Gap distribution among sequences arising in the theory of modular forms

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By

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Contents

Chapter 1

Introduction

The rich story of equidistribution started in the years 1909-1910 by the work of P. Bohl [5], H. Weyl [50] and W. Sierpinski [43] where they studied the distribution of the sequence $\{n\alpha\} \pmod 1$, (for an irrational α) in the unit interval. Let us recall that a sequence of real numbers $\{x_n\}$ lying in the interval $[0,1] \subseteq \mathbb{R}$ is said to be uniformly distributed or equidistributed with respect to Lebesgue measure if for any interval $[\alpha,\beta] \subseteq [0,1]$, we have

$$\lim_{V\to\infty} \frac{1}{V} \#\{n \leq V : x_n \in [\alpha,\beta]\} = \beta - \alpha,$$

where $\#$ denotes the cardinality. This subject attracted great attention of mathematicians from all branches of mathematics after Hermann Weyl related the study of equidistribution to the study of exponential sums in his 1916 paper [51]. Our work is partly motivated by the following result of Van der Corput (see [27], page no. 176): If for each positive integer $s,$ the sequence $\{x_{n+s} - x_n\}$ is uniformly distributed (mod 1), then the sequence $\{x_n\}$ is uniformly distributed (mod 1).

Let us consider the following classical question:

Question 1.0.1. *Is the converse of Van der Corput's result true?*

In other words, if $\{x_n\}$ is uniformly distributed (mod 1), then is it true that for any positive integer s, the sequence $\{x_{n+s} - x_n\}$ is uniformly distributed (mod 1)?

The answer to the above question is surprisingly no. For example, consider the well-studied sequence $\{n\alpha\}$ $(\text{mod } 1)$, α is irrational. Write the sequence as follows:

For a natural number N, define,

$$A_\alpha(N) = \{\alpha n \,(\text{mod})\; 1 : 1 \leq n \leq N\} \subset \{\{n\alpha\}\; (mod\; 1) : n \in \mathbb{N}\},$$

and write them as increasing order as follows:

$$A_\alpha(N) = \{0 < x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_N < 1\} \tag{1.1}$$

where $x_{N+1} = 1 + x_1$.

In 1957, Steinhaus conjectured the following fact:

$$\#\{x_{i+1} - x_i : 1 \leq i \leq N\} \leq 3,$$

where $\#$ denotes the cardinality of the set. There are several proofs of the above conjecture available in the literature, but the first proof was given by Vera Sós [44] and [45] in 1958. The above statement is popularly known as "The three gap theorem".

In 2002, Vâjâitu and Zaharescu [48] investigated the following question:

Question 1.0.2. *Let $A_\alpha(N)$ be as defined in (1.1). Remove as many elements of $A_\alpha(N)$ as one likes. Then, how large is the cardinality of the consecutive differences of the resulting set?*

More generally, they prove the following:

For any subset $\Omega(N)$ of $A_\alpha(N)$, there are no more than $(2+\sqrt{2})\sqrt{N}$ distinct consecutive differences, that is, if

$$B(\Omega(M)) = \{0 \leq y_1 \leq y_2 \leq \cdots \leq y_M < 1 \leq y_{m+1} := 1 + y_1\} \subset A_\alpha(N)$$

then

$$\#\{y_{i+1} - y_i : 1 \leq i \leq M,\ y_i \in \Omega(M)\} \leq 2\sqrt{2N} + 1.$$

In 2015, using additive combinatorics A. Balog, A.Granville and J. Solymosi [2] improved the bound of above result [48] to $2\sqrt{2N} + 1$ for any finite subset of $\mathbb{R}/\mathbb{Z}$, where $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers. In particular for our concerned sequence, they proved that

$$\#B(\Omega(N)) \leq 2\sqrt{2N} + 1.$$

From the above result, we can conclude that for any subsequence say $\{y_n\} = \{y_1, y_2, y_3, ...\}$ (arranged in ascending order) of $\{x_n\} = \{x_1, x_2, x_3, ...\}$, the consecutive difference $\{y_{i+1} - y_i\}$ is not uniformly distributed (mod 1). In this thesis we will show that, if a sequence is equidistributed in $[-\frac{1}{2}, \frac{1}{2}]$ with respect to a probability measure say $\mu = F(x)dx$ (for the definition see Section 2.1), then the fractional parts of gaps of all elements of the sequence will be equidistributed in $[0, 1]$ with respect to the measure $F(x) * F(x)\, dx$, where $*$ is the convolution of the measures.

More explicitly, if we have two sequences say $\{x_n\}_{n=1}^\infty$ and $\{y_m\}_{m=1}^\infty$ such that they are equidistributed with respect to probability measures $\mu_1 = F_1(x)dx$ and $\mu_2 = F_2(x)dx$ respectively in $[-\frac{1}{2}, \frac{1}{2}]$, then the sequence of fractional parts of gaps between elements of $\{x_n\}$ and $\{y_m\}$, that is, $\{x_n - y_m\}_{n,m=1}^\infty$ (mod 1) is equidistributed in $[0, 1]$ with respect to $F_1(x) * F_2(x)dx$. We are also able to predict quantitatively the rate of convergence of the following:

$$\lim_{V\to\infty} \frac{1}{V^2} \#\{1 \leq m, n \leq V : \{x_n - y_m\} \bmod 1 \in [\alpha, \beta]\},$$

where $[\alpha, \beta]$ is any subinterval of $[0, 1]$, whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_m\}_{m=1}^{\infty}$ satisfy some conditions that have been described in Theorem 4.1.6. More generally, we have results for r equidistributed sequences. These results are stated and proved in the Chapter 4 as Theorem 4.1.1, 4.1.4 and 4.1.6. We have discussed several applications of our results in Chapter 6.

Let $S(N, k)$ be the space of all holomorphic cusp forms of weight k with respect to $\Gamma_0(N)$. For any positive integer n, let $T_n(N, k)$ be the n^{th} Hecke operator acting on $S(N, k)$. Let $s(N, k)$ denote the dimension of the vector space $S(N, k)$. For a positive integer $n \geq 1$, let

$$\{a_{n,i,N}, 1 \leq i \leq s(N, k)\}$$

denote the eigenvalues of T_n, counted with multiplicity. For any positive integer n, let $T_n^{'}$ be the normalized Hecke operator acting on $S(N, k)$, defined as follows

$$T_n^{'} := \frac{T_n}{n^{\frac{k-1}{2}}}.$$

Consider

$$\left\{ \frac{a_{n,i,N}}{n^{\frac{k-1}{2}}}, 1 \leq i \leq s(N, k) \right\},$$

the eigenvalues of $T_n^{'}$ counted with multiplicity. Let p be a prime number such that p and N are coprime. Then by the theorem of Deligne (see [14]) proving the Ramanujan-Petersson inequality, we know that

$$a_{p,i,N} \in [-2p^{\frac{k-1}{2}}, 2p^{\frac{k-1}{2}}].$$

For each i, choose $\theta_{p,i,N} \in [0, \pi]$ such that

$$\frac{a_{p,i,N}}{p^{\frac{k-1}{2}}} = 2 \cos \theta_{p,i,N}.$$

The $\theta_{p,i,N}$ are called eigenangles.

In Chapter 2, we discuss all the preliminaries we require. In Chapter 3, we discuss Eichler-Selberg trace formula and Kuznetsov trace formula. In Chapter 4, we discuss the distribution of gaps between equidistributed sequences. In Chapter 5, we prove a variant of the Erdös-Turán Inequality. In Chapter 6, we study the distribution of gaps between eigenangles of Hecke operators. Using results of Murty and Sinha [32], Murty and Srinivas [33] have recently proved the following results

$$\begin{aligned}\#\{(i,j), 1 \leq i,j \leq s(N,k) : \theta_{p,i,N} \pm \theta_{p,j,N} = 0\} \\ = \mathrm{O}\left((s(N,k))^2 \left(\frac{\log p}{\log kN}\right)\right).\end{aligned}$$

Note that taking k and N sufficiently large, the above result gives a little evidence towards the Maeda and Tsaknias conjectures.

For N = 1, the famous Maeda conjecture (see [19]) predicts that the polynomial $\prod_{i=1}^{s(N,k)}(x - a_{p,i,1})$ is irreducible over $\mathbb{Q}$. It also predicts that the Galois group of minimal polynomial of T_n is the full symmetric group S_d, where d is the dimension of $S(1,k)$. Based on computational data, Tsaknias (see [46]) predicts that for a fixed level $N > 1$, the above polynomial is a product of bounded numbers of irreducible polynomials viewed as a function of k. In this thesis, we can get the measure with respect to which the differences of eigenangles of Hecke operators are equidistributed. As a special case to our result, we obtain the result of Murty and Srinivas. We could also get an error term (see Chapter 6, Theorems 6.1.1, 6.1.3). We also discuss similar results for primitive Maass forms (see Theorems 6.2.1, 6.2.2). In the case of primitive Maass forms, we have assumed the Ramanujan bound.

Chapter 7 is a report on recent joint work with M. Ram Murty, where we have given a multiplicity one theorem for Hecke eigenforms. We also discuss a prove of the Joint Sato-Tate conjecture. The contents of this chapter are independent of the content of the previous chapters.

Chapter 2

Preliminary concepts

The study of Fourier coefficients of modular forms is of great interest to number theorists and others because the coefficients carry interesting pieces of information about modular forms. In this thesis, we will study some statistical properties of the Fourier coefficients of modular forms and Maass forms.

2.1 Equidistribution and its Extensions

Definition 2.1.1. *A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be* ***uniformly distributed modulo 1*** *or equidistributed mod 1 with respect to the Lebesgue measure if for every pair of real numbers a, b with $0 \leq a \leq b \leq 1$, we have*

$$\lim_{N \to \infty} \frac{\sharp\{n \leq N : (x_n) \in [a, b]\}}{N} = b - a,$$

where $(x_n) := x_n - [x_n]$ denotes the fractional part of x_n.

Note that in the above expression, $b - a$ is the Lebesgue measure of the interval $[a, b]$. More generally equidistribution can be defined for any probability measure as follows:

Definition 2.1.2. *A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be* ***equidistributed (mod 1)*** *with respect to a probability measure μ if for every pair of real numbers a, b with $0 \leq a \leq b \leq 1$, we have*

$$\lim_{N\to\infty} \frac{\sharp\{n \leq N : (x_n) \in [a,b]\}}{N} = \mu([a,b]),$$

where $(x_n) := x_n - [x_n]$ denotes the fractional part of x_n.

The above definition can be generalized to any unit interval as follows:

Definition 2.1.3. *Let $[\alpha, \beta] \subset \mathbb{R}$ be any interval of unit length. A sequence of real numbers $\{x_n\}_{n=1}^{\infty}$ is said to be* ***equidistributed*** *in $[\alpha, \beta]$ with respect to a probability measure μ if for every pair of real numbers a, b with $\alpha \leq a \leq b \leq \beta$, we have*

$$\lim_{N\to\infty} \frac{\sharp\{n \leq N : (x_n) \in [a,b]\}}{N} = \mu([a,b]).$$

Remark 2.1.4. *In this thesis, the sequences of interest to us are equidistributed in the interval $[-\frac{1}{2}, \frac{1}{2}]$.*

The above definition can be rewritten as follows:
A sequence $\{x_n\}$ of real numbers is said to be uniformly $\bmod 1$ if and only if for every interval $I \subset [0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(x_n) = \int_0^1 \chi_I(x) d\mu(x).$$

Equivalently, for all (complex valued) Riemann integrable functions $f(x)$ of period 1,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x) d\mu(x).$$

In particular, whenever $d\mu(x) = dx$, for all non-zero $m \in \mathbb{Z}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i m x_n} = 0.$$

In 1916, Weyl (see [51]) proved that the above condition is sufficient criterion for uniform distribution. More explicitly, he proves the following result now known as Weyl's Criterion:

Theorem 2.1.5. *A sequence* $\{x_n\}$ *is uniformly distributed* $\bmod 1$ *if and only if*

$$c_m := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(mx_n) = 0$$

for every $m \in \mathbb{Z}$, $m \neq 0$, $e(t) = e^{2\pi i t}$.

The necessary part is clear from the previous discussion. To prove the sufficient part we use Weierstrass approximation theorem that says every continious function can be approximated by trigonometric polynomials. For detail of the proof, see [27], page 172.

As an application of Weyl's theorem, one can derive the following:

Theorem 2.1.6. *If* θ *is irrational then* $\{(n\theta)\}$ *is uniformly distributed* $\bmod 1$.

For a proof see [27], exercise 11.1.8. The definition of equidistribution of a sequence can be extended to the notion of equidistribution of sets as follows

Definition 2.1.7. *A sequence of finite multisets* A_n *with* $\sharp A_n \to \infty$ *counted with multiplicity is said to be equidistributed mod 1 with respect to a probability measure* μ *if for every continuous function* f *on* $[0,1]$*, we have*

$$\lim_{n\to\infty} \frac{1}{\sharp A_n} \sum_{t\in A_n} f(t) = \int_0^1 f(x) d\mu$$

In this case the Weyl limits can be defined as follows:

Definition 2.1.8. *For* $m \in \mathbb{Z}$,

$$c_m := \lim_{n\to\infty} \frac{1}{\sharp A_n} \sum_{t\in A_n} e(mt).$$

Remark 2.1.9. *When μ is the Lebesgue measure and $A_n = \{x_1, x_2, ..., x_n\}$, the above definition is same as the classical definition of uniform distribution $mod\ 1$ as in Definition 2.1.1*

In 1928, Schoenberg [37] and in 1924 Wiener [52] independently gave a necessary and sufficient criterion for equidistribution of a sequence in $[0, 1]$ with respect to some positive continuous measure in terms of the Weyl limit (see [22, Theorem 7.5] or [27, 11.3.3]). In the context of set equidistribution we also have the Wiener-Schoenberg theorem (see [27, Theorem 11.6.16]) as follows. We follow the presentation of page 195 of [27].

Theorem 2.1.10. *Let $\{A_n\}$ be a sequence of sets as mentioned above. The sequence $\{A_n\}$ is equidistributed with respect to some positive continuous measure if and only if for all $m \in \mathbb{Z}$ the Weyl limits c_m defined as above exist and*

$$\sum_{m=1}^{N} |c_m|^2 = \mathrm{o}(N) \text{ as } N \to \infty.$$

Proof. Consider the sequence of finite multisets $\{A_n\}$ such that $\#A_n \to \infty$ as $n \to \infty$ is equidistributed in $[0, 1]$ with respect to some positive contiinious measure μ. Then by definition, we know that for any continuous function $f : [0, 1] \to \mathbb{C}$,

$$\lim_{n\to\infty} \frac{1}{\sharp A_n} \sum_{t \in A_n} f(t) = \int_0^1 f(x) d\mu.$$

In particular, choosing $f = e(mx)$, we have

$$c_m = \lim_{n\to\infty} \frac{1}{\sharp A_n} \sum_{t \in A_n} e(mt) = \int_0^1 e(mx) d\mu.$$

Hence, the Weyl limit c_m exist for all $m \in \mathbb{Z}$. Moreover,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}|c_m|^2=\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}\int_0^1\int_0^1 e(m(x-y))d\mu(x)d\mu(y).$$

Interchanging the sum and integral, the above is equal to

$$\lim_{N\to\infty}\int_0^1\int_0^1\left(\frac{1}{N}\sum_{m=1}^{N}e(m(x-y))d\mu(x)d\mu(y)\right).$$

Note that,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}e(m(x-y))=\begin{cases}0 & \text{if } x-y\notin\mathbb{Z}\\ 1 & \text{otherwise.}\end{cases}$$

Since $\left|\frac{1}{N}\sum_{m=1}^{N}e(m(x-y))\right|$ is bounded by 1, by dominated convergence theorem , we can interchange the limit and integral. After interchanging the limit and integral the required limit is zero, because the measure of the set $\{(x,y)\in[0,1]^2 : x-y\in\mathbb{Z}\}$ is zero.

