Study of Poincaré-Hardy type inequalities and eigenvalue problems for second-order elliptic PDEs

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Doctor of Philosophy

by

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Dedicated to

My family

Certificate

Certified that the work incorporated in the thesis entitled "*Study of Poincaré-Hardy type inequalities and eigenvalue problems for second-order elliptic PDEs*", submitted by *Prasun Roychowdhury* was carried out by the candidate, under our supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

Date: March 02, 2022

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I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

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Contents

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Abstract

The major text of this thesis is studying Poincaré-Hardy and Hardy-Rellich type inequalities on one of the most discussed Cartan-Hadamard manifold namely hyperbolic space and studying eigenvalue problems for second-order elliptic PDEs. The thesis is divided into two parts. In the first part we have centralized our attention on the following three problems:

- On some strong Poincaré inequalities on Riemannian models and their improvements.
- On higher order Poincaré inequalities with radial derivatives and Hardy improvements on the hyperbolic space.
- Hardy-Rellich and second order Poincaré identities on the hyperbolic space via Bessel pairs.

In the second part we have focused our essence on the following two problems:

- Generalized principal eigenvalues of convex nonlinear elliptic operators in $\mathbb{R}^N$.
- On ergodic control problem for viscous Hamilton-Jacobi equations for weakly coupled elliptic systems.

Part I

Study of Poincaré-Hardy type inequalities

Chapter 1

Poincaré-Hardy type inequalities

Geometrical inequalities play a significant contribution in the study of function spaces and related partial differential equations (PDEs). Starting from the notes of David Hilbert [68] in 1906, the Hardy inequality has become a very fascinating branch of mathematics, in particular in functional inequality (see [82] for the prehistory of Hardy inequality). Apart from this, Poincaré and Rellich inequalities have also been played a crucial role in analysis and related branches. For many decades these inequalities were studied in great generality in Euclidean domains but its generalization to the Riemannian manifolds was provoked after the historical work of Carron [48] and become a popular area of research. However, in the past few decades, one of the main goals of nurturing these inequalities is finding the optimal version of the inequality and their improvements with positive L^2-type remainder terms.

The main contribution in this thesis lies in developing higher order Poincaré type inequalities and its improvements on the hyperbolic space and in some cases on model manifolds, under appropriate curvature assumptions. We also deal with higher order Poincaré inequalities with radial derivatives and we study a family of Hardy-Rellich identities involving radial derivative possibly with optimal constants in many cases. In this introduction, we shall briefly explain the main questions and results obtained in my doctoral studies. Moreover, all the details are covered in Chapters 3, 4, and 5 corresponds to the material obtained in the papers [29], [112], and [30] respectively.

1.1 Known literature based on inequalities

The main goal of this thesis is to study different types of functional inequalities and their improvements on the negative sectional curvature symmetric manifolds, in particular on the hyperbolic space. Let (M, g) be a Cartan-Hadamard manifold with dimension N (namely, a manifold which is complete, simply connected, and has everywhere non-positive sectional curvature). In addition, suppose Cartan-Hadamard manifolds whose sectional curvatures are bounded above by a strictly negative constant, then M is known to admit a Poincaré inequality which reads as follows, there exists $\Lambda > 0$ such that

$$\int_M |\nabla_g u|^2 \, \mathrm{d}v_g \geq \Lambda \int_M |u|^2 \, \mathrm{d}v_g \text{ for all } u \in \mathcal{C}_c^\infty(M), \tag{1.1.1}$$

where ∇_g and $\mathrm{d}v_g$ define the Riemannian gradient and volume element in (M, g).

Let $\mathbb{H}^N$ be the N-dimensional hyperbolic space which is one of the most discussed Cartan-Hadamard manifolds. Indeed it enjoys all the properties namely it is complete, simply connected, and has constant negative curvature. Now for the space $\mathbb{H}^N$, (1.1.1) holds true and Λ turns out to be $\left(\frac{N-1}{2}\right)^2$ and moreover $\left(\frac{N-1}{2}\right)^2$ coincides with the bottom of spectrum of the Laplace-Beltrami operator on $\mathbb{H}^N$. To be precise consider $\lambda_1(\mathbb{H}^N)$ denote the bottom of the spectrum of $\Delta_{\mathbb{H}^N}$ which is explicitly given by

$$\lambda_1(\mathbb{H}^N) = \inf_{u \in \mathcal{C}_c^\infty(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N}} = \left(\frac{N-1}{2}\right)^2 . \tag{1.1.2}$$

Analogous to (1.1.1), higher-order Poincaré inequality involving higher-order derivatives also holds in $\mathbb{H}^N$. In this context, a worthy reference on this inequality is [76, Lemma 2.4] where it has been shown that for k and l be non-negative integers with $0 \leq l < k$ there holds

$$\int_{\mathbb{H}^N} |\nabla^k_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla^l_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}, \tag{1.1.3}$$

for all $u \in \mathscr{W}^{k,2}(\mathbb{H}^N)$, where

$$\nabla^k_{\mathbb{H}^N} := \begin{cases} \Delta^{k/2}_{\mathbb{H}^N} & \text{if } k \text{ is even integer,} \\ \nabla_{\mathbb{H}^N} \Delta^{(k-1)/2}_{\mathbb{H}^N} & \text{if } k \text{ is odd integer.} \end{cases}$$

Also $\Delta^k_{\mathbb{H}^N}$ denotes the k-th iterated Laplace-Beltrami operator and $\nabla_{\mathbb{H}^N}$ represents the Riemannian gradient in $\mathbb{H}^N$. By constructing a minimizing sequence one can show the above constant in (1.1.3) is sharp (see [100]). Also it is worth to mention that the following infimum is not achieved

$$\inf_{u \in \mathscr{W}^{k,2}(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\nabla^k_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |\nabla^l_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}} = \left(\frac{N-1}{2}\right)^{2(k-l)} .$$

This marks an important step in the development of a comprehensive study of inequality (1.1.3) related to its improvement. We refer to [2,26,27,93,101] for more details.

Recently, there has been a constant effort to improve the Poincaré inequality in terms of adding possibly optimal Hardy weights, i.e. adding on the r.h.s. a term of the form $\int_M W \, u^2 \, \mathrm{d}v_g$ with $W \geq 0$ "as large as possible", see [53] for a general treatment of Hardy weights for second-order elliptic operators. Starting from the works of [2] and [27], where a Poincaré-Hardy inequality was shown with sharp constants, further generalisation to p-Laplacian and higher-order case have been obtained in [25] and [26], respectively. We also refer to [28] for more general improvements and the study of extremals. The kind of weights obtained in these papers, which are singular at a fixed point of M, makes this subject a sort of lateral branch of that very rich field of research originated from the seminal papers [41, 42], which dealt with possible improvements (not only of L^2 type) of the classical Hardy inequality on bounded euclidean domains or on curved spaces, see e.g. [1, 16–18, 51, 52, 59, 62, 63, 77–79, 81, 94, 117, 119] and reference therein.

Furthermore, Carron [48] derived the classical Hardy inequality on Riemannian manifolds which open up new directions in the study of Hardy inequality on non-trivial geometry. The classical Hardy inequality (see [48] for details) on Riemannian manifolds (M, g) reads as for $N \geq 3$, there holds

$$\int_M |\nabla_g u|^2 \, \mathrm{d}v_g \geq \left(\frac{N-2}{2} \right)^2 \int_M \frac{u^2}{r^2} \, \mathrm{d}v_g \tag{1.1.4}$$

for all $u \in \mathcal{C}_c^\infty(M)$, where manifolds satisfy a geometric condition $\Delta_g r \geq \frac{(N-1)}{r}$. Here $r = \varrho(x, x_o)$ denotes the geodesic distance between a point x and a fixed pole x_o in M and Δ_g denotes the Laplace-Beltrami operator on the Riemannian manifold (M, g). In recent days, a large part of the works dealt with an improvement of the above inequality with optimal Hardy type remainder terms. Among all the recent work in these directions, we are bringing up only a few of them [39, 40, 51, 77, 78, 90, 95, 98, 119] without a claim of completeness.

Drawing primary motivation from the above discussion regarding improvement of Hardy inequalities with L^2 reminder term, now one can talk about the improvement of (1.1.1) on $\mathbb{H}^N$, by means of a Hardy-type improvement. This has been considered in [27, Theorem 2.1] related to improvement of (1.1.3) in the case $k = 1$ and $l = 0$ with Hardy-type remainder terms which says for $N \geq 3$ there holds

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)(N-3)}{4} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \,, \tag{1.1.5}$$

for $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$. Also note that both constants $\left(\frac{N-1}{2}\right)^2$ and $\frac{1}{4}$ in (1.1.5) are sharp in a suitable sense.

In view of the work [102], where a similar question has been recently posed in the context of Hardy inequalities, one may wonder whether inequality (1.1.5) still holds if we replace $|\nabla_{\mathbb{H}^N} u|^2$ with its radial part $|\nabla_{r,\mathbb{H}^N} u|^2$. Here "$\nabla_{r,\mathbb{H}^N} u$" represents the radial part of the gradient in $\mathbb{H}^N$ and more details will be given in subsequent chapter. Since, by Gauss's lemma one has $|\nabla_{\mathbb{H}^N} u|^2 \geq |\nabla_{r,\mathbb{H}^N} u|^2$, and by this we will present our first result regarding stronger version of Poincaré-Hardy inequality (1.1.5) (see Subsection 1.2.1). We want to mention that our result is true for model manifold too.

Now, in view of (1.1.3) and for what previously discussed in the first-order case, it is natural to think about possible extensions of (1.1.3) to the second-order case i.e., $k = 2$ and $l = 0$. In this respect the following inequality from [102, Theorem 5.2] turns out to be meaningful:

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} (\Delta_{r,\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \tag{1.1.6}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ with $N \geq 5$, where "$\Delta_{r,\mathbb{H}^N}$" denotes the radial part of the Laplace-Beltrami operator $\Delta_{\mathbb{H}^N}$ on $\mathbb{H}^N$, see Chapter 2 for details. Clearly, inequality (1.1.6) suggests a possible stronger version of the second-order case and which might involve the operator $\Delta_{r,\mathbb{H}^N}$. We obtain a positive response to this question and see Subsection 1.2.1 and Chapter 3 for details.

We continue this discussion by mentioning that for the improvement of second-order Poincaré inequality the following result has been obtained in [27, Theorem 3.1]. For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ with $N \geq 5$ there holds

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{9}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)^2}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \,, \tag{1.1.7}$$

where the constant $\frac{(N-1)^2}{8}$ was proved to be sharp. There was an open conjecture that the constant $\frac{9}{16}$ in front of $\frac{1}{r^4}$ would be a sharp constant. We derive another improved version of the above inequality (1.1.7) with constant $\frac{(N-4)^2}{16}$ instead of $\frac{9}{16}$ in front of $\frac{1}{r^4}$. Eventually, this turns out that the conjecture becomes false for $N \geq 8$. Also, we mention that, unfortunately, neither (1.1.7) nor our version of the inequality solves the problem of sharpness of the constant in front of $\frac{1}{r^4}$ which is still open.

All the above discussions mainly focus on sharper version and improvement of (1.1.3), at least for the cases $k = 1, l = 0$ and $k = 2, l = 0$ in terms of the radial derivatives. So naturally one can ask whether an inequality of type (1.1.3) involving, only higher-order radial derivatives holds true. Indeed the answer is affirmative and we obtain a similar result like (1.1.3), only involving radial derivatives keeping the sharpness of the constant intact (see Subsection 1.2.2 and Chapter 4 for details).

It is interesting to discuss the improvement of the higher order Poincaré inequality (1.1.3). Recently it has been developed in [26, Theorem 2.1] which reads as for integer k, l with $0 \leq l < k$ and $N > 2k$, then there exist k positive constants $\alpha_{k,l}^j = \alpha_{k,l}^j(N)$ such that following inequality holds

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N}^l u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \alpha_{k,l}^1 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{j=2}^{k-1} \alpha_{k,l}^j \int_{\mathbb{H}^N} \frac{u^2}{r^{2j}} \, \mathrm{d}v_{\mathbb{H}^N} + \alpha_{k,l}^k \int_{\mathbb{H}^N} \frac{u^2}{r^{2k}} \, \mathrm{d}v_{\mathbb{H}^N}. \tag{1.1.8}$$

Also, note that $\alpha_{k,l}^1$ and $\alpha_{k,l}^k$ signify the coefficient for the leading term as $r \to 0$ and $r \to \infty$. In the same spirit, we obtain the improvement of higher-order radial Poincaré inequality in terms of Hardy-type remainder terms.

Surely it is not at all a straightforward generalization of the above. We need to devise all together a new strategy to obtain our desired results.

The next part of our work takes its origin from the following family of Hardy-Poincaré inequalities recently proved in [28]: for all $N-2 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$ and all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\begin{aligned} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \; &+ \; h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \left[\left(\frac{N-2}{2} \right)^2 - h_N^2(\lambda) \right] \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \gamma_N(\lambda) \, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} \, u^2 \, \mathrm{d}v_{\mathbb{H}^N} \end{aligned} \tag{1.1.9}$$

where $\gamma_N(\lambda) := \sqrt{(N-1)^2 - 4\lambda}$, and $h_N(\lambda) := \frac{\gamma_N(\lambda)+1}{2}$. The interest of (1.1.9) relies on the fact that it provides in a single inequality, proved by means of a unified approach, an optimal improvement (in the sense of adding non-negative terms in the right side of the inequality) of the Poincaré inequality (1.1.2) and an optimal improvement of the Hardy inequality. Indeed, for $\lambda = \lambda_1(\mathbb{H}^N)$ ($\gamma_N = 0$) inequality (1.1.9) becomes the improved Poincaré inequality (1.1.5) and for $\lambda = N-2$ ($\gamma_N = N-3$) (1.1.9) becomes the improved Hardy inequality:

$$\begin{aligned} \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + (N-2) \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ + \frac{(N-2)(N-3)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} \, u^2 \, \mathrm{d}v_{\mathbb{H}^N} \, , \end{aligned} \tag{1.1.10}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ with $N \geq 3$. This is the right place to address the connection between inequalities and the quadratic form of an operator. By this, we will obtain further directions to study the above inequality (1.1.9). Let P be a second-order non-negative elliptic operator with real coefficients which is defined on M and let q be the associated quadratic form defined on $\mathcal{C}_c^\infty(M)$. A Hardy-type inequality with a weight $V \geq 0$ has the form

$P - V \geq 0$, namely for all $u \in \mathcal{C}_c^\infty(M)$, there holds

$$< Pu, u >:= q(u) \geq \int_M V u^2 \, \mathrm{d}v_g. \tag{1.1.11}$$

A non-negative operator $P - V$ is called critical in M, i.e., V is an optimal Hardy weight for P, if inequality $P - V \geq 0$ cannot be improved, i.e., the above inequality is not true when V is replaced by $W \geq V$ and $W \neq V$.

We are interested to study optimal Hardy weight for the operator $P := -\Delta_{\mathbb{H}^N} - \frac{(N-1)^2}{4}$. Due to [27, Theorem 2.1] and (1.1.5), we found that the operator $H := P - \frac{1}{4r^2} - \frac{(N-1)(N-3)}{4} \frac{1}{\sinh^2 r}$ is critical in $\mathbb{H}^N \setminus \{x_o\}$. Now one can ask whether we have a family of optimal Poincaré-Hardy for the operator P_λ with $0 \leq \lambda \leq \frac{(N-1)^2}{4}$ where $P_\lambda := -\Delta_{\mathbb{H}^N} - \lambda$. Now denoted with V_λ the positive potential at the r.h.s. of (1.1.9), it has been established in [28] that the operator $-\Delta_{\mathbb{H}^N} - V_\lambda(r)$ is critical. In other words we found the answer is affirmative for $N - 2 \leq \lambda \leq \frac{(N-1)^2}{4}$. From (1.1.5) and (1.1.10), it is clear that Poincaré and Hardy inequalities are well studied for the case $\lambda = \lambda_1(\mathbb{H}^N)$. This has been done using some Criticality theory arguments and Liouville type estimates (see [7, 28, 107]). Inspired by these, we can ask whether the above is true for the higher-order cases. Precisely we are interested to study the family of optimal inequality for the higher-order Rellich type operator $P_\lambda := \Delta^2_{\mathbb{H}^N} + \lambda \Delta_{\mathbb{H}^N}$ with $0 \leq \lambda \leq \frac{(N-1)^2}{4}$. Unfortunately here we can not apply Criticality theory or Liouville type estimate for the higher-order cases and so we need to build a completely new setup for studying these types of operators.

We notice that (1.1.9) was proved by means of a unified approach based on criticality theory, well established for second-order operators only (see [53]), together with the exploitation of a family of explicit radial solutions to the associated equations. Therefore a similar approach seems not applicable in the higher-order case. Here, drawing primary motivation from the seminal paper [63], we extend (1.1.9) to the second-order by using the notion of Bessel pair. This notion has been very recently developed in [60] on Cartan-Hadamard manifolds to establish several interesting Hardy identities and inequalities which, in particular, generalize many well-known Hardy inequalities on Cartan-Hadamard manifolds. By combining some ideas from [60,63], and through delicate computations with spherical harmonics, in the present book, we develop the method of Bessel pairs to derive general abstract Rellich inequalities and identities on $\mathbb{H}^N$. We get either Poincaré and Hardy-Rellich identities or improved inequalities, by means of a unified way where the key ingredient is the clever construction of a family of Bessel pairs (see Chapter 5 for details). Finally, as a few applications of the obtained inequalities, we derive quantitative versions of the second-order Heisenberg-Pauli-Weyl uncertainty principle. As far as our known literature, the results provided represent the first examples of the second-order Heisenberg-Pauli-Weyl uncertainty principle in the Hyperbolic context.

1.2 Brief description of the obtained results

Here we will discuss the brief description of the problems and obtained results of the first part of the thesis.

1.2.1 Problem 1

In Chapter 3, we have studied a stronger version of sharp Poincarè-Hardy inequality on the hyperbolic space. The optimality issue of the constant has been delicately tackled by constructing minimizing sequence. Additionally, we find a new improvement of the second-order Rellich-Poincarè inequality on the hyperbolic space. Also, we presented such inequality holds true for some model manifolds with suitable curvature assumptions. Indeed, we prove that the following *stronger* version of (1.1.5) holds

$$\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)(N-3)}{4} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} \,, \tag{1.2.1}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ and $N \geq 3$. Clearly, (1.2.1) reproves inequality (1.1.5) but we derive it by using a different technique: here the main tool exploited is the spherical harmonics decomposition technique, while the proof of (1.1.5) was based on finding a ground state and on criticality theory, see [27, Theorem 2.1]. Furthermore, we argue quite differently also in proving the optimality of the constants appearing in (1.2.1). Notice that the sharpness of all constants in (1.2.1) can be derived by combining Gauss's lemma with the sharpness of the corresponding constants in (1.1.5). Nevertheless, in this thesis, we provide an alternative and more direct proof of the sharpness of the dominating term at infinity of (1.2.1), namely of the constant $\frac{1}{4}$, which is based on the delicate construction of a suitable minimizing sequence. This argument may have its own interest in the study of related partial differential equations, furthermore, it can be carried over to more general Riemannian models having negative sectional curvatures bounded above (see Chapter 3 for details).

Motivated by this, we prove the following improved Poincarè type inequality:

$$\int_{\mathbb{H}^N} (\Delta_{r,\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N^2-1)}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N}, \tag{1.2.2}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ with $N \geq 3$. We notice that (1.2.2) looks like the proper second-order analogue of (1.2.1). On the other hand, clever exploitation of (1.2.2), jointly with the spherical harmonics decomposition technique, yields the following improved version of (1.1.3) with $k = 2$ and $l = 1$:

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N^2-1)}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N}. \tag{1.2.3}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ with $N \geq 5$. It is worth mentioning that our obtained results above are true on general model manifolds satisfying suitable curvature bounds and having $\mathbb{H}^N$ as a remarkable particular case.

1.2.2 Problem 2

Chapter 4 deals with establishing the higher-order Poincaré inequality only in terms of radial derivatives involving the Riemannian gradient and Laplace-Beltrami operator on the hyperbolic space $\mathbb{H}^N$. Indeed we obtain for non-negative integers k and l with $0 \leq l < k$, there holds

$$\int_{\mathbb{H}^N} |\nabla^k_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla^l_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \tag{1.2.4}$$

for all $u \in \mathscr{W}^{k,2}(\mathbb{H}^N)$. Moreover, the constant $\left(\frac{N-1}{2}\right)^{2(k-l)}$ is sharp which is intercepted by the delicate use of integral representation of the volume of a ball in hyperbolic space $\mathbb{H}^N$ and by using some clever estimates derived in [100].

At the end of this chapter, we discuss the Hardy-type improvements of (1.2.4) by finding a detailed description of a coefficient related to asymptotic Hardy-type remainder terms. Broadly speaking here we derive for non-negative integer k and l with $0 \leq l < k$ for $N > 2k$ there exist k positive constants $C^j_{k,l}$ such that

$$\int_{\mathbb{H}^N} |\nabla^k_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla^l_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{j=1}^{k} C^j_{k,l} \int_{\mathbb{H}^N} \frac{u^2}{r^{2j}} \, \mathrm{d}v_{\mathbb{H}^N}. \tag{1.2.5}$$

In a similar fashion, like in [26], here also we calculate the explicit expression of $C^1_{k,l}$ and $C^k_{k,l}$ related to dominating term for $r \to 0$ and $r \to \infty$ respectively. It is worthy to note that we achieve another version of (1.2.5) with dimension restriction $N \geq 4k - 1$ but with a better constants in front of the leading order Hardy term.

1.2.3 Problem 3

In Chapter 5, we are concerned to prove a family of Hardy-Rellich and Poincaré identities and inequalities on the hyperbolic space having, as particular cases, improved Hardy-Rellich, Rellich, and second-order Poincaré inequalities. All remainder terms provided considerably improve those already known in the literature, and all identities hold with same constants for radial operators also. The main result in Chapter 5 we obtain the following abstract Rellich identity: Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0,\infty)$ with positive solution f on $(0,\infty)$. Then for all $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\begin{aligned}\int_{\mathbb{H}^N} V(r)|\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \int_{\mathbb{H}^N} W(r)|\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ (N-1)\int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r}\right)|\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &- (N-1)\int_{\mathbb{H}^N} V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)|\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \int_{\mathbb{H}^N} V(r)(f(r))^2\left|\nabla_{r,\mathbb{H}^N}\left(\frac{u_r}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}\end{aligned}$$

Moreover, under some added assumption we prove the above results hold true for non-radial function also and that time we obtain the inequality instead of identity. After building this identity, now by choosing the Bessel pair delicately we obtain the second-order analogue of the result (1.1.9). Finally, as by-product of our previous result we deduce the following version of Heisenberg-Pauli-Weyl principle in $\mathbb{H}^N$: Let $N \geq 5$. For all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$

and all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\left(\int_{\mathbb{H}^N} \left(|\Delta_{\mathbb{H}^N} u|^2 - \lambda |\nabla_{\mathbb{H}^N} u|^2\right) \mathrm{d}v_{\mathbb{H}^N}\right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \geq h_N^2(\lambda) \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)^2 .$$

——————— ◦ ———————

Chapter 2

Geometrical preliminaries

This chapter is devoted to the study of basic aspects of Riemannian geometry. We will revisit some basic tools regarding curvature on Riemannian manifolds. In the later part, we will see some special types of the Riemannian manifold namely the Riemannian model manifold. It is worth mentioning that hyperbolic space admits model manifold structure and we will discuss some necessary geometric tools on this space. For a more detailed study, we refer to [89, 105, 110]. Next, we will discuss some basic facts on spherical harmonics. This is a certain type of decomposition and one of the key ingredients which will be useful for most of the proofs. More details regarding spherical harmonics can be found in [114, Chapter 4] and [96].

2.1 Riemannian manifolds

An N-dimensional differential topological manifold M is a space that locally looks like some N-dimensional euclidean space $\mathbb{R}^N$.

Definition 2.1.1. *Let M be a Hausdorff, second countable topological space. Suppose that there exist a collection of open subsets $\{\mathscr{U}_\alpha : \alpha \in I\}$ of M, where I is the index set, satisfy the following properties:*

(i) $\cup_\alpha \mathscr{U}_\alpha = M$.

(ii) There exists homeomorphisms $\phi_\alpha : \mathscr{U}_\alpha \to \phi_\alpha(\mathscr{U}_\alpha)$*(open subset of* $\mathbb{R}^N$*).*

(iii) There holds the following compatibility conditions: $\phi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta)$ *is open for all* α, β *and there exists* $k \in \mathbb{N} \cup \{\infty\}$ *such that the following map (transition map),*

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\alpha(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \to \phi_\beta(\mathscr{U}_\alpha \cap \mathscr{U}_\beta) \text{ is } \mathcal{C}^k \text{ for all } \alpha, \beta \in I$$

whenever $\mathscr{U}_\alpha \cap \mathscr{U}_\beta$ *is non-empty.*

Then $(M, \{(\mathscr{U}_\alpha, \phi_\alpha) : \alpha \in I\})$ *is called a* $\mathcal{C}^k$*-manifold of dimension* N*. The collection* $\{(\mathscr{U}_\alpha, \phi_\alpha) : \alpha \in I\}$ *is called a* $\mathcal{C}^k$*-atlas on* M *and the members of the atlas are called charts for* M*. The coordinates of a point* $p \in M$*, related to* ϕ*, are the coordinates of the point* $\phi(p) \in \mathbb{R}^N$*.*

We call M is a smooth manifold if it is $\mathcal{C}^\infty$-manifold. Now onwards we will only work with smooth manifolds. For example, let $\mathscr{U}$ be any open subset of $\mathbb{R}^N$. Then $(\mathscr{U}, \{(\mathscr{U}, id)\})$ is a smooth N-dimensional manifold.

Let M be a smooth manifold and p be a point on M. Let $\mathcal{C}^\infty(p)$ be the set of all real valued smooth functions defined in neighbourhood of p. Hence, if $f \in \mathcal{C}^\infty(p)$, then there exists $\mathscr{U}_f$ (neighbourhood of p depends on f) such that f is defined on $\mathscr{U}_f$ and smooth.

Definition 2.1.2. *A tangent vector* v *at* p *is a real valued function defined on* $\mathcal{C}^\infty(p)$ *and for* $f, g \in \mathcal{C}^\infty(p)$ *it satisfy the following properties:*

(i) $v(f) = v(g)$ *if* $f = g$ *on* $\mathscr{U}_f \cap \mathscr{U}_g$.

(ii) $\mathbb{R}$*-linear property:* $v(\lambda f + \mu g) = \lambda v(f) + \mu v(g)$ *for* $\lambda, \mu \in \mathbb{R}$.

(iii) Leibniz rule: $v(fg) = f(p)v(g) + v(f)g(p)$.

We denote the $\mathbb{R}$-vector space T_pM be the tangent space at the point p and this is the collections of all tangent vectors at p. If $\{x^i\}_{i=1}^N$ is a set of local coordinates of p, then the basis of the vector space T_pM is $\{\partial_i\}_{i=1}^N$ where $\partial_i = \frac{\partial}{\partial x^i}|_p$. Also cotangent space is the dual of the tangent space and denoted by T_p^*M. In the above local coordinate, the basis of cotangent space T_p^*M will be $\{dx^i\}_{i=1}^N$ where $dx^i(\partial_j) = \delta_{ij}$, the celebrated Kronecker delta function.

Also note that tangent bundle $TM = \sqcup_{p\in M} T_pM$ and cotangent bundle $T^*M = \sqcup_{p\in M} T_p^*M$ are the vector bundle over M having the tangent space T_pM and cotangent space T_p^*M as fibre at the point p on M. A vector field X on M is an assignment $p \mapsto X_p \in T_pM$. We may consider it as on each point p it gives a direction X_p. For a local chart $(\mathscr{U}, x^i)$ around p, if the map $p \mapsto X_p(x^i)$ for each $1 \leq i \leq N$, is smooth then we say that the vector field X is smooth. In this chapter, we always consider a smooth vector field. Sometimes a section of the tangent bundle is called a vector field over M, while a section of the cotangent bundle is called a differential form. Let $\mathscr{X}(M)$ be the collection of all smooth vector fields on M and one can check that $\mathscr{X}(M)$ forms a vector space structure over $\mathbb{R}$ and it is a module over the ring $\mathcal{C}^\infty(M)$.

For a general finite dimensional vector space V, we define a tensor of type (α, β) by a multilinear map $T : \prod_{i=1}^{\alpha}(V^*) \times \prod_{j=1}^{\beta}(V) \to \mathbb{R}$, where V^* is the dual space of V.

Definition 2.1.3. *An N-dimensional Riemannian manifold is a pair (M, g) where M is a smooth N-dimensional manifold and $g(p) = g_p$ gives an inner product on T_pM for each $p \in M$. Also assume g_p varies smoothly. This is interpreted as for any two fixed $X, Y \in \mathscr{X}(M)$, the map $p \mapsto g_p(X_p, Y_p)$ is a smooth function.*

Suppose M is a smooth manifold, then one can always give a Riemann structure on it by the help of partition of unity. Let (M, g) be an N-dimensional Riemannian manifold. Then g_p is a $(0, 2)$ symmetric tensor on T_pM. For a local coordinate $\{x_i\}_{i=1}^N$ around p we can write the metric g_p in terms of the basis $\{dx^i \otimes dx^j\}$. The expression of the metric tensor is given by

$$g = g_{ij} dx^i \otimes dx^j,$$

where $g_{ij} = g_p(\partial_i, \partial_j)$ and we used all the evaluation at the point p. Also note that in the above standard Einstein summation convention is used and same will be used in the later part of the chapter.

Example 2.1.1. *The Euclidean space $\mathbb{R}^N$ admits Riemannian manifold structure by letting $g_p(u, v) = < u, v >$ be the canonical Euclidean inner product at each tangent space $T_p\mathbb{R}^N \cong \mathbb{R}^N$. Finally, the metric tensor will look like $g = \delta_{ij} dx^i \otimes dx^j$.*

Example 2.1.2. *Suppose (M, g_M) be a Riemannian manifold and $N \subset M$ be a smooth submanifold. Then (N, g_N) admits Riemannian manifold structure by denoting $g_N(u, v) = g_M(u, v)$.*

Example 2.1.3. *Let (M, g_M) be a Riemannian manifold and $\psi : N \to M$ be an immersion. Then we can give a Riemannian manifold structure on N by pulled back map $D\psi$ via ψ. Precisely, we define the metric by $g_N(u, v)_p = g_M(D\psi_p(u), D\psi_p(v))_{\psi(p)}$. Let $\mathbb{S}^{N-1} = \{x \in \mathbb{R}^N : ||x||^2 = 1\}$ be the unit sphere on $\mathbb{R}^N$. Then via stereographic projection we can see $(\mathbb{S}^{N-1}, g_{\mathbb{S}^{N-1}})$ becomes a Riemannian manifold.*

Example 2.1.4. *In $\mathbb{R}^2 \setminus \{(0, x) : x > 0\}$ we can give polar coordinate structure by $x = r\cos\theta$ and $y = r\sin\theta$. In this case the polar coordinate metric will look like $g = dr \otimes dr + r^2 d\theta \otimes d\theta$.*

2.2 Musical Isomorphisms

Suppose (M, g) be a Riemannian manifold then by the help of the metric g we can define an isomorphism between tangent bundle TM and cotangent bundle T^*M. The map from TM to T^*M is called the flat operator. Let X be a vector on TM then flat operator sends it to $X^\flat \in T^*M$, where $X^\flat(Y) := g(X, Y)$. In terms of local coordinate representation we have $X^\flat(\cdot) = g(X^i\partial_i, \cdot) = g_{ij}X^i dx^j$, where $X = X^i\partial_i$. We can also write as $X^\flat = X_j dx^j$ where $X_j = g_{ij}X^i$. This is why sometimes, we say that flat is obtained by lowering a index.

It is easy to notice that the matrix of the flat operator is itself the matrix of the Riemannian metric g and hence the flat operator is invertible. We call the inverse a sharp operator from T^*M to TM. Let the inverse of the matrix (g_{ij}) be (g^{ij}) and therefore this satisfy the relation $g^{ij}g_{jk} = \delta_{ik}$. Now let w be an element of T^*M then sharp operator send it to $w^\sharp \in TM$ and it satisfies $(w^\sharp)^\flat = w$. In terms of local coordinate representation if we have $w = w_j dx_j$, then one can check that $w^\sharp = w^i\partial_i$ where $w^i = g^{ij}w_j$. This is why one says the sharp operator is obtained by raising an index.

The above flat($\flat$) and sharp($\sharp$) operators are inverse of each other and the related isomorphism is called musical isomorphism between TM and T^*M. The most useful application of the sharp operator is obtaining the classical Riemannian gradient operator. Let f be a real valued smooth function on M. Then the differential operator of f is defined by df and in terms of basis it can be written as $df = \partial_i f dx^i$. Then the gradient of f is denoted as ∇f and it is defined by $(df)^\sharp$. Hence ∇f is characterized by

$$df(Y) = g(\nabla f, Y) \text{ for all } Y \in TM,$$

and in terms of local coordinate representation it is written by

$$\nabla f = g^{ij}\partial_i f \partial_j.$$

2.3 Levi-Civita Connection

Lie bracket is an very interesting algebraic operation on $\mathscr{X}(M)$. Let $X, Y \in \mathscr{X}(M)$, then Lie bracket $[X, Y] \in \mathscr{X}(M)$ and it is defined as follows:

$$[X, Y](p)(f) := X_p(Yf) - Y_p(Xf) \text{ for all } p \in M \text{ and } f \in \mathcal{C}^\infty(M).$$

In the above we need to make sure that Xf of Yf is smooth function on M. This can be easily checked by noting that $(Xf)(q) = X_q(f)$ and $(Yf)(q) = Y_q(f)$ for all $q \in M$. Let fix $X \in \mathscr{X}(M)$ and by the help of the above Lie bracket we can define a linear map $\mathscr{L}_X : \mathscr{X}(M) \to \mathscr{X}(M)$ by $\mathscr{L}_X(Y) := [X, Y]$ for all $Y \in \mathscr{X}(M)$. This quantity $\mathscr{L}_X(Y)$ is called Lie derivative of X in the direction of Y.

Now we will state the fundamental theorem of Riemannian geometry and for details refer [105, Chapter 2].

Theorem 2.3.1. *Let (M, g) be a Riemannian manifold. Then there exist a unique affine connection $D : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ and for $X, Y, Z \in \mathscr{X}(M)$ and $f \in \mathcal{C}^\infty(M)$ it satisfies the following conditions:*

(1) $D_X(Y + Z) = D_X Y + D_X Z$

(2) $D_{X+Y} Z = D_X Z + D_Y Z$

(3) $D_{fX} Y = f D_X Y$

(4) $D_X fY = f D_X Y + X(f)Y$

(5) $D_X Y - D_Y X = [X, Y]$

(6) $X(g(Y, Z)) = g(D_X Y, Z) + g(Y, D_X Z)$

The above connection is called Levi-Civita connection and it is uniquely defined. It is worth to mention that the uniqueness of of connection is established by the following relation:

$$g(D_Z X, Y) = \frac{1}{2}\Big(Z(g(X,Y)) - Y(g(Z,X)) + X(g(Y,Z))\Big) \tag{2.3.1}$$

We can think $D_X Y$ as covariant derivative of Y in the direction of X. Now if we have $X = \sum_i f_i \partial_i$ and $Y = \sum_j h_j \partial_i$, then using the conditions of connections and $D_{\partial_i}\partial_j$ one can easily calculate $D_X Y$. In the literature we write $D_{\partial_i}\partial_j = \sum_{k=1}^{N} \Gamma_{ij}^k \partial_k$, where Γ_{ij}^k are uniquely defined smooth functions and named the Christoffel symbols. One can easily compute Γ_{ij}^k from the relation (2.3.1) in respect of the metric g. We also want to mention that in terms of Christoffel symbols we have

$$D_X Y = \sum_k \Big\{ X(h_k) + \sum_{i,j} \Gamma_{ij}^k f_i h_j \Big\} \partial_k.$$

2.4 Curvature

After having the connection on the Riemannian manifold, it is natural to study the more geometric term and curvature tensor is the immediate thing to study.

Definition 2.4.1. *Let (M, g) be a Riemannian manifold and D be the Levi-Civita connection on M. Then the following relation*

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]} Z,$$

for $X, Y, Z \in \mathscr{X}(M)$, defines a function $R : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ and called the Riemannian curvature of M.

For the properties of the curvature R refer [105, Proposition 4, Chapter 2]. The Riemannian curvature R is $(1,3)$ tensor and it can be seen as $(0,4)$ tensor as follows:

$$R(X,Y,Z,W) := g(R(X,Y)Z, W) \text{ for all } X, Y, Z, W \in \mathscr{X}(M).$$

Example 2.4.1. *For Euclidean space $\mathbb{R}^N$ with standard metric, one can check that curvature tensor $R \equiv 0$ and this can be verified using $D_{\partial_i}\partial_j = 0$ Metrics for which this type of phenomenon holds is called flat.*

From curvature tensor, we can define several type of curvature. Before going further first we want to define the curvature operator on the space $\Lambda_p^2 M$, which is the space of bi-vectors defined on the vector space T_pM, for some $p \in M$. First note that inner product on $\Lambda_p^2 M$ is defined as follows:

$$g(x \wedge y, v \wedge w) = g(x,v)g(y,w) - g(x,w)g(y,v) \text{ for all } x \wedge y, v \wedge w \in \Lambda_p^2 M.$$

Next exploiting the curvature tensor one can define a symmetric bilinear map $R : \Lambda^2 M \times \Lambda^2 M \to \mathbb{R}$ as follows:

$$R(X \wedge Y, V \wedge W) := R(X, Y, W, V) \text{ for all } X \wedge Y, V \wedge W \in \Lambda^2 M.$$

We are now ready to define the curvature operator. It is a self-adjoint operator $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$, satisfies the following relation

$$g(\mathcal{R}(X \wedge Y), V \wedge W) = R(X \wedge Y, V \wedge W).$$

Let us assume $p \in M$ and $v, w \in T_pM$. Then consider the parallelogram $\Pi = \{tv + sw \ : \ 0 \le t, s \le 1\}$. Now the following quantity

$$\sec(v, w) = \frac{g(\mathcal{R}(v \wedge w), v \wedge w)}{(\text{area}(\Pi))^2}$$

is defined as sectional curvature of (v, w). A Riemannian manifold (M, g) is said to have constant curvature if its sectional curvature is constant for each plane on each points. By rescaling the metric one can show that the only possible constant value of sectional curvature is -1, or 0, or 1.

Example 2.4.2. *The unit sphere $(\mathbb{S}^{N-1}, g_{\mathbb{S}^{N-1}})$ has constant sectional curvature 1.*

Example 2.4.3. *The Euclidean space $\mathbb{R}^N$ with standard metric $g_{\mathbb{R}^N}$ has constant sectional curvature* 0*.*

Example 2.4.4. *Consider the Lorentz inner product $< \cdot, \cdot >_1$ on $\mathbb{R}^{N+1}$ as follows: for $x = (x_1, \cdots, x_{N+1})$ and $y = (y_1, \cdots, y_{N+1})$ we define*

$$< x, y >_1 = -x_1 y_1 + \sum_{i=2}^{N+1} x_i y_i.$$

Now consider the space

$$H^N = \{x = (x_1, \cdots, x_{N+1}) \in \mathbb{R}^{N+1} \ : \ < x, x >_1 = -1, \ x_1 > 0\} \subset \mathbb{R}^{N+1}.$$

On H^N, one can check that $< \cdot, \cdot >_1$ works as Riemannian metric and it is simply denoted as N-dimensional hyperbolic space $\mathbb{H}^N$. This space has constant sectional curvature -1*.*

Now for a point $p \in M$ consider the vector $v, w \in T_pM$. Then define the map $R_v^w : T_pM \to T_p^M$ by $R_v^w(x) := R(x, v)w$, where R is the Riemannian curvature tensor of (M, g). Next we define the Ricci curvature as follows:

$$\mathrm{Ric}(v, w) := \text{trace of } R_v^w = \sum_{i=1}^{N} g(R_v^w(e_i), e_i),$$

where $\{e_1, \cdots, e_N\}$ forms an orthonormal basis of T_pM. Finally we want to define another map $\mathrm{Ric} : T_pM \to T_pM$ by $\mathrm{Ric}(v) = \sum_{i=1}^{N} R(v, e_i)e_i$. Then we define the scalar curvature at each point $p \in M$ by

$$\mathrm{Scal}(p) := \text{trace of Ric} = \sum_{i=1}^{N} g(\mathrm{Ric}(e_i), e_i) = 2 * (\text{trace of } \mathcal{R}) = 2 \sum_{i<j} \sec(e_i, e_j).$$

2.5 Distance function and Geodesics

In this section we will see special type of function called distance function. Let $\mathcal{U} \subset (M, g)$ be an open set. Then function $r : \mathcal{U} \to \mathbb{R}$ is called distance function if $g(\nabla r, \nabla r) \equiv 1$.

We say a parametrized curve $\gamma : [a, b] \to M$ is geodesic if $D_{\gamma'(t)}\gamma'(t) = 0$, where D is the Levi-Civita connection on (M, g). Now for any curve γ, its length $L(\gamma)$ is given by

$$L(\gamma) := \int_a^b [g(\gamma'(t), \gamma'(t))]^{1/2} \, \mathrm{d}t.$$

Sometimes we need to consider a geodesic with the maximum possible domain. Let us first define maximal geodesic:

Definition 2.5.1. *Let $p \in M$ and $v \in T_pM$. Then $\gamma_{p,v}$, defined on an interval I (containing 0) is called a maximal geodesic if the following properties holds:*

(i) $\gamma(0) = p$ and $\gamma'(0) = v$.

(ii) If $\sigma : J \to M$ is a geodesic with $\sigma(0) = p$ and $\sigma'(0) = v$ then $J \subset I$.

By the help of ODE theory and domain of solution one can show that maximal geodesic exists uniquely on a manifold M. Then we can define the exponential map $\exp_p : T_pM \to M$ by $\exp_p(v) := \gamma_{p,v}(1)$ whenever 1 belongs to the domain of $\gamma_{p,v}$.

Finally we say a Riemannian manifold (M, g) is geodesically complete if the domain of every maximal geodesic is entire real line $\mathbb{R}$. Also we define the distance between two points p, q on M by

$$d(p, q) := \inf_{\gamma} L(\gamma),$$

where infimum is considered over curves joining the points p and q. The definition of completeness using geodesic and as metric (M, d) both are equivalent due the Hopf-Rinow theorem (see [105, Chapter 8, Theorem 16]).

2.6 Riemannian model manifolds

Lets start with a fixed point $p \in M$ and in the last section we have seen the exponential map $\exp_p$. If we define polar coordinate on N-dimensional manifold $T_pM(\cong \mathbb{R}^N)$ then, by the Gauss Lemma (see [83, Lemma 7.13]) we can see, the coordinates of T_pM will turn out as a polar coordinate on M around p by the map $\exp_p$. This is called geodesic polar coordinate on M. In polar coordinate r denote the distance of a point and a fixed pole o and there are $(N-1)$ coordinates $\{\theta_i\}_{i=1}^{N-1}$, which are orthogonal to r. In particular we can write the Riemannian metric as follows

$$g = dr \otimes dr + \sum_{i,j=1}^{N-1} G_{ij}^2(r, \theta_1, \ldots, \theta_{N-1})\, d\theta_i \otimes d\theta_j.$$

Definition 2.6.1. *An N-dimensional Riemannian manifold (M, g) is called Riemannian model manifold if it satisfies the following two conditions:*

(1) There exists a chart of M which covers entire M and its image in $\mathbb{R}^N$ is a $r_\circ$ radius ball $\mathscr{B}_{r_\circ} = \{x \in \mathbb{R}^N : ||x|| < r_\circ\}$, where $r_\circ \in [0, \infty]$ and if $r_\circ = \infty$ then $\mathscr{B}_{r_\circ}$ is entire $\mathbb{R}^N$.

(2) The polar coordinate (r, θ) representation of g in terms of above chart has the form

$$g = dr \otimes dr + \psi^2(r)\, g_{\mathbb{S}^{N-1}}, \tag{2.6.1}$$

where ψ is a $\mathcal{C}^\infty$ non-negative function on $[0, r_\circ)$, positive on $(0, r_\circ)$ such that $\psi'(0) = 1$ and $\psi^{(2k)}(0) = 0$ for all $k \geq 0$.

In the above definition, $r_\circ$ is called the radius of the Riemannian model manifold. These conditions on ψ ensure that the manifold is smooth and the metric at the pole o is given by the Euclidean metric [105, Chapter 1, 3.4]. The coordinate r, by construction, represents the Riemannian distance from the pole o, see e.g. [65, 66, 105] for further details. In particular, all the assumptions above are satisfied by $\psi(r) = r$ and by $\psi(r) = \sinh(r)$ with $r_\circ = \infty$: in the first case M coincides with the Euclidean space $\mathbb{R}^N$, in the latter with the hyperbolic space $\mathbb{H}^N$.

It is known that there exist an orthonormal frame $\{F_j\}_{j=1}^N$ on (M,g) where F_N corresponds to the radial coordinate, and $F_1,\dots,F_{N-1}$ to the spherical coordinates, for which $F_i \wedge F_j$ diagonalize the curvature operator $\mathcal{R}$:

$$\mathcal{R}(F_i \wedge F_N) = -\frac{\psi''}{\psi} F_i \wedge F_N, \quad i < N,$$
$$\mathcal{R}(F_i \wedge F_j) = -\frac{(\psi')^2 - 1}{\psi^2} F_i \wedge F_j, \quad i,j < N.$$

The quantities

$$K_{\pi,r}^{rad} := -\frac{\psi''}{\psi} \quad \text{and} \quad H_{\pi,r}^{tan} := -\frac{(\psi')^2 - 1}{\psi^2} \tag{2.6.2}$$

then coincide with the sectional curvatures w.r.t. planes containing the radial direction and, respectively, orthogonal to it.

Sometimes we need to assume that

$$K_{\pi,r}^{rad} \le -1 \quad \text{in } (0,+\infty). \tag{2.6.3}$$

Since, by the Sturm-Comparison Theorem, the above condition also implies that $H_{\pi,r}^{tan} \le -1$. We basically require the boundedness from above of both sectional curvatures. Let us recall the following useful lemma that can be easily derived from the Sturm-Comparison theorem, see [106, Lemma 2.1] and that we will repeatedly exploit in our proofs.

Lemma 2.6.1. *Let (M,g) be an N-dimensional Riemannian model manifold with metric g as given in* (2.6.1) *with smooth function ψ. If* (2.6.3) *holds, then*

$$\frac{\psi'(r)}{\psi(r)} \ge \coth r \quad \textit{and} \quad \psi(r) \ge \sinh r \quad \forall\, r > 0\,. \tag{2.6.4}$$

2.7 Laplace-Beltrami operator on manifolds

Now we will define the notion of volume element on Riemannian manifold and for the detail motivation refer [84]. Let (M,g) be a N-dimensional

Riemannian manifold and $(\mathcal{U}, \{x^i\}_{i=1}^N)$ be a chart around p on M. Then volume element is defined by $dv_g = \sqrt{G}\, dx^1 \otimes \ldots \otimes dx^N$, where $G = \det(g_{ij})$. If f is a smooth function with support within $\mathcal{U}$ then we define the integral $\int_{\mathcal{U}} f(x) dv_g$, where the integral is Riemann integral. One can check that this definition is well-defined and using partition of unity we can extend this definition on entire M for a compactly supported function. Finally, when M is compact then we define $\mathrm{vol}(M) := \int_M 1\, \mathrm{d}v_g$.

Next we will define divergence of a smooth vector field $X \in \mathscr{X}(M)$. It is a unique smooth real valued function on M denoted as "div X" and it is described by the following property

$$\int_M (\operatorname{div} X) u \, \mathrm{d}v_g = -\int_M g(X, \nabla u)\, \mathrm{d}v_g \text{ for all } u \in \mathcal{C}_c^\infty(M). \qquad (2.7.1)$$

The existence is followed by considering a local coordinate chart around some point and then patching those charts we can extend it for entire M and finally uniqueness is established by using some simple measure theoretic property. By exploiting the relation (2.7.1), we can check that for a smooth vector field $X = X^i \partial_i$, the expression of div X will be

$$\operatorname{div} X = \frac{1}{\sqrt{G}} \partial_i (X^i \sqrt{G}).$$

Finally consider a smooth function f on Riemannian manifold M. Then Laplace Beltrami operator is a smooth function on M and it is defined by $\Delta f := \operatorname{div}(\nabla f)$. The local coordinate representation of Laplacian will be

$$\Delta f = \frac{1}{\sqrt{G}} \sum_{i=1}^N \partial_i \Big(\sqrt{G} \sum_{j=1}^N g^{ij} \partial_j f \Big).$$

For the Riemannian model manifold (M, g) with the metric $g = dr \otimes dr + \psi^2(r)\, g_{\mathbb{S}^{N-1}}$, in terms of polar coordinate (r, θ) the volume form becomes

$$dv_g = \psi(r)^{N-1} dr \otimes d\theta,$$

where $d\theta$ denotes the $(N-1)$ volume form on $\mathbb{S}^{N-1}$. Next, we note that the Riemannian Laplacian of a scalar function u on M is given by

$$\Delta_g u(r,\theta_1,\ldots,\theta_{N-1}) = \frac{1}{\psi^2}\frac{\partial}{\partial r}\left[(\psi(r))^{N-1}\frac{\partial u}{\partial r}(r,\theta_1,\ldots,\theta_{N-1})\right] + \frac{1}{\psi^2}\Delta_{\mathbb{S}^{N-1}}u(r,\theta_1,\ldots,\theta_{N-1}), \tag{2.7.2}$$

where $\Delta_{\mathbb{S}^{N-1}}$ is the Riemannian Laplacian on the unit sphere $\mathbb{S}^{N-1}$. In particular, for radial functions, namely functions depending only on r, $\Delta_g u$ reads

$$\Delta_{r,g}u(r) = \frac{1}{(\psi(r))^{N-1}}\frac{\partial}{\partial r}\left[(\psi(r))^{N-1}\frac{\partial u}{\partial r}(r)\right] = u''(r) + (N-1)\frac{\psi'(r)}{\psi(r)}u'(r), \tag{2.7.3}$$

where from now on a prime will denote, for radial functions, derivative w.r.t the radial component r. Also, let us recall the Gradient in terms of the polar coordinate decomposition is given by

$$\nabla_g u(r,\theta_1,\ldots,\theta_{N-1}) = \left(\frac{\partial u}{\partial r}(r,\theta_1,\ldots,\theta_{N-1}), \frac{1}{\psi(r)}\nabla_{\mathbb{S}^{N-1}}u(r,\theta_1,\ldots,\theta_{N-1})\right), \tag{2.7.4}$$

where $\nabla_{\mathbb{S}^{N-1}}$ denotes the Gradient on the unit sphere $\mathbb{S}^{N-1}$. Again, the radial contribution of the Gradient, $\nabla_{r,g}u$, is defined as

$$\nabla_{r,g}u = \left(\frac{\partial u}{\partial r}, 0\right) = \big(u'(r), 0\big). \tag{2.7.5}$$

2.8 Spherical Harmonics

We want to point out that Spherical decomposition will be a key method in most of the proofs of the first part of the thesis. Over the years this method has become a remarkable tool in functional inequality and for the extensive study refer [114, Chapter 4, Lemma 2.18] and [96]. Let $u(x) = u(r,\sigma) \in \mathcal{C}_c^\infty(M)$, $r \in [0,\infty)$ and $\theta \in \mathbb{S}^{N-1}$, we can write

$$u(x) := u(r,\theta) = \sum_{n=0}^{\infty} d_n(r)P_n(\theta)$$

in the Hilbert space $L^2(M)$, where $\{P_n\}$ is an orthonormal system of spherical harmonics in the space $L^2(\mathbb{S}^{N-1})$ and

$$d_n(r) = \int_{\mathbb{S}^{N-1}} u(r,\theta) P_n(\theta)\, \mathrm{d}\theta\,.$$

A spherical harmonic P_n of order n is the restriction to $\mathbb{S}^{N-1}$ of a homogeneous harmonic polynomial of degree n. Moreover, it satisfies

$$-\Delta_{\mathbb{S}^{N-1}} P_n = \lambda_n P_n$$

for all $n \in \mathbb{N}_0$ where $\lambda_n = (n^2 + (N-2)n)$ are the eigenvalues of Laplace Beltrami operator $-\Delta_{\mathbb{S}^{N-1}}$ on $\mathbb{S}^{N-1}$ with corresponding eigenspace dimension c_n. We note that $\lambda_n \geq 0$, $\lambda_0 = 0$, $c_0 = 1$, $c_1 = N$ and

$$c_n = \binom{N+n-1}{n} - \binom{N+n-3}{n-2}$$

for $n \geq 2$.

Chapter 3

On some strong Poincaré inequalities on Riemannian models and their improvements

In this chapter, we will study second and fourth-order improved Poincaré type inequalities on the hyperbolic space involving Hardy-type remainder terms. Since theirs l.h.s. only involve the radial operator, so they can be seen as stronger versions of the classical Poincaré inequality. We will also show that such inequalities hold true on model manifolds as well, under suitable curvature assumptions, and the sharpness of some constants will be also discussed. The content of this chapter corresponds to the article [29].

3.1 Statement of results

We have already seen the structure of N-dimensional Riemannian model manifold (M, g) and assume the corresponding smooth function is ψ. A crucial quantity in our statements will be

$$\Lambda^{rad}_{\pi,r} := -2K^{rad}_{\pi,r} - (N-3)H^{tan}_{\pi,r}\,, \tag{3.1.1}$$

The quantity $\Lambda^{rad}_{\pi,r}$ is related to the bottom of the spectrum of the Laplacian, see inequality (3.1.2) below; in particular, when $\psi(r) = \sinh(r)$, then $\Lambda^{rad}_{\pi,r} = (N-1)$ hence $\frac{(N-1)}{4}\Lambda^{rad}_{\pi,r}$ coincides with the bottom of the spectrum of the Laplacian in $\mathbb{H}^N$. We will broadly see the results on this underlying space. This section is devoted to state the main results. Here is the first result in below:

Theorem 3.1.1. *Let (M, g) be an N-dimensional Riemannian model with $N \geq 3$ and with metric g as given in* (2.6.1). *Then, for all $u \in \mathcal{C}^\infty_c(M \setminus \{x_o\})$ there holds*

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \frac{(N-1)}{4}\int_M \Lambda^{rad}_{\pi,r}\, u^2 \,\mathrm{d}v_g + \frac{1}{4}\int_M \frac{u^2}{r^2}\,\mathrm{d}v_g + \frac{(N-1)(N-3)}{4}\int_M \frac{u^2}{\psi^2}\,\mathrm{d}v_g\,, \tag{3.1.2}$$

with $\Lambda^{rad}_{\pi,r}$ as defined in (3.1.1). *Moreover, if condition* (2.6.3) *holds and furthermore*

$$\frac{\psi'}{\psi} \sim C\, r^a \quad \text{as } r \to +\infty \tag{3.1.3}$$

for some $C > 0$ and $a \geq 0$, then the constant $\frac{1}{4}$ in (3.1.2) *is sharp, i.e.*

$$\frac{1}{4} = \inf_{\mathscr{W}^{1,2}(M)\setminus\{0\}} \frac{\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g - \frac{(N-1)}{4}\int_M \Lambda^{rad}_{\pi,r}\, u^2\ \mathrm{d}v_g}{\int_M \frac{u^2}{r^2}\ \mathrm{d}v_g}. \tag{3.1.4}$$

Remark 3.1.1. *A couple of remarks are in order about the further conditions required in the second part of the statement of Theorem 3.1.1. Condition* (2.6.3) *seems by no means technical as suggested by the following simple example. Consider the euclidean space, then $\psi(r) = r$ and* (2.6.3) *clearly fails. On the other hand, in this case, $\Lambda^{rad}_{\pi,r} \equiv 0$ and inequality* (3.1.2) *becomes the (strong) Hardy inequality:*

$$\int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{R}^N} \geq \frac{(N-2)^2}{4}\int_{\mathbb{R}^N} \frac{u^2}{r^2}\,\mathrm{d}v_{\mathbb{R}^N}\,,$$

hence $1/4$ is no more the sharp constant in front of the term $\frac{u^2}{r^2}$. As concerns condition (3.1.3), *it is needed to show that the quotient* (3.1.4) *is finite for* *our minimizing sequence, see Section 3.2. Nevertheless, we notice that this condition is not "too restrictive", in the sense that it allows unbounded curvatures from below as it happens, for instance, if* $\psi(r) = r\, e^{r^2}$ *for which* (3.1.3) *holds with* $C = 2$ *and* $a = 1$*, see Section 3.2 for a more detailed discussion and further examples.*

We point out that when (2.6.3) holds inequality (3.1.2) implies the following more explicit inequality:

Corollary 3.1.1. *Let* (M, g) *be an* N*-dimensional Riemannian model with* $N \geq 3$ *and with metric* g *as given in* (2.6.1) *and* (2.6.3) *holds. Then, for all* $u \in \mathcal{C}_c^\infty(M)$*, there holds*

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \left(\frac{N-1}{2}\right)^2 \int_M u^2 \,\mathrm{d}v_g + \frac{1}{4}\int_M \frac{u^2}{r^2}\,\mathrm{d}v_g + \frac{(N-1)(N-3)}{4}\int_M \frac{u^2}{\psi^2}\,\mathrm{d}v_g\,. \tag{3.1.5}$$

A remarkable particular case to which Theorem 3.1.1 applies is when $M = \mathbb{H}^N$, i.e. $\psi(r) = \sinh(r)$. In this case all constants in (3.1.2) are proved to be sharp.

