Topics in Motivic Homotopy Theory

A thesis

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of the degree of

Doctor of Philosophy

by

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Dedicated to
My Parents

Certificate

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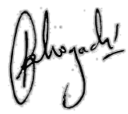

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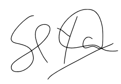

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Contents

1

Introduction

This thesis deals with two problems. The first one looks into the $\mathbb{A}^1$ connectivity of moduli stack of vector bundles on a curve. This project was borne out of the need to understand the relationship between an algebraic stack and its coarse moduli space in the motivic setting. While we do prove that the moduli stack in question is $\mathbb{A}^1$ connected, the broader question still remains. $\mathbb{A}^1$ connectivity of the aforementioned stack has some pleasant consequences. It allows us to classify projective bundles on any curve upto their $\mathbb{A}^1$ homotopy type. In fact we prove a stronger statement as we classify such bundles upto their $\mathbb{A}^1$-h cobordant class. Along with results of [2] this provides complete classification of projective bundles on curves upto their $\mathbb{A}^1$-h-cobordant class. Classification of projective bundles over higher dimensional varieties upto $\mathbb{A}^1$-h cobordism is not known. For a partial result over $\mathbb{P}^2$ see [1].
Another consequence of the theorem is that we are able to come up with an example of a scheme which while being $\mathbb{A}^1$-h cobordant to projective bundle, is not isomorphic to one. This answers a question raised in [1]. This is based on a joint work with Amit Hogadi [15].
The second problem in this thesis is regarding the Gersten resolution of an $\mathbb{A}^1$ invariant cohomology theory over a general base. Gersten resolutions are ubiquitous in Algebraic geometry and has a rich history, starting with Algebraic K theory. While Gersten complex always exists, proving it's exactness is the central question here. When the base scheme is a field, it's exactness is already known (see for instance [5]). When the base is a

DVR the conditional exactness was proved in [27]. The key ingredient in op. cit, is the authors' Gabber presentation lemma for DVR proved in [26]. Here we extend this result to schemes of arbitrary dimension, based on [7] and [9]. We also prove the exactness of Gersten complex for étale cohomology with finite coeffecients, known as Bloch Ogus theorem. This is based on joint work with Neeraj Deshmukh and Girish Kulkarni [8].
The main results are presented in Chapters 4 and 5. The raison d'être of other chapters is to provide prerequisites needed for these two chapters. We have eschewed proofs in these sections because most of the material is already standard and references have been provided. Chapter 2 and Section3.1 (with exception of section 2.3) are the foundational material for Chapter 4 while sections 2.3 and 3.2 go into chapter 5. Future directions are hinted in Remarks 5.1.9 and 5.2.6.

2

A primer on $\mathbb{A}^1$-homotopy theory

We first reveiw the basics of model categories and then define unstable and stable motivic homotopy categories.

2.1 Model categories

Following [17] or [12] we recall some basic definitions

Definition 2.1.1. Let $\mathcal{C}$ be a category with all small limits and colimits. A model category structure on $\mathcal{C}$ consists of three classes W, C, F of morphisms in $\mathcal{C}$ (called weak-equivalences, cofibrations, and fibrations respectively), satisfying the following.

M1 Given two composable morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, if any two of the morphisms f, g and $g \circ f$ are weak equivalences then so is the third.

M2 If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f (for the definition of retract see [17, Def. 1.1.1]).

M3 Given a diagram of solid arrows, a dotted arrow can be found making the following diagram commutative

$$\begin{array}{ccc} Z & \longrightarrow & E \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

if either

(a) p is a trivial fibration i.e. ($p \in W \cap F$) and i is a cofibration, or

(b) i is a trivial cofibration i.e. ($i \in W \cap C$) and p is a fibration.

M4 Any map $X \to Z$ in $\mathcal{C}$ admits two factorizations, $X \xrightarrow{f} E \xrightarrow{p} Z$ and $X \xrightarrow{i} Y \xrightarrow{g} Z$, such that f is an trivial cofibration, p is a fibration, i is a cofibration, and g is an trivial fibration.

Definition 2.1.2. An object X in a model category $\mathcal{C}$ is called **fibrant** if the map $X \to *$ to the final object is a fibration and **cofibrant** if the map $\emptyset \to X$ from the initial object is a cofibration. Given an object $X \in \mathcal{C}$, a **fibrant**(**cofibrant** replacement) is a trivial cofibration(fibration) $X \to RX$ ($QX \to X$) such that RX (QX) is fibrant (cofibrant).

The machinery of model category allows us to define homotopy category of a category with a given model structure while avoiding set theoretic pitfalls.

Definition 2.1.3. Let $\mathcal{C}$ be a model category and $X \in \mathcal{C}$ be an object. A **cylinder object** for X is an object $\mathrm{Cyl}(X)$ with the following diagram

$$X \coprod X \xrightarrow{i_0, i_1} \mathrm{Cyl}(X) \to X$$

which factors the fold map $X \coprod X \to X$ such that $\mathrm{Cyl}(X) \to X$ is a trivial fibration.

Definition 2.1.4. Let $f, g : X \to Y$ be morphisms for some objects X, Y in a model category $\mathcal{C}$. A **left homotopy** between f and g is a morphism $H : \mathrm{Cyl}(X) \to Y$ such that $H \circ i_0 = f$ and $H \circ i_1 = g$, for some cylinder object $\mathrm{Cyl}(X)$ for X. In such a scenario we say f and g are left homotopic.

Analogous to cylinder objects and left homotopies, there is a notion of path objects and right homotopies which we define below.

Definition 2.1.5. Let $\mathcal{C}$ be a model category and $Y \in \mathcal{C}$ be an object. A **path object** for Y is an object $\mathrm{P}(Y)$ with the following diagram

$$Y \to \mathrm{P}(Y) \xrightarrow{e_0, e_1} Y \times Y$$

which factors the diagonal map $Y \to Y \times Y$ such that $\mathrm{P}(Y) \to Y$ is a trivial cofibration.

Definition 2.1.6. Let $f, g : X \to Y$ be morphisms for some objects X, Y in a model category $\mathcal{C}$. A **right homotopy** between f and g is a morphism $H : X \to \mathrm{P}Y$ such that $e_0 \circ H = f$ and $e_1 \circ H = g$, for some path object $\mathrm{P}(Y)$ for Y. In such a scenario we say f and g are right homotopic.

Definition 2.1.7. Morphisms $f, g : X \to Y$ in $\mathcal{C}$ are said to be **homotopic** if they are left as well as right homotopic.

Definition 2.1.8. Let $\mathcal{C}$ be a model category. Its **homotopy category** denoted $\mathrm{Ho}(\mathcal{C})$ is a category such that

- objects of $\mathrm{Ho}(\mathcal{C})$ are same as $\mathcal{C}$
- Given $X, Y \in \mathcal{C}$, the morphisms in $\mathrm{Ho}(\mathcal{C})$ between them, denoted $[X, Y]$ are given by $\mathcal{C}(RQX, RQY)/\sim$, where $\sim$ denotes

Remark 2.1.9. *Given arbitrary X, Y in a model category $\mathcal{C}$, a left homotopy between two morphisms $f, g : X \to Y$ doesn't necessarily imply a right homotopy or vice-versa. Furthermore neither left homotopy nor right homotopy is an equivalence relation. However in the case X and Y are cofibrant-fibrant objects the previous two statements hold true.*

Definition 2.1.10. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction between model categories, with F and G left adjoint and right adjoint respectively. This is a **Quillen adjunction** if one of the following equivalent conditions is satisfied

- F preserves cofibrations and trivial cofibrations
- G preserves fibrations and trivial fibrations
- F preserves cofibrations and G preserves fibrations
- F preserves trivial cofibrations and G preserves trivial fibrations.

2.1.1 Simplicial sets

Let Δ be the category whose objects are finite ordered sets, denoted as $[n] := \{0 < 1 < \cdots < n\}$. A morphism between two objects $[n] \to [m]$ is an order preserving morphism of finite ordered sets. Any morphism in Δ $[n] \to [m]$ can be uniquely written as a composition of the morphisms of following type (for $0 \leq i \leq n$)

$$d^i : [n-1] \to [n], d^i(j) = j \text{ if } j < i \text{ and } d^i(j) = j+1 \text{ if } j \geq i$$
$$s^i : [n+1] \to [n], s^i(j) = j \text{ if } j \leq i \text{ and } s^i(j) = j-1 \text{ if } j > i$$

Definition 2.1.11. A **simplicial set** X is a functor

$$X : \Delta^{op} \to SET$$

where SET is the category of sets. Given a simplicial set X, its **n-simplices**, denoted X_n, is the set $X([n])$. Henceforth the category of simplicial sets will be denoted as $sSet$.

Remark 2.1.12. *Unravelling the definition of a simplicial set X we obtain that a simplicial set is a collection of sets $\{X_n\}$ along with morphisms $d_i : X_n \to X_{n-1}$ called the face maps , $s_i : X_n \to X_{n+1}$, called the degeneracy maps, satisfying the following simplicial identities*

$$d_i d_j = d_{j-1} d_i, \text{ for } i < j$$
$$d_i s_j = s_{j-1} d_i, \text{ for } i < j$$
$$d_j s_j = 1 = d_{j+1} s_j$$
$$d_i s_j = s_j d_{i-1} \text{ for } i > j+1$$
$$s_i s_j = s_{j+1} s_i \text{ for } i \leq j$$

Example 2.1.13. *One class of important example of simplicial sets is the representable functor $Hom_\Delta(\_, [n])$, henceforth denoted as Δ^n. Hence by Yoneda lemma, for a simplicial set X, we have $X_n = \mathrm{Hom}_{sSet}(\Delta^n, X)$.*

Example 2.1.14. *Given a small category $\mathcal{C}$ we can construct a simplicial set $N(\mathcal{C})$ called **nerve** of that category. Then zero simplices of $N(\mathcal{C})$ correspond to the objects of $\mathcal{C}$. $N(\mathcal{C})_n$ (the n simplices) correspond to the composable morphisms*

$$A_0 \to A_1 \to \cdots \to A_n$$

in $\mathcal{C}$. The face maps $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ send $A_0 \to A_1 \to \cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots \to A_n$ to $A_0 \to A_1 \to \cdots \to A_{i-1} \to A_{i+1}$ for $i \neq 0, n$. For $i = 0, n$ the face map d_i simply removes A_0 and A_n respectively.
The degeneracy map $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ sends $A_0 \to A_1 \to \cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots \to A_n$ to $A_0 \to A_1 \to \cdots \to A_{i-1} \to A_i \xrightarrow{id} A_i \to A_{i+1} \to \cdots \to A_n$

Let $|\Delta^n|$ be the standard n-simplex in $\mathbb{R}^{n+1}$ defined as a topolgical space in the following way

$$|\Delta^n| := \{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} | \sum_{i=0}^n t_i = 1, t_i \geq 0\}$$

This allows us to construct the **realization** functor

$$| - | : sSet \to Top$$

which sends Δ^n to $|\Delta^n|$. Since an arbitrary simplicial set X can be written as a colimit of representables i.e Δ^n, $|X|$ is defined by taking the colimit of the same diagram with respect to $|\Delta|^n$ in Top (where colimits exist). Moreover it can be shown that $|X|$ is a CW complex [12, Prop. 2.3.] .

Definition 2.1.15. A morphism $f : X \to Y$ of simplicial sets in a **weak equivalence** if $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces.

Theorem 2.1.16. *[12, Theorem 11.3] There is a model structure on sSet with weak equivalences as defined in Definition 2.2.12, cofibrations as monomorphisms of simplicial sets and fibrations the morphisms with lifting property with respect to trivial cofibrations.*

2.1.2 Chain complexes

Example 2.1.17. *Let $Ch_{\geq 0}$ be the category of bounded below co-chain complexes of abelian groups. Then we describe two model structures called **injective** and **projective** model structure.*

***Injective model structure**:*

- *$f : X \to Y$ is a weak equivalence if it is a quasi isomorphism, that is, the induced map $H^n(X) \to H^n(Y)$ is an isomorphism for all n.*
- *$f : X \to Y$ is a cofibration if $f_n : X_n \to Y_n$ is injective for all $n > 0$.*
- *$f : X \to Y$ is a fibration if $f_n : X_n \to Y_n$ is surjective and $\ker f_n$ is an injective group for all n.*

***Projective model structure**:*

- *$f : X \to Y$ is a weak equivalence if it is a quasi isomorphism, that is, the induced map $H^n(X) \to H^n(Y)$ is an isomorphism for all n.*
- *$f : X \to Y$ is a fibration if $f_n : X_n \to Y_n$ is surjective for all $n > 0$.*
- *$f : X \to Y$ is a cofibration if $f_n : X_n \to Y_n$ is injective and* cokerf_n *is a projective for all n.*

Let Ab denote the category of abelian groups.

Definition 2.1.18. A simplicial abelian group is a functor $X : \Delta^{op} \to Ab$

Given a simplicial abelian group X, let $NX_k := \bigcap_{i=0}^{k-1} ker(d_i) \subset X_k$, where d_i's are the boundary maps. One has the morphism $NX_k \xrightarrow{(-1)^k d_k} NX_{K-1}$ and this gives rise to a chain complex

$$\cdots \to NX_k \to NX_{k-1} \to \cdots$$

. Thus one has a functor

$$N : sAb \to Ch_{\leq 0}$$

from simplicial abelian groups to bounded above chain complexes of abelian groups. Moreover there is a functor $\Gamma : Ch_{\leq 0} \to sAb$ which is an inverse of N and gives rise to equivalence between these two categories. This is called **Dold Kan correspondence**. For more details see [12, III].

2.2 Unstable $\mathbb{A}^1$-homotopy category

In this section we recall some constructions from [22]. Let S be a Noetherian scheme of finite Krull dimension and $\mathcal{S}m_S$ denote the category of smooth schemes of finite type over S. The category of presheaves of simplicial sets on $\mathcal{S}m_S$ is denoted by $sPre(\mathcal{S}m_S)$.

Remark 2.2.1. *Note that any simplicial set S, considered as a constant presheaf is an object in $sPre(\mathcal{S}m_S)$. Any scheme $X \in \mathcal{S}m_S$ considered as presheaf of sets is also an object in $sPre(\mathcal{S}m_S)$.*

Definition 2.2.2. Let $\mathcal{S}$ be a Noetherian scheme of finite dimension. A finite family $\{U_i\} \to X$ family of étale morphisms is called a **Nisnevich cover** if one of the following equivalent conditions are satisfied

- For any point $x \in X$ there exists an i and a point $u \in U_i$ such that u maps to x and the induced morphism of residue fields $k(x)$ and $k(u)$ is an isomorphism
- for any point $x \in X$ the following morphism of schemes has a section

  $$\coprod_i (U_i \times_X \operatorname{Spec} \mathcal{O}^h_{X,x}) \to \operatorname{Spec}(\mathcal{O}^h_{X,x})$$

  where $\mathcal{O}^h_{X,x}$ is the Henselisation of local ring $\mathcal{O}_{X,x}$ at x.

