\zeta^j) & \cdots & \dfrac{\partial \rho}{\partial z_{n-1}}(\zeta^j) & \dfrac{\partial \rho}{\partial z_n}(\zeta^j) \end{pmatrix}. \tag{7.16}$$

Also, since

$$p^j = \zeta^j - \delta_j \frac{\nabla_{\overline{z}} \rho(\zeta^j)}{\big|\nabla_{\overline{z}} \rho(\zeta^j)\big|}, \tag{7.17}$$

we have

$$\phi_1^j(p^j) = \Big('0, -\delta_j \big|\nabla_{\overline{z}} \rho(\zeta^j)\big|\Big). \tag{7.18}$$

Therefore, from the definition of ϕ_2^j, we have

$$(\phi_2^j)'\big(\phi_1^j(p^j)\big) = \mathbb{I}_n.$$

Finally, recall that $\phi_3^j(z) = \big(A^j('z), z_n\big)$, where $A^j := A^{\zeta^j} : \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ are linear maps satisfying $A^j \to \mathbb{I}_{n-1}$. Therefore,

$$\phi_3^j = \begin{bmatrix} A_{p,q}^j & 0 \\ 0 & 1 \end{bmatrix},$$

where $A_{p,q}^j$ are the entries of the matrix of A^j. Thus,

$$h_j'(p^j) = \begin{pmatrix} A_{1,1}^j \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & \cdots & A_{1,n-1}^j \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & -\displaystyle\sum_{\nu=1}^{n-1} A_{1,\nu}^j \dfrac{\partial \rho}{\partial \overline{z}_\nu}(\zeta^j) \\ \vdots & \cdots & \vdots & \vdots \\ A_{n-1,1}^j \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & \cdots & A_{n-1,n-1}^j \dfrac{\partial \rho}{\partial \overline{z}_n}(\zeta^j) & -\displaystyle\sum_{\nu=1}^{n-1} A_{n-1,\nu}^j \dfrac{\partial \rho}{\partial \overline{z}_\nu}(\zeta^j) \\ \dfrac{\partial \rho}{\partial z_1}(\zeta^j) & \cdots & \dfrac{\partial \rho}{\partial z_{n-1}}(\zeta^j) & \dfrac{\partial \rho}{\partial z_n}(\zeta^j) \end{pmatrix} \to \mathbb{I}_n \tag{7.19}$$

in the operator norm.

We also note that as ϕ_2^j and ϕ_3^j fix points on the $\operatorname{Re} z_n$-axis, we have from (7.18),

$$q^j = h_j(p^j) = \Big('0, -\delta_j \big|\nabla_{\overline{z}} \rho(\zeta^j)\big|\Big).$$

As the normal to ∂D_j at 0 is the $\operatorname{Re} z_n$-axis and $\eta_j = \mathrm{d}(q^j, \partial D_j)$, we have $\eta_j = \delta_j \left|\nabla_{\overline{z}}\rho(\zeta^j)\right|$ and hence

$$\lim_{j\to\infty} \frac{\eta_j}{\delta_j} = 1. \tag{7.20}$$

Now recall that the Cayley transform Φ defined in (7.11) is a biholomorphism between D_∞ and $\mathbb{B}^n$. Also note that $b^* = ('0, -1) \in D_\infty$, $\Phi(b^*) = 0$, and

$$\Phi'(b^*) = -\operatorname{diag}\{1/\sqrt{2}, \ldots, 1/\sqrt{2}, 1/2\}. \tag{7.21}$$

We now present the proof of Theorem 1.0.7.

Proof of Theorem 1.0.7. Note that by the localization result Theorem 1.0.4, it is enough to compute the asymptotics for the domain $U \cap D$.