Conversely, suppose that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}|c_m|^2=0,$$

then by the Riesz representation theorem there exist a measurable function $g(x)$ such that

$$c_m=\int_0^1 e(mx)dg(x).$$

Hence,

$$\begin{aligned}0 &= \lim_{N\to\infty}\frac{1}{N}\sum_{m=1}^{N}|c_m|^2\\ &= \int_0^1\int_0^1\lim_{N\to\infty}\frac{1}{N}e(mx)e(-my)dg(x)dg(y)\\ &= \int_0^1\int_0^1\lim_{N\to\infty}\frac{1}{N}e(m(x-y))dg(x)dg(y).\end{aligned}$$

that is

$$\int_0^1 \int_0^1 f(x-y)dg(x)dg(y) = 0,$$

where

$$f(x-y) = \begin{cases} 0 & x-y \notin \mathbb{Z} \\ 1 & \text{otherwise.} \end{cases}$$

Now we will show that g is continuous. Let us assume that g is not continuous. In particular, let g has a jump discontinuity at $a \in (0,1)$. Then,

$$\int_0^1 \int_0^1 f(x-y)dg(x)dg(y) \geq [\mu(a^+) - \mu(a^-)]^2 > 0,$$

which is a contradiction to the fact that the double integral is zero. Hence, g is continuous. Now choosing $\mu(x) = \sum_{m=-\infty}^{\infty} c_m e(mx)$, we have for any $[\alpha, \beta]$ contained in $[0,1]$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : x_n \in [\alpha, \beta]\} = \int_\alpha^\beta \mu(x)dx.$$

Hence, the sequence of multisets $\{A_n\}$ is equidistributed with respect to the positive continuous measure μ. $\square$

2.2 Fourier Analysis

According to our need, let us recall some facts from Fourier analysis in this section. The reader may refer to [35] for detailed study. For our convenience, let us define:

$$e(x) := e^{2\pi i x}.$$

Let f be a periodic and integrable function of period 1 on $\mathbb{R}$. The Fourier series of f is given by

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e\left(nx\right),$$

where $\hat{f}(n)$ are called the Fourier coefficients, defined as

$$\hat{f}(n) := \int_0^1 f(x)e\left(-nx\right),\ n \in \mathbb{Z}.$$

Let $f_i, 1 \leq i \leq r$ be r integrable functions on $\mathbb{R}$ of period 1. Define the convolution of r periodic integrable functions of period 1 on $\mathbb{R}$, denoted as

$$f_1 * f_2 * \dots * f_r : \mathbb{R} \to \mathbb{C}$$

as follows:

$$f_1 * \dots * f_r(y) \tag{2.1}$$

$$:= \int_0^1 \dots \int_0^1 f_1(y_1)f_2(y_2)\dots f_r(y - y_1 - y_2 - \dots - y_{r-1})dy_{r-1}dy_{r-2}\dots dy_1.$$

Among the many interesting properties that convolution of periodic integrable functions satisfies, the following property serves our purpose:

$$(f_1 * f_2 * \dots * f_r)^{\wedge}(n) = \hat{f}_1(n)\hat{f}_2(n)\dots\hat{f}_r(n). \tag{2.2}$$

In particular, for $r = 2$,

$$(f_1 * f_2)^{\wedge}(n) = \hat{f}_1(n)\hat{f}_2(n).$$

As the above property is important for us, we will give a proof of the above fact for $r = 2$.

$$\begin{aligned} (f_1 * f_2)^{\wedge}(n) &= \int_0^1 \int_0^1 f_1(y_1)f_2(y - y_1)e^{-2\pi i n y}dy_1 dy \\ &= \int_0^1 \int_0^1 f_1(y_1)e^{-2\pi i n y_1} f_2(y - y_1)e^{-2\pi i n(y - y_1)}dy_1 dy \end{aligned}$$

Using Fubini's theorem, the above equals

$$\int_0^1 f_1(y_1)e^{-2\pi i n y_1}dy_1 \int_0^1 f_2(y - y_1)e^{-2\pi i n(y-y_1)}dy$$

$$= \hat{f}(n)\hat{g}(n).$$

For $r > 2$, we can prove similarly.

The following theorem can be concluded from the famous Riesz-Fischer theorem (see [35], page 91):

Theorem 2.2.1. *[Riesz-Fischer]*
Let $\{a_n\}$ be a sequence of real numbers. If

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty,$$

then there exists a unique periodic square Lebesgue integrable function f, that is $f \in L^2[0,1]$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} a_n e(nx), \quad \textit{that is } \hat{f}(n) = a_n.$$

2.3 Beurling-Selberg Polynomials

Let $\chi_I(x)$ be the characteristic function of the interval $I = [a, b]$ contained in $[0, 1]$. For a positive integer M, define $\triangle_M(x)$ to be the Fejer's Kernel, defined as below:

$$\triangle_M(x) = \sum_{|n|<M} \left(1 - \frac{|n|}{M}\right) e(nx) = \frac{1}{M}\left(\frac{sin \pi Mx}{\sin\ \pi x}\right)^2.$$

The Mth order Beurling polynomials are defined as follows:

$$B^*{}_M(x) = \frac{1}{M+1}\sum_{n=1}^{M}\left(\frac{n}{M+1} - \frac{1}{2}\right) \triangle_M\left(x - \frac{n}{M+1}\right)$$

$$+\frac{1}{2\pi(M+1)} \sin\left(2\pi(M+1)x\right) - \frac{1}{2\pi} \triangle_{M+1}(x) \sin 2\pi x + \frac{1}{2(M+1)} \triangle_{M+1}(x).$$

For an interval $[a, b]$, the Mth order Selberg polynomials are defined as below:

$$S_M^+(x) = b - a + B_M^*(x - b) + B_M^*(a - x)$$

and

$$S_M^-(x) = b - a + B_M^*(b - x) + B_M^*(x - a)$$

It is clear that both the above polynomials are trigonometric polynomial of degree at most M.

We will investigate some properties of Beurling-Selberg polynomials. The following result was observed by Beurling. It was Vaaler [47] who presented Beurling's result in the following language. We follow the presentation of [27, Theorem 11.4.3].

Theorem 2.3.1. *Let*

$$\operatorname{sgn}(z) = \begin{cases} 1 & \textit{if } \operatorname{Re}(z) \geq 0, \\ -1 & \textit{otherwise.} \end{cases}$$

If

$$B(z) = \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} + \frac{2}{z}\right)$$

then,

1. $B(z)$ *is entire.*

2. $B(x) \geq \operatorname{sgn}(x)$*, for all real x.*

3. $B(z) = \operatorname{sgn}(z) + \mathrm{O}\left(\frac{e^{2\pi|Imz|}}{|z|}\right)$.

4. $\int_{-\infty}^{\infty}(B(x) - \operatorname{sgn}(x))dx = 1$.

Proof: To prove **1.**, it is enough to notice that $(\sin\ \pi z)^2$ is entire with zeros of order 2 at all integers and

$$\left(\sum_{n=0}^{\infty}\frac{1}{(z-n)^2}-\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}+\frac{2}{z}\right)$$

is holomorphic in $\mathbb{C}-\{\mathbb{Z}-\{0\}\}$. It has poles of order two at all non-zero integers. All the poles of

$$\left(\sum_{n=0}^{\infty}\frac{1}{(z-n)^2}-\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}+\frac{2}{z}\right)$$

are getting cancelled by the zeros of $\sin\ \pi z$. Hence, $B(z)$ is entire.

Now we will prove **2.**. To prove the above claim we first observe the following:

$$\left(\frac{\sin\ \pi z}{\pi}\right)^2\sum_{n=-\infty}^{n=\infty}\frac{1}{(z-n)^2}=1, z\notin\mathbb{Z}.$$

For $x>0$ we also have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{(x+n)^2}&\leq\sum_{n=1}^{\infty}\int_{x+n-1}^{x+n+1}\frac{du}{u^2}\\ =\int_{x}^{\infty}\frac{du}{u^2}&=\frac{1}{x}=\sum_{n=0}^{\infty}\int_{x+n}^{x+n+1}\frac{du}{u^2}\\ &\leq\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}.\end{aligned}$$

Now for $Re\ z\geq 0$,

$$\begin{aligned}B(z)-\operatorname{sgn}(z)=\left(\frac{\sin\ \pi z}{\pi}\right)^2\sum_{n=0}^{\infty}\frac{1}{(z-n)^2}\\ -\left(\frac{\sin\ \pi z}{\pi}\right)^2\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}+\left(\frac{sin\ \pi z}{\pi}\right)^2\frac{2}{z}-\operatorname{sgn} z.\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} + \frac{2}{z}\right) - \operatorname{sgn}(z) \\
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\frac{2}{z} + \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2}\right) - 1 \\
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\frac{2}{z} + \sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - 2\sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2}\right) - 1 \\
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\frac{2}{z} - 2\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}\right).
\end{aligned}
$$

Now when z is real (say x), we have

$$
\begin{aligned}
B(x) - \operatorname{sgn}(x) &\geq \left(\frac{\sin\ \pi x}{\pi}^2\right)\left(\frac{2}{x} - \frac{2}{x}\right) \\
&\geq 0.
\end{aligned}
$$

Similarly, for $Re\ z < 0$,

$$
B(z) - \operatorname{sgn}(z)
$$

$$
\begin{aligned}
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\frac{2}{z} + 2\sum_{n=0}^{\infty} \frac{1}{(z-n)^2}\right) \\
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\frac{2}{z} + 2\left\{\sum_{n=0}^{\infty} \frac{1}{(z-n)^2} + \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2} - \sum_{n=-\infty}^{-1} \frac{1}{(z-n)^2}\right\}\right) \\
&= \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\frac{2}{z} - 2\sum_{n=1}^{\infty} \frac{1}{(z+n)^2}\right) + 2.
\end{aligned}
$$

Now if z is real (say x), we have

$$
\begin{aligned}
B(x) - \operatorname{sgn}(x) &\geq \left(\frac{\sin \pi x}{\pi}\right)^2 \left(\frac{2}{x} - \frac{2}{x}\right) + 2 \\
&\geq 0.
\end{aligned}
$$

Hence,

$$B(x) - \operatorname{sgn}(x) \geq 0 \text{ for real } x.$$

To prove **3.**, observe that

$$\sin^2 \pi z = \mathrm{O}(e^{2\pi|Im(z)|}).$$

In addition, for $x, y > 0$, we have

$$\sum_{n=0}^{\infty} \frac{1}{(x+n)^2 + y^2} \leq \frac{1}{x^2 + y^2} + \min\left(\int_0^{\infty} \frac{dt}{(x+t)^2}, \frac{dt}{(t^2+y^2)^2}\right)$$

$$= \frac{1}{x^2+y^2} + \min\left(\frac{1}{x}, \frac{\pi}{2y}\right).$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{|z+n|^2} = \mathrm{O}\left(\frac{1}{|z|}\right), \text{ for } \operatorname{Re}(z) \geq 0 \,,$$

and

$$\sum_{n=0}^{\infty} \frac{1}{|z-n|^2} = \mathrm{O}\left(\frac{1}{|z|}\right), \text{ for } \operatorname{Re}(z) < 0 \,.$$

Combining all above facts, we get **3.**.

We now prove **4.**.

Proof. We know,

$$\int_{-\infty}^{\infty} (B(x) - \operatorname{sgn}(x))dx = \lim_{A\to\infty} \int_{-A}^{A} (B(x) - \operatorname{sgn}(x))dx.$$

Now,

$$\int_{-A}^{A} (B(x) - \operatorname{sgn}(x))dx = \int_{-A}^{0} (B(x) - \operatorname{sgn}(x))dx + \int_{0}^{A} (B(x) - \operatorname{sgn}(x))dx$$

$$= \int_0^A (B(x) + B(-x))dx$$

$$
\begin{aligned}
&= \int_0^A \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \left(\sum_{n=0}^{\infty} \frac{1}{(x-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} + \frac{2}{x} \right) \right. \\
&\quad \left. + \left(\sum_{n=0}^{\infty} \frac{1}{(-x-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(-x+n)^2} - \frac{2}{x} \right) \right\} dx \\
&= \int_0^A \left(\frac{\sin \pi x}{\pi}\right)^2 \left\{ \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(x-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} \right. \\
&\quad \left. + \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} - \sum_{n=1}^{\infty} \frac{1}{(x-n)^2} \right\} dx \\
&= \int_0^A \left(\frac{\sin \pi x}{\pi}\right)^2 \left(\frac{2}{x^2}\right) dx \\
&= 2\int_0^A \left(\frac{\sin \pi x}{\pi}\right)^2 dx
\end{aligned}
$$

Hence,

$$
\lim_{A \to \infty} \int_0^A (B(x) - \operatorname{sgn}(x)) dx = 2\int_0^A \left(\frac{\sin \pi x}{\pi}\right)^2 dx = 1.
$$

$\square$

Selberg (see [38, page 213-218]) used Beurling polynomials to approximate the characteristic function of an interval. More precisely,

Theorem 2.3.2. ***Selberg,(1970)*** *Let $I = [a, b]$ be an interval and χ_I be its characteristic function. Then there are continuous functions $S^+(x)$ and $S^-(x)$ in $L^1(\mathbb{R})$ such that*

$$
S^-(x) \le \chi_I(x) \le S^+(x)
$$

with $\hat{S}^{\pm}(t) = 0$, for $|t| \ge 1$. In addition,

$$
\int_{-\infty}^{\infty} \big(\chi_I(x) - S^-(x)\big)\, dx = 1
$$

and

$$\int_{-\infty}^{\infty} \big(S^+(x) - \chi_I(x)\big)\, dx = 1.$$

Proof: We know that

$$B(z) = \left(\frac{\sin\ \pi z}{\pi}\right)^2 \left(\sum_{n=0}^{\infty} \frac{1}{(z-n)^2} - \sum_{n=1}^{\infty} \frac{1}{(z+n)^2} + \frac{2}{z}\right).$$

Let

$$S^+(x) = \frac{1}{2}(B(x-a) + B(b-x)).$$

Thus,

$$S^+(x) \geq \frac{1}{2}(\operatorname{sgn}(x-a) + \operatorname{sgn}(b-x)) = \chi_I(x)$$

as we have

$$B(x-a) \geq \operatorname{sgn}(x-a) \ \ \textit{and} \ \ B(b-x) \geq \operatorname{sgn}(b-x).$$

$$\frac{1}{2}(B(x-a) + B(x-b)) \geq \frac{1}{2}(\operatorname{sgn}(x-a) + \operatorname{sgn}(b-x))$$

and so

$$S^+(x) \geq \frac{1}{2}(\operatorname{sgn}(x-a) + \operatorname{sgn}(b-x)).$$

Hence,

$$\int_{-\infty}^{\infty} (S^+(x) - \chi_I(x)) dx = 1$$

as

$$\int_{-\infty}^{\infty} (S^+(x) - \chi_I(x)) dx = \frac{1}{2}\int_{-\infty}^{\infty} [(B(x-a) + B(x-b)) - \chi_I(x)] dx.$$

But again,

$$\chi_I(x) = \frac{1}{2}\{\operatorname{sgn}(x-a) + \operatorname{sgn}(b-x)\}. \tag{2.3}$$

Using (2.3), we have

$$\begin{aligned}\int_{-\infty}^{\infty}(S^+(x)-\chi_I(x))dx &= \frac{1}{2}\int_{-\infty}^{\infty}[(B(x-a)+B(x-b))-\{\text{sgn}(x-a)+\text{sgn}(b-x)\}]dx \\ &= \frac{1}{2}+\frac{1}{2} \\ &= 1.\end{aligned}$$

We get the above equality by using Theorem 2.3.1. And from above it is also clear that $S^+(x) \in L^1(R)$. Since $S^+(x)$ is a restriction of an entire function, it is continuous.

Now we will show that for $t > 1$,

$$\hat{S}^+(t) = \int_{-\infty}^{\infty} S^+(x)e(-tx)dx = 0. \tag{2.4}$$

Let $J(A,B) = \int_{-A}^{B} S^+(x)e(-tx)dx$.

To prove (2.4), we will begin by proving the following:

$$J(A,B) = \int_{-A}^{B} S^+(x)e(-tx)dx = \text{O}\left(\frac{1}{A}+\frac{1}{B}\right) \text{ as } A, B \to \infty.$$

By contour integration,

$$\begin{aligned}\int_{-A}^{B} S^+(x)e(-tx)dx = \int_{-A}^{-A-iT} S^+(x)e(-tx)dx \\ + \int_{-A-iT}^{B-iT} S^+(x)e(-tx)dx + \int_{B-iT}^{B} S^+(x)e(-tx)dx.\end{aligned} \tag{2.5}$$

First we will estimate the second integral on the right-hand side of (2.5).

$$\begin{aligned}\int_{-A-iT}^{B-iT} S^+(x)e(-tx)dx =& \frac{1}{2}\int_{-A-iT}^{B-iT}(B(x-a)+B(b-x))e(-tx)dx \\ =& \frac{1}{2}\int_{-A-iT}^{B-iT} B(x-a)e(-tx)dx + \frac{1}{2}\int_{-A-iT}^{B-iT} B(b-x)e(-tx)dx.\end{aligned} \tag{2.6}$$

Substituting $x - iT + a$ and $-x + iT + b$ in place of x in the first and second integral respectively of (2.6), we have

$$\begin{aligned}\int_{-A-iT}^{B-iT} S^{+}(x)e(-tx)dx &\ll \int_{A'}^{B'} |B(x-iT)|e^{-2\pi tT}dx\\ &\ll \int_{A'}^{B'} e^{2\pi T}e^{-2\pi tT}dx,\end{aligned}$$

where $A' = A + \max\{|a|, |b|\}$ and $B' = B + \max\{|a|, |b|\}$.