Corollary 3.1.2. *Let* $M = \mathbb{H}^N$*, the Hyperbolic space with* $N \geq 3$*. Then, for all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ *there holds*

$$\int_{\mathbb{H}^N} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} u^2 \,\mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4}\int_{\mathbb{H}^N} \frac{u^2}{r^2}\,\mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)(N-3)}{4}\int_{\mathbb{H}^N} \frac{u^2}{(\sinh r)^2}\,\mathrm{d}v_{\mathbb{H}^N} \tag{3.1.6}$$

with all constants sharp. More precisely, the Poincaré constant $\left(\frac{N-1}{2}\right)^2$ *is sharp in the sense that no inequality of the form*

$$\int_{\mathbb{H}^N} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{H}^N} \geq c \int_{\mathbb{H}^N} u^2 \,\mathrm{d}v_{\mathbb{H}^N}$$

holds, for $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$*, when* $c > \left(\frac{N-1}{2}\right)^2$*. The constant* $\frac{1}{4}$ *is sharp in the sense explained in Theorem 3.1.1 while the constant* $\frac{(N-1)(N-3)}{4}$ *is sharp in the sense that no inequality of the form*

$$\int_{\mathbb{H}^N} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + c \int_{\mathbb{H}^N} \frac{u^2}{(\sinh r)^2} \, \mathrm{d}v_{\mathbb{H}^N}$$

holds, for $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$*, when* $c > \frac{(N-1)(N-3)}{4}$*.*

By combining spherical harmonics decomposition technique with Theorem 3.1.1 we derive the following second order analogue of Theorem 3.1.1

Theorem 3.1.2. *Let* (M, g) *be an* N*-dimensional Riemannian model with* $N \geq 3$ *and with metric* g *as given in* (2.6.1)*. Then, for all* $u \in \mathcal{C}_c^\infty(M)$*, there holds*

$$\int_M (\Delta_{r,g} u)^2 \, \mathrm{d}v_g - \frac{(N-1)}{4} \int_M \left[\Lambda_{\pi,r}^{rad} + 4(K_{\pi,r}^{rad} - H_{\pi,r}^{tan})\right] \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \frac{1}{4} \int_M \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g + \frac{(N^2-1)}{4} \int_M \frac{1}{\psi^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \, , \tag{3.1.7}$$

with $\Lambda_{\pi,r}^{rad}$ *as defined in* (3.1.1)*.*

Under suitable curvature bounds inequality (3.1.7) implies the following more explicit inequality:

Corollary 3.1.3. *Let* (M, g) *be an* N*-dimensional Riemannian model with* $N \geq 3$ *and with metric* g *as given in* (2.6.1) *and* (2.6.3) *holds. If furthermore*

$$K_{\pi,r}^{rad} \geq H_{\pi,r}^{tan} \quad \text{in } (0, +\infty) \, , \tag{3.1.8}$$

then there holds

$$\int_M (\Delta_{r,g} u)^2 \, \mathrm{d}v_g \geq \left(\frac{N-1}{2}\right)^2 \int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g + \frac{1}{4} \int_M \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g + \frac{(N^2-1)}{4} \int_M \frac{1}{\psi^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \, , \tag{3.1.9}$$

for all $u \in \mathcal{C}_c^\infty(M)$*.*

Remark 3.1.2. *Condition* (3.1.8) *holds with the equality if* $M = \mathbb{H}^N$*; examples of models satisfying* (2.6.3) *and for which the strict inequality holds in* (3.1.8) *can be given by taking* $\psi(r) = re^{br^{a+1}}$ *for* r *large, with* $b > 0$ *and* $a > -1$*, see the proof of Corollary 3.2.2 in Section 3.2.*

When $M = \mathbb{H}^N$ (3.1.7) clearly coincides with (3.1.9) but we also have the optimality of the constant $\left(\frac{N-1}{2}\right)^2$. For the sake of clarity we give the precise statement here below:

Corollary 3.1.4. *Let* $M = \mathbb{H}^N$*, the Hyperbolic space with* $N \geq 3$*. Then, for all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ *there holds*

$$\int_{\mathbb{H}^N} (\Delta_{r,g} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{H}^N} + \frac{(N^2-1)}{4} \int_{\mathbb{H}^N} \frac{1}{(\sinh r)^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_{\mathbb{H}^N} \tag{3.1.10}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$*. Furthermore the constant* $\left(\frac{N-1}{2}\right)^2$ *in the above inequality turns out to be sharp in the sense that no inequality of the form*

$$\int_{\mathbb{H}^N} (\Delta_{r,g} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq c \int_{\mathbb{H}^N} \left(\frac{\partial u}{\partial r}\right)^2 \, \mathrm{d}v_{\mathbb{H}^N}$$

holds, for $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ *when* $c > \left(\frac{N-1}{2}\right)^2$.

Next we state a Rellich type improvement for the second order Poincaré inequality (5.1.1) with $\ell = 0$ but on more general model manifolds. The proof comes by exploiting either inequality (1.1.6) on Riemannian models (see Lemma 3.5.1 in the following) and inequality (3.1.7), the first brought the restriction $N \geq 5$ and the latter yields the curvature conditions (2.6.3) and (3.1.8) below.

Theorem 3.1.3. *Let* (M, g) *be an* N*-dimensional Riemannian model with* $N \geq 5$ *and with metric* g *as given in* (2.6.1) *with* ψ *satisfying* (2.6.3)*, and* (3.1.8)*. Then for all* $u \in \mathcal{C}_c^\infty(M)$ *there holds*

$$\int_M (\Delta_g u)^2 \, \mathrm{d}v_g \geq \left(\frac{N-1}{2}\right)^4 \int_M u^2 \, \mathrm{d}v_g + \frac{(N-4)^2}{16} \int_M \frac{u^2}{r^4} \, \mathrm{d}v_g$$
$$+ \frac{(N-1)^2}{16} \int_M \frac{u^2}{r^2} \mathrm{d}v_g \, . \tag{3.1.11}$$

Remark 3.1.3. *As already explained in the Introduction, inequality* (3.1.11) *with* $M = \mathbb{H}^N$ *must be compared with inequality* (1.1.7). *In particular, it gives rise to the interesting fact that the constant appearing in front of the Rellich term* $\frac{u^2}{r^4}$ *can be larger than* $\frac{9}{16}$.

We conclude by stating a Hardy-type improvement of the second order Poincaré inequality (5.1.1) with $\ell = 1$ on model manifolds. Here the main tools exploited in the proofs are spherical harmonics decomposition and reduction of dimension technique. The latter yields the restriction $N \geq 5$, while conditions (2.6.3) and (3.1.8) come again from inequality (3.1.7) that we apply for each component of the decomposition.

Theorem 3.1.4. *Let* (M, g) *be an* N*-dimensional Riemannian model with* $N \geq 5$ *and with metric* g *as given in* (2.6.1) *with* ψ *satisfying* (2.6.3) *and* (3.1.8). *Then for all* $u \in \mathcal{C}_c^\infty(M)$ *there holds*

$$\int_M (\Delta_g u)^2 \, \mathrm{d}v_g \geq \left(\frac{N-1}{2}\right)^2 \int_M |\nabla_g u|^2 \, \mathrm{d}v_g + \frac{1}{4} \int_M \frac{|\nabla_g u|^2}{r^2} \, \mathrm{d}v_g$$
$$+ \frac{(N^2-1)}{4} \int_M \frac{|\nabla_g u|^2}{\psi^2} \, \mathrm{d}v_g \, . \tag{3.1.12}$$

3.2 Some prototype model manifolds

In this section we discuss our first order results on N-dimensional Riemannian models (M, g) with $N \geq 3$ and with metric g as given in (2.6.1) with ψ satisfying the further condition for large r:

$$\psi(r) = Ae^{br^{a+1}} \qquad \text{for } r \geq R >> 1 \tag{3.2.1}$$

or

$$\psi(r) = Are^{br^{a+1}} \quad \text{for } r \geq R >> 1 \tag{3.2.2}$$

for some $R, A, b > 0$ and $a \geq -1$. The case (3.2.1) includes $\mathbb{H}^N$ (for $a = 0$), while the case (3.2.2) includes $\mathbb{R}^N$ (for $a = -1$). In both cases the sectional curvatures satisfy

$$K_{\pi,r}^{rad} \sim -b^2(a+1)^2 r^{2a} \quad \text{and} \quad H_{\pi,r}^{rad} \sim -b^2(a+1)^2 r^{2a} \quad \text{as } r \to +\infty\,.$$

Hence, for $a \geq 0$ unbounded curvatures from below are allowed. We refer to [67, Section 2.3] for further possible choices of ψ and their geometric interpretation.

In case (3.2.1) from Theorem 3.1.1 we derive the following improved Poincaré inequality for functions supported outside $\mathscr{B}_R(x_o)$:

Corollary 3.2.1. *Let (M, g) be an N-dimensional Riemannian model with $N \geq 3$ and with metric g as given in* (2.6.1) *with ψ satisfying condition* (3.2.1) *for some $R, A, b > 0$ and $a \geq 0$. Then, for all $u \in \mathcal{C}_c^\infty(M \setminus \mathscr{B}_R(x_o))$, there holds*

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \left(\frac{N-1}{2}\right)^2 (a+1)^2 b^2 \int_M r^{2a}\, u^2 \,\mathrm{d}v_g + \frac{1}{4}\int_M \frac{u^2}{r^2}\,\mathrm{d}v_g + \frac{2ba(a+1)(N-1)}{4}\int_M r^{a-1} u^2 \,\mathrm{d}v_g. \tag{3.2.3}$$

Notice that (3.2.3) can be seen as an improved Poincaré inequality since $\int_M r^{2a}\, u^2 \,\mathrm{d}v_g \geq \int_M u^2 \,\mathrm{d}v_g$ for $a \geq 0$ and $u \in \mathcal{C}_c^\infty(M \setminus \mathscr{B}_R(x_o))$ with $R \geq 1$. In particular, for $a = 0$ and $b = 1$ we recover the sharp Poincaré constant in $\mathbb{H}^N$, i.e. $\left(\frac{N-1}{2}\right)^2$.

Proof. For $r \geq R$ we compute:

$$K_{\pi,r}^{rad} = -b^2(a+1)^2 r^{2a} - b(a+1)(a+2)r^{a-1}\,, \quad H_{\pi,r}^{tan} = -b^2(a+1)^2 r^{2a} + \frac{1}{A^2 e^{2br^{a+1}}}$$

and

$$\Lambda_{\pi,r}^{rad} = 2ba(a+1)r^{a-1} + b^2(a+1)^2(N-1)r^{2a} - \frac{N-3}{A^2 e^{2br^{a+1}}}\,.$$

Then, the desired inequality comes by inserting the above term into (3.1.2), summing up and rearranging all terms. □

In case (3.2.2) from Theorem 3.1.1 we derive an improved Hardy inequality for functions supported outside $\mathscr{B}_R(x_o)$:

Corollary 3.2.2. *Let* (M,g) *be an* N*-dimensional Riemannian model with* $N \geq 3$ *and with metric* g *as given in* (2.6.1) *with* ψ *satisfying condition* (3.2.2) *for some* $R, A, b > 0$ *and* $a \geq -1$*. Then, for all* $u \in \mathcal{C}_c^\infty(M \setminus \mathscr{B}_R(x_o))$*, there holds*

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \frac{(N-2)^2}{4}\int_M \frac{u^2}{r^2}\,\mathrm{d}v_g + \frac{(a+1)^2 b^2 (N-1)^2}{4}\int_M r^{2a}\,u^2\,\mathrm{d}v_g + \frac{b(a+1)(N-1)(N-1+a)}{2}\int_M r^{a-1}u^2\,\mathrm{d}v_g. \tag{3.2.4}$$

When $a=-1$ the decay of the manifold is of euclidean type and (3.2.4) reduces to the standard Hardy inequality. On the other hand, when $a \geq 0$, since $\int_M r^{2a}\,u^2\,\mathrm{d}v_g \geq \int_M u^2\,\mathrm{d}v_g$ for all $u \in \mathcal{C}_c^\infty(M \setminus \mathscr{B}_R(x_o))$ with $R \geq 1$, (3.2.4) reads as an improved Hardy-Poincaré inequality.

Proof. For $r \geq R$ we compute:

$$K_{\pi,r}^{rad} = -b^2(a+1)^2 r^{2a} - ba(a+1)r^{a-1}\,,$$

$$H_{\pi,r}^{tan} = -b^2(a+1)^2 r^{2a} - \frac{1}{r^2} - 2b(a+1)r^{a-1} + \frac{1}{A^2 r^2 e^{2br^{a+1}}}\,,$$

and

$$\Lambda_{\pi,r}^{rad} = 2b(N-1+a)r^{a-1} + b^2(a+1)^2(N-1)r^{2a} + \frac{N-3}{r^2} - \frac{N-3}{A^2 r^2 e^{2br^{a+1}}}\,.$$

Then, the (3.2.4) comes by inserting the above term into (3.1.2), summing up and rearranging all terms. □

A further remarkable consequence of Theorem 3.1.1 is the following improved Hardy inequality:

Corollary 3.2.3. *Let* (M, g) *be an* N*-dimensional Riemannian model with* $N \geq 3$ *and with metric* g *as given in* (2.6.1) *with* $\psi(r) = re^{r^{2m}}$ *for some positive integer* m*. Then, for all* $u \in \mathcal{C}_c^\infty(M \setminus \{x_o\})$ *there holds*

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \frac{(N-2)^2}{4} \int_M \frac{u^2}{r^2} \mathrm{d}v_g + m^2(N-1)^2 \int_M r^{4m-2} u^2 \, \mathrm{d}v_g$$
$$+ m(N-1)(N-2+2m) \int_M r^{2m-2} u^2 \, \mathrm{d}v_g \,, \tag{3.2.5}$$

where the constant $\frac{(N-2)^2}{4}$ *is sharp.*

Remark 3.2.1. *It's worth noticing that, under the assumption of Corollary 3.2.3, we have*

$$\frac{(N-1)}{4}\Lambda_{\pi,r}^{rad} = -\frac{(N-1)K_{\pi,r}^{rad}}{4} + \frac{(N-1)(N-3)}{4}\frac{1}{r^2} - \frac{(N-1)(N-3)}{4}\frac{1}{\psi^2(r)} \,.$$

Once this observed, we readily infer that, if condition (2.6.3) *holds, then the constant in front of the term* $\frac{1}{r^2}$ *on the right hand side of* (3.1.2) *cannot be larger than* $\frac{1}{4}$ *otherwise we would contradict the sharpness of the Hardy constant* $\frac{(N-2)^2}{4}$*.*

Proof. It is readily seen that the function $\psi(r) = re^{r^{2m}}$ satisfies condition of being a model manifold, hence Theorem 3.1.1 applies. On the other hand, some computations yield:

$$K_{\pi,r}^{rad} = -(2m)^2 r^{4m-2} - 2m(2m+1)r^{2m-2} \,,$$
$$H_{\pi,r}^{tan} = -(2m)^2 r^{4m-2} - \frac{1}{r^2} - 4mr^{2m-2} + \frac{1}{r^2 e^{2r^{2m}}} \,,$$

and

$$\Lambda_{\pi,r}^{rad} = (2n)^2(N-1)r^{4m-2} + 4m(N-2+2m)r^{2m-2} + \frac{(N-3)}{r^2} - \frac{(N-3)}{r^2 e^{2r^{2m}}} \,.$$

Finally, inequality (3.2.3) comes by inserting the above term into (3.1.2), summing up and rearranging all terms. As for the sharpness of the constant $\frac{(N-2)^2}{4}$, it comes from [48]. $\square$

3.3 Proof of Theorem 3.1.1

This section is devoted to the proof of Theorem 3.1.1. The proof is divided into two steps.

Step 1. Let $u \in \mathcal{C}_c^\infty(M \setminus \{x_o\})$, where o denotes the pole, and define the transformation

$$v(x) = \psi(r)^{\frac{(N-1)}{2}} u(x) \text{ where } r = \rho(x, o) \text{ and } x = (r, \sigma) \in (0, \infty) \times \mathbb{S}^{N-1}\,.$$

Then clearly $v \in \mathcal{C}_c^\infty(M \setminus \{x_o\})$.

An easy calculation gives

$$\frac{\partial v}{\partial r} = \frac{\partial}{\partial r}\left(\psi(r)^{\frac{(N-1)}{2}} u\right) = \psi(r)^{\frac{(N-1)}{2}} \frac{\partial u}{\partial r} + \frac{(N-1)}{2}\frac{\psi'(r)}{\psi(r)}\, v\,.$$

By arranging the terms we obtain:

$$\frac{\partial u}{\partial r} = \frac{1}{\psi(r)^{\frac{(N-1)}{2}}}\left[\frac{\partial v}{\partial r} - \frac{(N-1)}{2}\frac{\psi'(r)}{\psi(r)}\, v\right]. \tag{3.3.1}$$

Step 2. Now, expanding v in terms of spherical harmonics:

$$v(x) = v(r, \sigma) = \sum_{n=0}^{\infty} d_n(r) P_n(\sigma)\,,$$

we find

$$\frac{\partial v}{\partial r} = \sum_{n=0}^{\infty} d_n'(r) P_n(\sigma)\,.$$

Furthermore, from (3.3.1) we observe that

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g = \int_M \frac{1}{\psi^{(N-1)}}\left(\frac{\partial v}{\partial r}\right)^2 \mathrm{d}v_g - (N-1)\int_M \frac{1}{\psi^{(N-1)}}\frac{\partial v}{\partial r}\frac{\psi'}{\psi}\, v\, \mathrm{d}v_g$$
$$+ \frac{(N-1)^2}{4}\int_M \frac{1}{\psi^{(N-1)}}\frac{\psi'^2}{\psi^2}\, v^2\, \mathrm{d}v_g\,. \tag{3.3.2}$$

We will evaluate each term of (3.3.2) separately. Using the integration by parts formula and the orthonormal properties of $\{P_n\}$ i.e $\int_{\mathbb{S}^{N-1}} P_n P_m \,\mathrm{d}\sigma = \delta_{nm}$, we find

$$\int_M \frac{1}{\psi^{(N-1)}} \left(\frac{\partial v}{\partial r}\right)^2 \,\mathrm{d}v_g = \sum_{n=0}^{\infty} \int_0^{\infty} (d_n')^2 \,\mathrm{d}r \tag{3.3.3}$$

$$\int_M \frac{1}{\psi^{(N-1)}} \frac{\partial v}{\partial r} \frac{\psi'}{\psi}\, v \,\mathrm{d}v_g = -\frac{1}{2} \sum_{n=0}^{\infty} \int_0^{\infty} d_n^2 \,\frac{\psi\psi'' - \psi'^2}{\psi^2} \,\mathrm{d}r \tag{3.3.4}$$

$$\int_M \frac{1}{\psi^{(N-1)}} \frac{\psi'^2}{\psi^2}\, v^2 \,\mathrm{d}v_g = \sum_{n=0}^{\infty} \int_0^{\infty} d_n^2 \,\frac{\psi'^2}{\psi^2} \,\mathrm{d}r\,. \tag{3.3.5}$$

By substituting (3.3.3), (3.3.4) and (3.3.5) into (3.3.2) and after simplifying, we obtain

$$\begin{aligned}\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g &= \sum_{n=0}^{\infty} \int_0^{\infty} (d_n')^2 \,\mathrm{d}r + \frac{(N-1)}{2} \sum_{n=0}^{\infty} \int_0^{\infty} d_n^2 \,\frac{\psi\psi'' - \psi'^2}{\psi^2} \,\mathrm{d}r \\ &\quad + \frac{(N-1)^2}{4} \sum_{n=0}^{\infty} \int_0^{\infty} d_n^2 \,\frac{\psi'^2}{\psi^2} \,\mathrm{d}r \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} (d_n')^2 \,\mathrm{d}r \\ &\quad - \frac{(N-1)}{4} \sum_{n=0}^{\infty} \int_0^{\infty} \Big[2K_{\pi,r}^{rad} + (N-3)H_{\pi,r}^{tan}\Big] d_n^2 \,\mathrm{d}r \\ &\quad + \frac{(N-1)(N-3)}{4} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d_n^2}{\psi^2} \,\mathrm{d}r\,. \end{aligned} \tag{3.3.6}$$

Next, by using the 1-dimensional Hardy inequality:

$$\int_0^{\infty} (d_n')^2 \,\mathrm{d}r \geq \frac{1}{4} \int_0^{\infty} \frac{d_n^2}{r^2} \,\mathrm{d}r$$

into (3.3.6) we get

$$\begin{aligned}\int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g &\geq \frac{1}{4} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d_n^2}{r^2} \,\mathrm{d}r \\ &\quad - \frac{(N-1)}{4} \sum_{n=0}^{\infty} \int_0^{\infty} \Big[2K_{\pi,r}^{rad} + (N-3)H_{\pi,r}^{tan}\Big] d_n^2 \,\mathrm{d}r \\ &\quad + \frac{(N-1)(N-3)}{4} \sum_{n=0}^{\infty} \int_0^{\infty} \frac{d_n^2}{\psi^2} \,\mathrm{d}r\,. \end{aligned} \tag{3.3.7}$$

Finally, inserting the equalities:

$$\int_M \Big[2K_{\pi,r}^{rad} + (N-3)H_{\pi,r}^{tan}\Big]u^2 \, \mathrm{d}v_g = \sum_{n=0}^{\infty}\int_0^{\infty} \Big[2K_{\pi,r}^{rad} + (N-3)H_{\pi,r}^{tan}\Big]d_n^2 \, \mathrm{d}r\,,$$

$$\int_M \frac{u^2}{r^2}\, \mathrm{d}v_g = \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d_n^2}{r^2}\, \mathrm{d}r \text{ and } \int_M \frac{u^2}{\psi^2}\, \mathrm{d}v_g = \sum_{n=0}^{\infty}\int_0^{\infty}\frac{d_n^2}{\psi^2}\, \mathrm{d}r$$

into (3.3.7) and recalling (3.1.1) we obtain the desired inequality (3.1.2) with $u \in \mathcal{C}_c^\infty(M \setminus \{x_o\})$.

Remark 3.3.1. *If* (2.6.3) *holds inequality* (3.1.2) *can be extended to functions belonging to* $\mathcal{C}_c^\infty(M)$ *by density arguments. Indeed, in this case* M *is a Cartan-Hadamard manifold with strictly negative curvatures and, since, for* $N > 2$*, the set* $\{x_o\}$ *is compact and has zero capacity, the following inclusion holds:* $\overline{\mathcal{C}_c^\infty(M)}^{\|\nabla\cdot\|_2} \subset \overline{\mathcal{C}_c^\infty(M\setminus\{x_o\})}^{\|\nabla\cdot\|_2}$ *(see [51, Proposition A.1 and Theorem 6.5]). Then, by using Gauss Lemma* $|\frac{\partial u}{\partial r}| \le |\nabla u|$*, one deduces the validity of* (3.1.2) *for all* $\mathcal{C}_c^\infty(M)$*.*

3.3.1 Optimality of the constant: sequential approach

We set

$$C_M := \inf_{\mathscr{W}^{1,2}(M)\setminus\{0\}} \frac{\int_M (\frac{\partial u}{\partial r})^2 \, \mathrm{d}v_g - \frac{(N-1)}{4}\int_M \Lambda_{\pi,r}^{rad}\, u^2 \, \mathrm{d}v_g}{\int_M \frac{u^2}{r^2}\, \mathrm{d}v_g}\,. \tag{3.3.8}$$

If (2.6.3) holds, then $\Lambda_{\pi,r}^{rad} > 0$ and, by combining density arguments with Fatou's Lemma, we infer that (3.1.2) holds in $\mathscr{W}^{1,2}(M)$ and, in turn, that $C_M \ge \frac{1}{4}$. So it remains to show that $C_M \le \frac{1}{4}$ and this will be done by giving a proper minimizing sequence. Again we divide the proof in some steps.

Step 1. Let us define the sequence $\{\phi_n\}$ for $n \in \mathbb{N}$ as follows

$$\phi_n(r) = \begin{cases} 0 & 0 < r \le 1 \\ n^{-\alpha}(r-1) & 1 \le r \le 2 \\ n^{-\alpha} & 2 \le r \le n \\ r^{-\alpha} & n \le r \end{cases} \tag{3.3.9}$$

where $\alpha > 1+a$ and $a \geq 0$ is as given in (3.1.3). Clearly, $\phi_n \in L^1_{loc}(0,+\infty)$ and its weak derivative writes

$$\phi_n'(r) = \begin{cases} 0 & 0 < r \leq 1 \\ n^{-\alpha} & 1 \leq r \leq 2 \\ 0 & 2 \leq r \leq n \\ -\alpha r^{-\alpha-1} & n \leq r\,. \end{cases} \tag{3.3.10}$$

Next we recall that, from the proof of Proposition 4.1 in [27], the function $u_0(r) := \frac{r^{\frac{1}{2}}}{\psi^{\frac{N-1}{2}}}$ satisfies:

$$-\Delta_{r,g}\, u_0 - \frac{(N-1)}{4}\Lambda^{rad}_{\pi,r}\, u_0 = \frac{1}{4}\frac{u_0}{r^2} + \frac{(N-1)(N-3)}{4}\,\frac{u_0}{\psi^2} \quad \text{for } r > 0\,. \tag{3.3.11}$$

Using polar coordinates and by exploiting (3.1.3), it follows that $u_0\phi_n$ and $u_0\phi_n^2$ both belong to $\mathscr{W}^{1,2}(M)$ for all $n \geq 3$ and for $\alpha > 1+a$. In particular, (3.1.3) assures $(u_0\phi_n)', (u_0\phi_n^2)' \in L^2(M)$. Indeed, for $r \geq n$, we have that

$$\begin{aligned}((u_0\phi_n)')^2\psi^{N-1} \sim \frac{1}{r^{2\alpha-1}}\Bigg[\left(\alpha - \frac{1}{2}\right)^2 \frac{1}{r^2} + \frac{(N-1)^2}{4}\, C^2\, r^{2a} \\ + (2\alpha-1)\frac{(N-1)}{2}\, C\, r^{a-1}\Bigg]\end{aligned} \tag{3.3.12}$$

and this term turns out to be integrable at infinity for $\alpha > 1+a$, if ψ satisfies the above condition; the term $(u_0\phi_n^2)'$ can be treated similarly.

Step 2. Let $R > n$, by multiplying the equation (3.3.11) by $u_0\phi_n^2$ and integrating by parts, we obtain

$$\begin{aligned}&\int_{\mathscr{B}_R(x_o)} \left(\frac{\partial u_0}{\partial r}\right)\left(\frac{\partial (u_0\phi_n^2)}{\partial r}\right) \mathrm{d}v_g - \int_{\partial\mathscr{B}_R(x_o)} \left(\frac{\partial u_0}{\partial r}\right) u_0\phi_n^2\, \mathrm{d}S_g \\ &- \frac{(N-1)}{4}\int_{\mathscr{B}_R(x_o)} \Lambda^{rad}_{\pi,r}\, (u_0\phi_n)^2\, \mathrm{d}v_g \\ &= \frac{1}{4}\int_{\mathscr{B}_R(x_o)} \frac{(u_0\phi_n)^2}{r^2}\, \mathrm{d}v_g + \frac{(N-1)(N-3)}{4}\int_{\mathscr{B}_R(x_o)} \frac{(u_0\phi_n)^2}{\psi^2}\, \mathrm{d}v_g\,,\end{aligned} \tag{3.3.13}$$

where we have exploited the fact that ϕ_n is supported outside $\mathscr{B}_1(x_o)$, hence no problem of integrability arises at $r = 0$.

Next we note that

$$\int_{\partial \mathscr{B}_R(x_o)} \left(\frac{\partial u_0}{\partial r}\right) u_0 \phi_n^2 \, \mathrm{d}S_g = \int_{\partial \mathscr{B}_R(x_o 0)} u_0'(R)\, u_0(R) \phi_n^2(R) \, \mathrm{d}S_g$$
$$= \frac{1}{2}\left(1 - (N-1) R \frac{\psi'(R)}{\psi(R)}\right) \frac{1}{R^\alpha} = o(1) \,\mathrm{as}\, R \to +\infty$$

where in the above we have exploited the fact that $|\partial \mathscr{B}_R(x_o)| = \omega_N \, \psi^{N-1}(R)$.

On the other hand, for $r > n$, we have that

$$\Big|\left(\frac{\partial u_0}{\partial r}\right)\left(\frac{\partial (u_0 \phi_n^2)}{\partial r}\right)\Big| = \left|(u_0')^2 \phi_n^2 + 2 u_0 u_0' \phi_n \phi_n'\right|$$
$$= \left|\frac{1}{r^{2\alpha-1}}\left(\frac{1}{4}\frac{1}{r^2} + \frac{(N-1)^2}{4}\left(\frac{\psi'}{\psi}\right)^2 + \left(\alpha - \frac{N-1}{2}\right)\frac{\psi'}{\psi}\frac{1}{r} - \alpha \frac{1}{r^2}\right)\frac{1}{\psi^{N-1}}\right|$$
$$\leq \frac{1}{r^{2\alpha-1}}\left(\frac{1}{4}\frac{1}{r^2} + \frac{(N-1)^2}{4} C^2\, r^{2a} + \left(\alpha + \frac{N-1}{2}\right) C\, r^{a-1} + \alpha \frac{1}{r^2}\right)\frac{1}{\psi^{N-1}}$$
$$\Big[\text{as } r \to +\infty \text{ (using (3.1.3))}\Big],$$

$$\frac{(u_0 \phi_n)^2}{r^2} = \frac{1}{r^{2\alpha+1}}\frac{1}{\psi^{N-1}},$$

$$\frac{(u_0 \phi_n)^2}{\psi^2} \leq \frac{1}{r^{2\alpha+1}}\frac{1}{\psi^{N-1}} \quad \text{(using (2.6.3) and (2.6.4))}.$$

As for the terms involving the curvatures, we first note that, if (2.6.3) holds then $\Lambda_{\pi,r}^{rad} > 0$ (see the proof of Corollary 3.1.1 below) and, by density arguments, (3.1.2) yields

$$\frac{(N-1)}{4} \int_M \Lambda_{\pi,r}^{rad} \, u^2 \, \mathrm{d}v_g \leq \int_M \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \quad \text{for all } u \in H^1(M)\,.$$

Hence, since $u_0 \phi_n \in \mathscr{W}^{1,2}(M)$, we deduce that

$$\frac{(N-1)}{4} \int_M \Lambda_{\pi,r}^{rad} \, (u_0 \phi_n)^2 \, \mathrm{d}v_g \leq \int_M \left(\frac{\partial (u_0 \phi_n)}{\partial r}\right)^2 \mathrm{d}v_g$$

where the boundedness of the latter term and, in turn, of the first, follows by (3.3.12).

Since for all terms listed above we have integrability at infinity whenever $\alpha > 1 + a$, by Lebesgue Theorem, we can pass to the limit in (3.3.13) as $R \to +\infty$ and we conclude that

$$\int_M \left(\frac{\partial u_0}{\partial r}\right)\left(\frac{\partial (u_0\phi_n^2)}{\partial r}\right) \mathrm{d}v_g - \frac{(N-1)}{4}\int_M \Lambda_{\pi,r}^{rad}\,(u_0\phi_n)^2 \,\mathrm{d}v_g$$
$$= \frac{1}{4}\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g + \frac{(N-1)(N-3)}{4}\int_M \frac{(u_0\phi_n)^2}{\psi^2}\,\mathrm{d}v_g\,. \tag{3.3.14}$$

Step 3. Next we observe that

$$\int_M \left(\frac{\partial u_0}{\partial r}\right)\left(\frac{\partial (u_0\phi_n^2)}{\partial r}\right) \mathrm{d}v_g = \int_M \left(\frac{\partial (u_0\phi_n)}{\partial r}\right)^2 \mathrm{d}v_g - \int_M u_0^2\left(\frac{\partial \phi_n}{\partial r}\right)^2 \mathrm{d}v_g\,.$$

Note that, by the above discussion, the first two terms are well defined. Furthermore, for $r \geq n$, we have that

$$u_0^2\left(\frac{\partial \phi_n}{\partial r}\right)^2 \psi^{N-1} = \frac{\alpha^2}{r^{2\alpha+1}}$$

which is in fact integrable for $\alpha > 0$.

By using this into (3.3.14) we have

$$\int_M \left(\frac{\partial (u_0\phi_n)}{\partial r}\right)^2 \mathrm{d}v_g - \frac{(N-1)}{4}\int_M \Lambda_{\pi,r}^{rad}\,(u_0\phi_n)^2\,\mathrm{d}v_g$$
$$= \frac{1}{4}\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g + \frac{(N-1)(N-3)}{4}\int_M \frac{(u_0\phi_n)^2}{\psi^2}\,\mathrm{d}v_g$$
$$+ \int_M u_0^2\left(\frac{\partial \phi_n}{\partial r}\right)^2 \mathrm{d}v_g \tag{3.3.15}$$

and by considering the quotient in (3.3.8) with $u = u_0\phi_n$, we obtain

$$\frac{\int_M u_0^2\left(\frac{\partial(u_0\phi_n)}{\partial r}\right)^2 \mathrm{d}v_g\,\mathrm{d}v_g - \frac{(N-1)}{4}\int_M \Lambda_{\pi,r}^{rad}\,(u_0\phi_n)^2\,\mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g}$$
$$= \frac{1}{4} + \frac{(N-1)(N-3)}{4}\,\frac{\int_M \frac{(u_0\phi_n)^2}{\psi^2}\,\mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g} + \frac{\int_M u_0^2\left(\frac{\partial \phi_n}{\partial r}\right)^2 \mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g}\,.$$

Then, from definition of C_M and the above, we infer

$$C_M \leq \frac{1}{4} + \frac{(N-1)(N-3)}{4} \frac{\int_M \frac{(u_0\phi_n)^2}{\psi^2} \, \mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2} \, \mathrm{d}v_g} + \frac{\int_M u_0^2 \left(\frac{\partial \phi_n}{\partial r}\right)^2 \, \mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2} \, \mathrm{d}v_g} . \tag{3.3.16}$$

Step 4. We estimate each term of the r.h.s. of (3.3.16). Note that ω_N denotes the N dimensional measure of unit sphere, hence a finite number.

First we estimate the denominator

$$\begin{aligned} \int_M \frac{(u_0\phi_n)^2}{r^2} \, \mathrm{d}v_g &= \omega_N \int_0^\infty \frac{(u_0\phi_n)^2}{r^2} (\psi(r))^{(N-1)} \, \mathrm{d}r = \omega_N \int_0^\infty \frac{\phi_n^2}{r} \, \mathrm{d}r \\ &\geq \omega_N \int_2^n \frac{\phi_n^2}{r} \, \mathrm{d}r + \omega_N \int_n^\infty \frac{\phi_n^2}{r} \, \mathrm{d}r \\ &= \omega_N \, n^{-2\alpha} \left[\log(\frac{n}{2}) + \frac{1}{2\alpha} \right]. \end{aligned} \tag{3.3.17}$$

Now we consider

$$\begin{aligned} \int_M u_0^2 \left(\frac{\partial \phi_n}{\partial r} \right)^2 \, \mathrm{d}v_g &= \omega_N \int_0^\infty u_0^2 (\phi_n')^2 (\psi(r))^{(N-1)} \, \mathrm{d}r = \omega_N \int_0^\infty r (\phi_n')^2 \, \mathrm{d}r \\ &= \omega_N \int_1^2 r (\phi_n')^2 \, \mathrm{d}r + \omega_N \int_n^\infty r (\phi_n')^2 \, \mathrm{d}r \\ &= \omega_N \, n^{-2\alpha} \left[\frac{3}{2} + \frac{\alpha}{2} \right]. \end{aligned} \tag{3.3.18}$$

Finally, using integration by parts, $\alpha > 1$, (2.6.4) and $\sinh r \geq r$, we obtain

$$
\begin{aligned}
&\int_M \frac{(u_0\phi_n)^2}{\psi^2}\,\mathrm{d}v_g \\
&= \omega_N \int_0^\infty \frac{(u_0\phi_n)^2}{\psi^2(r)} (\psi(r))^{(N-1)}\,\mathrm{d}r \\
&= \omega_N \int_0^\infty \frac{r(\phi_n)^2}{\psi^2(r)}\,\mathrm{d}r \\
&= \omega_N \int_1^2 \frac{r(\phi_n)^2}{\psi^2(r)}\,\mathrm{d}r + \omega_N \int_2^n \frac{r(\phi_n)^2}{\psi^2(r)}\,\mathrm{d}r + \omega_N \int_n^\infty \frac{r(\phi_n)^2}{\psi^2(r)}\,\mathrm{d}r \\
&\leq \omega_N \int_1^2 \frac{r\,n^{-2\alpha}(r-1)^2}{\sinh^2 r}\,\mathrm{d}r + \omega_N \int_2^n \frac{r\,n^{-2\alpha}}{\sinh^2 r}\,\mathrm{d}r + \omega_N \int_n^\infty \frac{r^{1-2\alpha}}{\sinh^2 r}\,\mathrm{d}r \\
&\leq \frac{\omega_N\, n^{-2\alpha}}{\sinh^2 1} \int_1^2 r(r-1)^2\,\mathrm{d}r + \omega_N\, n^{-2\alpha} \int_2^n (\operatorname{csch} r)\,\mathrm{d}r + \omega_N \int_n^\infty r^{-1-2\alpha}\,\mathrm{d}r \\
&= \omega_N\, n^{-2\alpha} \left[\frac{7}{12\,(\sinh^2 1)} + \log\left|\frac{\exp(-n)-1}{\exp(-n)+1}\right| - \log\left|\frac{\exp(-2)-1}{\exp(-2)+1}\right| + \frac{1}{\alpha} \right].
\end{aligned}
\tag{3.3.19}
$$

Step 5. We are now in the final stage. Using (3.3.19) and (3.3.17) we have for $n \to \infty$,

$$
\frac{\int_M \frac{(u_0\phi_n)^2}{\psi^2}\,\mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g} \leq \frac{\frac{7}{12\,(\sinh^2 1)} + \log\left|\frac{\exp(-n)-1}{\exp(-n)+1}\right| - \log\left|\frac{\exp(-2)-1}{\exp(-2)+1}\right| + \frac{1}{2\alpha}}{\log(\frac{n}{2}) + \frac{1}{2\alpha}} \to 0
$$

and, using (3.3.18) and (3.3.17), we deduce that

$$
\frac{\int_M u_0^2 \left(\frac{\partial \phi_n}{\partial r}\right)^2 \mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g} \leq \frac{\frac{3}{2} + \frac{\alpha}{2}}{\log(\frac{n}{2}) + \frac{1}{2\alpha}} \to 0 \quad \text{for } n \to \infty.
$$

Finally, combining these two and (3.3.16), we can say that,

$$
C_M \leq \frac{1}{4} + \frac{(N-1)(N-3)}{4} \frac{\int_M \frac{(u_0\phi_n)^2}{\psi^2}\,\mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g} + \frac{\int_M u_0^2 \left(\frac{\partial \phi_n}{\partial r}\right)^2 \mathrm{d}v_g}{\int_M \frac{(u_0\phi_n)^2}{r^2}\,\mathrm{d}v_g} = \frac{1}{4} + o(1)\,.
$$

Hence, $\{u_0\phi_n\}$ is the required minimizing sequence and, in turn, $C_M = \frac{1}{4}$.

Proof of Corollary 3.1.1. Recalling (3.1.2), we know that

$$\int_M \left(\frac{\partial u}{\partial r}\right)^2 \, \mathrm{d}v_g \geq \frac{N-1}{4} \int_M \Lambda_{\pi,r}^{rad} u^2 \, \mathrm{d}v_g + \frac{1}{4} \int_M \frac{u^2}{r^2} \, \mathrm{d}v_g$$
$$+ \frac{(N-1)(N-3)}{4} \int_M \frac{u^2}{\psi^2} \, \mathrm{d}v_g,$$

for all $u \in \mathcal{C}_c^\infty(M)$. Therefore, to prove the statement, it is enough to show that $\Lambda_{\pi,r}^{rad} \geq (N-1)$ for all $r > 0$. This follows by using (2.6.3) and (2.6.4), by which we deduce that

$$\Lambda_{\pi,r}^{rad}(r) = \left[2\frac{\psi''(r)}{\psi(r)} + (N-3)\frac{((\psi'(r))^2-1)}{\psi^2(r)}\right] \geq (2 + (N-3)) = N-1$$

for all $r > 0$. $\square$

Proof of Corollary 3.1.2. Let $M = \mathbb{H}^N$, the statement of Corollary 3.1.2 comes from Theorem 3.1.2 by taking $\psi(r) = \sinh(r)$, once proved the sharpness of the constants $\left(\frac{N-1}{2}\right)^2$ and $\frac{(N-1)(N-3)}{4}$. This issue readily follows from the sharpness of the same constants in inequality (1.1.5) otherwise, by Gauss's Lemma, we would get a contradiction. $\square$

3.4 Proof of Theorem 3.1.2

In this section we will prove the Theorem 3.1.2. For any $u \in \mathcal{C}_c^\infty(M)$ we have

$$\Delta_{r,g} u = \frac{\partial^2 u}{\partial r^2} + (N-1)\frac{\psi'}{\psi}\frac{\partial u}{\partial r}\,.$$

Exploiting the spherical harmonic decomposition, we write

$$u(x) := u(r,\sigma) = \sum_{n=0}^{\infty} a_n(r) P_n(\sigma)$$

and we obtain

$$(\Delta_{r,g} u)^2 = \left(\sum_{n=0}^{\infty}\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)P_n\right)^2 \text{ and } \left(\frac{\partial u}{\partial r}\right)^2 = \left(\sum_{n=0}^{\infty} a_n' P_n\right)^2.$$

Recalling that $\{P_n\}$ is orthonormal and by using polar coordinates and integrating by parts, we get

$$\int_M (\Delta_{r,g} u)^2 \, \mathrm{d}v_g - \frac{(N-1)}{4} \int_M \Lambda_{\pi,r}^{rad} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g$$
$$= \omega_N \sum_{n=0}^{\infty} \int_0^{\infty} \left[(a_n'')^2 - (N-1)(a_n')^2 \left(\frac{\psi'}{\psi}\right)' - \frac{(N-1)}{4} \Lambda_{\pi,r}^{rad} \, (a_n')^2 \right] \psi^{N-1} \, \mathrm{d}r \, .$$

Now, exploiting Theorem 3.1.1 for each a_n', we get

$$\int_0^{\infty} (a_n'')^2 \, \psi^{N-1} \, \mathrm{d}r \geq \frac{(N-1)}{4} \int_0^{\infty} \Lambda_{\pi,r}^{rad} (a_n')^2 \, \psi^{N-1} \, \mathrm{d}r$$
$$+ \frac{1}{4} \int_0^{\infty} \frac{(a_n')^2}{r^2} \, \psi^{N-1} \, \mathrm{d}r + \frac{(N-1)(N-3)}{4} \int_0^{\infty} \frac{(a_n')^2}{\psi^2} \, \psi^{N-1} \, \mathrm{d}r$$

and, in turn, we infer

$$\int_M (\Delta_{r,g} u)^2 \, \mathrm{d}v_g - \frac{(N-1)}{4} \int_M \Lambda_{\pi,r}^{rad} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \geq \frac{1}{4} \int_M \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g$$
$$+ \frac{(N^2-1)}{4} \int_M \frac{1}{\psi^2} \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g + (N-1) \int_M \left[K_{\pi,r}^{rad} - H_{\pi,r}^{tan} \right] \left(\frac{\partial u}{\partial r}\right)^2 \mathrm{d}v_g \, .$$

By rearranging we obtain the desired result. □

Proof of Corollary 3.1.3. The desired inequality follows at once by (3.1.7) simply noting that, when (2.6.3) and (3.1.8) holds, then

$$\Lambda_{\pi,r}^{rad} + 4(K_{\pi,r}^{rad} - H_{\pi,r}^{tan}) \geq N-1 \quad \text{for all } r > 0 \, .$$

See also the proof of Corollary 3.1.1. □

Proof of Corollary 3.1.4. When $M = \mathbb{H}^N$ all assumptions of Theorem 3.1.2 are satisfied and since $K_{\pi,r}^{rad} = H_{\pi,r}^{tan} = -1$, the desired inequality follows at once. Concerning the sharpness of the constant $\left(\frac{N-1}{2}\right)^2$, it follows by combining (5.1.1) with (1.1.6) and Gauss's lemma. □

For future purposes, we conclude the section by stating a further inequality that follows directly by combining Theorem 3.1.2 and Corollary 3.1.1:

Corollary 3.4.1. *Let $N \geq 5$ and M as given in* (2.6.1) *satisfying* (2.6.3) *and* (3.1.8)*. Then, for all $u \in \mathcal{C}_c^\infty(M)$ there holds*

$$\begin{aligned}\int_M (\Delta_{r,g}u)^2 \, \mathrm{d}v_g \geq & \left(\frac{N-1}{2}\right)^4 \int_M u^2 \, \mathrm{d}v_g + \frac{(N-1)^2}{16} \int_M \frac{u^2}{r^2} \, \mathrm{d}v_g \\ & + \frac{(N-1)^3(N-3)}{16} \int_M \frac{u^2}{\psi^2} \, \mathrm{d}v_g \\ & + \frac{1}{4} \int_M \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \, \mathrm{d}v_g + \frac{(N^2-1)}{4} \int_M \frac{1}{\psi^2} \left(\frac{\partial u}{\partial r}\right)^2 \, \mathrm{d}v_g \, .\end{aligned} \tag{3.4.1}$$

3.5 Proof of Theorem 3.1.3

A key ingredient in the proof of Theorem 3.1.3 will be Lemma 3.5.1 below which is proved buying the lines of [102, Theorem 5.2] where the same inequality is given in $\mathbb{H}^N$. Here we extend its statement to more general manifolds using Sturm Comparison Principle, i.e. our Lemma 2.6.1.

Lemma 3.5.1. *Let (M, g) be an N-dimensional Riemannian model with metric g as given in* (2.6.1) *with ψ satisfying* (2.6.3)*. Then, for $0 \leq \beta < N-4$, there holds*

$$\int_M (\Delta_g u)^2 r^{-\beta} \, \mathrm{d}v_g \geq \int_M (\Delta_{r,g}u)^2 r^{-\beta} \, \mathrm{d}v_g \quad \text{for all } u \in \mathcal{C}_c^\infty(M). \tag{3.5.1}$$

Moreover, the equality holds when u is a radial function.

Proof. Consider any $u \in \mathcal{C}_c^\infty(M)$, then

$$\Delta_{r,g}u = \frac{\partial^2 u}{\partial r^2} + (N-1)\frac{\psi'}{\psi}\frac{\partial u}{\partial r} \text{ and } \Delta_g u = \frac{\partial^2 u}{\partial r^2} + (N-1)\frac{\psi'}{\psi}\frac{\partial u}{\partial r} + \frac{1}{\psi^2}\Delta_{S^{N-1}} \, .$$

Now, by decomposing into spherical harmonics the above expressions, see the proof of Theorem 3.1.2 for more details, we write

$$u(x) := u(r,\sigma) = \sum_{n=0}^{\infty} a_n(r) P_n(\sigma)\,.$$

From this decomposition of u we have

$$\Delta_{r,g} u = \sum_{n=0}^{\infty} \Delta_g a_n(r) P_n(\sigma)$$

and

$$\Delta_g u = \sum_{n=0}^{\infty} \left(\Delta_g a_n(r) - \lambda_n \frac{a_n(r)}{\psi(r)^2} \right) P_n(\sigma)\,.$$

Hence, to prove (3.5.1) it is enough to show that

$$\lambda_n \int_M \frac{a_n^2}{r^\beta \psi^4}\, \mathrm{d}v_g - 2 \int_M \frac{a_n(\Delta_g a_n)}{r^\beta \psi^2}\, \mathrm{d}v_g \geq 0 \quad \text{for all } n \geq 1.$$

Since we have that $2a_n(\Delta_g a_n) = \Delta_g(a_n^2) - 2|\nabla_g a_n|^2$, using by parts formula, the above inequality is equivalent to

$$\lambda_n \int_M \frac{a_n^2}{r^\beta \psi^4}\, \mathrm{d}v_g - \int_M a_n^2 \Delta_g \left(\frac{1}{r^\beta \psi^2} \right) \mathrm{d}v_g + 2 \int_M \frac{|\nabla_g a_n|^2}{r^\beta \psi^2}\, \mathrm{d}v_g \geq 0 \text{ for all } n \geq 1. \tag{3.5.2}$$

Set $\kappa(r) := \frac{1}{r^\beta \psi^2}$, by exploiting (2.6.3), we have

$$\begin{aligned} -\Delta_g \kappa(r) &= \kappa(r) \left(2\frac{\psi''}{\psi} + 2(N-4)\frac{\psi'^2}{\psi^2} - \frac{\beta(\beta+1)}{r^2} + \beta(N-5)\frac{\psi'}{r\psi} \right) \\ &\geq \kappa(r) \left(2 + 2(N-4)\frac{\psi'^2}{\psi^2} - \frac{\beta(\beta+1)}{r^2} + \beta(N-5)\frac{\psi'}{r\psi} \right). \end{aligned} \tag{3.5.3}$$

We claim that for $f \in \mathcal{C}_c^\infty(M)$ radial there holds

$$\int_M \frac{|\nabla_g f|^2}{r^\beta \psi^2}\, \mathrm{d}v_g \geq \frac{(N-\beta-4)^2}{4} \int_M \frac{f^2}{r^\beta \psi^4}\, \mathrm{d}v_g \tag{3.5.4}$$

which, in polar coordinate, is equivalent to

$$\int_0^\infty f'^2 r^{-\beta} \psi^{N-3}\, \mathrm{d}r \geq \frac{(N-\beta-4)^2}{4} \int_0^\infty f^2 r^{-\beta} \psi^{N-5}\, \mathrm{d}r\,. \tag{3.5.5}$$

Integrating by parts we have the following

$$-\int_0^\infty ff' r^{N-\beta-4}\Big(\frac{\psi}{r}\Big)^{N-5}\,\mathrm{d}r = \frac{(N-\beta-4)}{2}\int_0^\infty f^2 r^{-\beta}\psi^{N-5}\,\mathrm{d}r + \frac{(N-5)}{2}\int_0^\infty f^2 r^{-\beta}\psi^{N-4}\left(\frac{\psi' r-\psi}{\psi^2}\right)\mathrm{d}r\,. \tag{3.5.6}$$

On the other hand, using Young's inequality and Lemma 2.6.1 by which $\frac{r}{\psi}\le 1$ for all $r>0$, we have

$$\begin{aligned}-\int_0^\infty ff'\, r^{N-\beta-4}\Big(\frac{\psi}{r}\Big)^{N-5}\,\mathrm{d}r &= -\int_0^\infty (fr^{-\beta/2}\psi^{(N-5)/2})(f' r^{-\beta/2}\psi^{(N-3)/2}\frac{r}{\psi})\,\mathrm{d}r\\ &\le \frac{(N-\beta-4)}{4}\int_0^\infty f^2 r^{-\beta}\psi^{N-5}\,\mathrm{d}r + \frac{1}{(N-\beta-4)}\int_0^\infty f'^2 r^{-\beta}\psi^{N-3}\,\mathrm{d}r\end{aligned}$$

and, by combining the above inequality with (3.5.6) and recalling that by Lemma 2.6.1 there holds $\left(\frac{\psi' r-\psi}{\psi^2}\right)\ge 0$ for all $r>0$, we obtain (3.5.5).

With the help of (3.5.3) and (3.5.4) into (3.5.2), it is enough to prove

$$\int_M \frac{a_n^2}{r^\beta\psi^4}\bigg[\lambda_n + \frac{(N-\beta-4)^2}{2} + 2\psi^2 + 2(N-4)\psi'^2 - \beta(\beta+1)\frac{\psi^2}{r^2} + \beta(N-5)\frac{\psi'\psi}{r}\bigg]\mathrm{d}v_g \ge 0$$

for all $n\ge 1$, which, in our assumptions, follows by showing that

$$\lambda_n + \frac{(N-\beta-4)^2}{2} + 2\psi^2 + 2(N-4)\psi'^2 - \beta(\beta+1)\frac{\psi^2}{r^2} + \beta(N-5)\frac{\psi'\psi}{r} \ge 0$$

for all $r>0$. Indeed, by (2.6.4) and $(\coth r)\ge \frac{1}{r}$, we can estimate the above term by below as follows

$$\lambda_n + \frac{(N-\beta-4)^2}{2} + 2\psi^2 + \bigg[2(N-4) - \beta(\beta+1) + \beta(N-5)\bigg]\frac{\psi^2}{r^2}\,.$$

Finally, the conclusion comes by noticing that

$$2(N-4) - \beta(\beta+1) + \beta(N-5) \geq 0$$

for all $0 \leq \beta < N-4$. □

Proof of Theorem 3.1.3. Let $u \in \mathcal{C}_c^\infty(M)$, by combining Corollary 3.4.1 and Lemma 3.5.1, we obtain

$$\begin{aligned}\int_M (\Delta_g u)^2 \, dv_g \geq & \left(\frac{N-1}{2}\right)^4 \int_M u^2 \, dv_g + \frac{(N-1)^2}{16} \int_M \frac{u^2}{r^2} \, dv_g \\ & + \frac{(N-1)^3(N-3)}{16} \int_M \frac{u^2}{\psi^2} \, dv_g \\ & + \frac{1}{4} \int_M \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \, dv_g + \frac{(N^2-1)}{4} \int_M \frac{1}{\psi^2} \left(\frac{\partial u}{\partial r}\right)^2 \, dv_g \, .\end{aligned} \tag{3.5.7}$$

Now, from [102, Theorem 3.1], we know that for all $u \in \mathcal{C}_c^\infty(M)$ with $N \geq 5$ there holds

$$\int_M \frac{1}{r^2} \left(\frac{\partial u}{\partial r}\right)^2 \, dv_g \geq \frac{(N-4)^2}{4} \int_M \frac{u^2}{r^4} \, dv_g$$

which, substituted into (3.5.7), gives

$$\begin{aligned}\int_M (\Delta_g u)^2 \, dv_g \geq & \left(\frac{N-1}{2}\right)^4 \int_M u^2 \, dv_g + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_g \\ & + \frac{(N-1)^2}{16} \int_M \frac{u^2}{r^2} \, dv_g + \frac{(N-1)^3(N-3)}{16} \int_M \frac{u^2}{\psi^2} \, dv_g \\ & + \frac{(N^2-1)}{4} \int_M \frac{1}{\psi^2} \left(\frac{\partial u}{\partial r}\right)^2 \, dv_g\end{aligned}$$

and the non-negativity of last two terms immediately gives (3.1.11). This concludes the proof. □

3.6 Proof of Theorem 3.1.4

Let $u \in \mathcal{C}_c^\infty(M)$, by spherical harmonics decomposition we have

$$u(x) := u(r, \sigma) = \sum_{n=0}^{\infty} a_n(r) P_n(\sigma) \, ,$$

see again the proof of Theorem 3.1.2 for more details. Then,

$$|\nabla_g u|^2 = \sum_{n=0}^{\infty} (a_n')^2 P_n^2 + \frac{a_n^2}{\psi^2} |\nabla_{\mathbb{S}^{N-1}} P_n|^2$$

and

$$\begin{aligned}(\Delta_g u)^2 = &\sum_{n=0}^{\infty} \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right)^2 P_n^2 + \sum_{n=0}^{\infty} \frac{a_n^2}{\psi^4} (\Delta_{\mathbb{S}^{N-1}} P_n)^2 \\ &+ 2\sum_{n=0}^{\infty} \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right) \frac{a_n}{\psi^2} (\Delta_{\mathbb{S}^{N-1}} P_n) P_n.\end{aligned}$$

Let us compute the r.h.s. of (3.1.12) in terms of a_n and P_n. Using the fact that $\int_{\mathbb{S}^{N-1}} P_n P_m \, d\sigma = \delta_{nm}$, we obtain

$$\begin{aligned}&\frac{1}{4}\int_M \frac{|\nabla_g u|^2}{r^2} \, \mathrm{d}v_g + \frac{(N^2-1)}{4} \int_M \frac{|\nabla_g u|^2}{\psi^2} \, \mathrm{d}v_g \\ &= \frac{\omega_N}{4} \sum_{n=0}^{\infty} \left[\int_0^{\infty} \frac{(a_n')^2}{r^2} \psi^{N-1} \, \mathrm{d}r + \lambda_n \int_0^{\infty} \frac{a_n^2}{r^2 \psi^2} \psi^{N-1} \, \mathrm{d}r \right. \\ &\left. + (N^2-1) \int_0^{\infty} \frac{(a_n')^2}{\psi^2} \psi^{N-1} \, \mathrm{d}r + (N^2-1)\lambda_n \int_0^{\infty} \frac{a_n^2}{\psi^4} \psi^{N-1} \, \mathrm{d}r \right]. \qquad (3.6.1)\end{aligned}$$

Next we consider the l.h.s. of (3.1.12), we have

$$\begin{aligned}&\int_M (\Delta_g u)^2 \, \mathrm{d}v_g - \left(\frac{N-1}{2}\right)^2 \int_M |\nabla_g u|^2 \, \mathrm{d}v_g \\ &= \omega_N \sum_{n=0}^{\infty} \int_0^{\infty} \left(\left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right)^2 \right) \psi^{N-1} \, \mathrm{d}r \\ &+ \omega_N \int_{\mathbb{S}^{N-1}} \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n^2}{\psi^4} (\Delta_{\mathbb{S}^{N-1}} P_n)^2 \right) \psi^{N-1} \, \mathrm{d}r\mathrm{d}\sigma \\ &+ 2\omega_N \int_{\mathbb{S}^{N-1}} \int_0^{\infty} \left(\sum_{n=0}^{\infty} \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right) \frac{a_n}{\psi^2} (\Delta_{\mathbb{S}^{N-1}} P_n) P_n \right) \psi^{N-1} \, \mathrm{d}r\mathrm{d}\sigma \\ &- \omega_N \left(\frac{N-1}{2}\right)^2 \int_0^{\infty} \left(\sum_{n=0}^{\infty} (a_n')^2 \right) \psi^{N-1} \, \mathrm{d}r \\ &- \omega_N \left(\frac{N-1}{2}\right)^2 \int_0^{\infty} \left(\sum_{n=0}^{\infty} \lambda_n \frac{a_n^2}{\psi^2} \right) \psi^{N-1} \, \mathrm{d}r. \qquad (3.6.2)\end{aligned}$$

We consider each term of the r.h.s. of (3.6.2) separately. First we use

(3.1.7) for each $a_n(r)$ and we get

$$\sum_{n=0}^{\infty}\int_0^{\infty}\left(a_n''+(N-1)\frac{\psi'}{\psi}a_n'\right)^2\psi^{N-1}\,\mathrm{d}r\geq\left(\frac{N-1}{2}\right)^2\sum_{n=0}^{\infty}\int_0^{\infty}(a_n')^2\psi^{N-1}\,\mathrm{d}r$$
$$+\frac{1}{4}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{(a_n')^2}{r^2}\psi^{N-1}\,\mathrm{d}r+\frac{(N^2-1)}{4}\sum_{n=0}^{\infty}\int_0^{\infty}\frac{(a_n')^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r\,. \tag{3.6.3}$$

Then, we exploit the equation $-\Delta_{\mathbb{S}^{N-1}}P_n=\lambda_nP_n$, the orthonormal properties of the $\{P_n\}$ and by parts formula to obtain

$$\int_{\mathbb{S}^{N-1}}\int_0^{\infty}\sum_{n=0}^{\infty}\frac{a_n^2}{\psi^4}(\Delta_{\mathbb{S}^{N-1}}P_n)^2\psi^{N-1}\,\mathrm{d}r\mathrm{d}\sigma=\omega_N\sum_{n=0}^{\infty}\lambda_n^2\int_0^{\infty}\frac{a_n^2}{\psi^4}\psi^{N-1}\,\mathrm{d}r \tag{3.6.4}$$

and

$$2\int_{\mathbb{S}^{N-1}}\int_0^{\infty}\sum_{n=0}^{\infty}\left(a_n''+(N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}(\Delta_{\mathbb{S}^{N-1}}P_n)P_n\psi^{N-1}\,\mathrm{d}r\mathrm{d}\sigma$$
$$=-2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}\left(a_n''+(N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}\psi^{N-1}\,\mathrm{d}r$$
$$=-2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}a_n''\,a_n\psi^{N-3}\,\mathrm{d}r\;-\omega_N(N-1)\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}(a_n^2)'\,\frac{\psi'}{\psi}\psi^{N-3}\,\mathrm{d}r$$
$$=2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}(a_n')^2\psi^{N-3}\,\mathrm{d}r\;-2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}(a_n^2)'\,\frac{\psi'}{\psi}\psi^{N-3}\,\mathrm{d}r$$
$$=2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}(a_n')^2\psi^{N-3}\,\mathrm{d}r\;+2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}a_n^2\left(\frac{\psi''}{\psi}-\frac{(\psi')^2}{\psi^2}\right)\psi^{N-3}\,\mathrm{d}r$$
$$+2(N-3)\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}a_n^2\left(\frac{\psi'}{\psi}\right)^2\psi^{N-3}\,\mathrm{d}r$$
$$=2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}(a_n')^2\psi^{N-3}\,\mathrm{d}r+2\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}a_n^2\frac{\psi''}{\psi}\psi^{N-3}\,\mathrm{d}r$$
$$+2(N-4)\,\omega_N\sum_{n=0}^{\infty}\lambda_n\int_0^{\infty}a_n^2\left(\frac{\psi'}{\psi}\right)^2\psi^{N-3}\,\mathrm{d}r\,. \tag{3.6.5}$$

Now we estimate the first term of (3.6.5) by using (3.1.5) for "$N-2$" dimension. Notice that, to this aim, we need $N-2\geq 3$, i.e. $N\geq 5$. For the remaining two terms we use (2.6.4), (2.6.3), $(\coth r)^2=1+\frac{1}{(\sinh r)^2}\geq 1+\frac{1}{\psi^2}$

and $\frac{1}{r^2} \geq \frac{1}{\psi^2}$ for $r > 0$, to obtain

$$
\begin{aligned}
&2\int_{\mathbb{S}^{N-1}}\int_0^\infty \sum_{n=0}^\infty \left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}(\Delta_{\mathbb{S}^{N-1}}P_n)P_n\psi^{N-1}\,\mathrm{d}r\mathrm{d}\sigma \\
&\geq 2\,\omega_N\frac{(N-3)^2}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty a_n^2\psi^{N-3}\,\mathrm{d}r + 2\,\omega_N\frac{1}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{r^2}\psi^{N-3}\,\mathrm{d}r \\
&+ 2\,\omega_N\frac{(N-3)(N-5)}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{\psi^2}\psi^{N-3}\,\mathrm{d}r \\
&+ 2\,\omega_N(N-3)\sum_{n=0}^\infty \lambda_n\int_0^\infty a_n^2\psi^{N-3}\mathrm{d}r + 2(N-4)\,\omega_N\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{\psi^2}\psi^{N-3}\,\mathrm{d}r \\
&= \omega_N\frac{(N-3)(N+1)}{2}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r + \frac{\omega_N}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{r^2\psi^2}\psi^{N-1}\,\mathrm{d}r \\
&+ \omega_N\frac{(2N^2-8N-1)}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{\psi^4}\psi^{N-1}\,\mathrm{d}r. \qquad (3.6.6)
\end{aligned}
$$

Therefore, taking into account (3.6.3), (3.6.4), (3.6.6) and using the fact that $\lambda_n \geq (N-1) = \lambda_1$ into (3.6.2), we infer that

$$
\begin{aligned}
&\int_M (\Delta_g u)^2\,\mathrm{d}v_g - \left(\frac{N-1}{2}\right)^2\int_M |\nabla_g u|^2\,\mathrm{d}v_g \geq \frac{\omega_N}{4}\sum_{n=0}^\infty\int_0^\infty \frac{(a_n')^2}{r^2}\psi^{N-1}\,\mathrm{d}r \\
&+ \frac{\omega_N}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{r^2\psi^2}\psi^{N-1}\,\mathrm{d}r + \omega_N\frac{(N^2-1)}{4}\sum_{n=0}^\infty\int_0^\infty \frac{(a_n')^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r \\
&+ \omega_N\frac{(2N^2-4N-5)}{4}\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{\psi^4}\psi^{N-1}\,\mathrm{d}r \\
&+ \omega_N\left[\frac{(N-3)(N+1)}{2} - \left(\frac{N-1}{2}\right)^2\right]\sum_{n=0}^\infty \lambda_n\int_0^\infty \frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r.
\end{aligned}
$$

Hence, by noticing that $\left[\frac{(N-3)(N+1)}{2} - \left(\frac{N-1}{2}\right)^2\right] \geq 0$ and $\frac{(2N^2-4N-5)}{4} \geq \frac{(N^2-1)}{4}$ for $N \geq 5$, and by combining the above inequality with (3.6.1), we conclude that

$$
\begin{aligned}
\int_M (\Delta_g u)^2\,\mathrm{d}v_g \geq \left(\frac{N-1}{2}\right)^2\int_M |\nabla_g u|^2\,\mathrm{d}v_g + \frac{1}{4}\int_M \frac{|\nabla_g u|^2}{r^2}\,\mathrm{d}v_g \\
+ \frac{(N^2-1)}{4}\int_M \frac{|\nabla_g u|^2}{\psi^2}\,\mathrm{d}v_g\,,
\end{aligned}
$$

which completes the chapter of the thesis. □

Chapter 4

On Higher order Poincaré Inequalities with radial derivatives and Hardy improvements on the hyperbolic space

In this chapter, we will prove higher-order Poincaré inequalities involving only radial derivatives. Then we will see the sharpness of the constant and in the end, we will see the improvement of these inequalities with Hardy-type remainder terms. The content of this chapter describes the paper [112].