(i) By invariance of the Kobayashi–Fuks metric,

$$\tau_{\tilde{B}, U\cap D}(p^j, u) = \tau_{\tilde{B}, \tilde{D}_j}\big(b^*, S_j'(p^j)u\big).$$

Note that

$$S_j'(p^j)u = \Lambda_j\big(h_j'(p^j)u\big) = \Big(\eta_j^{-1/2}\,'\big(h_j'(p^j)u\big), \eta_j^{-1}\big(h_j'(p^j)u\big)_n\Big),$$

and so by (7.19),

$$\eta_j S_j'(p^j)u \to ('0, u_n)$$

uniformly in unit vectors u. Therefore, by Proposition 7.3.3,

$$\lim_{j\to\infty} \delta_j\, \tau_{\tilde{B}, U\cap D}(p^j, u) = \lim_{j\to\infty} \frac{\delta_j}{\eta_j} \tau_{\tilde{B}, \tilde{D}_j}\big(b^*, \eta_j S_j'(p^j)u\big) = \tau_{\tilde{B}, D_\infty}\big(b^*, ('0, u_n)\big)$$

uniformly in unit vectors u. Now, all that is required is to compute the right hand side using the Cayley transform Φ from (7.11), its derivative from (7.21), and the transformation rule. Thus,

$$\tau_{\tilde{B}, D_\infty}\big(b^*, ('0, u_n)\big) = \tau_{\tilde{B}, \mathbb{B}^n}\left(0, \left('0, -\frac{u_n}{2}\right)\right) = \frac{1}{2}\sqrt{(n+1)(n+2)}|u_n|,$$

where the last equality follows from Proposition 5.1.1, and this proves (i).

(ii) For brevity, we write $u^j = u_H(p^j)$ and $u^0 = u_H(p^0)$. By invariance of the Kobayashi–Fuks metric,

$$\tau_{\tilde{B}, U\cap D}\big(p^j, u_H(p^j)\big) = \tau_{\tilde{B}, \tilde{D}_j}\big(b^*, S_j'(p^j)u^j\big).$$

Note that, since $u^j \in H_{q^j}(\partial D)$, we have from (7.19)

$$h_j'(p^j)u^j = (v_1^j, \ldots, v_{n-1}^j, 0),$$

where

$$v_l^j = \sum_{\nu=1}^{n-1} A_{l,\nu}^j \left(u_\nu^j \frac{\partial\rho}{\partial\overline{z}_n}(\zeta^j) - u_n^j \frac{\partial\rho}{\partial\overline{z}_\nu}(\zeta^j)\right), \quad l = 1, \ldots, n-1.$$

Therefore,

$$S_j'(p^j)u^j = \Lambda_j\big(h_j'(p^j)u^j\big) = \left(\frac{v_1^j}{\sqrt{\eta_j}}, \ldots, \frac{v_{n-1}^j}{\sqrt{\eta_j}}, 0\right).$$

Observe that $v_l^j \to u_l^0$ and the convergence is uniform on unit vectors u and so

$$\sqrt{\eta_j} S_j'(p^j) u^j \to \big(u_1^0, \ldots, u_{n-1}^0, 0\big)$$

and this convergence is also uniform in unit vectors u. Hence, by Proposition 7.3.3,

$$\sqrt{\delta_j} \tau_{\tilde{B}, U\cap D}\big(p^j, u_H(p^j)\big) = \sqrt{\frac{\delta_j}{\eta_j}} \tau_{\tilde{B}, \tilde{D}_j}\big(b^*, \sqrt{\eta_j} S_j'(p^j) u^j\big) \to \tau_{\tilde{B}, D_\infty}\big(b^*, ('u^0, 0)\big)$$

uniformly in unit vectors u. Again, using the Cayley transform Φ from (7.11) and its derivative from (7.21), the transformation rule gives

$$\tau_{\tilde{B}, D_\infty}\big(b^*, ('u^0, 0)\big) = \tau_{\tilde{B}, \mathbb{B}^n}\left(0, \left(-\frac{'u^0}{\sqrt{2}}, 0\right)\right) = \sqrt{\frac{1}{2}(n+1)(n+2)|'u^0|^2},$$

by Proposition 5.1.1. This proves (ii) once we observe from (7.7) that $\mathcal{L}_\rho\big(p^0, u_H(p^0)\big) = |'u^0|^2$.