One can then define a Grothendieck site with Nisnevich topology generated by Nisnevich covers. Stalk at a point x of a scheme X in Nisnevich topology is $\mathcal{O}^h_{X,x}$. One then obtains a **local model structure** on $sPre(\mathcal{S}m_S)$ such that

- Let $\mathcal{X}$, $\mathcal{Y}$ be simplicial presheaves. Then $\mathcal{X} \to \mathcal{Y}$ is weak equivalence if $\mathcal{X}(\operatorname{Spec}(\mathcal{O}^h_{X,x})) \to \mathcal{Y}(\operatorname{Spec}(\mathcal{O}^h_{X,x}))$ is a weak equivalence of simplicial sets for any point x in any scheme $X \in \mathcal{S}m_S$.
- Cofibrations are monomorphisms of simplicial presheaves.
- Fibrations are characterized by lifting property with respect to trivial cofibrations.

Definition 2.2.3. An **elementary distinguished square** is the following pullback diagram in $\mathcal{S}m_S$

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

such that where p is étale morphism, j is an open embedding and $p^{-1}(X-U) \to X-U$ is an isomorphism. Note that the closed subschemes $p^{-1}(X-U)$ and $X-U$ are assumed to have reduced induced scheme structure.

One has the following characterisation of fibrant objects in the local model structure on $sPre(\mathcal{S}m_S)$.

Proposition 2.2.4. *Let S be a Noetherian scheme of finite Krull dimension. $\mathcal{X} \in sPre(\mathcal{S}m_S)$ is projective fibrant if and only if for every elementary distinghished square*

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X \end{array}$$

the natural map

$$\mathcal{X}(X) \to \mathcal{X}(V) \times_{\mathcal{X}(U \times V)} \mathcal{X}(U)$$

is a weak equivalence of simplicial sets, $\mathcal{X}(X)$ *is a fibrant simplicial set and* $\mathcal{X}(\phi)$ *is an empty object.*

One can do the Bousefield localisation (see [14] for more details) of local model structure on $sPre(\mathcal{S}m_S)$ with respect to projection morphisms $U \times \mathbb{A}^1 \to U$ and obtain a new model structure denoted $sPre(\mathcal{S}m_S)^{\mathbb{A}^1}$ where all the morphisms of the form $\mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ (for any $\mathcal{X} \in sPre(\mathcal{S}m_S)$)are weak equivalences. The **unstable motivic homotopy category**, denoted $\mathcal{H}(S)$ is the homotopy category of $sPre(\mathcal{S}m_S)^{\mathbb{A}^1}$. A fibrant object $\mathcal{X}$ in $sPre(\mathcal{S}m_S)^{\mathbb{A}^1}$ is called an $\mathbb{A}^1$ **local** object. In addition to being a Nisnevich fibrant object as characterised in Proposition 2.2.4 it satisfies the following condition

$$\mathcal{X}(U) \to \mathcal{X}(U \times \mathbb{A}^1)$$

is a weak equivalence of simplicial sets for every $U \in \mathcal{S}m_S$.

Example 2.2.5. $\mathbb{G}_m$*, abelian varieties and their products are some examples of schemes which are* $\mathbb{A}^1$*-local.*

Remark 2.2.6. *A pointed simplicial presheaf consists of a simplicial presheaf* $\mathcal{X}$ *along with a morphism* $* \to \mathcal{X}$*, where* $*$ *is the final object in* $sPre(\mathcal{S}m_S)$*. There is a forgetful functor from the category of pointed simplicial presheaves to* $sPre(\mathcal{S}m_S)$*. All the constructions of this section can be carried out mutatis mutandis for the category of pointed simplicial presheaves on* $\mathcal{S}m_S$*. For instance a morphism of pointed simplicial presheaves is a weak equivalence if it is a weak equivalence after applying the forgetful functor. Moreover the forgetful functor admits a left adjoint which also induces an adjunction on resulting homotopy categories.*

2.2.1 $\mathbb{A}^1$ homotopy sheaves

There exists a functor $L^{\mathbb{A}^1} : sPre(\mathcal{S}m_S) \to sPre(\mathcal{S}m_S)$ such that given any simplicial presheaf $\mathcal{X}$, $L^{\mathbb{A}^1}(\mathcal{X})$ is a fibrant object in $\mathcal{H}(S)$ (for construction see [22, page 107]), this functor is called $\mathbb{A}^1$**-fibrant replacement functor**.

Let $\pi_i^s(X, x)$ denote the i^{th} homotopy group of a pointed simplicial set (X, x). Then the i^{th} $\mathbb{A}^1$**-homotopy sheaf** of a pointed simplicial sheaf $(\mathcal{X}, x)$ (denoted $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$) is defined to be the Nisnevich sheafification of the preseheaf

$$U \mapsto \pi_i^s(L^{\mathbb{A}^1}(\mathcal{X})(U), x) \simeq [U \wedge S^i, \mathcal{X}]_{\mathcal{H}(S)}$$

.

Definition 2.2.7. Let k be a field and $S = Spec(k)$

1. A sheaf of sets $\mathcal{F}$ on $\mathcal{S}m_S$ in the Nisnevich topology is said to be $\mathbb{A}^1$**-invariant** if for any $X \in \mathcal{S}m_S$, the map $\mathcal{F}(X) \to \mathcal{F}(\mathbb{A}^1 \times X)$ induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection.

2. A sheaf of groups G on $\mathcal{S}m_S$ in the Nisnevich topology is said to be **strongly** $\mathbb{A}^1$ **-invariant** if for any $X \in \mathcal{S}m_S$ the map

   $$H^i_{Nis}(X; G) \to H^i_{Nis}(X \times \mathbb{A}^1; G)$$

   induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection for $i \in \{0, 1\}$.

3. A sheaf M of abelian groups on $\mathcal{S}m_S$ in the Nisnevich topology is said to be **strictly** $\mathbb{A}^1$**-invariant** if for any $X \in \mathcal{S}m_S$ the map

   $$H^i_{Nis}(X; M) \to H^i_{Nis}(X \times \mathbb{A}^1; M)$$

   induced by the projection $\mathbb{A}^1 \times X \to X$, is a bijection for any $i \in \mathbb{N}$.

Remark 2.2.8. *Given any pointed simplicial presheaf $(\mathcal{X}, x)$, $\pi_0^{\mathbb{A}^1}(\mathcal{X}, x)$ is conjectured by F. Morel to be $\mathbb{A}^1$-invariant. In full generality this conjecture is open though it has been proved in some special cases.*

Remark 2.2.9. *When the base S is a field, $\pi_1^{\mathbb{A}^1}(\mathcal{X}, x)$ is a strongly $\mathbb{A}^1$ invariant sheaf of groups [21, Theorem 6.1] and with the additional condition of base being a perfect field, $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ $(i > 1)$ is a strictly $\mathbb{A}^1$ invariant sheaf of abelian groups [21, Corollary 6.2]. Moreover any strongly $\mathbb{A}^1$ invariant sheaf of abelian groups over a perfect field is strictly $\mathbb{A}^1$ invariant [21, Corollary 5.45].*

Definition 2.2.10. A pointed presheaf $\mathcal{X} \in sPre(\mathcal{S}m_S)^{\mathbb{A}^1}$ is called $\mathbb{A}^1$**- connected** if the canonical map $\mathcal{X} \to S$ induces an isomorphism $\pi_0^{\mathbb{A}^1}(\mathcal{X}) \xrightarrow{\simeq} \pi_0^{\mathbb{A}^1}(S) \simeq *$.

Example 2.2.11. *$\mathbb{A}^n$'s are $\mathbb{A}^1$-connected schemes (in fact they are $\mathbb{A}^1$ contractible). $\mathbb{P}^n$'s are $\mathbb{A}^1$-connected. Projective bundles over $\mathbb{A}^1$-connected proper schemes are $\mathbb{A}^1$-connected.*

For the remaining section assume S is a field, denoted k.

Lemma 2.2.12. *[20, Lemma 6.1.3.] Let $\mathcal{Y}$ be an $\mathbb{A}^1$ local simplicial sheaf such that $\pi_0\mathcal{Y}(F) = *$ for every finitely generated field extension F/k. Then $\mathcal{Y}$ is $\mathbb{A}^1$-connected.*

Proof. We have to prove for any smooth scheme X over k, $\pi_0\mathcal{Y}(X)$ is trivial. Since we are working with Nisnevich sheaves we need to prove that for a covering $X' \to X$, the morphism $X' \to \pi_0\mathcal{Y}$ is trivial. Surjectivity of sheaves $\mathcal{Y} \to \pi_0\mathcal{Y}$ implies any morphism $X' \to \pi_0\mathcal{Y}$ lifts to a morphism $U \to \mathcal{Y}$ for some Nisnevich covering of X'. So it's suffices to show for any irreducible smooth k scheme U, any morphism $U \to \pi_0\mathcal{Y}$ obtained after composing with $\mathcal{Y} \to \pi_0\mathcal{Y}$ is trivial.
Let F be the function field of U. Then the condition $\pi_0\mathcal{Y}(F) = *$ implies that there exists a dense open $U' \subset U$ such that $U' \to \pi_0\mathcal{Y}$ is trivial. By assuming $\mathcal{Y}$ to be simplicially fibrant we can prove that any morphism $U \to \pi_0\mathcal{Y}$ restricts to a trivial morphism $U' \to \pi_0\mathcal{Y}$ for some $U' \subset U$. Therefore we have a morphism $U/U' \to \pi_0\mathcal{Y}$. By Gabber presentation lemma one can prove that U/U' is $\mathbb{A}^1$- connected and hence $U/U' \to \pi_0\mathcal{Y}$ is trivial. This finishes the proof. □

For a simplicial presheaf $\mathcal{X}$ consider the presheaf $U \mapsto \mathcal{X}(U)/\sim$, where $\sim$ is the equivalence relation generated by "naive" $\mathbb{A}^1$ homotopies. x and y in $\mathcal{X}(U)$ are **naively $\mathbb{A}^1$ homotopic** if there exists $f : \mathbb{A}^1_U \to \mathcal{X}$ such that $f_0 = x$ and $f_1 = y$. Its sheafification will be denoted as $\mathcal{X}^{nv}$.

Let $\Delta^n_{\mathbb{A}^1}$ be **algebraic** n **simplex** defined as the following smooth scheme

$$\Delta^n_{\mathbb{A}^1} := \operatorname{Spec} k[x_0, \cdots, x_n]/(\Sigma_{i=0}^{i=n} x_i - 1)$$

Note that $\Delta^n_{\mathbb{A}^1}$ is isomorphic (non canonically) to $\mathbb{A}^n_k$. Given $\mathcal{X}, \mathcal{Y} \in sPre(\mathcal{S}m_S)^{\mathbb{A}^1}$ we can obtain a simplicial sheaf $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})$ such that $\underline{\mathrm{Hom}}(\mathcal{X}, \mathcal{Y})_n := \mathrm{Hom}_{sPre(\mathcal{S}m_S)^{\mathbb{A}^1}}(\mathcal{X} \times \Delta^n_{\mathbb{A}^1}, \mathcal{Y})$.

Definition 2.2.13. Let $\mathcal{X}$ be a simplicial sheaf. Then $\mathrm{Sing}_*^{\mathbb{A}^1}(\mathcal{X})$ is the diagonal of bisimplicial sheaf

$$\underline{\mathrm{Hom}}(\Delta^m_{\mathbb{A}^1}, \mathcal{X}_n)$$

Remark 2.2.14. *It follows from definitions that for any given simplicial sheaf* $\mathcal{X}$*,* $\pi_0^s(\mathrm{Sing}_*^{\mathbb{A}^1}(\mathcal{X})) \simeq \mathcal{X}^{nv}$

Corollary 2.2.15. *A simplicial sheaf* $\mathcal{X}$ *is* $\mathbb{A}^1$*-connected if* $\mathcal{X}^{nv}(F) = *$ *for every finitely generated field extension* F *over* k*.*

Proof. Consider $L^{\mathbb{A}^1}(\mathcal{X})$. As a consequence of unstable connectivity theorem [22, Section 2, Corollary 3.22] and Remark 2.2.14, the condition $\mathcal{X}^{nv}(F) = *$ implies $L^{\mathbb{A}^1}(\mathcal{X})(F) = *$ for every finitely generated field extension F/k. Then by Lemma 2.2.12 we have that $\mathcal{X}$ is $\mathbb{A}^1$-connected. □

2.3 Stable $\mathbb{A}^1$ homotopy theory

Let S^1 denote the simplicial set $\Delta^1/\partial\Delta^1$.

Definition 2.3.1. The category of **spectra**, *Spt*, consists of

- Objects called spectrum. A spectrum X consists of simplicial sets X_i $i \in \mathbb{N}$ along with morphisms $X_i \wedge S^1 \to X_{i+1}$ for all i.
- A morphism of spectra $\sigma : X \to Y$ consists of morphisms of simplicial sets $\sigma_i : X_i \to Y_i$ along with following commutative diagram

$$\begin{array}{ccc} X_i \wedge S^1 & \longrightarrow & X_{i+1} \\ \downarrow{\scriptstyle \sigma \wedge S^1} & & \downarrow{\scriptstyle \sigma_{i+1}} \\ Y_i \wedge S^1 & \longrightarrow & Y_{i+1} \end{array}$$

Remark 2.3.2. *By adjunction between suspension and loop functors a morphism $X_i \wedge S^1 \to X_{i+1}$ is equivalent to a morphism $X_i \to \Omega_{S^1} X_{i+1}$. This induces a natural map $\pi_k(X_i) \to \pi_{k+1}(X_{i+1})$, for all i and k.*

Definition 2.3.3. Given a spectrum X its n^{th} **stable homotopy group** $\pi_n(X)$ is the following colimit

$$\varinjlim_i \pi_{n+i}(X_i)$$

Definition 2.3.4. An S^1**-spectrum** E is the data of simplicial sheaves E_i, $i \in \mathbb{N}$ along with the maps $E_i \wedge S^1 \to E_{i+1}$.

