American Option Pricing in Regime Switching Models

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Certificate

This is to certify that this dissertation entitled American Option Pricing in Regime Switching Models towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Shamant Basidoni at Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami, Associate Professor, Department of Mathematics , during the academic year 2022-2023.

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This thesis is dedicated to my mother, Chetana Basidoni

Declaration

I hereby declare that the matter embodied in the report entitled American Option Pricing in Regime Switching Models are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Anindya Goswami and the same has not been submitted elsewhere for any other degree.

Shamant Basidoni

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Abstract

The objective of this project is to examine the literature on the pricing of American options in some theoretical market models. The initial motivation was to examine the pricing of American options in a semi-Markov regime-switching model, which did not become possible due to the time constraints. This thesis presents a survey of literature I have covered in this regard. In the first chapter, some theorems and results from stochastic calculus, needed for understanding the literature, are summarised. In the second chapter contingent claims, hedging, and stochastic representations of option prices are examined. The third chapter examines literature about pricing American options under a regime-switching model. Since it is often a difficult task to get closed-form solutions for pricing options, certain approximation methods are listed.

Contents

Definitions

Definition 0.1. *A filtered probability space is a probability space* $(\Omega, \mathscr{F}, P)$ *with a filtration defined on it. A filtration* $\{\mathscr{F}_t\}_t$ *is a family of sub-*σ*-algebras of* $\mathscr{F}$ *indexed by* $t \in \mathbb{R}^+$ *for which* $s \leq t \implies \mathscr{F}_s \subseteq \mathscr{F}_t$ *holds true.*

Definition 0.2. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *be a filtered probability space. A process* $X = \{X_t\}_t$ *defined on it is called a* $\mathscr{F}_t$ *martingale if the following hold for all* $0 \leq s \leq t < \infty$
(1) X *is* $\{\mathscr{F}_t\}_t$*-adapted* (2) $E(|X_t|) < \infty$ (3) $E(X_t \mid X_s) = X_s$ *almost surely.*
X *is called a sub(super)-martingale if (1) and (2) hold and the following holds.*
(4)$E(X_t \mid X_s) \geq (\leq)\ X_s$ *almost surely.*

Definition 0.3. *Consider a measure space* $(\Omega, \mathscr{F}, P)$*. Consider a family of real valued measurable functions* $f = (f)_{i \in I}$ *defined on it. A real valued measurable function* g *is called the essential supremum of* f*, if* $g \geq f_i \quad (a.s.)\ \forall i \in I$*, and for another real valued measurable function* h*,* $h \geq f_i \quad (a.s.)\ \forall i \in I$ *implies* $h \geq g$ *almost surely.*

Definition 0.4. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *be a filtered probability space. A non-negative valued random-variable* τ *is called a stopping time with respect to the filtration* $\{\mathscr{F}_t\}_t$ *if* $\tau^{-1}([0,t]) \in \mathscr{F}_t, \quad \forall 0 \leq t < \infty$*.*

Definition 0.5. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *be a filtered probability space. For* $0 \leq t \leq T \leq \infty$*,* $S_{t,T}$ *denotes the family of* $\mathscr{F}_t$*-stopping times* τ *that satisfy* $t \leq \tau \leq T$ *almost surely.*

Definition 0.6. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *be a filtered probability space. An* $\mathscr{F}_t$*- adapted process* $X = \{X_t\}_t$ *is called a local martingale if there exists a sequence of non-decreasing stopping times* $(\tau_n)_{n \in \mathbb{N}}$ *which satisfies the following.*
(1) $P(\lim_{n \to \infty} \tau_n = \infty) = 1$*.*
(2) The stopped process X^{τ_n}*. given by* $X^{\tau_n}_t = X_{\min(t, \tau_n)}$*, is an* $\{\mathscr{F}_t\}$*-martingale for all* $n \in \mathbb{N}$*.*

Definition 0.7. *Let* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ *be a filtered probability space. An* $\mathscr{F}_t$*- adapted process* $X = \{X_t\}_t$ *is called a semi-martingale if it admits the following decomposition*

$$X_t = M_t + A_t$$

here $M = \{M_t\}_t$ *is a local martingale, and* $A = \{A_t\}_t$ *is a* $\mathscr{F}_t$*-adapted RCLL process of bounded variation.*

Introduction

A contingent claim is a tradeable financial entity whose value, as the name suggests, is contingent on something else. Usually, this 'something else' is a stock. Since the behavior of the stock prices themselves are, to the most extent, unpredictable, it becomes a challenge to determine the price of a contingent claim with a stock as the underlying asset. One can, however, assume a specific mathematical distribution of stock prices and try to evaluate the price of a contingent claim. Under certain mathematical conditions imposed on the market model that make economic sense, it is a challenging problem to price the contingent claims. We look at two types of contingent claims here, namely European contingent claims and American contingent claims. Each claim is defined by a time of maturity, a payoff function, and an exercise criterion. Options are particular types of claims that interest us. European options can only be exercised at the time of maturity. In contrast, an American option can be exercised on or before maturity. This makes the problem of pricing an American option mathematically interesting due to the added layer of complexity with the early exercise option. Pricing of American options also has practical significance due to their extensive trading in real security markets.

A well-accepted theoretical framework for pricing the options was first developed for European-style options by Black, Scholes, and Merton [4, 18]. The first one to explicitly present a rigorous pricing theory for American-styled claims was Bensoussan [2]. The earlier works in this direction include McKean [17] and Moerbeke [19]. Bensoussan, in his paper, presented a set of properties motivated by an economic sense to be satisfied by the value process of the contingent claim, and showed that they characterize one and only one value function. Using "penalisation method," the optimal stopping problem for the exercise time was addressed. Unfortunately, this method limited the scope by putting restrictions on the regularity and boundedness of the payoff function that excluded even a simple American call option. Karatzas [15] built up on this model and employed the "martingale" treatment of the optimal stopping problem(refer to [9] for details). They presented a hedging strategy for American contingent claims where the fair value of the claim was the smallest value of wealth that allowed the construction of the said hedging strategy.

Extensive research has been done to study the extension of Black-Scholes-Merton market models with Markov-modulated regime switching. Regime-switching models allow certain simplistic random variability of market parameters within finitely many possible states. Such models have been heavily studied in finance literature following the influential work of Hamilton [12]. In these, generally, the market parameters are modeled using Markov pure jump processes, whose states correspond to various Market regimes. Moreover, if the asset price evolves as a geometric Brownian motion (GBM) during the inter-transition period, such processes are called Markov-modulated geometric Brownian motion (MMGBM). This model gives rise to an incomplete market, where the fair price of a derivative is not unique. Many authors have studied European style options under such a market model, including [1, 6, 8, 10]. The fair pricing of American-style options has also been addressed by several authors, including [5, 23, 14]. [5] provides an approximate solution to the pricing of American options. Zhang [23] gives an exact closed-form solution for a perpetual American put option with regime switching, an option with no expiry date. There are also some further generalizations of asset dynamics, considered by several authors by introducing jump discontinuities in the asset price process, with or without Markov regimes. See, for example, Merton [18], and [22] for pricing European style options on such market model. On the other hand, Huyên Pham, [20], and Zhang [24], for example, evaluate the price of an American option when the price of the underlying asset follows a jump-diffusion model.

Original contribution In this thesis we mention in detail certain parts of the literature mentioned above. We have tried to fill in small gaps of proofs that are left for one's own understanding, and provide corrections wherever we have found mathematical inaccuracies. We have compiled the results of stochastic calculus and the literature on American option pricing that would serve as an introduction to one trying to understand the pricing of American contingent claims.

Chapter 1

Pertinent Theorems and Results of Stochastic Calculus

1.1 Girsanov theorem.

Throughout this theorem we assume $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ to be a probability space equipped with a filtration which satisfies the usual conditions. All the processes are defined on it unless said otherwise.
We define some processes W, X, Z as below:

- $W = \{W_t^{(1)}, \ldots, W_t^{(d)}, \mathscr{F}_t; 0 \leq t < \infty\}$ is a d-dimensional Brownian motion.

- $X = \{X_t^{(1)}, \ldots, X_t^{(d)}, \mathscr{F}_t; 0 \leq t < \infty\}$ is a vector valued measurable and adapted process which satisfies $P\left[\int_0^T (X_s^{(i)})^2 ds < \infty\right] = 1$ for all $1 \leq i \leq d$, $\forall \quad 0 \leq T < \infty$.

- Define, $\forall \quad 0 \leq t < \infty$, $Z_t(X) = exp\left[\sum_{i=1}^d \int_0^t X_s^{(i)} dW_s^{(i)} - \frac{1}{2}\int_0^t ||X_s||^2 ds\right]$. Set $Z = \{Z_t(X)\}_t$.

Note that the term in the exponent is a semi-martingale, so Ito's rule applied on $\{Z_t(X)\}$ with $F(x) = e^x$ gives us :

$$Z_t(X) = 1 + \int_0^t \sum_{i=1}^d Z_s(X) X_s^{(i)} dW_s^{(i)} + \int_0^t Z_s(X) \left(-\frac{1}{2} ||X_s||^2 \right) ds + \frac{1}{2} \int_0^t Z_s(X) ||X_s||^2 ds. \tag{1.1}$$

Hence Z satisfies,

$$Z_t(X) = 1 + \int_0^t \sum_{i=1}^d Z_s(X) X_s^{(i)} dW_s^{(i)} \tag{1.2}$$

for all $0 \leq t < \infty$, and so is a continuous local martingale. It can be shown that if X is bounded then, $\{Z_t(X)\}$ is in fact a continuous martingale.

- Assume $\{Z_t(X)\}$ is a martingale. Define, for each T with $0 \leq T < \infty$ a new probability measure $\tilde{P}$ on $(\Omega, \mathscr{F}_T)$ as follows:

$$\forall\ A \in \mathscr{F}_T, \tilde{P}(A) = E(1_A Z_T(X)) \tag{1.3}$$

  we can check it is indeed a probability measure as $E(Z_t(X)) = E(Z_0(X)) = 1$ for all $0 \leq t < \infty$.