(iii) By the transformation rule for the Kobayashi–Fuks metric, we have

$$g_{\tilde{B}, U\cap D}(p^j) = g_{\tilde{B}, \tilde{D}_j}(b^*) \big|\det S_j'(p^j)\big|^2. \tag{7.22}$$

Note that

$$\det S_j'(p^j) = \det \Lambda_j \det h_j'(p^j) = \eta_j^{-(n+1)/2} \det h_j'(p^j),$$

and so by (7.19),

$$\eta_j^{n+1} |\det S_j'(p^j)|^2 \to 1.$$

Therefore,

$$\delta_j^{n+1} g_{\tilde{B}, U\cap D}(p^j) = \left(\frac{\delta_j}{\eta_j}\right)^{n+1} g_{\tilde{B}, \tilde{D}_j}(b^*) \eta_j^{n+1} \big|\det S_j'(p^j)\big|^2 \to g_{\tilde{B}, D_\infty}(b^*).$$

As before, using the Cayley transform Φ from (7.11) and its derivative from (7.21), we obtain from the transformation rule,

$$g_{\tilde{B}, D_\infty}(b^*) = g_{\tilde{B}, \mathbb{B}^n}(0) \, |\det \Phi'(b^*)|^2 = \frac{(n+1)^n (n+2)^n}{2^{n+1}},$$

by Proposition 5.1.1. This completes the proof of (iii), and the theorem. □

7.5 Existence of closed geodesics with prescribed homotopy class

This section can be considered as an application of studying the boundary behavior of invariant metrics on strongly pseudoconvex domains. The aim here is to prove Theorem 1.0.8. This result is motivated by a theorem of Herbort [28, Theorem 1.2] on the existence of closed geodesics for the Bergman metric on strongly pseudoconvex domains, given by:

Theorem 7.5.1 (Herbort, [28, Theorem1.1])**.** *Let $G \subset \mathbb{R}^N$ be a bounded domain such that $\pi_1(G)$ is nontrivial and the following conditions are satisfied:*

(i) *For each $p \in \overline{G}$ there is an open neighborhood $U \subset \mathbb{R}^n$, such that the set $G \cap U$ is simply connected.*

(ii) *The domain G is equipped with a complete Riemannian metric g which possesses the following property:*

(P) *For each $S > 0$ there is a $\delta > 0$ such that for every point $p \in G$ with $d(p, \partial G) < \delta$ and every $X \in \mathbb{R}^n$, $g(p, X) \geq S\|X\|^2$.*

Then every nontrivial homotopy class in $\pi_1(G)$ contains a closed geodesic for g.

Proof of Theorem 1.0.8. We will show that both the conditions in Theorem 7.5.1 hold for $G = D$ and $g = ds^2_{\tilde{B},D}$. By the smoothness of ∂D, it is evident that condition (i) is satisfied. For condition (ii), note that we have the following relation as given in (5.8),

$$\tau_{\tilde{B},D}(z, u) = \tau_{B,D}(z, u)\sqrt{n + 1 - \mathrm{Ric}_{B,D}(z, u)}$$

for every $z \in D$ and $u \in \mathbb{C}^n$. Again using the fact that $\mathrm{Ric}_{B,D}(z, u)$ approaches -1 near the boundary of a strongly pseudoconvex domain (see for example [36]), there exists $C = C(D) > 0$ such that

$$\tau_{\tilde{B},D}(z, u) \geq C\tau_{B,D}(z, u) \tag{7.23}$$

for z near the boundary of D and unit vectors u. As both the Bergman and Kobayashi–Fuks metrics are Kähler, this relation also holds for z on any compact subset of D and unit vectors u. Thus (7.23) holds for all $z \in D$ and $u \in \mathbb{C}^n$. This has the following two consequences. Firstly, since the Bergman metric dominates the Carathéodory metric on bounded domains (see Hahn [27]) and the Carathéodory metric is complete on strongly pseudoconvex domains, (7.23) implies that the Kobayashi–Fuks metric on D is complete. Secondly, as the Bergman metric on D satisfies property (P) which was observed in the proof of Theorem 1.2 in [28], (7.23) also implies that the Kobayashi–Fuks metric on D satisfies property (P) as well, and hence condition (ii) holds. This completes the proof of the theorem. ☐