The last inequality is by using the previous theorem, that is

$$B(z) = \operatorname{sgn}(z) + \mathrm{O}\left(\frac{e^{2\pi|Im(z)|}}{|z|}\right).$$

Hence,

$$\int_{-A-iT}^{B-iT} S^{+}(x)e(-tx)dx \ll \int_{A'}^{B'} e^{2\pi T}e^{-2\pi tT}dx.$$

Now, since $t > 1,$ the above integral goes to 0, whenever $T \to \infty$.
Now we estimate the first and third integrals.
For $z = -A + iy$ and $A > |a|$,

$$B(z-a) = B(-A+iy-a) = B(-A-a+iy) = \operatorname{sgn}(-A-a+iy)+\mathrm{O}\left(\frac{e^{-2\pi y}}{A}\right).$$

For $A > |b|$,

$$B(b-z) = B(b+A-iy) = \operatorname{sgn}(b+A-iy) + \mathrm{O}\left(\frac{e^{-2\pi y}}{A}\right).$$

So,

$$S^{+}(z) \ll \frac{e^{-2\pi y}}{A}.$$

The first integral on the right hand side of (2.5) is

$$\ll \frac{1}{A}\int_{-\infty}^{0} e^{-2\pi y} e^{2\pi t y} dy \ll \frac{1}{A}.$$

Similarly we can estimate the last integral on the right-hand side of (2.5).

$$J(A,B) = \mathrm{O}\left(\frac{1}{A}+\frac{1}{B}\right).$$

Now letting $A, B \to \infty$, we have shown that $\hat{S}^{+}(t) = 0$, for $t > 1$.
For $t < 1$, we can use the fact $\hat{S}^{+}(-t) = \bar{\hat{S}}^{+}(t)$ and deduce the desired result.
Using continuity of S^{+}, the proof for $t = \pm 1$ follows. Now defining

$$S^{-}(x) = -\frac{1}{2}(B(x-a) + B(x-b))$$

and proceeding analogously we get the required result.
Note that, for any $\delta > 0$, choosing $S^{\pm}(x)$ for the interval $[\delta a, \delta b]$, we have

$$\hat{S}^{\pm}(t) = 0 \text{ for } |t| > \delta \text{ and}$$
$$\int_{\infty}^{\infty}(\chi_I(x) - S^{-}(x)) = \int_{\infty}^{\infty}(S^{+}(x) - \chi_I(x)) = \frac{1}{\delta}.$$

Since $S^{\pm}$ is continuous, the crux of the above discussion is that the characteristic function of any interval in $\mathbb{R}$ can be approximated by a continuous function. For detail see Exercise 11.4.5 of [27]. Moreover, from the work of Vaaler (see [27]), we have the following facts:
For all $M \geq 1$

(a) For a subinterval $[a,b]$ of $[0,1]$,

$S_M^{-}(x) \leq \chi_I(x) \leq S_M^{+}(x)$.

$S^{\pm}_M(x) = \sum_{0\le|m|\le M} \hat{S}^{\pm}_m(m)e(mx)$

(b)

$\hat{S}^{\pm}_M(0) = b - a \pm \frac{1}{M+1}$.

When $M \neq 0$,

$|\hat{S}^{\pm}_M(m)| \le \frac{1}{M+1} + \min\{b-a, \frac{1}{\pi|m|}\}$.

(c)

$$||S^{\pm}_M(x) - \chi(x)||_{L_1} \le \frac{1}{M+1}.$$

Using above, we can conclude that

$$|\hat{S}^{\pm}_M(n) - \hat{\chi}_I(n)| \le ||S^{+}_M - \chi(x)||_{L_1} \le \frac{1}{M+1}.$$

(d)

For $n \neq 0$, note that

$$|\hat{\chi}_I(n)| = \left|\frac{\sin\ \pi n(b-a)}{\pi k}\right| \le \min\left(b-a, \frac{1}{\pi|n|}\right).$$

For $0 < |n| < M$,

$$|\hat{S}^{\pm}_M(n)| \le \frac{1}{M+1} + \min\left(b-a, \frac{1}{\pi|n|}\right).$$

The above result follows from following facts combining with **(c)**:

2.4 Erdös-Turán inequality

In 1948 Erdös and Turán gave an effective version of the Weyl's criterion. The following version of the Erdös-Turán inequality can be found in [27]. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence. For any natural number N and an interval (a,b) in $\mathbb{R}$, define the discrepancy D_N of the sequence $\{x_n\}_{n=1}^{\infty}$ as

$$D_N := \sup_{0 \leq a < b \leq 1} \left| \frac{\#\{n \leq N : a \leq x_n \leq b\}}{N} - (b-a) \right|.$$

Theorem 2.4.1. *(Erdös-Turán, 1948) For any integer $M \geq 1$,*

$$D_N \leq \frac{1}{M+1} + 3\sum_{m=1}^{M} \frac{1}{Nm} \Big| \sum_{n=1}^{N} e^{2\pi i m x_n} \Big|.$$

There are several proofs of the above inequality. In [25] Montgomery gives a proof using Beurling-Selberg polynomials (see [27, Theorem 11.4.8]). Generalizing the idea implicit in Montgomery's work, Murty and Sinha have proved the following variant of Erdös-Turán inequality in ([32, Theorem 8]).

Theorem 2.4.2. *[Murty-Sinha] Let $I = [a,b] \subset [0,1]$ and c_m's are the Weyl limits given by Definition 2.1.8. Let*

$$\mu = F(x)dx,$$

where

$$F(x) = \sum_{m=-\infty}^{\infty} c_m e(mx).$$

Define

$$N_I(V) := \#\{n \leq V : x_n \in I\}$$

and

$$D_{I,V}(\mu) := |N_I(V) - V\mu(I)|.$$

Then,

$$D_{I,V}(\mu) \leq \frac{V||\mu||}{M+1} + \sum_{1\leq|m|\leq M}\left(\frac{1}{M+1} + min\left(b-a, \frac{1}{\pi|m|}\right)\right)\left|\sum_{n=1}^{r} e(mx_n) - Vc_m\right|$$

where $||\mu|| = \underset{x\in[0,1]}{Sup}|F(x)|$.

We will follow the presentation of [32].

Proof. Let χ_I be the characteristic function of interval I. Then by Theorem 2.3.2

$$S_M^-(x_n) \leq \chi_I(x_n) \leq S_M^+(x_n). \tag{2.7}$$

Taking sum over all terms of (2.7), we get

$$\sum_{n\leq V} S_M^-(x_n) \leq \sum_{n\leq V}\chi_I(x_n) \leq \sum_{n\leq V} S_M^+(x_n).$$

Since $S_M^\pm(x)$ is a periodic and entire function, hence by Fourier's theory, we have

$$\sum_{n\leq V} S_M^\pm(x_n) = \sum_{|m|\leq M}\hat{S}_M^\pm(m)\sum_{n\leq V} e(mx_n).$$

Subtracting $\sum_{|m|\leq M}\hat{S}_M^\pm(m)Vc_m$ from both the sides, we get

$$\sum_{n\leq V} S_M^\pm(x_n) - V\sum_{|m|\leq M}\hat{S}_M^\pm(m)c_m = \sum_{1\leq|m|\leq M}\hat{S}_M^\pm(m)\sum_{n\leq V}\left(e(mx_n) - Vc_m\right).$$

Taking the absolute value on both sides, we get

$$\left|\sum_{n\leq V} S_M^\pm(x_n) - V\sum_{|m|\leq M}\hat{S}_M^\pm(m)c_m\right| = \left|\sum_{|m|\leq M}\hat{S}_M^\pm(m)\sum_{n\leq V}\left(e(mx_n) - Vc_m\right)\right|.$$

So,

$$\left|\sum_{n\leq V} S_M^\pm(x_n) - V\sum_{|m|\leq M}\hat{S}_M^\pm(m)c_m\right| \leq \left|\sum_{1\leq|m|\leq M\leq V}\hat{S}_M^\pm(m)\sum_{n\leq V}\left(e(mx_n) - Vc_m\right)\right|.$$

Since $\hat{S}_M^{\pm}(m)c_m = 0$ for $|m| > M$, without loss of generality we can extend the sum $\sum_{|m|\leq M} \hat{S}_M^{\pm}(m)c_m$ to $\sum_{m\in\mathbb{Z}} \hat{S}_M^{\pm}(m)$. Then,

$$\sum_m \hat{S}_M^{\pm}(m) = \sum_m \int_0^1 S_m^{\pm}(x)e(-mx)dx.$$

Now interchanging the sum and integral, we have

$$\sum_m \hat{S}_M^{\pm}(m) = \int_0^1 S_M^{\pm}(x)d\mu.$$

Now from **(c)**, we have

$$\left|\int_0^1 (S_M^{\pm}(x) - \chi_I(x))d\mu\right| \leq \frac{||\mu||}{M+1}$$

and

$$\hat{S}_M^{\pm}(m) \leq \frac{1}{M+1} + min\left(b-a, \frac{1}{\pi|n|}\right).$$

□

In Section 2.5 and Section 2.6 we review definitions and properties of cusp forms and Maass forms respectively. We also review the definition and properties of Hecke operators acting on all these spaces respectively. Based on our need, we will state several results without giving proof, but for a concerned reader, we will give reference for details.

2.5 Modular Forms and Hecke operators

In this section, we will review basic definitions and facts about modular forms. For details, the reader may consult [28] and [20]. Let us start with the following notation:

Let $\mathcal{H}$ be the upper half plane, defined as

$$\mathcal{H} := \{x+iy : x, y \in \mathbb{R}, y > 0\}.$$

For a commutative ring R with unity, define

$$GL_2^+(R) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in R \text{ and } ad - bc > 0 \right\}.$$

Let $\mathbb{Q}$ be the set of rational numbers. Define the action of $GL_2^+(\mathbb{Q})$ on $\mathcal{H}$ as follows:
For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q})$ and $z \in \mathcal{H}$,

$$\gamma z := \frac{az + b}{cz + d}.$$

Now, for any positive integer k and a meromorphic function f defined on $\mathcal{H}$, define

$$f|_k\gamma(z) = (det\ \gamma)^{\frac{k}{2}} f(\gamma z)(cz + d)^{-k}.$$

Define

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$. Let f be a holomorphic function on $\mathcal{H}$ such that

$$f|_k\gamma(z) = f(z) \text{ for all } \gamma \in \Gamma.$$

Since Γ has finite index in $SL_2(\mathbb{Z})$, there exists a positive integer, say l, such that

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^l = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \in \Gamma.$$

Hence,

$$f(z + l) = f(z) \;\text{ for all } z \in \mathcal{H}.$$

That is f is of period l. So, f has a Fourier series expansion at ∞,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(f) q^{\frac{n}{l}}, \;\; q = e^{2\pi i z}. \tag{2.8}$$

Definition 2.5.1. *A function f as defined above is said to be* ***holomorphic at infinity*** *if*

$$a_n = 0 \text{ for all } n < 0 \text{ in (2.8).}$$

Definition 2.5.2. *A function f as defined above is said to* ***vanish at infinity*** *if $a_0 = 0$ in (2.8).*

Since the above defined action is a transitive action in Γ, for any $\gamma^{'} \in SL_2(\mathbb{Z})$, $(\gamma^{'})^{-1}\Gamma\gamma^{'}$ has finite index in $SL_2(\mathbb{Z})$ and $f|_k\gamma^{'}(z)$ is also periodic. Hence it has a Fourier series expansion at infinity,

$$f|_k\gamma^{'}(z) = \sum_{n=-\infty}^{\infty} a_n(f|_k\gamma^{'})q^{\frac{n}{l}}.$$

With the above notations and definitions, we will define modular forms.

Definition 2.5.3. *Let $k \in \mathbb{Z}$. A function f on $\mathcal{H}$ is said to be a* ***modular form*** *of weight k with respect to Γ if*
(i) f is holomorphic on $\mathcal{H}$.
(ii) $f(\gamma z) = (cz+d)^k f(z)$ for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
(iii) $f_k|\gamma(z)$ is holomorphic at ∞ for all $\gamma \in SL_2(\mathbb{Z})$.

Remark 2.5.4. *Let Γ be a congruent subgroup of $SL_2(\mathbb{Z})$ of level N. Let $q_N = e^{\frac{2\pi iz}{N}}$ for $z \in \mathcal{H}$. Let $f : \mathcal{H} \Rightarrow \mathbb{C}$ satisfies (i) and (ii) in Definition 2.5.3 and in the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q_N^n$, the coefficients satisfies*

$$|a_n| \leq cn^r \text{for some constant } c, r \in \mathbb{R}^+.$$

Then $f_k|\gamma(z)$ is ***holomorphic at*** *∞ for all $\gamma \in SL_2(\mathbb{Z})$. (see Proposition 1.2.4 of [15].)*

Definition 2.5.5. *We say that f* ***vanishes at the cusps*** *if $f|_k\gamma^{'}$ vanishes at cusps for all $\gamma^{'} \in SL_2(\mathbb{Z})$.*

Definition 2.5.6. *A function f as defined above is said to be a* ***cusp form*** *of weight k with respect to Γ if*
(i) f is a modular form of weight k with respect to Γ.
(ii) f vanishes at the cusps.

Remark 2.5.7. *To check whether f vanishes at the cusps, it is sufficient to check that the constant term in the Fourier series expansion (2.8) of $f_k|\gamma$ is zero for all $\gamma \in SL_2(\mathbb{Z})$.*

Now we will define one of the most important families of linear operators in the study of modular forms known as Hecke operators.
We will define these operators using the theory of double cosets. Let Γ_1 and Γ_2 be two congruent subgroup of $SL_2(\mathbb{Z})$. Let α be an element of $GL_2^+(\mathbb{Q})$. A **double coset** in $GL_2^+(\mathbb{Q})$ is defined as follows:

$$\Gamma_1\alpha\Gamma_2 := \{g_1\alpha g_2 : g_1 \in \Gamma_1, g_2 \in \Gamma_2\}.$$

We can decompose the double coset as a disjoint union of Γ_1-orbits, that is

$$\Gamma_1\alpha\Gamma_2 = \coprod_j \Gamma_1\alpha\gamma_j \text{ for some } \gamma_j \in \Gamma_2.$$

Let $S^\times$ be a multiplicative subgroup of $(\mathbb{Z}/N\mathbb{Z})^\times$. Let S^+ be an additive subgroup of $\mathbb{Z}$.
For a positive integer n, let

$$X_n = X_n(N, S^\times, S^+) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = n, N|c, a \in S^\times, b \in S^+ \right\}.$$

Note that X_1 is a congruent subgroup of $SL_2(\mathbb{Z})$ and the set X_n is invariant under the left and right action of X_1. We also note that there is a finite set of orbit representatives $\alpha_i \in X_n$ such that

$$X_n = \coprod_i X_1\alpha_i$$

Definition 2.5.8. *For any modular form f of weight k on X_1, the n-th Hecke operator T_n that maps to a modular form of weight k on X_1 is defined as*

$$T_n(f) := n^{\frac{k}{2}-1} \sum_i f|_k \alpha_i.$$

These operators take modular forms to modular forms and cusp forms to cusp forms. In this thesis, we are interested in a particular kind of subgroup of $SL_2(\mathbb{Z})$:

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \ c \equiv \ 0 \ (mod \ N) \right\},$$

for some positive integer N. Note that $X_1(N, (\mathbb{Z}/N\mathbb{Z})^\times, \mathbb{Z}) = \Gamma_0(N)$ and it has index

$$\psi(N) := N \prod_{p|N} \left(1 + \frac{1}{p}\right)$$

in $SL_2(\mathbb{Z})$, for example (see Exercise 2.3.4, page 19 in [28]). Since the subgroup $\Gamma_0(N)$ is a finite index subgroup of $SL_2(Z)$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$, modular forms with respect to $\Gamma_0(N)$ are periodic with period 1 and will have Fourier series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n, \ \ q = e^{2\pi i z}.$$

The Fourier series expansion at $i\infty$ of a cusp form is

$$f(z) = \sum_{n=1}^{\infty} a_n(f) q^n.$$

For our further use, let $M(N, k)$ be the space of modular forms of weight k with respect to $\Gamma_0(N)$ and $S(N, k)$ be the space of cusp forms of weight k with respect to $\Gamma_0(N)$. We know that $M(N, k)$ and $S(N, k)$ are finite dimensional vector spaces over $\mathbb{C}$.

Our particular interest is when the Hecke operators act on the spaces $S(N, k)$. The above mentioned Hecke operators have several interesting properties. Here we will mention some of them as the following theorem:

Theorem 2.5.9. *(Hecke) For positive integers n, the Hecke operators T_n acting on $S(N, k)$ satisfy the following properties:*
(i) For $m, n \geq 1$,

$$T_m T_n = T_n T_m.$$

(ii) If $(m, n) = 1$, then

$$T_m T_n = T_{mn}.$$

(iii) For a prime power p^r, $r \geq 1$,

$$T_{p^r} T_p = \begin{cases} T_{p^{r+1}} + p^{k-1} T_{p^{r-1}} & \text{if } (p, N) = 1 \\ (T_p)^r & \text{otherwise.} \end{cases}$$

For a proof of Theorem 4.1.1, see [20], page 60.