4.1 Statement of main results

The N-dimensional hyperbolic space $\mathbb{H}^N$ admits *Riemannian Model* manifold structure (see Section 2.6) whose metric g is represented in spherical coordinates as (2.6.1) with $\psi(r) = \sinh r$. Let us first recall from (2.7.3) and

(2.7.5), the following two quantities. For $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ we write

$$\Delta_{r,\mathbb{H}^N} u := \frac{\partial^2 u}{\partial r^2} + (N-1)\coth r \frac{\partial u}{\partial r} \quad \text{and} \quad \nabla_{r,\mathbb{H}^N} u := \Big(\frac{\partial u}{\partial r}, 0\Big).$$

These two quantities are so called radial contribution of the Laplace-Beltrami operator and Riemannian gradient in $\mathbb{H}^N$ respectively.

For notational economy we will always use $\Delta_{r,\mathbb{H}^N} = \Delta_r$, $\nabla_{r,\mathbb{H}^N} = \nabla_r$ and finally for any non-negative integer k we denote $\nabla^k_{r,\mathbb{H}^N} = \nabla^k_r$, which is described as

$$\nabla^k_{r,\mathbb{H}^N} := \begin{cases} \Delta^{k/2}_{r,\mathbb{H}^N} & \text{if } k \text{ is even integer,} \\ \nabla_{r,\mathbb{H}^N} \Delta^{(k-1)/2}_{r,\mathbb{H}^N} & \text{if } k \text{ is odd integer.} \end{cases}$$

Before stating the main outcome we want to verify one useful tool in Partial Differential Equation namely, integration by parts formula.

Lemma 4.1.1. *Let f and $g \in \mathcal{C}_c^\infty(\mathbb{H}^N)$. Then it holds*

$$\int_{\mathbb{H}^N} (\Delta_r f)\ g \,\mathrm{d}v_{\mathbb{H}^N} = -\int_{\mathbb{H}^N} (\nabla_r f) \cdot (\nabla_r g) \,\mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} f\ (\Delta_r g) \,\mathrm{d}v_{\mathbb{H}^N}.$$

Proof. Exploiting polar coordinate transformation and by parts formula on first variable i.e., in radial coordinate we deduce

$$\int_{\mathbb{H}^N} (\Delta_r f)\ g \,\mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{S}^{N-1}} \int_0^\infty \left(\frac{\partial^2 f}{\partial r^2} + (N-1)\frac{\psi'}{\psi}\frac{\partial f}{\partial r} \right) \psi^{(N-1)}\ g \,\mathrm{d}r\ \mathrm{d}\sigma$$
$$= -\int_{\mathbb{S}^{N-1}} \int_0^\infty \frac{\partial f}{\partial r}\frac{\partial g}{\partial r} \psi^{(N-1)} \,\mathrm{d}r\ \mathrm{d}\sigma = -\int_{\mathbb{H}^N} (\nabla_r f) \cdot (\nabla_r g) \,\mathrm{d}v_{\mathbb{H}^N}$$
$$= \int_{\mathbb{S}^{N-1}} \int_0^\infty f \frac{\partial^2 g}{\partial r^2} \psi^{(N-1)} \,\mathrm{d}r\ \mathrm{d}\sigma + (N-1) \int_{\mathbb{S}^{N-1}} \int_0^\infty f \frac{\psi'}{\psi}\frac{\partial g}{\partial r} \psi^{(N-1)} \,\mathrm{d}r\ \mathrm{d}\sigma$$
$$= \int_{\mathbb{H}^N} f\ (\Delta_r g) \,\mathrm{d}v_{\mathbb{H}^N}.$$

□

We are now ready to state one of our main result.

Theorem 4.1.1. *For all non-negative integers l and k with $0 \leq l < k$ and for all $N \geq 3$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(k-l)} \int_{\mathbb{H}^N} |\nabla_r^l u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \quad \text{for all } u \in \mathscr{W}^{k,2}(\mathbb{H}^N). \tag{4.1.1}$$

Also the constant $\left(\frac{N-1}{2}\right)^{2(k-l)}$ is optimal in a sense that no inequality of the form

$$\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \Lambda \int_{\mathbb{H}^N} |\nabla_r^l u|^2 \, \mathrm{d}v_{\mathbb{H}^N}$$

holds, for $u \in \mathscr{W}^{k,2}(\mathbb{H}^N)$, when $\Lambda > \left(\frac{N-1}{2}\right)^{2(k-l)}$.

4.2 Proof of Theorem 4.1.1

We divide the proof into three steps. In the first step, we show the existence of the inequality (4.1.1) and in the rest of the two steps, we will tackle the optimality issues.

Step 1. Beginning with $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$, for the case $k = 1$ and $l = 0$, we already have from (3.1.6)

$$\int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N}. \tag{4.2.1}$$

Now we will arrive to the higher order Poincaré inequality in terms of radial derivatives by using Lemma 4.1.1 and Hölder inequality step by step.

$$\begin{aligned}
&\int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} (\nabla_r u) \cdot (\nabla_r u) \, \mathrm{d}v_{\mathbb{H}^N} \\
&= - \int_{\mathbb{H}^N} (\Delta_r u) \, u \, \mathrm{d}v_{\mathbb{H}^N} = - \int_{\mathbb{H}^N} (\nabla_r^2 u) \, u \, \mathrm{d}v_{\mathbb{H}^N} \\
&\leq \left(\int_{\mathbb{H}^N} |\nabla_r^2 u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}} \\
&\leq \frac{2}{(N-1)} \left(\int_{\mathbb{H}^N} |\nabla_r^2 u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}}.
\end{aligned}$$

By arranging these we deduce the case $k = 2$ and $l = 1$, which reads as

$$\int_{\mathbb{H}^N} |\nabla_r^2 u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N}. \tag{4.2.2}$$

Now we are ready to prove the higher order Poincaré inequality in terms of radial derivatives using induction. Suppose k be an even integer with $k \geq 2$, then using (4.2.1) we get

$$\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} |\Delta_r^{k/2} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\leq \left(\frac{N-1}{2}\right)^{-2} \int_{\mathbb{H}^N} |\nabla_r \Delta_r^{k/2} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \left(\frac{N-1}{2}\right)^{-2} \int_{\mathbb{H}^N} |\nabla_r^{k+1} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

Assume k be an odd integer with $k \geq 3$, then exploiting (4.2.2) we have

$$\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} |\nabla_r \Delta_r^{\frac{k-1}{2}} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\leq \left(\frac{N-1}{2}\right)^{-2} \int_{\mathbb{H}^N} |\nabla_r^2 \Delta_r^{\frac{k-1}{2}} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \left(\frac{N-1}{2}\right)^{-2} \int_{\mathbb{H}^N} |\nabla_r^{k+1} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

Finally use of (4.2.1) and (4.2.2) over and over yields the general case

$$\int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \leq \left(\frac{N-1}{2}\right)^{-2} \int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \leq \left(\frac{N-1}{2}\right)^{-4} \int_{\mathbb{H}^N} |\nabla_r^2 u|^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\leq \left(\frac{N-1}{2}\right)^{-6} \int_{\mathbb{H}^N} |\nabla_r^3 u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \leq \cdots \leq \left(\frac{N-1}{2}\right)^{-2k} \int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

Now if we commence with any non-negative integer k and l with $k > l$, then beginning with $\int_{\mathbb{H}^N} |\nabla_r^l u|^2 \, \mathrm{d}v_{\mathbb{H}^N}$ and repeatedly exploiting (4.2.1) and (4.2.2), we will get to $\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N}$ with appropriate constant. At the end density arguments establish the result (4.1.1).

Step 2. In the rest of the section we discuss the optimality issues. The argument runs similar like the proof of sharpness of constant in [100]. Radial behaviour of the operator on a radial function is crucially used here. Let us write the integral representation of the volume of a ball in hyperbolic space $\mathbb{H}^N$ as follows

$$G(r) := N\omega_N \int_0^r (\sinh s)^{N-1}\,\mathrm{d}s, \tag{4.2.3}$$

where ω_N denotes the surface measure of unit sphere $\mathbb{S}^{N-1}$ in the underlying N-dimensional Euclidean space $\mathbb{R}^N$. Observe that this function $G(r) : [0,\infty) \to [0,\infty)$ defines the hyperbolic volume of the ball with center at fixed pole x_o and radius $r = \varrho(x, x_o)$ i.e. $G(r) := \mathrm{Vol}(B(x_o\ ;\ \varrho(x, x_o)))$. Note that $G(r)$ is clearly continuous and strictly increasing function. Next choose $F(r)$ as inverse of $G(r)$ and it's clear that $F(r)$ will be continuous, strictly increasing function and satisfying

$$r = N\omega_N \int_0^{F(r)} (\sinh s)^{N-1}\,\mathrm{d}s \text{ for } r \geq 0. \tag{4.2.4}$$

In the above (4.2.4) using $(\sinh s) \leq (\cosh s)$ and exploiting L'Hospital's rule we deduce for any non-negative real number there hold

$$(N-1)r \leq N\omega_N(\sinh F(r))^{N-1} \text{ and } \lim_{r\to\infty} \frac{N\omega_N(\sinh F(r))^{N-1}}{(N-1)r} = 1.$$

So by the definition of limit we can say that for any $\epsilon > 0$ there exist real number R_0 such that whenever $r \geq R_0$ there holds

$$(N-1)r \leq N\omega_N(\sinh F(r))^{N-1} \leq (1+\epsilon)(N-1)r. \tag{4.2.5}$$

Now for $R > R_0$, let us define the radial function $f_R : [0,\infty) \to [0,\infty)$ as follows

$$f_R(r) := \begin{cases} R_0^{-\frac{1}{2}} & \text{if } r \in [0, R_0), \\ r^{-\frac{1}{2}} & \text{if } r \in [R_0, R), \\ R^{-\frac{1}{2}}\Big(2 - \dfrac{r}{R}\Big) & \text{if } r \in [R, 2R), \\ 0 & \text{if } r \in [2R, \infty). \end{cases}$$

Along with this function we define two more sequences of radial functions $\{v_{R,i}\}_{i\geq 0}$ and $\{g_{R,i}\}_{i\geq 1}$ for $i \geq 0$ in below :
(i) first define $v_{R,0}(r) := f_R(r)$;
(ii) next construct the maximal function $g_{R,i+1} := \frac{1}{r}\int_0^r v_{R,i}(t)\ \mathrm{d}t$;
(iii) finally we set $v_{R,i+1} := \int_r^\infty \frac{t\, g_{R,i+1}(t)}{(N\omega_N(\sinh F(t))^{N-1})^2}\ \mathrm{d}t$.

These two non-increasing functions $v_{R.i}$ and $g_{R,i}$ can be computed explicitly. We are skipping the details here. Without giving the proof, we are mentioning a key lemma which will crucially play an important role here. For details refer to [100, Proposition 2.1].

Lemma 4.2.1. *For any $\epsilon > 0$ and $i \geq 1$. there exist radial functions $h_{R,i}$ and $w_{R,i}$ such that the following holds*
(i) $v_{R,i} = h_{R,i} + w_{R,i}$;
(ii) there exist positive real number C independent of R,
such that $\int_0^\infty |w_{R,i}|^2\ \mathrm{d}s \leq C$;
(iii) and $\frac{1}{(1+\epsilon)^{2i}}\left(\frac{2}{N-1}\right)^{2i} f_R \leq h_{R,i} \leq \left(\frac{2}{N-1}\right)^{2i} f_R$.

Step 3. Let us define the radial function in terms of f_R,

$$u_R(x) := f_R(\mathrm{Vol}(B(x_o\ ;\ \varrho(x,x_o)))).$$

Now we will compute $\int_{\mathbb{H}^N} |u_R|^2\ \mathrm{d}v_{\mathbb{H}^N}$ and $\int_{\mathbb{H}^N} |\nabla_r u_R|^2\ \mathrm{d}v_{\mathbb{H}^N}$ separately and finiteness of those quantities will confirm, $u_R \in \mathscr{W}^{1,2}(\mathbb{H}^N)$. Shifting into polar coordinate and by exploiting change of variable it follows,

$$\int_{\mathbb{H}^N} |u_R(x)|^2\ \mathrm{d}v_{\mathbb{H}^N} = \int_0^\infty \int_{\mathbb{S}^{N-1}} |u_R(r,\sigma)|^2 (\sinh r)^{N-1}\ \mathrm{d}\sigma\ \mathrm{d}r$$
$$= N\omega_N \int_0^\infty (f_R(G(r)))^2 (\sinh r)^{N-1} \mathrm{d}r = \int_0^\infty (f_R(t))^2\ \mathrm{d}t = \ln\left(\frac{R}{R_0}\right) + \frac{4}{3}.$$

Use of (4.2.5) yields

$$\begin{aligned}
\int_{\mathbb{H}^N} |\nabla_r u_R(x)|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \int_0^\infty \int_{\mathbb{S}^{N-1}} \Big|\frac{\partial}{\partial r} u_R(r,\sigma)\Big|^2 (\sinh r)^{N-1} \, \mathrm{d}\sigma \, \mathrm{d}r \\
&= N\omega_N \int_0^\infty (f_R'(G(r))G'(r))^2 (\sinh r)^{N-1} \, \mathrm{d}r \\
&= (N\omega_N)^3 \int_0^\infty (f_R'(G(r)))^2 (\sinh r)^{3(N-1)} \, \mathrm{d}r \\
&= (N\omega_N)^2 \int_0^\infty (f_R'(t))^2 (\sinh F(t))^{2(N-1)} \, \mathrm{d}t \\
&\leq (1+\epsilon)^2 (N-1)^2 \int_0^\infty (f_R'(t))^2 t^2 \, \mathrm{d}t \\
&= \frac{(N-1)^2}{4}(1+\epsilon)^2 \left[\ln\left(\frac{R}{R_0}\right) + \frac{28}{3} \right].
\end{aligned}$$

Next considering the ratios of these two quantities we deduce

$$\begin{aligned}
\inf_{u \in \mathscr{W}^{1,2}(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |u|^2 \, \mathrm{d}v_{\mathbb{H}^N}} &\leq \liminf_{R\to\infty} \frac{\int_{\mathbb{H}^N} |\nabla_r u_R|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |u_R|^2 \, \mathrm{d}v_{\mathbb{H}^N}} \\
&\leq \left(\frac{N-1}{2}\right)^2 (1+\epsilon)^2.
\end{aligned}$$

So from the existence inequality (4.1.1), for the case $k = 1, l = 0$ and letting ϵ towards zero we can conclude $\left(\frac{N-1}{2}\right)^2$ is optimal constant. It's worth noticing that, by the help of *Gauss's Lemma* one can quickly infer that $\left(\frac{N-1}{2}\right)^2$ is the best constant whenever $k = 1$ and $l = 0$ but this method will help us to comment about the optimality of the other higher index cases.

Next we will deal with the case $k = 2$ and $l = 0$. In this context we define

$$u_R(x) := v_{R,1}(\mathrm{Vol}(B(x_o\ ;\ \varrho(x,x_o)))).$$

Due to the radial behaviour of $u_R(x)$ and by the definition of $v_{R,1}$ we can write

$$-\Delta_r u_R = -\Delta_{\mathbb{H}^N} u_R = f_R(\mathrm{Vol}(B(x_o\ ;\ \varrho(x,x_o)))).$$

Now like earlier we have that

$$\int_{\mathbb{H}^N} |\Delta_r u_R|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} |f_R(\mathrm{Vol}(B(x_o\ ;\ \varrho(x,x_o))))|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \ln\left(\frac{R}{R_0}\right) + \frac{4}{3}.$$

By the help of the Lemma 4.2.1, polar coordinate transformation and change of variable we deduce

$$\begin{aligned}\Big(\int_{\mathbb{H}^N} |u_R(x)|^2 \, \mathrm{d}v_{\mathbb{H}^N}\Big)^{\frac{1}{2}} &= \Big(\int_0^\infty |v_{R,1}(r)|^2 \mathrm{d}r\Big)^{\frac{1}{2}} \\ &\geq \Big(\int_0^\infty |h_{R,1}(r)|^2 \mathrm{d}r\Big)^{\frac{1}{2}} - \Big(\int_0^\infty |w_{R,1}(r)|^2 \mathrm{d}r\Big)^{\frac{1}{2}} \\ &\geq \frac{4}{(1+\epsilon)^2 (N-1)^2}\left[\ln\left(\frac{R}{R_0}\right) + \frac{4}{3}\right]^{\frac{1}{2}} - C.\end{aligned}$$

Again this implies

$$\begin{aligned}\inf_{u \in \mathscr{W}^{2,2}(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} |\Delta_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |u|^2 \, \mathrm{d}v_{\mathbb{H}^N}} &\leq \liminf_{R \to \infty} \frac{\int_{\mathbb{H}^N} |\Delta_r u_R|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |u_R|^2 \, \mathrm{d}v_{\mathbb{H}^N}} \\ &\leq \left(\frac{N-1}{2}\right)^4 (1+\epsilon)^4.\end{aligned}$$

So the inequality (4.1.1) and letting ϵ towards zero we obtain $\left(\frac{N-1}{2}\right)^4$ is the best constant for the case $k = 2$ and $l = 0$. Recall that, exploiting [29, Lemma 6.1] we can tackle the optimality issue but once again this method will help to speak about the other sharpness cases.

Now consider the case $k = 2m, l = 0$ and we define

$$u_R(x) := v_{R,m}(\mathrm{Vol}(B(x_o\ ;\ \varrho(x, x_o)))).$$

Again due to the radial nature of the function it is easy to see that

$$(-\Delta_r)^m u_R(x) = (-\Delta_{\mathbb{H}^N})^m u_R(x) = f_R(\mathrm{Vol}(B(x_o\ ;\ \varrho(x, x_o)))).$$

Exploiting Lemma 4.2.1 for the case $i = m$ and running similar argument like earlier we deduce the constant $\left(\frac{N-1}{2}\right)^{4m}$ is best possible.

Now consider the case $k = 2m+1, l = 0$ and if possible assume there exist a constant Θ such that for $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ there holds

$$\Theta \int_{\mathbb{H}^N} |u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \leq \int_{\mathbb{H}^N} |\nabla_r (\Delta_r^m u)|^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

But from earlier evaluation, we know the constant is sharp for the case $k = 2m+2, l = 0$. So using this and inequality (4.1.1) for the case $k = 2, l = 1$ we can write

$$\Theta \int_{\mathbb{H}^N} |u|^2 \, dv_{\mathbb{H}^N} \leq \int_{\mathbb{H}^N} |\nabla_r(\Delta_r^m u)|^2 \, dv_{\mathbb{H}^N} \leq \left(\frac{N-1}{2}\right)^{-2} \int_{\mathbb{H}^N} |\Delta_r^{m+1} u|^2 \, dv_{\mathbb{H}^N}.$$

This implies

$$\Theta \left(\frac{N-1}{2}\right)^2 \leq \left(\frac{N-1}{2}\right)^{2m+4} \implies \Theta \leq \left(\frac{N-1}{2}\right)^{2m+2}.$$

This and density argument proves that, for the case $k = 2m+1$ and $l = 0$, whenever $u \in \mathscr{W}^{2m+1,2}(\mathbb{H}^N)$, the constant $\left(\frac{N-1}{2}\right)^{2m+2}$ is optimum. Hence by the same technique we can prove that, constant $\left(\frac{N-1}{2}\right)^{2(k-l)}$ is sharp for any non-negative integer k and l with $k > l$, whenever $u \in \mathscr{W}^{k,2}(\mathbb{H}^N)$.

Remark 4.2.1. *From the result in Theorem 4.1.1, we can write*

$$\inf_{u \in \mathscr{W}^{k,2}(\mathbb{H}^N) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} |\nabla_r^l u|^2 \, dv_{\mathbb{H}^N}} = \left(\frac{N-1}{2}\right)^{2(k-l)}$$

and for this reason, always strict inequality holds in (4.1.1)*, except* $u = 0$*. So this observation opens the account for improvement of* (4.1.1) *and to support this we proceed to the subsequent sections.*

4.3 Preparatory results for improvement

In this section, we will mainly focus on some useful lemmas which will help to construct improvement of (4.1.1). We want to point out that *Spherical decomposition* (see Section 2.8) is the key method here. The first application of this method will be the establishment of weighted Hardy inequality in terms of radial derivatives. For a similar type of result, one can refer to [28, Theorem 5.1].

Theorem 4.3.1. *Assume that* $0 \leq 2\alpha < (N+3)$. *For all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, *there holds*

$$\int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-2-\alpha)^2}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)}{2} \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)(N-3-2\alpha)}{4} \int_{\mathbb{H}^N} g(r) \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N}, \quad (4.3.1)$$

where $g(r) = \frac{r \coth r - 1}{r^2}$ *is a positive function. Moreover, the constant* $\frac{(N-2-\alpha)^2}{4}$ *is optimal in the obvious sense.*

Proof. Start with $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ and we define

$$v(x) = (\sinh r)^{(N-1)/2} u(x) r^{-\alpha/2} \text{ where } x = (r, \sigma) \in (0, \infty) \times \mathbb{S}^{N-1}.$$

An easy calculation gives

$$\frac{1}{r^{\alpha/2}} \frac{\partial u}{\partial r} = \frac{1}{(\sinh r)^{\frac{(N-1)}{2}}} \left[\frac{\partial v}{\partial r} - \frac{(N-1)}{2} (\coth r) \, v + \frac{\alpha}{2} \frac{v}{r} \right].$$

After squaring the above term, we observe

$$\begin{aligned}
&\int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} \\
&= \int_{\mathbb{H}^N} \frac{1}{(\sinh r)^{(N-1)}} \left(\frac{\partial v}{\partial r} \right)^2 \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)^2}{4} \int_{\mathbb{H}^N} \frac{(\coth r)^2}{(\sinh r)^{(N-1)}} v^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{\alpha^2}{4} \int_{\mathbb{H}^N} \frac{1}{(\sinh r)^{(N-1)}} \frac{v^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} - (N-1) \int_{\mathbb{H}^N} \frac{(\coth r)}{(\sinh r)^{(N-1)}} \frac{\partial v}{\partial r} v \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \alpha \int_M \frac{1}{(\sinh r)^{(N-1)}} \frac{\partial v}{\partial r} \frac{v}{r} \, \mathrm{d}v_{\mathbb{H}^N} - \frac{\alpha(N-1)}{2} \int_M \frac{(\coth r)}{(\sinh r)^{(N-1)}} \frac{v^2}{r} \, \mathrm{d}v_{\mathbb{H}^N}.
\end{aligned}$$

Now expanding v in spherical harmonics $v(x) := v(r, \sigma) = \sum_{n=0}^\infty d_n(r) P_n(\sigma)$, we obtain

$$\int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha}\, \mathrm{d}v_{\mathbb{H}^N} = \sum_{n=0}^{\infty} \Bigg[\int_0^\infty {d'_n}^2 \, \mathrm{d}r + \frac{(N-1)^2}{4} \int_0^\infty (\coth r)^2 \, d_n^2 \, \mathrm{d}r$$
$$+ \frac{\alpha^2}{4} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r - (N-1) \int_0^\infty (\coth r) \, d'_n d_n \, \mathrm{d}r$$
$$+ \alpha \int_0^\infty \frac{d'_n d_n}{r} \, \mathrm{d}r - \frac{\alpha(N-1)}{2} \int_0^\infty (\coth r) \frac{d_n^2}{r} \, \mathrm{d}r \Bigg]$$
$$= \sum_{n=0}^{\infty} \Bigg[\int_0^\infty {d'_n}^2 \, \mathrm{d}r + \frac{(N-1)^2}{4} \int_0^\infty (\coth r)^2 \, d_n^2 \, \mathrm{d}r + \frac{\alpha^2}{4} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r$$
$$- \frac{(N-1)}{2} \int_0^\infty \frac{d_n^2}{(\sinh r)^2} \, \mathrm{d}r + \frac{\alpha}{2} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r - \frac{\alpha(N-1)}{2} \int_0^\infty (\coth r) \frac{d_n^2}{r} \, \mathrm{d}r \Bigg].$$

Observing that

$$\frac{(N-1)^2}{4} \int_0^\infty (\coth r)^2 \, d_n^2 \, \mathrm{d}r - \frac{(N-1)}{2} \int_0^\infty \frac{d_n^2}{(\sinh r)^2} \, \mathrm{d}r$$
$$= \frac{(N-1)^2}{4} \int_0^\infty d_n^2 \, \mathrm{d}r + \frac{(N-1)(N-3)}{4} \int_0^\infty \frac{d_n^2}{(\sinh r)^2} \, \mathrm{d}r,$$

and using 1-dimensional Hardy inequality and $(\coth r) \geq 1/r$, we infer

$$\int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha}\, \mathrm{d}v_{\mathbb{H}^N} \geq \sum_{n=0}^{\infty} \Bigg[\frac{(\alpha+1)^2}{4} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r + \frac{(N-1)^2}{4} \int_0^\infty d_n^2 \, \mathrm{d}r$$
$$+ \frac{(N-1)(N-3)}{4} \int_0^\infty \frac{d_n^2}{(\sinh r)^2} \, \mathrm{d}r - \frac{\alpha(N-1)}{2} \int_0^\infty (\coth r) \frac{d_n^2}{r} \, \mathrm{d}r \Bigg]$$
$$= \sum_{n=0}^{\infty} \Bigg[\frac{(\alpha+1)^2}{4} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r + \left[\frac{(N-1)^2}{4} - \frac{(N-1)(N-3)}{4} \right] \int_0^\infty d_n^2 \, \mathrm{d}r$$
$$+ \frac{(N-1)(N-3)}{4} \int_0^\infty (\coth r)^2 d_n^2 \, \mathrm{d}r - \frac{\alpha(N-1)}{2} \int_0^\infty (\coth r) \frac{d_n^2}{r} \, \mathrm{d}r \Bigg]$$
$$\geq \sum_{n=0}^{\infty} \Bigg[\frac{(\alpha+1)^2}{4} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r + \frac{(N-1)}{2} \int_0^\infty d_n^2 \, \mathrm{d}r$$
$$+ \left[\frac{(N-1)(N-3)}{4} - \frac{\alpha(N-1)}{2} \right] \int_0^\infty (\coth r) \frac{d_n^2}{r} \mathrm{d}r \Bigg]$$
$$= \sum_{n=0}^{\infty} \Bigg[\frac{(N-2-\alpha)^2}{4} \int_0^\infty \frac{d_n^2}{r^2} \, \mathrm{d}r + \frac{(N-1)}{2} \int_0^\infty d_n^2 \, \mathrm{d}r$$
$$+ \frac{(N-1)(N-3-2\alpha)}{4} \int_0^\infty g(r) \, d_n^2 \, \mathrm{d}r \Bigg].$$

Finally writing all the above terms w.r.t. u we establish our desired Theorem 4.3.1. Optimality of the constant $\frac{(N-2-\alpha)^2}{4}$ was already established in [102, Theorem 3.1]. $\square$

Remark 4.3.1. *The coefficient in front of the last term in* (4.3.1) *is negative whenever* $N-3<2\alpha$. *Note that* $g(r)\leq 1/3$ *for every* $r>0$, *we deduce*

$$\frac{(N-1)}{2}+\frac{(N-1)(N-3-2\alpha)}{12}=\frac{(N-1)(N+3-2\alpha)}{12}>0$$

for $N+3>2\alpha$. *Hence, the initial restriction of dimension in* (4.3.1) *is justified. Also note that exploiting Gauss's Lemma in* (4.3.1) *we can obtain different version of weighted Hardy inequality. Another implication of* (4.3.1) *is an immediate improvement of [102, Theorem 3.1] for the case* $p=2$.

By granting, $N-3\geq 2\alpha$ in (4.3.1), one has the following corollary:

Corollary 4.3.1. *Let* $0\leq 2\alpha\leq N-3$. *Then, for all* $u\in \mathcal{C}_c^\infty(\mathbb{H}^N\setminus\{x_o\})$, *there holds*

$$\int_{\mathbb{H}^N}\frac{|\nabla_r u|^2}{r^\alpha}\,\mathrm{d}v_{\mathbb{H}^N}\geq\frac{(N-2-\alpha)^2}{4}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{(N-1)}{2}\int_{\mathbb{H}^N}\frac{u^2}{r^\alpha}\,\mathrm{d}v_{\mathbb{H}^N}. \tag{4.3.2}$$

Furthermore, the constant $\frac{(N-2-\alpha)^2}{4}$ *is sharp in the obvious sense.*

Now we will develop weighted Rellich type inequality with Hardy type remainder terms which is an analogous result of [28, Theorem 5.2]. Before going into detail first recall another important lemma.

Lemma 4.3.1. *For all* $u\in\mathcal{C}_c^\infty(\mathbb{H}^N)$, *there holds* $\Delta_r(u^2)=2u(\Delta_r u)+2|\nabla_r u|^2$.

Proof. This follows from it's own definition and by simple calculation

$$\begin{aligned}\Delta_r(u^2)&=\frac{\partial^2u^2}{\partial r^2}+(N-1)(\coth r)\frac{\partial u^2}{\partial r}=2\frac{\partial}{\partial r}\Big(u\frac{\partial u}{\partial r}\Big)+2(N-1)(\coth r)\frac{\partial u}{\partial r}u\\&=2u\frac{\partial^2 u}{\partial r^2}+2\Big(\frac{\partial u}{\partial r}\Big)^2+2u(N-1)(\coth r)\frac{\partial u}{\partial r}=2u(\Delta_r u)+2|\nabla_r u|^2.\end{aligned}$$

$\square$

Exploiting Lemma 4.3.1 and Theorem 4.3.1, we state a weighted Rellich inequality.

Theorem 4.3.2. *Let α be a positive number and $N > \max\{\alpha+2, 2\alpha-3\}$. For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} \frac{|\Delta_r u|^2}{r^{\alpha-2}} \, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{(N-2-\alpha)(N-2+\alpha)(N-1)}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{(N-1)(N-3-2\alpha)(N-2-\alpha)(N-2+\alpha)}{8} \int_{\mathbb{H}^N} g(r) \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N}, \quad (4.3.3)$$

where $g(r)$ is as defined in (4.3.1). *Moreover, the constant $\frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16}$ is sharp in the obvious sense.*

Proof. This proof mainly relies on the inequality (4.3.1). Notice that, whenever $\alpha > 0$, there holds

$$-\Delta_r \frac{1}{r^\alpha} = -\Delta_{\mathbb{H}^N} \frac{1}{r^\alpha} \geq \frac{\alpha(N-2-\alpha)}{r^{\alpha+2}} \text{ for } r > 0.$$

First we multiply above by u^2 and after that performing by parts formula, Lemma 4.3.1 and Young's inequality with $\epsilon > 0$, we deduce

$$\int_{\mathbb{H}^N} \frac{|\Delta_r u|^2}{r^{\alpha-2}} \, \mathrm{d}v_{\mathbb{H}^N} \geq 2\epsilon \int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} + \left[\epsilon\alpha(N-2-\alpha) - \epsilon^2\right] \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\geq \left[\epsilon\alpha(N-2-\alpha) - \epsilon^2 + \epsilon\frac{(N-2-\alpha)^2}{2}\right] \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \epsilon(N-1) \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} + \epsilon\frac{(N-1)(N-3-2\alpha)}{2} \int_{\mathbb{H}^N} g(r) \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N}.$$

Now the coefficient in front of $\int_{\mathbb{H}^N} u^2/r^{\alpha+2} \, \mathrm{d}v_{\mathbb{H}^N}$ will be maximum when $\epsilon = \frac{(N-2-\alpha)(N-2+\alpha)}{4}$ and substituting this we obtain our required result. Optimality issue of the constant $\frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16}$ was already tackled in [102, Theorem 4.3]. □

Remark 4.3.2. *Exploiting [29, Lemma 6.1] in* (4.3.3) *we can deduce another version of weighted Rellich inequality with Hardy type remainder terms. Also* *it is important to notice that, (4.3.2) gives one more version of immediate improvement of [102, Theorem 4.3] for the case $p = 2$.*

Collecting the conditions in Theorem 4.3.2 and $2\alpha \leq N-3$, one has the following corollary:

Corollary 4.3.2. *Let $0 \leq 2\alpha \leq N-3$. Then for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} \frac{|\Delta_r u|^2}{r^{\alpha-2}} \,\mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \,\mathrm{d}v_{\mathbb{H}^N} + \frac{(N-2-\alpha)(N-2+\alpha)(N-1)}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha} \,\mathrm{d}v_{\mathbb{H}^N}. \quad (4.3.4)$$

Moreover, the constant $\frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16}$ is sharp in the obvious sense.

In the rest of the part, we will construct more weighted Hardy and Rellich type inequalities in terms of radial derivatives. Most of the ideas are taken from [119]. It is worth mentioning that, here we will only discuss the results for the case $p = 2$ but one can verify that, same things hold true in the case of L^p Hardy inequality on $\mathbb{H}^N$, with $p \geq 2$. First, we describe an important lemma below.

Lemma 4.3.2. *Let $N \geq 3$. For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} r^{2-N}|u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \leq 4 \int_{\mathbb{H}^N} r^{2-N}|\nabla_r u|^2 \,\mathrm{d}v_{\mathbb{H}^N}. \quad (4.3.5)$$

Proof. First observe that

$$\frac{[r^{2-N}(\sinh r)^{N-1}]'}{[r^{2-N}(\sinh r)^{N-1}]} = \frac{1}{r} + (N-1)\Big[\coth r - \frac{1}{r}\Big] \geq 1.$$

Indeed, the above inequality holds true. It is easy to see the above inequality is equivalent to the following

$$(N-1)(r\coth r - 1) \geq (r-1).$$

First consider the case, $r \geq 1$, then the above holds true follows from the fact that $N \geq 3$ and $\coth r > 1$, $r > 0$. In the remaining case, i.e., for $0 < r < 1$, inequality holds true follows from the fact that $[\coth r - \frac{1}{r}] \geq 0$ and $\frac{1}{r} > 1$. Then, exploiting by parts formula and Hölder inequality into above, we derive

$$\begin{aligned}
&\int_{\mathbb{H}^N} r^{2-N}|u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{S}^{N-1}} \int_0^\infty r^{2-N}(\sinh r)^{N-1}|u|^2 \, \mathrm{d}r \, \mathrm{d}\sigma \\
&\leq \int_{\mathbb{S}^{N-1}} \int_0^\infty [r^{2-N}(\sinh r)^{N-1}]'|u|^2 \, \mathrm{d}r \, \mathrm{d}\sigma \\
&= -2 \int_{\mathbb{S}^{N-1}} \int_0^\infty r^{2-N}(\sinh r)^{N-1} u \frac{\partial u}{\partial r} \, \mathrm{d}r \, \mathrm{d}\sigma \\
&\leq 2 \bigg(\int_{\mathbb{S}^{N-1}} \int_0^\infty r^{2-N}(\sinh r)^{N-1}|u|^2 \, \mathrm{d}r \, \mathrm{d}\sigma \bigg)^{1/2} \times \\
&\bigg(\int_{\mathbb{S}^{N-1}} \int_0^\infty r^{2-N}(\sinh r)^{N-1}|\nabla_r u|^2 \, \mathrm{d}r \, \mathrm{d}\sigma \bigg)^{1/2}.
\end{aligned}$$

Finally shifting in the original coordinate we get the desired result. □

Let us define the quantity

$$\mu_r(\mathbb{H}^N) = \inf_{u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\}) \setminus \{0\}} \frac{\int_{\mathbb{H}^N} r^{2-N}|\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} r^{2-N}|u|^2 \, \mathrm{d}v_{\mathbb{H}^N}}.$$

In turn of Lemma 4.3.2, we deduce $\mu_r(\mathbb{H}^N) \geq 1/4$. In fact better estimate of $\mu_r(\mathbb{H}^N)$ holds true.

Lemma 4.3.3. *Let $N \geq 3$. Then there holds $\mu_r(\mathbb{H}^N) \geq \frac{N-1}{4}$.*

Proof. Start with the function $u = \big(2\cosh^2(r/2)\big)^{(1-N)/2} v$. After going along with the exactly same estimate in [119, Theorem 5.2], we will deduce the result. Taking the advantage of radial function $\zeta(r) = \big(2\cosh^2(r/2)\big)^{(1-N)/2}$ and exploiting Lemma 4.1.1, we will arrive at the same conclusion. □

Now we are ready to establish the analogous version of [119, Theorem 4.2] and due to Gauss's Lemma, the following theorem comes out as a stronger version of it.

Theorem 4.3.3. *Let $0 \leq \alpha < N-2$ with $N \geq 3$. Then for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-2-\alpha)^2}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N}, \tag{4.3.6}$$

where the constant $\frac{(N-2-\alpha)^2}{4}$ is sharp in the obvious sense.

Proof. Start with the substitution $u = r^{(2+\alpha-N)/2} v$ and from a simple calculation, we have that

$$|\nabla_r u|^2 = \left(\frac{2+\alpha-N}{2}\right)^2 \left(\frac{u}{r}\right)^2 + r^{2+\alpha-N} |\nabla_r v|^2 + 2\left(\frac{2+\alpha-N}{2}\right) r^{1+\alpha-N} v \frac{\partial v}{\partial r}.$$

Before performing integration, first multiply above by $1/r^\alpha$ and we obtain

$$\begin{aligned} \int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} = {} & \left(\frac{N-2-\alpha}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}} \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} r^{2-N} |\nabla_r v|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ & - (N-\alpha-2) \int_{\mathbb{H}^N} r^{1-N} v \frac{\partial v}{\partial r} \, \mathrm{d}v_{\mathbb{H}^N}. \end{aligned}$$

Transferring into polar coordinate and using by parts rule, we deduce

$$\begin{aligned} & - (N-\alpha-2) \int_{\mathbb{H}^N} r^{1-N} v \frac{\partial v}{\partial r} \, \mathrm{d}v_{\mathbb{H}^N} \\ & = \frac{(N-\alpha-2)(N-1)}{2} \int_{\mathbb{H}^N} r^{1-N} v^2 \left[\coth r - \frac{1}{r}\right] \mathrm{d}v_{\mathbb{H}^N}. \end{aligned}$$

Exploiting Taylor series expansion of $\cosh r$ and $\sinh r$ near origin, for $0 < r \leq 1$, we deduce

$$\begin{aligned} (\coth r - \frac{1}{r}) & = \frac{1}{r \sinh r}\Big(r \cosh r - \sinh r\Big) \\ & = \frac{1}{r \sinh r}\left(r \sum_{n=0}^\infty \frac{r^{2n}}{(2n)!} - \sum_{n=0}^\infty \frac{r^{2n+1}}{(2n+1)!}\right) \\ & \geq \frac{1}{r \sinh r} \cdot \frac{r^3}{3} \\ & \geq \frac{r}{3 \sinh 1}. \end{aligned}$$

The last inequality follows from the fact that, the function $g(r) := r\sinh 1 - \sinh r$ is in $\mathcal{C}^2([0,1])$ and concave in $[0,1]$ with zero on the boundary and hence $g(r) \geq 0$ whenever $0 < r \leq 1$. By the above estimate along with the Lemma 4.3.3 and getting back into the form of u, we derive

$$\int_{\mathbb{H}^N} \frac{|\nabla_r u|^2}{r^\alpha}\, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-2-\alpha}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}}\, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{(N-1)}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha}\, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-\alpha-2)(N-1)}{6\sinh 1} \int_{B(o;1)} \frac{u^2}{r^\alpha}\, \mathrm{d}v_{\mathbb{H}^N}.$$

In the end, non-negativity of the last term immediately gives (4.3.3). $\square$

Taking Theorem 4.3.3 as weighted Hardy inequality and adopting the similar technique exploited in Theorem 4.3.2, one has the following version of weighted Rellich inequality.

Corollary 4.3.3. *Let $0 \leq \alpha < N-2$ with $N \geq 3$. Then for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\int_{\mathbb{H}^N} \frac{|\Delta_r u|^2}{r^{\alpha-2}}\, \mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+2}}\, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{(N-2-\alpha)(N-2+\alpha)(N-1)}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^\alpha}\, \mathrm{d}v_{\mathbb{H}^N}. \quad (4.3.7)$$

Moreover, the constant $\frac{(N-2-\alpha)^2(N-2+\alpha)^2}{16}$ is sharp in the obvious sense.

Observing into both the weighted Rellich inequalities (4.3.4) and (4.3.7), one can wonder, whether more better improvement possible near origin, precisely can we add one more Hardy type remainder term namely, $\int_{\mathbb{H}^N} u^2/r^{\alpha-2}\, \mathrm{d}v_{\mathbb{H}^N}$. To give affirmative answer of this question, first we develop the following lemma.

Lemma 4.3.4. *Let $-2 \leq \alpha < N-4$ and $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$. Then there* *holds*

$$\int_{\mathbb{H}^N} r^{-\alpha}|\Delta_r u + \frac{(N+\alpha)(N-\alpha-4)}{4}\frac{u}{r^2}|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$\leq \int_{\mathbb{H}^N} \frac{|\Delta_r u|^2}{r^\alpha}\,\mathrm{d}v_{\mathbb{H}^N} - \frac{(N+\alpha)(N-\alpha-4)}{2}\int_{\mathbb{H}^N}\frac{|\nabla_r u|^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{(N+\alpha)(N-3\alpha-8)(N-\alpha-4)^2}{16}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+4}}\,\mathrm{d}v_{\mathbb{H}^N}.$$

Proof. Exploiting $(\coth r) \geq 1/r$, we deduce

$$\Delta_r(r^{-\alpha-2}) \leq (\alpha+2)(\alpha+4-N)r^{-\alpha-4}.$$

Applying by parts formula and Lemma 4.3.1, in the above inequality, we infer

$$\int_{\mathbb{H}^N}\frac{u\Delta_r u}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N} \leq -\frac{(\alpha+2)(N-\alpha-4)}{2}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+4}}\,\mathrm{d}v_{\mathbb{H}^N} - \int_{\mathbb{H}^N}\frac{|\nabla_r u|^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}. \tag{4.3.8}$$

Finally, the conclusion comes by noting that

$$\int_{\mathbb{H}^N} r^{-\alpha}|\Delta_r u + \frac{(N+\alpha)(N-\alpha-4)}{4}\frac{u}{r^2}|^2 \,\mathrm{d}v_{\mathbb{H}^N}$$
$$= \int_{\mathbb{H}^N} \frac{|\Delta_r u|^2}{r^\alpha}\,\mathrm{d}v_{\mathbb{H}^N} + \frac{(N+\alpha)^2(N-\alpha-4)^2}{16}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+4}}\,\mathrm{d}v_{\mathbb{H}^N}$$
$$+ \frac{(N+\alpha)(N-\alpha-4)}{2}\int_{\mathbb{H}^N}\frac{u\Delta_r u}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}.$$

□

Now using Lemma 4.3.4 and weighted Hardy inequality (4.3.6), we obtain the following weighted Rellich type inequality. Also note that, exploiting [29, Lemma 6.1], this version will become stronger than [119, Theorem 4.4] and will give a quick improvement of [102, Theorem 4.3], for the case $p = 2$.

Theorem 4.3.4. *Let $0 \leq \alpha < N-4$. Then for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N}\frac{|\Delta_r u|^2}{r^\alpha}\,\mathrm{d}v_{\mathbb{H}^N} \geq \frac{(N+\alpha)^2(N-4-\alpha)^2}{16}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+4}}\,\mathrm{d}v_{\mathbb{H}^N} \tag{4.3.9}$$
$$+ \frac{(N-1)(N-2-\alpha)(N-2+\alpha)}{8}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)^2}{16}\int_{\mathbb{H}^N}\frac{u^2}{r^\alpha}\,\mathrm{d}v_{\mathbb{H}^N}.$$

Moreover, the constant $\frac{(N+\alpha)^2(N-4-\alpha)^2}{16}$ *is sharp in the obvious sense.*

Proof. Replacing the index α by $\alpha-2$ in (4.3.8) and then substituting (4.3.6) into it, we deduce

$$(\alpha+1)\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{(N-1)}{4}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha}}\,\mathrm{d}v_{\mathbb{H}^N}$$
$$\leq\int_{\mathbb{H}^N}\frac{u}{r^{\alpha}}\left[-\Delta_r u-\frac{(N+\alpha)(N-\alpha-4)}{4}\frac{u}{r^2}\right]\mathrm{d}v_{\mathbb{H}^N}.$$

Now we estimate the last term by Young's inequality with $\epsilon>0$ and taking $a=\left|\frac{u}{r^{\alpha/2}}\right|$ and $b=\left|r^{-\alpha/2}\left[-\Delta_r u-\frac{(N+\alpha)(N-\alpha-4)}{4}\frac{u}{r^2}\right]\right|$, we obtain

$$2\epsilon(\alpha+1)\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{\epsilon(N-1-2\epsilon)}{2}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha}}\,\mathrm{d}v_{\mathbb{H}^N}$$
$$\leq\int_{\mathbb{H}^N}r^{-\alpha}|\Delta_r u+\frac{(N+\alpha)(N-\alpha-4)}{4}\frac{u}{r^2}|^2\,\mathrm{d}v_{\mathbb{H}^N}.$$

Next we exploit the information that, function $f(\epsilon)=\epsilon(N-1-2\epsilon)/2$ attains maximum when $\epsilon=(N-1)/4$. Finally, applying Lemma 4.3.4 and Theorem 4.3.3, in the form of changed index α by $\alpha+2$, we achieve our desired result. □

We establish Theorem 4.3.4, using Theorem 4.3.3 and Lemma 4.3.4. On the other hand, in a similar way, we can deduce Corollary 4.3.4 using Corollary 4.3.1 instead of Theorem 4.3.3 in the proof of Theorem 4.3.4.

Corollary 4.3.4. *Let* $0\leq 2\alpha\leq N-7$. *Then for all* $u\in C_c^\infty(\mathbb{H}^N\setminus\{x_o\})$, *there holds*

$$\int_{\mathbb{H}^N}\frac{|\Delta_r u|^2}{r^{\alpha}}\,\mathrm{d}v_{\mathbb{H}^N}\geq\frac{(N+\alpha)^2(N-4-\alpha)^2}{16}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+4}}\,\mathrm{d}v_{\mathbb{H}^N} \tag{4.3.10}$$
$$+\frac{(N-1)(N-2-\alpha)(N-2+\alpha)}{4}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha+2}}\,\mathrm{d}v_{\mathbb{H}^N}+\frac{(N-1)^2}{4}\int_{\mathbb{H}^N}\frac{u^2}{r^{\alpha}}\,\mathrm{d}v_{\mathbb{H}^N}.$$

Moreover, the constant $\frac{(N+\alpha)^2(N-4-\alpha)^2}{16}$ *is sharp in the obvious sense.*

Remark 4.3.3. *If we compare both the weighted Hardy inequalities in Corollary 4.3.1 and Theorem 4.3.3, one can observe that coefficient in front of*

$\int_{\mathbb{H}^N} u^2/r^\alpha \, \mathrm{d}v_{\mathbb{H}^N}$ *in the equation* (4.3.2) *is better than* (4.3.6). *But it's also important to notice that,* (4.3.2) *demands larger dimension restriction than* (4.3.6). *Analogous observation also holds true for the case of weighted Rellich inequalities in Corollary 4.3.4 and Theorem 4.3.4. Moreover, for both the cases we are getting one instance, where* $\mu_r(\mathbb{H}^N) > (N-1)/4$ *is possible.*

Iterating inequality (4.3.9), we obtain the following improved weighted Rellich inequality on higher order radial derivation on $\mathbb{H}^N$ and this result will be used many times in the last part of the chapter.

Lemma 4.3.5. *Let* β *be a positive integer, which satisfy* $0 \leq \alpha < N - 4\beta$. *Then there exist positive constants* $\Xi^j_{\alpha,\beta}$, *for* $j = 0$ *to* 2β, *such that for all* $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$, *there holds*

$$\int_{\mathbb{H}^N} \frac{(\Delta_r^\beta u)^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} \geq \sum_{j=0}^{2\beta} \Xi^j_{\alpha,\beta} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+4\beta-2j}} \, \mathrm{d}v_{\mathbb{H}^N}. \tag{4.3.11}$$

Moreover, the coefficient corresponding to the leading terms namely $\Xi^0_{\alpha,\beta}$ *and* $\Xi^{2\beta}_{\alpha,\beta}$, *for* $r \to 0$ *and* $r \to \infty$ *respectively, can be explicitly given by as follows*

$$\Xi^0_{\alpha,\beta} = \prod_{j=0}^{\beta-1} \frac{(N + (\alpha+4j))^2 (N - (\alpha+4j) - 4)^2}{16}$$

and

$$\Xi^{2\beta}_{\alpha,\beta} = \left(\frac{N-1}{4}\right)^{2\beta} \quad \textit{for } \beta \geq 1 \textit{ and } \alpha \geq 0.$$

Finally after iterating (4.3.10), we deduce the following result but with a different initial condition.

Lemma 4.3.6. *Let* β *be a positive integer, which satisfy* $0 \leq 2\alpha \leq N - 8\beta + 1$. *Then there exist positive constants* $\zeta^j_{\alpha,\beta}$, *for* $j = 0$ *to* 2β, *such that for all* $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$, *there holds*

$$\int_{\mathbb{H}^N} \frac{(\Delta_r^\beta u)^2}{r^\alpha} \, \mathrm{d}v_{\mathbb{H}^N} \geq \sum_{j=0}^{2\beta} \zeta^j_{\alpha,\beta} \int_{\mathbb{H}^N} \frac{u^2}{r^{\alpha+4\beta-2j}} \, \mathrm{d}v_{\mathbb{H}^N}, \tag{4.3.12}$$

where $\zeta^0_{\alpha,\beta} = \Xi^0_{\alpha,\beta}$ *and* $\zeta^{2\beta}_{\alpha,\beta} = 4^\beta \, \Xi^{2\beta}_{\alpha,\beta}$, *for* $\beta \geq 1$ *and* $\alpha \geq 0$.

In the rest of the chapter for notational convention we will be assuming $\Xi^0_{n,0} = 1$ and $\zeta^0_{n,0} = 1$, for every integer n. Also we will assume $\sum_{j=m}^{n} = 0$ and $\prod_{j=m}^{n} = 1$, whenever integers satisfy $n < m$.

4.4 Improvement of Higher order radial Poincaré inequalities

This section is devoted to the proof of (1.2.5). In the same spirit to explore further in l.h.s. of (3.1.10), exploiting (4.3.6) for the case of $\alpha = 2$ into it, we deduce with a different constant than [26, Theorem 2.1] that, for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ and $N \geq 5$, there holds,

$$\int_{\mathbb{H}^N} |\Delta_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}. \quad (4.4.1)$$

Furthermore, using (3.1.6) in (4.4.1), we obtain for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ with $N \geq 5$ there holds

$$\int_{\mathbb{H}^N} |\Delta_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N(N-1)}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}. \quad (4.4.2)$$

Indeed, all these lower order improvements can be lifted into the general higher order indices scenario. In particular, applying these lower order indices results and induction we will approach towards the development of the result (1.2.5). In the coming part, we will mainly rely on the Lemma 4.3.5 and Lemma 4.3.6. We divide this section into a couple of subsections to cover up all the possible higher order indices k, l and side by side we will explicitly calculate the coefficients corresponding to the asymptotic terms $r \to 0$ and $r \to \infty$ also.

General integer k and $l = 0$. This part is divided into two steps based on the situation k is odd or even. First we state the results and after that we will give the details of the proof.

Theorem 4.4.1. *Let k be a positive integer and $N > 2k$. Then there exist k positive constants $C^i_{k,0}$ such that the following inequality holds*

$$\int_{\mathbb{H}^N} |\nabla_r^k u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{N-1}{2}\right)^{2k} \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \sum_{i=1}^{k} C^i_{k,0} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}, \quad (4.4.3)$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$. Moreover, the leading terms are explicitly given by

$$C^k_{k,0} = \begin{cases} \left(\frac{N-4}{2^{2m}}\right)^2 \displaystyle\prod_{j=1}^{m-1} (N+4j)^2(N-4j-4)^2 & \text{if } k = 2m, \\ \frac{1}{2^{4m+2}} \displaystyle\prod_{j=1}^{m} (N+4j-2)^2(N-4j-2)^2 & \text{if } k = 2m+1, \end{cases}$$

and

$$C^1_{k,0} = \begin{cases} \frac{N(N-1)}{2^{4m}} \displaystyle\sum_{j=1}^{m} (N-1)^{4m-2j-2} & \text{if } k = 2m, \\ \frac{N(N-1)}{2^{4m+2}} \displaystyle\sum_{j=1}^{m} (N-1)^{2m+2j-2} + \frac{(N-1)^{2m}}{2^{4m+2}} & \text{if } k = 2m+1. \end{cases}$$

Proof. Suppose $k = 2m$ even, we will apply induction m. For the basic step we already have the result in (4.4.2). Now assume it holds true for the case $k = 2m - 2 \geq 2$, which describes that for $N > 4m - 4$, there holds

$$\begin{aligned} &\int_{\mathbb{H}^N} (\Delta_r^{m-1} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{N-1}{2}\right)^{4m-4} \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &\geq \frac{N(N-1)}{2^{4m-4}} \sum_{j=1}^{m-1} (N-1)^{4m-2j-6} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{i=2}^{2m-3} C^i_{2m-2,0} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \left(\frac{N-4}{2^{2m-2}}\right)^2 \prod_{j=1}^{m-2} (N+4j)^2(N-4j-4)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^{4m-4}} \, \mathrm{d}v_{\mathbb{H}^N}. \end{aligned}$$

Next we will establish the inductive step and so starting with $N > 4m$, exploiting induction hypothesis above, (4.4.2) and (4.3.11), we deduce

$$\begin{aligned}
&\int_{\mathbb{H}^N} |\Delta_r^m u|^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} |\Delta_r^{m-1}(\Delta_r u)|^2 \, dv_{\mathbb{H}^N} \\
&\geq \left(\frac{N-1}{2}\right)^{4m-4} \int_{\mathbb{H}^N} (\Delta_r u)^2 \, dv_{\mathbb{H}^N} + \sum_{i=2}^{2m-3} C^i_{2m-2,0} \int_{\mathbb{H}^N} \frac{(\Delta_r u)^2}{r^{2i}} \, dv_{\mathbb{H}^N} \\
&+ \frac{N(N-1)}{2^{4m-4}} \sum_{j=1}^{m-1} (N-1)^{4m-2j-6} \int_{\mathbb{H}^N} \frac{(\Delta_r u)^2}{r^2} \, dv_{\mathbb{H}^N} \\
&+ \left(\frac{N-4}{2^{2m-2}}\right)^2 \prod_{j=1}^{m-2} (N+4j)^2 (N-4j-4)^2 \int_{\mathbb{H}^N} \frac{(\Delta_r u)^2}{r^{4m-4}} \, dv_{\mathbb{H}^N} \\
&\geq \left(\frac{N-1}{2}\right)^{4m-4} \left[\left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N} \right. \\
&\left. + \frac{N(N-1)}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} \right] + \sum_{i=2}^{2m-3} C^i_{2m-2,0} \int_{\mathbb{H}^N} \frac{(\Delta_r u)^2}{r^{2i}} \, dv_{\mathbb{H}^N} \\
&+ \frac{N(N-1)}{2^{4m-4}} \sum_{j=1}^{m-1} (N-1)^{4m-2j-6} \left[\sum_{\gamma=0}^{2} \Xi^\gamma_{2,1} \int_{\mathbb{H}^N} \frac{u^2}{r^{6-2\gamma}} \, dv_{\mathbb{H}^N} \right] \\
&+ \left(\frac{N-4}{2^{2m-2}}\right)^2 \prod_{j=1}^{m-2} (N+4j)^2 (N-4j-4)^2 \left[\sum_{\gamma=0}^{2} \Xi^\gamma_{4m-4,1} \int_{\mathbb{H}^N} \frac{u^2}{r^{4m-2\gamma}} \, dv_{\mathbb{H}^N} \right] \\
&= \left(\frac{N-1}{2}\right)^{4m} \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2m} C^i_{2m,0} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N}.
\end{aligned}$$

Substituting the value of $\Xi^2_{2,1}$ and in the end changing index from j to $j-1$, we obtain

$$\begin{aligned}
C^1_{2m,0} &= \frac{N(N-1)}{16} \left(\frac{N-1}{2}\right)^{4m-4} + \frac{N(N-1)}{2^{4m-4}} \sum_{j=1}^{m-1} (N-1)^{4m-2j-6} \, \Xi^2_{2,1} \\
&= \frac{N(N-1)}{2^{4m}} \sum_{j=0}^{m-1} (N-1)^{4m-2j-4} = \frac{N(N-1)}{2^{4m}} \sum_{j=1}^{m} (N-1)^{4m-2j-2}
\end{aligned}$$

and finally arranging the terms after plugging in the value of $\Xi^0_{4m-4,1}$, we deduce

$$\begin{aligned}
C^{2m}_{2m,0} &= \left(\frac{N-4}{2^{2m-2}}\right)^2 \prod_{j=1}^{m-2} (N+4j)^2 (N-4j-4)^2 \, \Xi^0_{4m-4,1} \\
&= \left(\frac{N-4}{2^{2m}}\right)^2 \prod_{j=1}^{m-1} (N+4j)^2 (N-4j-4)^2.
\end{aligned}$$

This gives that inequality holds for $k = 2m$ and completes the induction. Next we turn to the case $k = 2m + 1$ odd with the same idea to argue by induction on m. Notice, if $m = 0$, (4.4.3) follows directly from (3.1.6) with $C^1_{1,0} = 1/4$. Next in a similar manner, assuming result is true for $k = 2m-1 \geq 1$, we can extend it for the case $k = 2m+1$, by applying Lemma 4.3.5 and (4.4.2) suitably. For the brevity we are skipping the details. □

Remark 4.4.1. *Using Corollary 4.3.1 for $\alpha = 2$ in* (3.1.10) *we deduce that for $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$, with $N \geq 7$ there holds*

$$\int_{\mathbb{H}^N} |\Delta_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)}{8} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}. \tag{4.4.4}$$

Remark 4.4.2. *Exploiting* (3.1.6) *in the above inequality* (4.4.4)*, we deduce for all $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$, with $N \geq 7$ there holds*

$$\int_{\mathbb{H}^N} |\Delta_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N^2-1)}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}. \tag{4.4.5}$$

If we compare (4.4.4) *and* (4.4.5) *with the inequalities* (4.4.1) *and* (4.4.2) *respectively, then it is easy to observe that inequalities in the first case perform better when r approaching towards zero. In particular, this creates another interesting fact that if we compare* (4.4.5)*, after applying [29, Lemma 6.1] suitably, with [29, Theorem 2.3] in the manifold $M = \mathbb{H}^N$ with $N \geq 7$, then the constant appearing in front of the Hardy term $\frac{1}{r^2}$ can be larger than $\frac{(N-1)^2}{16}$ as proved in [29], keeping the constant in front of Rellich term unchanged. Also, we notice that unfortunately in both cases finding the best possible constant is still an open question.*

If we use above inequality (4.4.5), Lemma 4.3.6 and (3.1.6), then we will be obtaining the following corollary, where the constants are larger than (4.4.3) but demand more dimensional restriction.