Chapter 8

Future research plans on the Kobayashi–Fuks metric

Problem 8.0.1. One can try to derive the localization results for the holomorphic sectional curvature as well as the Ricci curvature of the Kobayashi–Fuks metric near local holomorphic peak points of bounded pseudoconvex domains in higher dimension. As a result of which, we can find out the boundary asymptotics for the associated holomorphic sectional curvature and the Ricci curvature on bounded strongly pseudoconvex domains in $\mathbb{C}^n$. Note that we have already established the required localization in dimension one, in which case both the stated curvatures coincide with the Gaussian curvature.

The approach in this direction would be to first express the holomorphic sectional curvature (similarly, the Ricci curvature) in terms of some cleverly chosen maximal domain functions in higher dimension. Then we try to localize all the domain functions involved using Hörmander's solution of certain weighted $\bar{\partial}$-problem. In this step, one might need to derive some monotonicity results for the quantities related to the Kobayashi–Fuks metric.

Here one of the issues will be the complexity of computations. Since the expression of the Kobayashi–Fuks metric involves computing certain determinant and its partial derivatives, on the top of that finding associated curvatures require computing even higher derivatives. Hence the computations become huge! Certain tricks like Jacobi's formula might be incorporated somehow to reduce the calculations in finding the partial derivatives of the determinant.

Problem 8.0.2. The boundary behavior of the Kobayashi–Fuks metric and its related invariants can be found out on more general class of pseudoconvex domains, for example – Levi corank one domains and h-extendible domains.

The first step in this direction would be to implement the scaling technique on respective domains for converting the boundary problem into interior problem. But in this step, establishing the inner-stability results under the scaling might be an issue, as we do not have a clear generalization of Ramanadov type stability result for general pseudoconvex domains. So one might look for some other machinery to settle issues related to the stability results.

Problem 8.0.3 (Existence of geodesic spirals for the Kobayashi–Fuks metric)**.** Herbort [28] showed that – If D is a strongly pseudoconvex domain in $\mathbb{C}^n$ such that the universal covering $\tilde{D}$ of D is infinitely sheeted, then for each point $z_0 \in D$ which does not lie on a closed geodesic there exists a geodesic spiral for the Bergman metric passing through z_0.

Since the Kobayashi–Fuks metric is closely related to the Bergman metric, it would be interesting to find such a result for the Kobayashi–Fuks metric.

Problem 8.0.4 (L^2-cohomology of the Kobayashi–Fuks metric)**.** Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^n$. We denote the space of square integrable harmonic (p, q) forms associated to the Bergman metric by $\mathcal{H}_2^{p,q}(D)$. Donnelly and Fefferman [16] proved the following (see also [17]):

Theorem 8.0.5. *If D is strongly pseudoconvex, then*

$$dim\,\mathcal{H}_2^{p,q}(D) = \begin{cases} 0, & \text{if } p+q \neq n, \\ \infty, & \text{if } p+q = n. \end{cases}$$

One can definitely ask whether similar statement holds true if we replace the space of square integrable harmonic (p, q) forms associated to the Bergman metric by that of the Kobayashi–Fuks metric. Using similar tools as in the proof of Theorem 8.0.5, we are hopeful to get an affirmative answer to the above question.

Problem 8.0.6. Consider a line bundle L over a complex manifold M. Let D be a relatively compact domain in M and consider sections of L over D. The Bergman metric can be defined for this collection of sections (assuming there are plenty of them). So can the Kobayashi–Fuks metric. It would be interesting to have versions of all the results we have obtained on the Kobayashi–Fuks metric in this setting.

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