Theorem 1.1. *Given that $\{Z_t(X)\}$ is a martingale, define $\tilde{W} = \{\tilde{W}_t^{(1)}, \ldots, \tilde{W}_t^{(d)}, \mathscr{F}_t; 0 \leq t < \infty\}$ as follows:*

$$\tilde{W}_t^{(i)} \triangleq W_t^{(i)} - \int_0^t X_s^{(i)} ds \quad 1 \leq i \leq d, \quad 0 \leq t < \infty.$$

Then for each $T \in [0, \infty)$, $\{\tilde{W}_t, \mathscr{F}_t; 0 \leq t \leq T\}$ is a d-dimensional Brownian motion on $(\Omega, \mathscr{F}_T, \tilde{P})$.

Lemma 1.2. *Assume $\{Z_t(X)\}_t$ is a martingale. Let, $\tilde{E}_T(.)$ be the expectation operator under the measure $\widetilde{P}_T(.)$, then for $0 \leq s \leq t \leq T$, for a $\mathscr{F}_t$ measurable random variable Y, which is integrable with respect to the measure $\widetilde{P}_T(.)$ the following holds:*

$$\tilde{E}_T \left[Y \mid \mathscr{F}_s \right] = \frac{1}{Z_s(X)} E \left[Y Z_t(X) \mid \mathscr{F}_s \right], \quad a.s.\ P\ and\ \tilde{P}_T. \tag{1.4}$$

Proof. Consider $A \in \mathscr{F}_s$, we have:

$$\begin{aligned}
\tilde{E}_T \left\{ 1_A \frac{1}{Z_s(X)} E\left[Y Z_t(X) \mid \mathscr{F}_s \right] \right\} &= E\left\{ 1_A \frac{Z_T(X)}{Z_s(X)} E\left[Y Z_t(X) \mid \mathscr{F}_s \right] \right\} \\
&= E\left\{ E\left\{ 1_A \frac{Z_T(X)}{Z_s(X)} E\left[Y E[Z_T(X) \mid \mathscr{F}_t] \mid \mathscr{F}_s \right] \mid \mathscr{F}_s \right\} \right\} \\
&= E\left\{ 1_A \frac{1}{Z_s(X)} E\left[Y Z_T(X) \mid \mathscr{F}_s \right] E\left\{ Z_T(X) \mid \mathscr{F}_s \right\} \right\} \\
&= E\left\{ 1_A E\left[Y Z_T(X) \mid \mathscr{F}_s \right] \right\} \\
&= E\left[1_A Y Z_T(X) \right] = \tilde{E}_T\left[1_A Y \right].
\end{aligned}$$

Since A was chosen arbitrarily from $\mathscr{F}_s$, by the definition of conditional expectation the lemma holds. □

Lemma 1.3. *Let $\mathscr{M}_T^{c,\,loc}$ be the set of all continuous local martingales $M = \{M_t, \mathscr{F}_t;\, 0 \le t \le T\}$ on $(\Omega, \mathscr{F}_T, P)$ with $M_0 = 0$ almost surely. Let $\tilde{\mathscr{M}}_T^{c,loc}$ be defined the same way, but on $(\Omega, \mathscr{F}_T, \tilde{P})$.*

Claim 1. *Fix $0 \le T < \infty$. If $M \in \mathscr{M}_T^{c,loc}$, then the process $\tilde{M}$, given by*

$$\tilde{M}_t \triangleq M_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\left\langle M, W^{(i)} \right\rangle_s, \mathscr{F}_t; \quad 0 \le t \le T \tag{1.5}$$

is in $\tilde{\mathscr{M}}_T^{c,loc}$.

Claim 2. *If $N \in \mathscr{M}_T^{c,loc}$ and*

$$\tilde{N}_t \triangleq N_t - \sum_{i=1}^{d} \int_0^t X_s^{(i)} d\left\langle N, W^{(i)} \right\rangle_s; \quad 0 \le t \le T, \tag{1.6}$$

then

$$\langle \tilde{M}, \tilde{N} \rangle_t = \langle M, N \rangle_t; \quad 0 \le t \le T, \qquad a.s.\ P \text{ and } \widetilde{P}_T. \tag{1.7}$$

Here the proof is provided for the case where X, $Z(X)$, M and N are bounded in t and ω. M and N are also assumed to have bounded quadratic variations. For the proof of the general case one can refer to [16].

Proof. From Kunita Watanabe theorem, [16]

$$\left|\int_0^t X_s^{(i)} d\left\langle M, W^{(i)}\right\rangle_s\right| \leq \left(\int_0^t 1.d\langle M\rangle_t\right)^{\frac{1}{2}} \cdot \left(\int_0^t \left(X_s^{(i)}\right)^2 d\langle W^{(i)}\rangle_s\right)^{\frac{1}{2}}, \tag{1.8}$$

$$\left|\int_0^t X_s^{(i)} d\left\langle M, W^{(i)}\right\rangle_s\right|^2 \leq \langle M\rangle_t \cdot \int_0^t \left(X_s^{(i)}\right)^2 ds. \tag{1.9}$$

Thus, $t \mapsto \left|\int_0^t X_s^{(i)} d\left\langle M, W^{(i)}\right\rangle_s\right|$ is bounded. Since, $\tilde{M}_t \triangleq M_t - \sum_{i=1}^d \int_0^t X_s^{(i)} d\left\langle M, W^{(i)}\right\rangle_s$, $\tilde{M}$ is also bounded. Consider the process $Z(X)\tilde{M}$. From integration by parts formula we have for each $0 \leq t < \infty$:

$$Z_t(X)\tilde{M}_t = \int_0^t Z_u(X) dM_u + \sum_{i=1}^d \int_0^t \tilde{M}_u X_u^{(i)} Z_u(X) dW_u^{(i)}. \tag{1.10}$$

Due to the boundedness of Z, M and X, the process $Z(X)\tilde{M}$ as is a P-martingale. Therefore, for $0 \leq s \leq t \leq T$, we have from Lemma 1.2:

$$\tilde{E}_T\left[\tilde{M}_t \mid \mathscr{F}_s\right] = \frac{1}{Z_s(X)} E\left[Z_t(X)\tilde{M}_t \mid \mathscr{F}_s\right] = \tilde{M}_s, \quad \text{a.s. } P \text{ and } \tilde{P}_T. \tag{1.11}$$

Thus, $\tilde{M}$ is a $\tilde{P}$-martingale. Which implies, $\tilde{M} \in \tilde{M}^{c,\text{ loc.}}$.

From integration by parts formula we have for each $0 \leq t < \infty$

$$\begin{aligned}\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t = \int_0^t \tilde{M}_u dN_u + \int_0^t \tilde{N}_u dM_u - \sum_{i=1}^d \Bigg[\int_0^t \tilde{M}_u X_u^{(i)} d\left\langle N, W^{(i)}\right\rangle_u \\ + \int_0^t \tilde{N}_u X_u^{(i)} d\left\langle M, W^{(i)}\right\rangle_u\Bigg]\end{aligned} \tag{1.12}$$

almost surely, we also have:

$$\begin{aligned}Z_t(X)\left[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t\right] = \int_0^t Z_u(X)\tilde{M}_u dN_u + \int_0^t Z_u(X)\tilde{N}_u dM_u \\ + \sum_{i=1}^d \int_0^t \left[\tilde{M}_u\tilde{N}_u - \langle M, N\rangle_u\right] X_u^{(i)} Z_u(X) dW_u^{(i)}.\end{aligned} \tag{1.13}$$

This last process is a P-martingale as $Z_s, \tilde{N}, \tilde{M}$ are bounded. We have from Lemma 1.2 for

$0 \leq s \leq t \leq T$,

$$\tilde{E}_T\left[\tilde{M}_t\tilde{N}_t - \langle M, N\rangle_t \mid \mathscr{F}_s\right] = \tilde{M}_s\tilde{N}_s - \langle M, N\rangle_s; \quad \text{a.s. } P \text{ and } \tilde{P}_T. \tag{1.14}$$

Hence, we have $\langle \tilde{M}, \tilde{N}\rangle_t = \langle M, N\rangle_t; 0 \leq t \leq T$, a.s. $\tilde{P}_T$ and P. □

We now give the proof of the Theorem 1.1 (following [16]).

Proof. We show that the continuous process $\tilde{W}$ on $\left(\Omega, \mathscr{F}_T, \tilde{P}_T\right)$ satisfies the hypotheses of P. Lévy's criteria for being a d-dimensional Brownian motion [16].
$\tilde{W} = \{\tilde{W}_t^{(1)}, \ldots, \tilde{W}_t^{(d)}, \mathscr{F}_t; 0 \leq t < \infty\}$ is a continuous $\mathbb{R}^{\mathrm{d}}$-valued process adapted to the filtration $\mathscr{F}_t$ by definition, as each component of $\tilde{W}$ is defined as :

$$\tilde{W}_t^{(k)} \triangleq W_t^{(k)} - \int_0^t X_s^{(k)} ds \quad 1 \leq k \leq d, \quad 0 \leq t < \infty. \tag{1.15}$$

For each component $1 \leq k \leq d$ the process $\tilde{M}_t^{(k)}$ is defined as follows :

$$\begin{aligned} \tilde{M}_t^{(k)} &= \tilde{W}_t^{(k)} - \tilde{W}_0^{(k)} \\ &= \tilde{W}_t^{(k)}. \end{aligned}$$

From Lemma 1.3 we have, $\{\tilde{W}_t^{(k)}\}$ is a $\{\mathscr{F}_t\}$ adapted continuous local martingale. Lemma 1.3 also implies that the cross variations of the components is :

$$\left\langle \tilde{W}^{(j)}, \tilde{W}^{(k)} \right\rangle_t = \left\langle W^{(j)}, W^{(k)} \right\rangle_t = \delta_{j,k}t; \quad 0 \leq t \leq T \text{ a.s. } \tilde{P}_T \text{ and } P. \tag{1.16}$$

Thus, for each $T \in [0, \infty)$, $\{\tilde{W}_t, \mathscr{F}_t; 0 \leq t \leq T\}$ is a d-dimensional Brownian motion on $(\Omega, \mathscr{F}_T, \tilde{P})$. □

1.2 Doob's regulartiy theorem

Theorem 1.4. *Consider a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$ with a filtration that satisfies usual conditions. Let $Y = \{Y_t\}_{t\geq 0}$ be a $\{\mathscr{F}_t\}$ adapted supermartingale defined on it. If the map $t \mapsto E(Y_t)$ is right continuous for $t \geq 0$, then there exists a version of Y whose paths are right continuous almost surely.*

The proof of the above theorem will be presented at the end of this subsection. First let's define a few terms and notations.