Definition 2.3.5. Given an S^1 spectrum E, its $\mathbf{n}^{th}$**- homotopy sheaf**, denoted $\pi_n(E)$ is the sheaf associated to the presehaf

$$X \mapsto \varinjlim_i \pi_{n+i}(E_i(X))$$

for $X \in \mathcal{S}m_S$.

Definition 2.3.6. The category of S^1- spectra, $Spt_{S^1}(\mathcal{S}m_S)$, has the following model structure called **stable model structure**

- $f : E \to F$ is a weak equivalence if f induces an isomorphism of homotopy sheaves $\pi_n(E) \to \pi_n(F)$ for all n.

- $E \to F$ is a cofibration if $E_0 \to F_0$ and the following maps are cofibrations

$$(S^1 \wedge F_n) \cup_{S^1 \wedge E_n} E_{n+1} \to F_{n+1}$$

The weak equivalences in the above model structure are called **stable weak equivalences**. The homotopy category of $Spt_{S^1}(\mathcal{S}m_S)$, denoted $\mathcal{SH}_{S^1}(S)$ is called stable homotopy category of spectra. $\mathcal{SH}_{S^1}(S)$ has the structure of a **triangulated category**. For definition of a triangulated category see [24] . As a consequence the following is true

- $\mathcal{SH}_{S^1}(S)$ is an additive category with the additive functor $S^1 \wedge \_ : \mathcal{SH}_{S^1}(S) \to \mathcal{SH}_{S^1}(S)$ an equivalence of additive category.

- an exact triangle is of the form

$$E \xrightarrow{f} F \to C(f) \to E[1]$$

where $C(f)$ is a spectrum such that $C(f)_n$ is the cone of morphism $f_n : E_n \to F_n$ and $E[1] := E \wedge S^1$.

Remark 2.3.7. *The category of simplicial sheaves embed inside $Spt_{S^1}(\mathcal{S}m_S)$ via the fully faithful functor Σ^∞, where*

$$\Sigma^\infty(\mathcal{F})_n := \mathcal{F} \wedge (S^1)^{\wedge n}$$

.

Definition 2.3.8. Given a spectrum $E \in Spt_{S^1}(\mathcal{S}m_S)$. We call the functor on Sm_S defined as $X \mapsto [\Sigma^\infty X, E]_{\mathcal{SH}_{S^1}(S)}$, for X/S as the cohomology theory defined by E.

Remark 2.3.9. *The Dold Kan correspondence defined in previous section extends to a functors between Spectra and unbounded chain complexes. However this is only an equivalence between the category of unbounded chain complexes and a subcategory of Spectra (spectra of modules over Eilenberg Maclane spectrum). This is called* ***Stable Dold Kan correspondence****.*

Definition 2.3.10.

- Let $E \in Spt_{S^1}(\mathcal{S}m_S)$. Then E is called **$\mathbb{A}^1$-local** if $E(X) \to E(X \times \mathbb{A}^1)$ is a weak equivalence.

- A morphism $F \to G \in Spt_{S^1}(\mathcal{S}m_S)$ is called a **stable** $\mathbb{A}^1$ **weak equivalence** if for any $\mathbb{A}^1$ local spectrum E the following map is an isomorphism

$$[G, E]_{\mathcal{SH}_{S^1}(S)} \to [F, E]_{\mathcal{SH}_{S^1}(S)}$$

- The **stable** $\mathbb{A}^1$ **homotopy category**, denoted $\mathcal{SH}^{\mathbb{A}^1}_{S^1}(S)$ is the homotopy category of $Spt^{\mathbb{A}^1}_{S^1}(\mathcal{S}m_S)$- which is Bousefield localization of $Spt_{S^1}(\mathcal{S}m_S)$ at stable $\mathbb{A}^1$ weak equivalences.

2.3.1 Gersten Complex

The main reference for this section is [27].

Let $f : X \to Y$ be a morphism, where $X, Y \in \mathcal{S}m_S$. This morphism induces the following adjunction

$$f^* : sPre(\mathcal{S}m_Y) \rightleftarrows sPre(\mathcal{S}m_X) : f_*$$

such that $f_*\mathcal{F}(U) = \mathcal{F}(X \times_Y U)$ for any $U \in \mathcal{S}m_Y$. Moreover $f^*Z = X \times_Y Z$ where X is considered as a representable sheaf. This adjunction extends to following adjunction

$$f^* : Spt_{S^1}(\mathcal{S}m_Y) \rightleftarrows Spt_{S^1}(\mathcal{S}m_X) : f_*$$

such that $f_*E(U) = E(X \times_Y U)$ for any $U \in \mathcal{S}m_Y$. In case $f : X \to Y$ is smooth f^* also admits a left adjoint functor denoted $f_\#$

$$f_\# \ : Spt_{S^1}(\mathcal{S}m_X) \rightleftarrows Spt_{S^1}(\mathcal{S}m_Y) : f^* \tag{2.1}$$

Let X_{Nis} denote the small Nisnevich site at X. Assume $f : X \to Y$ is smooth. This gives rise to the inclusion functor $i_{X/Y} : X_{Nis} \to \mathcal{S}m_Y$. Precomposing with this functor leads to the adjunction

$$i^*_{X/Y} : Spt_{S^1}(X_{Nis}) \rightleftarrows Spt_{S^1}(\mathcal{S}m_Y) : i_{X/Y,*}$$

It follows from definitions that $i_{X/Y,*}(E)(U) = E(U)$, for $U \in X_{Nis}$. From now on we will denote $i_{X/Y,*}(E)$ by E_X.

Remark 2.3.11. *Given $f : X \to Y$ a smooth morphism we will use the notation of 2.1 to also denote adjunction*

$$f_\# \ : Spt_{S^1}(X_{Nis}) \rightleftarrows Spt_{S^1}(Y_{Nis}) : f^*$$

Example 2.3.12. *Let X be a scheme and $x \in X$ be a point. Then corresponding to the morphism of schemes $\mathrm{j} : \mathrm{Spec}\, \mathcal{O}_{X,x} \to X$, we have the the following adjunction*

$$\mathrm{j}^* : Spt_{S^1}(X_{Nis}) \leftrightarrows Spt_{S^1}(\mathrm{Spec}\,(\mathcal{O}_{X,x})_{Nis}) : \mathrm{j}_*$$

Proposition 2.3.13. *Let $X \in \mathcal{S}m_S$ and $j : U \to X$ be an étale morphism. Then $j^* \circ E_X \simeq E_U$ for any $E \in Spt_{S^1}(\mathcal{S}m_S)$*

The previous proposition when applied to open immersions allows us to define the following spectrum.

Definition 2.3.14. Let $X \in \mathcal{S}m_S$ and $E \in Spt_{S^1}(\mathcal{S}m_S)$. Let $Z \hookrightarrow X$ be a closed immersion and denote by $j : X\backslash Z \hookrightarrow X$, the open immersion. Then $E_{Z/X} \in Spt_{S^1}(X_{Nis})$ is defined to be the homotopy fibre (which is not unique, but unique upto homotopy type) of the following

$$E_X \to j_*j^*E_X \simeq j^*E_{X\backslash Z}$$

Remark 2.3.15. ***(Excision)*** *[27, Lemma 3.6] Let $X \in \mathcal{S}m_S$ and $E \in Spt_{S^1}(\mathcal{S}m_S)$. Let $Z \hookrightarrow X$ be a closed immersion and denote by $j : U \to X$, an étale map and $U_Z := U \times_X Z$. Then $j^*E_{Z/X} \simeq E_{U\cap Z/U}$. In particular $E_{Z/X}(U) \simeq E_{U_Z/U}(U)$. In particular when U is an open immersion $E_{Z/X}(U) \simeq E_{U\cap Z/U}(U)$*

Definition 2.3.16. Let $E \in Spt_{S^1}(\mathcal{S}m_S)$ and $Z' \subseteq Z \subseteq X$ be closed subschemes of X. Then the induced map $E_{Z/X} \to E_{Z'/X}$ is called **forget support** map .

We define

$$E_{X^{(p)}} := \underset{\substack{Z\subset X \text{closed} \\ codim(Z,X)\geq p}}{colim} E_{Z/X}$$

in $Spt_{S^1}(X_{Nis})$, for $p \geq 0$ with structure maps are given by forget support maps.

Remark 2.3.17. *It's clear from the definitions that $E_{X^{(0)}} \simeq E_X$ and there are canonical morphisms $E_{X^{(p)}} \to E_{X^{(p-1)}}$.*

Definition 2.3.18. The spectrum $E_{X^{(p-1/p)}}$ is defined to be the homotopy cofibre of the map $E_{X^{(p)}} \to E_{X^{(p-1)}}$

Definition 2.3.19. Let $E \in Spt_{S^1}(\mathcal{S}m_S)$ then we define E^n as the presheaf $X \mapsto \pi_{-n}(E(X))$.

For rest of the section we will assume $E \in Spt_{S^1}(\mathcal{S}m_S)$ is a fibrant object in the stable model structure defined in Definition 2.3.6.

Remark 2.3.20. *Following isomorphism holds*

$$E_{X^{(p-1/p)}} \simeq \bigoplus_{z \in X^{(p-1)}} \mathfrak{j}_* \mathfrak{j}^* E_{Z/X}$$

where $\mathfrak{j}$ *is as defined in Example 2.3.12 and* $Z := \overline{z}$. *Analogous isomorphism holds for the presheaves* E^n, *that is*

$$E^n_{X^{(p-1/p)}} \simeq \bigoplus_{z \in X^{(p-1)}} \mathfrak{j}_* \mathfrak{j}^* E^n_{Z/X}$$

Moreover $\mathfrak{j}_* \mathfrak{j}^* E^n_{Z/X}$ *is a flabby sheaf of abelian groups on* X_{Nis}. *As a consequence* $E^n_{X^{(p-1/p)}}$ *is a flabby sheaf.*

The cofiber sequence in the Definition 2.3.18 gives rise to a long exact sequence of homotopy groups for each p. Using these long exact sequences for each p, we can construct a chain complex of flabby sheaves (with the exception of E^n_X) of abelian groups on X_{Nis},

$$0 \to E^n_X \xrightarrow{e} E^n_{X^{(0/1)}} \xrightarrow{d^0} E^{n+1}_{X^{(1/2)}} \xrightarrow{d^1} \dots \xrightarrow{d^{d-2}} E^{n+d-1}_{X^{(d-1/d)}} \xrightarrow{d^{d-1}} E^{n+d}_{X^{(d)}} \to 0 \tag{2.2}$$

Definition 2.3.21. Nisnevich Gersten complex of E in degree n, denoted as $\mathcal{G}^\bullet(E, n)$ is defined as follows

$$\mathcal{G}^p(E, n) := \bigoplus_{z \in X^{(p)}} \mathfrak{j}_* \mathfrak{j}^* E^n_{Z/X}$$

The differential $d^i : E^{n+i}_{X^{(i/i+1)}} \to E^{n+i+1}_{X^{(i+1/i+2)}}$ is constructed as follows. The homotopy cofibre sequence $E_{X^{(i+1)}} \to E_{X^{(i)}} \to E_{X^{(i/i+1)}}$ gives rise to a long exact sequence of homotopy groups, part of which looks like

$$\to E^{n+i}_{X^{(i+1)}} \to E^{n+i}_{X^{(i)}} \to E^{n+i}_{X^{(i/i+1)}} \xrightarrow{f} E^{n+i+1}_{X^{(i+1)}} \to$$

. Similarly homotopy cofibre sequence $E_{X^{(i+2)}} \to E_{X^{(i+1)}} \to E_{X^{(i+1/i+2)}}$ gives rise to

$$\to E^{n+i+1}_{X^{(i+2)}} \to E^{n+i+1}_{X^{(i+1)}} \xrightarrow{g} E^{n+i+1}_{X^{(i+1/i+2)}} \to$$

Then $d^i := g \circ f$
The above construction implies that the complex 2.2 is exact if following morphism are zero for all $p > 0$

$$E^n_{X^{(1)}} \to E^n_{X^{(0)}}$$
$$E^{n+p}_{X^{(p)}} \to E^{n+p}_{X^{(p-1)}}$$

However we will be interested in checking the exactness of above complex at a particular position p. The exactness of sheaves can be checked on stalks and stalks in Nisnevich topology are Henselian local rings. Thus we obtain the following conditions for the exactness of Gersten complex at position p, where $(E^n_X)^\sim$ is sheafification of (E^n_X)

- Exactness at $(E^n_X)^\sim$: if and only if for every $x \in X$ and all closed $Z \subset X$ of codimension greater than 0, the forget support map

$$E^n_{Z/X}(\operatorname{Spec} \mathcal{O}^h_{X,x}) \to E^n_X(\operatorname{Spec} \mathcal{O}^h_{X,x}) \tag{2.3}$$

  is trivial

- Exactness at $\mathcal{G}^p(E,n), p \geq 0$: If for all closed $Z \subseteq X$ with codimension greater than $p-1$ there exists closed $Z' \subseteq X$ of codimension greater than $p-2$ such that

$$E^{n+p}_{Z/X}(X) \to E^{n+p}_{Z'/X}(X) \tag{2.4}$$

  is trivial and for all closed Z of codimension greater than $p+1$, there exists closed Z' of codimension greater than p such that

$$E^{n+p+1}_{Z/X}(X) \to E^{n+p+1}_{Z'/X}(X) \tag{2.5}$$

  is trivial

3

Recollections from Algebraic Geometry

In this chapter we review some details about Algebraic stacks from [25] and étale cohomology from [19]

3.1 Stacks

Throughout Sch will denote the category of schemes and unless mentioned otherwise, we will be working with étale site in this section.

Definition 3.1.1. Let $p : F \to Sch$ be a functor. We say that F is **fibered category** over Sch if the following hold

- For every morphism $f : X \to Y$ in Sch and every lift y of Y (i.e $y \in F$ such that $p(y) = Y$) there is a lift $\phi : x \to y$ of f, i.e $p(x) = X$ and $p(\phi) = f$. Often we say that x is a pullback of y along f.

- For every pair of morphism $\phi : x \to z$, $\psi : y \to z$ and any morphism $f : p(x) \to p(y)$ such that $p(\psi) \circ f = p(\phi)$, there exists a unique lift $\chi : x \to y$ of f such that $\psi \circ \chi = \phi$.

For any $X \in Sch$, F_X is the category whose objects are objects in F lying over X and morphisms are morphisms lying over id_X.

Definition 3.1.2. Let $p : F \to Sch$ be a fibered category such that F_X is a groupoid, then $p : F \to Sch$ is called **category fibered in groupoids**.