Corollary 4.4.1. *Let k be a positive integer and $N \geq 4k-1$. Then there exist k positive constants $D^i_{k,0}$ such that the following inequality holds*

$$\int_{\mathbb{H}^N} |\nabla^k_r u|^2 \, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{N-1}{2}\right)^{2k} \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \sum_{i=1}^{k} D^i_{k,0} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}, \quad (4.4.6)$$

for all $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$. Moreover, the leading terms are given by $D^k_{k,0} = C^k_{k,0}$ and

$$D^1_{k,0} = \begin{cases} \dfrac{(N^2-1)}{16} \displaystyle\sum_{j=1}^{m} \left(\frac{N-1}{2}\right)^{4m-2j-2} & \text{if } k = 2m, \\ (N^2-1) \displaystyle\sum_{j=1}^{m} \frac{(N-1)^{2m+2j-2}}{2^{2m+2j+2}} + \frac{(N-1)^{2m}}{2^{2m+2}} & \text{if } k = 2m+1. \end{cases}$$

General case $k = 2m$ even and $l = 2h$ even. Here we will discuss the case when both index are even.

Theorem 4.4.2. *Let $k = 2m > l = 2h \geq 0$ be integers and $N > 2k$. Then there exist k positive constants $C^i_{k,l}$ such that for all $u \in \mathcal{C}^\infty_c(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} (\Delta^m_r u)^2 \, \mathrm{d}v_{\mathbb{H}^N} - \left(\frac{N-1}{2}\right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta^h_r u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \sum_{i=1}^{k} C^i_{k,l} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}, \quad (4.4.7)$$

where $C^k_{k,l} = C^{k-l}_{k-l,0} \; \Xi^0_{2(k-l),l/2}$ and $C^1_{k,l} = C^1_{k-l,0} \; \Xi^l_{2,l/2}$.

Proof. By applying first (4.4.3) with $k = 2(m-h)$, then (4.3.5) with $\alpha = 2i$

and $\beta = h$, we deduce

$$\int_{\mathbb{H}^N} (\Delta_r^m u)^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} (\Delta_r^{m-h}(\Delta_r^h u))^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\geq \left(\frac{N-1}{2}\right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{i=1}^{2(m-h)} C^i_{2(m-h),0} \int_{\mathbb{H}^N} \frac{(\Delta_r^h u)^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\geq \left(\frac{N-1}{2}\right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \sum_{i=1}^{2(m-h)} C^i_{2(m-h),0} \left[\sum_{j=0}^{2h} \Xi^j_{2i,h} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i+4h-2j}} \, \mathrm{d}v_{\mathbb{H}^N} \right]$$
$$= \left(\frac{N-1}{2}\right)^{4(m-h)} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{i=1}^{2m} C^i_{2m,2h} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}$$

with $C^1_{2m,2h} = C^1_{2(m-h),0} \ \Xi^{2h}_{2,h}$ and $C^{2m}_{2m,2h} = C^{2(m-h)}_{2(m-h),0} \ \Xi^0_{4(m-h),h}$. □

General case $k = 2m+1$ odd and $l = 2h$ even. Here one index is odd and other is even.

Theorem 4.4.3. *Let $k = 2m+1 > l = 2h \geq 0$ be integers and $N > 2k$. Then there exist k positive constants $C^i_{k,l}$ such that for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} |\nabla_r(\Delta_r^m u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{4(m-h)+2} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \sum_{i=1}^{k} C^i_{k,l} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}, \tag{4.4.8}$$

where $C^k_{k,l} = C^{k-l}_{k-l,0} \ \Xi^0_{2(k-l),l/2}$ and $C^1_{k,l} = C^1_{k-l,0} \ \Xi^l_{2,l/2}$.

Proof. Exploiting (4.4.3) with $k = 2(m-h)+1$ and Lemma 4.3.5 for $\alpha = 2i$

and $\beta = h$, we obtain

$$\int_{\mathbb{H}^N} |\nabla_r(\Delta_r^m u)|^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} |\nabla_r^{2(m-h)+1}(\Delta_r^h u)|^2 \, dv_{\mathbb{H}^N}$$
$$\geq \left(\frac{N-1}{2}\right)^{4(m-h)+2} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2(m-h)+1} C^i_{2(m-h)+1,0} \int_{\mathbb{H}^N} \frac{(\Delta_r^h u)^2}{r^{2i}} \, dv_{\mathbb{H}^N}$$
$$\geq \left(\frac{N-1}{2}\right)^{4(m-h)+2} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, dv_{\mathbb{H}^N}$$
$$+ \sum_{i=1}^{2(m-h)+1} C^i_{2(m-h)+1,0} \left[\sum_{j=0}^{2h} \Xi^j_{2i,h} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i+4h-2j}} \, dv_{\mathbb{H}^N} \right]$$
$$= \left(\frac{N-1}{2}\right)^{4(m-h)+2} \int_{\mathbb{H}^N} (\Delta_r^h u)^2 \, dv_{\mathbb{H}^N} + \sum_{i=1}^{2m+1} C^i_{2m+1,2h} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N}$$

with $C^{2m+1}_{2m+1,2h} = C^{2(m-h)+1}_{2(m-h)+1,0} \, \Xi^0_{4(m-h)+2,h}$ and $C^1_{2m+1,2h} = C^1_{2(m-h)+1,0} \, \Xi^{2h}_{2,h}$. □

General case $k = 2m$ even and $l = 2h+1$ odd. Here is the next case.

Theorem 4.4.4. *Let $k = 2m > l = 2h+1 \geq 1$ be integers and $N > 2k$. Then there exist k positive constants $C^i_{k,l}$ such that for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} (\Delta_r^m u)^2 \, dv_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{4(m-h)-2} \int_{\mathbb{H}^N} |\nabla_r(\Delta_r^h u)|^2 \, dv_{\mathbb{H}^N}$$
$$+ \sum_{i=1}^{k} C^i_{k,l} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, dv_{\mathbb{H}^N}, \tag{4.4.9}$$

where

$$C^1_{k,l} = \begin{cases} \dfrac{(N-1)^{2k-2l-1}}{2^{2k-2l+2}} \, \Xi^{l-1}_{2,(l-1)/2} + C^1_{k-l-1,0} \, \Xi^{l+1}_{2,(l+1)/2} & \text{if } k = 2m, l = 2h+1 \\ & \text{where } m-h \neq 1, \\ \dfrac{(N-1)}{16} \, \Xi^{l-1}_{2,(l-1)/2} & \text{if } k = 2h+2 \\ & \text{and } l = 2h+1, \end{cases}$$

and

$$C^k_{k,l} = \begin{cases} C^{k-l-1}_{k-l-1,0} \, \Xi^0_{2(k-l-1),(l+1)/2} & \text{if } k = 2m, \; l = 2h+1 \text{ and } m-h \neq 1, \\ \dfrac{(N-4)^2}{16} \, \Xi^0_{4,(l-1)/2} & \text{if } k = 2h+2 \text{ and } l = 2h+1. \end{cases}$$

Proof. Let $m-h \neq 1$. By applying first (4.4.3) with $k = 2(m-h-1)$, then (4.4.1) and in the end Lemma 4.3.5 with $\alpha = 2i, \beta = h+1$ once and another time with $\alpha = 2, \beta = h$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{H}^N} (\Delta_r^m u)^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} (\Delta_r^{m-h-1}(\Delta_r^{h+1} u))^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \left(\frac{N-1}{2}\right)^{4(m-h-1)} \int_{\mathbb{H}^N} (\Delta_r^{h+1} u)^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \sum_{i=1}^{2m-2h-2} C^i_{2m-2h-2,0} \int_{\mathbb{H}^N} \frac{(\Delta_r^{h+1} u)^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \left(\frac{N-1}{2}\right)^{4(m-h-1)} \left[\left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_r (\Delta_r^h u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right. \\
&\left. + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{(\Delta_r^h u)^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{(N-1)}{16} \int_{\mathbb{H}^N} \frac{(\Delta_r^h u)^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \right] \\
&+ \sum_{i=1}^{2m-2h-2} C^i_{2m-2h-2,0} \left[\sum_{j=0}^{2h+2} \Xi^j_{2i,h+1} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i+4h+4-2j}} \, \mathrm{d}v_{\mathbb{H}^N} \right] \\
&\geq \left(\frac{N-1}{2}\right)^{4(m-h-1)} \left[\left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_r (\Delta_r^h u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right. \\
&\left. + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{(\Delta_r^h u)^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} \right] \\
&+ \frac{(N-1)^{4(m-h)-3}}{2^{4(m-h)}} \left[\sum_{j=0}^{2h} \Xi^j_{2,h} \int_{\mathbb{H}^N} \frac{u^2}{r^{2+4h-2j}} \, \mathrm{d}v_{\mathbb{H}^N} \right] \\
&+ \sum_{i=1}^{2m-2h-2} C^i_{2m-2h-2,0} \left[\sum_{j=0}^{2h+2} \Xi^j_{2i,h+1} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i+4h+4-2j}} \, \mathrm{d}v_{\mathbb{H}^N} \right] \\
&= \left(\frac{N-1}{2}\right)^{4(m-h)-2} \int_{\mathbb{H}^N} |\nabla_r (\Delta_r^h u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{i=1}^{2m} C^i_{2m,2h+1} \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}.
\end{aligned}
$$

Furthermore, one can observe

$$
C^1_{2m,2h+1} = \frac{(N-1)^{4(m-h)-3}}{2^{4(m-h)}} \, \Xi^{2h}_{2,h} + C^1_{2m-2h-2,0} \, \Xi^{2h+2}_{2,h+1}
$$

and

$$
C^{2m}_{2m,2h+1} = C^{2m-2h-2}_{2m-2h-2,0} \, \Xi^0_{4m-4h-4,h+1}
$$

and this establishes the result.

If $m-h=1$, then exploiting (4.4.1) and Lemma 4.3.5 with $\alpha=4, \beta=h$ once and then $\alpha=2, \beta=h$, (4.4.9) holds with proper constants and this concludes the proof. □

General case $k=2m+1$ odd and $l=2h+1$ odd. Here is the final senario.

Theorem 4.4.5. *Let $k=2m+1>l=2h+1\geq 1$ be integers and $N>2k$. Then there exist k positive constants $C_{k,l}^i$ such that for all $u\in \mathcal{C}_c^\infty(\mathbb{H}^N\setminus\{x_o\})$, there holds*

$$\int_{\mathbb{H}^N} |\nabla_r(\Delta_r^m u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{4(m-h)-2} \int_{\mathbb{H}^N} |\nabla_r(\Delta_r^h u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{i=1}^{k} C_{k,l}^i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}, \tag{4.4.10}$$

where $C_{k,l}^k=\frac{1}{4}\ \Xi_{2,(k-1)/2}^0$ and

$$C_{k,l}^1 = \begin{cases} \frac{1}{4}\ \Xi_{2,(k-1)/2}^{k-1} + \frac{(N-1)^2}{4} C_{k-1,l}^1 & \text{if } k=2m+1, \text{ and } l=2h+1 \\ & \text{where } m-h\neq 1, \\ \frac{(N-1)^3}{2^6}\ \Xi_{2,(l-1)/2}^{l-1} + \frac{1}{4}\ \Xi_{2,(l+1)/2}^{l+1} & \text{if } k=2h+3 \text{ and } l=2h+1. \end{cases}$$

Proof. Assume $m-h\neq 1$. Exploiting first (3.1.6), then (4.4.9) for the index $k=2m, l=2h+1$ and finally Lemma 4.3.5 with $\alpha=2, \beta=m$, we have

$$\begin{aligned} &\int_{\mathbb{H}^N} |\nabla_r(\Delta_r^m u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} (\Delta_r^m u)^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4}\int_{\mathbb{H}^N} \frac{(\Delta_r^m u)^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \\ &= \left(\frac{N-1}{2}\right)^2 \left[\left(\frac{N-1}{2}\right)^{4(m-h)-2} \int_{\mathbb{H}^N} |\nabla_r(\Delta_r^h u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right. \\ &\left. + \sum_{i=1}^{2m} C_{2m,2h+1}^i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N} \right] + \frac{1}{4} \sum_{j=0}^{2m} \Xi_{2,m}^j \int_{\mathbb{H}^N} \frac{u^2}{r^{2+4m-2j}} \, \mathrm{d}v_{\mathbb{H}^N} \\ &= \left(\frac{N-1}{2}\right)^{4(m-h)} \int_{\mathbb{H}^N} |\nabla_r(\Delta_r^h u)|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \sum_{i=1}^{2m+1} C_{2m+1,2h+1}^i \int_{\mathbb{H}^N} \frac{u^2}{r^{2i}} \, \mathrm{d}v_{\mathbb{H}^N}, \end{aligned}$$

where constants are represented by $C_{2m+1,2h+1}^{2m+1}=\frac{1}{4}\ \Xi_{2,m}^0$ and $C_{2m+1,2h+1}^1 = \frac{1}{4}\ \Xi_{2,m}^{2m} + \frac{(N-1)^2}{4} C_{2m,2h+1}^1$.

If $m-h=1$, then using first (3.1.6), then inequality (4.4.1) and in the end applying Lemma 4.3.5 with $\alpha=2, \beta=h+1$ first and then $\alpha=2, \beta=h$, we deduce the results. □

Making use of Lemma 4.3.6, Corollary 4.4.1 and improved inequalities for lower indices namely (3.1.6), (4.4.4) and (4.4.5) with the preceding technique, we will be obtaining another version of (1.2.5). This result gives a better constant but requires more dimension restriction. Without detailing the proof, just by noting this result, we will finish this section.

Corollary 4.4.2. *Let $k>l$ be positive integers and $N\geq 4k-1$. There exist k positive constants such that for all $u\in \mathcal{C}_c^\infty(\mathbb{H}^N\setminus\{x_o\})$, there holds*

$$\int_{\mathbb{H}^N}|\nabla_r^k u|^2\,\mathrm{d}v_{\mathbb{H}^N}-\left(\frac{N-1}{2}\right)^{2(k-l)}\int_{\mathbb{H}^N}|\nabla_r^l u|^2\,\mathrm{d}v_{\mathbb{H}^N}\geq\sum_{j=1}^{k}D_{k,l}^j\int_{\mathbb{H}^N}\frac{u^2}{r^{2j}}\,\mathrm{d}v_{\mathbb{H}^N}. \tag{4.4.11}$$

Moreover, the leading terms for $r\to 0$ and $r\to\infty$, namely $D_{k,l}^k$ and $D_{k,l}^1$ are given by as follows

$$D_{k,l}^k:=\begin{cases} D_{k-l,0}^{k-l}\ \zeta^0_{2(k-l),l/2} & \text{if } l=2h \text{ and } k>l,\\ D_{k-l-1,0}^{k-l-1}\ \zeta^0_{2(k-l-1),(l+1)/2} & \text{if } k=2m,\ l=2h+1 \text{ and } m-h\neq 1,\\ \dfrac{(N-4)^2}{16}\ \zeta^0_{4,(l-1)/2} & \text{if } k=2h+2 \text{ and } l=2h+1,\\ \dfrac{1}{4}\ \zeta^0_{2,(l+1)/2} & \text{if } k=2m+1 \text{ and } l=2h+1,\end{cases}$$

and

$$D^1_{k,l} := \begin{cases} D^1_{k-l,0}\ \zeta^l_{2,l/2} & \textit{if } l = 2h \textit{ and } k > l, \\ \dfrac{(N-1)^{2k-2l-1}}{2^{2k-2l+1}}\ \zeta^{l-1}_{2,(l-1)/2} + D^1_{k-l-1,0}\ \zeta^{l+1}_{2,(l+1)/2} & \textit{if } k = 2m, l = 2h+1 \\ & \quad \textit{and } m - h \neq 1, \\ \dfrac{(N-1)}{8}\ \zeta^{l-1}_{2,(l-1)/2} & \textit{if } k = 2h+2 \\ & \quad \textit{and } l = 2h+1, \\ \dfrac{1}{4}\ \zeta^{k-1}_{2,(k-1)/2} + \dfrac{(N-1)^2}{4} D^1_{k-1,l} & \textit{if } k = 2m+1, \\ & l = 2h+1, \\ & \textit{and } m - h \neq 1, \\ \dfrac{(N-1)^3}{2^5}\ \zeta^{l-1}_{2,(l-1)/2} + \dfrac{1}{4}\ \zeta^{l+1}_{2,(l+1)/2} & \textit{if } k = 2h+3 \\ & \quad \textit{and } l = 2h+1. \end{cases}$$

Chapter 5

Hardy-Rellich and second order Poincaré identities on the hyperbolic space via Bessel pairs

In this chapter, we will prove a family of Hardy-Rellich and Poincaré identities and inequalities on the hyperbolic space having, as particular cases, improved Hardy-Rellich, Rellich, and second-order Poincaré inequalities. All remainder terms provided improvement of those already known in the literature, and all identities hold with the same constants for radial operators also. Furthermore, as applications of the main results, second-order versions of the uncertainty principle on the hyperbolic space will be derived. The material of this chapter corresponds to the article [30].

5.1 Introduction to main results

We have already seen that the N-dimensional hyperbolic space $\mathbb{H}^N$ admits a polar coordinate decomposition structure. From now onward, if nothing is specified, we will always assume $N \geq 2$. Also we have already introduced the radial version of Riemannian Laplacian and Gradient in earlier chapter. As concerns inequality (1.1.5), we recall that it has been shown first in [2] and then, with different methods, adapted to larger classes of manifolds in [27] where criticality has also been shown. Very recently, a further development has been done in [60] where, by using the notion of Bessel pairs, it has been proved that a further positive term of the form $\int_{\mathbb{H}^N} \frac{r}{\sinh^{N-1} r} \left| \nabla_{\mathbb{H}^N} \left(u \frac{\sinh^{\frac{N-1}{2}} r}{r} \right) \right|^2 dv_{\mathbb{H}^N}$ can be added at the r.h.s. of (1.1.5) so that the inequality becomes an equality. Clearly, this is not in contrast with the criticality proved in [27] since the added term is not of the form Vu^2. We refer the interested reader to [25] for the L^p version of (1.1.5), and to [36] for remainder terms of (1.1.2) involving the Green's function of the Laplacian.

Regarding (1.1.10), it's worth recalling that generalizations to Riemannian manifolds of the classical euclidean Hardy inequality have been intensively pursued after the seminal work of Carron [48]. In particular, on Cartan-Hadamard manifolds the optimal constant is known to be $\left(\frac{N-2}{2}\right)^2$ and improvements of the Hardy inequality have been given e.g., in [51,60,77–79,119]. This is in contrast to what happens in the Euclidean setting where the operator $-\Delta_{\mathbb{R}^N} - \left(\frac{N-2}{2}\right)^2 \frac{1}{|x|^2}$ is known to be critical in $\mathbb{R}^N \setminus \{0\}$ (see [53]). In particular, in inequality (1.1.10) the effect of the curvature allows to provide a remainder term of L^2-type, therefore of the same kind of that given in the seminal paper by Brezis-Vazquez [42] for the Hardy inequality on euclidean bounded domains.

The above mentioned results make it natural to investigate the existence of a family of inequalities extending (1.1.9) to the second order, that is an inequality including either improvement of the second order Poincaré inequalities:

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^{2(2-l)} \int_{\mathbb{H}^N} |\nabla^l_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N} \quad (l = 0 \text{ or } l = 1) \tag{5.1.1}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ $(N \geq 2)$, and improvement of the second order Hardy inequalities:

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \mathrm{d}v_{\mathbb{H}^N} \geq \frac{N^2}{4}\left(\frac{N-4}{2}\right)^{2(1-l)} \int_{\mathbb{H}^N} \frac{|\nabla^l_{\mathbb{H}^N} u|^2}{r^{4-2l}} \mathrm{d}v_{\mathbb{H}^N} \quad (l = 0 \text{ or } l = 1) \tag{5.1.2}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N)$ $(N \geq 5)$, i.e. the Rellich inequality which comes for $l = 0$ and the Hardy-Rellich inequality for $l = 1$. We recall that inequalities (5.1.1) are known from [100] and [76] with optimal constants, while improvements have been provided in [26, 27] and, for radial operators, in [29, 112]. Instead, inequalities (5.1.2) were firstly studied in [77] and in [119], where the optimality of the constants was proved together with the existence of some remainder terms. More recently, a stronger version of (5.1.2), only involving radial operators and still holding with the same constants, has been obtained in [102]. See also [80] for improved versions of (5.1.2) in the general framework of Finsler-Hadamard manifolds.

In the present chapter we complete the picture of results in $\mathbb{H}^N$ by proving a family of inequalities including either an improved version of (5.1.1) and an improved version of (5.1.2) when $l = 1$, therefore extending (1.1.9) to the second order, see Theorem 5.1.2 below. Furthermore, in Theorem 5.1.1, we show that the obtained family of inequalities reads as a family of *identities* for radial operators (also for non radial functions) giving a more precise understanding of the remainder terms provided. A fine exploitation of these results also allows to obtain improved versions of (5.1.1) and of (5.1.2) for

$l = 0$ in such a way to exhaust the second order scenario, see Corollaries 5.1.3 and 5.1.4. As far we are aware, all the improvements provided have a stronger positive impact, on the r.h.s. of (5.1.1) and of (5.1.2), than those already known in literature, see Remark 5.1.2 in the following.

5.1.1 Hardy-Rellich and Poincaré identities

Our main result for radial operators regarding Hardy-Rellich and Poincaré identities and improved version reads as follows

Theorem 5.1.1. *For all* $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$ *and all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ *there holds*

$$\begin{aligned}
&\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \lambda \int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left| \nabla_{r,\mathbb{H}^N} \left(\frac{u_r}{\Psi_\lambda(r)} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}
\end{aligned}$$

where $\gamma_N(\lambda) := \sqrt{(N-1)^2 - 4\lambda}$, $h_N(\lambda) := \frac{\gamma_N(\lambda)+1}{2}$ *and* $\Psi_\lambda(r) := r^{-\frac{N-2}{2}} \left(\frac{\sinh r}{r}\right)^{-\frac{N-1+\gamma_N(\lambda)}{2}}$. *Furthermore, for* $N \geq 5$ *and* λ *given, the constants* $h_N^2(\lambda)$ *and* $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ *are jointly sharp in the sense that, fixed* $h_N^2(\lambda)$*, the inequality does not hold if we replace* $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ *with a larger constant.*

Remark 5.1.1. *We remark that the the function* $\frac{r \coth r - 1}{r^2}$ *is positive, strictly decreasing and satisfies*

$$\frac{r \coth r - 1}{r^2} \sim \frac{1}{3} \quad \text{as} \quad r \to 0^+ \qquad \text{and} \qquad \frac{r \coth r - 1}{r^2} \sim \frac{1}{r} \quad \text{as} \quad r \to +\infty\,.$$

Furthermore, the map $[0, \lambda_1(\mathbb{H}^N)] \ni \lambda \mapsto h_N(\lambda)$ *is decreasing and* $\frac{1}{4} \leq h_N(\lambda) \leq \left(\frac{N}{2}\right)^2$.

Furthermore, for non radial operators we obtain the second order analogous to (1.1.9):

Theorem 5.1.2. *Let $N \geq 5$. For all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$ and all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\begin{aligned}
&\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \geq \lambda \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \\
&+ h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \,\mathrm{d}v_{\mathbb{H}^N} + \left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \,\mathrm{d}v_{\mathbb{H}^N} \\
&+ \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \\
&+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{\mathbb{H}^N} \left(\frac{u_r}{\Psi_\lambda(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}
\end{aligned}$$

where $\gamma_N(\lambda), h_N(\lambda)$ and $\Psi_\lambda(r)$ are as given in Theorem 5.1.1. Furthermore, for any given λ, the constants $h_N^2(\lambda)$ and $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ are jointly sharp in the sense explained in Theorem 5.1.1.

We notice that the dimension restriction $N \geq 5$ in Theorem 5.1.2 comes from assumption (5.3.2) in Theorem 5.3.2 below where we state our abstract Rellich inequalities, see also Remark 5.3.1 for some comments about this assumption that naturally comes when passing from the radial to the non radial framework. Theorems 5.1.1 and 5.1.2 yield a number of improved Poincaré and Hardy-Rellich inequalities that we state here below; a comparison with previous results is provided in Remark 5.1.2. More precisely, for $\lambda = 0$ we readily got the following improved Hardy-Rellich identity and inequality:

Corollary 5.1.1. *For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\begin{aligned}
\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} &= \left(\frac{N}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \,\mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \\
&+ \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left|\nabla_{r,\mathbb{H}^N} \left(\frac{(\sinh r)^{N-1} u_r}{r^{\frac{N}{2}}}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \,.
\end{aligned}$$

Moreover, if $N \geq 5$, for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N}{2}\right)^2 \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} \, |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left| \nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{N-1} \, u_r}{r^{\frac{N}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N} \,,$$

and the constant $\left(\frac{N}{2}\right)^2$ appearing in the L.H.S of both equations is the sharp constant.

For $\lambda = \lambda_1(\mathbb{H}^N)$ we got an improvement of the second order Poincaré identity (5.1.1) with $l = 0$, and the related inequality:

Corollary 5.1.2. *For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N^2-1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{r,\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N} .$$

Moreover, if $N \geq 5$, for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^2 \int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N^2-1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N} .$$

The constant $\left(\frac{N-1}{2}\right)^2$ appearing in the L.H.S of both equations is the sharp constant. Moreover, for $N \geq 5$, the constants $\frac{1}{4}$ and $\frac{N^2-1}{4}$ are jointly sharp in the sense explained in Theorem 5.1.1.

By combining Corollary 5.1.1 with [60, Corollary 3.2] we also get an improved Rellich inequality:

Corollary 5.1.3. *For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\begin{aligned}
\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \frac{N^2}{4}\left(\frac{N-4}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N^2(N-4)(N-1)}{8} \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^4} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N^2}{4} \int_{\mathbb{H}^N} \frac{1}{r^{N-2}} \left|\nabla_{r,\mathbb{H}^N}\left(r^{\frac{N-4}{2}} u\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \\
&+ \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left|\nabla_{r,\mathbb{H}^N}\left(\frac{(\sinh r)^{N-1}\, u_r}{r^{\frac{N}{2}}}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \,.
\end{aligned}$$

Moreover, if $N \geq 5$, for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\begin{aligned}
\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &\geq \frac{N^2}{4}\left(\frac{N-4}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N^2(N-4)(N-1)}{8} \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^4} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r\coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{N^2}{4} \int_{\mathbb{H}^N} \frac{1}{r^{N-2}} \left|\nabla_{\mathbb{H}^N}\left(r^{\frac{N-4}{2}} u\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \\
&+ \int_{\mathbb{H}^N} \frac{r^N}{(\sinh r)^{2(N-1)}} \left|\nabla_{\mathbb{H}^N}\left(\frac{(\sinh r)^{N-1}\, u_r}{r^{\frac{N}{2}}}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \,,
\end{aligned}$$

and the constant $\frac{N^2}{4}\left(\frac{N-4}{2}\right)^2$ appearing in the L.H.S of both equations is the sharp constant.

Instead, by combining Corollary 5.1.2 with [60, Theorem 1.4 and Corollary 3.2], we improve (5.1.1) with $l = 0$, i.e. we complete the second order scenario about Poincaré identities and inequalities:

Corollary 5.1.4. *For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\begin{aligned}
&\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&= \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \left(\frac{N-1}{4}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{(N-1)^3(N-3)}{16} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N^2-1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \left[\left(\frac{N-1}{2}\right)^2 + 1\right] \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{r,\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}.
\end{aligned}$$

Moreover, if $N \geq 5$, for all $u \in C_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\begin{aligned}
&\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&\geq \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \left(\frac{N-1}{4}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{(N-1)^3(N-3)}{16} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \frac{1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + \frac{N^2-1}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ \left[\left(\frac{N-1}{2}\right)^2 + 1\right] \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N}.
\end{aligned}$$

The constant $\left(\frac{N-1}{2}\right)^4$ appearing in the L.H.S of both equations is the sharp constant. Moreover, for $N \geq 5$, the constants $\frac{1}{4}$ and $\frac{N^2-1}{4}$ in both equations are jointly sharp in the sense explained in Theorem 5.1.1.

Remark 5.1.2. *As far as we are aware, improved second order Poincaré and Hardy-Rellich* equalities *in $\mathbb{H}^N$ were not known in literature; besides, the above inequalities yield improvements of Poincaré and Hardy-Rellich inequalities which are stronger than those already known in literature. As concerns the Hardy-Rellich and Rellich inequalities, improved versions were already known from [80], [102] and [119] on general manifolds but with fewer and smaller remainder terms. As a matter of example, if we compare Corollary 5.1.1 with [102, Theorem 4.2], the improvement of the Hardy-Rellich* *inequality provided there reads as* $\frac{3N(N-1)}{2}\int_{\mathbb{H}^N}\frac{|\nabla_{r,\mathbb{H}^N}u|^2}{\pi^2+r^2}\,\mathrm{d}v_{\mathbb{H}^N}$*, therefore it decays more rapidly, both as* $r\to 0^+$ *and as* $r\to+\infty$*, than the term* $\frac{N(N-1)}{2}\int_{\mathbb{H}^N}\frac{r\coth r-1}{r^2}\,|\nabla_{r,\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}$ *provided in Corollary 5.1.1. Similarly, if we compare Corollary 5.1.2 with [102, Theorem 4.3], again, the corrections of the Rellich inequality provided there decays more rapidly than ours, both as* $r\to 0^+$ *and as* $r\to+\infty$*. As concerns the improved second order Poincaré inequalities given by Corollaries 5.1.3 and 5.1.4, the gain with respect to the inequalities already known in [29] is in the adding of a further remainder term.*

5.1.2 Second order Heisenberg-Pauli-Weyl uncertainty principle

Another remarkable consequence of Theorem 5.1.2 is the following quantitative version of Heisenberg-Pauli-Weyl (in short HPW) principle in $\mathbb{H}^N$:

Theorem 5.1.3. *Let* $N\geq 5$*. For all* $0\leq\lambda\leq\lambda_1(\mathbb{H}^N)$ *and all* $u\in\mathcal{C}_c^\infty(\mathbb{H}^N\setminus\{x_o\})$ *there holds*

$$\left(\int_{\mathbb{H}^N}\left(|\Delta_{\mathbb{H}^N}u|^2-\lambda|\nabla_{\mathbb{H}^N}u|^2\right)\mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N}r^2|\nabla_{\mathbb{H}^N}u|^2\mathrm{d}v_{\mathbb{H}^N}\right) \geq h_N^2(\lambda)\left(\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}^N}u|^2\mathrm{d}v_{\mathbb{H}^N}\right)^2 \tag{5.1.3}$$

where $h_N(\lambda)$ *is as defined as in Theorem 5.1.1. In particular, for* $\lambda=0$*, we obtain*

$$\left(\int_{\mathbb{H}^N}|\Delta_{\mathbb{H}^N}u|^2\mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N}r^2|\nabla_{\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}\right)\geq\frac{N^2}{4}\left(\int_{\mathbb{H}^N}|\nabla_{\mathbb{H}^N}u|^2\,\mathrm{d}v_{\mathbb{H}^N}\right)^2, \tag{5.1.4}$$

for all $u\in\mathcal{C}_c^\infty(\mathbb{H}^N\setminus\{x_o\})$*.*

Remark 5.1.3. *In the Euclidean context the second order Heisenberg-Pauli-Weyl uncertainty principle has been only recently studied in [49, Theorem*

2.1-2.2] where it is proved that the best constant switches from $\frac{N^2}{4}$ *to* $\frac{(N+2)^2}{4}$ *when passing to the second order. Moreover, Duong-Nguyen in [56, Theorem 1.1] has studied the weighted version of inequality* (5.1.4) *in the Euclidean setting and discuss its sharp constants and extremals.*

As far as we know, inequality (5.1.3) *is the first example of second order Heisenberg-Pauli-Weyl uncertainty principle in the Hyperbolic context. For the first order case, we refer instead to [69] and [79] where the authors fully describe the influence of curvature to uncertainty principles in the Riemannian and Finslerian settings. It's worth mentioning that a finer exploitation of Theorem 5.1.2 yields the improved version of* (5.1.3) *below which supports the conjecture that the sharp constant* (5.1.3) *should be larger than* $h_N^2(\lambda)$*. More precisely, a small modification of the proof of Theorem 5.1.3 allows us to prove that, for all* $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$ *and all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$*, there holds*

$$
\begin{aligned}
&\left(\int_{\mathbb{H}^N} \left(|\Delta_{\mathbb{H}^N} u|^2 - \lambda |\nabla_{\mathbb{H}^N} u|^2\right) \mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \\
&\quad \geq h_N^2(\lambda)\left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)^2 \\
&\quad + \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right) \times \left\{\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N} \right. \\
&\quad \left. + \gamma_N(\lambda)\, h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right\}
\end{aligned}
$$

where $\gamma_N(\lambda)$ *and* $h_N(\lambda)$ *are defined as in Theorem 5.1.1. Therefore, for* $\lambda = 0$*, we obtain the improved version of* (5.1.4)*:*

$$
\begin{aligned}
&\left(\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right) \geq \frac{N^2}{4}\left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)^2 \\
&+ \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)\left(\frac{N(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)
\end{aligned}
$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$*. The above inequality should be compared with inequality* (5.3.4) *provided in Section 5.3 which also improves* (5.1.4)*.*

We conclude the section by stating the counterpart of Theorem 5.1.3 for radial operators:

Theorem 5.1.4. *For all* $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$ *and all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ *there holds*

$$\left(\int_{\mathbb{H}^N} \left(|\Delta_{r,\mathbb{H}^N} u|^2 - \lambda |\nabla_{r,\mathbb{H}^N} u|^2 \right) \mathrm{d}v_{\mathbb{H}^N} \right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N} \right) \geq h_N^2(\lambda) \left(\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N} \right)^2$$

where $h_N(\lambda)$ *is as defined as in Theorem 5.1.1. In particular, for* $\lambda = 0$, *we obtain*

$$\left(\int_{\mathbb{H}^N} |\Delta_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N} \right) \left(\int_{\mathbb{H}^N} r^2 |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right) \geq \frac{N^2}{4} \left(\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^2$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$.

5.2 Bessel pairs

Ghoussoub-Moradifam in [63] provided a very general framework to obtain various Hardy-type inequalities and their improvements on the Euclidean space (or bounded domain). It was based on the notion of Bessel pair that we recall in the following:

Definition 5.2.1. *We say that a pair* (V, W) *of* $\mathcal{C}^1$*-functions is a Bessel pair on* $(0, R)$ *for some* $0 < R \leq \infty$ *if the ordinary differential equation:*

$$(Vy')' + Wy = 0$$

admits a positive solutions f *on the interval* $(0, R)$.

Example 5.2.1. $\left(r^{N-1}, r^{N-1} \frac{(N-2)^2}{4} \frac{1}{r^2} \right)$ *forms a Bessel pair with the positive solution function* $f(r) = r^{\frac{(2-N)}{2}}$ *on the interval* $(0, \infty)$.

In [63] the authors proved the following inequality, for some positive constant $C > 0$:

$$\int_{\mathscr{B}_R} V(x) |\nabla u|^2 \, \mathrm{d}x \geq C \int_{\mathscr{B}_R} W(x) \, |u|^2 \, \mathrm{d}x \quad \forall \; u \in \mathcal{C}_c^\infty(\mathscr{B}_R), \tag{5.2.1}$$

subject to the constraints that the functions V and W are positive radial functions defined on the euclidean ball $\mathscr{B}_R$ and $(r^{N-1}V, r^{N-1}W)$ is a Bessel pairs such that $\int_0^R \frac{1}{r^{N-1}V(r)}\,\mathrm{d}r = \infty$ and $\int_0^R r^{N-1}V(r)\,\mathrm{d}r < \infty$ where $0 < R \leq \infty$ is the radius of the ball $\mathscr{B}_R$.

Remark 5.2.1. *The Bessel pair in the Example 5.2.1 and* (5.2.1) *produce the known Hardy inequality.*

In view of (5.2.1), with particular choices of (V, W), the results in [63] simplified and improved several known results concerning Hardy inequalities and their improvements. Recently, the notion of Bessel pairs has been exploited: in [87] to establish improved Hardy inequalities involving general distance functions, in [86] to sharpen several Hardy type inequalities on half spaces, and in [85] to prove Hardy inequalities on homogeneous groups.

Regarding Cartan-Hadamard manifolds, the notion of Bessel pairs has been very recently exploited to obtain improved Hardy inequalities in [60]; to our future purposes, we recall their Theorem 3.2 on $\mathbb{H}^N$:

Lemma 5.2.1. *[60, Theorem 3.2] Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0, R)$ with positive solution f on $(0, R)$. Then for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_0\})$, there holds*

$$\begin{aligned}
&\int_{\mathscr{B}_R} V(r)|\nabla_{\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N} \\
&= \int_{\mathscr{B}_R} W(r)|u|^2 \,\mathrm{d}v_{\mathbb{H}^N} + \int_{\mathscr{B}_R} V(r)(f(r))^2 \left|\nabla_{\mathbb{H}^N}\left(\frac{u}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \\
&- (N-1)\int_{\mathscr{B}_R} V(r)\frac{f'(r)}{f(r)}\left(\coth r - \frac{1}{r}\right)u^2 \,\mathrm{d}v_{\mathbb{H}^N}.
\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathscr{B}_R} V(r)|\nabla_{r,\mathbb{H}^N} u|^2 \,\mathrm{d}v_{\mathbb{H}^N}\\&= \int_{\mathscr{B}_R} W(r)|u|^2 \,\mathrm{d}v_{\mathbb{H}^N} + \int_{\mathscr{B}_R} V(r)(f(r))^2 \left|\nabla_{r,\mathbb{H}^N}\left(\frac{u}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}\\&- (N-1)\int_{\mathscr{B}_R} V(r)\frac{f'(r)}{f(r)}\left(\coth r - \frac{1}{r}\right) u^2 \,\mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

In view of Lemma 5.2.1 a natural question is to study higher order Hardy type inequalities in $\mathbb{H}^N$ using the notion of Bessel pairs. In the Euclidean space (or in bounded euclidean domains) these questions were studied in [63]. One of their results read as follows: let $0 < R \leq \infty, V$ and W be positive $\mathcal{C}^1$-functions on $\mathscr{B}_R \setminus \{0\}$ such that $(r^{N-1}V, r^{N-1}W)$ forms a Bessel pair. Then for all radial functions $u \in \mathcal{C}_c^\infty(\mathscr{B}_R)$ there holds

$$\begin{aligned}\int_{\mathscr{B}_R} V(x)|\Delta u|^2 \,\mathrm{d}x &\geq \int_{\mathscr{B}_R} W(x)\,|\nabla u|^2 \,\mathrm{d}x\\&+ (N-1)\int_{\mathscr{B}_R}\left(\frac{V(x)}{|x|^2} - \frac{V_r(x)}{|x|}\right)|\nabla u|^2 \,\mathrm{d}x, \qquad (5.2.2)\end{aligned}$$

where $r = |x|$. In addition, if $W(x) - 2\frac{V(x)}{|x|^2} + 2\frac{V_r(x)}{|x|} - V_{rr}(x) \geq 0$ on $(0,R)$, then the above is true for non radial function as well (we refer [63, Theorem 3.1-3.3] for more insight). We also refer to [57, 58, 85] for recent results on Hardy-Rellich inequalities and their improvements on the Euclidean space using the approach of Bessel pairs.

5.3 Abstract Rellich identities and inequalities via Bessel pairs

In the present section, we extend (5.2.2) to $\mathbb{H}^N$ by showing first the following:

Theorem 5.3.1. *Let $(r^{N-1}V, r^{N-1}W)$ be a Bessel pair on $(0,R)$ with positive solution f on $(0,R)$. Then for all radial function $u \in \mathcal{C}_c^\infty(\mathscr{B}_R \setminus \{x_o\})$ there* *holds*

$$\int_{\mathscr{B}_R} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \int_{\mathscr{B}_R} W(r)|\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$+ (N-1) \int_{\mathscr{B}_R} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$- (N-1) \int_{\mathscr{B}_R} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathscr{B}_R} V(r)(f(r))^2 \left| \nabla_{\mathbb{H}^N} \left(\frac{u_r}{f(r)} \right) \right|^2 dv_{\mathbb{H}^N} . \tag{5.3.1}$$

As a direct consequence of the above result, we tackle the non-radial scenario by spherical harmonic method and we prove:

Corollary 5.3.1. *Let* $(r^{N-1}V, r^{N-1}W)$ *be a Bessel pair on* $(0, \infty)$ *with positive solution* f *on* $(0, \infty)$*. Then for all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ *there holds*

$$\int_{\mathbb{H}^N} V(r)|\Delta_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \int_{\mathbb{H}^N} W(r)|\nabla_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$+ (N-1) \int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nabla_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$- (N-1) \int_{\mathbb{H}^N} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nabla_{r,\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} V(r)(f(r))^2 \left| \nabla_{r,\mathbb{H}^N} \left(\frac{u_r}{f(r)} \right) \right|^2 dv_{\mathbb{H}^N}$$

Now it is natural to ask whether there is a counterpart of Theorem 5.3.1 for any function, not necessarily radial. We give an affirmative answer in below provided that V satisfies the extra condition (5.3.2) below:

Theorem 5.3.2. *Let* $(r^{N-1}V, r^{N-1}W)$ *be a Bessel pair on* $(0, \infty)$ *with positive solution* f *on* $(0, \infty)$*. Also assume* $N \geq 5$ *and* V *satisfies*

$$(N-5)\frac{V(r)}{\sinh^2 r} + 3\frac{V_r(r) \cosh r}{\sinh r} - V_{rr}(r) + (N-4)V(r) \geq 0. \tag{5.3.2}$$

Then for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ *there holds*

$$\begin{aligned}\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq &\int_{\mathbb{H}^N} W(r)|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ (N-1)\int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r)\cosh r}{\sinh r}\right)|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &- (N-1)\int_{\mathbb{H}^N} V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \int_{\mathbb{H}^N} V(r)(f(r))^2\left|\nabla_{\mathbb{H}^N}\left(\frac{u_r}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} \,. \end{aligned} \tag{5.3.3}$$

Remark 5.3.1. *We remark that assumption* (5.3.2) *in Theorem 5.3.2 is not too restrictive to our purposes: we shall provide a remarkable family of* (V, W) *for which the assumption holds true in the proof of Theorem 5.1.1. On the other hand, an analogous assumption was even required in the Euclidean space as well, see* (5.2.2) *and the comments just below; this seems the natural prize to pay in order to pass to the higher order case.*

We conclude the section by stating an abstract version of Heisenberg-Pauli-Weyl uncertainty principle involving Bessel pairs which follows as a corollary from Theorem 5.3.2 and Corollary 5.3.1:

Theorem 5.3.3. *Let* $(r^{N-1}V, r^{N-1}W)$ *be a Bessel pair on* $(0,\infty)$ *with positive solution* f *on* $(0,\infty)$ *and set*

$$\tilde{W}(r) := W(r)+(N-1)\left(\frac{V(r)}{\sinh^2 r}-\frac{V_r(r)\cosh r}{\sinh r}\right)-(N-1)V(r)\frac{f'}{f}\left(\coth r-\frac{1}{r}\right).$$

Furthermore, let $N \geq 5$ *and assume that* V *satisfies 5.3.2 and that* $\tilde{W}(r) > 0$ *for all* $r > 0$*. Then for all* $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ *there holds*

$$\left(\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\tilde{W}(r)} \, \mathrm{d}v_{\mathbb{H}^N}\right) \geq \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)^2$$

and

$$\left(\int_{\mathbb{H}^N} V(r)|\Delta_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} \frac{|\nabla_{r,\mathbb{H}^N} u|^2}{\tilde{W}(r)} \mathrm{d}v_{\mathbb{H}^N}\right) \geq \left(\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N}\right)^2.$$

We want to mention that for the second inequality we do not require condition 5.3.2, whereas the other conditions and Corollary 5.3.1 are enough.

Remark 5.3.2. *A non trivial example of pairs satisfying the assumptions of Theorem 5.3.3 is given by the family of Bessel pairs* $(r^{N-1}, r^{N-1}W_\lambda)$*, for all* $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$*, defined in* (5.4.1) *below and exploited in the proof of Theorem 5.1.1. Indeed, they satisfy condition* (5.3.2) *and, in this case, the function* $\tilde{W}$ *reads*

$$\tilde{W}_\lambda(r) = \lambda + h_N^2(\lambda)\frac{1}{r^2} + \left(\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right)\frac{1}{\sinh^2 r} + \frac{\gamma_N(\lambda)\, h_N(\lambda)}{r}\left(\coth r - \frac{1}{r}\right)$$

which is positive in $(0, +\infty)$ *for all* $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$*. In particular, with this pair, taking* $\lambda = 0$ *for simplicity, Theorem 5.3.3 yields*

$$\left(\int_{\mathbb{H}^N} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)\left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\frac{N^2}{4}\frac{1}{r^2} + \frac{N(N-1)}{2r}(\coth r - \frac{1}{r})} \, \mathrm{d}v_{\mathbb{H}^N}\right) \ge \left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\right)^2, \tag{5.3.4}$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$*. The above inequality turns out to be more stringent than* (5.1.4) *thereby confirming the conjecture that* $\frac{N^2}{4}$ *is not the sharp constant in* (5.1.4)*.*

5.4 Proofs of Theorems 5.1.1 and 5.1.2

Proofs of Theorems 5.1.1 and 5.1.2. The proof follows, respectively, by applying Corollary 5.3.1 and Theorem 5.3.2 with the family of Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$ with $0 \le \lambda \le \lambda_1(\mathbb{H}^N)$ and

$$W_\lambda(r) := \lambda + h_N^2(\lambda)\frac{1}{r^2} + \left(\left(\frac{N-2}{2}\right)^2 - h_N^2(\lambda)\right)\frac{1}{\sinh^2 r} + \left(\frac{\gamma_N(\lambda)\, h_N(\lambda)}{r} + (N-1)\frac{\Psi_\lambda'(r)}{\Psi_\lambda(r)}\right)\left(\coth r - \frac{1}{r}\right) \qquad (r > 0), \tag{5.4.1}$$

where $\gamma_N(\lambda)$ and $h_N(\lambda)$ are as defined in the statement of Theorem 5.3.1 and

$$\Psi_\lambda(r) := r^{-\frac{N-2}{2}} \left(\frac{\sinh r}{r}\right)^{-\frac{N-1+\gamma_N(\lambda)}{2}} \qquad (r>0)\,.$$

In particular, by noticing that

$$\Psi_\lambda'(r) = \Psi_\lambda(r)\left[\frac{h_N(\lambda)}{r} + \frac{1-N-\gamma_N(\lambda)}{2}\coth r\right],$$

$$\begin{aligned}\Psi_\lambda''(r) = \Psi_\lambda(r)\Big[&\frac{(1-N-\gamma_N(\lambda))^2}{4} + \frac{\gamma_N^2(\lambda)-1}{r^2}\\ &- \frac{(1-N-\gamma_N(\lambda))(1+N+\gamma_N(\lambda))}{4\sinh^2 r} + \frac{(1-N-\gamma_N(\lambda))h_N(\lambda)\coth r}{r}\Big]\end{aligned}$$

and recalling the definition of $\gamma_N(\lambda)$, it follows that $\Psi_\lambda(r)$ satisfies

$$(r^{N-1}\Psi_\lambda'(r))' + r^{N-1}W_\lambda(r)\Psi_\lambda(r) = 0 \quad \text{for } r>0\,,$$

namely $(r^{N-1}, r^{N-1}W_\lambda)$ is a Bessel pair with positive solution $\Psi_\lambda(r)$. See also [28, Lemma 6.2] where the functions Ψ_λ were originally introduced but exploited with different purposes. Finally, from Corollary 5.3.1 we deduce that, for all function $u \in \mathcal{C}_c^\infty(\mathscr{B}_R \setminus \{x_o\})$, there holds

$$\begin{aligned}\int_{\mathscr{B}_R} |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = &\int_{\mathscr{B}_R} W_\lambda(r)|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\\ &+ (N-1)\int_{\mathscr{B}_R}\left(\frac{1}{\sinh^2 r}\right)|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\\ &- (N-1)\int_{\mathscr{B}_R}\frac{\Psi_\lambda'(r)}{\Psi_\lambda(r)}\left(\coth r - \frac{1}{r}\right)|\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N}\\ &+ \int_{\mathscr{B}_R}(\Psi_\lambda(r))^2\left|\nabla_{\mathbb{H}^N}\left(\frac{u_r}{\Psi_\lambda(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}\,.\end{aligned}$$

By this, recalling (5.4.1), the proof of Theorem 5.1.1 follows. The proof of Theorem 5.1.2 works similarly by applying Theorem 5.3.2 since condition (5.3.2) holds for the Bessel pair $(r^{N-1}, r^{N-1}W_\lambda)$ if $N \geq 5$.

As concerns the proof of the fact that the constants $h_N^2(\lambda)$ and $\left[\left(\frac{N}{2}\right)^2 - h_N^2(\lambda)\right]$ are jointly sharp when $N \geq 5$, this follows by noticing that as $r \to 0$ we have

$$h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N} + \left[\left(\frac{N}{2} \right)^2 - h_N^2(\lambda) \right] \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} \sim \frac{N^2}{4} \int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N} \, .$$

Therefore, locally, we recover inequality (5.1.2) for $l = 1$; by this we readily infer that, for $h_N^2(\lambda)$ fixed, any larger constant in front of the term $\frac{|\nabla_{\mathbb{H}^N} u|^2}{\sinh^2 r}$ would contradict the optimality of the constant $\frac{N^2}{4}$ in (5.1.2) (when $l = 1$). □

Proof of Corollary 5.1.3. The proof follows from Corollary 5.1.1 by evaluating the term $\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N}$ with the aid of [60, Corollary 3.2] from which we know that

$$\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, dv_{\mathbb{H}^N} = \left(\frac{N-4}{2} \right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N} + \frac{(N-4)(N-1)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^4} u^2 \, dv_{\mathbb{H}^N} + \int_{\mathbb{H}^N} \frac{1}{r^{N-2}} \left| \nabla_{\mathbb{H}^N} \left(r^{\frac{N-4}{2}} u \right) \right|^2 dv_{\mathbb{H}^N} \, .$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$. The proof for radial operators follows similarly since the above identity holds with the same constants for radial operators too. □

Proof of Corollary 5.1.4. Here the proof follows by combining Corollary 3.3 with [60, Theorem 1.4] according to which we know that

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} = \left(\frac{N-1}{2} \right)^2 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{1}{4} \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, dv_{\mathbb{H}^N} + \frac{(N-1)(N-3)}{4} \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, dv_{\mathbb{H}^N} + \int_{\mathbb{H}^N} \frac{r}{(\sinh r)^{N-1}} \left| \nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{\frac{N-1}{2}} u_r}{r^{\frac{1}{2}}} \right) \right|^2 dv_{\mathbb{H}^N} \, .$$

for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ and similarly for radial operators since the above identity holds with the same constants for radial operators too. □

Proof of Theorem 5.1.3. The proof is a simple application of Cauchy-Schwartz inequality combined with Theorem 5.1.2:

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathbb{H}^N} r|\nabla_{\mathbb{H}^N} u| \frac{|\nabla_{\mathbb{H}^N} u|}{r} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$\leq \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}} \underbrace{\left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}}}_{Using\ Theorem\ 5.1.2}$$
$$\leq \frac{1}{h_N(\lambda)} \left(\int_{\mathbb{H}^N} \left(|\Delta_{\mathbb{H}^N} u|^2 - \lambda |\nabla_{\mathbb{H}^N} u|^2 \right) \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}} \left(\int_{\mathbb{H}^N} r^2 |\nabla_{\mathbb{H}^N} u|^2 \mathrm{d}v_{\mathbb{H}^N} \right)^{\frac{1}{2}}.$$

□

5.5 Proofs of Theorems 5.3.1, 5.3.2 and 5.3.3

We shall begin with the proof of Theorem 5.3.1.

Proof of Theorem 5.3.1.

Let $u \in \mathcal{C}_c^\infty(\mathscr{B}_R \setminus \{x_o\})$ be a radial function, in terms of polar coordinates we have

$$\int_{\mathscr{B}_R} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = N\omega_N \Bigg[\int_0^R V(r) u_{rr}^2 (\sinh r)^{N-1} \, \mathrm{d}r$$
$$+ (N-1)^2 \int_0^R V(r)(\coth r)^2 u_r^2 (\sinh r)^{N-1} \, \mathrm{d}r$$
$$+ 2(N-1) \int_0^R V(r) u_{rr} u_r (\coth r)(\sinh r)^{N-1} \, \mathrm{d}r \Bigg].$$

Now, applying integration by parts in the last term and setting $\nu = u_r$, we deduce

$$\int_{\mathscr{B}_R} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathscr{B}_R} V(r)|\nabla_{\mathbb{H}^N} \nu|^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ (N-1) \int_{\mathscr{B}_R} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N}. \tag{5.5.1}$$

On the other hand, from Lemma 5.2.1 for the function ν we have

$$\int_{\mathscr{B}_R} V(r)|\nabla_{\mathbb{H}^N}\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \int_{\mathscr{B}_R} W(r)|\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathscr{B}_R} V(r)(f(r))^2 \left|\nabla_{\mathbb{H}^N}\left(\frac{\nu}{f(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N} - (N-1)\int_{\mathscr{B}_R} V(r)\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)|\nu|^2 \, \mathrm{d}v_{\mathbb{H}^N}.$$

By using this identity into (5.5.1) and writing back in terms of u we deduce (5.3.1). □

Before going to prove Corollary 5.3.1 and Theorem 5.3.2, we shall assume by spherical harmonics decomposition

$$u(r,\Theta) = \sum_{n=0}^{\infty} a_n(r)P_n(\Theta), \tag{5.5.2}$$

where $u(x) = u(r,\Theta) \in C_c^{\infty}(\mathbb{H}^N)$, $r \in (0,\infty)$ and $\Theta \in \mathbb{S}^{N-1}$.