Theorem 2.5.10. *Let $n \geq 1$ be coprime to N. Then the Hecke operators T_n are self-adjoint operators with respect to the Petersson inner product, where the Petersson inner product on $S(N, k)$ is defined as follows:*
For any $f, g \in S(N, k)$,

$$< f, g >= \int_{\mathcal{H}/\Gamma_0(N)} f(z) \bar{g}(z) y^k \frac{dxdy}{y^2}. \tag{2.9}$$

For a proof, see [20] page 75. From Theorem 2.5.10, we can conclude that the eigenvalues of the Hecke operators T_n are real, whenever $(n, N) = 1$. Moreover, the collection of all Hecke operators acting on $S(N, k)$ generate an algebra over $\mathbb{C}$. For a positive integer $n \geq 1$, let

$$\{\lambda_{n,i,N}, \ 1 \leq i \leq s(N, k)\}$$

denote the set of eigenvalues of T_n. For any positive integer n, let

$$T_n^{'} := \frac{T_n}{n^{\frac{k-1}{2}}}$$

be the normalized Hecke operator acting on $S(N,k)$ with eigenvalues

$$\{a_{n,i,N} = \frac{\lambda_{n,i,N}}{n^{\frac{k-1}{2}}}, 1 \leq i \leq s(N,k)\}.$$

By the celebrated theorem of Deligne [14], which proves the famous Ramanujan conjecture, we know that for any prime p not dividing N, the eigenvalues of $T_p^{'}$ lie in the interval $[-2,2]$. Since the Hecke operators are diagonalizable and commutative there exists an ordered basis for $S(N,k)$ such that every operator T_n is represented on that basis by a diagonal matrix. An element of such a basis is called an eigenform. A basis of $S(N,k)$ consisting of eigenforms is called a **Hecke eigenbasis**. An eigenform f is said to be a **normalized** eigenform, if the first Fourier coefficient $a_1(f) = 1$.

2.6 Maass Forms

In this section, we will follow the presentation of [24]. Let $\mathcal{H}$ be the upper half-plane. Consider $\Gamma = SL_2(\mathbb{Z})$. The non-Euclidean Laplace operator on the space of smooth functions on $\mathcal{H}$ is given by

$$\triangle := -y^2\left(\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2}\right).$$

If $f(z) = u + iv$, then

$$\triangle(f(z)) = v^2\left(\frac{\delta^2}{\delta x^2} + \frac{\delta}{\delta y^2}\right) f(z)$$

The operator $\triangle$ is invariant under the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$, that is, for any smooth function f on $\mathcal{H}$ and for any $z \in \mathcal{H}$,

$$(\triangle f)(\gamma z) = \triangle(f(\gamma z))$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and the action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ defined as follows:

$$\gamma z = \frac{az+b}{cz+d}.$$

Definition 2.6.1. *A smooth function $f \neq 0$ on $\mathcal{H}$ is called a **Maass form** with respect to the group Γ if*
(i) *for all $\gamma \in \Gamma$ and $z \in \mathcal{H}$;*

$$f(\gamma z) = f(z),$$

(ii) *f is an eigenfunction of $\triangle$, that is,*

$$\triangle f = \lambda f \text{ for } \lambda \in \mathbb{C};$$

(iii) *there exists a positive integer N such that*

$$f(z) \ll y^N, \text{ as } y \to \infty.$$

Definition 2.6.2. *A Maass form f is said to be a **cusp form** if, for all $z \in \mathcal{H}$,*

$$\int_0^1 f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} z\right) dx = \int_0^1 f(z+x)dx = 0.$$

We know that the Maass cusp forms span a subspace $\mathcal{C}(\Gamma \setminus \mathcal{H})$ in $L^2(\Gamma \setminus \mathcal{H})$, where $L^2(\Gamma \setminus \mathcal{H})$ denotes the square integrable functions on $\Gamma \setminus \mathcal{H}$ and the L^2 norm is induced by the Petersson inner product.

2.7 Hecke Operators on Maass forms

For any positive integer n, define

$$Y_n = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = n \right\}.$$

Note that there is a natural action of $Y_1 = SL_2(\mathbb{Z})$ on Y_n. For any positive integer n and $f \in \mathcal{C}(\Gamma\backslash\mathcal{H})$, define the n^{th} **Hecke operator** acting on $\mathcal{C}(\Gamma\backslash\mathcal{H})$ as follows:

$$T_n(f) := \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

In the case of Maass forms these Hecke operators also satisfy several interesting properties. According to our need, we record some of the properties.

Theorem 2.7.1. *For $n \ge 1$, let T_n be the n^{th} Hecke operator acting on $\mathcal{C}(\Gamma \setminus \mathcal{H})$. Then*
(i) *For $m, n \ge 1$,*

$$T_m T_n = \sum_{d|(m,n)} T_{\frac{mn}{d^2}}.$$

In particular,

$$T_m T_n = T_n T_m.$$

(ii) *The Hecke operators commute with the Laplace operator $\triangle$.*
(iii) *The Hecke operators are self-adjoint operators.*

Now again, since the Hecke operators T_n together with the Laplacian $\triangle$ form a commutative family $\hbar$ of Hermitian operators on $L^2(\Gamma \setminus \mathcal{H})$ and the Hecke operators are self-adjoint in $L(\Gamma \setminus \mathcal{H})$, there exists a basis of $\mathcal{C}(\Gamma \setminus \mathcal{H})$ consisting of simultaneous eigenfunctions for all T_n.
Consider $\{u_j : j \ge 0\}$ to be a complete orthonormal basis for the space $\mathcal{C}(\Gamma \setminus \mathcal{H})$ consisting of the common eigenfunctions of $\hbar$, where u_0 is a constant function. Then

$$\triangle u_j = \left(\frac{1}{4} + (t_j)^2\right) u_j \text{ and } T_n u_j = \lambda_j(n) u_j,$$

where $\lambda_j(n) \in \mathbb{R}$. Selberg predicted that $t_j > 0$. When the group is $SL_2(\mathbb{Z})$, it has been known for some time (Selberg, Roelcke) that $t_j > \frac{1}{4}$. The Fourier expansion of a Maass form, for $z = x + iy \in \mathcal{H}$

$$u_j(z) = \sqrt{y}\rho_j(1) \sum_{n \neq 0} \lambda_j(n) K_{it_j}(2\pi|n|y) e(nx),$$

where $\rho_j(1) \neq 0$ and K_v is the modified Bessel function of the third kind, that is

$$K_v = \frac{\pi}{2} \frac{I_{-v}(x) - I_v(x)}{\sin \pi\alpha},$$

where

$$I_v(x) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+v+1)} \left(\frac{x}{2}\right)^{2m+v}.$$

A **primitive Maass forms** is a Maass form normalized by $\rho_j(1)$, that is, $\frac{u_j(z)}{\rho_j(1)}$. We know :

$$\Omega(T) := \#\{j : 0 < t_j \leq T\} = \frac{1}{4\pi} vol(\Gamma \setminus \mathcal{H}) T^2 + \mathrm{O}(T \log T),$$

where $vol(\Gamma \setminus \mathcal{H})$ denote the volume of $\Gamma/\mathcal{H}$. The Ramanujan conjecture predicts that for any prime p,

$$|\lambda_j(p)| \leq 2.$$

At present we are far from the above bound. The best bound towards the Ramanujan's conjecture for Maass forms is due to Kim and Sarnak (see [21]), that is,

$$|\lambda_j(p)| \leq p^{\frac{7}{64}} + p^{-\frac{7}{64}}.$$

Assuming the Ramanujan's conjecture, we can write

$$\lambda_j(p) = 2\cos\theta_{p,j}, \quad \theta_{p,j} \in [0, \pi]. \tag{2.10}$$

2.8 Equidistribution of Hecke eigenvalues

Let $E = \{(x, y) \in \mathbb{R}^2 : y^2 = x^3 + Ax + B,\ A, B \in \mathbb{Q}\}$ be an elliptic curve with rational coefficients. Let us define the j-invariants of elliptic curves.

Definition 2.8.1. *For any elliptic curve E, the j**-invariant** of E is defined as:*

$$j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2}.$$

Definition 2.8.2. *The **discriminant** Δ_E of an elliptic curve E is defined as*

$$\Delta_E = -16(4A^3 + 27B^2).$$

The importance of j-invariant is that it determines E up to isomorphism over an algebraically closed field. Let E be an elliptic curve defined as in Definition 2.8.1 with integer coefficients, that is $A, B \in \mathbb{Z}$. For any prime p, let $N_p(E) = |E(\mathbb{F}_p)|$ be the number of solutions of (E mod p) and define an integer $a_p(E)$ such that

$$N_p(E) = p + 1 - a_p(E).$$

Definition 2.8.3. *A prime p is said to be of **good reduction** if p does not divide the discriminant Δ.*

From the result of Hasse we know that

$$|a_p(E)| \leq 2\sqrt{p}.$$

In the years 1950 and 1951, Sato and Tate independently studied the behaviour of a_p's varying p. Write

$$a_p(E) = 2\sqrt{p} \cos \theta_p(E), \theta_p(E) \in [0, \pi].$$

Sato and Tate independently formulated the following conjecture:

Conjecture (Sato-Tate): Suppose E has no complex multiplication. Then $a_p(E)$ (resp. $\theta_p(E)$) are equidistributed in $[-1,1]$ (resp. in $[0,\pi]$) as $p \to \infty$ with respect to the measure

$$\frac{2}{\pi}\sqrt{1-x^2}dx \text{ (resp. } \frac{2}{\pi}\sin^2\theta d\theta).$$

In [40], Serre generalized the above conjecture as follows:

Conjecture (Sato-Tate): For a fixed i such that $1 \leq i \leq s(N,k)$, Let $a_{p,i,N}$ be an eigenvalue of the p^{th} normalized Hecke operator T_p acting on a non C.M $f \in S(N,k)$ such that p is coprime to N. Then the family $\{a_{p,i,N}\}$ is equidistributed in $[-2,2]$ as $p \to \infty$ with respect to the Sato-Tate measure

$$d\mu_\infty = \frac{1}{2\pi}\sqrt{4-x^2}dx.$$

More precisely, the Sato-Tate conjecture is saying that for any continuous function $\phi : [-2,2] \to \mathbb{R}$ and interval $[\alpha,\beta] \subset [-2,2]$

$$\lim_{V\to\infty}\frac{1}{V}\sum_{p\leq V}\phi\left(a_{p,i,N}\right) = \int_\alpha^\beta \phi d\mu_\infty.$$

Recently the above conjecture has been proved by Barnet-Lamb, Geraghty, Harris and Taylor [3], [10], [18] in a series of papers. In 1997, Serre [39] studied the "vertical" Sato-Tate conjecture by fixing a prime p and varying N and k. In particular, he proved the following theorem:

Theorem 2.8.4 (Serre). *Let N_λ, k_λ be positive integers such that k_λ is even. $N_\lambda + k_\lambda \to \infty$ and p is a prime not dividing N_λ for any λ. Then the family of eigenvalues of the normalized p^{th} Hecke operator*

$$T_p'(N_\lambda,k_\lambda) = \frac{T_p(N_\lambda,k_\lambda)}{p^{\frac{k_\lambda-1}{2}}}$$

is equidistributed in the interval $\Omega = [-2,2]$ *with respect to the measure*

$$d\mu_p := \frac{p+1}{\pi} \frac{\sqrt{1-\frac{x^2}{4}}}{(p^{\frac{1}{2}}+p^{-\frac{1}{2}})^2 - x^2} dx.$$

Remark Also in 1997, Conrey, Duke and Farmer [13] studied a special case of above result by fixing $N = 1$. In 2009, Murty and Sinha [32] investigated the effective / quantitative version of Serre's results, in which they give an explicit estimate on the rate of convergence. They proved the following theorem:

Theorem 2.8.5 (Murty-Sinha)**.** *Let* p *be a fixed prime. Let* $\{(N,k)\}$ *be a sequence of pairs of positive integers such that* k *is even and* N *is not divisible by* p. *For any interval* $[\alpha, \beta] \subset [-2,2]$,

$$\frac{1}{s(N,k)} \sharp \{1 \leq i \leq s(N,k) : a_{p,i,N} \in [\alpha,\beta]\} = \int_{\alpha}^{\beta} d\mu_p + \mathrm{O}\left(\frac{\log\ p}{\log\ kN}\right),$$

where the constant can be computed effectively.

In the case of Maass forms, the Sato-Tate conjecture is still open. In 1987, Sarnak [36, Theorem 1.2] studied a vertical version of Sato-Tate conjecture for primitive Maass forms. Assuming Ramanujan's bound in the case of primitive Maass forms, Sarnak's theorem can be interpreted as follows:

Theorem 2.8.6. *[Sarnak] For a fixed prime* p, *the eigenvalues* $\lambda_j(p), 1 \leq j \leq r(T)$ *of Hecke operators are equidistributed in* $[-2,2]$ *with respect to the measure* μ_p *as* $T \to \infty$, *where* $r(T)$ *denotes the number of Laplacian eigenvalues up to* T^2 *and*

$$\mu_p = \frac{p+1}{\pi} \frac{\sqrt{1-\frac{x^2}{4}}}{\left(p^{\frac{1}{2}}+p^{-\frac{1}{2}}\right)^2 - x^2} dx.$$

Recently Lau and Wang [24], using the Kuznetsov trace formula made Sarnak's [36] result effective, that is, they proved the joint distribution of eigenvalues of the Hecke operators quantitatively for primitive Maass forms of level 1 and stated the same for primitive holomorphic cusp forms. They proved the following theorem:

Theorem 2.8.7 (Lau-Wang)**.** *Let p be a prime. Let k be positive even integer such that $\log(p) \leq \delta \log k$, for some small absolutely constant δ. Let $\{\lambda_j(p)\}$ be the eigenvalues of normalized Hecke operators $T_p^{'}$ acting on space of primitive Maass forms. For any $I = [\alpha, \beta] \subset [-2, 2]$*

$$\frac{1}{\Omega(T)} \sharp \{0 \leq t_j \leq T : (\lambda_j(p)) \in I\}$$

$$= \int_I d\mu_p + \mathrm{O}\left(\frac{\log(p)}{\log\, k}\right),$$

where $d\mu_p = \frac{p+1}{\pi} \frac{\sqrt{1-\frac{x^2}{4}}}{\left(p^{\frac{1}{2}}+p^{-\frac{1}{2}}\right)^2 - x^2} dx.$

They have mentioned that their methods will work for higher level.

Chapter 3

Trace formulas

3.1 Eichler-Selberg trace formula

In this thesis, we will use the Eichler-Selberg trace formula as one of our important tools to prove the main theorems. This is a formula for the trace of Hecke operators T_n acting on $S(N,k)$ in terms of class numbers of binary quadratic forms and certain arithmetic functions. In this section, we follow the presentation of [32]. Let $\triangle$ be a non-negative integer congruent to 0 or $1\ (mod\ 4)$, let $B(\triangle)$ be the set of all positive definite binary quadratic forms with discriminent $\triangle$, that is

$$B(\triangle) = \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{Z}, a > 0, b^2 - 4ac = \triangle\}.$$

We denote the set of primitive forms by

$$b(\triangle) = \{f(x,y) \in B(\triangle) : gcd(a,b,c) = 1\}.$$

Let us define an action of the full modular group $SL_2(\mathbb{Z})$ on $B(\triangle)$ as follows:

$$f(x,y)\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} := f(\alpha x + \beta y, \gamma x + \delta y).$$

Note that this action respects primitive forms. We know that the above action has finitely many orbits. We define $h(\triangle)$ to be the number of orbits of $b(\triangle)$.

Let h_w be defined as follows:

$$\begin{aligned} h_w(-3) &= \frac{1}{3} \\ h_w(-4) &= \frac{1}{2} \\ h_w(\triangle) &= h(\triangle) \text{ for } \triangle < -4. \end{aligned}$$

Theorem 3.1.1. *For any positive integer n, the trace Tr of T_n acting on $S(N,k)$ is given by*

$$Tr\ T_n = A_1(n) + A_2(n) + A_3(n) + A_4(n),$$

where $A_i(n)$s are as follows:

$$A_1(n) = \begin{cases} n^{\frac{k}{2}-1}\frac{k-1}{12}\psi(N) & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise.} \end{cases}$$

where $\psi(N) = N\prod_{p|N}(1+\frac{1}{p})$;

$$A_2(n) = -\frac{1}{2}\sum_{t\in\mathbb{Z}, t^2<4n} \frac{\varrho^{k-1}-\bar{\varrho}^{k-1}}{\varrho-\bar{\varrho}} \sum_f h_w\big(\frac{t^2-4n}{f^2}\big)\mu(t,f,n);$$

where ϱ and $\bar{\varrho}$ are the zeros of the polynomial x^2-tx+n and the inner sum runs over positive divisors f of $\frac{(t^2-4n)}{f^2} \in \mathbb{Z}$ is congruent to 0 or 1 $(mod\ 4)$. $\mu(t,f,n)$ is given by

$$\mu(t,f,n) = \frac{\psi(N)}{\psi(\frac{N}{N_f})} M(t,n,NN_f),$$

where $N_f = gcd(N,f)$ and $M(t,n,K)$ denote the number of solutions of the congruence $x^2-tx+n \equiv 0 \pmod{K}$;

$$A_3(n) = -\sum_{d|n, 0<d\leq\sqrt{n}} d^{k-1}\sum_{c|N}\phi\big(gcd(c,\frac{N}{c})\big) \tag{3.1}$$

Here, ϕ denotes Euler's function and in the first summation, if there is a contribution from the term $d=\sqrt{n}$, it should be multiplied by $\frac{1}{2}$. In the inner sum, we *also need the condition that* $gcd(c, \frac{N}{c})$ *divides* $gcd(N, \frac{n}{d} - d)$;

$$A_4(n) = \begin{cases} \sum_{t|n, t>0} t & \text{if } k = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Serre [39] and Murty-Sinha [32] used the above Eichler-Selberg trace formula to compute Weyl limits for the family of Hecke eigenangles. They also used it to estimate

$$\left| \sum_{i=1}^{s(N,k)} (2 \cos m\theta_{p,i_j,N} - c_m(s(N,k)) \right|.$$

3.2 Kuznetsov trace formula

The following weighted version of Kuznetsov trace formula can be found in [24, Lemma 3.1]:

Theorem 3.2.1. *[Kuznetsov] Let* m, n *be two positive integers. Then for any* $\epsilon > 0$,

$$\sum_{t_j \leq T} \alpha_j \lambda_j(m) \lambda_j(n) = \frac{T^2}{\pi^2} \delta_{m,n} + \mathrm{O}(T^{1+\epsilon}(mn)^{\frac{7}{64}} + (mn)^{\frac{1}{4}+\epsilon}),$$

where $\alpha_j = \frac{(\rho_j(1))^2}{\cosh \pi t_j}$ *and* $\delta_{m,n}$ *is the Kronecker symbol.*

Below we will state the unweighted version of Kuznetsov trace formula (see [24, Lemma 3.3])

Theorem 3.2.2. *Let* $k_0 = \frac{11}{155}$, $\eta_0 = \frac{43}{620}$. *Let* m, n *be positive integers. For* $\epsilon > 0$,

$$\sum_{t_j \leq T} \lambda_j(m) \lambda_j(n) = \frac{1}{12} T^2 \delta_{m,n=\square} \frac{\sigma((m,n))}{\sqrt{mn}} + \mathrm{O}_\epsilon \left(T^{2-k_0+\epsilon}(mn)^{\frac{7}{64}} + (mn)^{\eta_0+\epsilon} \right),$$

where $\sigma(l) = \sum_{d|l} d$ *and* $\delta_{l=\square} = 1$ *if* l *is a square and* $\delta_{l=\square} = 0$ *otherwise.*

In [24], Lau-Wang used the above Kuznetsov trace formula to get the estimates

$$\left|\sum_{i=1}^{\Omega(T)} 2\cos m\theta_i - c_m\Omega(T)\right|.$$

Chapter 4

Distribution of gaps between equidistributed sequences

4.1 Distribution of gaps between equidistributed sequences

Let us start with a result that predicts the Weyl limits of gaps of equidistributed families. Henceforth, we will use $[x]$ as fractional part of x.

$$[x] := x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ is the largest integer less than equal to x.