Remark 3.1.3. *Corresponding to any presheaf $F : Sch^{op} \to SET$, one can obtain a category fibered in groupoids $\mathcal{F} \to Sch$. The objects of $\mathcal{F}$ are the pairs (X, x) where $X \in Sch$ and $x \in F(X)$. A morphism $f : (X, x) \to (Y, y)$ corresponds to a morphism $f_1 : X \to Y$ such that $F(f_1)(y) = x$. In particular any scheme (regarded as a functor of points) is a category fibered in groupoids.*

Definition 3.1.4. Let $p : F \to Sch$ and $p' : G \to Sch$ be two categories fibered in groupoids. A morphism between F and G' is a functor of categories $\phi : F \to G$ such that $p' \circ \phi = p$. Given two categories fibered in groupoids F and G', the category $HOM(F, G)$ has as objects the morphisms between F and G, while the morphisms in $HOM(F, G')$ are the natural transformations of the functors between F and G'.

The following is a version of Yoneda lemma for categories fibered in groupoids.

Proposition 3.1.5. *Let $F \to Sch$ be a category fibered in groupoids and X be a scheme. Then*

$$HOM(X, F) \to F_X$$

given by $\phi \mapsto \phi(id_X)$ is an equivalence of categories.

Definition 3.1.6. A morphism $F \to G$ of categories fibered in groupoids is said to be **representable by schemes** if for any any scheme X and any morphism $X \to G$, the resulting fibered product $F \times_G X$ is a scheme.

Remark 3.1.7. *For the construction of fiber product of categories fibered in groupoids see [25, Section 3.4.9].*

Definition 3.1.8. Let $\mathcal{X} \to Sch$ be a category fibered in groupoids and X a scheme with x, y objects in F_X. Then we have the following presheaf

$$\mathbf{Isom}(x, y) : Sch/X \to Set$$

such that

$$\mathbf{Isom}(x, y)(Y \xrightarrow{f} X) := \mathrm{Isom}_{F_Y}(f^*x, f^*y)$$

where f^*x and f^*y is the pullback of x and y respectively to F_Y along f.

Roughly speaking a stack is a category fibered in groupoids satisfying *sheaf condition*.

Definition 3.1.9. Let $p : \mathcal{X} \to Sch$ be a category fibered in groupoids and $\{X_i \to X\}_{i\in I}$ be morphisms in Sch. Let $\mathcal{X}(\{X_i \to X\})$ be category of collection of data $(\{E\}_{i\in I}, \{\sigma_{ij}\}_{i,i\in I})$, where $E_i \in \mathcal{X}(X_i)$ and for each $i, j \in I$, $\sigma_{i,j} : pr_1^* E_i \to pr_2^* E_j$ is an isomorphism in $\mathcal{X}(X_i \times_X X_j)$ such that the composition

$$pr_{12}^* pr_1^* E_i \xrightarrow{pr_{12}^* \sigma_{ij}} pr_{12}^* pr_2^* E_j = pr_{23}^* pr_1^* E_j \xrightarrow{pr_{23}^* \sigma_{jk}} pr_{23}^* pr_2^* E_k$$

equals the composition

$$pr_{12}^* pr_1^* E_i = pr_{13}^* pr_1^* E_i \xrightarrow{pr_{13}^* \sigma_{ik}} pr_{13}^* pr_2^* E_k = pr_{23}^* pr_2^* E_k$$

in $\mathcal{X}(X_i \times_X X_j \times_X X_k)$ The set of isomorphisms $\{\sigma_{ij}\}$is called **descent data** on $\{E_i\}_{i\in I}$. There is a natural functor

$$\epsilon : \mathcal{X}(X) \to \mathcal{X}(\{X_i \to X\})$$

A given descent data $\{\sigma_{ij}\}$ for $\{E_i\}_{i\in I}$ is called **effective** if it's in essential image of ϵ.

Definition 3.1.10. A category fibered in groupoids $p : \mathcal{X} \to Sch$ is called a **stack** if it satisfies the following conditions

- For all schemes $U \in Sch$ and for all $x, y \in \mathcal{X}(U)$, $\mathbf{Isom}(x, y)$ is a sheaf on $(Sch/U)_{ét}$
- Given any covering $\{X_i \to X\}$, for $X \in Sch$, any descent data with respect to it is effective.

Definition 3.1.11. Let S be a scheme. An **algebraic space** X/S is a functor $X : (Sch/S)^{op} \to Set$ satisfying the following

- X is an sheaf on the site $(Sch/S)_{ét}$
- The diagonal morphism $\Delta : X \to X \times X$ is representable by schemes.
- There exist a surjective étale morphism $U \to X$, where U is a scheme over S.

Definition 3.1.12. A morphism of stacks $\mathcal{X} \to \mathcal{Y}$ is **representable** if for every scheme U and a morphism $U \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} U$ is an algebraic space.

Definition 3.1.13. An **algebraic stack** $\mathcal{X}$ is a stack satisfying the following

- The diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.
- There exists a smooth surjective morphism $X \to \mathcal{X}$, where X is a scheme.

Example 3.1.14. *One important example, and one which this thesis is concerned with is moduli stack of vector bundles on a curve. Let C be a smooth projective curve of genus g over a field k. Fix a line bundle $\mathcal{L} \in \mathrm{Pic}(C)$. Then the following category fibered in groupoids, denoted $Bun_{n,\mathcal{L}}$, is indeed an algebraic stack*

$$Bun_{n,\mathcal{L}}(Y) = \{ \textit{category of rank } n \textit{ vector bundles on } C \times Y \textit{ with an isomorphism of its determinant to } p^*(\mathcal{L}), \textit{ where } p : C \times Y \to C \}$$

Example 3.1.15. *Let X be a scheme over a fixed base scheme S and G be a smooth group scheme over S with an action on X. Then the functor of groupoids $[X/G]$ is defined for any scheme Y/S as the groupoid with objects*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \big\downarrow & & \\ Y & & \end{array}$$

where $P \to Y$ is a principal G bundle and $P \to X$ is a G equivariant map. $[X/G]$ is known to be an algebraic stack and is often called a quotient stack.

Definition 3.1.16. Let $\mathcal{X}/S$ be an algebraic stack. A **coarse moduli space** for $\mathcal{X}$ is a morphism $\pi : \mathcal{X} \to X$, with X an algebraic space such that

- Given a morphism $\mathcal{X} \to Y$ with Y an algebraic space, it factors uniquely through a morphism $X \to Y$. In other words, π is initial among maps to algebraic spaces.
- $|\mathcal{X}(k)| \to X(k)$ is a bijection for every algebraically closed field k. $|\mathcal{X}(k)|$ denotes the set of isomorphism classes in the groupoid $\mathcal{X}(k)$.

The following theorem is an important result about existence of coarse moduli spaces

Theorem 3.1.17. ***(Keel-Mori theorem)*** *Assume S is locally noetherian and $\mathcal{X}/S$ is an algebraic stack locally of finite presentation with finite diagonal. Then there exists a coarse moduli $\mathcal{X} \to X$ for $\mathcal{X}$.*

3.1.1 Stacks as simplicial sheaves

Intuitively any category fibered in groupoids $p : F \to Sch$ can be thought of as a "functor" $F : Sch \to Grpd$, where $Grpd$ is the category of groupoids. However strictly speaking such an F is what is called a **lax 2-functor** as composition of two morphism is satisfied only upto a coherence condition.

Definition 3.1.18. [6, Lemma 2.4, Remark 2.5, Def. 2.6] Let $p : F \to C$ be a category fibered in groupoids. Then the functor $X \mapsto NF(X)$, for a smooth scheme X (where N is the nerve functor defined in Example 2.1.14) gives a simplicial presheaf and hence an object in unstable motivic homotopy category.

3.2 Étale cohomology

We will mostly follow [19]. All cohomology groups in this chapter are étale cohomology groups unless specified otherwise.

Definition 3.2.1. A **site** is a category $\mathcal{C}$ along with set of families of maps $(U_i \to U)_{i \in I}$, called coverings, for each object U in $\mathcal{C}$ satisfying the following

- For any morphism $V \to U$, $V \times_U U_i$ exists and is a covering of V
- given a coverings $U_i \to U$ and $(V_{ij} \to U_i)_{j \in J_i}$ for each U_i, $(V_{ij} \to U)_{i,j}$ is a covering of U.

We will denote the site with $(\mathcal{C}, \tau)$.

Definition 3.2.2. A sheaf on a site $(\mathcal{C}, \tau)$. is a presheaf $\mathcal{F}$ satisfying the following sheaf condition

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I \times I} \mathcal{F}(U_i \times U_j)$$

for any covering$(U_i \to U)_{i \in I}$ for any object U in $\mathcal{C}$.

Example 3.2.3. *Let X be a scheme. The **small étale site** on X, denoted $X_{\text{ét}}$, is category with objects $f : Y \to X$ such that f is étale. Morphism between objects $f : Y \to$*

*X and $f' : Y' \to X$ is a morphism $g : Y \to Y'$ such that $f = f' \circ g$ and covering for an object U is a surjective étale map $V \to U$. An **étale sheaf** is sheaf on $X_{\text{ét}}$.*

Remark 3.2.4. *The category of sheaves of abelian groups on $X_{\text{ét}}$ will be denoted $Sh(X_{\text{ét}})$. It is known to be an abelian category with enough injectives.*

Definition 3.2.5. Let X be Noetherian. A sheaf $\mathcal{F} \in Sh(X_{\text{ét}})$ is called **locally constant** on X if there exists an étale cover $U_i \to X$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf. $\mathcal{F} \in Sh(X_{\text{ét}})$ is called **constructible** if $X = \coprod_{i \in I} Z_i$, where Z_i are locally closed subschemes and I finite, such that $\mathcal{F}|_{Z_i}$ is locally constant with finite stalks.

Definition 3.2.6. In light of previous remark, given $\mathcal{F} \in Sh(X_{\text{ét}})$, its **étale cohomology** $H^i(X, \mathcal{F})$ is the i^{th} right derived functor of the (left exact) global section functor evaluated at $\mathcal{F}$

$$\Gamma : Sh(X_{\text{ét}}) \to Ab$$

where $\Gamma(\mathcal{F}) := \mathcal{F}(X)$

Example 3.2.7. *In étale topology all presheaves representable by schemes are sheaves. Therefore $\mathbb{G}_m := \mathbb{G}_m(U) := \Gamma(\text{Spec}\,(U), \mathcal{O}_U)^\times$ and μ_n where $\mu_n(U)$ is the subgroup of roots of unity in $\Gamma(\text{Spec}\,(U), \mathcal{O}_U)$, for $U \to X$, affine and étale, are étale sheaves. In fact, assuming n is invertible in every residue field of X, μ_n is a $\Lambda := \mathbb{Z}/n\mathbb{Z}$ module, constructible and locally isomorphic to it.*

Example 3.2.8. *Let X be a scheme and n be an integer invertible over any residue field of X. Let $\Lambda = \mathbb{Z}/n\mathbb{Z}$. Then we have a constructible sheaf, denoted $\Lambda(r)$, $r \in \mathbb{Z}$, on $X_{\text{ét}}$ defined as*

$$\Gamma(U, \Lambda(r)) = \begin{cases} \mu_n(\Gamma(U, \mathcal{O}_U))^{\otimes r}, & r > 0 \\ \Lambda, & r = 0 \\ \text{Hom}_\Lambda(\mu_n(\Gamma(U, \mathcal{O}_U))^{\otimes -r}, \Lambda), & r < 0 \end{cases}$$

*There is a short exact sequence of sheaves called **Kummer sequence***

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

The resulting long exact sequence of cohomology groups gives a morphism $H^1(X, \mathbb{G}_m) \simeq$ $\text{Pic}(X) \xrightarrow{c_1} H^2(X, \Lambda(1)) = H^2(X, \mu_n)$

Definition 3.2.9. For any sheaf $\mathcal{F}$ of locally constant Λ module on X and an integer r, we define $\mathcal{F}(r) := \mathcal{F} \otimes \Lambda(r)$.

Definition 3.2.10. Let $Z \hookrightarrow X$ be a closed subscheme and U be its complement. Then we have the left exact functor

$$\Gamma_Z(X, \_) : Sh(X_{\text{ét}}) \to Ab$$

such that $\Gamma_Z(X, \mathcal{F}) = \ker(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}))$. Its i^{th} right derived functor is denoted as $H^i_Z(X_{\text{ét}}, \_)$

Theorem 3.2.11. ***(Gabber purity)*** *Let $i : Z \hookrightarrow X$ be a closed immersion of regular Noetherian schemes, pure of codimension c. For any sheaf $\mathcal{F}$ of locally constant Λ module on X there are canonical isomorphisms*

$$H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r_Z(X, \mathcal{F})$$

for all $r \geq 0$.

For more details about Gabber purity and its proof see [10].

Definition 3.2.12. Assume the setup of the previous theorem. Then $cl_X(Z)$ called **class map** denotes the image of 1 in the following map

$$\Lambda = H^0(Z, \Lambda) \to H^{2c}_Z(X, \Lambda(c)) \to H^{2c}(X, \Lambda(c))$$

The first map is an isomorphism from the previous theorem for $\mathcal{F} = \Lambda(c)$, $r = 2c$ and second morphism is the forget support map.