For $a \geq b \in \mathbb{R}$, let $\lim\limits_{a\downdownarrows b}$ be the notation for the right limit, limit as a approaches b from the right. Similarly $\lim\limits_{a\upuparrows b}$ for left limits with $a \leq b \in \mathbb{R}$.

A function $F : \mathbb{Q}^+ \to \mathbb{R}$ is regularisable if

$$a)\lim_{u\downdownarrows t} F_u \quad \text{exists for all} \quad t \geq 0$$
$$b)\lim_{u\upuparrows t} F_u \quad \text{exists for all} \quad t > 0.$$

Let $G = \{\omega \in \mathscr{F} : \text{the map t} \mapsto Y_t(\omega)\text{is regularisable}\}$. Define $X = \{X_t\}_{t\geq 0}$ as follows:

$$X_t(\omega) = \begin{cases} \lim\limits_{u\downdownarrows t} Y_u & \text{if} \quad \omega \in G \\ 0 & \text{otherwise} \end{cases} \tag{1.17}$$

It can be shown that $P(G) = 1$. Check Theorem 65.1 of [21] for the proof.

Lemma 1.5. *Let $\{q(n) : n \in -\mathbb{N}\}$ be a sequence of rationals and $q(n) \downdownarrows t$ as $n \downdownarrows -\infty$. With Y defined as in Theorem 1.4. The following holds:*

$$\mathit{lim}_{q(n)\downdownarrows t} Y_{q(n)} \qquad \textit{exists a.s. and in } L^1(P). \tag{1.18}$$

Proof. Notice that $(Y_{q(n)}, \mathscr{F}_{q(n)})$ is a supermartingale with $\sup\limits_n E(Y_{q(n)})$ finite. Thus we can apply Levy-Doob downward lemma (Lemma 63.7 in [21]) to the supermartingale $Y_{q(n)}$. The result follows. $\square$

Proof of Theorem 1.4. We first prove X as defined above is a $\mathscr{F}_t$ adapted supermartignale. Next we show that $X_t = Y_t$ almsot surely.

Consider $v \geq t \in \mathbb{R}$ and a sequence $q(n)$ of rational numbers $v > q(n) \downdownarrows t$.

$$\begin{aligned} E(Y_v \mid \mathscr{F}_{q(n)}) &\leq Y_{q(n)} && \text{(as Y is a supermartingale)} \\ \implies E(Y_v \mid \mathscr{F}_{t^+}) &\leq \lim_{q(n)\downdownarrows t} Y_{q(n)} = X_t && \text{(from Lemma 1.5)} \\ \implies E(Y_v \mid \mathscr{F}_t) &\leq X_t && \text{(from right continuity of } \mathscr{F}_t\text{).} \end{aligned}$$

Now consider $u \geq t \in \mathbb{R}$ and a sequence $q(n)$ of rational numbers $u \leq q(n) \downdownarrows u$. Again using Lemma 1.5 and L^1 right-continuity of conditional expectation we get from above

$$\begin{aligned} &\lim_{n\to-\infty} E(Y_{q(n)} \mid \mathscr{F}_t) \leq \lim_{n\to-\infty} X_t \\ &E(\lim_{n\to-\infty} Y_{q(n)} \mid \mathscr{F}_t) \leq X_t \\ &E(X_u \mid \mathscr{F}_t) \leq X_t. \end{aligned}$$

Thus, X is a $\{\mathscr{F}_t\}$ supermartingale.

Consider $u \geq t \in \mathbb{R}$ we have,

$$\begin{aligned} &E(Y_u \mid \mathscr{F}_t) \leq Y_t \\ &\lim_{u\to t} E(Y_u \mid \mathscr{F}_t) \leq \lim_{u\to t} Y_t \\ &E(\lim_{u\to t} Y_u \mid \mathscr{F}_t) \leq Y_t \\ &E(X_t \mid \mathscr{F}_t) \leq Y_t. \end{aligned}$$

Since X is an $\mathscr{F}_t$ adapted process, we get $X_t \leq Y_t$ almost surely.

Given that $t \mapsto E(Y_t)$ is right continuous and by Lemma 1.5, we have, for $t \in [0, \infty)$

$$\begin{aligned} &\lim_{u \downarrow\downarrow t} E(Y_u) = E(Y_t) \quad \text{and} \\ &\lim_{u \downarrow\downarrow t} E(Y_u) = E(X_t) \\ \Longrightarrow\ &E(X_t) = E(Y_t). \end{aligned} \tag{1.19}$$

From equation (1.19) and the result that $X_t \leq Y_t$ almost surely, we get $X_t = Y_t$ almost surely. By construction X has all paths right continuous. Thus the theorem is proved. $\square$

1.3 Fakeev's results on optimal stopping

Theorem 1.6. *Consider a filtered probability space* $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$*, and a right continuous adapted process* X *defined on it. Assume* X *satisfies,* $E(X_t^-) < \infty$*, where* $X^- = max\,[0, -X]$ *is the negative part of the process. Let,* S_s *denote the class of stopping times with respect to* $\{\mathscr{F}_t\}_t$ *that satisfy* $s \leq \tau(\omega) < \infty$ *almost surely. Let,* f *denote the minimal right continuous supermartingale process that satisfies* $f_t \geq X_t$ *almost surely, for all* $0 \leq t < \infty$*. The following hold:*

$$f_s = \underset{\tau \in S_s}{es.sup} E(X_\tau | \mathscr{F}_s) \tag{1.20}$$

$$\underset{\tau \in S_s}{sup} E(X_\tau) = E(f_s). \tag{1.21}$$

Lemma 1.7. *A family of functions is said to admit needle-like variation if the functions* $\{h_i\}_{i \in I}$ *satisfy the following condition: for any* h_l *and* h_k *in the family, and any* $B \in \mathscr{F}$*, the function* $h_d = h_l 1_B + h_k 1_{B^c}$ *is also in the family. Here* 1_B *represents the indicator function on a set* B*.*
Consider a family of integrable functions $\{h_i\}_{i \in I}$ *admitting needle-like variation defined on a measure space* $(\Omega, \mathscr{F}, P)$*. Let,* $P(\Omega) < \infty$*. Then,* $\Pi(A),\ A \in \mathscr{F}$*, as defined below*

$$\Pi(A) = \sup_{i \in I} \int_A h_i(x) dP(x), \tag{1.22}$$

is a σ*-additive,* P*-continuous set function on* $\mathscr{F}$*. The Radon-nikodym derivative* $d\Pi/dP$ *is the essential upper bound of* $(h_i)_{i \in I}$ *:*

$$\frac{d\Pi}{dP} = es.\sup_{i \in I}\ h_i(x). \tag{1.23}$$

Proof. We will first prove the result for h_i that are uniformly bounded from above: $h_i(x) \leq K < \infty$. We get,

$$-\infty < \int_A h_i(x) dP(x) \leq \Pi(A) \leq KP(A) < \infty. \tag{1.24}$$

Hence, $\Pi(A)$ is finite for all $A \in \mathscr{F}$, which implies the σ-finiteness. It is trivially P-continuous. For additivity, let $A \cap B = \emptyset$. Then,

$$\Pi(A \cup B) = \sup_i \int_{A \cup B} h_i dP \leq \sup_i \int_A h_i dP + \sup_i \int_B h_i dP = \Pi(A) + \Pi(B). \tag{1.25}$$

For the reverse inequality consider, arbitrary h_i and h_j. We have, $h_k = 1_{A^c}h_i + 1_A h_j$ also in the family by construction.

$$\int_A h_i dP + \int_B h_j dP = \int_{A\cup B} h_k dP \leq \Pi(A \cup B), \tag{1.26}$$

Since this holds for any arbitrary h_i and h_j we choose, it follows,

$$\Pi(A) + \Pi(B) \leq \Pi(A \cup B), \tag{1.27}$$

this implies Π is additive. The additivity, P-continuity, and σ-finiteness of $\Pi(A)$ imply that $\Pi(A)$ is a σ-additive set function. Thus from Radon-Nikodym theorem we get a measurable function $f(x)$ for which the following holds.

$$\Pi(A) = \int_A f(x)dP(x) \tag{1.28}$$

for all $A \in \mathscr{F}$. We see (1.9) is true for any $A \in \mathscr{F}$, thus, we have $f(x) \geq h_i(x)$ (a.e.) for all $i \in I$.

Consider another function $\psi(x) \geq h_i(x)$ (a.e.) $\forall\ i \in I$, then for some $A \in \mathscr{F}$

$$\int_A \psi dP \geq \sup_i \int_A h_i dP = \int_A f dP. \tag{1.29}$$

By monotonicity of Lebesgue integrals we have $\psi(x) \geq f(x)$ (a.e.). Thus $f = \operatorname*{es.sup}_{i\in I} h_i(x)$. For the general case, we define new functions by putting bounds on the functions h_i as follows.

For $n \in \mathbb{N}$, define $h_i^{[n]} = \min(h_i, N)$ and $f_i^{[n]} = \operatorname*{es.sup}_{i\in I} h_i^{[n]}(x)$. Lemma holds for these functions as they are bounded. That is, $\forall A \in \mathscr{F}$, $\sup_{i\in I} \int_A h_i^{[n]} dP = \int_A f^{[n]} dP$.
Taking limits on both side as $n \to \infty$, we get

$$\lim_{n\to\infty} \left(\sup_{i\in I} \int_A h_i^{[n]} dP \right) = \lim_{n\to\infty} \left(\int_A f^{[n]} dP \right). \tag{1.30}$$

Notice that for $n_1, n_2 \in \mathbb{N}$ with $n_1 > n_2$, $h_i^{[n_1]} \geq h_i^{[n_2]}$ for all $\omega \in \Omega$. Thus,

$$\int_A h_i^{[n_1]} dP \geq \int_A h_i^{[n_2]} dP \tag{1.31}$$

$$\implies \sup_{i \in I} \int_A h_i^{[n_1]} dP \geq \sup_{i \in I} \int_A h_i^{[n_2]} dP \tag{1.32}$$

due to the monotonicity of integrals.