Theorem 4.1.1. *Consider* $\{X_{1_n}\}_{n=1}^{\infty}, \{X_{2_n}\}_{n=1}^{\infty}, \ldots, \{X_{r_n}\}_{n=1}^{\infty}$ *to be collection of* r *sequences of multisets in* $[0, 1]$ *such that* $1 \leq i \leq r$, $\#X_{i_n} \to \infty$ *as* $n \to \infty$. *For every* $m \in \mathbb{Z}$, *let* c_{i_m} *be the* m^{th} *Weyl limit of* X_{i_n} *respectively, that is ,*

$$c_{i_m} := \lim_{n \to \infty} \frac{1}{\#X_{i_n}} \sum_{t \in X_{i_n}} e(mt).$$

Assume that c_{i_m} *exists for each* i. *If* C_m *is the* m^{th}*Weyl limit of the family*

$$\{[x_1 + x_2 + \cdots + x_r],\ x_i \in X_{i_n},\ 1 \leq i \leq r\}$$

that is, for $m \in \mathbb{Z}$,

$$C_m := \lim_{n\to\infty} \frac{1}{\prod_{i=1}^{r} \#X_{i_n}} \sum_{\substack{x_i \in X_{i_n}\\ 1\le i\le r}} e(m[x_1 + x_2 + \cdots + x_r]).$$

Then the Weyl limit

$$C_m = \prod_{i=1}^{r} c_{i_m}. \tag{4.1}$$

Remark 4.1.2. *To consider the gaps, we can take the family*

$$\{[x_1 - x_2 - \cdots - x_r],\ x_i \in X_{i_n},\ 1 \le i \le r\}.$$

in the above theorem.

Further, if we consider the multisets such that if $x \in A_{i_n}$ then $-x \in A_{i_n}$. For simplicity, let us write $A_{i_n} = \{\pm x_i\}$, we get the following corollary.

Corollary 4.1.3. *In particular, if we consider the family $A_{i_n} = \{\pm x_i\} \subseteq [-\frac{1}{2}, \frac{1}{2}]$, then we have*

$$c_{i_m} = \lim_{n\to\infty} \frac{1}{\#A_{i_n}} \sum_{t\in A_{i_n}} e(m[t]).$$

Let C_m be the m^{th} Weyl limit of the family

$$\{[x_1 + x_2 + \cdots + x_r],\ x_i \in A_{i_n},\ 1 \le i \le r\},$$

that is for $m \in \mathbb{Z}$,

$$C_m := \lim_{n\to\infty} \frac{1}{\prod_{i=1}^{r} \#A_{i_n}} \sum_{\substack{x_i \in A_{i_n}\\ 1\le i\le r}} e(m[x_1 + x_2 + \cdots + x_r]).$$

Then the Weyl limit

$$C_m = \prod_{i=1}^{r} c_{i_m} \tag{4.2}$$

Proof. By the definition of the Weyl limits, we know

$$C_m := \lim_{n\to\infty} \frac{1}{\prod_{i=1}^{r} \#X_{i_n}} \sum_{\substack{x_i \in X_{i_n} \\ i=1,2,\ldots,r}} e\left(m\left[x_1 + x_2 + \cdots + x_r\right]\right)$$

Observe that

$$e(mx) = e(m[x]).$$

Using the above observation, we have

$$\begin{aligned} C_m &= \lim_{n\to\infty} \frac{1}{\prod_{i=1}^{r} \#X_{i_n}} \prod_{i=1}^{r} \sum_{x_i \in X_{i_n}} e(mx_i) \\ &= \prod_{i=1}^{r} c_{i_m}. \end{aligned}$$

In particular, for $A_{i_n} = \{\pm x_i\} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ the above calculation follows immediately, that is for any non-zero integer m, if c_{i_m} be the m^{th} Weyl limit of the family $\{\pm x_i,\ x_i \in X_{i_n},\ 1 \leq i \leq r\}$ and C_m be the m^{th} Weyl limit of the family $\{[x_1 + x_2 + \cdots + x_r],\ x_i \in A_{i_n},\ 1 \leq i \leq r\}$, then

$$C_m = \prod_{i=1}^{r} c_{i_m}.$$

$\square$

In the next theorem, we will predict the measure with respect to which the above mentioned family in Theorem 4.1.1 is equidistributed.

Theorem 4.1.4. *Consider* $\{A_{1_n}\}_{n=1}^{\infty}, \{A_{2_n}\}_{n=1}^{\infty}, \ldots, \{A_{r_n}\}_{n=1}^{\infty} \subseteq [-\frac{1}{2}, \frac{1}{2}]$ *to be sequences of multisets such that* $-x_i \in A_{i_n}$ *whenever* $x_i \in A_{i_n}$*, and* $\#A_{i_n} \to \infty$ *as* $n \to \infty$ *for* $i = 1, 2, 3, \ldots, r$*. If* $\{A_{i_n}\}$*,* $i = 1, 2, \ldots, r$ *are equidistributed in* $[-\frac{1}{2}, \frac{1}{2}]$ *with respect to the measure* $F_i(x)dx$ *respectively, where*

$$F_i(x) = \sum_{m=-\infty}^{\infty} c_{i_m} e(mx),$$

then the family

$$\{[x_1 + x_2 + \cdots + x_r],\ x_i \in A_{i_n}\}$$

is equidistributed in $[0,1]$ *with respect to the measure*

$$\mu = F(x)dx,$$

where

$$F(x) = \sum_{m=-\infty}^{\infty} C_m e(mx).$$

Moreover, if

$$\sum_{m=-\infty}^{\infty} |c_{i_m}|^2 < \infty \text{ for all } 1 \leq i \leq r, \tag{4.3}$$

then the above function $F(x)$ *equals*

$$F_1 * F_2 * \cdots * F_r(x),$$

where

$$F_1 * F_2 * \cdots * F_r(y)$$
$$= \int_0^1 \cdots \int_0^1 F_1(y_1)F_2(y_2)\ldots F_r(y - y_1 - y_2 - \cdots - y_{r-1})dy_{r-1}dy_{r-2}\ldots dy_1.$$

Remark 4.1.5. *In general an equidistributed family may not satisfy (4.3).*

Proof. Since $\{A_{i_n}\}$ are equidistributed in $[-\frac{1}{2}, \frac{1}{2}]$, with respect to the measure $F_i(x)dx$, by Theorem 2.1.10

$$\lim_{V\to\infty} \frac{1}{V} \sum_{|m|\leq V} |c_{i_m}|^2 = 0 \text{ for } 1 \leq i \leq r.$$

Observe that, if there exist a subsequence say $\{y_n\}$ of $\{|c_{i_m}|^2\}$, such that $|y_n| \geq 1$ for all n, then

$$\sum_{n\leq V} |y_n|^2 \geq V$$

and hence,

$$\sum_{n\le V} |c_{i_m}|^2 \ge V$$

which is a contradiction. Hence, $|c_{i_m}| < 1$, except possibly for finitely many m.

Now using Theorem 4.1.1 and above fact, we have

$$|C_m| < |c_{i_m}|$$

except possibly for finitely many m and for all $1 \le i \le r$. Hence,

$$\sum_{|m|\le V} |C_m|^2 \le \sum_{|m|\le V} |c_{i_m}|^2.$$

Now taking the limit, we have

$$\lim_{V\to\infty} \frac{1}{V} \sum_{|m|\le V} |C_m|^2 = 0. \tag{4.4}$$

Hence, by Theorem 2.1.10, we can conclude that

$$\{[x_1 + x_2 + \cdots + x_r]\}$$

is equidistributed in $[0, 1]$ with respect to the measure

$$\mu = F(x)dx,$$

where

$$F(x) = \sum_{m=-\infty}^{\infty} C_m e(mx).$$

In addition, if the concerned family satisfies (4.3), that is

$$\sum_{m=-\infty}^{\infty} |c_{i_m}|^2 < \infty, 1 \le i \le r,$$

then $|c_{i_m}| < 1$, except possibly for finitely many m.
Using Theorem 4.1.1 and the last line, we have

$$|C_m|^2 \leq |c_{i_m}|^2 \text{ except possibly finitely many } m.$$

Hence,

$$\sum_{m=-\infty}^{\infty} |C_m|^2 < \infty.$$

Hence, by Theorem 2.2.1, there exist a function $F \in L^2([0,1])$ such that

$$\hat{F}(m) = C_m \text{ for all } \text{ m.}$$

By uniqueness of F, the above function has to be

$$F(x) = \sum_m C_m e(mx).$$

But note that

$$C_m = \prod_{i=1}^{r} c_{i_m} \text{ and } \prod_{i=1}^{r} \hat{F}_i(m) = (F_1 * F_2 * \cdots * F_r)^{\wedge}(m).$$

Hence,

$$F(x) = F_1 * F_2 * \cdots * F_r(x).$$

□

Let $\{A_{i_n}\}_{n=1}^{\infty}$ be equidistributed sequences of finite multisets. If we know the distribution effectively, then our next result will help us to predict the effective equidistribution of family of gaps of equidistributed families.

Theorem 4.1.6. *Let $\{A_{1_n}\}_{n=1}^{\infty}, \{A_{2_n}\}_{n=1}^{\infty}, ..., \{A_{r_n}\}_{n=1}^{\infty}$ be sequences of finite multisets as defined in Theorem 4.1.4. Let $\{A_{i_n}\}_{n=1}^{\infty}, 1 \leq i \leq r$ be equidistributed sequences in $[-\frac{1}{2}, \frac{1}{2}]$ with respect to the measure $F_i(x)dx$, where*

$$F_i(x) = \sum_{m=-\infty}^{\infty} c_{i_m} e(mx)$$

and c_{i_m} are as defined in Theorem 4.1.1.

Consider $\underline{x} = (x_1, x_2, .., x_r)$, $A_{\underline{n}} = A_{1_n} \times A_{2_n} \times \cdots \times A_{r_n}$.

Then, for any positive integer M and any $I = [\alpha, \beta] \subseteq [0, 1]$, we have

$$\left| \frac{1}{\prod_{i=1}^{r}(\# A_{i_n})} \# \left\{ \underline{x} \in A_{\underline{n}} : [x_1 + x_2 + \cdots + x_r] \in I \right\} - \int_I \mu \right|$$

$$\leq \frac{\prod_{i=1}^{r} \# A_{i_n}}{M+1}$$

$$+ \sum_{|m| \leq M} \left(\frac{1}{M+1} + min \left(\beta - \alpha, \frac{1}{\pi |m|} \right) \right) \left(\left| \prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(m x_i) - \prod_{i=1}^{r} \# A_{i_n} c_{i_m} \right| \right),$$

where μ is as defined in Theorem 4.1.4.

For proof of the above theorem see the next chapter.

Chapter 5

A Variant of the Erdös-Turán Inequality

The following variant of Erdös-Turán inequality will be very useful for getting the effective equidistribution of eigenvalues of Hecke operators.

Theorem 5.0.7. *Let $I = [a, b]$ be an interval of $\mathbb{R}$. For $1 \leq i \leq r$, let $\{A_{i_n}\}$ be defined as in Theorem 4.1.4. Let $V_n := \prod_{i=1}^{r} \# A_{i_n}$ and*

$$x_{\underline{n}} := [\pm x_1 \pm x_2 \pm ... \pm x_r], x_i \in A_{i_n}.$$

Let $C_{\underline{m}}$ the Weyl limit defined as in Definition 2.1.8. Let

$$\mu = F(x)dx,$$

where

$$F(x) = \sum_{m=-\infty}^{\infty} C_m e(mx).$$

Define

$$N_I(V_n) := \# \{(x_1, x_2, ..., x_r) \in A_{1_n} \times A_{2_n} \times ... \times A_{r_n} : x_{\underline{n}} \in I\}$$

and

$$D_{I,V_n}(\mu) := |N_I(V_n) - V_n \mu(I)|.$$

Then,

$$D_{I,V_n}(\mu) \leq \frac{V_n \|\mu\|}{M+1}$$

$$+ \sum_{1 \leq |m| \leq M} \left(\frac{1}{M+1} + min\left(b-a, \frac{1}{\pi|m|}\right)\right) \left| \prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(\pm m x_i) - \prod_{i=1}^{r} \# A_{i_n} c_{i_m} \right|,$$

where $||\mu|| = Sup_{x \in [0,1]} |F(x)|$.

Proof. Let χ_I be the characteristic function of the interval I. Then by **(a)** of Section 2.3, we have

$$\sum_{\substack{x_i \in A_{i_n} \\ 1 \leq i \leq r}} S_M^-(x_{\underline{n}}) \leq \sum_{\substack{x_i \in A_{i_n} \\ 1 \leq i \leq r}} \chi_I(x_{\underline{n}}) \leq \sum_{\substack{x_i \in A_{i_n} \\ 1 \leq i \leq r}} S_M^+(x_{\underline{n}}).$$

Now using the Fourier expansion of $S_M^{\pm}(x_{\underline{n}})$, we know that

$$\sum_{\substack{x_i \in A_{i_n} \\ 1 \leq i \leq r}} S_M^{\pm}(x_{\underline{n}}) = \sum_{|m| \leq M} \hat{S}_M^{\pm}(m) \left(\prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(\pm m x_i) \right).$$

Subtracting $\prod_{i=1}^{r} \# A_{i_n} c_{i_m}$ from the inner exponential sums, we get

$$\sum_{\substack{x_i \in A_{i_n} \\ 1 \leq i \leq r}} S_M^{\pm}(x_{\underline{n}}) - \left(\prod_{i=1}^{r} \# A_{i_n} \right) \sum_{|m| \leq M} \hat{S}_M^{\pm}(m) C_m \tag{5.1}$$

$$= \sum_{|m| \leq M} \hat{S}_M^{\pm}(m) \left(\prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(\pm m x_i) - \prod_{i=1}^{r} \# A_{i_n} c_{i_m} \right).$$

Since $c_{i_0} = 1$,

$$\left(\prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(\pm m x_i) - \prod_{i=1}^{r} \# A_{i_n} c_{i_m} \right) = 0 \text{ for } m = 0.$$

Taking the absolute value on both sides we get

$$\left| \sum_{\substack{x_i \in A_{i_n} \\ 1 \le i \le r}} S_M^{\pm}(x_{\underline{n}}) - \prod_{i=1}^{r} \#A_{i_n} \sum_{|m| \le M} \hat{S}_M^{\pm}(m) C_m \right|$$

$$\le \sum_{|m| \le M} |\hat{S}_M^{\pm}(m)| \left| \prod_{i=1}^{r} \sum_{x_i \in A_{i_n}} e(\pm m x_i) - \prod_{i=1}^{r} \#A_{i_n} c_{i_m} \right|.$$

Now let us consider the sum

$$\sum_{|m| \le M} \hat{S}_M^{\pm}(m) C_m.$$

Since for all $|m| > M$, $\hat{S}_M^{\pm}(m) = 0$, without loss of generality let us extend the range of sums to $\mathbb{Z}$. Then, we have

$$\sum_{m} \hat{S}_M^{\pm}(m) C_m = \sum_{m} C_m \int_0^1 S_M^{\pm}(x) e(-mx) dx.$$