Remark 3.2.13. *Let $i : Z \hookrightarrow X$ be a closed immersion of regular Noetherian schemes and Z is of codimension 1. Then $cl_X(Z) = c_1(Z)$ in $H^2(X, \Lambda(1))$ where cl_X is the class map of previous definition and c_1 is the map defined in Example 3.2.8.*

The following result is known in literature as Gabber rigidity or Gabber affine proper base change theorem, see [11, Theorem 1]

Theorem 3.2.14. *Let A be a Henselian local ring with residue field k. Then for any constructible sheaf $\mathcal{F}$ on $\operatorname{Spec} A$, with the order of torsion coprime to char(k), $H^n(\operatorname{Spec} A, \mathcal{F}) \simeq H^n(\operatorname{Spec} k, \mathcal{F}|_k)$ for all $n \geq 0$.*

3.2.1 Derived categories

Let $Ch(Sh(X_{\text{ét}}))$ and $Ch^+(Sh(X_{\text{ét}}))$ be the category of co-chain complexes in the abelian category $Sh(X_{\text{ét}})$ and co-chain complexes $\mathcal{A}^\bullet$ such that $A^i = 0$ for $i << 0$ respectively. Then akin to model structure described in section 2.1.2, there is a model structure on $Ch(Sh(X_{\text{ét}}))$ and $Ch^+(Sh(X_{\text{ét}}))$ where weak equivalence are quasi-isomorphisms and fibrant objects are complexes with injective objects. See [18, Theorem 2.2]. The homotopy categories thus obtained are called **derived category**, denoted $\mathcal{D}(X)$ and $\mathcal{D}^+(X)$ respectively. While these derived categories are not abelian they are triangulated categories. $\mathcal{D}^b(X)$ will denote the derived category of chain complexes whose cohomology groups are bounded. $\mathcal{D}^b_c(X)$ will denote the derived category of chain complexes whose cohomology groups are constructible sheaves and $\mathcal{D}^b_c(X, \Lambda)$ the derived category of chain complexes whose cohomology groups are constructible sheaves of Λ modules. Morever given any abelian category $\mathcal{C}$ one can construct $\mathcal{D}^+(\mathcal{C})$ in a similar fashion.
Let $\mathcal{F} : Sh(X_{\text{ét}}) \to \mathcal{C}$, be a left exact functor and $\mathcal{C}$ be any abelian category. Then it's **right derived functor**, denoted $R\mathcal{F}$ is the functor

$$R\mathcal{F} : \mathcal{D}^+(X) \to \mathcal{D}^+(\mathcal{C})$$

such that $R\mathcal{F}(\mathcal{A}^\bullet) = \mathcal{F}(\mathcal{I}^\bullet)$ where $\mathcal{I}^\bullet$ is a complex of injective objects and quasi isomorphic to $\mathcal{A}^\bullet$. In other words $\mathcal{I}^\bullet$ is a fibrant replacement of $\mathcal{A}^\bullet$. We denote by $R^i\mathcal{F} := H^i(R\mathcal{F})$.

Example 3.2.15. *Let $i : Z \hookrightarrow X$ be a closed subscheme and $j : X \setminus Z = U \hookrightarrow X$ be the open immersion. For a sheaf $\mathcal{F} \in Sh(X_{\text{ét}})$, $i^!(\mathcal{F}) := i^*(\ker(\mathcal{F} \to j_*j^*\mathcal{F}))$. Moreover we have the following adjunction*

$$i_* : Sh(Z_{\text{ét}}) \rightleftarrows Sh(X_{\text{ét}}) : i^!$$

Hence $i^!$ is left exact. Hence we have an induced derived functor

$$Ri^! : \mathcal{D}^b(X,\Lambda) \to \mathcal{D}^b(Z,\Lambda)$$

,Then Gabber purity implies the following isomorphism

$$R^{2c} i^! \mathcal{A}^\bullet \simeq i^*(\mathcal{A}^\bullet)(-c)[2c]$$

for any complex $\mathcal{A}^\bullet \in \mathcal{D}^b(X,\Lambda)$

Example 3.2.16. *The left exact functor $\Gamma_Z(X,\_)$ defined in Definition3.2.10 leads to a derived functor*

$$R\Gamma_Z(X,\_) : \mathcal{D}(X) \to \mathcal{D}(Ab)$$

Let $\mathcal{F} \in Sh(X_{\text{ét}})$ considered as an object in $\mathcal{D}(X)$, as complex concentrated in degree 0. Then $R^i\Gamma_Z(X,\mathcal{F}) \simeq H^i_Z(X,\mathcal{F})$.

Remark 3.2.17. *Observe the isomorphism $\operatorname{Hom}_{\mathcal{D}^b(X,\Lambda)}(\Lambda[-2],\Lambda(1)) \cong H^2(X,\Lambda)$. Therefore if a cohomology class $[c] \in H^2(X,\Lambda)$ is trivial then the corresponding morphism $\Lambda[-2] \to \Lambda(1)$ is trivial in $\mathcal{D}^b(X,\Lambda)$. For instance if X is a local scheme then $\operatorname{Pic}(X)$ is trivial and so $cl_{Z/X}(1)$ will be zero in $H^2(X,\Lambda)$.*

Remark 3.2.18. *Let $\mathcal{A}^\bullet$ be any complex in $\mathcal{D}^b(S,\Lambda)$. By stable Dold Kan correspondence alluded to in Remark 2.3.9, we have a corresponding spectrum denoted $E^{\text{ét}}(\mathcal{A}^\bullet) \in Spt_{S^1}(\mathcal{S}m_{S,\text{ét}})$. Then $E(\mathcal{A}^\bullet) := R\epsilon_* E^{\text{ét}}(\mathcal{A}^\bullet) \in Spt_{S^1}(\mathcal{S}m_S)$, where $\epsilon : Spt_{S^1}(\mathcal{S}m_{S,\text{ét}}) \to Spt_{S^1}(\mathcal{S}m_S)$ is induced by canonical functor from étale site to Nisnevich site. By [27, Lemma 6.3], $E(\mathcal{A}^\bullet)$ is $\mathbb{A}^1$ local.*

4

$\mathbb{A}^1$-connectivity of $Bun_{n,\mathcal{L}}$ and its consequences

In this chapter we prove the first main result of the thesis namely Theorem 4.0.1 about $\mathbb{A}^1$ connectivity of the moduli stack of vector bundles on a curve. We then explore some consequences of the theorem, more specifically, classification of projective bundles on a curve upto $\mathbb{A}^1$-h-cobordism. Throughout, unless mentioned otherwise, the base field k is assumed to be infinite.

Theorem 4.0.1. *$Bun_{n,\mathcal{L}}$ is $\mathbb{A}^1$ connected for any curve C and $\mathcal{L} \in \mathrm{Pic}(C)$.*

The proof of the Theorem 4.0.1 relies on finding an explicit $\mathbb{A}^1$-concordance (see [1, Definition 5.1] or definition 4.1.1) between a vector bundle $\mathcal{E}$ of rank n and determinant $\mathcal{L}$ to the vector bundle $\mathcal{O}_C^{n-1} \oplus \mathcal{L}$. This is achieved by induction on n. The results produced here are from [15].

4.1 $\mathbb{A}^1$-concordances of vector bundles and their classification

Definition 4.1.1. Given a scheme X/k. Then two vector bundles $\mathcal{E}_0$ and $\mathcal{E}_1$ on X are said to be directly $\mathbb{A}^1$-concordant if there exists a vector bundle $\mathcal{E}$ on $X \times \mathbb{A}^1$ such that $i_0^*\mathcal{E} = \mathcal{E}_0$ and $i_1^*\mathcal{E} = \mathcal{E}_1$, where $i_k : X \times \{k\} \hookrightarrow X \times \mathbb{A}^1$, for $k = 0, 1$. $\mathcal{E}_0$ and $\mathcal{E}_1$ on X are $\mathbb{A}^1$-concordant if they are equivalent under equivalence relation generated by direct $\mathbb{A}^1$ concordance.

Lemma 4.1.2. *Let $\mathcal{E}_0$ and $\mathcal{E}_1$ are $\mathbb{A}^1$-concordant vector bundles on a normal variety X and $\mathcal{V}$ be a vector bundle on $X \times \mathbb{A}^1$. Then $(i_0^*(\mathcal{V}) \otimes \mathcal{L}) \oplus \mathcal{E}_0$ and $(i_1^*(\mathcal{V}_1) \otimes \mathcal{L}) \oplus \mathcal{E}_1$ are $\mathbb{A}^1$-concordant, for any $\mathcal{L} \in \mathrm{Pic}(X)$.*

Proof. It is enough to prove the lemma in the case when $\mathcal{E}_0$ and $\mathcal{E}_1$ are directly $\mathbb{A}^1$-concordant. Let the direct $\mathbb{A}^1$- concordance be given by a vector bundle $\mathcal{E}$ on $X \times \mathbb{A}^1$. Note that $p^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X \times \mathbb{A}^1)$ (where $p : X \times \mathbb{A}^1 \to X$) is an isomorphism (see [13, II, Prop. 6.6]) with the inverse given by $i_0^* = i_1^*$. Then the lemma immediately follows from the definition by considering the vector bundle $(\mathcal{V} \otimes p^*\mathcal{L}) \oplus \mathcal{E}$ and exactness of pullback functor for vector bundles. □

In the the light of previous lemma, the following corollary is rather obvious but we state it nevertheless, keeping in mind its direct application in the proof of Theorem 4.0.1.

Corollary 4.1.3. *Let $\mathcal{E}_0$ and $\mathcal{E}_1$ are $\mathbb{A}^1$-concordant vector bundles on a projective normal variety X Then following statements hold*

1. *$\mathcal{O}_X^n \oplus \mathcal{E}_0$ and $\mathcal{O}_X^n \oplus \mathcal{E}_1$ are $\mathbb{A}^1$-concordant for any $n \geq 0$.*
2. *$\mathcal{O}_X(m) \oplus \mathcal{E}_0$ and $\mathcal{O}_X(m) \oplus \mathcal{E}_1$ are $\mathbb{A}^1$-concordant for any m.*

Proof. For the first statement take $\mathcal{V} = O^n_{X \times \mathbb{A}^1}$, keeping notation of previous lemma in mind.
For the second statement take $\mathcal{V} = O_{X \times \mathbb{A}^1}$ and $\mathcal{L} = p^*\mathcal{O}_X(m)$. □

Proposition 4.1.4. *Let $0 \to \mathcal{E}_0 \to \mathcal{E} \to \mathcal{E}_1 \to 0$ be any short exact sequence of vector bundles on a projective scheme X. Then $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{E}_0 \oplus \mathcal{E}_1$.*

Proof. Consider $\mathcal{E}$ as an element in $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_0)$. In case $\mathcal{E}$ is trivial then our claim is obvious, so assume to the contrary. Consider the the moduli functor $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_0)$ given by $Y \mapsto \mathrm{Ext}^1(\mathrm{p}^*\mathcal{E}_1, \mathrm{p}^*\mathcal{E}_0)$, where $p : X \times Y \to X$. It's well known that this functor is representable by $\mathbb{A}^n_k$(see [16, Proposition 3.1]), where $n = \dim(\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{E}_0))$ as a vector space over k. Note that $n > 0$ by assumption that $\mathcal{E}$ is non trivial. Therefore by representability there is a universal class $\mathcal{V}$ (of vector bundle) on $X \times \mathbb{A}^n$ whose pullback to $X \times t_i \hookrightarrow X \times \mathbb{A}^n$, $i = 0, 1$ is $\mathcal{E}$ and $\mathcal{E}_0 \oplus \mathcal{E}_1$ respectively for some $t_i \in \mathbb{A}^n$. We connect these t_i's via an $\mathbb{A}^1$ and restrict $\mathcal{V}$ to $X \times \mathbb{A}^1$ to obtain a direct $\mathbb{A}^1$ concordance between $\mathcal{E}$ and $\mathcal{E}_0 \oplus \mathcal{E}_1$. □

Theorem 4.1.5. *Let $\mathcal{E}$ and $\mathcal{F}$ be rank n vector bundles on the curve C. Then the following hold*

1. *$\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-1} \oplus \det(\mathcal{E})$.*

2. *$\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{F}$ iff $\det(\mathcal{E}) \cong \det(\mathcal{F})$.*

Proof. Clearly, (1) implies one direction of (2). We first prove (1) for the case when $n = 2$. For general case we will use induction.

Case 1: $n = 2$. First assume $\mathcal{E}$ is globally generated. Then by [13, II, Exercise 8.2], we have the following exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{E}' \to 0$$

$\mathcal{E}'$ is a line bundle and by the additivity property of first chern class of vector bundles over exact sequence

$$c_1(\mathcal{L}) = c_1(\mathcal{E}) = c_1(\mathcal{O}_C).c_1(\mathcal{E}') = c_1(\mathcal{E}').$$

Therefore by Proposition 4.1.4, $\mathcal{E}$ is directly $\mathbb{A}^1$- concordant to $\mathcal{O}_C \oplus \mathcal{L}$. For a general $\mathcal{E}$, choose $m >> 0$ such that $\mathcal{E}(m)$, $\mathcal{L}(m)$ are globally generated. Then again by applying [13, II, Exercise 8.2] we get a short exact sequence for $\mathcal{E}(m)$ which we tensor by $\mathcal{O}(-m)$ to obtain the following

$$0 \to \mathcal{O}_C(-m) \to \mathcal{E} \to \mathcal{L}(m) \to 0$$

This proves $\mathcal{E}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$. As the final step we now prove that $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$. Note that m is choosen such that $\mathcal{L}(m)$ is globally generated, therefore $\mathcal{O}_C(m) \oplus \mathcal{L}(m)$ is globally generated. Hence we have a short exact which shows $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$.

$$0 \to \mathcal{O}_C(-m) \to \mathcal{O}_C \oplus \mathcal{L} \to \mathcal{L}(m) \to 0$$

Therefore, $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C \oplus \mathcal{L}$.

Case 2: Now we handle the general case. So assume $n > 2$ and choose m such that $\mathcal{E}(m)$ and $\mathcal{L}(m)$ are globally generated. Then we have a short exact sequence giving a direct $\mathbb{A}^1$-concordance between $\mathcal{E}$ and $\mathcal{O}_C(-m) \oplus \mathcal{E}'$, where $\mathcal{E}'$ is a vector bundle of rank $n-1$ with determinant $\mathcal{L}(m)$. By induction, $\mathcal{E}'$ is $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$. Therefore by second statement of Corollary 4.1.3 we have an $\mathbb{A}^1$-concordance between

$\mathcal{O}_C(-m) \oplus \mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$ and $\mathcal{O}_C(-m) \oplus \mathcal{E}'$. Hence $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}(-m) \oplus \mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$. Now by the last short exact sequence in the previous case $\mathcal{O}_C \oplus \mathcal{L}$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C(-m) \oplus \mathcal{L}(m)$, which implies – by first statement of Corollary 4.1.3 – that $\mathcal{O}(-m) \oplus \mathcal{O}_C^{n-2} \oplus \mathcal{L}(m)$ is directly $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-1} \oplus \mathcal{L}$, thus finishing the proof of (1).

For proving (2), we note that the reverse implication directly follows from (1). It is enough to show that when $\mathcal{E}$ is directly $\mathbb{A}^1$ concordant to $\mathcal{F}$, $\det(\mathcal{E}) \cong \det(\mathcal{F})$. So we have a vector bundle $\mathcal{E}'$ on $C \times \mathbb{A}^1$ such that $i_0^*\mathcal{E}' = \mathcal{E}$ and $i_1^*\mathcal{E}' = \mathcal{F}$. Then by property of chern classes we have $(i_0)_*(c_1(i_0^*\mathcal{E})) = c_1(\mathcal{E}') = (i_1)_*(c_1(i_1^*\mathcal{F}))$. But $c_1(i_0^*\mathcal{E}') = c_1(\det\mathcal{E})$ and $c_1(i_1^*\mathcal{E}') = c_1(\det\mathcal{F})$. Moreover, $(i_0)_* = (i_1)_*$ are isomorphisms on Picard group, hence on CH^1, as $\mathrm{Pic}(C) \cong \mathrm{Pic}(C \times \mathbb{A}^1)$. Therefore $\det(\mathcal{E}) \cong \det(\mathcal{F})$.