This implies,

$$\lim_{n \to \infty} \left(\sup_{i \in I} \int_A h_i^{[n]} dP \right) = \sup_{n \in \mathbb{N}} \left(\sup_{i \in I} \int_A h_i^{[n]} dP \right). \tag{1.33}$$

Thus, we can rewrite (1.30) as

$$\begin{aligned}
&= \sup_{n \in \mathbb{N}} \left(\sup_{i \in I} \int_A h_i^{[n]} dP \right) = \lim_{n \to \infty} \int_A f^{[n]} dP \\
&= \sup_{i \in I} \left(\sup_{n \in \mathbb{N}} \int_A h_i^{[n]} dP \right) = \lim_{n \to \infty} \int_A f^{[n]} dP \\
&= \sup_{i \in I} \left(\lim_{n \to \infty} \int_A h_i^{[n]} dP \right) = \lim_{n \to \infty} \int_A f^{[n]} dP.
\end{aligned}$$

Since $h_i^{[n]}$s differ from h_is only in the domain where $h_i^{[n]}$s take positive values, the integrals can be separated into positive and negative parts. This allows us to apply monotone convergence theorem on the limits to get,

$$\sup_{i \in I} \int_A h_i(x) dP(x) = \int_A f dP \tag{1.34}$$

where $f = \underset{i \in I}{\text{es.sup}}\ h_i(x)$. □

Proof of the Theorem 1.6. Notice that by the definition of conditional expectation we can write

$$\sup_{\tau \in S_s} E(X_\tau) = \sup_{\tau \in S_s} \int_\Omega E(X_\tau | \mathscr{F}_s). \tag{1.35}$$

It can be shown that the family of integrable functions $(E(X_\tau | \mathscr{F}_s))_{\tau \in S_s}$ admits needle-like variation. Thus, by Lemma 1.7 above equation can be written as

$$\sup_{\tau \in S_s} E(X_\tau) = \int_\Omega \underset{\tau \in S_s}{\text{es.sup}}\ E(X_\tau | \mathscr{F}_s). \tag{1.36}$$

If we define $\underset{\tau \in S_s}{\text{es.sup}}\ E(X_\tau | \mathscr{F}_s) \triangleq f_s$ we get

$$\sup_{\tau \in S_s} E(X_\tau) = E(f_s) \tag{1.37}$$

To prove f is a supermartingale, consider $s \geq t$ and $A \in \mathscr{F}_t$. Then,

$$\int_A f_s dP = \sup_{\tau \in S_s} \int_A X_\tau dP \leq \sup_{\tau \in S_t} \int_A X_\tau dP = \int_A f_t dP.$$

Thus, $f_t \geq E\left(f_s \mid \mathscr{F}_t\right)$ a.e. and $(f_t, \mathscr{F}_t)$ is a supermartingale.

Now we prove the minimality and right continuity of the supermartingale f. We first prove the results for when $X_t, t \geq 0$ is bounded by an integrable function $B(\omega)$. Later, for the general case we bound X by $n \in \mathbb{N}$, and prove the results hold as $n \to \infty$.

By Theorem 1.4, to prove the right continuity of the paths of f we only need to prove right continuity of the function $v_t : t \mapsto E(f_t)$. Since, f is a supermartingale, v_t is non-increasing. So, $v_t^+ \leq v_t$. Here v_t^+ denotes the right limit of the function v_t. We need to prove the reverse inequality.

Note that from equation (1.37) $v_t = \sup_{\tau \in S_t} E(X_\tau)$. Thus, for any $\epsilon > 0$ we can find a stopping time $\tau_\epsilon \in S_t$ such that

$$v_t < EX_{\tau_\epsilon} + \frac{\epsilon}{2}. \tag{1.38}$$

For $n \in \mathbb{N}$ define $\tau_e^{[n]}$ as follows:

$$\tau_e^{[n]} = \frac{i}{2^n} \text{ if } \frac{i-1}{2^n} \leq \tau_\varepsilon < \frac{i}{2^n}. \tag{1.39}$$

$\tau_{\varepsilon^{[n]}}$ is a stopping time belonging to the family $S_{t^{[n]}}$ and $\lim_{n\to\infty} \tau_\epsilon^{[n]} = \tau_\epsilon$ (a.e.). Since, $\tau_\epsilon^{[n]} > \tau_\varepsilon$ for each $n \in \mathbb{N}$, $\tau_\epsilon^{[n]}$ approach τ_ϵ from right. From the right continuity of X we have,

$$\lim_{n\to\infty} X_{\tau_\epsilon^{[n]}} = X_{\tau_\epsilon}. \tag{1.40}$$

Since $\{|X_t|, t \geq 0\}$ is bounded by an integrable function $B(\omega)$, we have from bounded convergence theorem

$$\lim_{n\to\infty} E\left(X_{\tau_\epsilon^{[n]}}\right) = E\left(X_{\tau_\epsilon}\right). \tag{1.41}$$

Thus there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$

$$\left| E(X_{\tau_\epsilon^{[n]}}) - E(X_{\tau_\epsilon}) \right| < \frac{\varepsilon}{2}. \tag{1.42}$$

Thus, equations (1.42) and (1.38) imply,

$$v_t < E(X_{\tau_\epsilon^{[n]}}) + \epsilon \leq v_{t+} + \epsilon. \tag{1.43}$$

Since this holds for any $\epsilon > 0$, we have $v_t \leq v_t^+$. This proves the right continuity of f.

To prove the minimality of f, consider $g = g(t, \omega)$, a right continuous supermartingale that majorizes x_t. Then,

$$g_t \geq x_t \geq E\left[Y \mid \mathscr{F}_t\right], \tag{1.44}$$

where $Y(\omega) = -\sup_t X_t^-(\omega)$. By the regularity of the right continuous supermartingale g_t, for any $\tau \in S_t$, we have,

$$g_t \geq E\left(g_\tau \mid \mathscr{F}_t\right) \geq E\left(x_\tau \mid \mathscr{F}_t\right) \text{ (a.e.)}. \tag{1.45}$$

Since, equation (1.45) holds for any $\tau \in S_t$, from (1.37) we get $g_t \geq f_t$ almost everywhere. Thus, minimality is proved.

For the general case with the condition $E(X_t^-) < \infty$, for $n \in \mathbb{N}$ we put

$$X_t(n) = \min\left(X_t, n\right), \quad f_t(n) = \operatorname*{es.sup}_{\tau \in S_t}\left(E\left(x_\tau(n) \mid \mathscr{F}_t\right)\right). \tag{1.46}$$

For each $n \in \mathbb{N}$, $|X_t|$ is bounded by the integrable random variable $\max(n, X_t^-)$, thus from the previous result $f_t(n)$ is right continuous. We have $\lim_{n\to\infty} f_t(n) = f_t$ $(a.e.)$. Since the limit of monotonously increasing right continuous supermartingales is a right continuous supermartingale, right continuity of f is proved. The minimality is proved the same as for the bounded case mentioned above. □

Remark 1.8. *Given a $t > 0$, in Theorem 1.6, the class S_t has been introduced. In light of Definition 0.5, $S_t = \cup_{T \geq t} S_{t,T}$. Let $S_t^* := S_{t,\infty}$ denote the class of stopping times τ (not necessarily finite) such that $P(\tau \geq t) = 1$. We state the following results from [9].*

- *Theorem 1.6 holds true even when the class of stopping times S_t is replaced by S_t^*.*

- *If the process X satisfies the additional conditions that*
  (a) $E(\sup_t |X_t|) < \infty$ *and* *(b) the map* $t \mapsto X_t$ *is continuous almost surely then the random variable*

$$\tau_0 \triangleq \begin{cases} \inf(s \geq t : f_s = X_s) \\ \infty \quad \textit{if there is no such } s \end{cases}$$

  is the optimal stopping time. That is $E(X_{\tau_0}) = \sup_{\tau \in S_t^*} E(X_\tau)$.

Chapter 2

Fair Pricing and Hedging of Option Contracts

The goal of this chapter is to introduce the concept of hedging, mainly for two specific types of contingent claims: European and American contingent claims. In the first section, we present some terminologies and their mathematical descriptions, which will serve as tools for building the language to talk about options. We also introduce the market models and asset price dynamics in this section. In the second section, we use the general structure built for the hedging in the previous section to address the specific problem of pricing a European option and arrive at the famous Black-Scholes equation. The third section will have the details of the concept of Hedging as introduced in [15] for European and American contingent claims. The optimal stopping problem for the exercise of the American option will be analysed through a martingale approach.

2.1 Asset Price Dynamics, Wealth Process, and Hedging

2.1.1 Asset Price Dynamics

Consider a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_t, P)$. Let W be a d-dimensional Brownian motion: $\{W(t) = (W_1(t), \ldots, W_d(t))^*, \mathscr{F}_t; 0 \leq t < \infty\}$ defined on it, where * denotes trans-

position. While defining the filtration, $\mathscr{F}_t$ is chosen such that it is the P-augmentation of the filtration generated by the Brownian motion. This will be the space on which all the processes in this chapter will be defined unless mentioned otherwise.
Consider a market where financial entities called *assets* are traded. Each asset has a *price* at which the asset can be bought or sold at any time $t \geq 0$. Our market consists of $d+1$ assets, where d is a positive integer. We will define some terms and parameters of the market and specify the restrictions on those below.

1. One of the assets is called *the bond.* It represents a money market account that allows one to invest in or get a loan from an ideal bank at the instantaneous interest rate, $r(t)$ at time t. The price of the bond is denoted by $P_0(t)$.

2. The remaining d assets are termed *stocks.* These are the 'risky' assets.

3. $\{r(t); 0 \leq t < \infty\}$ is the interest rate of the bond.

4. $\{b_i(t); 0 \leq t < \infty\}$ is the appreciation rate of the ith stock for each $1 \leq i \leq d$.

5. $\{\mu_i(t); 0 \leq t < \infty\}$ is the dividend rate of the ith stock for each $1 \leq i \leq d$.

6. $\{\sigma_{ij}(t); 0 \leq t < \infty\}$ is the dispersion coefficient of the ith stock due to the jth component of the Brownian motion for each $1 \leq i, j \leq d$.

7. We define matrices $\sigma(t) = \{\sigma_{ij}(t)\}_{i \leq i, j \leq d}$ and $D(t) = \sigma(t)\sigma^*(t)$. We assume there exists $\epsilon > 0$ such that, the condition

$$x^* D(t, \omega) x \geq \varepsilon \|x\|^2, \quad \forall x \in \mathbb{R}^{\mathrm{d}}, \tag{2.1}$$

holds for every $(t, \omega) \in [0, \infty) \times \Omega$.