Now interchanging the sum and integral and using the definition of μ, the above quantity equals

$$\int_0^1 S_M^{\pm}(x) d\mu.$$

Using **(c)** of Section 2.3, we have

$$\left| \int_0^1 (S_M^{\pm}(x) - \chi_I(x)) d\mu \right| \le \frac{\|\mu\|}{M+1}. \tag{5.2}$$

Note that

$$D_{I,V_n}(\mu) = \left| N_I(V) - \left(\prod_{i=1}^{r} A_{i_n} \right) \mu(I) \right|$$

$$= \left| \sum_{\substack{x_i \in A_{i_r} \\ 1 \le i \le r}} \chi_I(x_{\underline{n}}) - \prod_{i=1}^{r} (\#A_{i_n}) \int_0^1 \chi_I(x) d\mu \right|.$$

Now adding and subtracting $\prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu$ to the above expression, we get

$$= \left|\prod_{i=1}^{r}\sum_{x_i\in A_{i_n}}\chi_I(x_{\underline{n}}) - \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu\right.$$

$$\left. + \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu - \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 \chi_I(x)d\mu\right|.$$

Using triangle inequality, we get

$$D_{I,V}(\mu) \leq \left|\prod_{i=1}^{r}\sum_{x_i\in A_{i_n}}\chi_I(x_{\underline{n}}) - \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu\right|$$

$$+ \left|\prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu - \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 \chi_I(x)d\mu\right|$$

$$\leq \left|\prod_{i=1}^{r}\sum_{x_i\in A_{i_n}}\chi_I(x_{\underline{n}}) - \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu\right|$$

$$+ \left|\prod_{i=1}^{r}(\#A_{i_n})\int_0^1 (S_M^+(x) - \chi_I(x))d\mu\right|.$$

Now using (5.2), the above is

$$\leq \frac{\prod_{i=1}^{r}(\#A_{i_n}\|\mu\|)}{M+1} + \left|\prod_{i=1}^{r}\sum_{x_i\in A_{i_n}}\chi_I(x_{\underline{n}}) - \prod_{i=1}^{r}(\#A_{i_n})\int_0^1 S_M^+(x)d\mu\right|.$$

Using **(a)** of Section 2.3, we have

$$\leq \frac{\prod_{i=1}^{r}(\#A_{i_n}\|\mu\|)}{M+1}$$

$$+ \left|\prod_{i=1}^{r}\sum_{x_i\in A_{i_n}} S_M^+(x_{\underline{n}}) - \prod_{i=1}^{r}\#A_{i_n}\int_0^1 S_M^+(x)d\mu\right|.$$

Using (5.1) and the fact that

$$\int_0^1 S_M^+(x)d\mu = \sum_{|m|\le M} \hat{S}_M^+(m)C_m,$$

$D_{I,V}(\mu)$ is

$$\le \frac{\prod_{i=1}^r (\#A_{i_n}\|\mu\|)}{M+1} + \sum_{|m|\le M} \hat{S}_M^{\pm}(m)\left(\prod_{i=1}^r \sum_{x_i\in A_{i_n}} e(\pm m x_i) - \prod_{i=1}^r \#A_{i_n} c_{i_m}\right).$$

Now using **(b)** of Section 2.3, we have

$$|D_{I,V}(\mu)| \le \frac{\prod_{i=1}^r (\#A_{i_n}\|\mu\|)}{M+1}$$

$$+ \sum_{|m|\le M} \frac{1}{M+1} + min\left(b-a, \frac{1}{\pi|m|}\right)\left|\prod_{i=1}^r \sum_{x_i\in A_{i_n}} e(m x_i) - \prod_{i=1}^r (\#A_{i_n})C_m\right|.$$

$\square$

Chapter 6

Distribution of gaps between eigenangles of Hecke operators

In the following sections we will state several applications of Theorem 4.1.1, 4.1.4 and 4.1.6.

6.1 Distribution of gaps between eigenangles for cusp forms

In this section, we use the notations from the Section 2.5. In this section we study the distribution of gaps between eigenangles of Hecke operators acting on the space $S(N,k)$. In this case, let $n = s(N,k)$ and for all $1 \leq j \leq r$,

$$A_{j_n} = \left\{ \frac{\theta_{p,i,N}}{2\pi},\ 1 \leq i \leq s(N,k) \right\}.$$

Note that

$$\#A_{j_n} = n = s(N,k).$$

So as $N+k \to \infty$, we have $n \to \infty$ that is we have infinite number of multi sets. Since each sets are same and we are going to study the distributions of gaps between the elements of the multisets A_{j_n}, it is better to introduce another subscript j to i that is we will write

$$A_{j_n} = \left\{ \frac{\theta_{p,i_j,N}}{2\pi},\ 1 \leq i \leq s(N,k) \right\}.$$

The subscript j to i is for the convenience of the reader.

Theorem 6.1.1. *Let N be a positive integer and p a prime not dividing N. For an interval $[\alpha, \beta] \subseteq [0,1]$, $r \leq s(N,k)$,*

$$\frac{1}{s(N,k)^r} \# \left\{ 1 \leq i_1, ..., i_r \leq s(N,k) : \left[\frac{\pm\theta_{p,i_1,N} \pm \cdots \pm \theta_{p,i_r,N}}{2\pi} \right] \in [\alpha, \beta] \right\}$$
$$= \int_{[\alpha,\beta]} \nu_p + \mathrm{O}\left(\frac{\log\, p}{\log\, kN} \right),$$

where

$$\nu_p = F(x) * F(x) * \cdots * F(x) dx,$$

$\theta_{p,i,N} \in [0,\pi]$ such that $a_{p,i,N} = 2\cos\theta_{p,i,N}$ and

$$F(x) = 4(p+1) \frac{\sin^2 2\pi x}{\left(p^{\frac{1}{2}} + p^{-\frac{1}{2}}\right)^2 - \cos^2 2\pi x}.$$

Here the implied constant is effectively computable.

Remark 6.1.2. *For $r = 2$, the above mentioned measure is*

$$\nu_p = \frac{2(1+\cos 4\pi x)(1-\frac{1}{p^2}) + \frac{4}{p}(\frac{1}{p^2} - \cos 4\pi x)}{1 + \frac{1}{p^4} - \frac{2}{p^2}\cos 4\pi x} dx.$$

The following theorem can be deduced from Theorem 6.1.1.

Theorem 6.1.3. *For any $\alpha \in [0,1]$,*

$$\# \left\{ 1 \leq i_1, i_2, \ldots, i_r \leq s(N,k) : [\pm\theta_{p,i_1,N} \pm \theta_{p,i_2,N} \pm \cdots \pm \theta_{p,i_r,N}] = \alpha \right\}$$
$$= \mathrm{O}\left((s(N,k))^r \left(\frac{\log\, p}{\log\, kN} \right) \right),$$

where the implied constant is effectively computable.

In the above theorem for $r = 2$, we have an interesting consequence, namely

Theorem 6.1.4. *For any* $\alpha \in [0, 1]$,

$$\#\left\{(i_1, i_2) : \left[\frac{\pm\theta_{p,i_1,N} \pm \theta_{p,i_2,N}}{2\pi}\right] = \alpha\right\} = \mathrm{O}\left((s(N,k))^2 \left(\frac{\log p}{\log kN}\right)\right).$$

In particular, taking $\alpha = 0$ and using the fact that

$$\#\left\{(i_1, i_2) : \left(\frac{\pm\theta_{p,i_1,N} \pm \theta_{p,i_2,N}}{2\pi}\right) = \alpha\right\} \leq \#\left\{(i_1, i_2) : \left\{\frac{\pm\theta_{p,i_1,N} \pm \theta_{p,i_2,N}}{2\pi}\right\} = \alpha\right\},$$

we have the following corollary which recovers Theorem 1 in [33].

Corollary 6.1.5.

$$\begin{aligned} \#\{(i_1, i_2), 1 \leq i_1, i_2 \leq s(N,k) : (\theta_{p,i_1,N} \pm \theta_{p,i_2,N}) = 0\} \\ = \mathrm{O}\left((s(N,k))^2 \left(\frac{\log p}{\log kN}\right)\right). \end{aligned}$$

Using the results from Chapter 2, we know that $\left\{\frac{\pm\theta_{p,i_j,N}}{2\pi}, 1 \leq j \leq r\right\}$ is equidistributed in $[-\frac{1}{2}, \frac{1}{2}]$ with respect to the measure

$$\mu_p = F(x)dx,$$

where

$$F(x) = 4(p+1)\frac{\sin^2 2\pi x}{\left(p^{\frac{1}{2}} + p^{-\frac{1}{2}}\right)^2 - 4\cos^2 2\pi x}. \tag{6.1}$$

So using Theorem 4.2, we can conclude that, the concerned family

$$\left[\frac{\pm\theta_{p,i_1,N} \pm \cdots \pm \theta_{p,i_r,N}}{2\pi}\right]$$

is equidistributed in $[0, 1]$ with respect to the measure $\underbrace{F * F \cdots * F}_{r \text{ times}} dx$.

To proceed further, let us prove the following proposition:

Proposition 6.1.6. *For any positive integer m,*

$$\left| \prod_{j=1}^{r} \sum_{\substack{i \\ 1 \le i \le s(N,k)}} 2\cos m\theta_{p,i_j,N} - C_m(s(N,k))^r \right|$$

$$\ll p^{\frac{3rm}{2}} m^r (\log\, p)^r 2^{r\nu(N)} + (\sqrt{N} d(N))^r,$$

where $\nu(N)$ is the number of distinct prime factor of N and $d(N)$ is the divisor function. The constant is effectively computable.

Proof. Estimating each term of the Eichler-Selberg trace formula, Murty and Sinha (see [32] Theorem 18 and (11)) prove the following: For any positive integer m, let c_{i_m} be the Weyl limits of the family

$$\{\pm\theta_{p,i_j,N}, 1 \le j \le s(N,k)\}.$$

For $1 \le i \le r$, the Weyl limits c_{i_m} are given by

$$c_{i_m} = \begin{cases} 1 & \text{if } m = 0 \\ \left(\frac{1}{p^{\frac{m}{2}}} - \frac{1}{p^{\frac{m-2}{2}}}\right) & \text{if } m \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{6.2}$$

Moreover, for $m \ge 1$ and $1 \le j \le r$,

$$\left| \sum_{i=1}^{s(N,k)} (2\cos m\theta_{p,i_j,N} - c_{i_m}(s(N,k)) \right| \ll p^{\frac{3m}{2}} 2^{\nu(N)} \log\, p^m + d(N)\sqrt{N}.$$

Using the fact that

$$2\cos m\theta_{p,i_j,N} = X_m(2\cos\theta_{p,i_j,N}) - X_{m-2}(2\cos\theta_{p,i_j,N}), \;\; m \ge 2, \tag{6.3}$$

where

$$X_m(2\cos\theta) = \frac{\sin(m+1)\theta}{\sin\theta}$$

we have

$$\prod_{j=1}^{r} \sum_{\substack{i \\ 1 \leq i \leq s(N,k)}} 2\cos m\theta_{p,i_j,N} = (TrT'_{p^m} - TrT'_{p^{m-2}})^r.$$

Now, using the estimates of Eichler-Selberg trace formula on page 696 of [32] and the well-known inequality

$$(a-b)^r \leq r(a^r + b^r), \tag{6.4}$$

we have

$$(TrT'_{p^m} - TrT'_{p^{m-2}})^r \ll r\left(\left(\left(\frac{k-1}{12}\right)\psi(N)\left(\frac{1}{p^{\frac{m}{2}}} - \frac{1}{p^{\frac{m-1}{2}}}\right)\right)^r + \left(p^{\frac{3m}{2}} m(\log\, p)2^{\nu(N)} + \sqrt{N}d(N)\right)^r.\right.$$

Using (6.4) and the fact that $s(N,k) = \mathrm{O}(kN)$, we get

$$\left|\prod_{j=1}^{r} \sum_{\substack{i \\ 1 \leq i \leq s(N,k)}} 2\cos m\theta_{p,i_j,N} - (c_m(s(N,k)))^r\right|$$
$$\ll p^{\frac{3rm}{2}} m^r (\log\, p)^r 2^{r\nu(N)} + (\sqrt{N}d(N))^r.$$

□

Proof of Theorem 6.1.1 Using Theorem 4.1.6, the concerned quantity is

$$\ll \frac{(s(N,k))^r}{M+1} + p^{\frac{3rM}{2}} M^r (\log\, p)^r 2^{r\nu(N)} + (\sqrt{N}d(N))^r.$$

Now we want to choose M such that

$$\frac{(s(N,k))^r}{M+1} \sim p^{\frac{3rM}{2}}.$$

And that can be achieved by choosing $M = c\frac{\log\, kN}{\log\, p}$ for a sufficiently small constant c. The result then follows.

6.2 Distribution of gaps between eigenangles for primitive Maass forms

For notations used in this section, the reader may see Section 2.6. In this case, for $1 \leq i \leq r$,

$$A_{j_i} = \left\{ 0 < t_j \leq T : \frac{\pm\theta_j}{2\pi} \right\}.$$

Note the cardinality of each A_{j_i} are $\Omega(T)$. Since each multi sets are same, for our convenience let us introduce a suffix to j that is

$$A_{j_i} = \left\{ 0 < t_{j_i} \leq T : \frac{\pm\theta_{j_i}}{2\pi} \right\}.$$

Theorem 6.2.1. *There exists a small constant $\delta > 0$ such that for all large T,*

$$\frac{1}{(\Omega(T))^r} \# \left\{ 0 < t_{j_i} \leq T, 1 \leq i \leq r : \left[\frac{\pm\theta_{j_1}(p) \pm \theta_{j_2}(p) \cdots \pm \theta_{j_r}(p)}{2\pi} \right] \in I \right\}$$

$$= \int_I \nu_p dx + \mathrm{O}\left(\frac{\log\, p}{\log\, T} \right)$$

holds uniformly for integers $1 \leq r \leq \Omega(T)$ and for primes p satisfying

$$r \log\, p \leq \delta \log\, T,$$

and uniformly for any interval $I = [a, b] \subseteq [0, 1]$. Here

$$\nu_p = \underbrace{F_p(x) * F_p(x) * \cdots * F_p}_{r \text{ times}}(x),$$

and $F_p(x)$ is as defined in (6.1). Here the implied constant is effectively computable and $\theta_{j_i}(p)$ are called as eigenangles of Hecke operators acting on the space of primitive Maass forms.

As a consequence of the above Theorem 6.2.1, we have the following result

Theorem 6.2.2. *For any $\alpha \in [0,1]$,*

$$\#\left\{0 < t_{j_i} \leq T, 1 \leq i \leq r : \left[\frac{\pm\theta_{j_1}(p) \pm \theta_{j_2}(p) \cdots \pm \theta_{j_r}(p)}{2\pi}\right] = \alpha\right\}$$
$$= \mathrm{O}\left((\Omega(T))^r \left(\frac{\log p}{\log T}\right)\right).$$

Corollary 6.2.3. *For any $\alpha = 0$ and $r = 2$,*

$$\#\left\{0 < t_{j_i} \leq T, 1 \leq i \leq r : \left[\frac{\pm\theta_{j_1}(p) \pm \theta_{j_2}(p)}{2\pi}\right] = 0\right\}$$
$$= \mathrm{O}\left((\Omega(T))^r \left(\frac{\log p}{\log T}\right)\right).$$

Remark 6.2.4. *From above corollary, we can derive similar result like Corollary 6.1.5.*

In order to prove Theorem 6.2.1, we first need the following result.

Proposition 6.2.5. *For $m \geq 1,\ 1 \leq i \leq r \leq \Omega(T)$, $0 < k < \frac{11}{155}$ and $\eta > \frac{43}{620}$,*

$$\left|\prod_{j=1}^{r} \sum_{i=1}^{\Omega(T)} \cos m\theta_{j_i}(p) - \Omega(T)^r C_m\right|$$
$$\ll (T^{2-k} p^{m\eta})^r,$$

where the implied constant depends only on η.

Proof. The following result is a special case of [24], Lemma 4.1:
For any positive integer m, let c_m be the Weyl limits of the family

$$\left\{\frac{\pm\theta_i(p)}{2\pi}, 1 \leq i \leq \Omega(T)\right\}.$$

Then the Weyl limits c_m are given by

$$c_m = \begin{cases} 1 & \text{if } m = 0 \\ \left(\frac{1}{p^{\frac{m}{2}}} - \frac{1}{p^{\frac{m-2}{2}}}\right) & \text{if } m \text{ is even} \\ 0 & \text{otherwise.} \end{cases} \tag{6.5}$$

Moreover, for $m \geq 1$,

$$\left| \sum_{i=1}^{\Omega(T)} 2\cos m\theta_i - c_m \Omega(T) \right| \ll T^{2-k} p^{m\eta}.$$

Lau and Wang [24] proved the above result using Kuznetsov trace formula. Now proceeding as in the proof of Propositions 6.1.6, the result follows. $\square$

Proof. From Sarnak's theorem (see [36, Theorem 1.2]), we can conclude that the family $\left\{ \frac{\pm\theta_i(p)}{2\pi}, 1 \leq i \leq r \right\}$ is equidistributed in $[-\frac{1}{2}, \frac{1}{2}]$ with respect to $F_p(x)dx$.