□

Proof of Theorem 4.0.1. We regard $Bun_{n,\mathcal{L}}$ as a simplicial sheaf. By definition, any two F valued points of $Bun_{n,\mathcal{L}}$ are two rank n (with determinant condition) vector bundles, say $\mathcal{E}_0$ and $\mathcal{E}_1$ on C. A morphism $\mathbb{A}^1_F \to Bun_{n,\mathcal{L}}$ is a vector bundle $\mathcal{E}$ on $C \times \mathbb{A}^1$. Then $\mathcal{E}_0$ and $\mathcal{E}_1$ are naively $\mathbb{A}^1$-homotopic if and only if they are $\mathbb{A}^1$-concordant. By Theorem 4.1.5 both $\mathcal{E}_0$ and $\mathcal{E}_1$ are $\mathbb{A}^1$-concordant to $\mathcal{O}_C^{n-1} \oplus \mathcal{L}$. Hence they are $\mathbb{A}^1$-concordant to each other. Therefore by Lemma 2.2.12, $Bun_{n,\mathcal{L}}$ is $\mathbb{A}^1$-connected. □

Motivated by the question of $\mathbb{A}^1$ connectivity of moduli stack of stable vector bundles, we observe in the example below that there does not seem to be an immediate way of concluding $\mathbb{A}^1$-connectivity of a stack by looking at its coarse moduli space.

Example 4.1.6. *Let C be a curve of genus 2 over $\mathbb{C}$. In particular it is a hyperelliptic curve (See [13, IV, Exercise 1.7(a)]). Therefore, there is a finite morphism $f : C \to \mathbb{P}^1$ of degree 2 and we have an action of the finite group $\mathbb{Z}_2$ on C. By [13, IV, Exercise 2.2(a)], such a morphism is unramified at all but 6 points (denoted as closed subscheme Z') of C. So the action of $\mathbb{Z}_2$ is free on $C \setminus Z'$. Let Z denote the closed subset in $\mathbb{P}^1$ corresponding to the 6 branched points. The quotient stack $\left[C/\mathbb{Z}_2\right]$ (see Example 3.1.15) has coarse moduli space $\mathbb{P}^1$ and the morphism $\pi : \left[C/\mathbb{Z}_2\right] \to \mathbb{P}^1$ gives an isomorphism of an open subscheme of $\left[C/\mathbb{Z}_2\right]$ with $\mathbb{P}^1 \setminus Z$.*

We now claim that $\left[C/\mathbb{Z}_2\right]$ is not $\mathbb{A}^1$ connected. This follows from [21, Theorem 6.50] applied to the morphism $E(G) \times C \to \left[C/\mathbb{Z}_2\right]$. This morphism is a G-torsor over $\left[C/\mathbb{Z}_2\right]$

and $\pi_0^{\mathbb{A}^1}(\mathbb{Z}_2) \cong \mathbb{Z}_2$ being a finite abelian group is a strictly $\mathbb{A}^1$-invariant sheaf. Hence we can apply the theorem to aforementioned morphism and obtain a long exact sequence of $\mathbb{A}^1$ homotopy groups/sets. But on the account of $E(G)$ being simplicially contractible and C being $\mathbb{A}^1$-rigid (as all curves of genus $g > 0$ are) $\pi_0^{\mathbb{A}^1}(E(G) \times C) \cong \pi_0^{\mathbb{A}^1}(C) \cong C$. So by long exact sequence, $\left[C/\mathbb{Z}_2\right]$ being $\mathbb{A}^1$-connected would imply surjection of finite group $\mathbb{Z}_2$ on C, which can not happen.

4.2 Applications

Definition 4.2.1. [2, Definition 3.1.1] Let X_0 and X_1 be smooth and proper varieties over k. They are **directly $\mathbb{A}^1$-h cobordant** if there exists a smooth scheme X with $f : X \to \mathbb{A}^1$ a proper surjective morphism such that

1. fibers of f over 0 and 1 are X_0 and X_1 respectively
2. the natural maps $X_i \hookrightarrow X$ for $i = 0, 1$ are $\mathbb{A}^1$-weak equivalences.

X_0 and X_1 are $\mathbb{A}^1$-h cobordant if they are equivalent under the equivalence relation generated by direct $\mathbb{A}^1$-h-cobordance.

Remark 4.2.2. *While $\mathbb{A}^1$-concordance is a relation between vector bundles, $\mathbb{A}^1$-h cobordism a relation between proper schemes. Note that by [1, Lemma 6.4], projectivization of $\mathbb{A}^1$-concordant vector bundles are $\mathbb{A}^1$-h cobordant.*

As applications of results in the previous section we first classify projective bundles on a curve upto $\mathbb{A}^1$-h-cobordism.

Theorem 4.2.3. *Let $X = \mathbb{P}_C(\mathcal{E})$ and $Y = \mathbb{P}_C(\mathcal{F})$ be $\mathbb{P}^n$-bundles over C. Then the following are equivalent :*

1. *X and Y are $\mathbb{A}^1$- weakly equivalent.*
2. *X and Y are $\mathbb{A}^1$-h cobordant.*
3. *$\det(\mathcal{E}) \otimes \det(\mathcal{F})^{-1} = \mathcal{L}^{\otimes n+1}$ for some $\mathcal{L} \in \mathrm{Pic}(C)$.*

In another application of our theorem we answer a question raised in [1]: whether a variety which is $\mathbb{A}^1$-h-cobordant to a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ has a structure of $\mathbb{P}^1$-bundle over $\mathbb{P}^2$. The answer is no and we prove in following theorem that the suggested example in op. cit. indeed works.

Theorem 4.2.4. *Let $X := \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{O}(1)$ on $\mathbb{P}^1$. Then X is $\mathbb{A}^1$-h-cobordant to $\mathbb{P}^1 \times \mathbb{P}^2$ but doesn't have the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.*

We now paraphrase the classification of $\mathbb{P}^n$-bundles on $\mathbb{P}^1$ up to $\mathbb{A}^1$-weak equivalence proved in [2] to highlight that Theorem 4.2.3 is its direct generalization to a general curve.

Proposition 4.2.5. *[2, Proposition 3.2.10] $\mathbb{P}(\mathcal{O}^n_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a))$ and $\mathbb{P}(\mathcal{O}^n_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(b))$ are $\mathbb{A}^1$-weakly equivalent if and only if they are $\mathbb{A}^1$-h-cobordant if and only if $n+1$ divides $a-b$.*

Note that in case of $C = \mathbb{P}^1$ the condition $\det(\mathcal{E}) \otimes \det(\mathcal{F})^{-1} = \mathcal{L}^{\otimes n+1}$ in Theorem 4.2.3 exactly translates to the fact that $n+1$ divides $a-b$ as stated in the previous proposition. This is due to $\mathrm{Pic}(\mathbb{P}^1)$ being isomorphic to $\mathbb{Z}$. For a general curve Picard group is much more complicated and humongous (think Jacobian variety of a curve) so one doesn't get any further simplification. We now prove Theorem 4.2.3, which is the extension of previous proposition.

Proof of 4.2.3. (3) $\implies$ (2): By Theorem 4.1.5, $\mathcal{E}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}^n_C \oplus \mathcal{L}_1$, where $\mathcal{L}_1 = \det(\mathcal{E})$ and $\mathcal{F}$ is $\mathbb{A}^1$-concordant to $\mathcal{O}^n_C \oplus \mathcal{L}_2$, where $\mathcal{L}_2 = \det(\mathcal{F})$. Hence X and $\mathbb{P}_C(\mathcal{O}^n_C \oplus \mathcal{L}_1)$ are $\mathbb{A}^1$-h cobordant. In the exact same manner, Y and $\mathbb{P}_C(\mathcal{O}^n_C \oplus \mathcal{L}_2)$ are $\mathbb{A}^1$-h cobordant. Suppose $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} = \mathcal{L}^{\otimes n+1}$. That implies $\mathcal{L}_1 = \mathcal{L}^{\otimes n+1} \otimes \mathcal{L}_2$ for some $\mathcal{L} \in \mathrm{Pic}(C)$. Let $\mathcal{E}' = (\mathcal{O}^n_C \oplus \mathcal{L}_2) \otimes \mathcal{L}$. Then $\det(\mathcal{E}') = \mathcal{L}_1$. Therefore $\mathbb{P}(\mathcal{E}')$ is $\mathbb{A}^1$-h-cobordant to $\mathbb{P}(\mathcal{O}^n_C \oplus \mathcal{L}_1)$. Furthermore, $\mathbb{P}(\mathcal{E}')$ is isomorphic (as a scheme) to $\mathbb{P}(\mathcal{O}^n_C \oplus \mathcal{L}_2)$ by the general fact that tensoring a vector bundle by a line bundle gives an isomorphism of projectivization of the two vector bundles. This proves X and Y are $\mathbb{A}^1$-h-cobordant.

(2) $\implies$ (1): this is immediate from the definition of $\mathbb{A}^1$-h-cobordism.

(1) $\implies$ (3) : Since Chow rings factor through $\mathbb{A}^1$-equivalence, it is enough to show

that the Chow rings of $\mathbb{P}(\mathcal{O}_C^n \oplus \mathcal{L}_1)$ and $\mathbb{P}(\mathcal{O}_C^n \oplus \mathcal{L}_2)$ are not isomorphic if $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} \neq \mathcal{L}^{\otimes n+1}$ for any $\mathcal{L} \in \mathrm{Pic(C)}$. The Chow ring of C – which is simply $\mathbb{Z} \oplus \mathrm{Pic}(C)$, with product of any two line bundles under the ring structure being zero – is denoted R. The Chow ring of $\mathbb{P}(\mathcal{E}_1)$ is $R_1 := R[\zeta]/(\zeta^{n+1} + c_1(\mathcal{E}_1)\zeta^n)$. But $c_1(\mathcal{E}_1) = c_1(\mathcal{L}_1)$. In R_1, both ζ and any element $x \in \mathrm{Pic}(C)$ have grading 1 with $x.y = 0$ for $x, y \in \mathrm{Pic}(C)$. Let's assume we have a graded ring isomorphism ϕ between R_1 and $R_2 := R[\sigma]/(\sigma^{n+1} + c_1(\mathcal{L}_2)\sigma^n)$. Then such an isomorphism has to respect the grading and hence $\phi(\zeta) = x + a.\sigma$, where $x \in \mathrm{Pic}(C)$ and $a \in \mathbb{Z}$. Furthermore by graded ring structure of R_1, as discussed before, $x^i = 0$ for any $i > 1$. Moreover $\phi(\zeta^{n+1} + c_1(\mathcal{L}_1)\zeta^n)$ has to be divsible by $\sigma^{n+1} + c_1(\mathcal{L}_2)\sigma^n$ in R_2. We expand $\phi(\zeta^{n+1} + c_1(\mathcal{L}_1)\zeta^n)$ as $a^{n+1}.\sigma^{n+1} + a^n\sigma^n((n+1).x + c_1(\mathcal{L}_1))$ and this expression is divisible by $\sigma^{n+1} + c_1(\mathcal{L}_2)\sigma^n$. Comparing coeffecients we conclude that $a = \pm 1$ and $c_1(\mathcal{L}_1) - c_1(\mathcal{L}_2) = \pm(n+1)x$. This implies that $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} = \mathcal{L}^{\otimes n+1}$, where $c_1(\mathcal{L}) = \pm x$. □

Now, we answer a question raised in [1], negatively.

Question 4.2.6. [1, Question 6.9.1] If X is any smooth projective variety that is $\mathbb{A}^1$-h-cobordant to a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, does X have the structure of a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$?

The authors further add the answer is possibly no and non-trivial rank three vector bundles over $\mathbb{P}^1$ deformable to the trivial one are the likely counterexamples. We now prove Theorem 4.2.4 which shows that the example alluded to above is indeed a correct counterexample.

Proof of 4.2.4. : By Theorem 4.1.5, $X := \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} \mathbb{P}^1$ is $\mathbb{A}^1$-h-cobordant (though not isomorphic) to trivial $\mathbb{P}^2$- bundle on $\mathbb{P}^1$, namely, $\mathbb{P}^1 \times \mathbb{P}^2$. Suppose $X \simeq \mathbb{P}_{\mathbb{P}^2}(\mathcal{E}') := Y \xrightarrow{\phi} \mathbb{P}^2$ for some rank 2 vector bundle $\mathcal{E}'$ on $\mathbb{P}^2$. We thus have the following diagram

$$\begin{array}{ccc} X \simeq Y & \xrightarrow{\phi} & \mathbb{P}^2 \\ \big\downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^1 & & \end{array}$$

Without loss of generality we can assume (by twisting $\mathcal{E}'$ with a suitable line bundle in $\mathrm{Pic}(\mathbb{P}^2)$), $c_1(\mathcal{E}') \in \{0, 1\}$. Since Y is $\mathbb{A}^1$-weakly equivalent to trivial bundle on $\mathbb{P}^2$, their Chow rings are isomorphic. By [1, Lemma 4.5], we have $c_1(\mathcal{E}')^2 - 4c_2(\mathcal{E}') = 0$. So

$c_1(\mathcal{E}') = 0 = c_2(\mathcal{E}')$. It thus suffices to show that $\mathcal{E}'$ splits as a direct sum of line bundles as this will prove that $\mathcal{E}' \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$ which will be a contradiction to our assumption.

To that end we will prove that ϕ has a section. That section will correspond to a short exact sequence of the form

$$0 \to \mathcal{L}_1 \to \mathcal{E}' \to \mathcal{L}_2 \to 0$$

As both chern classes of $\mathcal{E}'$ vanish, by property of chern classes over short exact sequences, both $\mathcal{L}_1$ and $\mathcal{L}_2$ will be trivial. Therefore such a short exact sequence has to be a split one.