8. The appreciation rates, the dividend rates, the dispersion coefficients, and the interest rate are all assumed to be $\{\mathscr{F}_t\}$ adapted measurable processes that are uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$, for every finite $T > 0$. These will be termed as "coefficients of the market model" henceforth.

The price of the bond follows the equation:

$$dP_0(t) = r(t) P_0(t) dt, \quad P_0(0) = p_0 = 1. \tag{2.2}$$

It determines the discount factor $\beta(t)$ given by: $\beta(t) = (P_0(t))^{-1}$. The prices of the stocks are assumed to follow time in-homogeneous geometric Brownian motions. The SDE is given below.

$$\begin{aligned} dP_i(t) &= P_i(t)\left[b_i(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad 0 \le t < \infty, \\ P_i(0) &= p_i > 0, \quad 1 \le i \le d. \end{aligned} \tag{2.3}$$

A direct derivation shows that the discounted stock prices obey the following equation

$$d\left[\beta(t)P_i(t)\right] = \beta(t)P_i(t)\left[\left(b_i(t) - r(t)\right)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_j(t)\right], \quad 0 \le t < \infty. \tag{2.4}$$

2.1.2 Portfolio-Consumption Processes and Wealth Process

Definition 2.1. *An $\mathbb{R}^d$-valued, $\{\mathscr{F}_t\}$-adapted, measurable process $\pi = \{\pi(t) = (\pi_1(t), \ldots, \pi_d(t))^*, \mathscr{F}_t; 0 \le t \le \infty\}$ is called a portfolio process if*

$$\sum_{i=1}^{d} \int_0^T \pi_i^2(s)ds < \infty \qquad a.s.\ P \tag{2.5}$$

is satisfied for all $0 \le T < \infty$.

Definition 2.2. *A progressively measurable, $\{\mathscr{F}_t\}$-adapted process $C = \{C_t, \mathscr{F}_t; 0 \le t < \infty\}$ taking values in $[0, \infty)$ is called a consumption process if the following conditions hold.*

(i) $C_0(\omega) = 0$.

(ii) For P-a.e. $\omega \in \Omega$, the path $t \mapsto C_t(\omega)$ is nondecreasing and right-continuous.

Now we shall construct a wealth process. Consider an investor whose investments are not big enough to affect the changes in the prices. If $N_i(t)$ is the number of ith asset the investor holds at time t, $i = 0, 1, \ldots, d$, then the wealth, given by X at time t is

$$X_t = \sum_{i=0}^{d} N_i(t)P_i(t), \quad 0 \le t < \infty. \tag{2.6}$$

Note that we assume that partial stocks can be held. Thus, $N_i(t) \in \mathbb{R}^+$. Let C_t denote the cumulative consumption by the investor from his wealth till time t. If the investor decides to redistribute their wealth among the assets by consumption after an increment h in time, the change in wealth will be

$$X_{t+h} - X_t = \sum_{i=0}^{d} N_i(t)\left[P_i(t+h) - P_i(t)\right] + h\sum_{i=1}^{d} N_i(t)P_i(t)\mu_i(t) - \left(C_{t+h} - C_t\right). \tag{2.7}$$

Let the amount invested in the ith asset be denoted by $\pi_i(t) \triangleq N_i(t)P_i(t), i = 0, 1, \ldots, d$ and let the $\mathbb{R}^d$ valued process $\pi \triangleq \{\pi(t) = (\pi_1(t), \ldots, \pi_d(t))^*\}$. We can then write the continuous-time version of (2.7) with the help of (2.3), (2.2) and (2.6) as

$$\begin{aligned} dX_t = &\left[r(t)X_t + \sum_{i=1}^{d} \pi_i(t)\left(b_i(t) + \mu_i(t) - r(t)\right)\right] dt - dC_t \\ &+ \sum_{i=1}^{d}\sum_{j=1}^{d} \pi_i(t)\sigma_{ij}(t)dW_j(t); \quad 0 \le t < \infty. \end{aligned} \tag{2.8}$$

Definition 2.3. *Given, a portfolio process π and a consumption process C, the solution to* (2.8)*, the wealth process corresponding to (π, C), can be written as*

$$\begin{aligned} X_t = P_o(t)\Bigg[x + &\int_0^t \beta(s)\pi^*(s)(b(s) + \mu(s) - r(s)\mathbf{1})ds - \int_0^t \beta(s)dC_s \\ &+ \int_0^t \beta(s)\pi^*(s)\sigma(s)dW(s)\Bigg], \quad 0 \le t < \infty. \end{aligned} \tag{2.9}$$

Here $b(s) = (b_1(s), b_2(s), \ldots, b_d(s))^*$ is the column vector of appreciation rates. $\mu(s)$ is also defined the same way. $\mathbf{1}$ is the vector in $\mathbb{R}^d$ with all it's entries equal to one.

Observe that in (2.9) discounted value of portfolio process is integrated with respect to time and Brownian motion in two different integrals. We can use Girsanov transformation to reduce these two integrals into one stochastic integral. This transformation allows for the discounted stock prices of the stocks that don't pay dividend to be treated as martingales under $\tilde{P}$. Which makes the problem of pricing of contingent claims in the setting of continuous trading more tractable. Refer to [13] for elucidation of this point.

Define $\theta(t)$ as follows:

$$\theta(t) \triangleq \sigma^*(t)D^{-1}(t)(b(t) + \mu(t) - r(t)\mathbf{1}), \mathscr{F}_t, \quad 0 \le t < \infty. \tag{2.10}$$

Define $Z_t(\theta)$ as follows:

$$Z_t \triangleq \exp\left\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\}, \mathscr{F}_t, \quad 0 \le t < \infty. \tag{2.11}$$

Note that because of the conditions on the parameters $\sigma_{i,j}(t), b_i(t), \mu_i(t), r(t)$ and matrix $D(t)$, presented in point number 8 in Subsection 2.1.1, $\theta(t)$ is uniformly bounded in $(t, \omega) \in [0, T] \times \Omega$ for every finite $T > 0$. Thus, the continuous local martingale in (2.11) becomes a continuous martingale as shown in Theorem 1.1.

Fix a time $T > 0$. This will usually be the time of maturity when dealing with contingent claims in later sections. If we define a new probability measure

$$\tilde{P}_T(A) \triangleq E\left(Z_T 1_A\right), \quad A \in \mathscr{F}_T, \tag{2.12}$$

then by Theorem 1.1 we have that

1. P and $\tilde{P}_T$ are equivalent measures on $\mathscr{F}_T$, and
2. the process $\tilde{W}$ as defined below is an $\mathbb{R}^d$-valued Brownian motion on $\left(\Omega, \mathscr{F}_T, \tilde{P}_T\right)$.

$$\tilde{W}(t) \triangleq W(t) + \int_0^t \theta(s)ds, \mathscr{F}_t, \quad 0 \le t \le T. \tag{2.13}$$

Thus, equations (2.8) and (2.9) can be rewritten as

$$dX_t = r(t)X_t dt - dC_t + \sum_{i=1}^d \sum_{j=1}^d \pi_i(s)\sigma_{ij}(s)d\tilde{W}_j(s), \tag{2.14}$$

$$\beta(t)X_t + \int_0^t \beta(s)dC_s = x + \int_0^t \beta(s)\pi^*(s)\sigma(s)d\tilde{W}(s), \tag{2.15}$$

respectively.

Definition 2.4. *Given a finite initial wealth $X_0 = x$ and a finite time T, we say a strategy, i.e., a pair of portfolio and consumption processes (π, C) is admissible on (T, x) if the wealth process of (π, C) given by (2.15) satisfies*

$$X_0 = x \quad \textit{and} \quad X_t \geq 0, \quad 0 \leq t \leq T \quad \textit{almost surely.}$$

The set of all admissible strategies for the initial wealth x and time horizon T, is denoted by $\mathscr{A}(T, x)$.

Evidently, due to (2.5) and Point 8 in Subsection 2.1.1, the RHS of (2.15) is a local martingale. Again notice that, as C is nondecreasing (see Definition 2.2), for an admissible strategy (π, C), LHS of (2.15) is non-negative. Now, it can further be proved that for an admissible strategy, the continuous non-negative local martingale in RHS of (2.15) is in fact a supermartingale as follows.

For any local martingale M we have, a sequence of non-decreasing stopping times s_n that that diverge almost surely, and the stopped process M^{s_n} given by $M_t^{s_n} = M_{\min\,(t, s_n)}$ is a martingale for all $n \in \mathbb{N}$. In addition to that if M is non-negative, then we have, from Fatou's lemma, for $0 \leq u \leq t < \infty$

$$\begin{aligned}(1)\ E(|M_t|) = E(M_t) =& E(\lim_{n\to\infty} M_t^{s_n}) \\ & \leq \liminf_{n\to\infty} E(M_t^{s_n}) = E(M_0^{s_n}) < \infty,\end{aligned}$$

$$\begin{aligned}(2)\ E(M_t \mid \mathscr{F}_u) =& E(\lim_{n\to\infty} M_t^{s_n} \mid \mathscr{F}_u) \\ & \leq \liminf_{n\to\infty} E(M_t^{s_n} \mid \mathscr{F}_u) = \liminf_{n\to\infty} M_u^{s_n} = M_u.\end{aligned}$$

Thus, M is a supermartingale.

Thus, the RHS of (2.15), a continuous non-negative local martingale is indeed a supermartingale. Therefore, the left side is also a supermartingale. Therefore we can apply optional sampling theorem to (2.15) to write

$$\tilde{E}_T\left[\beta(\tau)X_\tau + \int_0^\tau \beta(s)dC_s\right] \leq x, \quad \forall \tau \in S_{0,T}. \tag{2.16}$$

2.1.3 Hedging and Fair Price of Contingent Claims

Contingent claims are financial entities whose prices are contingent upon some underlying security. We define two types of contingent claims here.