In a continuation let us also describe the Gradient and Laplace Beltrami operator in this setting. Now onward, to shorten the notations, we will always assume $\psi(r) = \sinh r$. They will be

$$|\nabla_{\mathbb{H}^N} u|^2 = \sum_{n=0}^{\infty} {a_n'}^2 P_n^2 + \frac{a_n^2}{\psi^2}|\nabla_{\mathbb{S}^{N-1}} P_n|^2$$

and

$$(\Delta_{\mathbb{H}^N} u)^2 = \sum_{n=0}^{\infty}\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)^2 P_n^2 + \sum_{n=0}^{\infty}\frac{a_n^2}{\psi^4}(\Delta_{\mathbb{S}^{N-1}}P_n)^2 + 2\sum_{n=0}^{\infty}\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}(\Delta_{\mathbb{S}^{N-1}}P_n)P_n. \tag{5.5.3}$$

Along with this the radial contribution of the operators will look like as follows:

$$|\nabla_{r,\mathbb{H}^N} u|^2 = \sum_{n=0}^{\infty} {a_n'}^2 P_n^2$$

and

$$(\Delta_{r,\mathbb{H}^N} u)^2 = \sum_{n=0}^{\infty}\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)^2 P_n^2 \,.$$

Proof of Corollary 5.3.1. By spherical harmonics, we decompose u as in (5.5.2). Now, exploiting Theorem 5.3.1 for each a_n, we deduce

$$\begin{aligned}
&\int_{\mathbb{H}^N} V(r)|\Delta_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \sum_{n=0}^{\infty} \int_0^\infty V(r) \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right)^2 \psi^{N-1} \, \mathrm{d}r \\
&= \sum_{n=0}^{\infty} \Bigg[\int_0^\infty W {a_n'}^2 \psi^{N-1} \, \mathrm{d}r + \int_0^\infty V f^2 \left[\left(\frac{a_n'}{f} \right)' \right]^2 \psi^{N-1} \, \mathrm{d}r \\
&- (N-1) \int_0^\infty V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) (a_n')^2 \psi^{N-1} \, \mathrm{d}r \\
&+ (N-1) \int_0^\infty V {a_n'}^2 \psi^{N-3} \, \mathrm{d}r - (N-1) \int_0^\infty V_r \psi' (a_n')^2 \psi^{N-2} \, \mathrm{d}r \Bigg] \\
&= \int_{\mathbb{H}^N} W(r) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} V(r)(f(r))^2 \left| \nabla_{r,\mathbb{H}^N} \left(\frac{u_r}{f(r)} \right) \right|^2 \mathrm{d}v_{\mathbb{H}^N} \\
&- (N-1) \int_{\mathbb{H}^N} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \\
&+ (N-1) \int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \, .
\end{aligned}$$

This completes the proof. □

Proof of Theorem 5.3.2. Again, by spherical decomposition we can write u as in (5.5.2). Granting the advantage of $\psi(r) = \sinh r$, we can write some relations like $\frac{\psi'^2}{\psi^2} = 1 + \frac{1}{\psi^2}$ and $\psi'^2 = 1 + \psi^2$ and we will use these identities in the proof frequently. Now, using (5.5.3) for the decomposed function u, we get

$$\begin{aligned}
&\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \sum_{n=0}^{\infty} \Bigg[\int_0^\infty V(r) \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right)^2 \psi^{N-1} \, \mathrm{d}r \\
&+ \lambda_n^2 \int_0^\infty V(r) \frac{a_n^2}{\psi^4} \psi^{N-1} \, \mathrm{d}r - 2\,\lambda_n \int_0^\infty V(r) \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right) \frac{a_n}{\psi^2} \psi^{N-1} \, \mathrm{d}r \Bigg].
\end{aligned}$$

Exploiting Corollary 5.3.1 for each a_n, we deduce

$$\int_{\mathbb{H}^N} V(r)|\Delta_{\mathbb{H}^N} a_n|^2 \, dv_{\mathbb{H}^N} = N\omega_N \int_0^\infty V(r) \left(a_n'' + (N-1)\frac{\psi'}{\psi} a_n' \right)^2 \psi^{N-1} \, dr$$
$$= N\omega_N \Bigg[\int_0^\infty W {a_n'}^2 \psi^{N-1} \, dr + \int_0^\infty V f^2 \left[\left(\frac{a_n'}{f} \right)' \right]^2 \psi^{N-1} \, dr$$
$$- (N-1) \int_0^\infty V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) (a_n')^2 \psi^{N-1} \, dr$$
$$+ (N-1) \int_0^\infty V {a_n'}^2 \psi^{N-3} \, dr - (N-1) \int_0^\infty V_r \psi' (a_n')^2 \psi^{N-2} \, dr \Bigg].$$

On the other hand, the r.h.s of inequality (5.3.3) in terms of spherical decomposition is

$$\int_{\mathbb{H}^N} W(r)|\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N} + \int_{\mathbb{H}^N} V(r)(f(r))^2 \left| \nabla_{\mathbb{H}^N} \left(\frac{u_r}{f(r)} \right) \right|^2 dv_{\mathbb{H}^N}$$
$$- (N-1) \int_{\mathbb{H}^N} V(r) \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$+ (N-1) \int_{\mathbb{H}^N} \left(\frac{V(r)}{\sinh^2 r} - \frac{V_r(r) \cosh r}{\sinh r} \right) |\nabla_{\mathbb{H}^N} u|^2 \, dv_{\mathbb{H}^N}$$
$$= \sum_{n=0}^{\infty} \Bigg[\int_0^\infty W {a_n'}^2 \psi^{N-1} \, dr + \lambda_n \int_0^\infty W a_n^2 \psi^{N-3} \, dr$$
$$+ \int_0^\infty V f^2 \left[\left(\frac{a_n'}{f} \right)' \right]^2 \psi^{N-1} \, dr + \lambda_n \int_0^\infty V {a_n'}^2 \psi^{N-3} \, dr$$
$$- (N-1) \int_0^\infty V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) {a_n'}^2 \psi^{N-1} \, dr$$
$$- (N-1)\lambda_n \int_0^\infty V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) a_n^2 \psi^{N-3} \, dr$$
$$+ (N-1) \int_0^\infty \left(\frac{V(r)}{\psi^2} - \frac{\psi'}{\psi} V_r(r) \right) (a_n')^2 \psi^{N-1} \, dr$$
$$+ (N-1)\lambda_n \int_0^\infty \left(\frac{V(r)}{\psi^2} - \frac{\psi'}{\psi} V_r(r) \right) \frac{a_n^2}{\psi^2} \psi^{N-1} \, dr \Bigg].$$

Therefore, we will be done if we prove that the following quantity $\mathcal{B}$ is non-negative:

$$\begin{aligned}\mathcal{B} := \sum_{n=0}^{\infty} \Bigg[& \lambda_n^2 \int_0^\infty V(r)\frac{a_n^2}{\psi^4}\psi^{N-1}\,\mathrm{d}r \\ & - 2\,\lambda_n \int_0^\infty V(r)\left(a_n'' + (N-1)\frac{\psi'}{\psi}a_n'\right)\frac{a_n}{\psi^2}\psi^{N-1}\,\mathrm{d}r \\ & - \lambda_n \int_0^\infty W(r)\frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r - (N-1)\lambda_n \int_0^\infty \left(\frac{V(r)}{\psi^2} - \frac{\psi'}{\psi}V_r(r)\right)\frac{a_n^2}{\psi^2}\psi^{N-1}\,\mathrm{d}r \\ & - \lambda_n \int_0^\infty V(a_n')^2\psi^{N-3}\,\mathrm{d}r + (N-1)\lambda_n \int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)a_n^2\psi^{N-3}\,\mathrm{d}r \Bigg].\end{aligned} \tag{5.5.4}$$

To show that $\mathcal{B}$ is non-negative, we establish some preliminary identities. Let $b_n(r) := \frac{a_n(r)}{\psi(r)}$, by Leibniz rule we have $a_n' = b_n'\psi + b_n\psi'$. Using this and by parts formula, we obtain

$$\begin{aligned}\int_0^\infty V{a_n'}^2\psi^{N-3}\,\mathrm{d}r = & \int_0^\infty V{b_n'}^2\psi^{N-1}\,\mathrm{d}r - (N-3)\int_0^\infty Vb_n^2\psi^{N-3}\,\mathrm{d}r \\ & - \int_0^\infty V_r b_n^2\psi'\psi^{N-2}\,\mathrm{d}r - (N-2)\int_0^\infty Vb_n^2\psi^{N-1}\,\mathrm{d}r.\end{aligned} \tag{5.5.5}$$

Then applying Lemma 5.2.1 for b_n, we deduce

$$\begin{aligned}\int_0^\infty V{b_n'}^2\psi^{N-1}\,\mathrm{d}r = & \int_0^\infty Wb_n^2\psi^{N-1}\,\mathrm{d}r + \int_0^\infty Vf^2\left[\left(\frac{b_n}{f}\right)'\right]^2\psi^{N-1}\,\mathrm{d}r \\ & - (N-1)\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)b_n^2\psi^{N-1}\,\mathrm{d}r.\end{aligned}$$

Using this estimate into (5.5.5) and writing b_n in terms of a_n, we have

$$\begin{aligned}&\int_0^\infty V{a_n'}^2\psi^{N-3}\,\mathrm{d}r = \int_0^\infty Wa_n^2\psi^{N-3}\,\mathrm{d}r + \int_0^\infty Vf^2\left[\left(\frac{a_n}{f\psi}\right)'\right]^2\psi^{N-1}\,\mathrm{d}r \\ & - (N-1)\int_0^\infty V\frac{f'}{f}\left(\coth r - \frac{1}{r}\right)a_n^2\psi^{N-3}\,\mathrm{d}r - (N-3)\int_0^\infty Va_n^2\psi^{N-5}\,\mathrm{d}r \\ & - \int_0^\infty V_r a_n^2\psi'\psi^{N-4}\,\mathrm{d}r - (N-2)\int_0^\infty Va_n^2\psi^{N-3}\,\mathrm{d}r.\end{aligned} \tag{5.5.6}$$

Before proving $\mathcal{B}$ is non-negative, first exploiting by parts formula, we evaluate separately some terms. First there holds

$$\int_0^\infty V a_n'' a_n \psi^{N-3}\,\mathrm{d}r = \frac{1}{2}\int_0^\infty V_{rr} a_n^2 \psi^{N-3}\,\mathrm{d}r + \frac{(N-3)}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r - \int_0^\infty V (a_n')^2 \psi^{N-3}\,\mathrm{d}r - (N-3)\int_0^\infty V a_n' a_n \psi' \psi^{N-4}\,\mathrm{d}r \tag{5.5.7}$$

and then

$$\int_0^\infty V a_n' a_n \psi' \psi^{N-4}\,\mathrm{d}r = -\frac{1}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r - \frac{(N-4)}{2}\int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r - \frac{(N-3)}{2}\int_0^\infty V a_n^2 \psi^{N-3}\,\mathrm{d}r. \tag{5.5.8}$$

Next, using (5.5.6), (5.5.7) and (5.5.8) into (5.5.4), we derive

$$\begin{aligned}
\mathcal{B} = \sum_{n=0}^{\infty}\Bigg[&\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r - 2\lambda_n \int_0^\infty V a_n'' a_n \psi^{N-3}\,\mathrm{d}r \\
&- 2(N-1)\lambda_n \int_0^\infty V a_n' a_n \psi' \psi^{N-4}\,\mathrm{d}r - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&- (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r + (N-1)\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r \\
&- \lambda_n \int_0^\infty V (a_n')^2 \psi^{N-3}\,\mathrm{d}r + (N-1)\lambda_n \int_0^\infty V \frac{f'}{f}\left(\coth r - \frac{1}{r}\right) a_n^2 \psi^{N-3}\,\mathrm{d}r\Bigg] \\
= \sum_{n=0}^{\infty}\Bigg[&\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r - 2\lambda_n\Bigg\{\frac{1}{2}\int_0^\infty V_{rr} a_n^2 \psi^{N-3}\,\mathrm{d}r \\
&+ \frac{(N-3)}{2}\int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r - \int_0^\infty V (a_n')^2 \psi^{N-3}\,\mathrm{d}r \\
&- (N-3)\int_0^\infty V a_n' a_n \psi' \psi^{N-4}\,\mathrm{d}r\Bigg\} - 2(N-1)\lambda_n \int_0^\infty V a_n' a_n \psi' \psi^{N-4}\,\mathrm{d}r \\
&- \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\,\mathrm{d}r - (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\,\mathrm{d}r \\
&+ (N-1)\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\,\mathrm{d}r - \lambda_n \int_0^\infty V (a_n')^2 \psi^{N-3}\,\mathrm{d}r \\
&+ (N-1)\lambda_n \int_0^\infty V \frac{f'}{f}\left(\coth r - \frac{1}{r}\right) a_n^2 \psi^{N-3}\,\mathrm{d}r\Bigg].
\end{aligned}$$

Further calculation gives

$$
\begin{aligned}
&\mathcal{B}\\
&= \sum_{n=0}^{\infty} \Bigg[\lambda_n^2 \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - 2\lambda_n \Bigg\{ \frac{1}{2} \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r - \int_0^\infty V {a_n'}^2 \psi^{N-3}\, \mathrm{d}r \Bigg\}\\
&- 4\lambda_n \Bigg\{ -\frac{1}{2} \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - \frac{(N-4)}{2} \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r\\
&- \frac{(N-3)}{2} \int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r \Bigg\} - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&- (N-1)\lambda_n \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r + 2\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r\\
&- \lambda_n \int_0^\infty V (a_n')^2 \psi^{N-3}\, \mathrm{d}r + (N-1)\lambda_n \int_0^\infty V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) a_n^2 \psi^{N-3}\, \mathrm{d}r \Bigg]\\
&= \sum_{n=0}^{\infty} \Bigg[\lambda_n (\lambda_n + N - 7) \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&+ 2\lambda_n \int_0^\infty V {a_n'}^2 \psi^{N-3}\, \mathrm{d}r + 4\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r\\
&+ 2\lambda_n (N-3) \int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&- \lambda_n \int_0^\infty V (a_n')^2 \psi^{N-3}\, \mathrm{d}r + (N-1)\lambda_n \int_0^\infty V \frac{f'}{f} \left(\coth r - \frac{1}{r} \right) a_n^2 \psi^{N-3}\, \mathrm{d}r \Bigg]\\
&= \sum_{n=0}^{\infty} \Bigg[\lambda_n (\lambda_n + N - 7) \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&+ 4\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r + \lambda_n \Bigg\{ \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&+ \int_0^\infty V f^2 \left[\left(\frac{a_n}{f\psi} \right)' \right]^2 \psi^{N-1}\, \mathrm{d}r - (N-3) \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r\\
&- \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r - (N-2) \int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r \Bigg\}\\
&+ 2\lambda_n (N-3) \int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r - \lambda_n \int_0^\infty W a_n^2 \psi^{N-3}\, \mathrm{d}r \Bigg]\\
&= \sum_{n=0}^{\infty} \Bigg[\lambda_n (\lambda_n - 4) \int_0^\infty V a_n^2 \psi^{N-5}\, \mathrm{d}r - \lambda_n \int_0^\infty V_{rr} a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&+ 3\lambda_n \int_0^\infty V_r a_n^2 \psi' \psi^{N-4}\, \mathrm{d}r + \lambda_n (N-4) \int_0^\infty V a_n^2 \psi^{N-3}\, \mathrm{d}r\\
&+ \lambda_n \int_0^\infty V f^2 \left[\left(\frac{a_n}{f\psi} \right)' \right]^2 \psi^{N-1}\, \mathrm{d}r \Bigg].
\end{aligned}
$$

Finally continuing the calculation we obtain

$$\begin{aligned}\mathcal{B} &= \sum_{n=0}^{\infty} \lambda_n \Bigg[(\lambda_n - 4) \int_0^{\infty} V a_n^2 \psi^{N-5} \, \mathrm{d}r + \int_0^{\infty} \left\{ \frac{3V_r \psi'}{\psi} - V_{rr} \right\} a_n^2 \psi^{N-3} \, \mathrm{d}r \\ &+ (N-4) \int_0^{\infty} V a_n^2 \psi^{N-3} \, \mathrm{d}r + \int_0^{\infty} V f^2 \left[\left(\frac{a_n}{f\psi} \right)' \right]^2 \psi^{N-1} \, \mathrm{d}r \Bigg] \\ &\geq \sum_{n=0}^{\infty} \lambda_n \Bigg[\int_0^{\infty} \left\{ (N-5) \frac{V}{\psi^2} + 3 \frac{V_r \psi'}{\psi} - V_{rr} + (N-4) V \right\} a_n^2 \psi^{N-3} \, \mathrm{d}r \\ &+ \int_0^{\infty} V f^2 \left[\left(\frac{a_n}{f\psi} \right)' \right]^2 \psi^{N-1} \, \mathrm{d}r \Bigg].\end{aligned}$$

In the last line we have used $\lambda_n \geq N - 1$ for all $n \geq 1$. Hence, $\mathcal{B}$ eventually turns out to be non-negative due to the hypothesis (5.3.2) and the non negativity of the last term. □

Proof of Theorem 5.3.3. The idea of the proof is similar to that of Theorem 5.1.3. First, exploiting the given conditions into Theorem 5.3.2, we deduce that for all $u \in \mathcal{C}_c^{\infty}(\mathbb{H}^N \setminus \{x_o\})$ there holds

$$\int_{\mathbb{H}^N} V(r) |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \geq \int_{\mathbb{H}^N} \tilde{W}(r) |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \, .$$

Finally, we use Hölder inequality and the above to get:

$$\begin{aligned}\left(\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right)^2 &= \left(\int_{\mathbb{H}^N} \sqrt{\tilde{W}(r)} |\nabla_{\mathbb{H}^N} u| \frac{|\nabla_{\mathbb{H}^N} u|}{\sqrt{\tilde{W}(r)}} \, \mathrm{d}v_{\mathbb{H}^N} \right)^2 \\ &\leq \left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\tilde{W}(r)} \, \mathrm{d}v_{\mathbb{H}^N} \right) \left(\int_{\mathbb{H}^N} \tilde{W}(r) |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right) \\ &\leq \left(\int_{\mathbb{H}^N} \frac{|\nabla_{\mathbb{H}^N} u|^2}{\tilde{W}(r)} \, \mathrm{d}v_{\mathbb{H}^N} \right) \left(\int_{\mathbb{H}^N} V(r) |\Delta_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} \right).\end{aligned}$$

The proof of Heisenberg-Pauli-Weyl uncertainty principle involving radial part of the operator is in a similar line. □

5.6 Appendix: a family of improved Hardy-Poincaré equalities

In this appendix we present a family of improved Hardy-Poincaré equalities which follows as a corollary from [60, Theorem 3.2], i.e. Lemma 5.2.1 above, by exploiting the family of Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$ introduced in Section 5.4 for all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N)$. If $\lambda = \lambda_1(\mathbb{H}^N)$ the identity we got is already known from [60, Theorem 3.2] while for $0 \leq \lambda < \lambda_1(\mathbb{H}^N)$ it is new and improves (1.1.9), i.e. [28, Theorem 2.1], with the presence of an exact remainder term. The precise statement of the result reads as follows:

Theorem 5.6.1. *Let $N \geq 2$. For all $0 \leq \lambda \leq \lambda_1(\mathbb{H}^N) = \left(\frac{N-1}{2}\right)^2$ and for all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \left[\frac{(N-2)^2}{4} - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \gamma_N(\lambda) h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{\mathbb{H}^N} \left(\frac{u}{\Psi_\lambda(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}$$

and for the radial operator we have

$$\int_{\mathbb{H}^N} |\nabla_{r,\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} = \lambda \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} + h_N^2(\lambda) \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \left[\frac{(N-2)^2}{4} - h_N^2(\lambda)\right] \int_{\mathbb{H}^N} \frac{u^2}{\sinh^2 r} \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \gamma_N(\lambda) h_N(\lambda) \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} u^2 \, \mathrm{d}v_{\mathbb{H}^N}$$
$$+ \int_{\mathbb{H}^N} (\Psi_\lambda(r))^2 \left|\nabla_{r,\mathbb{H}^N} \left(\frac{u}{\Psi_\lambda(r)}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N},$$

where $\gamma_N(\lambda) := \sqrt{(N-1)^2 - 4\lambda}$, $h_N(\lambda) := \frac{\gamma_N(\lambda)+1}{2}$ and $\Psi_\lambda(r) := r^{-\frac{N-2}{2}} \left(\frac{\sinh r}{r}\right)^{-\frac{N-1+\gamma_N(\lambda)}{2}}$.

Proof. The proof follows by applying [60, Theorem 3.2], i.e. Lemma 5.2.1

above, with the Bessel pairs $(r^{N-1}, r^{N-1}W_\lambda)$, where W_λ is as given in (5.4.1).

□

In particular, for $\lambda = N-2$ Theorem 5.6.1 yields the Hardy identity below which improves (1.1.10):

Corollary 5.6.1. *Let $N \geq 3$. For all $u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_o\})$ there holds*

$$\begin{aligned}\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u|^2 \, \mathrm{d}v_{\mathbb{H}^N} &= \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{H}^N} \frac{u^2}{r^2} \, \mathrm{d}v_{\mathbb{H}^N} + (N-2) \int_{\mathbb{H}^N} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \frac{(N-2)(N-3)}{2} \int_{\mathbb{H}^N} \frac{r \coth r - 1}{r^2} u^2 \, \mathrm{d}v_{\mathbb{H}^N} \\ &+ \int_{\mathbb{H}^N} \left(\frac{r^{1/2}}{\sinh r}\right)^{2(N-2)} \left|\nabla_{\mathbb{H}^N} \left(\frac{(\sinh r)^{N-2} u}{r^{(N-2)/2}}\right)\right|^2 \mathrm{d}v_{\mathbb{H}^N}.\end{aligned}$$

Part II

Study of eigenvalue problems for second-order elliptic PDEs

Chapter 6

Eigenvalue problems for second-order elliptic PDEs

The second part of the thesis devoted to study the generalized eigenvalue problem in $\mathbb{R}^N$ for a general convex nonlinear elliptic operator which is locally elliptic and positively homogeneous. Also in the later part of this we study ergodic problems in the whole space $\mathbb{R}^N$ for a weakly coupled systems of viscous Hamilton-Jacobi equations with coercive right-hand sides. One can find the details in the Chapter 7 and 8 covering the material of [38] and [9] respectively.

6.1 Brief description of the obtained results

Here we will discuss the brief description of the problems and obtained results of the second part of the thesis.

6.1.1 Problem 4

This main contribution in Chapter 7 of the thesis is to the study of eigenvalue problem of the form

$$F(D^2\psi, D\psi, \psi, x) = \lambda\psi \quad \text{in } \mathbb{R}^N,$$

where F is a fully nonlinear, convex, positively 1-homogeneous elliptic operator with measurable coefficients. We establish the existence of half (or demi) eigenvalues and characterize the set of all eigenvalues with positive and negative eigenfunctions. This generalizes a recent work of Berestycki and Rossi [35] which considers linear elliptic operators. We also derive necessary and sufficient conditions for the validity maximum principles in $\mathbb{R}^N$ and discuss the uniqueness of principal eigenfunctions.

It has long been known that certain types of positively homogeneous operators possess two principal eigenvalues (one corresponds to a positive eigenfunction and the other one corresponds to a negative eigenfunction). In fact, this first appeared in the work of Pucci [108] who computed these eigenvalues explicitly for certain extremal operators in the unit ball. Later it also appeared in a work of Berestycki [31] while studying the bifurcation phenomenon for some nonlinear Sturm-Liouville problem and Berestycki referred them as half eigenvalues. In connection to this work of Berestycki, Lions used a stochastic control approach in [92] to characterize these eigenvalues (he called it demi-eigenvalues) for operators which are the supremum of linear operators with $\mathcal{C}^{1,1}$-coefficients, and relate them to certain bifurcation problem. In their seminal work [33] Berestycki, Nirenberg and Varadhan introduced the notion of Dirichlet generalized principal eigenvalue for linear operators in non-smooth bounded domains and also established a deep connection between sign of the principal eigenvalue and validity of maximum principles. This work serves as a founding stone of the modern eigentheory and has been used to study eigenvalue problems for general nonlinear operators, including degenerate ones. We are in particular, attracted by the works [13,14,32,37,43,44,73,74,104,109]. We owe much to the work of Quass and Sirakov [109] who study the Dirichlet principal eigenvalue problem for convex, fully nonlinear operators in bounded domains.

All the above mentioned works deal with bounded domains. It is then natural to ask how the eigentheory changes for unbounded domains. In fact, the necessity for studying eigenvalue problems in $\mathbb{R}^N$ becomes important to understand the existence and uniqueness of solutions for certain semi-linear elliptic operators. See for instance, the discussion in [34, 35] and references therein. Principal eigenvalue is a key ingredient to find the rate functional for the large deviation estimate of empirical measures of diffusions [54,55,75]. Recently, eigenvalue problems in $\mathbb{R}^N$ have got much attention due to its application in the theory of risk-sensitive controls [4, 5, 10] (see the discussion in Subsection 7.1). Presently we are motivated by a recent study of Berestycki and Rossi [35] where the authors consider non-degenerate linear elliptic operators and develop an eigentheory for unbounded domains. Monotonicity property of the principal eigenvalue (with respect to the potentials) in $\mathbb{R}^N$ and its relation with the stability property of the twisted process is established in [10]. The paper [4] considers a class of semi-linear elliptic operators and obtains a variational representation of the principal eigenvalue under the assumption of geometric stability. Here we will develop an eigentheory for fully nonlinear positively homogeneous operators in the whole space $\mathbb{R}^N$ (see Chapter 7 for details). Similar kind of generalized eigenvalue problem has been studied in [8] on the whole Euclidean space for a class of integro-differential elliptic operators.

6.1.2 Problem 5

In Chapter 8 of the thesis we study the existence and uniqueness of solution $(\boldsymbol{u}, \lambda) = (u_1, u_2, \lambda)$ to the equation

$$\begin{aligned} -\Delta u_1(x) + H_1(x, \nabla u_1(x)) + \alpha_1(x)(u_1(x) - u_2(x)) &= f_1(x) - \lambda \quad \text{in } \mathbb{R}^N, \\ -\Delta u_2(x) + H_2(x, \nabla u_2(x)) + \alpha_2(x)(u_2(x) - u_1(x)) &= f_2(x) - \lambda \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{6.1.1}$$

where $H_i : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ denote the Hamiltonians, and $\alpha_i : \mathbb{R}^N \to \mathbb{R}_+$ are the switching rate parameters for $i = 1, 2$. The system of equations (6.1.1) arises as the Hamilton-Jacobi equations (HJE) in certain ergodic control problems of diffusions in a switching environment. The ergodic control problems for scalar second order elliptic equations has been studied extensively by several mathematicians and therefore, it is almost impossible to list all the important works in this direction. Nevertheless, we mention some of them that, in our opinion, are milestones in this topic. Ergodic control problems with quadratic Hamiltonian were first studied by Bensoussan and Freshe [23, 24] where the authors established the existence and uniqueness of unbounded solutions in $\mathbb{R}^N$. For space-time periodic Hamiltonians, the existence and uniqueness are considered by Barles and Souganidis [22]. Ichihara [70–72] considers the problem for a general class of Hamiltonians and recurrence/transience properties of the optimal feedback controls are also discussed. We also mention the work of Cirant [50] who investigates the ergodic control problem in $\mathbb{R}^N$ for a fairly general family of Hamiltonians. It is shown in [50] that the problem in $\mathbb{R}^N$ can be approximated by the ergodic control problems in bounded domains with Neumann boundary condition. Recently, the uniqueness of unbounded solutions for a general family of right-hand sides is established by Barles and Meireles [19], which is then further improved by the first two authors and Caffarelli [6] in the subcritical case. There are also several important works studying long-time behaviour of the solutions to certain parabolic equations and its convergence to the solutions to the ergodic control problems: see for instance, Barles-Souganidis [22], Fujita-Ishii-Loreti [61], Tchamba [116], Ichihara [71], Barles-Porretta-Tchamba [20], Barles-Quaas-Rodríguez [21].

On the other hand, number of works on the ergodic control problem for weakly coupled elliptic systems are very few. All existing results in this direction consider the domain to be torus and the Hamiltonians to be quadratic. See for instance, Mitake-Tran [97], Filippo-Gomes-Mitake-Tran [47] and references therein. However, if one assumes the action set to be compact then similar problems have been addressed in detail, see Ghosh-Arapostathis-Marcus [12], Arapostathis-Borkar-Ghosh [11, Chapter 5]. One of the main challenges in studying the weakly coupled systems is in establishing appropriate gradient estimates and bounds on the term $|u_1 - u_2|$ (see Proposition 8.4.1 below and the Chapter 8 for details).

Chapter 7

Generalized principal eigenvalues of convex nonlinear elliptic operators in $\mathbb{R}^N$

In this chapter we will study the generalized eigenvalue problem in $\mathbb{R}^N$ for a general convex nonlinear elliptic operator which is locally elliptic and positively homogeneous. Generalizing Berestycki and Rossi [35] we consider three different notions of generalized eigenvalues and compare them. We also discuss the maximum principles and uniqueness of principal eigenfunctions. The content of this chapter corresponds to the article [38].

7.1 Motivations behind the problem

One of the important examples of F comes from the control theory. In particular, we may consider

$$\begin{aligned} F(D^2\phi, D\phi, \phi, x) &= \sup_{\alpha}\{\operatorname{trace}(a_\alpha(x)D^2\phi(x)) + b_\alpha(x)\cdot D\phi(x) + c_\alpha(x)\phi(x)\} \\ &= \sup_{\alpha}\{L_\alpha\phi + c_\alpha\phi\}\,, \end{aligned} \tag{7.1.1}$$

where α varies over some index set $\mathcal{J}$, $\lambda(x)I \leq a_\alpha(x) \leq \Lambda(x)I$, and $\sup_{\alpha\in\mathcal{J}} |b_\alpha(x)|$, $\sup_{\alpha\in\mathcal{J}} |c_\alpha(x)|$ are locally bounded. The eigenvalue problem corresponding to the operator F appears in the study of risk-sensitive controls. See for instance, [5, 10] and references therein. To elaborate, suppose that $\mathcal{J}$ is a compact subset subset of $\mathbb{R}^m$. Let $\mathfrak{U}$ be the collection of Borel measurable maps $\alpha : \mathbb{R}^N \to \mathcal{J}$. Note that constant functions are also included in $\mathfrak{U}$. This set $\mathfrak{U}$ represents the collection of all Markov controls. Given $\alpha \in \mathfrak{U}$, suppose that X_α is the Markov diffusion process with generator L_α. Denote the law of X_α by $\mathbb{P}_\alpha$ and $\mathbb{E}_\alpha[\cdot]$ is the expectation operator associated with it. Consider the maximization problem

$$\Lambda = \sup_{\alpha\in\mathfrak{U}} \limsup_{T\to\infty} \frac{1}{T} \log \mathbb{E}_\alpha \left[e^{\int_0^T c_\alpha(X_t)\mathrm{d}t} \right].$$

Then under reasonable hypothesis, one can show that Λ is an eigenvalue of F (i.e. $\Lambda \in \mathcal{E}^+$) and for many practical reasons it is desirable that $\Lambda = \lambda_1^+(F)$. Also, simplicity of $\lambda_1^+(F)$ is important to find an optimal strategy or control. We refer the readers to [5, 10] for more details on this problem.

7.2 Preliminary model and assumptions

In this section we will first introduce our model and elaborate some basic assumptions. Let $\lambda, \Lambda : \mathbb{R}^N \to (0, \infty)$ be two locally bounded functions with the property that for any compact set $K \subset \mathbb{R}^N$ we have

$$0 < \inf_{x\in K} \lambda(x) \leq \sup_{x\in K} \Lambda(x) < \infty.$$

Choosing $K = \{x\}$ it follows from above that $0 < \lambda(x) \leq \Lambda(x)$ for all $x \in \mathbb{R}^N$. These two functions will be treated as the bounds of the ellipticity constants at point x. By $\mathcal{S}_N$ we denote the set of all $N \times N$ real symmetric matrices. The extremal Pucci operators corresponding to λ, Λ are defined as follows. For $M \in \mathcal{S}_N$ the extremal operators at $x \in \mathbb{R}^N$ are defined to be

$$\mathcal{M}^+_{\lambda,\Lambda}(x, M) = \sup_{\lambda(x)I \leq A \leq \Lambda(x)I} \text{trace}(AM) = \Lambda(x) \sum_{\beta_i \geq 0} \beta_i + \lambda(x) \sum_{\beta_i < 0} \beta_i,$$

$$\mathcal{M}^-_{\lambda,\Lambda}(x, M) = \inf_{\lambda(x)I \leq A \leq \Lambda(x)I} \text{trace}(AM) = \lambda(x) \sum_{\beta_i \geq 0} \beta_i + \Lambda(x) \sum_{\beta_i < 0} \beta_i,$$

where $\beta_1, \ldots, \beta_n$, denote the eigenvalues of the matrix M.

Our operator F is a Borel measurable function $F : \mathcal{S}_N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, with the following properties:

(H1) F is positively 1-homogeneous in the variables $(M, p, u) \in \mathcal{S}_N \times \mathbb{R}^N \times \mathbb{R}$ i.e., for every $t > 0$ we have we have

$$F(tM, tp, tu, x) \; = \; tF(M, p, u, x) \quad \text{for all } x \in \mathbb{R}^N.$$

In particular, $F(0, 0, 0, x) \equiv 0$.

(H2) F is convex in the variables $(M, p, u) \in \mathcal{S}_N \times \mathbb{R}^N \times \mathbb{R}$.

(H3) There exist locally bounded functions $\gamma, \delta : \mathbb{R}^N \to [0, \infty)$ satisfying

$$\begin{aligned} &\mathcal{M}^-_{\lambda,\Lambda}(x, M - N) - \gamma(x)|p - q| - \delta(x)|u - v| \\ &\leq F(M, p, u, x) - F(N, q, v, x) \\ &\leq \mathcal{M}^+_{\lambda,\Lambda}(x, M - N) + \gamma(x)|p - q| + \delta(x)|u - v|, \end{aligned}$$

for all $M, N \in \mathcal{S}_N$, $p, q \in \mathbb{R}^N, u, v \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

(H4) The function $(M, x) \in \mathcal{S}_N \times \mathbb{R}^N \mapsto F(M, 0, 0, x)$ is continuous.

Throughout this chapter we assume the conditions (H1)-(H4) without any further mention. Also, observe that due to our hypotheses the operator F satisfies the conditions in [109] which studies the Dirichlet eigenvalue problem for F in bounded domains. Therefore the results of [109] holds for F in smooth bounded domains.

Let us now define the principal eigenvalues of F in a smooth domain $\Omega \subset \mathbb{R}^N$, possibly unbounded. For any real number λ we define the following sets

$$\Psi^+(F, \Omega, \lambda) = \{\psi \in \mathscr{W}^{2,N}_{\text{loc}}(\Omega) \ : \ F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ and } \psi > 0 \ \text{ in } \ \Omega\},$$

and

$$\Psi^-(F, \Omega, \lambda) = \{\psi \in \mathscr{W}^{2,N}_{\text{loc}}(\Omega) \ : \ F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \ \text{ and } \ \psi < 0 \text{ in } \Omega\}.$$

By sub or super-solution we always mean L^N-strong solution. The (half) eigenvalues are defined to be

$$\lambda_1^+(F, \Omega) = \sup\{\lambda \in \mathbb{R} \ : \ \Psi^+(F, \Omega, \lambda) \neq \emptyset\},$$
$$\lambda_1^-(F, \Omega) = \sup\{\lambda \in \mathbb{R} \ : \ \Psi^-(F, \Omega, \lambda) \neq \emptyset\}.$$

Using the convexity of F and [109, Proposition 4.2] it follows that $\lambda_1^+(F, \Omega) \leq \lambda_1^-(F, \Omega) < \infty$. For F linear we also have $\lambda_1^+(F, \Omega) = \lambda_1^-(F, \Omega)$. In this chapter we would be interested in the case $\Omega = \mathbb{R}^N$ and for notational economy we denote $\lambda_1^{\pm}(F, \mathbb{R}^N) = \lambda_1^{\pm}(F)$.

Remark 7.2.1. *We can replace the L^N-strong super and sub-solutions in $\Psi^{\pm}(F, \Omega, \lambda)$ by L^N-viscosity super and sub-solutions, respectively.*

7.3 Statement of main results

We now state our main results. Most of the results obtained here are generalization of its linear counterpart in [35]. Recall from [35, Theorem 1.4] that for F linear and $\lambda \in (-\infty, \lambda_1(F)]$ there exists a positive function $\varphi \in \mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N), p > N$, satisfying $F(D^2\varphi, D\varphi, \varphi, x) + \lambda\varphi = 0$ in $\mathbb{R}^N$. Thus there is a continuum of eigenvalues with the largest one being the principal eigenvalue. This leads us to the following sets of eigenvalues.

Definition 7.3.1. *We say $\lambda \in \mathbb{R}$ is an eigenvalue with a positive eigenfunction if there exists $\phi \in \mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N), p > N$, such that*

$$F(D^2\phi, D\phi, \phi, x) = -\lambda\phi \quad \text{in}\, \mathbb{R}^N\,, \quad \text{and} \quad \phi > 0 \quad \text{in}\, \mathbb{R}^N.$$

We denote the collection of all eigenvalues with positive eigenfunctions by $\mathcal{E}^+$. Analogously, we define $\mathcal{E}^-$ as the collection of all eigenvalues with negative eigenfunctions.

Our first result generalizes [35, Theorem 1.4].

Theorem 7.3.1. *We have $\mathcal{E}_+ = (-\infty, \lambda_1^+(F)]$ and $\mathcal{E}_- = (-\infty, \lambda_1^-(F)]$.*

It is well known that for bounded domains it is also possible to define principal eigenvalues through sub-solutions (cf. [109, Theorem 1.2]). However, this situation is bit different for unbounded domains. To explain we introduce the following quantities.

$$\begin{aligned}\lambda_1^{\prime,+}(F) \coloneqq \inf\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N),\\ \psi > 0, F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \text{ in } \mathbb{R}^N\},\end{aligned}$$

$$\begin{aligned}\lambda_1^{\prime,-}(F) \coloneqq \inf\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N),\\ \psi < 0, F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ in } \mathbb{R}^N\},\end{aligned}$$

and

$$\begin{aligned}\lambda_1^{\prime\prime,+}(F) \coloneqq \sup\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N),\\ \inf_{\mathbb{R}^N}\psi > 0,\; F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ in } \mathbb{R}^N\}\,,\end{aligned}$$

$$\begin{aligned}\lambda_1^{\prime\prime,-}(F) \coloneqq \sup\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N),\\ \sup_{\mathbb{R}^N}\psi < 0,\; F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \text{ in } \mathbb{R}^N\}\,.\end{aligned}$$

We remark that in case of bounded domains one has

$$\lambda_1(F,\Omega) = \lambda_1^{\prime,+}(F,\Omega) = \lambda_1^{\prime\prime,+}(F,\Omega)$$

and

$$\lambda_1(F,\Omega) = \lambda_1^{\prime,-}(F,\Omega) = \lambda_1^{\prime\prime,-}(F,\Omega),$$

provided we required the sub-solution (super-solution) to vanish on $\partial\Omega$ in the definition of $\lambda_1^{\prime,+}$ ($\lambda_1^{\prime,-}$, resp.) (cf. [109]). But the same might fail to hold in unbounded domains (counter-example in [34, p. 201]). However, we could prove the following relation which generalizes [35, Theorem 1.7].

Theorem 7.3.2. *The following hold.*

(i) We have $\lambda_1^{\prime,+}(F) \le \lambda_1^{+}(F)$ and $\lambda_1^{\prime,-}(F) \le \lambda_1^{-}(F)$.

(ii) Suppose that

$$\sup_{\mathbb{R}^N} \delta(x) < \infty, \quad \limsup_{|x|\to\infty} \frac{\gamma(x)}{|x|} < \infty, \quad \text{and} \quad \limsup_{|x|\to\infty} \frac{\Lambda(x)}{|x|^2} < \infty\,. \tag{7.3.1}$$

Then we have $\lambda_1^{\prime\prime,+}(F) \le \lambda_1^{\prime,+}(F)$ and $\lambda_1^{\prime\prime,-}(F) \le \lambda_1^{\prime,-}(F)$.

In view of Theorem 7.3.2 we see that $\lambda_1^{\prime\prime,+}(F) \le \lambda_1^{\prime,+}(F) \le \lambda_1^{+}(F)$ and $\lambda_1^{\prime\prime,-}(F) \le \lambda_1^{\prime,-}(F) \le \lambda_1^{-}(F)$, provided (7.3.1) holds. Again, due to the convexity of F we have $\lambda_1^{+}(F) \le \lambda_1^{-}(F)$. One might wonder if there is any natural relation between "plus" and "minus" eigenvalues. We now argue that this might not be possible, in general. If we consider F to be linear then we have $\lambda_1^{\cdot,+}(F) = \lambda_1^{\cdot,-}(F)$, and therefore if (7.3.1) holds, then $\lambda_1^{+}(F) \ge \lambda_1^{\prime\prime,-}(F)$, by Theorem 7.3.2. We now produce an example where the reverse inequality holds.

Example 7.3.1. *Consider two linear elliptic operators of the form*

$$L_\alpha u \;=\; \Delta u + b_\alpha(x)\cdot Du + c_\alpha(x)u,$$

for $\alpha \in \{1,2\}$ *with the properties that*

$$\lambda_1''(L_2,\mathbb{R}^N) \,>\, \lambda_1''(L_1,\mathbb{R}^N) \quad \text{and} \quad \lambda_1''(L_1,\mathbb{R}^N) = \lambda_1'(L_1,\mathbb{R}^N) = \lambda_1(L,\mathbb{R}^N).$$

Now define a nonlinear operator

$$F(D^2u, Du, u, x) \coloneqq \Delta u + \max_{\alpha\in\{1,2\}}\{b_\alpha(x)\cdot Du\} + c_\alpha(x)u\}.$$

It is then easily seen that

$$\lambda_1''^{-}(F) \,\geq\, \max\{\lambda_1''(L_1,\mathbb{R}^N), \lambda_1''(L_2,\mathbb{R}^N)\}\,,$$

and

$$\lambda_1^{+}(F) \,\leq\, \min\{\lambda_1(L_1,\mathbb{R}^N), \lambda_1(L_2,\mathbb{R}^N)\}.$$

Combining we obtain

$$\lambda_1^{'',-}(F) \geq \lambda_1''(L_2) > \lambda_1''(L_1,\mathbb{R}^N) = \lambda_1(L_1,\mathbb{R}^N) \geq \lambda_1^{+}(F).$$

Next we list a few class of operators for which these three eigenvalues coincide (compare them with [35, Theorem 1.9]). We only provide the result for "plus" eigenvalues and the analogous result for "minus" eigenvalues are easy to guess.

Theorem 7.3.3. *The equality* $\lambda_1^+(F) = \lambda_1^{'',+}(F)$ *holds in each of the following cases:*

(i) $F = \tilde{F} + \tilde{\gamma}(x)$, *where* $\tilde{F}$ *is a nonlinear operator with an additional property* $\lambda_1^+(\tilde{F},\mathbb{R}^N) = \lambda_1^{'',+}(\tilde{F},\mathbb{R}^N)$, *and* $\tilde{\gamma} \in L^\infty(\mathbb{R}^N)$ *is a non-negative function satisfying* $\lim_{|x|\to\infty}\tilde{\gamma}(x) = 0$.

(ii) $\lambda_1^+(F) \,\leq\, -\limsup_{|x|\to\infty}\, F(0,0,1,x)$.

(iii) Assume that $\lambda_0 \leq \lambda(x) \leq \Lambda(x) \leq \Lambda_0$ *for all* $x \in \mathbb{R}^N$, $\lim_{|x|\to\infty}\gamma(x) = 0$ *and for all* $r > 0$ *and all* β *such that* $\beta < \limsup_{|x|\to\infty} F(0,0,1,x)$, *there exists* $\mathscr{B}_r(x_0)$ *satisfying* $\inf_{\mathscr{B}_r(x_0)} F(0,0,1,x) > \beta$.

(iv) There exists a $V \in \mathcal{C}^2(\mathbb{R}^N)$ *with* $\inf_{\mathbb{R}^N} V > 0$ *and*

$$F(D^2V, DV, V, x) \leq -\lambda_1^+(F)V \quad \text{for all } x \in \mathscr{B}^c,$$

for some ball $\mathscr{B}$.

Now we turn our attention towards maximum principles. It was observed in the seminal work of Berestycki, Nirenberg and Varadhan [33] that the sign of the principle eigenvalue determines the validity of maximum principles in bounded domains. Extension of this result for nonlinear operators are obtained by Quaas and Sirakov [109] and Armstrong [14]. Further generalization in smooth bounded domains for a class of degenerate, nonlinear elliptic operators are obtained by Berestycki et. al. [32], Birindelli and Demengel [37]. Recently, Berestycki and Rossi [35] establish the maximum principles in unbounded domains for linear elliptic operators. Here we extend their results to our nonlinear setting.

Definition 7.3.2 (Maximum principles)**.** *We say that the operator* F *satisfies* β^+*-MP with respect to a positive function* β *if for any function* $u \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ *satisfying*

$$F(D^2u, Du, u, x) \;\geq\; 0 \quad \text{in } \mathbb{R}^N, \quad \text{and} \quad \sup_{\mathbb{R}^N} \frac{u}{\beta} < \infty\,,$$

we have $u \leq 0$ *in* $\mathbb{R}^N$*. For* $\beta = 1$*, we simply mention this property as* $+$*MP.*

We say that the operator F *satisfies* β^-*-MP with respect to a negative function* β *if for any function* $u \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ *satisfying*

$$F(D^2u, Du, u, x) \;\leq\; 0 \quad \text{in } \mathbb{R}^N, \quad \text{and} \quad \sup_{\mathbb{R}^N} \frac{u}{\beta} < \infty\,,$$

we have $u \geq 0$ *in* $\mathbb{R}^N$*. For* $\beta = -1$*, we simply mention this property as* $-$*MP.*

Note that $\beta \equiv 1$ corresponds to the well known maximum principle. We would be interested in a function $\beta : \mathbb{R}^N \to (0, \infty)$ which satisfies either

$$\exists\, \sigma > 0, \quad \limsup_{|x|\to\infty}\, \beta(x)|x|^{-\sigma} = 0\,, \tag{7.3.2}$$

or

$$\exists\, \sigma > 0, \quad \limsup_{|x|\to\infty} \beta(x) \exp(-\sigma|x|) = 0\,. \tag{7.3.3}$$

Generalizing [35, Definition 1.2] we now consider the following quantities.

Definition 7.3.3. *Given a positive function $\beta : \mathbb{R}^N \to \mathbb{R}$, we define*

$$\lambda''^{,+}_{\beta}(F) := \sup\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N), \\ \psi \geq \beta,\; F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ in } \mathbb{R}^N\}\,,$$

$$\lambda''^{,-}_{\beta}(F) := \sup\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N), \\ \psi \leq -\beta,\; F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \text{ in } \mathbb{R}^N\}\,.$$

Our maximum principles would be established under the following growth conditions on the coefficients.

$$\sup_{\mathbb{R}^N} \delta(x) < \infty, \quad \limsup_{|x|\to\infty} \frac{\gamma(x)}{|x|} < \infty, \quad \text{and} \quad \limsup_{|x|\to\infty} \frac{\Lambda(x)}{|x|^2} \;<\; \infty\,. \tag{7.3.4}$$

$$\sup_{\mathbb{R}^N} \delta(x) < \infty, \quad \sup_{\mathbb{R}^N} \gamma(x) < \infty, \quad \text{and} \quad \sup \mathbb{R}^N\; \Lambda(x) < \infty\,. \tag{7.3.5}$$

Next we state our maximum principle

Theorem 7.3.4. *Suppose that either* (7.3.2) *and* (7.3.4) *or* (7.3.3) *and* (7.3.5) *hold. Then the following hold:*

(i) The operator F satisfies β^+-MP in $\mathbb{R}^N$ if $\lambda''^{,+}_{\beta}(F) > 0$.

(ii) The operator F satisfies $(-\beta)^-$-MP in $\mathbb{R}^N$ if $\lambda''^{,-}_{\beta}(F) > 0$.

As a consequence of Theorem 7.3.4 we obtain the following corollaries.

Corollary 7.3.1. *Suppose that either* (7.3.4) *or* (7.3.5) *holds. Then we have*

(i) The operator F satisfy $+$MP in $\mathbb{R}^N$ if $\lambda''^{,+}_{1}(F) > 0$.

(ii) *The operator F satisfy $-MP$ in $\mathbb{R}^N$ if $\lambda_1^{\prime\prime,-}(F) > 0$.*

(iii) *Suppose that $\lambda_1^{\prime\prime,+}(F) > 0$ (therefore, $\lambda_1^{\prime\prime,-}(F) > 0$). Let $u \in \mathscr{W}^{2,N}_{\mathrm{loc}}(\mathbb{R}^N) \cap Ł^\infty(\mathbb{R}^N)$ satisfy $F(D^2u, Du, u, x) = 0$ in $\mathbb{R}^N$. Then $u \equiv 0$.*

Corollary 7.3.2. *Suppose that F satisfies β^+-MP. Let $u, v \in \mathscr{W}^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ be such that*

$$F(D^2u, Du, u, x) \geq 0, \quad F(D^2v, Dv, v, x) \leq 0 \text{ in } \mathbb{R}^N,$$

and $\sup_{\mathbb{R}^N} \frac{u-v}{\beta} < \infty$. Then we have $u \leq v$ in $\mathbb{R}^N$.

Proof. Denote by $w = u - v$. By using the convexity of F it follows that

$$F(D^2w, Dw, w, x) \geq F(D^2u, Du, u, x) - F(D^2v, Dv, v, x) \geq 0 \quad \text{in } \mathbb{R}^N.$$

Hence the result follows from β^+-MP. □

Generalizing $\lambda_1^{\prime,+}(F)$ and $\lambda_1^{\prime,-}(F)$ we define the following quantities. Let β be a positive valued function and

$$\lambda_\beta^{\prime,+}(F) := \inf\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\mathrm{loc}}(\mathbb{R}^N), \\ \beta \geq \psi > 0,\; F(D^2\psi, D\psi, \psi, x) + \lambda\psi \geq 0 \text{ in } \mathbb{R}^N\},$$

and

$$\lambda_\beta^{\prime,-}(F) := \inf\{\lambda \in \mathbb{R} \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\mathrm{loc}}(\mathbb{R}^N), \\ -\beta \leq \psi < 0,\; F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \text{ in } \mathbb{R}^N\}.$$

As a necessary condition for the validity of maximum principles we deduce the following.

Theorem 7.3.5. *The following hold.*

(i) *If F satisfies the β^+-MP then $\lambda_\beta^{\prime,+}(F) \geq 0$. In particular, if F satisfies $+MP$ then we have $\lambda_1^{\prime,+}(F) \geq 0$.*

(ii) If F satisfies the $(-\beta)^-$-MP then $\lambda'^{,-}_{\beta}(F) \geq 0$. In particular, if F satisfies the $-$MP then we have $\lambda'^{,-}_1(F) \geq 0$.

Finally, we discuss about simplicity of the principal eigenvalues. For linear F uniqueness of principal eigenfunctions can be established imposing *Agmon's minimal growth condition at infinity* [35, Definition 8.2] on the eigenfunctions. But such criterion does not seem to work well for nonlinear F. Recently, in [7, Theorem 2.1] it is shown that Agmon's minimal growth criterion is equivalent to *monotonicity of the principal eigenvalue on the right*. Our next result establish simplicity of principal eigenvalue under certain monotonicity condition of principal eigenvalue *at infinity*.

Theorem 7.3.6. *Suppose that there exists a positive $V \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$F(D^2V, DV, V, x) \leq -(\lambda^+_1(F) + \varepsilon)V \quad \textit{for all}\, x \in K^c, \tag{7.3.6}$$

for some compact ball K and $\varepsilon > 0$. Then $\lambda^+_1(F)$ is simple i.e. the positive eigenfunction is unique upto a multiplicative constant.

We remark that (7.3.6) is equivalent to

$$\lambda^+_1(F) < \lim_{r\to\infty} \lambda^+_1(F, \bar{\mathscr{B}}^c_r).$$

Our next result is about simplicity of $\lambda^-_1(F)$.

Theorem 7.3.7. *Suppose that there exists a positive $V \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying*

$$F(D^2V, DV, V, x) \leq -(\lambda^-_1(F) + \varepsilon)V \quad \textit{for all}\, x \in K^c, \tag{7.3.7}$$

for some compact ball K and $\varepsilon > 0$. Then $\lambda^-_1(F)$ is simple.

7.4 Proofs of main results

In this section we prove Theorem 7.3.1-7.3.7. Let us start by recalling the following Harnack inequality from [109, Theorem 3.6] which will be crucial for our proofs. The result in [109, Theorem 3.6] is stated for L^N-viscosity solutions and also applies to L^N-strong solutions due to [46, Lemma 2.5].

Theorem 7.4.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded. Let $u \in \mathcal{C}(\bar{\Omega}) \cap \mathscr{W}^{2,N}_{\text{loc}}(\Omega)$ and $f \in L^N(\Omega)$ satisfy $u \geq 0$ in Ω and*

$$\mathcal{M}^{+}_{\lambda,\gamma}(x, D^2u) + \gamma|Du| + \delta u \geq f \quad \text{in}\, \Omega,$$
$$\mathcal{M}^{-}_{\lambda,\gamma}(x, D^2u) - \gamma|Du| - \delta u \leq f \quad \text{in}\, \Omega.$$

Then for any compact set $K \Subset \Omega$ we have

$$\sup_K u \;\leq\; C\,[\inf_K u + \|f\|_{L^N(\Omega)}]\,,$$

for some constant C dependent on $K, \Omega, N, \gamma, \delta$, $\min_\Omega \lambda$ and $\max_\Omega \Lambda$.

Next we prove Theorem 7.3.1. The idea is the following: we show using the Harnack inequality and stability estimate that the Dirichlet principal eigenpair in $\mathscr{B}_n$ converges to a principal eigenpair in $\mathbb{R}^N$. For any $\lambda < \lambda_1^+(F)$ or $\lambda < \lambda_1^-(F)$ we use the refined maximum principle in bounded domains and then stability estimate to pass the limit. We spilt the proof of Theorem 7.3.1 in Lemma 7.4.1 and Lemma 7.4.2.

Lemma 7.4.1. *It holds that $\mathcal{E}^+ = (-\infty, \lambda_1^+(F)]$.*

Proof. Let $\lambda_1^+(F, \mathscr{B}_n)$ be the Dirichlet principal eigenvalue in $\mathscr{B}_n$ corresponding to the positive principal eigenfunction. Existence of $\lambda_1^+(F, \mathscr{B}_n)$ follows from [109, Theorem 1.1]. For notational economy we denote $\lambda_1^+(F, \mathscr{B}_n) = \lambda^+_{1,n}$ and $\lambda_1^+(F) = \lambda_1^+$. We also set $E_p(\Omega) = \mathscr{W}^{2,p}_{\text{loc}}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. We divide the proof into two steps.

Step 1. We show that $\lim_{n\to\infty}\lambda^+_{1,n} = \lambda^+_1$ and $\lambda^+_1 \in \mathcal{E}^+$. It is obvious from the definition that $\lambda^+_{1,n}$ is decreasing in n and bounded below by λ^+_1. Thus if $\lim_{n\to\infty}\lambda^+_{1,n} = -\infty$, we also have $\lambda^+_1 = -\infty$ and there is nothing to prove. So we assume $\lim_{n\to\infty}\lambda^+_{1,n} \coloneqq \tilde{\lambda} > -\infty$. It is then obvious that $\tilde{\lambda} \geq \lambda^+_1$. From [109, Theorem 1.1] we have $\psi^+_{1,n} \in E_p(\mathscr{B}_n)$, $\forall\, p < \infty$, such that $\psi^+_{1,n} > 0$ in $\mathscr{B}_n$, $\psi^+_{1,n} = 0$ on $\partial\mathscr{B}_n$ and satisfies

$$F(D^2\psi^+_{1,n}, D\psi^+_{1,n}, \psi^+_{1,n}, x) \;=\; -\lambda^+_{1,n}\psi^+_{1,n} \quad \text{in } \mathscr{B}_n\,, \tag{7.4.1}$$

for all $n \geq 1$. Normalize each $\psi^+_{1,n}$ by choosing $\psi^+_{1,n}(0) = 1$. Fix any compact $K \subset \mathbb{R}^N$ such that $0 \in K$ and choose n_0 large so that $K \Subset \mathscr{B}_m$ for all $m \geq n_0$. Applying Theorem 7.4.1 on (7.4.1) we find a constant $C = C(n_0)$ satisfying

$$\sup_K \psi^+_{1,n} \;\leq\; C\, \inf_K \psi^+_{1,n} \leq C\, \psi^+_{1,n}(0) \;=\; C.$$

Thus applying [109, Theorem 3.3] we obtain, for $p > N$, that

$$\|\psi^+_{1,n}\|_{\mathscr{W}^{2,p}(K)} \;\leq\; C \quad \forall\, n > n_0\,.$$

Since K is arbitrary, using a standard diagonalization argument we can find a non-negative $\varphi^+ \in E_p(\mathbb{R}^N)$, $\forall\, p < \infty$, such that $\psi^+_{1,n} \to \varphi^+$ in $\mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N)$, upto a subsequence. Hence by [46, Theorem 3.8 and Corollary 3.7] we obtain

$$F(D^2\varphi^+, D\varphi^+, \varphi^+, x) = -\tilde{\lambda}\varphi^+ \quad \text{in } \mathbb{R}^N, \quad \varphi^+(0) = 1.$$

Again, applying Theorem 7.4.1 we have $\varphi^+ > 0$. Thus, $\tilde{\lambda} \leq \lambda^+_1$. This shows $\tilde{\lambda} = \lambda^+_1$ and $\lambda^+_1 \in \mathcal{E}^+$.

Step 2. We show that $\mathcal{E}^+ = (-\infty, \lambda^+_1]$. It is obvious that $\mathcal{E}^+ \subset (-\infty, \lambda^+_1]$. To show the reverse relation we consider $\lambda < \lambda^+_1$. We choose a sequence $\{f_n\}_{n\geq 1}$ of continuous, non positive, non-zero functions satisfying

$$\text{support}(f_n) \subset \mathscr{B}_n \setminus \overline{\mathscr{B}}_{n-1} \quad \text{for all } \; n \in \mathbb{N}.$$

Denote by $\tilde{F} = F + \lambda$. Then $\lambda_1^+(\tilde{F}, \mathscr{B}_n) = \lambda_{1,n}^+ - \lambda \geq \lambda_1^+ - \lambda > 0$. Therefore, by [109, Theorem 1.5 and Theorem 1.8], there exists a unique non-negative $u^n \in E_p(B_n)$, for all $p \geq N$, which satisfies

$$\tilde{F}(D^2u^n, Du^n, u^n, x) = f_n \quad \text{in } \mathscr{B}_n, \quad \text{and} \quad u^n = 0 \quad \text{on } \partial\mathscr{B}_n . \qquad (7.4.2)$$

By the strong maximum principle [109, Lemma 3.1] it follows that $u^n > 0$ in $\mathscr{B}_n$. For natural number $n \geq 2$ we define

$$v^n(x) := \frac{u^n(x)}{u^n(0)} .$$

Clearly, $v^n \in E_p(\mathscr{B}_{n-1})$, $\forall\, p < \infty$, positive in $\mathscr{B}_{n-1}$ and $v^n(0) = 1$. Also, by (7.4.2),

$$F(D^2v^n, Dv^n, v^n, x) = -\lambda v^n \quad \text{in } \mathscr{B}_{n-1} .$$

Now we continue as in Step 1 and extract a subsequence of v^n that converges in $\mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N)$ to some positive $\varphi \in E_p(\mathbb{R}^N)$, $\forall\, p < \infty$, and satisfies

$$F(D^2\varphi, D\varphi, \varphi, x) = -\lambda\varphi \quad \text{in } \mathbb{R}^N .$$

This gives us $\lambda \in \mathcal{E}_+$. Thus $\mathcal{E}_+ = (-\infty, \lambda_1^+]$. □

Next lemma concerns the eigenvalues with negative eigenfunctions.

Lemma 7.4.2. *It holds that $\mathcal{E}^- = (-\infty, \lambda_1^-(F)]$.*

Proof. Idea of the proof is similar to Lemma 7.4.1. Let $\lambda_1^-(F, \mathscr{B}_n)$ be the Dirichlet principal eigenvalue in $\mathscr{B}_n$ corresponding to the negative principal eigenfunction [109, Theorem 1.1]. For simplicity we denote

$$\lambda_1^-(F, \mathscr{B}_n) = \lambda_{1,n}^- \text{ and } \lambda_1^-(F) = \lambda_1^- .$$

We divide the proof of into two steps.

Step 1. We show that $\lim_{n\to\infty} \lambda_{1,n}^- = \lambda_1^-$ and $\lambda_1^- \in \mathcal{E}^-$. It is obvious from the definition that $\lambda_{1,n}^-$ in decreasing in n and bounded below by λ_1^-. Thus if $\lim_{n\to\infty} \lambda^-_{1,n} = -\infty$, we also have $\lambda^-_1 = -\infty$ and there is nothing to prove. So we assume $\lim_{n\to\infty} \lambda^-_{1,n} := \widehat{\lambda} > -\infty$. It is then obvious that $\widehat{\lambda} \geq \lambda^-_1$. From [109, Theorem 1.1], for all $n \in \mathbb{N}$, we have $\psi^-_{1,n} \in E_p(\mathscr{B}_n)$, $\forall\, p < \infty$, such that $\psi^-_{1,n} < 0$ in $\mathscr{B}_n$, $\psi^-_{1,n} = 0$ in $\partial\mathscr{B}_n$, and

$$F(D^2\psi^-_{1,n}, D\psi^-_{1,n}, \psi^-_{1,n}, x) \;=\; -\lambda^-_{1,n}\psi^-_{1,n} \quad \text{in } \mathscr{B}_n\,. \tag{7.4.3}$$

Normalize each $\psi^-_{1,n}$ by fixing $\psi^-_{1,n}(0) = -1$. Denoting $G(M,p,u,x) = -F(-M,-p,-u,x)$ we find from (7.4.3)

$$G(D^2\phi^-_{1,n}, D\phi^-_{1,n}, \phi^-_{1,n}, x) \;=\; -\lambda^-_{1,n}\phi^-_{1,n} \quad \text{in } \mathscr{B}_n\,,$$

for $\phi^-_{1,n} = -\psi^-_{1,n} \geq 0$. Since G satisfies conditions (H1), (H3) and (H4), Theorem 7.4.1 applies. Then using (7.4.3) and [109, Theorem 3.3], we can obtain locally uniform $\mathscr{W}^{2,p}_{\text{loc}}$ bounds on $\phi^-_{1,n}$. Now apply the arguments of Step 1 in the proof of Lemma 7.4.1 to show that $\lim_{n\to\infty} \lambda^-_{1,n} = \lambda^-_1$ and $\lambda^+_1 \in \mathcal{E}^-$.