Let $F \hookrightarrow X$ be the fiber of π over a point in $\mathbb{P}^1$, say x. We claim ϕ maps F isomorphically onto $\mathbb{P}^2$. First we claim that $\phi_{|F}$ is surjective. Suppose not, then $Z := \phi(F)$ is either a point or an irreducible curve (not necessarily smooth) in $\mathbb{P}^2$. The former possibility can be easily discounted on the account of ϕ being a $\mathbb{P}^1$-bundle map over $\mathbb{P}^2$. The latter case implies existence of a smooth $\mathbb{P}^1$ in $\mathbb{P}^2$ which contracts to a smooth point of Z. By taking another smooth point of Z we have another $\mathbb{P}^1$ in $\mathbb{P}^2$, which contracts to this point. However any two lines in $\mathbb{P}^2$ intersect, so they can not contract to two different points. So $\phi|_{\mathbb{P}^2}$ is a degree d morphism to $\mathbb{P}^2$ with $d \geq 1$.

It suffices to show that $d = 1$. We prove this by comparing the graded ring isomorphism induced on Chow rings of X and Y. Chow ring of X is $R_1 := \mathbb{Z}[x, y]/(x^2, y^3)$, where

(i) x is the divisor $\mathbb{P}^2$ as a fiber over a point of $\mathbb{P}^1$

(ii) y corresponds to a divisor D', such that the pushforward $\pi_*(\mathcal{O}_X(D'))$ to $\mathbb{P}^1$ is the vector bundle $\mathcal{E}$.

Similarly Chow ring of Y is $R_2 := \mathbb{Z}[s, t]/(s^2, t^3)$ where

(i) t corresponds to fiber of $\mathbb{P}^1$ (as a degree 1 curve in $\mathbb{P}^2$) via ϕ

(ii) s corresponds to a divisor D, such that the pushforward $\phi_*(\mathcal{O}_Y(D))$ to $\mathbb{P}^2$ is the rank two vector bundle $\mathcal{E}'$.

A simple calculation shows that the graded ring isomoprhism between R_1 and R_2 is given by $x \mapsto \pm s$ and $y \mapsto \pm t$. This implies s is equivalent (in Chow ring) to $\mathbb{P}^2$. Grauert's theorem ([13, III, Corollary 12.9]) implies that intersection multiplicity of $S = \mathbb{P}^2$ with any fiber of the map ϕ is 1. This can not happen unless $d = 1$ because if not, one can consider a point z in $\mathbb{P}^2$ such that the set $\phi|_{\mathbb{P}^2}^{-1}(z)$ has more than one point. This will force $\phi^{-1}(z) = \mathbb{P}^1$ to intersect $\mathbb{P}^2$ in more than one point, which as we just proved can not happen. This finishes the proof. $\square$

5

Exactness of Gersten complex over a general base

In this chapter we prove that Nisnevich Gersten complex, for $E \in Spt_{S^1}(\mathcal{S}m_S)$, defined in Definition 2.3.21 is exact barring at few places, which is dictated by the dimension of the base S. The idea is to verify the conditions stated in 2.3, 2.4 and 2.5. Main ingredient in the proof is Gabber presentation lemma proved in [7] and [9], which allows us to check those conditions. We also give the conditions under which the Gersten complex is exact at all places. When dimension of the base is 1 all the results produced here were proved in [27] whereas the infinite field case follows from [5] and [4]. The general base case presented here is from [8].

5.1 Gersten resolution

As a consequence of Gabber presentation lemma proved in [7] and [9] we have the following lemma

Lemma 5.1.1. *[9, Remark 3] Let X be an essentially smooth henselian local scheme over a Henselian local scheme S with the closed point s and let $Z \subset X$ be a closed subscheme of positive relative codimension($\dim Z_s < \dim X_s$). Then there is a map $p : X \to \mathbb{A}^1_V$, where $V = (\mathbb{A}^{\dim X-1}_S)^h_0$ is the henselisation at the point 0 and $\dim X$ is relative dimension of X/S, such that p is étale, p induces an isomorphism $Z \simeq p(Z)$, and $p(Z)$ is finite over V. Consequently, giving the following Nisnevich distinguished square:*

$$\begin{array}{ccc} X \setminus Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathbb{A}^1_V \setminus p(Z) & \longrightarrow & \mathbb{A}^1_V \end{array}$$

Remark 5.1.2. *Note that in the Nisnevich distinguished square in the above lemma V is a limit of Nisnevich neighborhood of $\mathbb{A}^{\dim X-1}$, whereas in [26] and [7] it is a Zariski neighborhood in $\mathbb{A}^{\dim X-1}$.*

The following proposition generalises [27, Proposition 5.9] to a more general base. The proof is exactly the same, except for the input from the presentation lemma.

Proposition 5.1.3. *Let $E \in Spt^1_S(Sm_s)$ be a $\mathbb{A}^1$-Nisnevich local fibrant spectrum. Let $X \in \mathcal{S}m_S$ be irreducible scheme, $Z \hookrightarrow X$ be a closed subscheme and x be a point in Z lying above $s \in S$, such that $\dim(Z_s) < \dim(X_s)$. Then Nisnevich-locally around x there exist*

1. *$V \in \mathcal{S}m_S$ a smooth relative curve $p : X \to V$ with Z finite over V*
2. *a closed subscheme $Z' \hookrightarrow X$ containing Z such that $\operatorname{codim}(Z', X) = \operatorname{codim}(Z, X) - 1$.*

and the forget support map induces the trivial morphism

$$p_* E_{Z/X} \to p_* E_{Z'/X}$$

in the homotopy category. In particular $E_{Z/X}(X) \to E_{Z'/X}(X)$ is trivial.

Proof. From Lemma 5.1.1 (and using a standard limiting argument) we can find a Nisnevich distinguished square

$$\begin{array}{ccc} X \setminus Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathbb{A}^1_V \setminus f(Z) & \longrightarrow & \mathbb{A}^1_V \end{array}$$

such that $Z \hookrightarrow X \xrightarrow{f} \mathbb{A}^1_V \xrightarrow{\pi} V$ is finite, after possibly shrinking X Nisnevich locally around x. Let $p = \pi \circ f$, $\overline{Z} = p(Z)_{red}$ and $Z' = p^{-1}(\overline{Z})$. Since π and f are flat, so is p

hence it follows that $\mathrm{codim}(Z', X) = \mathrm{codim}(Z, X) - 1$. By the excision 2.3.15 it follows that the upper horizontal morphism in the following diagram

$$\begin{array}{ccc} E_{f(Z)/\mathbb{A}^1_V} & \xrightarrow{\simeq} & f_* E_{Z/X} \\ \downarrow & & \downarrow f \\ E_{\mathbb{A}^1_Z/\mathbb{A}^1_V} & \longrightarrow & f_* E_{Z'/X} \end{array}$$

is an equivalence. In the above diagram the vertical maps are respective forget support maps and $f^{-1}f(Z) = Z$. Applying π_* to the above diagram we get the following diagram:

$$\begin{array}{ccc} \pi_* E_{f(Z)/\mathbb{A}^1_V} & \xrightarrow{\simeq} & p_* E_{Z/X} \\ \downarrow & & \downarrow f \\ \pi_* E_{\mathbb{A}^1_Z/\mathbb{A}^1_V} & \longrightarrow & p_* E_{Z'/X} \end{array}$$

From [27, Lemma 5.8], the left vertical map is trivial. Hence the right vertical map is also trivial thereby proving the proposition. □

Corollary 5.1.4. *Under the assumptions of the previous proposition, the forget support map*

$$E_{Z/X}(\mathrm{Spec}\,(\mathcal{O}^h_{X,x})_\eta) \to E_X(\mathrm{Spec}\,(\mathcal{O}^h_{X,x})_\eta)$$

is trivial, $\mathrm{Spec}\,(\mathcal{O}^h_{X,x})_\eta$ *is the generic fiber of the Henselian local scheme at* x*.*

Proof. By Lemma 5.1.1, we can find a cofinal family of Nisnevich neighbourhoods (W, w) of x each admitting a Nisnevich distinguished square as in Proposition 5.1.3. Since, $E(\mathrm{Spec}\,(\mathcal{O}^h_{X,x})_\eta)$ is $\mathrm{colim}_{(W,w)} E(W_\eta)$, where W_η is the generic fiber it is sufficient show that for such neighbourhoods the forget support map is trivial. So we assume $W = X$. As $X_\eta = \underset{X_\eta \subset T \subset X}{colim}\, T$ and $Z_\eta = \underset{X_\eta \subset T \subset X}{colim}\, T \cap Z$, where T is open subscheme of X, we have the following distinguished square

$$\begin{array}{ccc} T \setminus Z \cap T & \longrightarrow & T \\ \downarrow & & \downarrow \\ \mathbb{A}^1_V \setminus f(Z \cap T) & \longrightarrow & \mathbb{A}^1_V \end{array}$$

Now by previous proposition $E_{Z/X}(T) \to E_X(T)$ is trivial. Hence $E_{Z/X}(\mathrm{Spec}\,(\mathcal{O}_{X,x,})_\eta) \to E_X(\mathrm{Spec}\,(\mathcal{O}_{X,x,})_\eta)$ is trivial , as $E_X(X_\eta) = \mathrm{colim}_T\, E_X(T)$. In a similar fashion $E_{Z/X}(\mathrm{Spec}\,(\mathcal{O}^h_{X,x})_\eta) \to E_X(\mathrm{Spec}\,(\mathcal{O}^h_{X,x})_\eta)$ is trivial.

□

Theorem 5.1.5. *Let S be a Noetherian irreducible scheme of finite type of dimension p and let $E \in Spt_{S^1}(Sm_S)$ be a fibrant object in $Spt^{\mathbb{A}^1}_{S^1}(\mathcal{S}m_S)$ and $X \in Sm_S$ of dimension d. Then the complex*

$$0 \to (E^n_X)^\sim \xrightarrow{e} \bigoplus_{z\in X^{(0)}} \mathrm{j}_*\mathrm{j}^* E^n_{Z/X} \xrightarrow{d^0} \bigoplus_{z\in X^{(1)}} \mathrm{j}_*\mathrm{j}^* E^{n+1}_{Z/X} \xrightarrow{d^1} \cdots$$
$$\cdots \xrightarrow{d^{d-2}} \bigoplus_{z\in X^{(d-1)}} \mathrm{j}_*\mathrm{j}^* E^{n+d-1}_{Z/X} \xrightarrow{d^{d-1}} \bigoplus_{z\in X^{(d)}} \mathrm{j}_*\mathrm{j}^* E^{n+d}_{Z/X} \to 0 \quad (5.1)$$

is exact with possible exceptions at $(E^n_X)^\sim$ and $\bigoplus_{z\in X^{(i)}} \mathrm{j}_*\mathrm{j}^* E^{n+i}_{Z/X}$ for $1 \le i \le p$. Furthermore, the above complex is exact everywhere if for each $x \in X$ which lies over $s \in S$ and for any irreducible closed subset $Z \subset X$ of codimension k satisfying either

1. *$X_x \subseteq Z \subset X$ or*

2. *Z is an irreducible component of X_x*

there exists $Z' \supset Z$ of codimension $k-1$ such that following (forget support) map is trivial

$$E_{Z/X}(\mathrm{Spec}\,\mathcal{O}^h_{X,x}) \to E_{Z'/X}(\mathrm{Spec}\,\mathcal{O}^h_{X,x}).$$

In fact, this gives us a resolution of $(E^n_X)^\sim$ by flabby Nisnevich sheaves, which implies the following isomorphism

$$H^k(Y, (E^n_X)^\sim) \cong H^k(\mathcal{G}^\bullet(E,n)(Y))$$

for $Y \in X_{Nis}$, which vanishes for $k > d$.

Proof. As we can check exactness stalkswise, we assume S to be spectrum of a Henselian local ring. Let σ be the closed point. By conditions 2.4 and 2.5 the theorem follows by showing for a given closed subscheme $Z \subset X$ of codimension $\ge p+1$, there exists $Z \subseteq Z' \subseteq X$ with $\mathrm{codim}(Z', X) < \mathrm{codim}(Z, X)$, such that forget support map

$E^{n+s}_{Z/X}(\mathrm{Spec}\,(\mathcal{O}^h_{X,x})) \to E^{n+s}_{Z'/X}(\mathrm{Spec}\,(\mathcal{O}^h_{X,x}))$ is trivial. We can assume X to be a Henselian local scheme.

If Z does not contain the special fibre X_σ, then by Proposition 5.1.3 we are done. So now suppose Z contains the special fibre. If Z is irreducible, then by hypothesis there is a Z' such that $\mathrm{codim}(Z', X) < \mathrm{codim}(Z, X)$ and the forget support map $E_{Z/X}(X) \to E_{Z'/X}(X)$ is trivial. If Z is not irreducible, then we can write $Z = \cup_i Z_i$ where Z_i's are the irreducible components of Z. Without loss of generality assume $i = 2$. Hence, by hypothesis (and in case one of the irreducible component doesn't entirely lie over the closed point of S, by Proposition 5.1.3) there exist $Z_1 \subset T_1$ and $Z_2 \subset T_2$ such that forget support maps $E_{Z_1/X}(X) \to E_{T_1/X}(X)$ and $E_{Z_2/X}(X) \to E_{T_2/X}(X)$ are trivial.

Writing $T = T_1 \cup T_2$ we prove the forget support map $E_{Z/X}(X) \to E_{T/X}(X)$ is trivial. Note that as $E_{Z_i/X}(X) \to E_{T_i/X}(X)$ is trivial so is the composition $E_{Z_i/X}(X) \to E_{T_i/X}(X) \to E_{T/X}(X)$, for $i = 1, 2$. Since we have the triangle $E_{Z/X}(X) \xrightarrow{\mathrm{f}} E_{T/X}(X) \xrightarrow{\mathrm{g}} E_{(T\setminus Z)/(X\setminus Z)}(X \setminus Z)$, by a general fact about triangulated categories, proving f is trivial is equivalent to proving g is a monomorphism. Now using the isomorphism (from Remark 2.3.15) $E_{Z/X}(U) \cong E_{(U\cap Z)/U}(U)$ for any open subscheme U in X, we have $E_{T/X}(X \setminus Z) \cong E_{(T\setminus Z)/(X\setminus Z)}(X \setminus Z)$. This implies that g factors as $E_{T/X}(X) \to E_{T/X}(X \setminus Z_1) \to E_{(T\setminus Z)/(X\setminus Z)}(X \setminus Z)$. We will prove that both these morphisms are monomorphisms. We have the following exact triangle for Z_1

$$E_{Z_1/X}(X) \to E_{T/X}(X) \to E_{(T\setminus Z_1)/(X\setminus Z_1)}(X \setminus Z_1)$$

Therefore, $E_{T/X}(X) \to E_{(T\setminus Z_1)/(X\setminus Z_1)}(X \setminus Z_1) \cong E_{T/X}(X \setminus Z_1)$ is a monomorphism. Observing the triangle corresponding to Z_2

$$E_{Z_2/X}(X \setminus Z_1) \to E_{T/X}(X \setminus Z_1) \to E_{(T\setminus Z_2)/(X\setminus Z_2)}(X \setminus Z)$$

we conclude that $E_{T/X}(X \setminus Z_1) \to E_{(T\setminus Z_2)/(X\setminus Z_2)}(X \setminus Z) \cong E_{(T\setminus Z)/(X\setminus Z)}(X \setminus Z)$ is a monomorphism. This proves that composition $g : E_{T/X}(X) \to E_{T/X}(X \setminus Z_1) \to E_{(T\setminus Z)/(X\setminus Z)}(X \setminus Z)$ is a monomorphism.