Definition 2.5. *A contingent claim is a trade-able contract constituted by the triplet of parameters* (T, f, g)

- $T \in [0, \infty)$ *is the time of maturity.*
- $f(t, \omega)$ *is the terminal payoff on exercise.*
- $g(t, \omega)$ *is the payoff rate function. It represents the rate of continuous payoff from the claim up until the time the claim is exercised.*

Remark 2.6. *A European contingent claim (ECC) can only be exercised at the time of maturity. Thus,* $f(t, \omega)$ *is only defined for* $t = T$ *for an ECC. An American contingent claim (ACC) can be exercised on or before the maturity date, so one also needs to select an 'exercise time'* $\tau \in S_{0,T}$.

Remark 2.7. *For the case of ACC, the process* f *is assumed to have continuous paths. The processes* f *and* g *are assumed to be non-negative and progressively measurable, and there exists* $\mu > 1$ *such that the following holds*

$$E\left(\sup_{0 \le s \le t} f_s + \int_0^t g_s ds\right)^{\mu} < \infty \quad \textit{for every } 0 \le t < \infty. \tag{2.17}$$

A hedging strategy against a contingent claim is basically a pair of of portfolio and consumption processes (π, C) that give the same payoff as a contingent claim over the time period till the maturity. We formally define hedging strategies for European and American contingent claims below.

Here we mention a result from [15] that will be useful later.

Proposition 2.8. *Fix* $x, T \in [0, \infty)$*. If there is a consumption process* C *such that the following condition holds*

$$\tilde{E}_T \int_0^T \beta(s) dC_s \le x. \tag{2.18}$$

then, there exists a portfolio process π such that $(\pi, C) \in \mathscr{A}(T,x)$ and the wealth process generated by this pair is given by

$$X_t = \tilde{E}_T\left(\int_t^T \exp\left(-\int_t^s r(u)du\right) dC_s \mid \mathscr{F}_t\right) + \left(x - \tilde{E}_T \int_0^T \beta(s) dC_s\right) P_0(t). \tag{2.19}$$

Here $\tilde{E}_T$ represents the expectation with respect to $\tilde{P}$ as defined in (2.12).

Definition 2.9. *A pair of portfolio and consumption processes $(\pi, C) \in \mathscr{A}(T,x)$ is a **hedging strategy** against a European contingent claim with parameters $(T, f_T(\omega), g(t,\omega))$, if the following hold almost surely:*

$$X_T = f_T, \text{ and} \tag{2.20}$$

$$C_t = \int_0^t g_s ds, 0 \le t \le T \tag{2.21}$$

where $\{X_t\}$ is the wealth process generated by (π, C) according to (2.9).

Definition 2.10. *A pair of portfolio and consumption processes $(\pi, C) \in \mathscr{A}(T,x)$ is a **hedging strategy** against an American contingent claim with parameters $(T, f(t,\omega), g(t,\omega))$, if the following hold almost surely:*

$$A_t(\omega) \triangleq C_t(\omega) - \int_0^t g_s(\omega) ds, \quad 0 \le t \le T, \quad \text{is a continuous, non-decreasing function.} \tag{2.22}$$

$$X_t(\omega) \ge f_t(\omega), \quad \forall t \in [0,T]. \tag{2.23}$$

$$X_T(\omega) = f_T(\omega). \tag{2.24}$$

$$A_t(\omega) = A_{\tau_t(\omega)}(\omega) \quad \text{for every fixed number} \quad t \in [0,T], \tag{2.25}$$

where τ_t is defined as $\tau_t \triangleq \inf\{t \le s \le T; \quad X_s = f_s\}$.

The continuity of A in (2.22) implies continuity of the consumption process C. This implies the progressive measurability of the wealth process w.r.t. the filtration $\{\mathscr{F}_t\}_t$. This makes τ_t as defined above a $\{\mathscr{F}_t\}_t$ stopping time.

2.2 Fair pricing of European claims

Definition 2.11. *The fair price of a European Contingent Claim with parameters* (T, f_T, g) *is the smallest finite amount* $x > 0$ *for which* $\mathscr{A}(T, x)$*, the class of admissible strategies, contains a hedging strategy.*

Theorem 2.12. *Define the process* $Q = \{Q_t\}_t$*, where* Q_t *is the cumulative payoff of the claim at time* t *as follows:*

$$Q_t \triangleq \beta(t) f_t + \int_0^t \beta(s) g(s) ds, \quad \mathscr{F}_t, \quad 0 \leq t \leq T. \tag{2.26}$$

The fair price for the ECC (T, f_T, g) *is the expected value of the discounted price of the cumulative payoff at maturity. This is given by:*

$$\tilde{E}(Q_T) = \tilde{E}_T \left[f_T \exp\left(-\int_0^T r(u) du \right) + \int_0^T g_t \exp\left(-\int_0^t r(u) du \right) dt \right]. \tag{2.27}$$

Also there exists a hedging strategy $(\pi, C) \in \mathscr{A}(T, \tilde{E}(Q_T))$*, with a continuous adapted wealth process* $X = \{X_t, \mathscr{F}_t; 0 \leq t \leq T\}$ *given by,*

$$X_t = \tilde{E}_T \left[f_T \exp\left(-\int_t^T r(u) du \right) + \int_t^T g_s \exp\left(-\int_t^s r(u) du \right) ds \mid \mathscr{F}_t \right], \; 0 \leq t \leq T \tag{2.28}$$

a.s. $\tilde{P}$*.*

Proof. First of all we prove the proposed fair price $\tilde{E}(Q_T)$ is finite. Let K_T be an upper bound on both $\|\theta(t, \omega)\|$ and $\beta(t, \omega)$, where $(t, \omega) \in [0, T] \times \Omega$. We prove a stronger result that for $p > 1$, $\tilde{E}\left(\max_{0 \leq t \leq T} Q_t \right)^p$ is finite.

$$\begin{aligned}
&\tilde{E}\left(\max_{0 \leq t \leq T} Q_t \right)^p \\
&= E\left(Z_T(\theta) \left[\max_{0 \leq t \leq T} \left(\beta(t) f_t + \int_0^t \beta(s) g_s ds \right) \right] \right)^p \\
&\leq E\left(Z_T(\theta) \left[\max_{0 \leq t \leq T} \beta(t) f_t + \int_0^T \beta(s) g_s ds \right] \right)^p.
\end{aligned}$$

Define $\mu = p^2$ and $q = 1 - \frac{1}{p}$. Using Hölder's inequality we can write

$$E\left((Z_T(\theta))^p\left[\max_{0\le t\le T}\beta(t)f_t + \int_0^T \beta(s)g_s ds\right]^p\right)$$
$$\le (E(Z_T^p(\theta))^q)^{\frac{1}{q}} . \left(E\left[\max_{0\le t\le T}\beta(t)f_t + \int_0^T \beta(s)g_s ds\right]^{p^2}\right)^{\frac{1}{p}}$$
$$= (E(Z_T(\theta))^{qp})^{\frac{1}{q}} . \left(K_T^\mu . E\left(\max_{0\le t\le T} f_t + \int_0^T g_s ds\right)^\mu\right)^{\frac{1}{p}}.$$

By condition (2.17) $E\left(\max_{0\le t\le T} f_t + \int_0^T g_s ds\right)^\mu$ is finite. We have the result, if we show $E(Z_T(\theta))^\rho$ is finite for all $1 < \rho < \infty$.

$$E(Z_T(\theta))^\rho = E\left(\exp\left(-\int_0^t \rho\theta^*(s)dW(s) - \frac{1}{2}\int_0^t \rho\|\theta(s)\|^2 ds\right)\right)$$
$$= E\left(\exp\left(-\int_0^t \rho\theta^*(s)dW(s) - \left(\frac{1}{2}\int_0^t \rho\|\theta(s)\|^2 ds - \int_0^t \|\rho\theta(s)\|^2 ds + \int_0^t \|\rho\theta(s)\|^2 ds\right)\right)\right)$$
$$= E\left(\exp\left(-\int_0^t \rho\theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\rho\theta(s)\|^2 ds\right) . \exp\left(\frac{1}{2}\rho(\rho-1)\int_0^t \|\theta(s)\|^2 ds\right)\right)$$
$$\le E\left(\exp\left(-\int_0^t \rho\theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\rho\theta(s)\|^2 ds\right)\right) . \exp\left(\frac{1}{2}\rho(\rho-1)K_T^2 T\right)$$
$$= 1. \exp\left(\frac{1}{2}\rho(\rho-1)K_T^2 T\right) < \infty.$$

The last equality comes from the martingale property of $\{Z_t(\rho\theta)\}_t$.

Having proved the finiteness of $\tilde{E}(Q_T)$ we now prove that it is the fair price. Consider a hedging strategy $(\pi, C) \in \mathscr{A}(T, x)$ and X, the wealth process generated by it. From (2.16) we have

$$\tilde{E}_T\left[\beta(T)X_T + \int_0^T \beta(s)dC_s\right] \le x.$$

Since (π, C) is a hedging strategy, this implies

$$\tilde{E}\left[\beta(T)f_T + \int_0^T \beta(s)g_s ds\right] \le x.$$

Since the choice of the hedging strategy was arbitrary, the initial wealth x for any hedging strategy is bounded below by $\tilde{E}(Q_T)$. If we show the existence of a hedging strategy with initial wealth $x = \tilde{E}(Q_T)$, we prove that $\tilde{E}(Q_T)$ is the fair price of the ECC.

Consider a consumption process C with $C_t = \int_0^t g(s)ds$. Since f_T is non-negative we have,

$$\tilde{E}\left(\int_0^T \beta(s)dC_s\right) = \tilde{E}\left(\int_0^T \beta(s)g(s)ds\right) \leq \tilde{E}\left[\beta(t)f_T + \int_0^T \beta(s)g_s ds\right] = \tilde{E}(Q_T).$$

Therefore, (2.18) holds true with $x = \tilde{E}(Q_T)$. Hence, by Result 2.8, there exists a portfolio process π such that $(\pi, C) \in \mathscr{A}(T, \tilde{E}(Q_T))$ whose wealth process is given according to the equation (2.19) by

$$\begin{aligned}
X_t &= \tilde{E}_T\left(\int_t^T \exp\left(-\int_t^s r(u)du\right) g(s)ds \mid \mathscr{F}_t\right) + \left(\tilde{E}(Q_T) - \tilde{E}_T\int_0^T \beta(s)g(s)ds\right) P_0(t) \\
&= \tilde{E}_T\left(\int_t^T \frac{\beta(s)}{\beta(t)} g(s)ds \mid \mathscr{F}_t\right) + \left(\tilde{E}\left(\beta(T)f_T + \int_0^T \beta(s)g(s)ds\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. - \tilde{E}_T\int_0^T \beta(s)g(s)ds\right)(\beta(t))^{-1} \\
&= \left(\tilde{E}_T\left(\int_t^T \beta(s)g(s)ds \mid \mathscr{F}_t\right) + \tilde{E}\left(\beta(T)f_T\right)\right)(\beta(t))^{-1}.
\end{aligned}$$

Note that this wealth process, along with the consumption process C_t defined above satisfy the conditions of Definition 2.9 to be a hedging strategy. Thus Theorem 2.12 is proved. □

Valutaion process: We call X the valuation process because if there exists another hedging strategy $(\pi', C') \in \mathscr{A}(T, x)$ for $ECC(T, f_T, g)$, with initial wealth $x = \tilde{E}_T(Q_T)$, then $X'_t = X_t$ a.s. $\tilde{P}_T$. Where X' is the wealth process generated by (π', C').