Step 2. As discussed in Lemma 7.4.1, it is enough to show that for any $\lambda < \lambda^-_1$ we have $\lambda \in \mathcal{E}^-$. Consider a sequence $\{f_n\}_{n\geq 1}$ of continuous, non negative, non-zero functions satisfying

$$\text{support}(f_n) \subset \mathscr{B}_n \setminus \overline{\mathscr{B}}_{n-1} \quad \text{for all } \; n \in \mathbb{N}.$$

Denote by $\tilde{F} = F + \lambda$. Then $\lambda^-_1(\tilde{F}, \mathscr{B}_n) = \lambda^-_{1,n} - \lambda \geq \lambda^-_1 - \lambda > 0$. Therefore, by [109, Theorem 1.9], there exists a non-zero, non positive $u^n \in E_p(\mathscr{B}_n)$, for all $p \geq N$, satisfying

$$\tilde{F}(D^2u^n, Du^n, u^n, x) = f_n \quad \text{in } \mathscr{B}_n, \qquad \text{and} \qquad u^n = 0 \quad \text{in } \; \partial\mathscr{B}_n.$$

Since G satisfies (H3) we can apply the strong maximum principle [109, Lemma 3.1] to obtain that $u^n < 0$ in $\mathscr{B}_n$. Now repeat the arguments of Step 2 in the proof of Lemma 7.4.1 to conclude that $\lambda \in \mathcal{E}^-$. This completes the proof. $\square$

Proof of Theorem 7.3.1. The proof follows from Lemma 7.4.1-7.4.2. □

The following (standard) existence result will be required.

Lemma 7.4.3. *Suppose that $\underline{u}, \bar{u} \in E_p(\Omega)$, for some $p \geq N$ and Ω is a smooth bounded domain, and $\bar{u}$ ($\underline{u}$) is a super-solution(sub-solution) of $F(D^2u, Du, u, x) = f(x, u)$ in Ω for some $f \in L^{\infty}_{\text{loc}}(\bar{\Omega} \times \mathbb{R})$. Assume that f is locally Lipschitz in its second argument uniformly (almost surely) with respect to the first argument and $\underline{u} \leq 0, \bar{u} \geq 0$ on $\partial\Omega$. Then there exists $u \in E_p(\Omega)$ with $\underline{u} \leq u \leq \bar{u}$ in Ω and satisfies*

$$F(D^2u, Du, u, x) = f(x, u) \quad \text{in } \Omega,$$
$$v = 0 \qquad \text{on } \partial\Omega\,.$$

Proof. The proof is based on monotone iteration method. See also [109, Lemma 4.3] for a similar argument. Define the operator $\tilde{F} = F - \theta$ in such a way that $\lambda_1^+(\tilde{F}, \Omega) > 0$. We may choose θ large enough so that

$$\theta > \text{Lip}(f(x, \cdot) \text{ on } [\inf_{\Omega} \underline{u}, \sup_{\Omega} \bar{u}]) \quad \text{almost surely for } x \in \Omega,$$

and $\tilde{F}$ is proper i.e., decreasing in u. Also, note that $\tilde{F}$ satisfying (H1)-(H4). Now we define the monotone sequence. Denote by $v_0 = \underline{u}$, and for each $n \geq 0$, we define

$$\begin{cases} \tilde{F}(D^2v_{n+1}, Dv_{n+1}, v_{n+1}, x) = f(x, v_n) - \theta v_n & \text{in } \Omega\,, \\ v_{n+1} = 0 & \text{on } \partial\Omega\,. \end{cases}$$

Existence of $v_{n+1} \in E_p$ follows from [109, Theorem 3.4]. Also, since $\tilde{F}$ is proper, we can apply comparison principle [109, Theorem 3.2] to obtain $v_0 \leq v_1 \leq v_2 \leq \cdots \leq \bar{u}$. It is then standard to show that $v_n \to u$ in $\mathcal{C}(\bar{\Omega})$ for some $u \in E_p(\Omega)$ and u is our required solution (see for instance, [109, Lemma 4.3]). This completes the proof. □

Applying Lemma 7.4.3 we obtain the following.

Theorem 7.4.2. *It holds that* $\lambda_1^{\prime,+}(F) \le \lambda_1^+(F)$ *and* $\lambda_1^{\prime,-}(F) \le \lambda_1^-(F)$.

Proof. We divide the proof into two steps.

Step 1. We show that $\lambda_1^{\prime,+}(F) \le \lambda_1^+(F)$. Replacing F by $F - \lambda_1^+(F)$ we may assume that $\lambda_1^+(F) = 0$. Considering any λ satisfying $\lambda > 0$ we show that $\lambda_1^{\prime,+}(F) \le \lambda$. Recall from Lemma 7.4.1 that $\lambda_1^+(F, \mathscr{B}_n) \searrow \lambda_1^+(F)$ as $n \to \infty$. Thus we can find k large enough satisfying $\lambda > \lambda_1^+(F, \mathscr{B}_k) > \lambda_1^+(F) = 0$. Let $\psi_k^+ \in E_p(\mathscr{B}_k), p < \infty$, satisfy

$$F(D^2\psi_k^+, D\psi_k^+, \psi_k^+, x) = -\lambda_{1,k}^+ \psi_k^+ \quad \text{in } \mathscr{B}_k,$$
$$\psi_1^+ > 0 \text{ in } \mathscr{B}_k, \quad \psi_k^+ = 0 \text{ in } \partial\mathscr{B}_k,$$

where $\lambda_1^+(F, \mathscr{B}_k) = \lambda_{1,k}^+$. Let $\tilde{\delta} = \sup_{\mathscr{B}_k} \delta$ where δ is given by (H3). Normalize ψ_k^+ so that

$$\|\psi_k^+\|_{L^\infty(\mathscr{B}_k)} = \min\left\{1, \frac{\lambda - \lambda_{1,k}^+}{\lambda + \tilde{\delta}}\right\}.$$

Now we plan to find a bounded, positive solution of

$$F(D^2u, Du, u, x) = (\lambda + c^+(x))u^2 - \lambda u \quad \text{in } \mathbb{R}^N, \tag{7.4.4}$$

where $c(x) = F(0, 0, 1, x) \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. This would imply $F(D^2u, Du, u, x) \ge -\lambda u$, and therefore, $\lambda_1^{\prime,+}(F) \le \lambda$. Thus to complete the proof of Step 1 we only need to establish (7.4.4).

Let $\bar{u} = 1$ and $\underline{u} = \psi_k^+$. Note that $\bar{u}$ is a super-solution in $\mathbb{R}^N$ and $\underline{u}$ is a sub-solution in $\mathscr{B}_k$. Now fix any ball $\mathscr{B}$ containing $\mathscr{B}_k$. Since 0 is a sub-solution, by Lemma 7.4.3, we find $v \in E_p(\mathscr{B}), p < \infty$, with $0 \le v \le 1$ and satisfies

$$F(D^2v, Dv, v, x) = (\lambda + \tilde{\delta})v^2 - \lambda v \quad \text{in } \mathscr{B}, \quad v = 0 \text{ on } \partial\mathscr{B}.$$

The proof of Lemma 7.4.3 also reveals that $v \ge \psi_k^+$ in $\mathscr{B}_k$. Now choosing a sequence of $\mathscr{B}$ increasing to $\mathbb{R}^N$, and the interior estimate [109, Theorem 3.3] we can find a subsequence locally converging to a solution u of (7.4.4). Positivity of u follows from Theorem 7.4.1.

Step 2. We next show that $\lambda_1'^{,-}(F) \leq \lambda_1^-(F)$. Replacing F by $F - \lambda_1^-(F)$ we may assume that $\lambda_1^-(F) = 0$. Considering any λ satisfying $\lambda > 0$ we show that $\lambda_1'^{,-}(F) \leq \lambda$. As done in Step 1, we can choose k large enough so that $\lambda > \lambda_1^-(F, \mathscr{B}_k) \coloneqq \lambda_{1,k}^-$ and there exists $\psi_k^- \in E_p(\mathscr{B}_k)$ satisfying

$$\begin{aligned} F(D^2\psi_k^-, D\psi_k^-, \psi_k^-, x) &= -\lambda_{1,k}^-\psi_k^+ \quad \text{in } \mathscr{B}_k, \\ \psi_k^- < 0 \text{ in } \mathscr{B}_k, \quad \psi_k^- &= 0 \text{ in } \partial\mathscr{B}_k. \end{aligned}$$

Normalize ψ_k^- so that

$$\|\psi_k^-\|_{L^\infty(\mathscr{B}_k)} = \min\left\{1, \frac{\lambda - \lambda_{1,k}^-}{\lambda + \tilde{\delta}}\right\},$$

where $\tilde{\delta}$ is same as in Step 1. Then

$$F(D^2\psi_k^-, D\psi_k^-, \psi_k^-, x) \leq -(\lambda + c^-(x))(\psi_k^-)^2 - \lambda\psi_k^- \quad \text{in } \mathscr{B}_k.$$

Thus, using Lemma 7.4.3 and the arguments of Step 1, we obtain a negative, bounded solution $u \in \mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N)$, $p < \infty$, to

$$F(D^2u, Du, u, x) = -(\lambda + c^-(x))u^2 - \lambda u \leq -\lambda u.$$

This of course, implies $\lambda_1'^{,-}(F) \leq \lambda$. Hence the theorem. □

Theorem 7.3.2(ii) will be proved using Theorem 7.3.4. Thus we prove Theorem 7.3.4 first.

Theorem 7.4.3. *Suppose that either* (7.3.2) *and* (7.3.4) *or* (7.3.3) *and* (7.3.5) *hold. Then F satisfies β^+-MP in $\mathbb{R}^N$ provided $\lambda_\beta''^{,+}(F) > 0$.*

Proof. Let $u \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ be a function satisfying

$$F(D^2u, Du, u, x) \geq 0 \quad \text{in } \mathbb{R}^N, \quad \text{and} \quad \sup_{\mathbb{R}^N} \frac{u}{\beta} < \infty.$$

Also, since $\lambda_\beta''^{,+}(F) > 0$, there exists $\lambda > 0$ and $\psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ with the property that $\psi \geq \beta$ and

$$F(D^2\psi, D\psi, \psi, x) + \lambda\psi \leq 0 \quad \text{in } \mathbb{R}^N.$$

Multiplying ψ with a suitable constant we may assume that $\psi \geq u$.

For this proof we follow the idea of [35, Theorem 4.2]. Choose a smooth positive function $\chi : \mathbb{R}^N \to \mathbb{R}$ such that, for $|x| > 1$,

$$\chi(x) = \begin{cases} |x|^\sigma & \text{if } \beta \text{ satisfies (7.3.2)}, \\ \exp(\sigma|x|) & \text{if } \beta \text{ satisfies (7.3.3)}. \end{cases}$$

Using (H3) and an easy computation we obtain for $x \in \mathscr{B}_1^c$

$$F(D^2\chi, D\chi, \chi, x) \leq \begin{cases} \left[(\sigma^2 + N\sigma - 2\sigma)\frac{\Lambda(x)}{|x|^2} + \sigma\frac{\gamma(x)}{|x|} + \delta(x)\right]\chi & \text{or,} \\ \left[\sigma\left(\sigma + \frac{N-1}{|x|}\right)\Lambda(x) + \sigma\gamma(x) + \delta(x)\right]\chi \end{cases}$$

if β satisfies (7.3.2) or (7.3.3) respectively. Hence for both the cases, using (7.3.4) and (7.3.5) accordingly on $\overline{\mathscr{B}}_1^c$, there exists a positive constant C such that

$$F(D^2\chi, D\chi, \chi, x) \leq C\chi. \tag{7.4.5}$$

Modifying C, if required, we can assume (7.4.5) to hold in $\mathbb{R}^N$. Now set $\psi_n = \psi + \frac{1}{n}\chi$ and define $\kappa_n = \sup_{\mathbb{R}^N} \frac{u}{\psi_n}$. If $\kappa_n \leq 0$ then there is nothing to prove. Thus we assume $\kappa_n > 0$ to reach a contradiction. Since $\psi \geq u$ it follows that $\kappa_n \leq 1$ and $\kappa_n \leq \kappa_{n+1}$ for all $n \geq 1$. Moreover, by (7.3.2) and (7.3.3),

$$\limsup_{|x|\to\infty} \frac{u(x)}{\psi_n(x)} \leq n \sup_{\mathbb{R}^N} \frac{u}{\beta} \limsup_{|x|\to\infty} \frac{\beta(x)}{\chi(x)} = 0.$$

Hence there exist $x_n \in \mathbb{R}^N$ such that $\kappa_n = \frac{u(x_n)}{\psi_n(x_n)}$.

Let us now estimate the term $\frac{\chi(x_n)}{n}$. Note that

$$\frac{1}{\kappa_{2n}} \leq \frac{\psi_{2n}(x_n)}{u(x_n)} = \frac{1}{\kappa_n} - \frac{\chi(x_n)}{2n\, u(x_n)},$$

which implies

$$\frac{\chi(x_n)}{n} \leq 2\left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right)u(x_n) \leq 2\left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right)\psi(x_n).$$

Hence for each natural number n there exist a small positive η_n such that

$$\frac{\chi(x)}{n} \leq \left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right)\psi(x) \quad \text{in } \mathscr{B}_{\eta_n}(x_n). \tag{7.4.6}$$

On the other hand, using convexity of F with (7.4.5) and (7.4.6), we get

$$\begin{aligned} F(D^2\psi_n, D\psi_n, \psi_n, x) &\leq F(D^2\psi, D\psi, \psi, x) + \frac{1}{n}F(D^2\chi, D\chi, \chi, x) \\ &\leq \left[-\lambda + C\left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right)\right]\psi(x)\,, \end{aligned}$$

in $\mathscr{B}_{\eta_n}(x_n)$. Since $\{\kappa_n\}$ is a convergent sequence, we can choose m large enough so that

$$F(D^2\psi_m, D\psi_m, \psi_m, x) \;<\; 0 \quad \text{in } \mathscr{B}_{\eta_m}(x_m)\,. \tag{7.4.7}$$

Now note that $w = \kappa_m\psi_m - u$ is non-negative and by (H3), there exist positive a, b such that in $\mathscr{B}_{\eta_m}(x_m)$ we have

$$\begin{aligned} &\mathcal{M}^-_{\lambda,\Lambda}(x, D^2w) - a|Dw| - bw \\ &\leq\; \kappa_m F(D^2\psi_m, D\psi_m, \psi_m, x) - F(D^2u, Du, u, x) \\ &<\; 0\,. \end{aligned}$$

By the strong maximum principle [109, Lemma 3.1] we then obtain $w \equiv 0$ in $\mathscr{B}_{\eta_m}(x_m)$. But this contradicts (7.4.7) as

$$0 \leq F(D^2u, Du, u, x) \;=\; \kappa_m F(D^2\psi_m, D\psi_m, \psi_m, x) < 0 \quad \text{in } \mathscr{B}_{\eta_m}(x_m)\,.$$

Therefore, $\kappa_n \leq 0$ for large n and hence $u \leq 0$. □

In the same spirit of Theorem 7.4.3 we can also prove β^--MP.

Theorem 7.4.4. *Suppose that either* (7.3.2) *and* (7.3.4) *or* (7.3.3) *and* (7.3.5) *hold for the function* β. *Then* F *satisfies* $(-\beta)^-$*-MP in* $\mathbb{R}^N$ *provided* $\lambda''^{,-}_{\beta}(F) > 0$.

Proof. As done in Theorem 7.4.3, we choose $\lambda \in (0, \lambda''^{,-}_{\beta}(F))$ and $\psi \in \mathscr{W}^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ satisfying $\psi \leq -\beta$ and

$$F(D^2\psi, D\psi, \psi, x) + \lambda\psi \,\leq\, 0 \quad \text{in } \mathbb{R}^N .$$

Let $u \in \mathscr{W}^{2,N}_{\mathrm{loc}}(\mathbb{R}^N)$ be a function satisfying

$$F(D^2u, Du, u, x) \,\leq\, 0 \quad \text{in } \mathbb{R}^N, \qquad \text{and} \qquad \sup_{\mathbb{R}^N} \frac{u}{(-\beta)} \,<\, \infty .$$

We need to show that $u \geq 0$. To the contrary, we suppose that u is negative somewhere in $\mathbb{R}^N$. Multiplying ψ with a suitable positive constant we may assume $\psi \leq u$. Consider the function χ from Theorem 7.4.3 and note that (7.4.5) holds. Set $\psi_n(x) = \psi(x) - \frac{1}{n}\chi(x)$ and $\kappa_n := \sup_{\mathbb{R}^N} \frac{u}{\psi_n}$. It can be easily checked that $(\kappa_n)_{n\in\mathbb{N}}$ is positive, increasing and bounded by 1. Furthermore, $\kappa_n = \frac{u(x_n)}{\psi_n(x_n)}$ for some $x_n \in \mathbb{R}^N$. Then repeating a similar calculation we find that for each natural number n there exist a small positive η_n satisfying

$$-\frac{\chi(x)}{n} \geq \left(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\right)\psi(x) \quad \text{in } \mathscr{B}_{\eta_n}(x_n) .$$

Then using convexity, (7.4.5) and above estimate, we obtain

$$\begin{aligned} F(D^2\psi_n, D\psi_n, \psi_n, x) &\geq F(D^2\psi, D\psi, \psi, x) - \frac{1}{n}F(D^2\chi, D\chi, \chi, x) \\ &\geq \left[-\lambda\psi(x) - C\frac{\chi(x)}{n}\right] \\ &\geq \left[-\lambda + C\Big(\frac{1}{\kappa_n} - \frac{1}{\kappa_{2n}}\Big)\right]\psi(x) , \end{aligned}$$

in $\mathscr{B}_{\eta_n}(x_n)$. As $\psi(x)$ is negative and $\{\kappa_n\}$ is convergent, we can choose m large enough such that

$$F(D^2\psi_m, D\psi_m, \psi_m, x) \,>\, 0 \quad \text{in } \mathscr{B}_{\eta_m}(x_m). \tag{7.4.8}$$

Note that $w := \kappa_m\psi_n - u$ is a non-positive function vanishing at x_m. Repeating the arguments of Theorem 7.4.3 we find positive constants a_1, b_1 satisfying

$$\mathcal{M}^+_{\lambda,\Lambda}(x, D^2w) + a_1|Dw| - b_1 w \,\geq\, 0 ,$$

in $\mathscr{B}_{\eta_m}(x_m)$. This of course, implies $w \equiv 0$ in $\mathscr{B}_{\eta_m}(x_m)$ which is a contradiction to (7.4.8). Thus it must hold that $u \geq 0$. □

Proof of Theorem 7.3.4. The proof follows by combining Theorem 7.4.3-7.4.4. □

Now we prove Theorem 7.3.5.

Proof of Theorem 7.3.5. First we consider (i). To the contrary, suppose that $\lambda_\beta^{\prime,+}(F) < 0$. Then there exists $\lambda < 0$ such that $\lambda_\beta^{\prime,+}(F) < \lambda < 0$ and there exists $\psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$0 < \psi \leq \beta, \quad F(D^2\psi, D\psi, \psi, x) + \lambda\psi \, \geq \, 0\,.$$

This of course, implies $F(D^2\psi, D\psi, \psi, x) \geq -\lambda\psi \, > \, 0$ and $\sup \frac{\psi}{\beta} \leq 1$. This clearly violates β^+-MP.

Next we consider (ii). Again, we suppose that $\lambda_\beta^{\prime,-}(F) < 0$. Then there exists $\lambda < 0$ such that $\lambda_\beta^{\prime,-}(F) < \lambda < 0$ and there exists $\psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N)$ satisfying

$$0 > \psi \geq -\beta, \quad F(D^2\psi, D\psi, \psi, x) + \lambda\psi \, \leq \, 0\,.$$

This gives $F(D^2\psi, D\psi, \psi, x) \, \leq \, -\lambda\psi \, < \, 0$ and $\sup \frac{\psi}{(-\beta)} \, \leq \, 1$. This clearly violates $(-\beta)^-$-MP. □

Now we can prove Theorem 7.3.2(ii).

Theorem 7.4.5. *Assume that either* (7.3.4) *or* (7.3.5) *holds. Then we have*

$$\lambda_1^{\prime\prime,+}(F) \, \leq \, \lambda_1^{\prime,+}(F), \quad \textit{and} \quad \lambda_1^{\prime\prime,-}(F) \, \leq \, \lambda_1^{\prime,-}(F)\,.$$

Proof. Let us first show that $\lambda_1^{\prime\prime,+}(F) \, \leq \, \lambda_1^{\prime,+}(F)$. To the contrary, suppose that there exists λ with $\lambda < \lambda_1^{\prime\prime,+}(F)$ and $\lambda_1^{\prime,+}(F) < \lambda$. Then there exists positive $\psi \in \mathscr{W}^{2,N}_{\text{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $F(D^2\psi, D\psi, \psi, x) + \lambda\psi \, \geq \, 0$. Also, note that $\lambda_1^{\prime\prime,+}(F+\lambda) = \lambda_1^{\prime\prime,+} - \lambda > 0$. By Theorem 7.4.3, the operator

$F + \lambda$ satisfies +MP. Therefore, $\psi \le 0$ which contradicts the fact $\psi > 0$. Hence we must have $\lambda_1^{\prime\prime,+}(F) \le \lambda_1^{\prime,+}(F)$.

We prove the second claim. To the contrary, suppose that there exists λ with $\lambda < \lambda_1^{\prime\prime,-}(F)$ and $\lambda_1^{\prime,-}(F) < \lambda$. Then there exists a negative function $\psi \in \mathscr{W}^{2,N}_{\rm loc}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that $F(D^2\psi, D\psi, \psi, x) + \lambda\psi \le 0$. Also, we have $\lambda_1^{\prime\prime,-}(F + \lambda) = \lambda_1^{\prime\prime,-}(F) - \lambda > 0$, and therefore, the operator $F + \lambda$ satisfies $-$MP. This gives $\psi \ge 0$ which contradicts the fact $\psi < 0$. Hence we must have $\lambda_1^{\prime\prime,-}(F) \le \lambda_1^{\prime,-}(F)$. □

Proof of Theorem 7.3.2. The proof follows by combining Theorem 7.4.2-7.4.5. □

Our next result should be compared with [35, Theorem 7.6]. Recall that for a smooth domain Ω

$$\lambda_1^{\prime\prime,+}(F,\Omega) = \sup\{\lambda \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\rm loc}(\Omega),\\ \inf_{\Omega} \psi > 0 \text{ and } F(D^2\psi, D\psi, \psi, x) + \lambda\psi \le 0 \ \text{ in } \ \Omega\},$$

$$\lambda_1^{\prime\prime,-}(F,\Omega) = \sup\{\lambda \;:\; \exists\, \psi \in \mathscr{W}^{2,N}_{\rm loc}(\Omega),\\ \sup_{\Omega} \psi < 0 \text{ and } F(D^2\psi, D\psi, \psi, x) + \lambda\psi \ge 0 \ \text{ in } \ \Omega\}.$$

Theorem 7.4.6. *It holds that*

$$\lambda_1^{\prime\prime,+}(F) \;=\; \min\{\lambda_1^+(F), \lim_{r\to\infty} \lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c)\}.$$

Proof. Notice that the function $\lambda_1^{\prime\prime,+}(r) := \lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c)$ is an increasing function with respect to r and

$$\lambda_1^{\prime\prime,+}(F) \;\le\; \lim_{r\to\infty} \lambda_1^{\prime\prime,+}(r).$$

Also, from definition we already have $\lambda_1^{\prime\prime,+}(F) \le \lambda_1^+(F)$. Combining these two we obtain

$$\lambda_1^{\prime\prime,+}(F) \;\le\; \min\{\lambda_1^+(F), \lim_{r\to\infty} \lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c)\}.$$

Let us now show that the above inequality can not be strict. That is, for every

$$\lambda < \min\{\lambda_1^+(F), \lim_{r\to\infty} \lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c)\},$$

we have $\lambda_1^{\prime\prime,+}(F) \geq \lambda$. To do this we need to construct a positive super-solution of the operator $F+\lambda$ in the admissible class of $\lambda_1^{\prime\prime,+}(F)$. Choose a positive number R so that $\lambda < \lambda_1^{\prime\prime,+}(R)$. Then there exists positive function $\phi \in \mathscr{W}^{2,N}_{\rm loc}(\overline{\mathscr{B}}^c_R)$ with $\inf_{\mathscr{B}^c_R} \phi > 0$ and $F(D^2\phi, D\phi, \phi, x) + \lambda\phi \leq 0$ in $\overline{\mathscr{B}}^c_R$. We claim that there exists a function $\varphi \in \mathscr{W}^{2,p}_{\rm loc}(\mathscr{B}^c_{R+1}), p > N$, with $\inf_{\mathscr{B}^c_{R+1}} \varphi \geq 1$ and satisfies

$$F(D^2\varphi, D\varphi, \varphi, x) + \lambda\varphi \leq 0 \quad \text{in } \mathscr{B}^c_{R+1}\,. \tag{7.4.9}$$

Let us first complete the proof assuming (7.4.9). By Morrey's inequality we see that $\varphi \in \mathcal{C}^1(\bar{\mathscr{B}}^c_{R+1})$. Consider a positive eigenfunction $\psi \in \mathscr{W}^{2,N}_{\rm loc}(\mathbb{R}^N)$ associated to $\lambda_1^+(F)$. Choose a non-negative function $\chi \in \mathcal{C}^2(\mathbb{R}^N)$ such that $\chi = 0$ in $\mathscr{B}_{R+2}$ and $\chi = 1$ in $\mathscr{B}^c_{R+3}$. For $\epsilon > 0$, define $u := \psi + \epsilon\chi\varphi$. Using convexity of F we can write

$$F(D^2u, Du, u, x) \leq F(D^2\psi, D\psi, \psi, x) + \epsilon F(D^2(\chi\varphi), D(\chi\varphi), (\chi\varphi), x)\,.$$

From the construction we can immediately say that $F(D^2u, Du, u, x)+\lambda u \leq 0$ in $\mathscr{B}_{R+2} \cup \mathscr{B}^c_{R+3}$. We are left with the annuals region $\bar{\mathscr{B}}_{R+3} \setminus \mathscr{B}_{R+2}$. In this compact set we have

$$\begin{aligned}
&F(D^2u, Du, u, x) + \lambda u \\
&\leq\ (\lambda - \lambda_1^+(F))\psi + \epsilon\left[F(D^2(\chi\varphi), D(\chi\varphi), (\chi\varphi), x) + \lambda\chi\varphi\right] \\
&= (\lambda - \lambda_1^+(F))\psi + \epsilon\Big[F(\chi D^2\varphi + 2D\chi\cdot D\varphi + \varphi D^2\chi, \chi D\varphi + \varphi D\chi, \chi\varphi, x) \\
&+ \lambda\chi\varphi\Big] \\
&\leq (\lambda - \lambda_1^+(F))\psi + \epsilon\chi\Big[F(D^2\varphi, D\varphi, \varphi, x) + \lambda\varphi\Big] \\
&+ \epsilon F(2D\chi\cdot D\varphi + \varphi D^2\chi, \varphi D\chi, 0, x) \\
&\leq (\lambda - \lambda_1^+(F))\psi + \epsilon C < 0,
\end{aligned}$$

for ϵ small enough, where we have again used convexity of F. This of course, implies $\lambda_1^{\prime\prime,+}(F) \geq \lambda$, as required.

To complete the proof we only need to show (7.4.9). To this end, we may assume that $\inf \phi \geq 2$. Let $c(x) = F(0,0,1,x) + \lambda$ and define $f(x,u) = |c(x)| f(u)$ where $f : \mathbb{R} \to (-\infty, 0]$ is a Lipschitz function with the property that $f(1) = -1$, $f(t) = 0$ for $t \geq 2$. Then $\bar{u} = \phi$ is super-solution to

$$F(D^2u, Du, u, x) + \lambda u = f(x,u) \quad \text{in } \mathscr{B}_R^c,$$

and $\underline{u} = 1$ is a sub-solution. The existence of a solution to (7.4.9) follows by constructing solutions (squeezed between $\bar{u}$ and $\underline{u}$) in an increasing sequence of bounded domains in $\mathscr{B}_R^c$ and the passing to the limit using local stability bound [109, Theorem 3.3]. To construct a solution in any smooth bounded domain we may follow the idea of Lemma 7.4.3 with the help of general existence results from [118, Theorem 4.6] which deals with non-zero boundary condition. □

Now we would like to see if a result analogous to Theorem 7.4.6 holds for $\lambda_1^{\prime\prime,-}(F)$. Denote by

$$G(S,p,u,x) = -F(-M, -p, -u, x).$$

It is easily seen that G is a concave operator and $\lambda_1^{\prime\prime,-}(F) = \lambda_1^{\prime\prime,+}(G)$. But we can not apply the arguments of Theorem 7.4.6 for concave operators. To obtain the results we impose a mild condition at *infinity*.

Theorem 7.4.7. *Suppose that*

$$\lim_{r\to\infty} \lambda_1^{\prime\prime,-}(F, \overline{\mathscr{B}}_r^c) \;=\; \lim_{r\to\infty} \lambda_1^{\prime\prime,-}(G, \overline{\mathscr{B}}_r^c)\,. \tag{7.4.10}$$

Then we have

$$\lambda_1^{\prime\prime,-}(F) \;=\; \min\left\{\lambda_1^-(F), \lim_{r\to\infty} \lambda_1^{\prime\prime,-}(F, \overline{\mathscr{B}}_r^c)\right\}\,.$$

Proof. It is easy to see that

$$\lambda_1^{\prime\prime,-}(F) \leq \min\left\{\lambda_1^-(F), \lim_{r\to\infty} \lambda_1^{\prime\prime,-}(F, \overline{\mathscr{B}}_r^c)\right\}.$$

As done in Theorem 7.4.6, we show that the above inequality can be strict. So we consider any

$$\lambda < \min\left\{\lambda_1^-(F), \lim_{r\to\infty} \lambda_1^{\prime\prime,-}(F, \overline{\mathscr{B}}_r^c)\right\}, \tag{7.4.11}$$

and show that $\lambda_1^{\prime\prime,-}(F) \geq \lambda$. We now construct a sub-solution of the operator $F + \lambda$ in the admissible class of $\lambda_1^{\prime\prime,-}(F)$. Using (7.4.10) and (7.4.11) we find a positive R so that

$$\lambda < \lambda_1^{\prime\prime,-}(G, \overline{\mathscr{B}}_R^c).$$

Hence repeating the arguments of Theorem 7.4.6 we can find $\varphi \in \mathscr{W}^{2,p}_{\rm loc}(\mathscr{B}^c_{R+1})$, $p > N$, with $\sup_{\mathscr{B}^c_{R+1}} \varphi < 0$ and $G(D^2\varphi, D\varphi, \varphi, x) + \lambda\varphi \geq 0$ in $\mathscr{B}^c_{R+1}$. By Morrey's inequality $\varphi \in \mathcal{C}^1(B^c_{R+1})$. Also, consider a negative eigenfunction $\psi \in \mathscr{W}^{2,N}_{\rm loc}(\mathbb{R}^N)$ associated to $\lambda_1^-(F)$. Let χ be the cut-off function in Theorem 7.4.6 and define $u = \psi + \epsilon\chi\varphi$ for $\epsilon > 0$. Since, by convexity,

$$F(D^2u, Du, u, x) \geq F(D^2\psi, D\psi, \psi, x) + \epsilon G(D^2(\chi\phi), D(\chi\phi), (\chi\phi), x),$$

repeating a calculation analogous to Theorem 7.4.6 we find that for some ϵ small $F(D^2u, Du, u, x) + \lambda u \geq 0$ in $\mathbb{R}^N$. Thus we get $\lambda_1^{\prime\prime,-}(F) \geq \lambda$. □

To this end, we define $c(x) = F(0, 0, 1, x)$ and $d(x) = F(0, 0, -1, x)$. Our next result is a generalization to [35, Proposition 1.11].

Proposition 7.4.1. *Define* $\zeta = \limsup_{|x|\to\infty} c(x)$ *and* $\xi = \limsup_{|x|\to\infty} d(x)$. *Then the following hold.*

(i) Suppose that $\zeta < 0$, *and either* (7.3.4) *or* (7.3.5) *holds. Then* F *satisfies the* $+MP$ *if and only if* $\lambda_1^+(F) > 0$.

(ii) Suppose that $\xi > 0$, *and either* (7.3.4) *or* (7.3.5) *holds. Furthermore, assume* (7.4.10). *Then* F *satisfies the* $-MP$ *if and only if* $\lambda_1^-(F) > 0$.

We need a small lemma to prove Proposition 7.4.1.

Lemma 7.4.4. *The following hold for any smooth domain* Ω.

(i) $-\sup_\Omega\ c(x)\ \leq\ \inf_\Omega d(x)$.

(ii) $-\sup_\Omega\ c(x)\ \leq\ \lambda_1^{\prime\prime,+}(F,\Omega)$.

(iii) $\inf_\Omega\ d(x)\ \leq\ \lambda_1^{\prime\prime,-}(F,\Omega)$.

Proof. Part (i) follows from convexity property of F. Note that for $\lambda = -\sup_\Omega\ c(x)$, $\psi = 1$ is an admissible function for $\lambda_1^{\prime\prime,+}(F,\Omega)$. This gives us (ii). In a similar fashion we get (iii). □

Now we prove Proposition 7.4.1

Proof of Proposition 7.4.1. First consider (i). Assume that $\lambda_1^+(F) > 0$. Using Lemma 7.4.4 we obtain

$$0\ <\ -\zeta\ =\ \lim_{r\to\infty}\left(-\sup_{\bar{\mathscr{B}}_r^c}\ c(x)\right)\leq\ \lim_{r\to\infty}\ \lambda_1^{\prime\prime,+}(F,\overline{\mathscr{B}}_r^c)\,. \tag{7.4.12}$$

By Theorem 7.4.6, we obtain $\lambda_1^{\prime\prime,+}(F) > 0$, and therefore, using Theorem 7.4.3 we see that F satisfies the +MP. To show the converse direction we assume that F satisfies +MP. Then Theorem 7.3.5 implies that $\lambda_1^{\prime,+}(F) \geq 0$. Using Theorem 7.4.2 we then have $\lambda_1^+(F) \geq 0$. If possible, suppose that $\lambda_1^+(F) = 0$. We show that there exists a bounded principal eigenfunction φ which would give a contradiction to the validity of +MP, and hence we must have $\lambda_1^+(F) > 0$. Consider a smooth positive function ϕ satisfying $\phi = 1$ in $\mathscr{B}_r^c$ for some large r. Since $\zeta < 0$, we have a compact set K satisfying

$$c(x) < \frac{\zeta}{2}\phi(x), \quad \phi(x) = 1, \quad x \in K^c.$$

Recall the Dirichlet principal eigenfunction $\psi^+_{1,n}$ from (7.4.1). Choose $\kappa_n = \max_{\mathscr{B}_n} \frac{\psi^+_{1,n}}{\phi}$ and define $v_n = \kappa_n^{-1}\psi^+_{1,n}$. Observe that $\phi - v_n$ must vanish in K. Indeed, in $\mathscr{B}_n \setminus K$ we have

$$\begin{aligned}
&\mathcal{M}^-_{\lambda,\Lambda}(x, D^2(\phi - v_n)) - \gamma(x)|D\phi - v_n| - \delta(x)(\phi - v_n) \\
&\le F(0,0,\phi,x) - F(D^2 v_n, Dv_n, v_n, x) \\
&\le \frac{\zeta}{2} + \lambda^+_{1,n} v_n \\
&\le \frac{\zeta}{2} + \lambda^+_{1,n} < 0\,,
\end{aligned}$$

for all large n, and therefore, by strong maximum principle [109, Lemma 3.1], $\phi - v_n$ can not vanish in $\mathscr{B}_n \setminus K$. Now applying Harnack's inequality and standard $\mathscr{W}^{2,p}$ estimates we can extract a convergent subsequence of v_n converging to a positive eigenfunction φ. This completes the proof.

The proof for (ii) would be analogous. □

Next we prove Theorem 7.3.3

Proof of Theorem 7.3.3. (i) From the definition it follows that

$$\lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c) \ge \lambda_1^{\prime\prime,+}(\tilde{F}, \overline{\mathscr{B}}_r^c) - \sup_{\bar{\mathscr{B}}_r^c} \tilde{\gamma}(x)\,,$$

and then letting r towards infinity we have

$$\lim_{r\to\infty} \lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c) \ \ge\ \lim_{r\to\infty} \lambda_1^{\prime\prime,+}(\tilde{F}, \overline{\mathscr{B}}_r^c) \ge \lambda_1^{\prime\prime,+}(\tilde{F}) = \lambda_1^+(\tilde{F})\,.$$

Since $\tilde{\gamma}(x) \ge 0$, it gives us $\lambda_1^+(\tilde{F}) \ge \lambda_1^+(F)$. Combining it with above calculation, we find

$$\lim_{r\to\infty} \lambda_1^{\prime\prime,+}(F, \bar{\mathscr{B}}_r^c) \ge \lambda_1^+(F).$$

Applying Theorem 7.4.6 we obtain $\lambda_1^+(F) = \lambda_1^{\prime\prime,+}(F)$.

(ii) Using Lemma 7.4.4 and the given hypothesis we find

$$\lambda_1^+(F) \ \le\ -\limsup_{|x|\to\infty}\ c(x) \ =\ \lim_{r\to\infty}\ \Big(-\sup_{\overline{\mathscr{B}}_r^c}\ c(x)\Big) \ \le\ \lim_{r\to\infty} \lambda_1^{\prime\prime,+}(F, \overline{\mathscr{B}}_r^c)\,.$$

Hence, by Theorem 7.4.6, we get $\lambda_1^+(F) = \lambda_1^{\prime\prime,+}(F)$.

(iii) We show that under the given condition we have (ii). Hence it is enough to show that if $\sigma < \limsup_{|x|\to\infty} c(x)$ then $\lambda_1^+(F) \le -\sigma$. Now define a positive function

$$\psi(x) = \exp\left(-\frac{1}{1-|\varepsilon x|^2}\right)$$

on the ball $\mathscr{B}_{\frac{1}{\varepsilon}}$ where an appropriate ε will be chosen later. It is easily checked that

$$\begin{aligned} D_{x_i}\psi &= \frac{-2\varepsilon^2 x_i}{(1-|\varepsilon x|^2)^2}\psi, \\ D_{x_i x_j}\psi &= \left[\frac{4\varepsilon^4}{(1-|\varepsilon x|^2)^4}x_i x_j - \frac{2\varepsilon^2}{(1-|\varepsilon x|^2)^2}\delta_{ij} - \frac{8\varepsilon^4}{(1-|\varepsilon x|^2)^3}x_i x_j\right]\psi. \end{aligned}$$

For $x_0 \in \mathbb{R}^N$, define $\phi(x) = \psi(x - x_0)$. We will choose ε and x_0 such that

$$F(D^2\phi, D\phi, \phi, x) - \sigma\phi > 0 \quad \text{in } \mathscr{B}_{\frac{1}{\varepsilon}}(x_0). \tag{7.4.13}$$

Since all the notions of eigenvalues of F coincide in bounded domains (cf. [109]), using (7.4.13) we deduce

$$-\sigma \ge \lambda_1^{\prime,+}(F, \mathscr{B}_{\frac{1}{\varepsilon}}(x_0)) \;=\; \lambda_1^{+}(F, \mathscr{B}_{\frac{1}{\varepsilon}}(x_0)) \ge \lambda_1^+(F).$$

Thus we only need to establish (7.4.13). For a different way to construct such sub-solutions we refer [111]. Using (H3) we see that

$$\begin{aligned} & F(D^2\phi, D\phi, \phi, x) - \sigma\phi \qquad (7.4.14)\\ &= F(D^2\phi, D\phi, \phi, x) - F(0,0,\phi,x) + F(0,0,1,x)\phi - \sigma\phi \\ &\ge \mathcal{M}^-_{\lambda,\Lambda}(x, D^2\phi) - \gamma(x)|D\phi| + c(x)\phi - \sigma\phi \\ &\ge \left[\frac{4\lambda_0\varepsilon^2|\varepsilon(x-x_0)|^2}{(1-|\varepsilon(x-x_0)|^2)^4} - \frac{2N\Lambda_0\varepsilon^2}{(1-|\varepsilon(x-x_0)|^2)^2} - \frac{8\Lambda\varepsilon^2|\varepsilon(x-x_0)|^2}{(1-|\varepsilon(x-x_0)|^2)^3}\right. \\ &\qquad \left. - \frac{2\epsilon^2|x-x_0|\gamma(x)}{(1-|\epsilon(x-x_0)|^2)^2} + c(x) - \sigma\right]\phi\,. \qquad (7.4.15) \end{aligned}$$

Given ε we choose R such that $|\gamma(x)| \le \varepsilon$ for $|x| \ge R$ and then choose $x_0 \in \mathbb{R}^N$ satisfying $|x_0| \ge R + 2\varepsilon^{-1}$. Furthermore, due to our hypothesis, we can choose x_0 such that

$$\inf_{\mathscr{B}_{\frac{1}{\varepsilon}}(x_0)} c(x) \, > \, \sigma \,. \tag{7.4.16}$$

We now compute (7.4.14) in two steps.

Step 1. Suppose $1-\delta < |\varepsilon(x-x_0)|^2 < 1$ where δ is very close to zero and will be chosen later. It then follows from (7.4.14) that

$$\begin{aligned} &F(D^2\phi, D\phi, \phi, x) - \sigma\phi \\ &\geq \frac{\varepsilon^2}{(1-|\varepsilon(x-x_0)|^2)^4}\Big[4\lambda(1-\delta) - 2N\Lambda\delta^2 - 8\Lambda(1-\delta)\delta - 2\delta^2\Big]\phi \\ &\quad + \Big(c(x)-\sigma\Big)\phi \,. \end{aligned}$$

Now we can choose small positive δ, independent of ε, so that

$$4\lambda(1-\delta) - 2N\Lambda\delta^2 - 8\Lambda(1-\delta)\delta - 2\delta^2 > 0.$$

This proves (7.4.13) in the annulus.

Step 2. Now we are left with the part $0 \leq |\varepsilon(x-x_0)|^2 \leq 1-\delta$ where δ is already chosen in Step 1. An easy calculation reveals

$$F(D^2\phi, D\phi, \phi, x) - \sigma\phi \, \geq \, \Big[\Big(c(x)-\sigma\Big) - \frac{2N\Lambda\varepsilon^2}{\delta^2} - \frac{8\Lambda(1-\delta)\varepsilon^2}{\delta^3} - \frac{2\varepsilon^2}{\delta^2}\Big]\phi \,.$$

Using (7.4.16), we can choose ε small enough so that the RHS becomes positive.

Combining the above steps we obtain (7.4.13), completing the proof of part (iii).

(iv) This follows from Theorem 7.4.6. Let us also provide a more direct proof. Let φ^* be an eigenfunction corresponding to $\lambda_1^+(F) = \lambda_1^+$. For $\delta, \varepsilon > 0$ we define $\phi_\varepsilon = \varphi^* + \varepsilon V$. Choose ε small enough so that

$$\delta \min_{\bar{\mathscr{B}}} \varphi^* \, > \, \varepsilon \max_{\bar{\mathscr{B}}} [F(D^2V, DV, V, x) + \lambda_1^+ V]. \tag{7.4.17}$$

By using convexity and homogeneity it follows that

$$\begin{aligned}F(D^2\phi_\varepsilon, D\phi_\varepsilon, \phi_\varepsilon, x) &\leq F(D^2\varphi^*, D\varphi^*, \varphi^*, x) + \varepsilon F(D^2V, DV, V, x)\\ &= -\lambda_1^+\varphi^* + \varepsilon \mathbb{1}_{\mathscr{B}}(x) F(D^2V, DV, V, x) - \varepsilon\lambda_1^+ \mathbb{1}_{\mathscr{B}^c}(x)V(x)\\ &\leq -\lambda_1^+\phi_\varepsilon + \varepsilon \max_{\bar{\mathscr{B}}}[F(D^2V, DV, V, x) + \lambda_1^+ V]\\ &\leq -(\lambda_1^+ - \delta)\phi_\varepsilon,\end{aligned}$$

using (7.4.17). Hence $\lambda_1^{\prime\prime,+}(F) \geq \lambda_1^+(F) - \delta$ and from the arbitrariness of δ the result follows. □

Thus it remains to prove Theorem 7.3.6-7.3.7. Let us first attack Theorem 7.3.6.

Proof of Theorem 7.3.6. Without any loss of generality, we assume that $\lambda_1^+(F) = 0$. Recall from Lemma 7.4.1 that the pair $(\psi_{1,n}^+, \lambda_{1,n}^+)$ solving the Dirichlet eigenvalue problem with positive eigenfunction in $\mathscr{B}_n$. That is,

$$\begin{aligned}F(D^2\psi_{1,n}^+, D\psi_{1,n}^+, \psi_{1,n}^+, x) \;=\; -\lambda_{1,n}^+\psi_{1,n}^+ \quad \text{in } \mathscr{B}_n\,,\\ \psi_{1,n}^+ > 0 \text{ in } \mathscr{B}_n, \text{ and } \psi_{1,n}^+ = 0 \text{ on } \partial\mathscr{B}_n\,.\end{aligned} \tag{7.4.18}$$

Let $\kappa_n > 0$ be such that $\kappa_n\psi_{1,n}^+ \leq V$ in $\mathscr{B}_n$ and it touches V at some point in $\mathscr{B}_n$. We claim that $\kappa_n\psi_{1,n}^+$ has to touch V inside K. Note that, by (H3), if $w = V - \kappa_n\psi_{1,n}^+$ then

$$\begin{aligned}&\mathcal{M}_{\lambda,\Lambda}^-(x, w) - \gamma|Dw| - \delta w \leq -\varepsilon V + \lambda_{1,n}^+(\kappa_n\psi_{1,n}^+)\\ &\leq (-\varepsilon + \lambda_{1,n}^+)(\kappa_n\psi_{1,n}^+)\\ &\leq 0 \quad \text{in } K^c \cap \mathscr{B}_n\,,\end{aligned}$$

for large n, using (7.3.6) and (7.4.18). Thus, if w vanishes in $K^c \cap \mathscr{B}_n$, then it must be identically 0 in $K^c \cap \mathscr{B}_n$, by the strong maximum principle [109, Lemma 3.1]. But this is not possible since $w > 0$ on $\partial\mathscr{B}_n$. Now onwards we denote $\kappa_n\psi_{1,n}^+$ by $\psi_{1,n}^+$. By the above normalization, $\psi_{1,n}^+$ would converge, up to a subsequence, to a positive function $\varphi \in \mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N), p < \infty$, an eigenfunction corresponding to $\lambda_1^+(F) = 0$. See for instance, the argument in Lemma 7.4.1.

We now show that any other principal eigenfunction is a multiple to φ. For η, a small positive number, we define $\Xi_\eta = \psi^+_{1,n} - \eta V$. Using convexity of F we note that, in $\mathscr{B}_n \cap K^c$,

$$\begin{aligned} F(D^2\Xi_\eta, D\Xi_\eta, \Xi, x) &\geq F(D^2\psi^+_{1,n}, D\psi^+_{1,n}, \psi^+_{1,n}, x) - \eta F(D^2V, DV, V, x) \\ &\geq (-\lambda^+_{1,n}\psi^+_{1,n} + \eta\varepsilon V) \\ &\geq (-\lambda^+_{1,n} + \eta\varepsilon)V > 0\,, \end{aligned}$$

provided we choose n large (depending on η). Let ψ be any principal eigenfunction satisfying

$$F(D^2\psi, D\psi, \psi, x) = 0 \text{ in } \mathbb{R}^N.$$

Define

$$\delta = \delta(\eta) = \min_K \frac{\psi}{\Xi_\eta}.$$

Then $\delta\Xi_\eta \leq \psi$ on K. Since, by the Harnack inequality,

$$0 < \inf_n \inf_K \psi^+_{1,n} \leq \sup_n \sup_K \psi^+_{1,n} < \infty,$$

we can choose η_0 small enough (independent of n) so that

$$0 < \inf_{\eta\in(0,\eta_0]} \inf_n \inf_K \Xi_\eta \leq \sup_{\eta\in(0,\eta_0]} \sup_n \sup_K \Xi_\eta < \infty,$$

Thus, δ remains bounded and positive as $n \to \infty$ and $\eta \to 0$. Since $F(D^2\psi, D\psi, \psi, x) = 0$ in $\mathscr{B}_n \cap K^c$ and $\lambda_1^+(F, \mathscr{B}_n \cap K^c) > 0$, it follows from [109, Theorem 1.5], that

$$\delta\Xi_\eta \leq \psi \quad \text{in } \mathscr{B}_n\,.$$

Furthermore, there exists $x_\eta \in K$ so that $\delta\Xi_\eta(x_\eta) = \psi(x_\eta)$. Now letting $n \to \infty$ first, and then $\eta \to 0$, we can extract a subsequence so that $\delta \to \theta >$

0, and $x_\eta \to \hat{x} \in K$ and $\theta\varphi(\hat{x}) = \psi(\hat{x})$ with $\theta\varphi \le \psi$ in $\mathbb{R}^N$. Let $u = \psi - \theta\varphi$. It is easy to see that

$$\mathcal{M}^-_{\lambda,\Lambda}(x,u) - \gamma|Du| - \delta u \le 0 \quad \text{in } \mathbb{R}^N.$$

By the strong maximum principle we must have $u = 0$ and hence the proof.

□

Finally, we prove Theorem 7.3.7.

Proof of Theorem 7.3.7. The main idea of the proof is the same as that of the proof of Theorem 7.3.6. Without any loss in generality, we assume that $\lambda_1^-(F) = 0$. Let $(\psi^-_{1,n}, \lambda^-_{1,n})$ be the pair satisfying the Dirichlet eigenvalue problem in the ball $\mathscr{B}_n$ i.e.,

$$\begin{aligned} F(D^2\psi^-_{1,n}, D\psi^-_{1,n}, \psi^-_{1,n}, x) \;&=\; -\lambda^-_{1,n}\psi^-_{1,n} \quad \text{in } \mathscr{B}_n\,, \\ \psi^-_{1,n} < 0 \text{ in } \mathscr{B}_n, \text{ and } \psi^-_{1,n} &= 0 \text{ on } \partial\mathscr{B}_n\,. \end{aligned} \tag{7.4.19}$$

By Lemma 7.4.2, $\psi^-_{1,n} \searrow 0$ as $n \to \infty$. Recall that $G(M,p,u,x) := -F(-M,-p,-u,x)$. Denote by $\phi_n = -\psi^-_{1,n}$. Then we get from (7.4.19) that

$$G(D^2\phi_n, D\phi_n, \phi_n, x) \;=\; -\lambda^-_{1,n}\phi_n \quad \text{in } \mathscr{B}_n\,, \psi_n < 0 \text{ in } \mathscr{B}_n, \text{ and } \phi_n = 0 \text{ on } \partial\mathscr{B}_n\,. \tag{7.4.20}$$

Note that G satisfies (H1), (H2) and (H3) but it is a concave operator. So need some extra care to apply the proof of Theorem 7.3.6. Since F is convex it follows from (7.3.7) that

$$G(D^2V, DV, V, x) \;\le\; F(D^2V, DV, V, x) \le -(\lambda_1^-(F) + \varepsilon)V \quad \text{for all } x \in K^c. \tag{7.4.21}$$

As done in Theorem 7.3.6, using (7.4.21), we can normalize ϕ_n to touch V from below and it would touch V somewhere in K. Therefore, we can apply the Harnack inequality (see Lemma 7.4.2) to find a positive function φ such that $\phi_n \to \varphi$ in $\mathscr{W}^{2,p}_{\rm loc}(\mathbb{R}^N), p > N$, along some subsequence and

$$0 = -\lambda_1^-(F)\varphi = G(D^2\varphi, D\varphi, \varphi, x) = -F(-D^2\varphi, -D\varphi, -\varphi, x) \quad \text{in } \mathbb{R}^N .$$

It is enough to show that φ agrees with any other positive eigenfunction (up to a multiplicative constant) of G with eigenvalue 0.

Next we define $\Xi_\eta(x) = \phi_n - \eta V$. Since $\|\phi_n - \varphi\|_{L^\infty(K)} \to 0$, it is evident that $\Xi_\eta > 0$ for all η small, independent of n. Using (7.3.7) and (7.4.19), we see that, in $K^c \cap \mathscr{B}_n$,

$$\begin{aligned} F(-D^2\Xi_\eta, -D\Xi_\eta, -\Xi, x) &\le F(-D^2\phi_n, -D\phi_n, -\phi_n, x) + \eta F(D^2V, DV, V, x) \\ &\le (\lambda_{1,n}^-\phi_n - \eta\varepsilon V) \\ &\le (|\lambda_{1,n}^-| - \eta\varepsilon)V < 0, \end{aligned} \tag{7.4.22}$$

for all large n. Now consider any positive eigenfunction $\psi \in \mathscr{W}^{2,p}_{\rm loc}(\mathbb{R}^N)$ satisfying

$$F(-D^2\psi, -D\psi, -\psi, x) \;=\; 0\,,$$

and let

$$\delta = \delta(\eta) = \min_K \frac{\psi}{\Xi_\eta}\,.$$

Then $-\delta\Xi_\eta \ge -\psi$ on $\partial K \cup \partial\mathscr{B}_n$ for all n. From (7.3.7) if follows that $\lambda_1^+(F, K^c) \ge \varepsilon$. Since

$$\lambda_1^+(F, K^c \cap \mathscr{B}_n) \to \lambda_1^+(F, K^c) > 0 \text{ as } n \to \infty,$$

we can apply the maximum principle [109, Theorem 1.5] in $\mathscr{B}^c \cap K$ for all large n. From (7.4.22) we therefore get $\psi \ge \delta\Xi_\eta$ and $\delta\Xi_\eta$ touches ψ at some point in K. Now we can follow the arguments in Theorem 7.3.6 we show that $\varphi = t\psi$ for some $t > 0$. Hence the proof. □

We conclude the chapter with a remark on the eigenvalue problem in a general smooth unbounded domain.

Remark 7.4.1. *For the case of an unbounded domain with smooth boundary all the results developed here hold true and the proofs would be somewhat similar. As mentioned in [35], in case of general unbounded domains, one needs the boundary Harnack property to control the behaviour of eigenfunctions near the boundary. For the operator F, the boundary Harnack property has been obtained recently by Armstrong, Sirakov and Smart in [15, Appendix A]. Therefore one can easily adopt the techniques of [35] along with our results to deal with general unbounded domains.*

Chapter 8

On ergodic control problem for viscous Hamilton-Jacobi equations for weakly coupled elliptic systems

In this chapter we will study ergodic problems in the whole space $\mathbb{R}^N$ for a weakly coupled systems of viscous Hamilton-Jacobi equations with coercive right-hand sides. The Hamiltonians are assumed to have a fairly general structure and the switching rates need not be constant. We prove the existence of a critical value λ^* such that the ergodic eigenvalue problem has a solution for every $\lambda \leq \lambda^*$ and no solution for $\lambda > \lambda^*$. Moreover, the existence and uniqueness of non-negative solutions corresponding to the value λ^* are also established. We also exhibit the implication of these results to the ergodic optimal control problems of controlled switching diffusions. The detail of this chapter is covered from [9].

8.1 Motivations behind the problem

Consider the controlled dynamics is given by a pair (X, S) where $\{X_t\}$ denotes the continuous part governed by a controlled diffusion

$$\mathrm{d}X_t = \boldsymbol{b}(X_t, S_t)\,\mathrm{d}t - U_t\,\mathrm{d}t + \mathrm{d}W_t,$$

where W is a standard N-dimension Brownian motion, U is an admissible control, and $\{S_t\}$ is a two state Markov process, taking values in $\{1, 2\}$, responsible for random switching. The functions α_1, α_2 corresponds to the switching rates which is also allowed to be state dependent, that is,

$$\mathbb{P}(S_{t+\delta t} = j | S_t = i, X_s, S_s, s \le t) = \begin{cases} \alpha_1(X_t)\delta t + \mathfrak{o}(\delta t) & \text{if } j = 2,\ i = 1, \\ \alpha_2(X_t)\delta t + \mathfrak{o}(\delta t) & \text{if } j = 1,\ i = 2. \end{cases}$$

The HJE in (6.1.1) corresponds to the minimization problem

$$\lambda^* = \inf_{U\in\mathfrak{U}} \liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T (\boldsymbol{f}(X_t) + \boldsymbol{\ell}(X_t, S_t))\mathrm{d}t\right],$$

where $\mathfrak{U}$ denotes the set of all admissible controls. For a more precise description see Section 8.5. Because of the presence of both continuous dynamics and discrete jumps, regime-switching systems are capable of describing complex systems and randomness of the environment. We refer to the book of Yin and Zhu [120] for more detail on regime-switching dynamics and its application to the theory of stochastic control. Note that our equations (EP) includes the stochastic LQ ergodic control problem (that is, $\gamma_1 = \gamma_2 = 2$) for regime-switching dynamics which are quite popular models in portfolio selection problems (cf. [121, Chapter 6]). One of our main results establishes the existence of a unique optimal stationary Markov control (see Theorem 8.5.1) for the above optimization problem.

8.2 Basic assumptions

In this part we study the existence and uniqueness of solution $(\boldsymbol{u}, \lambda) = (u_1, u_2, \lambda)$ to the equation

$$\begin{aligned} -\Delta u_1(x) + H_1(x, \nabla u_1(x)) + \alpha_1(x)(u_1(x) - u_2(x)) &= f_1(x) - \lambda \quad \text{in } \mathbb{R}^N, \\ -\Delta u_2(x) + H_2(x, \nabla u_2(x)) + \alpha_2(x)(u_2(x) - u_1(x)) &= f_2(x) - \lambda \quad \text{in } \mathbb{R}^N, \end{aligned} \tag{EP}$$

where $H_i : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ denote the Hamiltonians, and $\alpha_i : \mathbb{R}^N \to \mathbb{R}_+$ are the switching rate parameters for $i = 1, 2$. We will move on this section by making the following assumptions:

Assumption 8.2.1. *The functions $\alpha_i : \mathbb{R}^N \to \mathbb{R}_+$ are continuously differentiable and for some constant $\alpha_0 > 0$ we have*

$$\alpha_0^{-1} \le \alpha_i(x) \le \alpha_0, \quad \sup_x |\nabla \alpha_i(x)| \le \alpha_0 \quad \text{for } i = 1, 2. \tag{8.2.1}$$

Also, the following hold.

(A1) There exist $\ell_i \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N)$, $\xi \mapsto \ell_i(x, \xi)$ strictly convex, and

$$H_i(x, p) = \sup_{\xi \in \mathbb{R}^N} \{\xi \cdot p - \ell_i(x, \xi)\}, \quad i = 1, 2.$$

Moreover, $H_i \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R}^N)$ and the functions $\xi \mapsto H_i(x, \xi)$ are strictly convex for $i = 1, 2$.

(A2) For some constants $\gamma_i > 1, i = 1, 2$, we have for $(x, p) \in \mathbb{R}^N \times \mathbb{R}^N$,

$$C_1^{-1} |p|^{\gamma_i} - C_1 \le H_i(x, p) \le C_1(|p|^{\gamma_i} + 1), \tag{HP1}$$

$$|\nabla_p H_i(x, p)| \le C_1(1 + |p|^{\gamma_i - 1}), \tag{HP2}$$

for some constant C_1 and $i = 1, 2$.

Since $\zeta \mapsto H_i(x,\zeta)$ is convex, it follows from (HP1) that

$$|\nabla_p H_i(x,p)| \leq \tilde{C}_1(1+|p|^{\gamma_i - 1}) \text{ for } (x.p) \in \mathbb{R}^N \times \mathbb{R}^N,\ i=1,2, \qquad \text{(RHP2)}$$

for some positive constant $\tilde{C}_1$. In fact, for $|p|>0$ we see that

$$\begin{aligned} |\nabla_p H_i(x,p)| = \max_{|e|=1} \nabla_p H_i(x,p)\cdot e &= \max_{|z|=|p|} \frac{1}{|p|}\nabla_p H_i(x,p)\cdot z \\ &\leq \max_{|z|=|p|} \frac{1}{|p|}(H_i(x,p+z) - H_i(p)) \end{aligned}$$

using convexity. This gives that (RHP2) follows from (HP1).