□

We can greatly simplify the condition for exactness of the Nisnevich Gersten complex in Theorem 5.1.5 when S is J-2. In this case, it suffices to check the triviality of the forget support maps for *regular* irreducible closed subschemes. The following is the precise statement:

Proposition 5.1.6. *In the setting of Theorem 5.1.5 assume S to be a J-2 ring. Then if for every regular irreducible closed subscheme $Z \subset X$ of codimension k satisfying either*

1. $X_\sigma \subseteq Z \subset X$ *or*
2. Z *is an irreducible component of* X_σ

there exists $Z' \supset Z$ of codimension $k-1$ such that the forget support map $E_{Z/X}(\operatorname{Spec}(\mathcal{O}^h_{X,x})) \to E_{Z'/X}(\operatorname{Spec}(\mathcal{O}^h_{X,x}))$ is trivial, the complex (5.1) of Theorem 5.1.5 is exact at all places.

Proof. As S is J-2, every closed subscheme $\overline{z} = Z$ has an open neighbourhood U containing z such that $U \cap Z = Z^{reg}$ is regular. Further $E_{Z/X}(\operatorname{Spec}(\mathcal{O}^h_{X,x})) \cong E_{Z^{reg}/X}(\operatorname{Spec}(\mathcal{O}^h_{X,x}))$ and we proceed in the same manner as in the proof of previous theorem. □

Example 5.1.7. *We now give an example of a spectrum $E' \in Spt_{S^1}(Sm_S)$ for which the Gersten resolution (5.1) is not exact. In fact, Ayoub's counterexample to Morel's conjecture on $\mathbb{A}^1$-connectivity [3] works for us. We give a brief description here.*
Fix a perfect field k. Let $\mathcal{K}^M_1$ denote the Nisnevich sheaf (on smooth schemes over k) respresenting Milnor K-theory. This sheaf, in fact, has transfers. Let S be a normal surface in $\mathbb{P}^3$ given by equation $w(x^3 - y^2z) + f(x, y, z) = 0$ with f a general homogeneous degree 4 polynomial. Then S is non singular outside the point $[0 : 0 : 0 : 1]$. Denote by $i : S \hookrightarrow \mathbb{P}^3_k$ the inclusion map and by $\pi : \mathbb{P}^3_k \to \operatorname{Spec} k$ the structure map of $\mathbb{P}^3_k$.
We will consider $\mathcal{K}^M_{1,S} := i^! \pi^*(\mathcal{K}^M_1)$. It follows from Section 3 of op. cit *that the Nisnevich sheafification(denoted cl_S) of the presheaf $U \mapsto H^1_{Nis}(U, \mathcal{K}^M_{1,S})$ on Sm_S is not strictly $\mathbb{A}^1$-invariant. In particular, it cannot be zero. Therefore, the Gersten resolution of $\mathcal{K}^M_{1,S}$ is not exact.*

Next we construct an $\mathbb{A}^1$-local fibrant spectrum with $(E'^0)^\sim \cong i^! \pi^* \mathcal{K}^M_1$. As $\mathcal{K}^M_1$ is an $\mathbb{A}^1$-invariant sheaf with transfers, it is also strictly $\mathbb{A}^1$-invariant. This implies that the associated Eilenberg-Maclane spaces $K(\mathcal{K}^M_1, n)$ are $\mathbb{A}^1$-local for all $n \geq 0$. Therefore, the spectrum E with $E_n := K(\mathcal{K}^M_1, n)$ is an $\mathbb{A}^1$- Nisnevich local fibrant spectrum in $Spt_{S^1}(Sm_k)$ with $(E^0)^\sim \cong \mathcal{K}^M_1$. Moreover, $E' := i^! \pi^*(E)$ is also an $\mathbb{A}^1$-Nisnevich local fibrant spectrum in $Spt_{S^1}(Sm_S)$ because $i^!$ and π^* both preserve fibrant objects in our situation. Finally $(E'^0)^\sim \cong i^! \pi^* \mathcal{K}^M_1$.

Remark 5.1.8. *While S defined in the previous example is not regular, the same example* *shows exactness of Gersten resolution fails for* $i_*(E')$ *in* $Spt_{S^1}(Sm_{\mathbb{P}^3_k})$*. This provides us with a counterexample over a regular base.*

Remark 5.1.9. *In the light of above counterexample, Gersten resolution of a cohomology theory over a general base is not exact. However conditions 2.3,2.4 and 2.5 suggest a way to remedy this malady. If one could further localize* $Spt^{\mathbb{A}^1}_{S^1}(\mathcal{S})$ *such that morphism in above conditions become trivial then in the resulting category Gersten complex will always be exact. If such a construction is possible, the resulting category will serve as a better model for doing motivic homotopy category over a general base. However constructing such a localization is not a straightforward Bousefield localization which is about making a set of maps isomorphisms, whereas what we need is to make a certain set of maps trivial or equivalently monomorphism. This line of thought is currently under investigation in a joint project.*

5.2 Bloch Ogus Theorem

In this section, we specialise to the étale cohomology and prove Theorem 5.2.5. All cohomology groups in this section, unless specified otherwise, are étale cohomology groups. We fix the $X \in \mathcal{S}m_S$ of finite relative dimension and $\mathcal{A}^\bullet \in \mathcal{D}^b_c(S, \Lambda)$ which we will call an l.c.c complex. Given such an l.c.c complex, by abuse of notation, we will also denote by $\mathcal{A}^\bullet \in \mathcal{D}^b_c(\mathcal{S}m_S, \Lambda)$. Restriction of $\mathcal{A}^\bullet$ to $\mathcal{D}^b_c(X, \Lambda)$ will be denoted as $\mathcal{A}^\bullet|_X$. We will also use notation from Remark 3.2.18.

To prove Theorem 5.2.5 we will verify the conditions stated in 2 and 3 about the vanishing of forget support maps. Given any $\mathcal{A}^\bullet \in \mathcal{D}^b_c(X, \Lambda)$ we have to verify the triviality of forget support maps for $E(\mathcal{A}^\bullet)$. Unwinding the definitions that is tantamount to verifying

$$R\Gamma_Z(X, \mathcal{A}^\bullet) \to R\Gamma_{Z'}(X, \mathcal{A}^\bullet) \tag{5.2}$$

is trivial Nisnevich locally. To verify these conditions we use Gabber purity for étale cohomology. As Gabber purity requires the schemes to be regular, we have to put some extra hypothesis on our base scheme such as regularity and J-2. Note that [27] assume their base to a DVR, hence the condition of regularity and J-2 is implicit in their hypothesis.

As a first step, we verify 5.2 for $\mathcal{A}^\bullet = \Lambda$ using following Lemma.

Lemma 5.2.1. *Let X/S be a Henselian regular local ring with $\sigma : k(x) \to X$, the closed point. Assume $\sigma_Z : Z \hookrightarrow X$, $\sigma_{Z'} : Z' \hookrightarrow X$ be regular closed subschemes (containing special fiber) such that $Z \subset Z'$ and $c = codim(X, Z') = codim(X, Z) - 1$. Then the following morphism*

$$\sigma^* \sigma_{Z*}((R\sigma_Z^!)(\Lambda)) \to \sigma^* \sigma_{Z'*}((R\sigma_{Z'}^!)(\Lambda))$$

is trivial in $\mathcal{D}^b(k(x), \Lambda)$)

Proof. We reduce the question to Z' (which is Henselian local because X is) and its codimension 1 closed subscheme Z. Denote $R\sigma_{Z'}^!(\Lambda)$ by $\mathcal{F}$ and consider the closed point $\sigma : k(x) \xrightarrow{\sigma'} Z' \hookrightarrow X$. Then purity for the closed immersion $\sigma_{Z/Z'} : Z \hookrightarrow Z'$ implies that $R\sigma_{Z/Z'}^! \mathcal{F} \cong R\sigma_Z^!(\Lambda) \cong \mathcal{F}(-1)[-2]$. Now by Lemma 6.6 of [27], $\sigma'^* \sigma_{Z/Z'*}((R\sigma_Z^!)(\Lambda)) \to \sigma'^*((R\sigma_{Z'}^!)(\Lambda))$ is trivial in $\mathcal{D}^b(k(x)_{et}, \Lambda))$.
We finish the proof by noting the isomorphisms $\sigma^* \sigma_{Z'*} \cong \sigma'^*$ and $\sigma^* \sigma_{Z*} \cong \sigma'^* \sigma_{Z/Z'*}$. ☐

Since étale cohomology is invariant for Henselian pairs(Theorem 3.2.14), the previous lemma immediately yields the following corollary.

Corollary 5.2.2. *In the setting of Lemma 5.2.1, the canonical morphism $R\Gamma_Z(X, \Lambda) \to R\Gamma_{Z'}(X, \Lambda)$ is trivial.*

Corollary 5.2.3. *In the setting of Lemma 5.2.1 the canonical morphism $R\Gamma_Z(X, \mathcal{A}^\bullet|_X) \to R\Gamma_{Z'}(X, \mathcal{A}^\bullet|_X)$ is trivial for any $\mathcal{A}^\bullet \in \mathcal{D}_c^b(S, \Lambda)$.*

Now we are in a position to prove the next theorem which will yield Bloch-Ogus theorem as its corollary. The key ingredients for the proof are Theorem 5.1.5 (and Proposition 5.1.6) and Corollary 5.2.3. We will merely sketch the proof as it follows the one given in [27], once all the essential ingredients are in place.

Theorem 5.2.4. *Let S be a J-2 Noetherian irreducible regular scheme of finite type. Let X/S be smooth, $\dim(X) = d$ and $\mathcal{A}^\bullet$ an l.c.c. complex in $\mathcal{D}_c^b(S_{\text{et}}, \Lambda)$. Then the Nisnevich Gersten complex $\mathcal{G}^\bullet(E(\mathcal{A}^\bullet), n)$ is a flasque resolution of the Nisnevich sheafification $R^n \varepsilon_* \mathcal{A}^\bullet|_X$ of étale cohomology with coefficients $\mathcal{A}^\bullet$. In particular, we get the exact* *sequence*

$$0 \to R^n\varepsilon_*\mathcal{A}^\bullet|_X \to \bigoplus_{z\in X^{(0)}} \mathrm{j}_*\mathrm{H}^n(k(z), \mathcal{A}^\bullet|_{k(z)}) \to \dots$$
$$\dots \to \bigoplus_{z\in X^{(d)}} \mathrm{j}_*\mathrm{H}^{n-d}(k(z), \mathcal{A}^\bullet|_{k(z)}(-d)) \to 0.$$

Proof. $E(\mathcal{A}^\bullet)$ is $\mathbb{A}^1$-local by Remark 3.2.18. Therefore we can apply Theorem 5.1.5. Next by Corollary 5.2.3 the morphism $\sigma^*\sigma_{Z*}((R\sigma_Z^!)(\mathcal{A}^\bullet|_X)) \to \sigma^*\sigma_{Z'*}((R\sigma_{Z'}^!)(\mathcal{A}^\bullet|_X))$ is trivial. Hence by Proposition 5.1.6, $\mathcal{G}^\bullet(E(\mathcal{A}^\bullet), n)$ is a flasque resolution of $R^n\varepsilon_*\mathcal{A}^\bullet|_X$. This proves the first part of the theorem.
Then one proves $\mathrm{j}^*E(\mathcal{A}^\bullet)^{n+s}_{Z/X} \cong \mathrm{H}^{n-s}(k(z), \mathcal{A}^\bullet|_{k(z)}(-s))$. To do so, by excision we have $\mathrm{j}^*E(\mathcal{A}^\bullet)^{n+s}_{Z/X} \cong z_*z^*E(\mathcal{A}^\bullet)^{n+s}_{z/X_z}$, where $\overline{z} = Z$ and $X_z := \mathrm{Spec}\,(O_{X,z})$. Moreover $z^*E(\mathcal{A}^\bullet)^{n+s}_{z/X_z} \cong \mathrm{H}^{n+s}_z(X_z, \mathcal{A}^\bullet)$. By purity the latter group is isomorphic to $\mathrm{H}^{n-s}(k(z), \mathcal{A}^\bullet|_{k(z)}(-s))$. As $\mathcal{G}^s(E(\mathcal{A}^\bullet), n) = \bigoplus_{z\in X^{(s)}} \mathrm{j}_*\mathrm{j}^*E(\mathcal{A}^\bullet)^{n+s}_{Z/X}$, this concludes the proof.

□

Theorem 5.2.4 immediately yields Theorem 5.2.5 after taking the Nisnevich stalks of the spectrum.

Theorem 5.2.5. *Let S be a J-2 Noetherian irreducible regular scheme of finite type. Fix a point $s \in S$. Let X/S be smooth of finite type, $d = \dim(X)$ and $\mathcal{A}^\bullet$ an l.c.c. complex in $\mathcal{D}^b_c(S_{\mathrm{et}}, \Lambda)$. Let x be a point of X lying over s and $Y = X^h_x$ the Nisnevich local scheme at x. Then there is an exact sequence*

$$0 \to \mathrm{H}^n(Y_{\mathrm{et}}, \mathcal{A}^\bullet|_Y) \xrightarrow{e} \bigoplus_{z\in Y^{(0)}} \mathrm{H}^n(k(z), z^*\mathcal{A}^\bullet|_Y) \xrightarrow{d^0} \cdots$$
$$\cdots \xrightarrow{d^{d-1}} \bigoplus_{z\in Y^{(d)}} \mathrm{H}^{n-d}(k(z), z^*\mathcal{A}^\bullet|_Y(-d)) \to 0. \quad (5.3)$$

Remark 5.2.6. *Any orientable cohomology theory satisfies Gabber purity(see [23]). However it's not yet known if such a cohomology theory (with finite coeffecients) satisfies Gabber rigidity. If Gabber rigidity holds then for such cohomology theories Gersten complex should be exact over a general base. This is currently a joint work in progress.*

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