Now proceeding just like proof of Theorems 6.1.1 and choosing $M = c\frac{\log p}{\log T}$, we have Theorem 6.2.1. $\square$

Chapter 7

Distinguishing Hecke eigenforms

The content of this chapter is joint work with M. Ram Murty (see [30]). This work is independent of the previous chapters. In this chapter, we will follow closely the presentation of [30].

7.1 Introduction

Let $E = \{(X, Y) : Y^2 = X^3 + AX + B, A, B \in \mathbb{Z}\}$ be an elliptic curve. For any prime p, let $|E(\mathbb{F}_p)|$ be the number of solutions of

$$Y^2 = X^3 + AX + B \ \ (mod\, p)$$

together with the point at ∞.

Definition 7.1.1. *A set of primes $\mathcal{P}$ is said to have positive upper density if*

$$\limsup_{x\to\infty} \frac{\#\{p \in \mathcal{P} : p \leq x\}}{\#\{p \leq x\}} > 0.$$

Recently, using Galois theory, Kulkarni, Patankar and Rajan [23] proved that if E_1 and E_2 are two elliptic curves defined over a number field K with at least one of them not of CM type such that

$$\#E_1(\mathbb{F}_p) = \#E_2(\mathbb{F}_p),$$

for a set of primes of positive lower density, then E_1 and E_2 become isogenous after base change. Given two normalized Hecke eigenforms f_1 and f_2 of weights k_1, k_2 and levels N_1, N_2 respectively, let

$$f_i(z) = \sum_{n=1}^{\infty} a_n(f_i) n^{(k_i-1)/2} q^n, \qquad q = e^{2\pi i z}, \qquad i = 1, 2, \tag{7.1}$$

be the Fourier expansions at infinity. By the work of Deligne [14], we can write

$$a_n(f_i) = 2\cos\ \theta_n^{(i)},\ \theta_n^{(i)} \in [0, \pi].$$

Definition 7.1.2. *A Dirichlet character is a group homomorphism from $(\mathbb{Z}/n\mathbb{Z})^*$ to $\mathbb{C}^*$, where $(\mathbb{Z}/n\mathbb{Z})^*$ is the group of units of $(\mathbb{Z}/n\mathbb{Z})$ and $\mathbb{C}^*$ is the set of non zero complex numbers.*

Definition 7.1.3. *An eigenform f is said to be of CM type if there exists an imaginary quadratic field $K/\mathbb{Q}$ such that $a_p(f) = 0$ whenever p is inert in K.*

In this chapter our goal is to prove the following theorem:

Theorem 7.1.4. *Suppose that at least one of f_1, f_2 is not of CM type. If*

$$\limsup_{x\to\infty} \frac{\#\{p \le x :\ a_p(f_1) = a_p(f_2)\}}{x/\log x} > 0,$$

then $f_1 = f_2 \otimes \chi$ for some Dirichlet character χ.

Using the celebrated work on the modularity of elliptic curves over $\mathbb{Q}$ due to Wiles [53], Breuil, Conrad, F. Diamond and Taylor [7], we get the result of Kulkarni, Patankar and Rajan [23] for elliptic curves over $\mathbb{Q}$.

7.2 Preliminaries

In this section, we collect the relevant facts that will be needed in various stages of our proof.

Proposition 7.2.1. *If f_1, f_2 are normalized Hecke eigenforms, with at least one not of CM type, such that $f_1 \neq f_2 \otimes \chi$ for some Dirichlet character χ, then for any two positive integers m, n,*

$$\sum_{p \leq x} \frac{\sin(m+1)\theta_p^{(1)}}{\sin \theta_p^{(1)}} \frac{\sin(n+1)\theta_p^{(2)}}{\sin \theta_p^{(2)}} = o(x/\log x),$$

as x tends to infinity. Here the summation is over primes.

Proof. This is essentially Theorem 2.4 of [17] combined with the standard Tauberian theorem. However, for the sake of completeness, we give a proof in section 7.3. □

Proposition 7.2.2. *Let $0 < \delta < \pi$. Let $f_\delta(x)$ be the "tent" function defined on $[-\pi, \pi]$ be given by*

$$f_\delta(x) = \begin{cases} 1 - |x|/\delta & \textit{if } |x| \leq \delta, \\ 0 & \textit{if } |x| > \delta. \end{cases}$$

Then, for any $M \geq 1$, we have

$$f_\delta(x) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{M} \frac{1 - \cos n\delta}{\pi n^2 \delta} \cos nx + O\left(\frac{1}{M\delta}\right),$$

where the implied constant is absolute.

Proof. To prove the above proposition, let us compute the Fourier expansions of $f_\delta(x)$. The Fourier expansions of $f_\delta(x)$ is given by

$$\frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty} \frac{1 - \cos n\delta}{\pi n^2 \delta} \cos nx.$$

For any $n \neq 0$,

$$\hat{f_\delta}(n) = \frac{1}{2\pi} \int_{-\delta}^{\delta} \left(1 - \frac{|x|}{\delta} e^{-inx}\right) dx$$

$$= \frac{1}{2\pi}\int_{-\delta}^{0}\left(1+\frac{x}{\delta}\right)e^{-inx}dx + \frac{1}{2\pi}\int_{0}^{\delta}\left(1-\frac{x}{\delta}\right)e^{-inx}dx.$$

Replacing x by $-x$ in the first integral, the above equals

$$\frac{1}{2\pi}\int_{0}^{\delta}\left(1-\frac{x}{\delta}\right)(2\cos nx)dx = \frac{\sin n\delta}{\pi n} - \frac{1}{\pi\delta}\left(\delta\frac{\sin n\delta}{n} + \frac{\cos n\delta}{n^2} - \frac{\cos n0}{n^2}\right) \tag{7.2}$$

$$= \left(\frac{1-\cos n\delta}{n^2\pi\delta}\right).$$

Now

$$\sum_{n=-\infty}^{\infty}\left(\hat{f}(n)e^{inx} + \hat{f}(-n)e^{-inx}\right) = \frac{\delta}{2\pi}\sum_{n=1}^{\infty}\left(\frac{1-\cos n\delta}{n^2\pi\delta}2\cos nx\right).$$

Note that the above sum equals:

$$\sum_{n=1}^{\infty}\left(\frac{2\cos nx}{n^2\pi\delta} - \frac{\cos n\delta\, 2\cos nx}{n^2\pi\delta}\right) = \sum_{n=1}^{\infty}\frac{2\cos nx}{n^2\pi\delta} - \sum_{n=1}^{\infty}\frac{\cos n\delta\, 2\cos nx}{n^2\pi\delta}. \tag{7.3}$$

Let us estimate the truncation of the first sum on the right-hand side of (7.3) at M, namely

$$\sum_{n=M+1}^{\infty}\frac{2\cos nx}{n^2\pi\delta}.$$

Since, $\cos nx \leq 1$, the above sum is less than or equal to

$$\sum_{n=M+1}^{\infty}\frac{2}{n^2\delta}.$$

We will use Euler Maclaurin summation formula (see [27, Theorem 2.1.9]) to estimate the above sum. Choosing $f(x) = \frac{1}{\delta x^2}$ in the Euler Maclaurin summation formula, we have

$$\sum_{n=M+1}^{\infty}\frac{2}{n^2\pi\delta} = \int_{M+1}^{\infty}\frac{1}{\delta x^2} = \frac{2}{\delta(M+1)}.$$

Hence

$$\sum_{n=M+1}^{\infty} \frac{2\cos nx}{n^2\pi\delta} = \mathrm{O}\left(\frac{1}{M\delta}\right).$$

Similarly, the second sum also will be $\mathrm{O}\left(\frac{1}{M\delta}\right)$ after truncating at M. So we can conclude that the tail terms will be $\mathrm{O}\left(\frac{1}{M\delta}\right)$ and we get the required result.

□

Proof of Theorem 7.1.4 Let us assume that $f_1 \neq \chi \otimes f_2$ for any character χ. Let $\pi > \delta > 0$ and take $f_\delta(x)$ as in Proposition 7.2.2. Note that if $\theta_p^{(1)} = \theta_p^{(2)}$, then

$$f_\delta(\theta_p^{(1)} - \theta_p^{(2)}) = 1.$$

Hence,

$$\#\{p \leq x : \theta_p^{(1)} = \theta_p^{(2)}\} \leq \sum_{p \leq x} f_\delta(\theta_p^{(1)} - \theta_p^{(2)}) + f_\delta(\theta_p^{(1)} + \theta_p^{(2)}).$$

By Proposition 7.2.2, the right hand side is equal to

$$\frac{\delta\pi(x)}{\pi} + 4\sum_{n=1}^{M} \frac{1-\cos n\delta}{\pi n^2\delta} \sum_{p \leq x} \cos n\theta_p^{(1)} \cos n\theta_p^{(2)} + O\left(\frac{\pi(x)}{M\delta}\right)$$

upon using the trigonometric identity

$$\cos(A+B) + \cos(A-B) = 2\cos A \cos B.$$

The inner sum corresponding to $n = 1$ is

$$\sum_{p \leq x} \cos\theta_p^{(1)} \cos\theta_p^{(2)}.$$

In [26, Lemma 5], Murty has showed that if $f_i, i = 1, 2$ is defined as in (7.1) and $N_1 = N_2 = 1$, then there there exists a v such that $h = x^\theta, v < \theta < 1$,

$$\sum_{x \leq p \leq x+h} a_p(f_1)a_p(f_2) = \mathrm{o}(h).$$

In particular, writing $a_p(f_i) = 2\cos\theta_p^{(i)}$, we have

$$\sum_{p\leq x} \cos\theta_p^{(1)} \cos\theta_p^{(2)} = \mathrm{o}(x).$$

To treat $n \geq 2$, we use the identity

$$2\cos n\theta = \frac{\sin(n+1)\theta}{\sin\theta} - \frac{\sin(n-1)\theta}{\sin\theta}, \tag{7.4}$$

so that we can rewrite our sum as

$$\sum_{n=2}^{M} \frac{1-\cos n\delta}{\pi n^2\delta}$$

$$\times \sum_{p\leq x} \left(\frac{\sin(n+1)\theta_p^{(1)}}{\sin\theta_p^{(1)}} - \frac{\sin(n-1)\theta_p^{(1)}}{\sin\theta_p^{(1)}} \right) \left(\frac{\sin(n+1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} - \frac{\sin(n-1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} \right).$$

Dividing by $\pi(x)$ and taking lim sup as x tends to infinity, we obtain upon applying Proposition 7.2.1, that the inner sums go to zero. Thus, we obtain

$$\limsup_{x\to\infty} \frac{\#\{p \leq x : \theta_p^{(1)} = \theta_p^{(2)}\}}{\pi(x)} \leq \frac{\delta}{\pi} + O\left(\frac{1}{\delta M}\right).$$

Letting M tend to infinity, we see that this density can be made arbitrarily small since δ is arbitrary. This contradicts our hypothesis and completes the proof. $\square$

7.3 Joint Sato-Tate distribution for two Hecke eigenforms

For the sake of completeness, let us review the Sato-Tate conjecture. For detail see Section 2.8.

Conjecture: If $f(p)$ is a p^{th} normalized Hecke eigenvalue then the family

$\{f(p)\}$ is equidistributed in $[-2, 2]$ as $p \to \infty$ with respect to the Sato-Tate measure

$$d\mu_\infty = \frac{1}{2\pi}\sqrt{4 - x^2}dx.$$

There are already several readable expositions of the proof of the Sato-Tate conjecture deduced from the potential automorphy of symmetric power L-functions (i.e the symmetric power L-function will be automorphic after a suitable change of the base field) (see for example, [17] and section 6 of Chapter 12 of [29]). What has not been explicitly presented in the literature is that the joint Sato-Tate distribution holds for two Hecke eigenforms, provided that one is not the Dirichlet twist of the other. In this section we prove the following theorem:

Theorem 7.3.1. *For any rectangle $I \subset [-2, 2]^2$,*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : (f_1(p), f_2(p)) \in I \textit{ where } f_1(p) \textit{ is not a character multiple of } f_2(p)\}$$

$$= \int_I d\mu \times d\mu,$$

where $d\mu$ is the Sato-Tate measure.

Assume that $f_1(p)$ is not a character multiple of $f_2(p)$ and $f_i(p) = 2\cos\theta_i(p)$, $\theta_i(p) \in [0, \pi]$. By a change of variable, the above theorem is equivalent to the following:

For any rectangle $I \subset [0, \pi]^2$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : (\theta_1(p), \theta_2(p)) \in I\}$$

$$= \int_I d\nu \times d\nu,$$

where $\nu = \frac{2}{\pi}\sin^2\theta$. Note that if $\theta_i(p)$ is in the family then $-\theta_i(p)$ is in the family. Now to prove the above theorem, we will compute the Weyl limits of the family $\{(\pm\theta_1(p), \pm\theta_2(p))\}$ and that is for any two positive integer m and n,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} e(\pm m\theta_1(p) \pm n\theta_2(p)) = \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} 2\cos m\theta_1(p) 2\cos n\theta_2(p).$$

Using (7.4), the above equals

$$= \sum_{p\leq x} \left(\frac{\sin(m+1)\theta_p^{(1)}}{\sin\theta_p^{(1)}} - \frac{\sin(m-1)\theta_p^{(1)}}{\sin\theta_p^{(1)}} \right) \left(\frac{\sin(n+1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} - \frac{\sin(n-1)\theta_p^{(2)}}{\sin\theta_p^{(2)}} \right).$$

From the above discussion, it is clear that to prove Theorem 7.3.1 it is sufficient to prove Proposition 7.2.1. To do so, according to our need, let us review some basic representation theory. Let G be a finite group and V be a complex vector space. For details, the reader may consult [8] and [41].

Definition 7.3.2. *A group homomorphism π of G to $GL(V)$ is said to be a **representation** (π, V) of G, where $GL(V)$ is the group of isomorphisms of V on to itself.*

Definition 7.3.3. *A **character** χ associated to a representation (π, V) of G is defined by*

$$\chi_\pi(g) = Tr_V(\pi(g)).$$

Let W be a subspace of V. The subspace W is said to be G**-invariant** if

$$\pi(g)W \subset W \text{ for all } g \in G.$$

Definition 7.3.4. *A representation of G on W, $\pi_W : G \to GL(W)$ is said to be a **subrepresentation** of (π, V) if*

$$\pi_W(g) = \pi(g)|W \text{ for all } g \in G.$$

Definition 7.3.5. *Any representation (π, V) of G is said to be an **irreducible representation** if the only G-invariant subspaces of G are $\{0\}$ and V.*

Definition 7.3.6. *Any character of an irreducible representation is called* ***irreducible character****.*

A linear combination of characters is known as a **generalised character**. A group G is said to be an **elementary group** if it is a direct product of cyclic groups and groups whose order is a power of a prime.

Definition 7.3.7. *Let G be a topological group and V be an Hilbert space. A* ***unitary representation*** *of G is an isometric action of G on V so that the action map $G \times V \to V$ is continuous.*

To define automorphic representations, we will closely follow [8, page 92] Let F be a number field and $\mathbb{A} = \mathbb{A}_F$ be the adele ring of F. Let $Z_{\mathbb{A}}$ be the group of scalar matrices with entries in the group $\mathbb{A}^*$, where $\mathbb{A}^*$ be the set of units of $\mathbb{A}$. The group $GL_n(F)$ is a discrete subgroup of $GL_n(\mathbb{A})$ and the quotient $GL_n(F)/GL_n(\mathbb{A})$, and the quotient $Z_{\mathbb{A}}GL_n(F)/GL_n(\mathbb{A})$ has finite volume. Let w be a unitary character of $\mathbb{A}^*/F^*$ i.e $|w(z)| = 1$ for all $z \in \mathbb{A}^*$. Let $L^2\left(GL_n(F)/GL_n(\mathbb{A}), w\right)$ be the space of measurable functions f on $GL_n(\mathbb{A})$ such that

$$f\left(\begin{pmatrix} z & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & z \end{pmatrix} g\right) = w(z)f(g) \text{ for } z \in \mathbb{A}^*,$$

and such that

$$\int_{Z_{\mathbb{A}}GL_n(F)/GL_n(\mathbb{A})} |f(g)|^2 dg < \infty,$$

where dg is the Haar measure.

Definition 7.3.8. *A representation (π, V) is said to be an* ***automorphic representation*** *if π is an irreducible unitary representation of $GL_n(\mathbb{A})$ that occurs in $L^2\left(GL_n(F)/GL_n(\mathbb{A}), w\right)$.*

Theorem 7.3.9 (Brauer). *Let G be a finite group, and let χ be a generalized character. Then, there exist elementary subgroups $E_1, E_2, ...$ and irreducible characters $\psi_1, \psi_2, ...$ of the E_i and integers $a_i \in \mathbb{Z}$ such that*

$$\chi = \sum a_i \chi_i^G.$$

Let (π, V) and $(\sigma, V^{'})$ be two representations of the group G. Let us denote the set of G-invariant maps from V to $V^{'}$ as $Hom_G(\pi, \sigma)$.