A typical example of H_i satisfying the above assumptions would be

$$H_i(x,p) = \frac{1}{\gamma_i}\langle p, a_i(x)p\rangle^{\gamma_i/2} + b_i(x)\cdot p,$$

where $a_i : \mathbb{R}^N \to \mathbb{R}^{N\times N}, b_i : \mathbb{R}^N \to \mathbb{R}^N$ are bounded functions with bounded derivatives and a_i are uniformly elliptic for $i=1,2$. In this case,

$$\ell_i(x,\xi) = \frac{1}{\gamma_i'}\langle \xi - b_i(x), a_i^{-1}(x)(\xi - b_i(x))\rangle^{\gamma_i'/2} \quad \text{where } \frac{1}{\gamma_i} + \frac{1}{\gamma_i'} = 1,$$

for $i=1,2$. The source terms f_i are assumed to satisfy the following

Assumption 8.2.2. *The functions $f_i : \mathbb{R}^N \to \mathbb{R}, i=1,2,$ are continuously differentiable and for some constant C_2 we have*

$$|\nabla f_i(x)| \leq C_2(1+|f_i(x)|^{2-\frac{1}{\gamma_i}}) \quad x \in \mathbb{R}^N, \qquad \text{(F1)}$$

for $i=1,2$. We also assume that for some $r>0$ we have

$$[|f_i(x)|+1]^{-1} \sup_{B_r(x)} |f_i(x)| < C_3, \quad \textit{for } x \in \mathbb{R}^N, \qquad \text{(F2)}$$

for some constant C_3 and $i=1,2$.

Without any loss of generality, we would assume that $r=1$. Note that (F1)-(F2) hold if we have $\sup_{x\in\mathbb{R}^N} |\nabla \log f_i(x)| < \infty$ and f_i are positive outside a compact set. Some other type of examples include $f_i(x) =$

$|x|^{\beta_1}(2+\sin((1+|x|^2)^{\beta_2}))$ for $\beta_i > 0$ and $(\beta_1+2\beta_2-1)\frac{\gamma_i}{2\gamma_i-1} \le \beta_1$. From (F2) we also see that

$$|f_i(x)| \le C_3(|f_i(y)|+1) \quad \text{whenever } |x-y| \le 1,$$

which readily gives

$$|f_i(x)| \le C_3 \left(\inf_{B_1(x)} |f_i(y)| + 1 \right) \quad \text{for all } x \in \mathbb{R}^N. \tag{8.2.2}$$

(F2) will be used to obtain certain estimate on the gradient of $\boldsymbol{u}$ (see Lemma 8.4.1).

Throughout the chapter, if $\mathcal{X}(\mathbb{R}^N)$ is a subspace of real-valued functions on $\mathbb{R}^N$ then we define the corresponding space $\mathcal{X}(\mathbb{R}^N \times \{1,2\}) := \left(\mathcal{X}(\mathbb{R}^N)\right)^2$, and endow it with the product topology, if applicable. Thus, a function $g \in \mathcal{X}(\mathbb{R}^d \times \{1,2\})$ is identified with the vector-valued function

$$\boldsymbol{g} := (g_1, g_2) \in \left(\mathcal{X}(\mathbb{R}^d)\right)^2, \quad \text{where } f_k(\cdot) := f(\cdot, k)\,, \quad k = 1,2\,.$$

With a slight abuse in notation we write $\boldsymbol{g} \in \mathcal{X}(\mathbb{R}^N \times \{1,2\})$.

8.3 Statement of main results

Our chief goal in this chapter is to find solutions corresponding to the critical value λ^* defined by

$$\lambda^* = \sup\{\lambda \in \mathbb{R} \;:\; \exists\, \boldsymbol{u} \in \mathcal{C}^2(\mathbb{R}^N \times \{1,2\}) \text{ such that } (\boldsymbol{u}, \lambda) \text{ is a subsolution to (EP)}\}. \tag{8.3.1}$$

The above definition is quite standard and has been used before by several authors [19, 20, 70, 116]. Our first main result is the following.

Theorem 8.3.1. *Grant Assumption 8.2.1 and also assume that for $i = 1, 2$, $\inf_{x\in\mathbb{R}^N} f_i(x) > -\infty$. Then for every $\lambda \le \lambda^*$ there exists $\boldsymbol{u} \in \mathcal{C}^2(\mathbb{R}^N \times \{1,2\})$ such that $(\boldsymbol{u}, \lambda)$ solves* (EP).

For a proof see Theorem 8.4.3 below. We should mention that the proof of Theorem 8.3.1 relies on an appropriate gradient estimate and bounds on the quantity $|u_1 - u_2|$ (see Proposition 8.4.1). In fact, these estimates are crucial for most of our proofs.

We say a function $g : \mathbb{R}^N \to \mathbb{R}$ is *coercive* if

$$g(x) \to \infty, \quad \text{as } |x| \to \infty.$$

Given a set $\mathcal{X}$ and two functions $g_1, g_2 : \mathcal{X} \to \mathbb{R}$, we say $g_1 \asymp g_2$ in $\mathcal{X}$ if there exist positive constants κ_1, κ_2 satisfying

$$\kappa_1 g_1 \leq g_2 \leq \kappa_2 g_1 \quad \text{in } \mathcal{X}.$$

Next we show that there exists a solution $\boldsymbol{u}$, bounded from below, corresponding to the eigenvalue λ^*.

Theorem 8.3.2. *Suppose Assumption 8.2.1 holds. Also, assume that $f_i, i = 1, 2,$ are coercive. Then there exists a solution $(\boldsymbol{u}, \lambda^*)$ to* (EP) *where $\inf_{\mathbb{R}^N} u_i > -\infty$ for $i = 1, 2$.*

For a proof see Theorem 8.4.4. Our next result concerns the uniqueness of solutions.

Theorem 8.3.3. *Grant Assumptions 8.2.1, and 8.2.2. In addition, we also assume that $f_1 \asymp f_2$ outside a compact set, and $f_i, i = 1, 2,$ are coercive. Let $(\boldsymbol{u}, \lambda)$ and $(\tilde{\boldsymbol{u}}, \tilde{\lambda})$ be two solutions to* (EP) *with $\inf_{\mathbb{R}^N} u_i > -\infty$, $\inf_{\mathbb{R}^N} \tilde{u}_i > -\infty$ for $i = 1, 2$. Then we must have $\lambda = \tilde{\lambda} = \lambda^*$ and $u_i = \tilde{u}_i + C$ for some constant C and $i = 1, 2$.*

Proof of Theorem 8.3.3 follows from Theorem 8.4.1. As can be seen from above that Assumption 8.2.2 is a bit stronger than the usual hypotheses used to establish uniqueness in the super-critical case (i.e., $\gamma_i \geq 2$) for scalar model (cf. [19]). In the scalar case, one generally uses an exponential transformation together with the coercive property of the solutions to establish uniqueness [19, 23]. Similar transformation does not seem to work in the present setting because of the presence of the coupling terms. So for the uniqueness we rely on the convex analytic approach of [6] and the estimates in Proposition 8.4.1. Also, the condition $f_1 \asymp f_2$ can be relaxed provided f_i satisfy certain polynomial growth hypothesis. See Theorem 8.4.2 for further detail.

Remark 8.3.1. *Above results correspond to a switching Markov process having two states , that is, the solution $\boldsymbol{u}$ is given by a tuple (u_1, u_2) of length 2. All the results of this chapter continue to hold if the weakly coupled system has any finite number of states, provided Assumption 8.2.1, and 8.2.2 are modified accordingly.*

8.4 Proofs of main results

In this section we prove Theorem 8.3.1, 8.3.2, and 8.3.3. We start by proving a gradient estimate which is a key ingredient for most of the proofs below.

Proposition 8.4.1. *Let Assumption 8.2.1 hold. Let $\varepsilon \in [0, 1]$. Suppose $\mathscr{B}_1 \Subset \mathscr{B}_2 \Subset D$ be two given concentric balls, centred at z, in $\mathbb{R}^N$. Consider a solution $\boldsymbol{u} \in \mathcal{C}^2(D \times \{1, 2\})$ to the system of equations*

$$\begin{aligned} -\Delta u_1(x) + H_1(x, \nabla u_1) + \alpha_1(x)(u_1(x) - u_2(x)) + \varepsilon u_1(x) &= f_1(x) \quad \text{in } D, \\ -\Delta u_2(x) + H_2(x, \nabla u_2) + \alpha_2(x)(u_2(x) - u_1(x)) + \varepsilon u_2(x) &= f_2(x) \quad \text{in } D. \end{aligned} \tag{8.4.1}$$

Then there exists a constant $C > 0$, dependent only on $\operatorname{dist}(\mathscr{B}_1, \partial\mathscr{B}_2), \gamma_i, C_1, N$ and $\sup_{B_2}(|\alpha_i| + |\nabla\alpha_i|)$ for $i = 1, 2$, satisfying

$$\begin{aligned} \sup_{\mathscr{B}_1}\{|\nabla u_1|^{2\gamma_1}, |\nabla u_2|^{2\gamma_2}\} \leq C\Big(1 + \sup_{\mathscr{B}_2}\sum_{i=1}^{2}(f_i)_+^2 \\ + \sup_{\mathscr{B}_2}\sum_{i=1}^{2}|\nabla f_i|^{2\gamma_i/(2\gamma_i-1)} + \sup_{\mathscr{B}_2}\sum_{i=1}^{2}(\varepsilon u_i)_-^2\Big). \end{aligned} \tag{8.4.2}$$

Furthermore, for some constant $\tilde{C}$*, dependent only on* $\mathrm{dist}(\mathscr{B}_1, \partial\mathscr{B}_2), \gamma_i, C_1, N, \alpha_0$*, we have*

$$|u_1(z)-u_2(z)|^2 \leq \tilde{C}\Big(1+\sup_{\mathscr{B}_2}\sum_{i=1}^{2}(f_i)_+^2+\sup_{\mathscr{B}_2}\sum_{i=1}^{2}|\nabla f_i|^{2\gamma_i/(2\gamma_i-1)}+\sup_{\mathscr{B}_2}\sum_{i=1}^{2}(\varepsilon u_i)_-^2\Big). \tag{8.4.3}$$

The proof of this Proposition is quite long and therefore, is deferred to Appendix 8.6.

Next we show that any solution of (EP) which is bounded from below, is actually coercive. This lemma should be compared with [19, Proposition 3.4] and [6, Lemma 2.1]. Our proof does not use Harnack's inequality like these previous works. Our proof is based on the comparison principle.

Lemma 8.4.1. *Grant Assumptions 8.2.1, and 8.2.2. Let* $\boldsymbol{u} = (u_1, u_2)$ *be a non-negative solution to*

$$-\Delta u_1 + H_1(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) = f_1 \quad \text{in } \mathbb{R}^N,$$
$$-\Delta u_2 + H_2(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) = f_2 \quad \text{in } \mathbb{R}^N.$$

Also, assume that $f_i, i = 1, 2,$ *are coercive. Then for some constants* M_1, M_2 *we have*

$$u_i(x) \geq M_1[f_i(x)]^{1/\gamma_i} - M_2 \quad x \in \mathbb{R}^N,\ i = 1, 2. \tag{8.4.4}$$

Moreover, if $f_1 \asymp f_2$ *outside a compact set, then* $\frac{1}{u_i(x)}|\nabla u_i|^2 \leq M_3[f_i(x)]^{1/\gamma_i}$ *outside a compact set, for some constant* M_3.

Proof. Choose $R > 0$ so that $f_i(x) > 1$ for $|x| \geq R$. Fix a point $x_0 \in \mathscr{B}^c_{R+1}(0)$ and define

$$\psi_i(y) = \theta|f_i(x_0)|^{1/\gamma_i}(1 - |y - x_0|^2),$$

where $\theta > 0$ is to chosen later and $i = 1, 2$. Then, using (HP1), we have in

$B_1(x_0)$

$$\begin{aligned}
&\Delta\psi_1(y) - H_1(y, \nabla\psi_1(y)) + \alpha_1(y)(\psi_2 - \psi_1) + f_1(y) \\
&\quad \geq \Delta\psi_1(y) - C_1|\nabla\psi_1|^{\gamma_1} - C_1 + \alpha_1(y)(\psi_2 - \psi_1) + f_1(y) \\
&\quad \geq -2N\theta|f_1(x_0)|^{1/\gamma_1} - 2^{\gamma_1}\theta^{\gamma_1}C_1|f_1(x_0)||y - x_0|^{\gamma_1} \\
&\quad - C_1 - \alpha_1(y)\theta|f_1(x_0)|^{1/\gamma_1} + f_1(y) \\
&\quad \geq f_1(x_0)\Big[- 2N\theta|f_1(x_0)|^{1/\gamma_1 - 1} - 2^{\gamma_1}\theta^{\gamma_1}C_1 \\
&\quad - C_1(f_1(x_0))^{-1} - \alpha_0\theta|f_1(x_0)|^{1/\gamma_1 - 1} + \kappa\Big],
\end{aligned} \tag{8.4.5}$$

where

$$\left[\inf_{|x|\geq R+1}\inf_{y\in B_1(x)} f(y)\right](|f(x)| + 1)^{-1} \geq \kappa > 0 \quad \text{for } R \text{ large enough, by (8.2.2).}$$

Since f_1 is coercive, we can choose θ small and R large so that the r.h.s. of (8.4.5) is positive. Similarly, we can also show that for some small θ and large R

$$\Delta\psi_2(y) - H_2(y, \nabla\psi_2) + \alpha_2(x)(\psi_1 - \psi_2) + f_2(y) \geq 0 \quad \text{in } \mathscr{B}_1(x_0),$$

whenever $|x_0| > R$. We can now apply comparison principle, Theorem 8.7.1, in $\mathscr{B}_1(x_0)$ to conclude that $(u_1, u_2) \geq (\psi_1, \psi_2)$ in $\mathscr{B}_1(x_0)$ implying $u_i(x_0) \geq \theta[f_i(x_0)]^{1/\gamma_i}$ for $i = 1, 2$ and for all $|x_0| > R$. This gives (8.4.4). Again, from (F1)-(F2) we have

$$\max\{|Du_1(x)|^{2\gamma_1}, |Du_2(x)|^{2\gamma_2}\} \leq C(1 + |f_1(x)|^2 + |f_2(x)|^2),$$

for some constant C and for all x outside a compact set. Since $f_1 \asymp f_2$ outside a compact set, the second conclusion follows from the above display and (8.4.4). Hence the proof. □

We now first establish the uniqueness and then discuss the existence results, that is, we assume Theorems 8.3.1, and 8.3.2 and prove Theorem 8.3.3 first, and then we prove Theorem 8.3.1, and 8.3.2.

8.4.1 Uniqueness

We begin by introducing a few notations. By $\boldsymbol{g} = (g_1, g_2) \in \mathcal{C}^2(\mathbb{R}^N \times \{1,2\})$ we mean $g_i \in \mathcal{C}^2(\mathbb{R}^N)$ for $i = 1,2$. Define the operator $\boldsymbol{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2)$: $\mathcal{C}^2(\mathbb{R}^N \times \{1,2\}) \to \mathcal{C}^2(\mathbb{R}^N \times \mathbb{R}^N \times \{1,2\})$ for $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$, $k = 1,2$ by

$$\mathcal{A}_k \boldsymbol{g}(x,\xi) \coloneqq \Delta g_k(x) - \xi \cdot \nabla g_k(x) + \alpha_k(x) \sum_{j=1}^{2} (g_j(x) - g_k(x)),$$

with $\boldsymbol{g} = (g_1, g_2) \in \mathcal{C}^2(\mathbb{R}^N \times \{1,2\})$. Also, $\mathcal{C}^2_c(\mathbb{R}^N \times \mathcal{S})$ denotes the class of functions in $\mathcal{C}^2(\mathbb{R}^N \times \mathcal{S})$ with compact support. Let $\mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S})$ denote the set of Borel probability measures $\boldsymbol{\mu} = (\mu_1, \mu_2)$, with $\mu_i = \boldsymbol{\mu}(\cdot \times \{i\})$ being a sub-probability measure. For a function $\boldsymbol{h} \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^2$ we use the notation

$$\boldsymbol{\mu}(\boldsymbol{h}) \coloneqq \int_{\mathbb{R}^N \times \mathbb{R}^N} \Big\langle \boldsymbol{h}(x,\xi)\,, \boldsymbol{\mu}(\mathrm{d}x, \mathrm{d}\xi) \Big\rangle = \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} h_k(x,\xi) \mu_k(\mathrm{d}x, \mathrm{d}\xi)\,.$$

We define

$$\mathcal{M} \coloneqq \Big\{ \boldsymbol{\mu} \in \mathcal{P}(\mathbb{R}^N \times \mathbb{R}^N \times \{1,2\}) : \boldsymbol{\mu}\Big(\boldsymbol{\mathcal{A}} \boldsymbol{g}\Big) = 0 \quad \forall\, \boldsymbol{g} \in \mathcal{C}^2_c(\mathbb{R}^N \times \{1,2\}) \Big\}\,.$$

Let

$$F_k(x,\xi) \coloneqq f_k(x) + \ell_k(x,\xi) \quad k = 1,2, \tag{8.4.6}$$

where ℓ_k is given by Assumption 8.2.1. Now define

$$\mathcal{M}_{\boldsymbol{F}} \coloneqq \Big\{ \boldsymbol{\mu} \in \mathcal{M} : \boldsymbol{\mu}(\boldsymbol{F}) < \infty \Big\},$$

and

$$\bar{\lambda} \coloneqq \inf_{\mu \in \mathcal{M}} \boldsymbol{\mu}(\boldsymbol{F}) = \inf_{\mu \in \mathcal{M}_{\boldsymbol{F}}} \boldsymbol{\mu}(\boldsymbol{F})\,. \tag{LP}$$

In Lemma 8.4.3 below we show that $\mathcal{M}_{\boldsymbol{F}}$ is non-empty. Our next result shows that λ^* is smaller that $\bar{\lambda}$.

Lemma 8.4.2. *Consider the setting of Theorem 8.3.3. Then we must have* $\lambda^* \leq \bar{\lambda}$.

Proof. We only consider the case when $\bar{\lambda} < \infty$, otherwise there is nothing to prove. Let $\boldsymbol{\mu} \in \mathcal{M}$ be such that $\boldsymbol{\mu}(\boldsymbol{F}) < \infty$. Since $\boldsymbol{\mu} \in \mathcal{M}$ we have

$$\boldsymbol{\mu}(\boldsymbol{\mathcal{A}g}) = \sum_{k=1}^{2} \int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}_k \boldsymbol{g}(x,\xi)\mu_k(\mathrm{d}x, \mathrm{d}\xi) = 0 \quad \text{for all } \boldsymbol{g} \in \mathcal{C}_c^2(\mathbb{R}^N \times \{1,2\}). \tag{8.4.7}$$

Let $\boldsymbol{u} = (u_1, u_2)$ be a non-negative solution to (EP) corresponding to λ^*, that is,

$$\begin{aligned} -\Delta u_1(x) + H_1(x, \nabla u_1(x)) + \alpha_1(x)(u_1(x) - u_2(x)) &= f_1(x) - \lambda^* \quad \text{in } \mathbb{R}^N, \\ -\Delta u_2(x) + H_2(x, \nabla u_2(x)) + \alpha_2(x)(u_2(x) - u_1(x)) &= f_2(x) - \lambda^* \quad \text{in } \mathbb{R}^N. \end{aligned} \tag{8.4.8}$$

Existence of u follows from Theorem 8.3.2. From Lemma 8.4.1 we also know that $u_i, i = 1, 2$, are coercive. We would modify $\boldsymbol{u}$ suitably so that it can used in (8.4.7) as a test function. To do so, we consider a family of concave functions.

For $r > 0$, we let χ_r be a concave $\mathcal{C}^2(\mathbb{R})$ function such that $\chi_r(t) = t$ for $t \le r$, and $\chi_r'(t) = 0$ for $t \ge 3r$. Then χ_r' and $-\chi_r''$ are non-negative, and the latter is supported on $[r, 3r]$. In addition, we select χ_r so that

$$|\chi_r''(t)| \le \frac{2}{t} \qquad \forall\, t > 0\,. \tag{8.4.9}$$

In particular, we may define χ_r by specifying

$$\chi_r''(t) = \begin{cases} \frac{4}{3}\frac{r-t}{r^2} & \text{if } r \le t \le \frac{3r}{2}, \\ -\frac{2}{3r} & \text{if } \frac{3r}{2} \le t \le \frac{5r}{2}, \\ \frac{4}{3}(\frac{t}{r^2} - \frac{3}{r}) & \text{if } \frac{5r}{2} \le t \le 3r. \end{cases}$$

Using (8.4.8) we now compute

$$\begin{aligned}
&\Delta\chi_r(u_k) - \xi\cdot\nabla\chi_r(u_k) + \alpha_k\sum_{j=1}^{2}(\chi_r(u_j)-\chi_r(u_k)) \\
&= \chi_r''(u_k)|\nabla u_k|^2 + \chi_r'(u_k)\Big(\Delta u_k - \xi\cdot\nabla u_k\Big) + \alpha_k\sum_{j=1}^{2}(\chi_r(u_j)-\chi_r(u_k)) \\
&= \chi_r''(u_k)|\nabla u_k|^2 + \chi_r'(u_k)\Big(\lambda^* + H_k(x,\nabla u_k) - f_k - \xi\cdot\nabla u_k\Big) \\
&\qquad + \alpha_k\sum_{j=1}^{2}\Big(\chi_r(u_j)-\chi_r(u_k)-\chi_r'(u_k)(u_j-u_k)\Big) \\
&= \chi_r''(u_k)|Du_k|^2 + \chi_r'(u_k)\Big(\lambda^* - f_k - \ell_k(x,\xi)\Big) \\
&\quad + \chi_r'(u_k)\Big(\ell_k(x,\xi) - \xi\cdot\nabla u_k + H_k(x,\nabla u_k)\Big) \\
&+ \alpha_k\sum_{j=1}^{2}\Big(\chi_r(u_j)-\chi_r(u_k)-\chi_r'(u_k)(u_j-u_k)\Big)\,.
\end{aligned} \tag{8.4.10}$$

Thus, defining

$$G_{r,k}[\boldsymbol{u}](x) \coloneqq \alpha_k\sum_{j=1}^{2}\Big(\chi_r(u_j)-\chi_r(u_k)-\chi_r'(u_k)(u_j-u_k)\Big)\,,$$

and integrating (8.4.10) with respect to a $\boldsymbol{\mu}$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{n}\int_{\mathbb{R}^N\times\mathbb{R}^N}\chi_r'\Big(u_k(x)\Big)\Big(f_k(x)+\ell_k(x,\xi)-\lambda^*\Big)\,\mu_k(\mathrm{d}x,\mathrm{d}\xi) \\
&= \sum_{k=1}^{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\chi_r'\Big(u_k(x)\Big)\Big(\ell_k(x,\xi)-\xi\cdot\nabla u_k + H_k(x,\nabla u_k)\Big)\,\mu_k(\mathrm{d}x,\mathrm{d}\xi) \\
&\qquad + \sum_{k=1}^{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\Big(\chi_r''\Big(u_k(x)\Big)\Big|Du_k(x)\Big|^2 + G_{r,k}[\boldsymbol{u}](x)\Big)\,\mu_k(\mathrm{d}x,\mathrm{d}\xi)\,.
\end{aligned} \tag{8.4.11}$$

Next we show that the last term on the r.h.s. of (8.4.11) goes to 0 as $r\to\infty$. Since $f_1 \asymp f_2$ outside a compact set and

$$\boldsymbol{\mu}(\boldsymbol{f}) = \sum_{k=1}^{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} f_k(x)\mu_k(\mathrm{d}x,\mathrm{d}\xi) < \infty,$$

we obtain

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (|f_1(x)| + |f_2(x)|)\mu_1(\mathrm{d}x, \mathrm{d}\xi) < \infty, \quad \text{and}$$
$$\int_{\mathbb{R}^N\times\mathbb{R}^N} (|f_1(x)| + |f_2(x)|)\mu_2(\mathrm{d}x, \mathrm{d}\xi) < \infty. \tag{8.4.12}$$

Therefore, using Lemma 8.4.1 and (8.4.9), we get

$$\begin{aligned}
&\sum_{k=1}^{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} |\chi_r''\big(u_k(x)\big)|\Big|Du_k(x)\Big|^2 \mu_k(\mathrm{d}x, \mathrm{d}\xi) \\
&\le \sum_{k=1}^{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathbb{1}_{\{r<u_k(x)<3r\}} \frac{2}{u_k(x)}\Big|Du_k(x)\Big|^2 \mu_k(\mathrm{d}x, \mathrm{d}\xi) \\
&\le \kappa \sum_{k=1}^{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathbb{1}_{\{r<u_k(x)<3r\}} |f_k(x)|^{1/\gamma_i} \mu_k(\mathrm{d}x, \mathrm{d}\xi),
\end{aligned}$$

for some constant κ. Since $u_k, k = 1, 2$, are coercive, using dominated convergence theorem it follows that the r.h.s. of the above display tends to 0 as $r \to \infty$. Again, since $\chi' \le 1$, it follows that

$$|G_{r,k}[\boldsymbol{u}](x)| \le 2\alpha_0 \mathbb{1}_{A_r^c}(x)|u_1(x) - u_2(x)| \quad \text{for all } x \in \mathbb{R}^N,\ k = 1, 2,$$

where $A_r = \{x \ :\ u_2(x) \vee u_1(x) \le r\}$. Using (F1)-(F2) and (8.4.3) we then have

$$|G_{r,k}[\boldsymbol{u}](x)| \le \kappa_1 \mathbb{1}_{A_r^c}(x)(|f_1(x)| + |f_2(x)|) \quad \text{for all } x \in \mathbb{R}^N,\ k = 1, 2,$$

for some constant κ_1. Again using (8.4.12) and dominated convergence theorem we thus get

$$\lim_{r\to\infty} \sum_{k=1}^{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} G_{r,k}[\boldsymbol{u}](x)\, \mu_k(\mathrm{d}x, \mathrm{d}\xi) = 0\,.$$

From our construction, it also follows that χ'_{3^n} is an increasing sequence. Therefore, letting $r = 3^n \to \infty$ in (8.4.11) and applying monotone convergence theorem we obtain

$$\boldsymbol{\mu}(\boldsymbol{F}) - \lambda^* = \sum_{k=1}^{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} \Big(\ell_k(x, \xi) - \xi\cdot\nabla u_k + H_k(x, \nabla u_k)\Big)\, \mu_k(\mathrm{d}x, \mathrm{d}\xi) \ge 0\,. \tag{8.4.13}$$

Since $\boldsymbol{\mu}$ is arbitrary, this proves the lemma. □

Next we show that $\mathcal{M}_{\boldsymbol{F}}$ is non-empty.

Lemma 8.4.3. *Suppose that $\boldsymbol{u}$ is a coercive, non-negative solution to* (EP) *with eigenvalue λ. Define*

$$\xi_k(x) = \nabla_p H_k(x, \nabla u_k(x)) \quad k = 1, 2.$$

Then there exists a Borel probability measure $\boldsymbol{\nu} = (\nu_1, \nu_2)$ on $\mathbb{R}^N \times \{1, 2\}$ so that

$$\boldsymbol{\mu_u} = (\mu_{1,u}, \mu_{2,u}) \in \mathcal{M}_{\boldsymbol{F}} \quad \textit{where} \quad \mu_{k,u} := \nu_k(\mathrm{d}x)\delta_{\xi_k(x)}(\mathrm{d}\xi).$$

Furthermore, $\bar{\lambda} \le \lambda$.

Proof. Since H_k is the Fenchel–Legendre transformation of ℓ_k, it is well known that

$$H_k(x, p) = p \cdot \xi - \ell_k(x, \xi) \quad \text{for } \xi = \nabla_p H_k(x, p), \tag{8.4.14}$$

for $k = 1, 2$. Therefore, we can rewrite (EP) in $\mathbb{R}^N$ as

$$\begin{cases} \Delta u_1(x) - \xi_1(x) \cdot \nabla u_1(x) - \alpha_1(x)(u_1(x) - u_2(x)) & = \lambda - F_1(x, \xi_1(x)), \\ \Delta u_2(x) - \xi_2(x) \cdot \nabla u_2(x) - \alpha_2(x)(u_2(x) - u_1(x)) & = \lambda - F_2(x, \xi_2(x)), \end{cases} \tag{8.4.15}$$

where $\boldsymbol{F}$ is given by (8.4.6). We define the extended generator $\boldsymbol{\mathcal{A}_u} = (\mathcal{A}_{1,\boldsymbol{u}}, \mathcal{A}_{2,\boldsymbol{u}}) : \mathcal{C}^2(\mathbb{R}^N \times \{1, 2\}) \to \mathcal{C}^2(\mathbb{R}^N \times \{1, 2\})$ for $(x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N$, $k = 1, 2$ by

$$\mathcal{A}_{k,\boldsymbol{u}}\boldsymbol{g}(x) := \Delta g_k(x) - \xi_k(x) \cdot \nabla g_k(x) + \alpha_k(x) \sum_{j=1}^{2} (g_j(x) - g_k(x)),$$

Since $\boldsymbol{u}, \boldsymbol{F}$ are coercive, there exists a switching diffusion (X_t, S_t) associated to the generator $\boldsymbol{\mathcal{A}_u}$ (cf. [11, Chapter 5]). Furthermore, the mean empirical measures of (X_t, S_t) will be tight and therefore, should have a limit point (cf. [11, Lemma 2.5.3]). Let $\boldsymbol{\nu} = (\nu_1, \nu_2)$ be one such limit points. It is also standard to show that

$$\sum_{k=1}^{2} \int_{\mathbb{R}^N} \mathcal{A}_{k,\boldsymbol{u}}\boldsymbol{g}(x) \nu_k(\mathrm{d}x) = 0 \tag{8.4.16}$$

for all $g \in \mathcal{C}_c^2(\mathbb{R}^N \times \{1,2\})$. Hence it follows that $\boldsymbol{\mu_u} \in \mathcal{M}$.

To prove the second part, we consider the concave function χ_r from Lemma 8.4.2. Since χ_r is concave we have $\chi_r'' \leq 0$ and

$$\chi_r(u_j) - \chi_r(u_k) - \chi_r'(u_k)(u_j - u_k) \leq 0.$$

Thus, the calculation of (8.4.10) and (8.4.14)-(8.4.15) gives

$$\begin{aligned}&\Delta\chi_r(u_k) - \xi_k \cdot \nabla\chi_r(u_k) + \alpha_k \sum_{j=1}^{2}(\chi_r(u_j) - \chi_r(u_k))\\ &\leq \chi_r'(u_k)(\lambda - F_k(x, \xi_k(x))).\end{aligned}$$

Integrating both sides with ν_k and summing over k, we obtain from (8.4.16) that

$$\sum_{k=1}^{2}\int_{\mathbb{R}^N} \chi_r'(u_k)F_k(x, \xi_k(x))\nu_k(\mathrm{d}x) \leq \lambda.$$

Now letting $r \to \infty$ and using Fatou's lemma we obtain

$$\boldsymbol{\mu_u}(\boldsymbol{F}) \leq \lambda.$$

Thus, $\boldsymbol{\mu_u} \in \mathcal{M}_{\boldsymbol{F}}$ and $\bar{\lambda} \leq \lambda$. □

We note that the proof of Lemma 8.4.3 also works for $\mathcal{C}^2$ super-solutions. Combining the above result with Lemma 8.4.2 we get the following corollary.

Corollary 8.4.1. *Under the assumptions of Theorem 8.3.3 we have*

$$\lambda^* = \inf\{\lambda \in \mathbb{R} \,:\, \exists \text{ nonnegative } \boldsymbol{u} \in \mathcal{C}^2(\mathbb{R}^N \times \{1,2\}) \text{ such that } (\boldsymbol{u}, \lambda) \text{ is a super-solution to (EP)}\}.$$

Note that the existence of a non-negative solution $\boldsymbol{u}$ for the value λ^* follows from Theorem 8.3.2.

Now we are ready to establish our uniqueness result.

Theorem 8.4.1. *Grant the setting of Theorem 8.3.3. Let $(\boldsymbol{u}, \lambda)$ be a solution to* (EP) *and $\boldsymbol{u}$ is non-negative. Then* *(a)* $\lambda = \lambda^* = \bar{\lambda} = \boldsymbol{\mu}_{\boldsymbol{u}}(\boldsymbol{F})$, *where* $\boldsymbol{\mu}_{\boldsymbol{u}}$ *is given by Lemma 8.4.3*

(b) Suppose that $(\tilde{\boldsymbol{u}}, \tilde{\lambda})$ *is another solution to* (EP) *and* $\tilde{\boldsymbol{u}}$ *is non-negative, then* $\tilde{\lambda} = \lambda^*$ *and* $\tilde{\boldsymbol{u}} = \boldsymbol{u} + C$ *for some constant* C.

Proof. (a) follows from Lemma 8.4.2, and 8.4.3 and (8.4.13). So we consider (b). Using Lemma 8.4.3, we find a Borel probability measure $\tilde{\boldsymbol{\nu}} = (\tilde{\nu}_1, \tilde{\nu}_2)$ such that for

$$\tilde{\boldsymbol{\mu}}_{\tilde{\boldsymbol{u}}} = (\tilde{\mu}_{1,\tilde{\boldsymbol{u}}}, \tilde{\mu}_{2,\tilde{\boldsymbol{u}}}) \quad \text{with} \quad \tilde{\mu}_{k,\tilde{u}} \coloneqq \tilde{\nu}_k(\mathrm{d}x)\delta_{\tilde{\xi}_k(x)}(\mathrm{d}\xi), \quad \tilde{\xi}_k(x) = \nabla_p H_k(x, \nabla \tilde{u}_k),$$

we have $\tilde{\lambda} = \tilde{\boldsymbol{\mu}}_{\tilde{\boldsymbol{u}}}(\boldsymbol{F}) = \lambda^*$. Again, by [11, Theorem 5.3.4], there exist strictly positive Borel measurable functions $\boldsymbol{\rho} = (\rho_1, \rho_2)$ and $\tilde{\boldsymbol{\rho}} = (\tilde{\rho}_1, \tilde{\rho}_2)$ satisfying

$$\nu_k(\mathrm{d}x) = \rho_k(x)\mathrm{d}x, \quad \tilde{\nu}_k(\mathrm{d}x) = \tilde{\rho}_k(x)\mathrm{d}x \quad \text{for } k = 1, 2.$$

Let us now define

$$\zeta_k = \frac{\rho_k}{\rho_k + \tilde{\rho}_k}, \quad \tilde{\zeta}_k = \frac{\tilde{\rho}_k}{\rho_k + \tilde{\rho}_k}, \quad v_k(x) = \xi_k(x)\zeta_k(x) + \tilde{\xi}_k(x)\tilde{\zeta}_k(x),$$
$$\widehat{\mu}_k(\mathrm{d}x, \mathrm{d}\xi) = \frac{1}{2}(\nu_k(\mathrm{d}x) + \tilde{\nu}_k(\mathrm{d}x))\delta_{v_k(x)}(\mathrm{d}\xi), \quad \text{for } k = 1, 2.$$

We claim that $\widehat{\boldsymbol{\mu}} = (\widehat{\mu}_1, \widehat{\mu}_2) \in \mathcal{M}$. Consider $\boldsymbol{g} = (g_1, g_2) \in \mathcal{C}^2_c(\mathbb{R}^N \times \{1, 2\})$. Also, we note that

$$\frac{1}{2}(\nu_k(\mathrm{d}x) + \tilde{\nu}_k(\mathrm{d}x)) = \frac{1}{2}(\rho_k(x) + \tilde{\rho}_k(x))\mathrm{d}x \quad \text{for } k = 1, 2.$$

A simple computation then yields

$$\begin{aligned}
&\int_{\mathbb{R}^N \times \mathbb{R}^N} \mathcal{A}_k(x, \xi)\, \widehat{\mu}_k(\mathrm{d}x, \mathrm{d}\xi) \\
&= \int_{\mathbb{R}^N} \Big(\Delta g_k(x) - v_k(x) \cdot \nabla g_k(x) + \alpha_k(x) \sum_{j=1}^{2} (g_j(x) - g_k(x))\Big) \\
&\qquad\qquad \times \frac{1}{2}(\nu_1(\mathrm{d}x) + \tilde{\nu}_1(\mathrm{d}x)) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \Big((\rho_k(x) + \tilde{\rho}_k(x))\Delta g_k(x) - (\xi_k(x)\rho_k(x) + \tilde{\xi}_k(x)\tilde{\rho}_k(x)) \cdot \nabla g_k(x) \\
&\qquad\qquad + (\rho_k(x) + \tilde{\rho}_k(x))\alpha_k(x) \sum_{j=1}^{2} (g_j(x) - g_k(x))\Big)\mathrm{d}x \\
&= \frac{1}{2} \int_{\mathbb{R}^N} \mathcal{A}_{k,u}\boldsymbol{g}(x)\, \nu_k(\mathrm{d}x) + \frac{1}{2} \int_{\mathbb{R}^N} \mathcal{A}_{k,\tilde{u}}\boldsymbol{g}(x)\, \tilde{\nu}_k(\mathrm{d}x).
\end{aligned}$$

Therefore,

$$\sum_{k=1}^{2}\int_{\mathbb{R}^N\times\mathbb{R}^N}\mathcal{A}_k(x,\xi)\,\widehat{\mu}_k(\mathrm{d}x,\mathrm{d}\xi)=\frac{1}{2}\left[\boldsymbol{\mu}_{\boldsymbol{u}}(\boldsymbol{\mathcal{A}}_{\boldsymbol{u}}\boldsymbol{g})+\boldsymbol{\mu}_{\tilde{\boldsymbol{u}}}(\boldsymbol{\mathcal{A}}_{\tilde{\boldsymbol{u}}}\boldsymbol{g})\right]=0.$$

This proves the claim. Using the convexity of ℓ_k in ξ it is also easily seen that $\widehat{\boldsymbol{\mu}}(\boldsymbol{F})<\infty$. Now from Lemma 8.4.2, and 8.4.3 we see that $\boldsymbol{\mu}_{\boldsymbol{u}}$ and $\boldsymbol{\mu}_{\tilde{\boldsymbol{u}}}$ are optimal for (LP). Thus we have

$$\begin{aligned}
0 &\le \widehat{\boldsymbol{\mu}}(\boldsymbol{F})-\frac{1}{2}\boldsymbol{\mu}_{\boldsymbol{u}}(\boldsymbol{F})-\frac{1}{2}\boldsymbol{\mu}_{\tilde{\boldsymbol{u}}}(\boldsymbol{F})\\
&=\frac{1}{2}\sum_{k=1}^{2}\Bigg[\int_{\mathbb{R}^N}\ell_k(x,v_k(x))(\rho_k(x)+\tilde{\rho}_k(x))\mathrm{d}x-\int_{\mathbb{R}^N}\ell_k(x,\xi_k(x))\rho_k(x)\mathrm{d}x\\
&\qquad-\int_{\mathbb{R}^N}\ell_k(x,\tilde{\xi}_k(x))\tilde{\rho}_k(x)\mathrm{d}x\Bigg]\\
&=\frac{1}{2}\sum_{k=1}^{2}\Bigg[\int_{\mathbb{R}^N}\Big(\ell_k(x,v_k(x))-\ell_k(x,\xi_k(x))\zeta_k(x)\\
&\qquad-\ell_k(x,\tilde{\xi}_k(x))\tilde{\zeta}_k\Big)(\rho_k(x)+\tilde{\rho}_k(x))\mathrm{d}x\Bigg]\le 0,
\end{aligned}$$

where the last line follows from the convexity of ℓ_k in ξ. Therefore,

$$\begin{aligned}
\sum_{k=1}^{2}\Bigg[\int_{\mathbb{R}^N}\Big(\ell_k(x,v_k(x))-\ell_k(x,\xi_k(x))\zeta_k(x)-\ell_k(x,\tilde{\xi}_k(x))\tilde{\zeta}_k\Big)(\rho_k(x)+\tilde{\rho}_k(x))\mathrm{d}x\Bigg]\\
=0.
\end{aligned}$$

Since $\rho_k,\tilde{\rho}_k$ are strictly positive, and ℓ_k is strictly convex, it the follows that $\xi_k=\tilde{\xi}_k$ for $k=1,2$. Since $H_k(x,\cdot)$ is strictly convex, by (A1), given ξ there exists a unique p satisfying

$$H_k(x,p)=p\cdot\xi-\ell_k(x,\xi).$$

Thus, from (8.4.14), we obtain $\nabla u_k=\nabla\tilde{u}_k$ in $\mathbb{R}^N$, for $k=1,2$. This, of course, implies $u_i=\tilde{u}_i+C_i$ for some constant C_i, $i=1,2$. Again, subtracting the equation of $\boldsymbol{u}$ from the equations of $\tilde{\boldsymbol{u}}$ we see that $\alpha_1(C_1-C_2)=0$ implying $C_1=C_2$. This completes the proof. □

The proof of uniqueness in Theorem 8.4.1 requires f_1 to be comparable to f_2 outside a compact set. This property is crucially used in Lemma 8.4.1, and 8.4.2. However, if we impose more structural assumption on $\boldsymbol{f}$ then we could relax the requirement of $f_1 \asymp f_2$.

(F3) Suppose that there exists $\beta_1, \beta_2 > 1$ satisfying

$$C^{-1}|x|^{\beta_i} - C \le f_i(x) \le C(|x|^{\beta_i} + 1), \quad x \in \mathbb{R}^N,$$

where

$$\beta_2 \le \beta_1 \frac{\gamma_1 + 1}{2}, \quad \beta_1 \le \beta_2 \frac{\gamma_2 + 1}{2},$$

and

$$\max\left\{\frac{\beta_1(\gamma_1+1)}{2\gamma_1}, \frac{\beta_2(\gamma_2+1)}{2\gamma_2}\right\} \le \beta_1 \wedge \beta_2 - 1\,.$$

As a consequence of (F3) it follows that

$$|f_2(x)|^{2/\gamma_1} \le \kappa(1 + |f_1(x)|^{1+\gamma_1^{-1}}) \quad \text{and} \quad |f_1(x)|^{2/\gamma_2} \le \kappa(1 + |f_2(x)|^{1+\gamma_2^{-1}}) \tag{8.4.17}$$

for some $\kappa > 0$. Theorem 8.4.1 can be improved as follows.

Theorem 8.4.2. *Suppose that Assumptions 8.2.1, and 8.2.2 and (F3) hold. Then the conclusions of Theorem 8.4.1 hold true.*

Proof. We only need to modify Lemma 8.4.1, and 8.4.2. Note that (8.4.4) holds. Using (F1),(F2),(8.4.2) and (8.4.17) it follows that

$$|\nabla u_i(x)|^2 \le \kappa_1(1 + |f_i(x)|^{1+\gamma_i^{-1}}) \tag{8.4.18}$$

for some constant κ_1. Therefore, for some compact set $\mathcal{K}$ and a constant κ_3, we obtain from (8.4.4) that

$$\frac{|\nabla u_i|^2}{u_i(x)} \le \kappa_3 |f_i(x)| \quad x \in \mathcal{K}^c. \tag{8.4.19}$$

Again, using (F3) and (8.4.18) we see that

$$|\nabla u_i(x)| \le \kappa_4 \left(1 + |x|^{\frac{\beta_i(1+\gamma_i)}{2\gamma_i}}\right) \quad \text{for some } \kappa_4, \quad i = 1, 2.$$

Using (F3) this also implies

$$\max\{u_1(x), u_2(x)\} \le \kappa_5 \min\{1 + |f_1(x)|, 1 + |f_2(x)|\} \tag{8.4.20}$$

for some κ_5. Using (8.4.19) and (8.4.20) we can complete the proof of Lemma 8.4.2. Rest of the argument of Theorem 8.4.1 follows without any change. □

8.4.2 Existence

First we establish Theorem 8.3.1. We see that if $\inf_{\mathbb{R}^N} f_i > -\infty$, then set of sub-solution in (8.3.1) is non-empty. In particular, if we choose $\lambda = \min_i \inf_{\mathbb{R}^N} f_i$, then $\boldsymbol{u} = (1, 1)$ is a sub-solution to (EP) with eigenvalue λ.

Lemma 8.4.4. *Grant Assumption 8.2.1 and also assume that $\boldsymbol{f} \in \mathcal{C}^1(\mathbb{R}^N \times \{1, 2\})$. Suppose that $\boldsymbol{u}$ is a $\mathcal{C}^2$ sub-solution to* (EP) *with some eigenvalue λ_1. Then* (EP) *has a $\mathcal{C}^2$ solution for every $\lambda \le \lambda_1$.*

Proof. Since $\boldsymbol{u}$ is also a sub-solution for any $\lambda \le \lambda_1$, it is enough to show that there exists a solution $\boldsymbol{w}$ to (EP) with eigenvalue λ_1. For a $n \in \mathbb{N}$, fix $D = \mathscr{B}_n(0)$. Applying Theorem 8.7.3, we can find a function $\boldsymbol{w}^n = (w_1^n, w_2^n) \in \mathcal{C}^2(D \times \{1, 2\})$ that satisfies

$$\begin{aligned} -\Delta w_1^n(x) + H_1(x, \nabla w_1^n(x)) + \alpha_1(x)(w_1^n(x) - w_2^n(x)) &= f_1(x) - \lambda_1 \quad \text{in } \mathscr{B}_n(0), \\ -\Delta w_2^n(x) + H_2(x, \nabla w_2^n(x)) + \alpha_2(x)(w_2^n(x) - w_1^n(x)) &= f_2(x) - \lambda_1 \quad \text{in } \mathscr{B}_n(0). \end{aligned} \tag{8.4.21}$$

We translate $\boldsymbol{w}^n$ to satisfy $w_1^n(0) = 0$. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^N$. Then, by Proposition 8.4.1, we get $\sup_n\{|w_1^n(0)|, |w_2^n(0)|\}$ bounded and

$$\sup_{\mathcal{K}}\{|\nabla w_1^n|, |\nabla w_2^n|\} < C_{\mathcal{K}},$$

for all n satisfying $\mathscr{B}_n(0) \Supset \mathcal{K}$. Thus, $\{\boldsymbol{w}^n\}$ is locally bounded in $\mathscr{W}^{2,p}_{\text{loc}}$, uniformly in n. Applying a diagonalization argument, we can find a subsequence of $\{\boldsymbol{w}^n\}$, converging to some $\boldsymbol{w} \in \mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N \times \{1, 2\})$ for $p > N$.

Passing limit in (8.4.21) gives

$$
\begin{aligned}
-\Delta w_1(x) + H_1(x, \nabla w_1(x)) + \alpha_1(x)(w_1(x) - w_2(x)) &= f_1(x) - \lambda_1 \quad \text{in } \mathbb{R}^N,\\
-\Delta w_2(x) + H_2(x, \nabla w_2(x)) + \alpha_2(x)(w_2(x) - w_1(x)) &= f_2(x) - \lambda_1 \quad \text{in } \mathbb{R}^N.
\end{aligned}
$$

We can now bootstrap the regularity of $\boldsymbol{w}$ to $\mathcal{C}^2$ using standard elliptic regularity theory (cf. [64]). □

Now we can complete the proof of Theorem 8.3.1.

Theorem 8.4.3. *Grant Assumption 8.2.1. For $i = 1, 2$, suppose that $f_i \in \mathcal{C}^1(\mathbb{R}^N)$ are bounded below. Then λ^* is finite and* (EP) *has solution for the eigenvalue λ^*. In particular, by Lemma 8.4.4,* (EP) *has a solution for every $\lambda \le \lambda^*$.*

Proof. From the discussion preceding Lemma 8.4.4 we see that

$$\lambda^* \ge \min_{i=1,2} \inf_{\mathbb{R}^N} f_i\,.$$

We first show that $\lambda^* < \infty$. Suppose, on the contrary, that $\lambda^* = \infty$. Then, in view of Lemma 8.4.4, there exists a sequence of solutions $\{(\boldsymbol{\phi}^k, \lambda_k)\} = \{(\phi_1^k, \phi_2^k, \lambda_k)\}$ of (EP) satisfying $\lambda_k \to \infty$ as $k \to \infty$. We can translate $\boldsymbol{\phi}^k$ to satisfy $\phi_1^k(0) = 0$. Since

$$
\begin{aligned}
-\Delta \phi_1^k(x) + H_1(x, \nabla \phi_1^k(x)) + \alpha_1(x)(\phi_1^k(x) - \phi_2^k(x)) &= f_1(x) - \lambda_k \quad \text{in } \mathbb{R}^N,\\
-\Delta \phi_2^k(x) + H_2(x, \nabla \phi_2^k(x)) + \alpha_2(x)(\phi_2^k(x) - \phi_1^k(x)) &= f_2(x) - \lambda_k \quad \text{in } \mathbb{R}^N,
\end{aligned}
\tag{8.4.22}
$$

and $(f_i - \lambda_k)_+ \le (f_i)_+$ for large k, it follows from Proposition 8.4.1 that

$$\sup_k \sup_{\mathcal{K}} \{|H_1(x, \nabla \phi_1^k)|, |H_2(x, \nabla \phi_2^k)|\} < \infty, \quad \sup_k \sup_{\mathcal{K}} \{|\phi_1^k|, |\phi_1^k|\} < \infty, \tag{8.4.23}$$

for every compact set $\mathcal{K}$ in $\mathbb{R}^N$. Setting

$$\psi_i^k := \lambda_k^{-1} \phi_i^k \quad \text{for } i = 1, 2,$$

we see from (8.4.22) that in $\mathbb{R}^N$,

$$-\Delta\psi_1^k(x) + \lambda_k^{-1} H_1(x, \nabla\hat{\phi}_1^k(x)) + \alpha_1(x)(\psi_1^k(x) - \psi_2^k(x)) = \lambda_k^{-1} f_1(x) - 1,$$
$$-\Delta\psi_2^k(x) + \lambda_k^{-1} H_2(x, \nabla\hat{\phi}_2^k(x)) + \alpha_2(x)(\psi_1^k(x) - \psi_2^k(x)) = \lambda_k^{-1} f_2(x) - 1.$$

Using (8.4.23) we see that $\{\boldsymbol{\psi}^k\}$ is locally bounded in $\mathscr{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for $p > N$. Therefore, we can find a convergence subsequence, converging to some $\boldsymbol{\psi}$. (8.4.23) also shows that $|\nabla\psi_i| = 0$ implying $\boldsymbol{\psi}$ to be a constant. Then passing limit in the above display we get a contradiction. Hence λ^* must be finite.

Now choose $\lambda_n < \lambda^*$ such that $\lambda_n \to \lambda^*$ as $n \to \infty$. Then, using Lemma 8.4.4, we get a solution $(u_1^n, u_2^n, \lambda_n)$ to (EP). Applying an argument similar to Lemma 8.4.4 we can extract a convergent subsequence, converging locally to $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{u}$ solves (EP) with the eigenvalue λ^*. This completes the proof. □

Rest of this section is devoted to the proof of Theorem 8.3.2, that is, we construct a non-negative solution to (EP) corresponding to the eigenvalue λ^*. The broad idea of the proof is the following: We solve the ergodic control problem (EP) on an increasing sequence of balls $\mathscr{B}_n$ and find solution pairs $(\boldsymbol{u}^n, \lambda_n)$ in the balls. We then show that λ_n decreases to λ^* and $\boldsymbol{u}^n \to \boldsymbol{u}$. Using the coercivity of $\boldsymbol{f}$, we can confine the minimizer of $\boldsymbol{u}^n$ inside a fixed compact set, independent of n. This also makes $\boldsymbol{u}$ bounded from below. For this idea to work it is important that $\boldsymbol{u}^n$ attends its minimum inside B_n. This can be achieved if we set $\boldsymbol{u}^n = +\infty$ on $\partial\mathscr{B}_n$. For $\gamma_i \leq 2$, this can be done using the arguments of Lasry-Lions in [88]. But for $\gamma_i > 2$, we need to modify $\boldsymbol{f}$ to *attend* the boundary data.

Let $\boldsymbol{f}$ be a $\mathcal{C}^1$ function. Let $B = \mathscr{B}_r(0)$ be the ball of radius $r \geq 1$ around

0. Let $\varrho : (0,\infty) \to (0,\infty)$ be a smooth, non-negative function satisfying

$$\varrho(x) = \begin{cases} x^{-1} & \text{for } x \in (0, \frac{1}{2}), \\ 0 & \text{for } x \geq 1. \end{cases}$$

Define

$$f_{i,\alpha}(x) = f_i(x) + [\varrho(r^2 - |x|^2)]^\alpha \quad x \in B,\ i = 1, 2,$$

for some α to be fixed later. Let $\beta > \max\{2, \gamma_1, \gamma_2\}$ be such that $(\beta+1)(\gamma_i \wedge 2) > \beta + 2$. Choose $\alpha > 0$ to satisfy $\beta < \alpha < (\beta+1)(\gamma_i \wedge 2)$. With no loss of generality, we also assume that $1 < \gamma_2 \leq \gamma_1$. Our next result concerns discounted problem in B.

Lemma 8.4.5. *Grant Assumption 8.2.1. Then, for any $\varepsilon \in (0,1)$, the system*

$$\begin{aligned} -\Delta w_1^\varepsilon + H_1(x, \nabla w_1^\varepsilon) + \alpha_1(x)(w_1^\varepsilon - w_2^\varepsilon) + \varepsilon w_1^\varepsilon = f_{1,\alpha} \quad \text{in } B, \\ -\Delta w_2^\varepsilon + H_2(x, \nabla w_2^\varepsilon) + \alpha_2(x)(w_2^\varepsilon - w_1^\varepsilon) + \varepsilon w_2^\varepsilon = f_{2,\alpha} \quad \text{in } B, \end{aligned} \tag{8.4.24}$$

admits a solution $(w_1^\varepsilon, w_2^\varepsilon)$ in $\mathcal{C}^2(\mathscr{B}_r \times \{1,2\})$ with $w_i^\varepsilon \to \infty$ as $x \to \partial\mathscr{B}_r$. Moreover, the set $\{\varepsilon w_i^\varepsilon(0) : \varepsilon \in (0,1)\}$ is bounded for $i = 1, 2$.

Proof. To find a solution of (8.4.24) first we find appropriate sub and super-solutions of (8.4.24). Define $\xi^\delta(x) = -\log(r^2 - \delta|x|^2)$ and let $(\xi_1^\delta, \xi_2^\delta) = (\kappa_1\xi^\delta, \kappa_1\xi^\delta)$. It can be easily checked that, for some $\delta_0 > 0$ and $\delta \in (\delta_0, 1)$,

$$\begin{aligned} -\Delta\xi_1^\delta + C_1(|\nabla\xi_1^\delta|^{\gamma_1} + 1) + \alpha_1(x)(\xi_1^\delta - \xi_2^\delta) + \varepsilon\xi_1^\delta \leq f_{1,\alpha} \quad \text{for } r - \delta_1 \leq |x| < r, \\ -\Delta\xi_2^\delta + C_1(|\nabla\xi_2^\delta|^{\gamma_2} + 1) + \alpha_2(x)(\xi_2^\delta - \xi_1^\delta) + \varepsilon\xi_2^\delta \leq f_{2,\alpha} \quad \text{for } r - \delta_1 \leq |x| < r \end{aligned}$$

for some appropriate constant κ_1, dependent on γ_1, γ_2. κ_1, δ_1, and δ can be chosen independent of ε. Now choose M suitably large, independent of ε, δ, so that $(\kappa_1\xi_1^\delta - \frac{M}{\varepsilon}, \kappa_1\xi_2^\delta - \frac{M}{\varepsilon})$ forms a sub-solution of (8.4.24).

Next we construct a super-solution. To this end, we consider the approximating function ψ_n from Lemma 8.7.1. More precisely, we consider a sequence of functions $\boldsymbol{\psi}_n = (\psi_n^1, \psi_n^2)$ where $\psi_n^i(x) = x$ if $\gamma_i \le 2$, otherwise $\psi_n^i = \psi_n$ from Lemma 8.7.1.

We define $(\zeta_1^\delta, \zeta_2^\delta) = (\kappa_2\zeta, \kappa_2\zeta)$ where

$$\zeta = (r^2 - \delta|x|^2)^{-\beta} \quad \text{for } i = 1, 2.$$

Using the condition $\beta < \alpha < (\beta+1)(\gamma_i \wedge 2)$, and choosing M large, independent of n, ε, δ, we see that $(\kappa_2\zeta_1^\delta + \frac{M}{\varepsilon}, \kappa_2\zeta_2^\delta + \frac{M}{\varepsilon})$ forms a super-solution to the equation

$$\begin{aligned} -\Delta w_1^\varepsilon + \psi_n^1(H_1(x, \nabla w_1^\varepsilon)) + \alpha_1(x)(w_1^\varepsilon - w_2^\varepsilon) + \varepsilon w_1^\varepsilon &= f_{1,\alpha} \quad \text{in } B, \\ -\Delta w_2^\varepsilon + \psi_n^2(H_2(x, \nabla w_2^\varepsilon)) + \alpha_2(x)(w_2^\varepsilon - w_1^\varepsilon) + \varepsilon w_2^\varepsilon &= f_{2,\alpha} \quad \text{in } B, \end{aligned}$$

for all n. From the argument of Theorem 8.7.3, we find a solution $\boldsymbol{w}^\delta = (w_1^\delta, w_2^\delta)$ of

$$\begin{aligned} -\Delta w_1^\delta + H_1(x, \nabla w_1^\delta) + \alpha_1(x)(w_1^\delta - w_2^\delta) + \varepsilon w_1^\delta &= f_{1,\alpha} \quad \text{in } B, \\ -\Delta w_2^\delta + H_2(x, \nabla w_2^\delta) + \alpha_2(x)(w_2^\delta - w_1^\delta) + \varepsilon w_2^\delta &= f_{2,\alpha} \quad \text{in } B, \end{aligned}$$

and

$$\kappa_1\xi_i^\delta - \frac{M}{\varepsilon} \le w_{i,n}^\delta \le \kappa_2\zeta_i^\delta + \frac{M}{\varepsilon} \quad \text{in } B,\ i = 1, 2.$$

Using the estimates in Proposition 8.4.1, we can now let $\delta \to 1$ and find a solution to

$$\begin{aligned} -\Delta w_1^\varepsilon + H_1(x, \nabla w_1^\varepsilon) + \alpha_1(x)(w_1^\varepsilon - w_2^\varepsilon) + \varepsilon w_1^\varepsilon &= f_{1,\alpha} \quad \text{in } B, \\ -\Delta w_2^\varepsilon + H_1(x, \nabla w_1^\varepsilon) + \alpha_2(x)(w_2^\varepsilon - w_1^\varepsilon) + \varepsilon w_2^\varepsilon &= f_{2,\alpha} \quad \text{in } B, \end{aligned}$$

satisfying

$$-\kappa_1 \log(r^2 - |x|^2) - \frac{M}{\varepsilon} \le w_i^\varepsilon \le \kappa_2(r^2 - |x|^2)^{-\beta} + \frac{M}{\varepsilon} \quad \text{in } B,\ i = 1, 2. \tag{8.4.25}$$

From (8.4.25) we also obtain

$$\sup_{\varepsilon \in (0,1)} \sup_{\mathscr{B}_{1/2}} |\varepsilon w_i^\varepsilon| < \infty.$$

This completes the proof. $\square$

Now we can provide a proof of Theorem 8.3.2.

Theorem 8.4.4. *Grant Assumption 8.2.1 and assume* $f_i, i = 1, 2$, *to be coercive. Then there exists a non-negative solution to* (EP) *corresponding to the eigenvalue* λ^*.