Proof. Consider the equivalent of equation (2.15) for (π', C') :

$$\beta(t)X'_t + \int_0^t \beta(s)dC'_s = x + \int_0^t \beta(s)(\pi'(s))^*\sigma(s)d\tilde{W}(s). \tag{2.29}$$

Let M'_t denote the supermartingale shown in (2.15). We can write $\tilde{E}_T(x + M'_0) = x = \tilde{E}_T(Q_T) = \tilde{E}_T(x + M'_T)$. The last equality follows from the hedging. Thus, $\{M'_t, \mathscr{F}_t; 0 \leq t \leq T\}$ is a $\tilde{P}_T$-martingale. We can write (2.29), for $t \in [0, T]$ as

$$X'_t = P_0(t)\left\{x + \int_0^t \beta(s)(\pi'(s))^*\sigma(s)d\tilde{W}(s) - \int_0^t \beta(s)dC'_S\right\}$$

$$\begin{aligned} X'_t &= P_0(t)\left\{x + \tilde{E}_T\left(M'_T \mid \mathscr{F}_t\right) - \int_0^t \beta(s)g_s ds\right\} \\ &= \tilde{E}_T\left[f_T \exp\left(-\int_t^T r(u)du\right) + \int_t^T g_s \exp\left(-\int_t^s r(u)du\right) ds \mid \mathscr{F}_t\right] \\ &= X_t \text{ a.s. } \tilde{P}_T. \end{aligned}$$

□

2.3 Fair pricing of American claims

Definition 2.13. *The fair price of a American Contingent Claim with parameters* (T, f, g) *is the smallest finite amount* $x > 0$ *for which a hedging strategy* $(\pi, C) \in \mathscr{A}(T, x)$ *as defined in Definition 2.10 exists.*

Theorem 2.14. *The fair price for the* $ACC(T, f, g)$*, at* $t = 0$ *is given by*

$$\sup_{\tau \in S_{0,T}} \tilde{E}_T\left[f_\tau \exp\left(-\int_0^\tau r(u)du\right) + \int_0^\tau g_s \exp\left(-\int_0^s r(u)du\right) ds\right].$$

Let the process Q *be as defined in the last section. Then there exists a hedging strategy strategy* $(\pi, C) \in \mathscr{A}\left(T, \sup_{\tau \in S_{0,T}} \tilde{E}(Q_\tau)\right)$ *with continuous adapted wealth process* $X = \{X_t, \mathscr{F}_t; 0 \leq t \leq T\}$ *given by,*

$$\begin{aligned} X_t = \operatorname*{ess\,sup}_{\tau \in S_{t,T}} \tilde{E}_T\Bigg[f_\tau \exp\left(-\int_t^\tau r(u)du\right) \\ + \int_t^\tau g_s \exp\left(-\int_t^s r(u)du\right) ds \mid \mathscr{F}_t\Bigg] \quad , 0 \leq t \leq T \ a.s. \end{aligned} \tag{2.30}$$

From the same arguments as in the last section we notice that $\sup_{\tau \in S_{0,T}} \tilde{E}(Q_\tau)$ is finite for the $ACC(t, f, g)$. Consider, for some $x, T \in [0, \infty)$, a hedging strategy $(\pi, C) \in \mathscr{A}(T, x)$ and

X the wealth process generated by it. From (2.16) we have

$$\tilde{E}_T\left[\beta(\tau)X_\tau + \int_0^\tau \beta(s)dC_s\right] \le x, \quad \tau \in S_{0,T}.$$

Since (π, C) is a hedging strategy, this implies

$$\tilde{E}\left[\beta(\tau)f_\tau + \int_0^\tau \beta(s)g_s ds\right] \le x, \quad \tau \in S_{0,T}$$
$$\Longrightarrow \sup_{\tau\in S_{0,T}} \tilde{E}(Q_\tau) \le x.$$

Since the choice of the hedging strategy was arbitrary, the initial wealth x for any hedging strategy is bounded below by $\sup_{\tau\in S_{0,T}} \tilde{E}(Q_\tau)$, if we show the existence of a hedging strategy with initial wealth $x = \sup_{\tau\in S_{0,T}} \tilde{E}(Q_\tau)$, we prove that $\sup_{\tau\in S_{0,T}} \tilde{E}(Q_\tau)$ is the fair price of the ACC.

Remark 2.15. *We first notice from Theorem 1.6 and [9] that*

- $\sup_{\tau\in S_{0,T}} \tilde{E}(Q_\tau) = \tilde{E}(Y_t)$, *where Y is the minimal RCLL supermartingale that satisfies $Y_t \ge Q_t$ a.s.for all $t \in [0,\infty)$.*
- $Y_t = \operatorname{ess\,sup}_{\tau\in S_{t,T}} \tilde{E}_T\left(Q_\tau \mid \mathscr{F}_t\right) \quad$ *a.s.* $\tilde{P}_T$.
- *The stopping time $\rho_t = \inf\{t \le s \le T; Y_s = Q_s\}$ is the optimal stopping time. That is, $\tilde{E}(Q_{\rho_t}) = \tilde{E}(Y_t)$.*

It turns out the supermartingale Y is regular and of class $D[0,T]$. We prove that it is of class $D[0,T]$. For the proof of regularity one can refer to [9] and [3].

Lemma 2.16. *The family of random variables $\{Y_\tau\}_{\tau\in S_{0.T}}$ is uniformly integrable with respect to $\tilde{P}_T$.*

Proof. Let m be an RCLL modification process of the martingale $\tilde{E}_T\left(\max_{0\le\theta\le T} Q_\theta \mid \mathscr{F}_t\right)$. Note that $\tilde{E}(Y_\tau)_{\tau\in S_{0,T}} \le \tilde{E}_T\left(\sup_{0\le t\le T} Y_t\right)$. If we prove the latter term in the inequality is finite we are done.

We have from Jensen's inequality, for $p > 1$,

$$\tilde{E}_T \left(\sup_{0 \le t \le T} Y_t \right)^p \le \tilde{E}_T \left(\sup_{0 \le t \le T} m_t^p \right).$$

applying Doob's L^p inequality to the RHS of the inequality we get

$$\tilde{E}_T \left(\sup_{0 \le t \le T} m_t^p \right) \le \left(\frac{p}{p-1} \right)^p \cdot \tilde{E}_T \left(m_t^p \right) \le \left(\frac{p}{p-1} \right)^p \cdot \tilde{E}_T \left(\max_{0 \le t \le T} Q_t \right)^p.$$

The last term is shown to be finite in the last section 2.2. This completes the proof. □

D-M decomposition of Y: From Lemma 2.16 and the regularity condition on Y we can write the supermartingale Y as follows:

$$Y_t = Y(0) + M_t - \Lambda_t, \quad 0 \le t \le T, \quad a.s.\tilde{P}_T \tag{2.31}$$

Here Λ is a continuous nondecreasing process and M is a $\tilde{P}_T$-martingale with RCLL paths and $M_0 = \Lambda_0 = 0, \tilde{E}_T (\Lambda_T) = Y(0) - \tilde{E}_T (Q_T)$.

From theroem 3.4.15 of [16] about representation of Brownian square integrable martingales as stochastic integrals, and Baye's rule we can write M_t as follows:

$$M_t = \sum_{j=1}^{d} \int_0^t \psi_j(s) d\tilde{W}_j(s), \quad 0 \le t \le T, \tag{2.32}$$

where $\{\psi_j(t), \mathscr{F}_i; 0 \le t \le T\}$ are measurable and adapted processes which satisfy

$$\sum_{j=1}^{d} \int_0^T \psi_j^2(t) dt < \infty \quad \text{a.s. } \tilde{P}_T. \tag{2.33}$$

proof of the Theorem 2.14. Define the process X as follows:

$$X_t \triangleq \frac{1}{\beta(t)} \left[Y_t - \int_0^t \beta(s) g_s ds \right], \mathscr{F}_t, \quad 0 \le t \le T \tag{2.34}$$

From the equation (2.31) and (2.32) we can write X_t as follows:

$$\beta(t)X_t + \int_0^t \beta(s)g_s ds + \Lambda_t = Y(0) + \int_0^t \psi^*(s)d\tilde{W}(s) \tag{2.35}$$

Comparing equation (2.35) with equation (2.15) we can construct a consumption process C given by,

$$C_t = \int_0^t g_s ds + \int_0^t P_0(s)d\Lambda_s \tag{2.36}$$

and because of (2.33) the process π given by,

$$\pi^*(s) = \frac{1}{\beta(s)}\left(\psi^*(s).\sigma^*(s).D^{-1}(s)\right) \tag{2.37}$$

becomes the portfolio process.