Theorem 7.3.10 (Frobenius reciprocity law). *Let H be a subgroup of a finite group G and σ be a representation of H. Let π be a representation of G. Then*

$$Hom_G\left(\pi, Ind_H^G(\sigma)\right) \equiv Hom_H\left(\pi|_H, \sigma\right),$$

and

$$Hom_G\left(Ind_H^G(\sigma), \pi\right) \equiv Hom_H\left(\sigma, \pi|_H\right),$$

where Ind_H^G is the induced representation of H on G.

The following theorem can be found in [27] or [49]:

Theorem 7.3.11. *[Tauberian theorem] Let $F(s) = \sum_{n=0}^{\infty} \frac{a_n}{n^s}$ be a Dirichlet series with non negative real coefficients a_n such that*
(a)$F(s)$ is absolutely convergent for $\mathrm{Re}(s) > 1$.

(b) $F(s)$ can be extends meromorphically to the region $\mathrm{Re}(s) \geq 1$ *with a possible pole at $s = 1$ with residue R.*

(c) $\sum_{n \leq x} a_n = \mathrm{O}(x)$. Then

$$\sum_{n \leq x} a_n = Rx + \mathrm{o}(x).$$

we now need to understand the ℓ-adic Galois representation attached to a cusp form of level N and weight k. Let ℓ be a prime. K be a finite extension of $\mathbb{Q}_l$. Let V be finite-dimensional K vector space.

Definition 7.3.12. *A continuous representation $\rho : Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_K(V)$ is said to be a ℓ-**adic** representation of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$.*

Definition 7.3.13. *The normal subgroup $I_p = \bar{\mathbb{Q}}/\mathbb{Q}_p(\zeta_m : p \nmid m)$ defined by any intermediate extensions of $\mathbb{Q}_p \subset \mathbb{Q}_p(\zeta_m : p \nmid m) \subset \bar{\mathbb{Q}}$ is called the **inertia subgroup**.*

Definition 7.3.14. *An ℓ-adic representation is said to be **unramified** at p if the restriction to the inertia group is trivial.*

Definition 7.3.15. *Let $f(z) = \sum_n a_n e^{2\pi i n z}$ be a normalized Hecke eigenform of level N and weight k. A 2-dimensional ℓ-adic representation V over K is said to be associated to f if, for every $p \nmid N\ell$, V is unramified at p and*

$$Tr(\phi_p : V) = a_p(f),$$

where $\phi_p \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,

$$\phi_p(a) = a^p.$$

Let $f = \sum_n \frac{a_n}{n^s}$ is an Hecke eigenform. Then by the famous work of Deligne [14], for any prime p,

$$a(p) = \alpha(p) + \beta(p) \text{ and } |\alpha(p)| = |\beta(p)| = |\alpha(p)\beta(p)| = 1.$$

Now we will define the corresponding Symmetric m-th power (sym^m) L-function attached to f.

Definition 7.3.16. *For any positive integer m, the m**-th Symmetric power** L-function attached to f is defined as*

$$L(s, Sym^m f) := \prod_p \sum_{i=0}^{m} \left(1 - \frac{\alpha(p)^i \beta(p)^{m-i}}{p^s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$.

Definition 7.3.17. *For two positive integers m, n, the **Rankin-Selberg convolution** of L-functions attached to Sym^m and Sym^n is defined as*

$$L(s, Sym^m f \times Sym^n f) := \prod_p \prod_{i=0}^{m} \prod_{j=0}^{n} \left(1 - \frac{\alpha(p)^i \beta(p)^{m-i} \alpha(p)^j \beta(p)^{n-j}}{p^s}\right)^{-1}$$

for $\mathrm{Re}(s) > 1$.

The following theorem can be found in [12, Proposition 5.1];

Theorem 7.3.18. *[Cogdel-Michel] Let f_1 and f_2 be two holomorphic primitive forms of even weight $k = 2l$ and trivial nebentypus, one of them, say f_1, not of CM type. Suppose that, for some given $m \geq 1$,*

$$L_p(s, Sym^m f_1) = L_p(s, Sym^n f_2)$$

for every prime p outside a set of density 0. Then there exists a character of order at most 2 such that $f_2 = f_1 \otimes \chi$. (If l is odd, then χ is trivial.)

Remark 7.3.19. *In case of non-trivial Nebentypus, Rajan (see [34, Corollary 5.1]) proved the above result.*

In 1981, Shahidi [42] proved the following non-vanishing theorem for Rankin-Selberg L-function.

Theorem 7.3.20. *[Shahidi] The Rankin-Selberg L-function $L(s, f_1 \times f_2)$ does not vanish on $\mathbb{R}(s) = 1$.*

Proof of Proposition 7.3.1 Let ρ_1 and ρ_2 be the associated (ℓ-adic) Galois representations of f_1, f_2 respectively. Let F/F_1 be solvable. Then there is a chain of fields $F \supset F_m \supset F_{m-1} \supset \cdots \supset F_1$ such that F/F_m and F_j/F_{j-1} for $2 \leq j \leq m$ are all cyclic of prime degree. By [1], the automorphic induction map exists from F to F_m and successively from F_j to F_{j-1} for $2 \leq j \leq m$ which at the final stage is F_1. Here most important thing for us is that both the Galois and automorphic representations obtained by descent are Galois invariant at every step (see for example, the comment at the bottom of [9, page 11]). By the work of [3], both $\mathrm{Sym}^n(\rho_1)$ and $\mathrm{Sym}^m(\rho_2)$ are potentially automorphic over a totally real Galois extension F over $\mathbb{Q}$. By the Arthur-Clozel theory of base change [1], we see that for any sub field F_1 of F with F/F_1 solvable, both $\mathrm{Sym}^m(\rho_1)|_{F_1}$ and $\mathrm{Sym}^n(\rho_2)|_{F_1}$ are also automorphic over F_1. Let $G = \mathrm{Gal}(F/\mathbb{Q})$. By Brauer induction, we can write

$$1 = \sum_i a_i \mathrm{Ind}_{H_i}^G \psi_i,$$

where the a_i's are integers and ψ_i's are one-dimensional characters of nilpotent subgroups H_i of G. Thus,

$$L(s, (\mathrm{Sym}^m(\rho_1) \otimes \mathrm{Sym}^n(\rho_2)) \otimes 1) = \prod_i L(s, (\mathrm{Sym}^m \rho_1 \otimes \mathrm{Sym}^n \rho_2) \otimes \mathrm{Ind}_{H_i}^G \psi_i)^{a_i}.$$

By Frobenius reciprocity,

$$(\mathrm{Sym}^m \rho_1 \otimes \mathrm{Sym}^n \rho_2) \otimes \mathrm{Ind}_{H_i}^G \psi_i = \mathrm{Ind}_{H_i}^G (((\mathrm{Sym}^m \rho_1) \otimes (\mathrm{Sym}^n \rho_2))|_{L^{H_i}} \otimes \psi_i) Gal(\bar{\mathbb{Q}}/\mathbb{Q}).$$

Since $(\mathrm{Sym}^m \rho_1)|_{L^{H_i}}$ and $(\mathrm{Sym}^n \rho_2)|_{L^{H_i}}$ are both automorphic over L^{H_i}, and ψ_i is a Hecke character of L^{H_i} by Artin reciprocity, we can form the Rankin-Selberg convolution:

$$L(s, (\mathrm{Sym}^m(\rho_1))|_{L^{H_i}} \otimes (\mathrm{Sym}^n(\rho_2))|_{L^{H_i}} \otimes \psi_i). \tag{7.5}$$

By theorems of Cogdel-Michel (see Theorem 7.3.18) and Rajan (see Remark 7.3.19) which says that the m-th symmetric powers of the Galois representations associated with f_1 and f_2 are equal then f_1 is a Dirichlet character twist of f_2. Thus, if $f_1 \neq \chi \otimes f_2$ for any Dirichlet character χ, we have

$$\pi_1 := (Sym^m(\rho_1))|_{F^{H_i}} \text{ and } \pi_2 := (Sym^n(\rho_2))|_{F^{H_i}} \otimes \psi_i$$

are such that $\pi_2 \ncong \pi \otimes |det|^{it}$ for any real number t.
By our hypothesis, f_1 and f_2 are analytic and non-vanishing for $\mathrm{Re}(s) \geq 1$.
By standard Rankin-Selberg theory (see [11, page 69 or 225]) and Shahidi's result (see Theorem 7.3.20), the L-function (7.5) is analytic and non-vanishing in the region $\mathrm{Re}(s) \geq 1$.
Thus,

$$L(s, (\mathrm{Sym}^m \rho_1) \otimes \mathrm{Sym}^n \rho_2)$$

extends to an analytic function to $\mathrm{Re}(s) \geq 1$ and is non-vanishing there. By the standard Tauberian theorem (see Theorem 7.3.11) applied to the logarithmic derivative of this L-function, we deduce Proposition 7.2.1.

Remark 7.3.21. *Our argument extends easily to imply a corresponding result for any two modular forms f_1 and f_2 over a totally real field since the results of [3] apply in this context also.*

Bibliography

[1] J. Arthur and L. Clozel, *Simple algebras, base change and the advanced theory of the trace formula,* Annals of Math. Studies, Vol. 120, Princeton University Press, Princeton, 1989.

[2] A. Balog, A. Granville, J. Solymosi, *Gaps between fractional parts and additive combinatorics,* Q. J. Math. (2015), 136-147.

[3] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy II*, Publ. RIMS Kyoto Univ. 47 (2011), 29-98.

[4] D. Blasius, *Hilbert modular forms and the Ramanujan conjecture,* Non commutative Geometry and Number Theory, Vieweg, Wiesbaden, 2006, 35-56.

[5] P. Bohl, *Über ein in der Theorie der säkutaren Störungen vorkommendes Problem,* J. reine angew. Math. 135, (1909), 189-283.

[6] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations,* Proc. Sympos. Pure Math. 33, Part 1, Amer. Math. Soc., Providence, RI, 1979, 189-207.

[7] C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises,* Journal of the American Math. Society, 14 (2001), 843-939.

[8] D. Bump *Automorphic forms and representations,* Vol. 55. Cambridge University Press, 1998.

[9] L. Clozel, *The Sato-Tate conjecture,* Current developments in mathematics, 2006, 1-34, International Press, Somerville, MA. 2008.

[10] L. Clozel, M. Harris and R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations,* Publ. Math. Inst. Hautes tudes Sci. No. 108 (2008), 1181.

[11] J. W. Cogdell, H. H. Kim and M. R. Murty, *Lectures on automorphic L-functions,* Vol. 20. American Mathematical Soc., 2004.

[12] J. W. Cogdell and P. Michel, *On the complex moments of symmetric power L-functions at s= 1,* International Mathematics Research Notices, no. 31 (2004), 1561-1617.

[13] J. B. Conrey, W. Duke and D. W. Farmer,*The distribution of the eigenvalues of Hecke operators*, Acta Arith. 78, no. 4 (1997), 405 - 409.

[14] P. Deligne, *La conjecture de Weil I, IHES Publ*. Math. No. 43 (1974), 273-307.

[15] F. Diamond and J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Mathematics, Springer (2005).

[16] P. Erdös and P. Turán, *On a problem in the theory of uniform distribution I, II*, Indag. Math 10 (1948), 370-378.

[17] M. Harris, *Galois representations, automorphic forms and the Sato-Tate conjecture,* Indian Journal of Pure and Applied Mathematics, 4 (5) (2014), 707-746.

[18] M. Harris, N. Shepherd-Barron and R. Taylor, *A family of Calabi-Yau varieties and potential automorphy,* Ann. of Math. (2) 171 (2010), no. 2, 779-813.

[19] H. Hida and Y. Maeda, *Non-abelian base change for totally real fields,* Pacific J. Math. Special Issue (1997), 189-217.

[20] L.J.P Kilford, *Modular forms: A classical and computational introduction*, Imperial College Press, 2008, 236 PP, ISBN 13 978-1-84816-213-6, ISBN 10.

[21] H.H. Kim, P. Sarnak, Appendix 2 in H.H. Kim, *Functoriality for the exterior square of GL4 and the symmetric fourth of GL2,* J. Amer. Math. Soc., 16 (2003), 139-183.

[22] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences,* Wiley-Interscience [John Wiley and Sons], New York-London-Sydney, 1974.

[23] M. Kulkarni, V. Patankar and C.S. Rajan, *Locally potentially equivalent two dimensional Galois representations and Frobenius fields of elliptic curves,* J. Number Theory, 164 (2016), 87-102.

[24] Y.-K. Lau and Y. Wang, *Quantitative version of the joint distribution of eigenvalues of the Hecke operators*, J. Number Theory 131 (2011), 2262-2281.

[25] H. L. Montgomery, *Ten lectures on the interface between analytic number theory and harmonic analysis,* CBMS Regional Conference Series in Mathematics, 84, 1994, American Mathematical Society, Providence, Rhode Island.

[26] M. R. Murty, *Oscillations of Fourier coefficients of modular forms,* Math. Ann., 262 (1983), 431-446.

[27] M. R. Murty, *Problems in Analytic Number Theory,* second ed., Grad. Texts in Math., vol. 206, Springer-Verlag, 2007.

[28] M.R. Murty, M. Dewar, H. Graves, *Problems in the Theory of Modular Forms,* Hindustan book agency, Institute of Mathematical Sciences Lecture Notes-1, 2015.

[29] M. R. Murty and V. Kumar Murty, *The mathematical legacy of Srinivasa Ramanujan,* Springer India, 2013.

[30] M. R. Murty and S. Pujahari, *Distinguishing Hecke eigenforms*, (to appear in Proc. Amer. Math. Soc.).

[31] M. R. Murty and C.S. Rajan, *Stronger multiplicity theorems for forms of general type on $GL(2)$,* in Analytic Number Theory, Proceedings of the Number Theory Conference in honour of H. Halberstam, (edited by B. Berndt, H. Diamond and A. Hildebrand), Progress in Mathematics, Vol. 139, 669-683, Birkhäuser, 1996.

[32] M. R. Murty and K. Sinha, *Effective equidistribution of eigenvalues of Hecke operators*, J. Number Theory, 129 (2009), no. 3, 681-714.

[33] M. R. Murty and K. Srinivas, *Some remarks related to Maeda's conjecture*, Proc. Amer. Math. Soc. 144 (2016), 4687-4692.

[34] C. S. Rajan, *Recovering modular forms and representations from tensor and symmetric powers,* Algebra and number theory, 281-298, Hindustan Book Agency, Delhi, 2005.

[35] W. Rudin, *Real and Complex Analysis,* Third ed., Tata Mcgraw-Hill edition.

[36] P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators,* Analytic Number Theory and Diophantine Problems, Stillwater, 1984, in: Progr. Math., vol. 70, Birkhuser, Basel, 1987, 321-331.

[37] I. Schoenberg, *Über die asymptotische Verteilung reeller Zahlen mod 1,* Mathematische Zeitschrift 28.1 (1928), 171-199.

[38] A. Selberg, *Collected papers,* Volume II, Springer-Verlag, Berlin, (1991), 213-218.

[39] J.-P. Serre, *Répartition asymptotique des valeurs propres de l'opérateur de Hecke T_p*, J. Amer. Math. Soc., 10 (1997), no. 1, 75-102.

[40] J.-P. Serre, *Abelian ℓ-adic representations and elliptic Curves,* W. A. Benjamin, New York, 1968.

[41] J.-P. Serre, *Linear representations of finite groups,* Springer Verlag, New York-Heidelberg, 1977.

[42] F. Shahidi, *On certain L-functions,* Amer. J. Math., 103, (2), (1981), 297-355.

[43] W. Sierpinski, *Sur la valeur asymptotique d'une certaine somme,* Bull Intl. Acad. Polonmaise des Sci. et des Lettres (Cracovie) series A, (1910), 9-11.

[44] V. T. Sós, *On the theory of diophantine approximations,* Acta Math. Acad. Sci. Hungar. 8 (1957), 461-472.

[45] V. T. Sós, *On the distribution mod 1 of the sequence* $n\alpha$, Ann. Univ. Sci. Budapest Etvs Sect. Math. 1 (1958), 127-134.

[46] P. Tsaknias, *A possible generalization of Maeda's conjecture,* Computations with Modular Forms, Contributions in Mathematical and Computational Sciences, Vol. 6 (Springer, 2014), 317-329.

[47] J. D. Vaaler, *Some extremal functions in Fourier analysis,* Bull. Amer. Math. Soc. 12 (1985), 183-216.

[48] M. Vâjâitu and A. Zaharescu, *Distinct gaps between fractional parts of sequences,* Proc. Amer. Math. Soc. 130 (2002), 3447-3452.

[49] A. Vatwani, *A simple proof of the Wiener-Ikehara Tauberian theorem,* Math. Student, 84 (2015), 127-134.

[50] H. Weyl, *Über die Gibbs'sche Erscheinung und verwandte KonvergenzphÄnomene,* Rendiconti del Circolo Matematico di Palermo 330, (1910), 377-407.

[51] H. Weyl, *Über die Gleichverteilung von Zahlen mod Eins,* Math. Ann, Vol. 77 (1916), no. 3, 313-352.

[52] N. Wiener, *The quadratic variation of a function and its Fourier coefficients,* Journal of Mathematics and Physics 3.2 (1924), 72-94.

[53] A. Wiles, *Modular elliptic curves and Fermat's last theorem,* Ann. of Math., 142 (1995), 443-551.