Proof. First we find a pair $(\boldsymbol{u}^n, \lambda_n)$ solving

$$\begin{aligned} -\Delta u_1^n + H_1(x, \nabla u_1^n) + \alpha_1(x)(u_1^n - u_2^n) &= f_{1,\alpha}^n - \lambda_n \quad \text{in } \mathscr{B}_n(0), \\ -\Delta u_2^n + H_2(x, \nabla u_2^n) + \alpha_2(x)(u_2^n - u_1^n) &= f_{2,\alpha}^n - \lambda_n \quad \text{in } \mathscr{B}_n(0), \end{aligned} \tag{8.4.26}$$

with $\boldsymbol{u}^n \to \infty$, as $x \to \partial\mathscr{B}_n(0)$, where

$$f_{i,\alpha}^n = f_i + [\varrho(n^2 - |x|^2)]^\alpha,$$

and α is same as in Lemma 8.4.5. Fix $n \in \mathbb{N}$ and denote by $B = \mathscr{B}_n(0)$. Consider the solution $\boldsymbol{w}^\varepsilon$ from Lemma 8.4.5. We set $v_1^\varepsilon = w_1^\varepsilon(x) - w_1^\varepsilon(0)$ and $v_2^\varepsilon(x) = w_2^\varepsilon(x) - w_1^\varepsilon(0)$. From (8.4.24) we then find

$$\begin{aligned} -\Delta v_1^\varepsilon + H_1(x, \nabla v_1^\varepsilon) + \alpha_1(x)(v_1^\varepsilon - v_2^\varepsilon) + \varepsilon w_1^\varepsilon &= f_{1,\alpha}^n \quad \text{in } B, \\ -\Delta v_2^\varepsilon + H_2(x, \nabla v_2^\varepsilon) + \alpha_2(x)(v_2^\varepsilon - v_1^\varepsilon) + \varepsilon w_2^\varepsilon &= f_{2,\alpha}^n \quad \text{in } B. \end{aligned} \tag{8.4.27}$$

From our choice of α and (8.4.25) we see that $f_{i,\alpha} - \varepsilon w_i^\varepsilon \geq \frac{1}{2} f_{i,\alpha}$ near the boundary, and since $\max_{B_{1/2}}\{|v_1^\varepsilon|, |v_2^\varepsilon|\}$ is bounded uniformly in ε (by Proposition 8.4.1), we can see that $v_i^\varepsilon \geq \kappa_3 \xi_i^\delta - M$ for some κ_3, using Theorem 8.7.1, where $\boldsymbol{\xi}^\delta$ is same as in Lemma 8.4.5. Now let $\delta \to 1$ to get a lower bound that blows up at the boundary. Using Proposition 8.4.1 and the fact $\{\varepsilon \boldsymbol{w}^\varepsilon(0)\}$ is bounded, we let $\varepsilon \to 0$ in (8.4.27) to find a solution to (8.4.26).

Now consider the sequence of solutions $\{\boldsymbol{u}^n, \lambda_n\}$ solving (8.4.26). We claim that $\lambda_n \geq \lambda_{n+1} \geq \lambda^*$. Suppose, on the contrary, that $\lambda_n < \lambda_{n+1}$. Choose a constant κ so that $\boldsymbol{u}^{n+1} + \kappa$ touches $\boldsymbol{u}^n$ from below in $\mathscr{B}_n$. This is possible as $\boldsymbol{u}^n$ blows up at the boundary. Let $\boldsymbol{v}^n = \boldsymbol{u}^n - \boldsymbol{u}^{n+1}$. Also, note that

$$f_{i,\alpha}^{n+1}(x) = f_i(x) \leq f_{i,\alpha}^n \quad \text{in } \mathscr{B}_n.$$

Choose $D \Subset \mathscr{B}_n$ so that $\boldsymbol{v}^n$ vanishes inside D. From (8.4.26) we then have

$$\begin{cases} -\Delta v_1^n + h_1^n \cdot \nabla v_1^n + \alpha_1(x)(v_1^n - v_2^n) & \geq \lambda_{n+1} - \lambda_n > 0 \quad \text{in } D, \\ -\Delta v_2^n + h_2^n \cdot \nabla v_1^n + \alpha_2(x)(v_2^n - v_1^n) & \geq \lambda_{n+1} - \lambda_n > 0 \quad \text{in } D, \end{cases}$$

where

$$h_i^n(x) = \int_0^1 \nabla_p H_i(x, \nabla u_i^{n+1} + t(\nabla u_i^n - \nabla u_i^{n+1}))\,\mathrm{d}t, \quad i = 1, 2.$$

By strong maximum principle we obtain $\boldsymbol{v}^n = 0$ in D. Since D is arbitrary, we must have $\boldsymbol{v}^n = 0$ in $\mathscr{B}_n$ which is a contradiction. Thus we have $\lambda_n \geq \lambda_{n+1}$. An analogous argument also shows $\lambda_n \geq \lambda^*$.

Using the estimates in Proposition 8.4.1, we can now find a subsequence of $\{\boldsymbol{u}^n\}$ converging weakly in $\mathscr{W}^{2,p}_{\text{loc}}(\mathbb{R}^N)$ to some $\boldsymbol{u}$. Passing limit in (8.4.26) we see that $\boldsymbol{u}$ solves (EP) with the eigenvalue λ^* (since $\lim_{n\to\infty} \lambda_n$ is equal to λ^*). To see that $\boldsymbol{u}$ is bounded from below, we consider a point $(x_n, i_u) \in \mathscr{B}_n \times \{1, 2\}$ so that $u^n_{i_n}(x_n)$ is the minimum of $\boldsymbol{u}^n$ in $\mathscr{B}_n$. From (8.4.26) we then obtain

$$\lambda_1 \geq \lambda_n \geq f^n_{i_n}(x_n) \geq f_{i_n}(x_n) \geq \min\{f_1(x_n), f_2(x_n)\}.$$

Since f_i is coercive, we can find a compact set $\mathcal{K}$, independent of n, so that $x_n \in \mathcal{K}$. Thus $\boldsymbol{u}^n \geq \min_{\mathcal{K}}\{u_1^n, u_2^n\}$. This, of course, implies that $\boldsymbol{u}$ is bounded from below. We can now translate $\boldsymbol{u}$ to make it non-negative. This completes the proof. □

We complete the section by mentioning few properties of $\lambda^* = \lambda^*(\boldsymbol{f})$.

Proposition 8.4.2. *Let $\boldsymbol{f}, \tilde{\boldsymbol{f}}$ be two $\mathcal{C}^1$ functions. Then*

(i) For any $c \in \mathbb{R}$ we have $\lambda^*(\boldsymbol{f} + c) = \lambda^*(f) + c$.

(ii) $\boldsymbol{f} \mapsto \lambda^*(\boldsymbol{f})$ is concave, that is, for $t \in [0, 1]$ we have

$$\lambda^*(t\boldsymbol{f} + (1-t)\boldsymbol{f}) \geq t\lambda^*(\boldsymbol{f}) + (1-t)\lambda^*(\tilde{\boldsymbol{f}}).$$

(iii) If $\boldsymbol{f} \leq \tilde{\boldsymbol{f}}$*, then* $\lambda^*(\boldsymbol{f}) \leq \lambda^*(\tilde{\boldsymbol{f}})$*. Furthermore, if we assume the setting of Theorem 8.4.1 or Theorem 8.4.2, then for* $\boldsymbol{f} \lneq \tilde{\boldsymbol{f}}$ *we have* $\lambda^*(\boldsymbol{f}) < \lambda^*(\tilde{\boldsymbol{f}})$*.*

Proof. (i) is obvious. (ii) follows from the convexity of H_i and the definition (8.3.1). Also, first part of (iii) follows from the definition (8.3.1). To Prove the second part, we suppose, on the contrary, that $\lambda^*(\boldsymbol{f}) = \lambda^*(\tilde{\boldsymbol{f}})$. Let $\tilde{\boldsymbol{u}}$ be a non-negative solution to (EP) with right-hand side $\tilde{\boldsymbol{f}}$ and eigenvalue $\lambda^*(\tilde{\boldsymbol{f}})$. Then $\tilde{\boldsymbol{u}}$ would be a super-solution to (EP) with right-hand side $\boldsymbol{f}$. From Lemma 8.4.3 we know that for

$$\tilde{\xi}_k(x) = \nabla_p H_k(x, \nabla \tilde{u}_k(x)) \quad k = 1, 2,$$

there exists a Borel probability measure $\tilde{\boldsymbol{\nu}} = (\tilde{\nu}_1, \tilde{\nu}_2)$ so that

$$\tilde{\boldsymbol{\mu}}_{\tilde{\boldsymbol{u}}} = (\tilde{\mu}_{1,u}, \tilde{\mu}_{2,u}) \quad \text{with} \quad \tilde{\mu}_{k,\tilde{\boldsymbol{u}}} := \tilde{\nu}_k(\mathrm{d}x)\delta_{\tilde{\xi}_k(x)}(\mathrm{d}\xi) \in \mathcal{M}_{\boldsymbol{F}}.$$

Moreover, $\tilde{\boldsymbol{\mu}}_{\tilde{\boldsymbol{u}}}(\boldsymbol{F}) \leq \lambda^*(\boldsymbol{f})$. By Theorem 8.4.1 or Theorem 8.4.2 we must have $\tilde{\boldsymbol{\mu}}_{\tilde{\boldsymbol{u}}}(\boldsymbol{F}) = \lambda^*(\boldsymbol{f})$. Again, using (8.4.13), we obtain

$$\sum_{k=1}^{2} \int_{\mathbb{R}^N} \Big(\ell_k(x, \tilde{\xi}_k(x)) - \tilde{\xi}_k(x) \cdot \nabla u_k + H_k(x, \nabla u_k) \Big) \tilde{\nu}_k(\mathrm{d}x) = 0.$$

Since $\tilde{\nu}_k$ has strictly positive densities (cf. [11, Theorem 5.3.4]), it follows that $\nabla u_k = \nabla \tilde{u_k}$. Thus $u_k = \tilde{u}_k + c_k$ for some constants c_k for $k = 1, 2$. Subtracting the equation satisfied by $\boldsymbol{u}$ and $\tilde{\boldsymbol{u}}$ we obtain

$$\alpha_1(x)(c_2 - c_1) = \tilde{f}_1(x) - f_1(x), \quad \text{and} \quad \alpha_2(x)(c_1 - c_2) = \tilde{f}_2(x) - f_2(x),$$

which implies

$$\frac{\tilde{f}_1(x) - f_1(x)}{\alpha_1(x)} + \frac{\tilde{f}_2(x) - f_2(x)}{\alpha_2(x)} = 0.$$

But this is not possible as $\boldsymbol{f} \lneq \tilde{\boldsymbol{f}}$. Hence we must have $\lambda^*(\boldsymbol{f}) < \lambda^*(\tilde{\boldsymbol{f}})$. □

8.5 Application to optimal ergodic control

In this section we describe the optimal ergodic control problem associated to the problem (EP). Denote by $\mathcal{S} = \{1, 2\}$, the state space of the switching continuous time Markov process. We introduce the regime switching controlled diffusion process on a given complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. This is a process (X_t, S_t) in $\mathbb{R}^N \times \mathcal{S}$ governed by the following stochastic differential equations:

$$\begin{aligned} \mathrm{d}X_t &= \boldsymbol{b}(X_t, S_t)\mathrm{d}t - U_t\,\mathrm{d}t + \sqrt{\mathrm{d}W_t}\,, \\ \mathrm{d}S_t &= \int_{\mathbb{R}} h(X_t, S_{t-}, z)\wp(\mathrm{d}t, \mathrm{d}z)\,, \end{aligned} \tag{8.5.1}$$

for $t \geq 0$, where

(i) (X_0, S_0) are prescribed deterministic initial data;

(ii) W is an N-dimensional standard Wiener process;

(iii) $\wp(\mathrm{d}t, \mathrm{d}z)$ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}t \times \mathfrak{m}(\mathrm{d}z)$, where $\mathfrak{m}$ is the Lebesgue measure on $\mathbb{R}$;

(iv) $\wp(\cdot, \cdot)$, $W(\cdot)$ are independent;

(v) The function $h \colon \mathbb{R}^N \times \mathcal{S} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$h(x, i, z) \coloneqq \begin{cases} j - i & \text{if } z \in \Delta_{ij}(x), \\ 0 & \text{otherwise,} \end{cases}$$

where for $i, j \in \mathcal{S}, i \neq j$, and fixed x, $\Delta_{ij}(x)$ are left closed right open disjoint intervals of $\mathbb{R}$ having length $m_{ij}(x)$, and

$$m_{11}(x) = -\alpha_1(x),\ m_{12} = \alpha_1(x),\ m_{21}(x) = \alpha_2(x),\ m_{22}(x) = -\alpha_2(x).$$

Note that $\boldsymbol{M}(x) := (m_{ij})$ can be interpreted as the rate matrix of the Markov chain S_t given that $X_t = x$. In other words,

$$\mathbb{P}(S_{t+h} = j \mid X_t, S_t) \ = \begin{cases} m_{S_t j}(X_t)h + \mathfrak{o}(h) & \text{if } S_t \neq j\,, \\ \\ 1 + m_{S_t j}(X_t)h + \mathfrak{o}(h) & \text{if } S_t = j\,, \end{cases}$$

and X behaves like an ordinary diffusion process governed by (8.5.1) between two consecutive jumps of S.

We assume $\boldsymbol{b} : \mathbb{R}^N \times \mathcal{S} \to \mathbb{R}^N$ to be a bounded $\mathcal{C}^1$ function with bounded first derivatives. The process $\{U_t\}$ takes values in $\mathbb{R}^N$ and non-anticipative in nature, that is, the sigma fields

$$\sigma\{X_0, S_0, W_s, U_s, \wp(A,B) \ : \ A \in \mathscr{B}([0,s]),\, B \in \mathscr{B}(\mathbb{R}),\, s \le t\}$$

and

$$\sigma\{W_s - W_t, \wp(A,B) \ : \ A \in \mathscr{B}([s,\infty)),\, B \in \mathscr{B}(\mathbb{R}),\, s \ge t\},$$

are independent. To introduce the admissible class of controls we set $\gamma_1 = \gamma_2 = \gamma$ and define

$$\mathfrak{U} = \left\{ U \ : \ \mathbb{E}\left[\int_0^T |U_t|^{\gamma'} \mathrm{d}t\right] < \infty \quad \text{for all } T > 0 \right\},$$

where γ' is the Hölder conjugate of γ. We also assume $\tilde{\ell}_i$ to satisfy the following bound

$$\kappa^{-1}|\xi|^{\gamma'} - \kappa \le \tilde{\ell}_i(x,\xi) \le \kappa(1 + |\xi|^{\gamma'}),$$

for some $\kappa > 0$ and $\xi \mapsto \ell_i(x,\xi)$ are strictly convex, $i = 1, 2$. We let

$$H_i(x,p) = -b_i(x)\cdot p + \sup_{\xi \in \mathbb{R}^N} \{p \cdot \xi - \ell_i(x,\xi)\} \quad i = 1, 2.$$

Also, assume that $H_i \in C^1(\mathbb{R}^N \times \mathbb{R}^N)$ and the functions $\xi \mapsto H_i(x,\xi)$ are strictly convex for $i = 1, 2$. It can be easily shown that (8.5.1) has a unique strong solution for $U \in \mathfrak{U}$. This can be verified using Picard iterations. Now we can state the main result of this section.

Theorem 8.5.1. *Consider the setting of Theorem 8.4.1 or Theorem 8.4.2. We also assume that* $\gamma_1 = \gamma_2 = \gamma$. *Then*

$$\inf_{U\in\mathfrak{U}} \liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \Big(\boldsymbol{f}(X_t,S_t) + \boldsymbol{\ell}(X_t,S_t,U_t)\mathrm{d}t\Big)\right] \;=\; \lambda^*. \tag{8.5.2}$$

Furthermore, the stationary Markov control

$$(\nabla_p H_1(x,\nabla u_1(x)), \nabla_p H_2(x,\nabla u_2(x))) + \boldsymbol{b}$$

is optimal where $\boldsymbol{u}$ *is a non-negative solution to* (EP) *corresponding to the eigenvalue* λ^*. *Furthermore, from* (8.4.13)*, we also see that this is the only optimal stationary Markov control.*

Proof. We only show that the l.h.s. of (8.5.2) is larger than λ^*. Rest of the proof follows from Theorem 8.4.1 or Theorem 8.4.2. Consider $U \in \mathfrak{U}$ so that

$$\begin{aligned}
&\liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \Big(\boldsymbol{f}(X_t,S_t) + \boldsymbol{\ell}(X_t,S_t,U_t)\mathrm{d}t\Big)\right] \\
&= \liminf_{T_n\to\infty} \frac{1}{T_n}\,\mathbb{E}\left[\int_0^{T_n} \Big(\boldsymbol{f}(X_t,S_t) + \boldsymbol{\ell}(X_t,S_t,U_t)\mathrm{d}t\Big)\right] < \infty.
\end{aligned} \tag{8.5.3}$$

We define the mean empirical measure as on $\mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}$ as follows

$$\boldsymbol{\mu}^n(A_1{\times}A_2{\times}C) = \frac{1}{T_n}\,\mathbb{E}\left[\int_0^{T_n} \mathbb{1}_{A_1\times C\times A_2}(X_t,U_t,S_t)\mathrm{d}t\Big)\right],\ A_i \in \mathscr{B}(\mathbb{R}^N), C\subset\mathcal{S}.$$

From the definition of $\boldsymbol{\mu}^n$ it follows that

$$\boldsymbol{\mu}^n(\boldsymbol{F}) = \frac{1}{T_n}\,\mathbb{E}\left[\int_0^{T_n} \Big(\boldsymbol{f}(X_t,S_t) + \boldsymbol{\ell}(X_t,S_t,U_t)\mathrm{d}t\Big)\right],$$

where $\boldsymbol{F}$ is given by (8.4.6). From the coercivity property of $\boldsymbol{F}$ it can be easily seen that $\{\boldsymbol{\mu}^n\}$ is tight. Let $\boldsymbol{\mu}$ be a sub-sequential limit of $\{\boldsymbol{\mu}^n\}$. Using [11, Lemma 2.5.3] and the lower-semi continuity property of weak convergence we see that $\boldsymbol{\mu} \in \mathcal{M}_{\boldsymbol{F}}$. Again, from (8.5.3), we get

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \Big(\boldsymbol{f}(X_t,S_t) + \boldsymbol{\ell}(X_t,S_t,U_t)\mathrm{d}t\Big)\right] \ge \boldsymbol{\mu}(\boldsymbol{F}).$$

By Lemma 8.4.2 we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left[\int_0^T \Big(\boldsymbol{f}(X_t,S_t) + \boldsymbol{\ell}(X_t,S_t,U_t)\mathrm{d}t\Big)\right] \ge \lambda^*.$$

This completes the proof. □

8.5.1 Some results from stochastic calculus

Here we recall some results from stochastic analysis which will be useful for the proof of Theorem 8.4.1. We are going to state few results for the above mentioned process $Z_t = (X_t, S_t)$ in (8.5.1) and for more details one can look in [11, Chapter 5] and [103, 120].

Next we define the extended controlled generator. For $\psi \in \mathcal{C}^2(\mathbb{R}^N \times \mathcal{S})$, we define the operators $\mathcal{L}^u$ and Π^u for a fixed $u \in \mathbb{R}^N$ as follows:

$$(\mathcal{L}^u\psi)(x,y) := \frac{1}{2}\Delta_x\psi(x,y) + \sum_{i=1}^{N}\left(b^i(x,y) - u^i\right)\frac{\partial\psi}{\partial x_i}(x,y),$$

and for $y \neq \tilde{y}$,

$$(\Pi^u\psi)(x,y) := -\alpha_y(x)\psi(x,y) + \alpha_y(x)\psi(x,\tilde{y}).$$

Lemma 8.5.1. *Let $\psi \in \mathcal{C}_c^2(\mathbb{R}^N \times \mathcal{S})$. Then for $T > 0$, there holds*

$$\mathbb{E}_z^{U_s}[\psi(Z_T)] - \psi(z) = \mathbb{E}_z\left[\int_0^T \left[\mathcal{L}^{U_s}\psi(Z_s) + \Pi^{U_s}\psi(Z_s)\right]\mathrm{d}s\right].$$

Proof of this lemma can be found in [11, Chapter 5, Lemma 5.1.4].

We write the operator $\mathcal{A}$ as follows

$$(\mathcal{A}\phi)_k = a_k^{ij}\partial_{ij}\phi_k + b_k^i\partial_i\phi_k + \lambda_k^k\phi_k + \sum_{l\neq k}\lambda_k^l\phi_l,$$

where a_k^{ij} are locally Lipschitz and other coefficients are in $L^\infty_{loc}(\mathbb{R}^N)$. Also we have $\lambda_k^l \geq 0$ for $k \neq l$ and $\sum_l \lambda_k^l = 0$. If $\vec{\mu} = (\mu_1, \cdots, \mu_N) \in \mathscr{P}(\mathbb{R}^N \times \mathcal{S})$, then we denote by

$$< \phi(x), \vec{\mu}(\mathrm{d}x) > = \sum_{k=1}^{N}\phi_k(x)\mu_k(\mathrm{d}x).$$

Theorem 8.5.2. *Suppose μ is a Borel probability measure on $\mathscr{P}(\mathbb{R}^N \times \mathcal{S})$ satisfying*

$$\int_{\mathbb{R}^N} < \mathcal{A}f(x), \vec{\mu}(\mathrm{d}x) = 0 \quad \textit{for all } f \in \mathcal{C}_c^2(\mathscr{P}(\mathbb{R}^N \times \mathcal{S})).$$

Then μ is absolutely continuous with respect to the Lebesgue measure. Let $\hat{\Lambda} \in \{0,1\}^{N\times N}$ is defined by

$$\hat{\Lambda}_{ij} := \begin{cases} 0 & \text{if } i = j \text{ or } \lambda_i^j = 0 \text{ a.e.}, \\ 1 & \text{otherwise}. \end{cases}$$

Then provided $\hat{\Lambda}$ is an irreducible matrix, ψ is strictly positive on $\mathscr{P}(\mathbb{R}^N \times \mathcal{S})$.

The proof of this theorem can be found in [11, Chapter 5, Theorem 5.3.4].

8.6 Proof of gradient estimate: Proposition 8.4.1

Part of the proof of this Proposition is inspired from [71].

Proof. With no loss of generality, we assume that $z = 0$, $B_1 = \mathscr{B}_1(0)$, $B_2 = \mathscr{B}_2(0)$, and $B_{\frac{1}{2}} = \mathscr{B}_{\frac{1}{2}}(0)$. We first show that

$$\sup_{B_1}\{|\nabla u_1|^{2\gamma_1}, |\nabla u_2|^{2\gamma_2}\} \leq C\Big(1 + \sup_{B_2}\sum_{i=1}^{2}(f_i)_+^2 + \sup_{B_2}\sum_{i=1}^{2}|\nabla f_i|^{2\gamma_i/(2\gamma_i-1)} + |u_1(0) - u_2(0)|^2 + \sup_{B_2}\sum_{i=1}^{2}(\varepsilon u_i)_-^2\Big). \tag{8.6.1}$$

Let $\rho : B_2 \to [0,1]$ be smooth, radial function which is decreasing along the radius, $\rho = 1$ in B_1, and support$(\rho) \subset B_2$. We take $\gamma = \min\{\gamma_1, \gamma_2\}$ and define $\eta = \rho^{\frac{4\gamma}{\gamma-1}}$. Without loss of generality we may assume that

$$\max_{B_2}\{\eta|\nabla u_1|^2, \eta|\nabla u_2|^2\} = \eta(x_0)|\nabla u_1(x_0)|^2 \quad \text{for some } x_0 \text{ in } B_2.$$

Define $\theta(x) = \eta(x)|\nabla u_1(x)|^2 = \eta(x)w(x)$ where $w(x) = |\nabla u_1(x)|^2$. Then we have $\nabla\theta(x_0) = 0$ and $\Delta\theta(x_0) \leq 0$. We may also assume that $\theta(x_0) > 1$. Otherwise, if $\theta(x_0) \leq 1$, we get

$$\max_{B_1}\{\eta|\nabla u_1|^2, \eta|\nabla u_2|^2\} \leq \theta(x_0) \leq 1,$$

and (8.6.1) follows. Therefore, we work with $\theta(x_0) > 1$. We see that

$$0 = \nabla\theta(x_0) = \eta(x_0)\nabla w(x_0) + w(x_0)\nabla\eta(x_0). \tag{8.6.2}$$

Now onward we shall evaluate everything at the point $x = x_0$ without explicitly mentioning the point x_0. Then

$$\begin{aligned}
\Delta w &= \mathrm{Tr}[(D^2u_1)^2] + \nabla(\Delta u_1)\cdot\nabla u_1 \\
&= \mathrm{Tr}[(D^2u_1)^2] + \nabla(H_1(x,\nabla u_1) + \alpha_1(u_1-u_2) + \varepsilon u_1 - f_1)\cdot\nabla u_1 \\
&= \mathrm{Tr}[(D^2u_1)^2] + \Bigg[\nabla_x H_1 + (\nabla_p H_1)D^2u_1 + (u_1-u_2)\nabla\alpha_1 \\
&\qquad + \alpha_1(\nabla u_1 - \nabla u_2) + \varepsilon\nabla u_1 - \nabla f_1\Bigg]\cdot\nabla u_1.
\end{aligned}$$

Using (8.6.2) we then obtain

$$\begin{aligned}
0 &\geq \Delta\theta = \eta\Delta w + 2\nabla\eta\cdot\nabla w + w\Delta\eta \\
&= \eta\Bigg[\mathrm{Tr}[(D^2u_1)^2] + \nabla_x H_1\cdot\nabla u_1 + (-2w\eta^{-1})\nabla\eta\cdot\nabla_p H_1 + (u_1-u_2)\nabla\alpha_1\cdot\nabla u_1 \\
&\quad + \alpha_1(\nabla u_1-\nabla u_2)\cdot\nabla u_1 + \varepsilon w - \nabla f_1\cdot\nabla u_1\Bigg] - 2\eta^{-1}w|\nabla\eta|^2 + w\Delta\eta \\
&\geq \eta\Bigg[\mathrm{Tr}[(D^2u_1)^2] - |\nabla_x H_1||\nabla u_1| - 2w\eta^{-1}|\nabla_p H_1||\nabla\eta| + (u_1-u_2)\nabla\alpha_1\cdot\nabla u_1 \\
&\quad + \alpha_1(\nabla u_1-\nabla u_2)\cdot\nabla u_1 - |\nabla f_1||\nabla u_1|\Bigg] - 2\eta^{-1}w|\nabla\eta|^2 - w|\Delta\eta|.
\end{aligned}$$

Using (8.4.1) , (HP1) and the inequality $(t_1+t_2+t_3+t_4)^2 \geq \frac{1}{4}t_1^2 - [(t_2)_-^2 + (t_3)_-^2 + (t_4)_-^2]$, we get (taking $t_1 = H_1 + C_1 \geq 0$)

$$N\,\mathrm{Tr}[(D^2u_1)^2] \geq (\Delta u_1)^2 \geq \left(\frac{1}{4C_1^2}|\nabla u_1|^{2\gamma_1} - (f_1+C_1)_+^2 - \alpha_1^2(u_1-u_2)^2 - (\varepsilon u_1)_-^2\right).$$

Since $N \geq 1$ and $\eta \leq 1$, we obtain

$$\begin{aligned}
&\frac{1}{4NC_1^2}\eta|\nabla u_1|^{2\gamma_1} \\
&\leq \eta\,\mathrm{Tr}[(D^2u_1)^2] + (f_1 + C_1)_+^2 + \eta\alpha_1^2(u_1 - u_2)^2 + (\varepsilon u_1)_-^2 \\
&\leq (f_1 + C_1)_+^2 + \eta\alpha_1^2(u_1 - u_2)^2 + (\varepsilon u_1)_-^2 + \eta|\nabla_x H_1||\nabla u_1| + 2w|\nabla_p H_1||\nabla\eta| \\
&\quad - \eta(u_1 - u_2)\nabla\alpha_1 \cdot \nabla u_1 - \eta\alpha_1(\nabla u_1 - \nabla u_2) \cdot \nabla u_1 \\
&\qquad + \eta|\nabla f_1||\nabla u_1| + 2\eta^{-1}w|\nabla\eta|^2 + w|\Delta\eta|. \qquad (8.6.3)
\end{aligned}$$

We observe that

$$\begin{aligned}
&\eta(x_0)\alpha_1(x_0)(|\nabla u_1(x_0)|^2 - \nabla u_2(x_0) \cdot \nabla u_1(x_0)) \\
&\geq \eta(x_0)\alpha_1(x_0)(|\nabla u_1(x_0)|^2 - |\nabla u_2(x_0)||\nabla u_1(x_0)|) \\
&\geq 0.
\end{aligned}$$

Also, by Mean Value Theorem, there exist $\zeta \in B_2$, with $|\zeta| < |x_0|$, and a constant $\kappa_1 > 0$, dependent on $\sup_{B_2} |\alpha_1|$, such that

$$\begin{aligned}
\eta(x_0)\alpha_1^2(u_1(x_0) - u_2(x_0))^2 &\leq \eta(x_0)\kappa_1\Big(|\nabla u_1(\zeta) - \nabla u_2(\zeta)|^2 + |u_1(0) - u_2(0)|^2\Big) \\
&\leq \eta(\zeta)\kappa_1\Big(|\nabla u_1(\zeta) - \nabla u_2(\zeta)|^2 + |u_1(0) - u_2(0)|^2\Big) \\
&\leq \kappa_1\Big(4\theta(x_0) + |u_1(0) - u_2(0)|^2\Big),
\end{aligned}$$

where in the second line we use the fact that η is radially decreasing. Another application of the Mean Value Theorem and a similar estimate as above gives us, for some ζ_1 with $|\zeta_1| < |x_0|$,

$$\begin{aligned}
&- \eta(x_0)(u_1(x_0) - u_2(x_0))\nabla\alpha_1(x_0) \cdot \nabla u_1(x_0) \\
&\leq \eta(x_0)|u_1(x_0) - u_2(x_0)||\nabla\alpha_1(x_0)||\nabla u_1(x_0)| \\
&\leq \kappa_2\sqrt{\eta(x_0)}\Big(|\nabla u_1(\zeta_1)| + |u_1(0) - u_2(0)|\Big)\sqrt{\theta(x_0)} \\
&\leq \kappa_2\Big(\sqrt{\eta(\zeta_1)}|\nabla u_1(\zeta_1)| + |u_1(0) - u_2(0)|\Big)\sqrt{\theta(x_0)} \\
&\leq \kappa_2\Big(2\theta(x_0) + |u_1(0) - u_2(0)|^2\Big),
\end{aligned}$$

for some constant κ_2 dependent on $\sup_{B_2}|\nabla\alpha_1|$, where in the last part we used $ab \leq 2^{-1}(a^2+b^2)$. Again, using (HP2)-(RHP2) and above three estimates in (8.6.3) we deduce that for some constant κ_3, dependent only on the bounds of α_1, it holds

$$\begin{aligned}
&\frac{1}{4NC_1^2}\eta|\nabla u_1|^{2\gamma_1} \\
&\leq 2(f_1)_+^2 + 2C_1^2 + (\varepsilon u_1)_-^2 + C_1\eta(1+|\nabla u_1|^{\gamma_1})|\nabla u_1| \\
&+ 2C_1(1+|\nabla u_1|^{\gamma_1-1})|\nabla u_1|^2|\nabla\eta| + \kappa_3\Big(\eta|\nabla u_1|^2 + |u_1(0)-u_2(0)|^2\Big) \\
&+ \eta|\nabla f_1||\nabla u_1| + |\nabla u_1|^2(2\eta^{-1}|\nabla\eta|^2 + |\Delta\eta|). \qquad (8.6.4)
\end{aligned}$$

Using Young's inequality for appropriate $\delta > 0$ to $|\nabla u_1||\nabla f_1|$, we obtain $\kappa_\delta > 0$ satisfying

$$|\nabla u_1||\nabla f_1| \leq \delta|\nabla u_1|^{2\gamma_1} + \kappa_\delta|\nabla f_1|^{2\gamma_1/(2\gamma_1-1)}.$$

Since $|\nabla u_1(x_0)| \geq 1$ and $\gamma_1 > 1$, we also have

$$(1+|\nabla u_1|^{\gamma_1})|\nabla u_1| \leq 2|\nabla u_1|^{\gamma_1+1}, \quad \text{and} \quad (1+|\nabla u_1|^{\gamma_1-1})|\nabla u_1|^2 \leq 2|\nabla u_1|^{\gamma+1}.$$

Thus, from (8.6.4) we obtain a constant $\kappa_4 > 0$, dependent on $N, C_1, \kappa_1, \kappa_2, \kappa_3, \kappa_\delta$, such that

$$\begin{aligned}
\eta|\nabla u_1|^{2\gamma_1} \leq \kappa_4\bigg(&1 + (f_1)_+^2 + |u_1(0)-u_2(0)|^2 + (\varepsilon u_1)_-^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1-1)} \\
&+ |\nabla u_1|^{\gamma_1+1}|\nabla\eta| + |\nabla u_1|^2\Big(2\eta^{-1}|\nabla\eta|^2 + |\Delta\eta|\Big)\bigg).
\end{aligned}$$

Now we define $V(x_0) = \eta(x_0)|\nabla u_1(x_0)|^{2\gamma_1}$ and $\beta = \frac{\gamma_1+1}{2\gamma_1} \in (\frac{1}{\gamma_1}, 1)$. Then

$$\begin{aligned}
\eta|\nabla u_1|^{2\gamma_1} &\leq \kappa_4\bigg(1 + (f_1)_+^2 + |u_1(0)-u_2(0)|^2 + (\varepsilon u_1)_-^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1-1)} \\
&\qquad + V^\beta\eta^{-\beta}|\nabla\eta| + V^{1/\gamma_1}\Big(2\eta^{-(\gamma_1+1)/\gamma_1}|\nabla\eta|^2 + \eta^{-1/\gamma_1}|\Delta\eta|\Big)\bigg) \\
&\leq \kappa_4\bigg(1 + (f_1)_+^2 + |u_1(0)-u_2(0)|^2 + (\varepsilon u_1)_-^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1-1)}\bigg) \\
&\qquad + \kappa_4 V^\beta\bigg(\eta^{-\beta}|\nabla\eta| + 2\eta^{-2\beta}|\nabla\eta|^2 + \eta^{-\beta}|\Delta\eta|\bigg),
\end{aligned}$$

where in the last line we used $V(x_0) \geq (\eta(x_0)|\nabla u_1|^2)^{\gamma_1} > 1$, $\eta \leq 1$ and $\frac{1}{\gamma_1} < \beta$. To conclude the proof of (8.6.1) it is enough to show that $\eta^{-\beta}|\nabla\eta|$ and $\eta^{-\beta}|\Delta\eta|$ are bounded quantities. Recall that $\eta = \rho^\tau$ where $\tau = \frac{4\gamma}{\gamma-1}$ with $\gamma = \min\{\gamma_1, \gamma_2\}$. It is easily seen that $\tau = \max\{\frac{4\gamma_1}{\gamma_1-1}, \frac{4\gamma_2}{\gamma_2-1}\}$. A simple calculation yields

$$\eta^{-\beta}|\nabla\eta| = \tau\rho^{\tau-1-\tau\beta}|\nabla\rho|,$$
$$\eta^{-\beta}|\Delta\eta| \leq \tau\{\rho^{\tau-1-\tau\beta}|\Delta\rho| + (\tau-1)\rho^{\tau-2-\tau\beta}|\nabla\rho|^2\}.$$

We observe that $1-\beta = \frac{\gamma_1-1}{2\gamma_1}$, and thus,

$$\tau(1-\beta)-1 \geq \frac{\gamma_1-1}{2\gamma_1}\frac{4\gamma_1}{\gamma_1-1} - 1 = 1, \quad \text{and} \quad \tau(1-\beta)-2 \geq 0.$$

Hence, there exist constant $C > 0$ satisfying

$$\eta(x_0)|\nabla u_1|^{2\gamma_1} \leq C\bigg(1 + (f_1)_+^2 + |u_1(0)-u_2(0)|^2 + (\varepsilon u_1)_-^2 + |\nabla f_1|^{2\gamma_1/(2\gamma_1-1)}\bigg).$$

Now taking supremum over B_2, we can write

$$\sup_{B_1}\{|\nabla u_1|^{2\gamma_1}, |\nabla u_2|^{2\gamma_2}\} \leq C\Big(1 + \sup_{B_2}(f_1)_+^2 + \sup_{B_2}|\nabla f_1|^{2\gamma_1/(2\gamma_1-1)}$$
$$+ |u_1(0)-u_2(0)|^2 + \sup_{B_2}(\varepsilon u_1)_-^2\Big).$$

If the maximum is attained at the second component we can repeat an analogous argument. This gives us (8.6.1).

Next we prove (8.4.3). Suppose, on the contrary, that there exists $\{(u_i^n, f_i^n, \alpha_i^n, \varepsilon_n)\}_n$ with α_i^n satisfying (8.2.1), and in D,

$$\begin{cases} -\Delta u_1^n(x) + H_1(x,\nabla u_1^n) + \alpha_1^n(x)(u_1^n(x)-u_2^n(x)) + \varepsilon u_1^n(x) &= f_1^n(x), \\ -\Delta u_2^n(x) + H_2(x,\nabla u_2^n) + \alpha_2^n(x)(u_2^n(x)-u_1^n(x)) + \varepsilon_n u_2^n(x) &= f_2^n(x), \end{cases} \tag{8.6.5}$$

and

$$|u_1^n(0)-u_2^n(0)|^2 > n\Big(1+\sup_{B_2}\sum_{i=1}^2(f_i^n)_+^2+\sup_{B_2}\sum_{i=1}^2|\nabla f_i^n|^{2\gamma_i/(2\gamma_i-1)}+\sup_{B_2}\sum_{i=1}^2(\varepsilon u_i)_-^2\Big). \tag{8.6.6}$$

First of all note that we can always set $u_1^n(0) = 0$. Therefore, by (8.6.6), we see that $|u_2^n(0)| \to \infty$. Suppose that there is a subsequence, denoted by the actual sequence, along which $u_2^n(0) \to \infty$. Define $v_i^n = \frac{1}{u_2^n(0)} u_i^n$. Since $a^2 \leq \kappa_i + a^{2\gamma_i}$ for some κ_i, for all $a \geq 0$, using (8.6.1) and (8.6.6) we find that

$$\sup_{B_1} \{|\nabla v_1^n|^{2\gamma_1}, |\nabla v_2^n|^{2\gamma_2}\} < C \quad \text{for all } \ n.$$

Since $(v_1^n(0), v_2^n(0)) = (0, 1)$, from above estimate if follows that $\sup_{B_1}(|v_1^n| + |v_2^n|)$ uniformly bounded in n. Using (HP1) and (8.6.6) we also get

$$\sup_n \sup_{B_1} [\frac{1}{u_2(0)} |H_1(x, \nabla u_1)| + \frac{1}{u_2(0)} |H_2(x, \nabla u_1)|] < \widehat{C}. \tag{8.6.7}$$

Therefore, it follows from (8.6.5) that $\|v_1^n\|_{\mathscr{W}^{2,p}(B_{\frac{1}{2}})}$, $\|v_2^n\|_{\mathscr{W}^{2,p}(B_{\frac{1}{2}})}$ are uniformly bounded in n (cf. [64, Theorem 9.11]) for any $p > N$, and hence we can extract a weakly convergence subsequence converging to some $v = (v_1, v_2) \in \mathscr{W}^{2,p}(B_{\frac{1}{2}}) \times \mathscr{W}^{2,p}(B_{\frac{1}{2}})$. From the Sobolev embedding we also see that $v_2^n \to v_2$ in $\mathcal{C}^{1,\alpha}(B_{\frac{1}{2}})$. Since $|\nabla v_i^n| \to |\nabla v_i|$ in $B_{\frac{1}{2}}$ and $\sup_n \sup_{B_{\frac{1}{2}}} \frac{1}{|u_2^n(0)|} |\nabla u_i^n|^{\gamma_i}$ is bounded, by (HP1) and (8.6.7), it follows that $\nabla v_i = 0$ in $B_{\frac{1}{2}}$. Thus, $v = (0, 1)$ in $B_{\frac{1}{2}}$. Now from the second equation of (8.6.5) we get

$$-\Delta v_2^n + \alpha_2^n (v_2^n - v_1^n) = \frac{1}{u_2^n(0)} f_2^n - \frac{1}{u_2^n(0)} H_2(x, \nabla u_2^n) \leq \frac{1}{u_2^n(0)} f_2^n + \frac{C_1}{u_2^n(0)},$$

by (HP1). Let φ be a non-zero, non-negative test function supported in $B_{\frac{1}{2}}$. Multiplying the above equation by φ, integrating over $B_{\frac{1}{2}}$ and letting $n \to \infty$ we obtain

$$\begin{aligned} \alpha_0^{-1} \int_{B_{\frac{1}{2}}} \varphi(x) \mathrm{d}x &\leq \liminf_{n\to\infty} \int_{B_{\frac{1}{2}}} \alpha_2^n(x) v_2^n(x) \varphi(x) \mathrm{d}x \\ &\leq \liminf_{n\to\infty} \int_{B_{\frac{1}{2}}} \varphi \Big[\Delta v_2^n + \frac{1}{u_2^n(0)} f_2^n + \alpha_2^n v_1^n + \frac{C_1}{u_2^n(0)}\Big] \mathrm{d}x = 0, \end{aligned}$$

where we use the fact that $\sup_{B_{1/2}} |\alpha_2^n v_1^n| \leq \alpha_0 \sup_{B_{1/2}} |v_1^n| \to 0$. Thus we arrive at a contradiction.

A similar contradiction is also arrived is $u_2^n(0) \to -\infty$ along some subsequence. This establishes (8.4.3).

(8.4.2) follows from (8.4.3) and (8.6.1). This completes the proof. □

8.7 Existence results in bounded domains

By D we denote a bounded $\mathcal{C}^{1,1}$ domain in $\mathbb{R}^N$.

Theorem 8.7.1 (Comparison principle)**.** *Let* $H_i \in \mathcal{C}^1(\mathbb{R}^N \times \mathbb{R}^N), i = 1, 2$ *be given functions. Let* $\boldsymbol{u} = (u_1, u_2) \in \mathcal{C}^2(D \times \{1, 2\}) \cap \mathcal{C}^1(\bar{D} \times \{1, 2\})$ *be a sub-solution to*

$$\begin{aligned} -\Delta u_1 + H(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) &= f_1 \quad \text{in } D, \\ -\Delta u_2 + H(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) &= f_2 \quad \text{in } D, \end{aligned} \tag{8.7.1}$$

and $\boldsymbol{v} = (v_1, v_2) \in \mathcal{C}^2(D \times \{1, 2\}) \cap \mathcal{C}^1(\bar{D} \times \{1, 2\})$ *be a super-solution to* (8.7.1)*. Moreover, assume that* $\boldsymbol{v} \geq \boldsymbol{u}$ *on* ∂D*. Then we have* $\boldsymbol{v} \geq \boldsymbol{u}$ *in* $\bar{D}$*.*

Proof. Write $w_i = v_i - u_i$. Then it follows from (8.7.1) that

$$\begin{aligned} -\Delta w_1 + h_1(x) \cdot \nabla w_1 + \alpha_1(x)(w_1 - w_2) &\geq 0 \quad \text{in } D, \\ -\Delta w_2 + h_2(x) \cdot \nabla w_2 + \alpha_2(x)(w_2 - w_1) &\geq 0 \quad \text{in } D, \end{aligned}$$

where

$$h_i(x) = \int_0^1 \nabla_p H_i(x, \nabla u_i(x) + t(\nabla v_i(x) - \nabla u_i(x)))\mathrm{d}t, \quad i = 1, 2.$$

The result follows by applying the maximum principle, Busca-Sirakov [45, Theorem 3.1], Sirakov [113, Theorem 1]. □

We next recall an existence result from [3]. Let $F_i : \bar{D} \times \mathbb{R}^N \to \mathbb{R}, i = 1, 2,$ be two continuous functions satisfying

$$|F_i(x, \xi)| \leq \kappa(1 + |\xi|^2) \quad \text{for all } (x, \xi) \in \bar{D} \times \mathbb{R}^N, \ i = 1, 2,$$

for some constant κ. We also assume that $\xi \mapsto F_i(x, \xi)$ is continuously differentiable.

Theorem 8.7.2. *Let $\bar{\boldsymbol{v}}, \underline{\boldsymbol{v}} \in \mathcal{C}^2(\bar{D} \times \{1,2\})$ be respectively a sub-solution and super-solution to*

$$-\Delta u_1 + F_1(x, \nabla u_1) + \alpha_1(u_1 - u_2) = 0 \quad \text{in } D,$$
$$-\Delta u_2 + F_2(x, \nabla u_2) + \alpha_1(u_2 - u_1) = 0 \quad \text{in } D,$$
$$u_1, u_2 = 0 \quad \text{on } \partial D.$$

Also, assume that $\underline{\boldsymbol{v}} \leq \bar{\boldsymbol{v}}$ in D. Then there exists a solution $\boldsymbol{u} \in \mathscr{W}^{2,p}(D \times \{1,2\}) \cap \mathcal{C}(\bar{D} \times \{1,2\})$ of the above equations satisfying $\underline{\boldsymbol{v}} \leq \boldsymbol{u} \leq \bar{\boldsymbol{v}}$.

Proof. This can be established by mimicking the arguments of Amann-Crandall [3, Theorem 1]. □

Note that Theorem 8.7.2 can be applied to find solution for our model provided the Hamiltonian has at-most quadratic growth in the gradient. To apply the theorem for a general Hamiltonian we need to introduce certain approximations.

Lemma 8.7.1. *Suppose that $\gamma > 2$. Given $C_1 > 0$, there exists a sequence of increasing $\mathcal{C}^{1,1}$ functions $\psi_n : [-C_1, \infty) \to [-C_1, \infty)$ satisfying the following*

(i) $\psi_n(x) \leq x$ *for all* $x \geq -C_1$,

(ii) $\psi_n(x) \geq \eta_1 x^{\frac{2}{\gamma}} - \eta_2$,

(iii) $0 \leq \psi_n'(x) \leq 1$,

where η_1, η_2 are positive constants independent of n. Furthermore,

$$\sup_x \frac{\psi_n(x)}{1 + |x|^2} < \infty,$$

and $\psi_n(x) \to x$ as $n \to \infty$, uniformly on compact sets.

Proof. Define for each $n \in \mathbb{N}$,

$$\psi_n(x) = \begin{cases} x & \text{for } x \leq n, \\ n - \frac{\gamma}{2} + \frac{\gamma}{2}\Big(x - n + 1\Big)^{\frac{2}{\gamma}} & \text{for } x > n. \end{cases}$$

Differentiating ψ_n we get that

$$\psi_n'(x) = \begin{cases} 1 & \text{for } x \le n, \\ \Big(x-n+1\Big)^{\frac{2}{\gamma}-1} & \text{for } x > n. \end{cases}$$

(i) and (iii) are obvious. To see (ii), we note that $\psi_n(x) \ge x^{\frac{2}{\gamma}} - (1 + C_1^{\frac{2}{\gamma}} + C_1)$ for $x \in [-C_1, n]$. For $x > n$ we also note that

$$\begin{aligned} n - \frac{\gamma}{2} + \frac{\gamma}{2}\Big(x-n+1\Big)^{\frac{2}{\gamma}} &\ge (n-1)^{\frac{2}{\gamma}} + \Big(x-n+1\Big)^{\frac{2}{\gamma}} - \frac{\gamma}{2} \\ &\ge x^{\frac{\gamma}{2}} - \frac{\gamma}{2}. \end{aligned}$$

This gives us (ii). □

We also require the following gradient estimate which follows by repeating the arguments in the proof of Proposition 8.4.1.

Lemma 8.7.2. *Grant Assumption 8.2.1. Let* $\epsilon \in [0,1)$ *and* $f_1, f_2 \in \mathcal{C}^1(\mathbb{R}^d)$. *Let* $\boldsymbol{u}$ *be a* $\mathcal{C}^2$ *function satisfying*

$$\begin{aligned} -\Delta u_1(x) + \psi_n^1(H_1(x,\nabla u_1)) + \alpha_1(x)(u_1(x) - u_2(x)) + \varepsilon u_1(x) &= f_1(x) \quad \text{in } \bar{\mathscr{B}}_2, \\ -\Delta u_2(x) + \psi_n^2(H_2(x,\nabla u_2)) + \alpha_2(x)(u_2(x) - u_1(x)) + \varepsilon u_2(x) &= f_2(x) \quad \text{in } \bar{\mathscr{B}}_2, \end{aligned}$$

where ψ_n^i *is the approximating sequence in Lemma 8.7.1 if* $\gamma_i > 2$, *otherwise* $\psi_n^i(x) = x$. *Suppose that* $\mathscr{B}_1 \Subset \mathscr{B}_2$ *and* $\mathscr{B}_1, \mathscr{B}_2$ *are concentric. Then there exists a constant* $C > 0$, *dependent on* $\operatorname{dist}(\mathscr{B}_1, \partial\mathscr{B}_2), \gamma_i$, $d, \eta_1, \eta_2, \alpha_0$ *but not on* n *and* $\boldsymbol{u}$, *satisfying*

$$\begin{aligned} &\sup_{\mathscr{B}_1}\{[\psi_n^1(H_1(x,\nabla u_1))]^2, [\psi_n^2(H_2(x,\nabla u_2))]^2\} \\ &\quad \le C\Big(1 + \sup_{\mathscr{B}_2}\sum_{i=1}^2 (f_i)_+^2 + \sup_{\mathscr{B}_2}\sum_{i=1}^2 |\nabla f_i|^{4/3} + |u_1(0) - u_2(0)|^2 + \sup_{\mathscr{B}_2}\sum_{i=1}^2 (\varepsilon u_i)_-^2\Big). \end{aligned}$$

Now we can prove our existence result.

Theorem 8.7.3. *Grant Assumption 8.2.1. Suppose* $\varepsilon \in [0,1]$ *and* $\boldsymbol{f} = (f_1, f_2) \in \mathcal{C}^1(\bar{D} \times \{1,2\})$. *Let* $\underline{\boldsymbol{v}} \in \mathcal{C}^2(\bar{D} \times \{1,2\})$ *be a sub-solution to*

$$\begin{aligned} -\Delta u_1 + H_1(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) + \varepsilon u_1 &= f_1 \quad \text{in } D, \\ -\Delta u_2 + H_1(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) + \varepsilon u_2 &= f_2 \quad \text{in } D. \end{aligned} \tag{8.7.2}$$

There there exists a solution $\boldsymbol{u} \in \mathcal{C}^2(D \times \{1,2\})$ *to* (8.7.2) *satisfying* $\boldsymbol{u} \geq \underline{\boldsymbol{v}}$ *in* D.

Proof. The main idea of proof is to use the existence result from Theorem 8.7.2 by making use of the approximation sequence in Lemma 8.7.1. A similar method was also used by Lions in [91] for scalar equations. In fact, the method of Lions uses more sophisticated tools like Bony maximum principle to obtain an up to the boundary bounds of the gradient. We do not use such results. We split the proof in two steps.

Step 1. Fix $n \geq 1$ and consider the system of equations

$$\begin{aligned} -\Delta w_1 + \psi_n^1(H_1(x, \nabla w_1)) + \alpha_1(x)(w_1 - w_2) + \varepsilon w_1 &= f_1 \quad \text{in } D, \\ -\Delta w_2 + \psi_n^2(H_1(x, \nabla w_2)) + \alpha_2(x)(w_2 - w_1) + \varepsilon w_2 &= f_2 \quad \text{in } D, \end{aligned} \tag{8.7.3}$$

where ψ_n^i is the approximating sequence from Lemma 8.7.1 if $\gamma_i > 2$, otherwise $\psi_n^i(x) = x$. By Lemma 8.7.1(i), we note that $\underline{\boldsymbol{v}}$ is a sub-solution to (8.7.3). So to apply Theorem 8.7.2 we need to find a super-solution. Denote by $M = \max_{\partial D}\{\underline{v}_1, \underline{v}_2\}$. Let $\bar{\boldsymbol{v}} \in \mathcal{C}^2(\bar{D} \times \{1,2\})$ be the unique solution to

$$\begin{aligned} -\Delta \bar{v}_1 + \alpha_1(x)(\bar{v}_1 - \bar{v}_2) + \varepsilon \bar{v}_1 &= f_1 + \eta_2 \wedge C_1 \quad \text{in } D, \\ -\Delta \bar{v}_2 + \alpha_2(x)(\bar{v}_2 - \bar{v}_1) + \varepsilon \bar{v}_2 &= f_2 + \eta_2 \wedge C_1 \quad \text{in } D, \\ \bar{v}_1, \bar{v}_2 &= M \quad \text{on } \partial D, \end{aligned} \tag{8.7.4}$$

where η_2 is given by Lemma 8.7.1(ii). In fact, using Sweers [115, Theorem 1.1], we can find a unique solution of (8.7.4) in $\mathscr{W}^{2,p}_{\text{loc}}(D) \times \mathcal{C}(\bar{D})$ and then using a standard bootstrapping argument we can improve the regularity. Using Lemma 8.7.1(ii) and (HP1) we then obtain from (8.7.4) that

$$\begin{aligned} -\Delta\bar{v}_1 + \psi_n^1(H_1(x,\nabla\bar{v}_1)) + \alpha_1(x)(\bar{v}_1 - \bar{v}_2) + \varepsilon\bar{v}_1 \geq f_1 &\quad \text{in } D,\\ -\Delta\bar{v}_2 + \psi_n^2(H_2(x,\nabla\bar{v}_2)) + \alpha_2(x)(\bar{v}_2 - \bar{v}_1) + \varepsilon\bar{v}_2 \geq f_2 &\quad \text{in } D,\\ \bar{v}_1, \bar{v}_2 = M &\quad \text{on } \partial D. \end{aligned}$$

This gives us the super-solution. By Theorem 8.7.1 we also have $\underline{\boldsymbol{v}} \leq \bar{\boldsymbol{v}}$ in $\bar{D}$. Now we can apply Theorem 8.7.2 to find a solution $\boldsymbol{w}^n = (w_1^n, w_2^n) \in \mathcal{C}^2(D\times\{1,2\}) \cap \mathcal{C}(\bar{D}\times\{1,2\})$ to (8.7.3) satisfying $\underline{\boldsymbol{v}} \leq \boldsymbol{w}^n \leq \bar{\boldsymbol{v}}$ in $\bar{D}$ for all n. It should also be noted that $\bar{\boldsymbol{v}}$ is independent of n.

Step 2. We now pass to the limit in (8.7.3) with the help of the gradient estimate in Lemma 8.7.2. From step 1 we notice that $\sup_D |w_1^n - w_2^n| < \infty$ uniformly in n. Thus, for any compact $\mathcal{K} \subset D$ we have $\max_{\mathcal{K}}\{|\nabla w_1^n|, |\nabla w_2^n|\} < \infty$ uniformly in n, by Lemma 8.7.2. Using (8.7.3) and standard elliptic estimates, we get

$$\sup_n \left\{\|w_1^n\|_{\mathscr{W}^{2,p}(\mathcal{K})}, \|w_2^n\|_{\mathscr{W}^{2,p}(\mathcal{K})}\right\} < \infty \quad \text{for every compact } \mathcal{K} \subset D.$$

Using a standard diagonalization argument we can find a subsequence, denoted by the actual one, so that $w_i^n \to u_i$ in $\mathscr{W}^{2,p}_{\mathrm{loc}}(D)$ for $p > N$ and $w_i^n \to u_i$ in $\mathcal{C}^1_{\mathrm{loc}}(D)$, as $n \to \infty$. Thus passing to the limit in (8.7.3) we obtain

$$\begin{aligned} -\Delta u_1 + H_1(x,\nabla u_1) + \alpha_1(x)(u_1 - u_2) + \varepsilon u_1 = f_1 &\quad \text{in } D,\\ -\Delta u_2 + H_1(x,\nabla u_2) + \alpha_2(x)(u_2 - u_1) + \varepsilon u_2 = f_2 &\quad \text{in } D, \end{aligned}$$

and $\underline{\boldsymbol{v}} \leq \boldsymbol{u} \leq \bar{\boldsymbol{v}}$ in D. Moreover, using standard theory of elliptic pde we obtain $\boldsymbol{u} \in \mathcal{C}^2(D\times\{1,2\})$. This completes the proof. □

Open-problems

In this part we will discuss few open-problems.

- From the Theorem 3.1.3 in Chapter 3, we know for $N \geq 5$ there holds

$$\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} \geq \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N} + \frac{(N-4)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N} + \frac{(N-1)^2}{16} \int_{\mathbb{H}^N} \frac{u^2}{r^2} dv_{\mathbb{H}^N} \; \forall\, u \in \mathcal{C}_c^\infty(\mathbb{H}^N \setminus \{x_0\}).$$

  This result immediately gives

$$\inf_{H^2(\mathbb{H}^N)\setminus\{0\}} \frac{\int_{\mathbb{H}^N} (\Delta_{\mathbb{H}^N} u)^2 \, dv_{\mathbb{H}^N} - \left(\frac{N-1}{2}\right)^4 \int_{\mathbb{H}^N} u^2 \, dv_{\mathbb{H}^N}}{\int_{\mathbb{H}^N} \frac{u^2}{r^4} \, dv_{\mathbb{H}^N}} \geq \frac{(N-4)^2}{16},$$

  which instantly implies for $N \geq 8$, the constant in front of the Rellich term $\frac{u^2}{r^4}$ can be larger than $\frac{9}{16}$ which was conjectured as a possible optimal constant in [27]. But what will be the best constant that is still unknown and this can be a further research topic.

- In the Chapter 5, we have seen from the Theorem 5.3.2 that the abstract Rellich type inequality holds true for N-dimensional hyperbolic space $\mathbb{H}^N$. It will be interesting to study this result on general Riemannian manifold (M, g) of dimension N.

- The Chapter 8 presents the existence and uniqueness results of the weakly coupled systems of viscous Hamilton-Jacobi equations (EP) under suitable assumptions. It will be interesting to study the parabolic coupled system of viscous Hamilton-Jacobi equations. More precisely, we are interested in the following system of equations

$$\begin{aligned} \frac{\partial u_1}{\partial t} - \Delta u_1 + H_1(x, \nabla u_1) + \alpha_1(x)(u_1 - u_2) = f_1(x) \text{ in } \mathbb{R}^N \times (0, +\infty), \\ \frac{\partial u_2}{\partial t} - \Delta u_2 + H_2(x, \nabla u_2) + \alpha_2(x)(u_2 - u_1) = f_2(x) \text{ in } \mathbb{R}^N \times (0, +\infty), \\ u_1(x, 0) = g_1(x) \text{ in } \mathbb{R}^N, \\ u_2(x, 0) = g_2(x) \text{ in } \mathbb{R}^N, \end{aligned}$$

  with the necessary assumption on the functions. The key interest will be in studying the large-time behaviour of the solutions after finding the existence and uniqueness of solutions.

- In the Chapter 7 we have seen the study of generalized principle eigenvalue for nonlinear operator. Now it will be interesting to do similar kind study for coupled system of equations with square gradient non-linearity. In particular, one can study the following system of eigenequations

$$\begin{aligned} \Delta u_1 - \frac{1}{u_1}|\nabla u_1|^2 - \alpha_1(x)(u_1 - u_2) + (f_1(x) - \lambda)u_1 = 0 \quad \text{in } \mathbb{R}^N, \\ \Delta u_2 - \frac{1}{u_2}|\nabla u_2|^2 - \alpha_2(x)(u_2 - u_1) + (f_2(x) - \lambda)u_2 = 0 \quad \text{in } \mathbb{R}^N, \end{aligned}$$

  with appropriate measurable coefficients and (u_1, u_2) are strictly positive functions. This kind of risk sensitive problem arises in exponential linear quadratic Gaussian control (LQG) problem. The scalar version was studied by Nagai [99] but the same problem for switching is much challenging due to gradient estimate.

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