We see from construction that the wealth process generated by these is the same as the equation (2.34). We see they also hedge the $ACC(T, f, g)$ of Definition 2.10. (2.22) is satisfied trivially. (2.23) and (2.24) are satisfied from the way Y is defined. (2.25) is satisfied due to the following:

$$\tilde{E}_T\left(Y_t\right) = \tilde{E}_T\left(Q_{\rho t}\right) = \tilde{E}_T\left(Y_{\rho t}\right). \tag{2.38}$$

Which implies, from (2.31) that $\tilde{E}_T\left(\Lambda_t\right) = \tilde{E}_T\left(\Lambda_{\rho t}\right)$. Thus, we get $\Lambda_t = \Lambda_{\rho t}$ a.s. $\tilde{P}_T$. Note that the optimal stopping time for Y from the remark 2.15 and stopping time of Definition 2.10 are the same for this choice of X . Hence we can claim that the hedging strategy (π, C) with portfolio and consumption processes as mentioned above have the initial wealth $\sup_{\tau \in S_{0,T}} \tilde{E}(Q_\tau)$, and thus the theorem is proved. □

Chapter 3

Explicit Pricing of claims

The hedging of contingent claims as mentioned in the second chapter demonstrates that the fair price exists for the European and American contingent claims as defined in (2.11) and (2.13), and gives a stochastic representation for the values of fair price and the hedging wealth process. Though this gives us the tools to analyse the properties of the fair price and the hedging wealth process, it does not provide explicit solutions representing the portfolio process and the fair price of the claims.

One can get explicit solutions representing these quantities if one makes certain concessions on the generality of the market models. In this chapter we describe a few ways to do that. In the first two sections we present the modified market model and show explicit solutions for the fair price of an ECC under restricted conditions on the market model. In the third section explicit formula for computing the portfolio process by using the Feynman-Kac formula is given. In the fourth section we show an explicit formula for pricing a perpetual American put option along with providing a stopping rule for the option.

3.1 Market model

Consider a market model with asset price dynamics as in section 2.1. Instead of having the coeffcients of the market as adapted processes, we assume them to be constants. For $t \geq 0$, let $r(t) = r$, $\mu_i(t) = \mu_i, \sigma_{ij}(t) = \sigma_{ij}$, with $r, \mu_i, \sigma_{ij} \in \mathbb{R}^+, 1 \leq i, j \leq d$.

Consider a contingent claim with terminal payoff function $f_t = L(P(t))$, and payoff rate function $g_t = 0$. Here $P(t)$ is the vector of the stock prices $P(t) = (P_1(t), \ldots, P_d(t))^*$. Let, $L : \mathbb{R}_+^d \to [0, \infty)$ be a continuous function. The prices of the stocks obey the following equations.

$$dP_i(t) = P_i(t) \left[(r - \mu_i)\, dt + \sum_{j=1}^{d} \sigma_{ij} d\tilde{W}_j(t) \right], \quad 1 \le i \le d. \tag{3.1}$$

$$\text{Or,} P_i(t) = p_i \exp \left[\left(r - \mu_i - \frac{1}{2} D_{ii} \right) t + \sum_{j=1}^{d} \sigma_{ij} \tilde{W}_j(t) \right]. \tag{3.2}$$

Let, for $z \in \mathbb{R}^d$

$$K_i(q, t, z) = q \exp \left[\left(r - \mu_i - \frac{1}{2} D_{ii} \right) t + \sum_{j=1}^{d} \sigma_{ij} z_j \right]$$

and for $s \in \mathbb{R}^d$, $K(s, t, z)$ be the vector $K(s, t, z) = (K_1(s_1, t, z), \ldots, K_d(s_d, t, z))^*$.

3.2 Fair price formula for an ECC

For an $ECC(T, f_T, 0)$ the valuation process is given by equation (2.28). Substituting the f and g from above we get the valuation process

$$\begin{aligned} X_t &= \frac{1}{\beta(t)} \tilde{E}_T \left[e^{-r(T)} L(P(T)) + 0 \mid \mathscr{F}_t \right] \\ &= \tilde{E}_T \Big[e^{-\tau(T-t)} L\Big(K_1 \left(P_1(t), T - t, \tilde{W}(T - t) \right), \\ &\qquad \ldots, K_d \left(P_d(t), T - t, \tilde{W}(T - t) \right) \Big) \mid \mathscr{F}_t \Big] \\ &= e^{-\tau(T-t)} \int_{\mathbb{R}^d} L(K(P(t), T - t, z)) \Gamma_{T-t}(z) dz \end{aligned} \tag{3.3}$$

a.s. $\tilde{P}_T$, for $t \in [0, T)$, where

$$\Gamma_t(z) \triangleq (2\pi t)^{-d/2} \exp \left\{ -\frac{\|z\|^2}{2t} \right\}, \quad z \in \mathbb{R}^d, \quad t > 0,$$

is the fundamental Gaussian kernel. Define $H(t,p)$ as follows:

$$H(t,p) \triangleq \begin{cases} e^{-r(T-t)} \int_{\mathbb{R}^d} L(K(T-t,p,z))\Gamma_{T-t}(z)dz, & 0 \le t < T, \quad p \in \mathbb{R}_+^d, \\ L(p), & t = T, \quad p \in \mathbb{R}_+^d, \end{cases} \tag{3.4}$$

the valuation process for the European claim can be written as

$$X_t = H(t, P(t)) \tag{3.5}$$

We can even integrate equation (3.4), if we set $d = 1$ and $L(p) = (p-c)^+$ where $c > 0$ is called the strike price. This ECC is called a European call option. This leads to the renown Black-Scholes Formula for pricing an option.

3.3 Valuation of the portfolio process

The hedging strategy introduced in Chapter 2 for an ECC show the existence of a portfolio process, but it does not provide a way to evaluate the portfolio process in terms of known quantities at a time $t \ge 0$.

Here we show that we can compute the portfolio process for the wealth process of (3.5) that hedges the ECC introduced in the subsection. 3.1

Applying Ito's rule to the wealth process of (3.5) we get

$$dH = \frac{\partial H}{\partial t} \cdot dt + \sum_{i=1}^{d} \frac{\partial H}{\partial P_i} \cdot dP_i + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 H}{\partial P_i \partial P_j} \cdot d\langle P_i, P_j \rangle \tag{3.6}$$

We can calculate $d\langle P_i, P_j \rangle$ from the 'multiplication table' [16, page number 154] and equation (3.1). We get,

$$\begin{aligned} d\langle P_i, P_j \rangle &= \left(\sum_{k=1}^{d} \sigma_{ik} d\tilde{W}_k \cdot P_i \right) \cdot \left(\sum_{n=1}^{d} \sigma_{jn} d\tilde{W}_n \cdot P_j \right) \\ &= P_i P_j D_{ij} dt \end{aligned}$$

Substituting in the above equation, we get

$$dH = \frac{\partial H}{\partial t} \cdot dt + \sum_{i=1}^{d} \frac{\partial H}{\partial P_i} \cdot P_i \left[(r - \mu_i)\, dt + \sum_{j=1}^{d} \sigma_{ij} d\tilde{W}_j \right] + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 H}{\partial P_i \partial P_j} \cdot P_i P_j D_{ij} dt \tag{3.7}$$

$$= \left(\frac{\partial H}{\partial t} + \sum_{i=1}^{d} \frac{\partial H}{\partial P_i} \cdot P_i (r - \mu_i) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 H}{\partial P_i \partial P_j} \cdot P_i P_j D_{ij} \right) dt + \sum_{i=1}^{d} \frac{\partial H}{\partial P_i} \cdot P_i \sum_{j=1}^{d} \sigma_{ij} d\tilde{W}_j(t). \tag{3.8}$$

Here we see that $H(t, p)$ as defined in (3.4) is a $C^{1,2}$ function. We also notice L is a continuous function, $r \geq 0$ is a constant, and $g = 0$. If L satisfies conditions of growth, such that $H(t, p)$ satisfies the following growth condition, for some constants $M, a \in \mathbb{R}$,

$$\underset{0 \leq t \leq T}{|H(t, p)|} \leq M e^{a\|p\|^2} \qquad \forall p \in \mathbb{R}^d \tag{3.9}$$

then the stochastic representation of $H(t, P(t)) = X_t$ as given in (3.3) gives that $H(t, P(t))$ is a solution to the following PDE

$$\frac{\partial H}{\partial t} + \sum_{i=1}^{d} \frac{\partial H}{\partial P_i} \cdot P_i (r - \mu_i) + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 H}{\partial P_i \partial P_j} \cdot P i P_j D_{ij} - rH = 0. \tag{3.10}$$

This result is a direct application of the Feynman-Kac formula [16, page number 267].

From (3.8) and (3.10) we get that the valuation process $X_t = H(t, P(t))$ satisfies the following

$$dX_t = rX_t dt + \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial H}{\partial P_i} (t) \cdot P_i(t) \sigma_{ij} d\tilde{W}_j(t). \tag{3.11}$$

Comparing this with (2.14) we conclude that the portfolio process can be given by

$$\pi(t) = \frac{\partial H}{\partial P_i} (t) \cdot P_i(t), \quad 1 \leq i \leq d, \quad 0 \leq t < \infty. \tag{3.12}$$

3.4 Pricing of a perpetual American put option.

An American perpetual put option is an ACC with the time of maturity of the Definition 2.5, $T = \infty$. To talk about pricing this claim the market model needs to be changed. This is to take into account that the equivalent martingale measure $\tilde{P}$ defined in (2.12) is for a finite T. This needs to be extended to be defined for all $t \geq 0$.

Karatzas [15] shows that this extension of the equivalent martingale measure $\tilde{P}$ is indeed possible under the assumption that all the coefficients of the market model and the process f and g of Definition 2.5 are progressively measurable Brownian functionals. The fair price of an ACC in that setting is given as:

$$V_0 = \sup_{\tau \in S_{0,\infty}} \tilde{E}\left[f_\tau \exp\left(-\int_0^\tau r(u)du \right) + \int_0^\tau g_s \exp\left(-\int_0^s r(u)du \right) ds \right]. \tag{3.13}$$

with the optimal stopping time $\rho_0 = \inf\{t \geq 0 : Y_t = Q_t\}$. Here Y and Q are defined the same way as in Section 2.3.

Here the stopping times $\tau \in S_{0.\infty}$ may take value $\tau = \infty$. For any progressively measurable non-negative process X,
$X_\infty(\omega) \triangleq \limsup_{t\to\infty} X_t(\omega)$ and $\tilde{E}(X_\tau) \triangleq \tilde{E}(X_\tau 1_{\tau<\infty} + X_\infty 1_{\tau=\infty})$

Consider the market model of Section 3.1. Set $d = 1$, and consider an ACC with $f_t = (c - P(t))^+, g_t = 0, \mu = 0$ and $T = \infty$. Here $c \in [0, \infty)$. We will call this a perpetual put option. For this option we will derive the value of the fair price, and optimal stopping boundary as presented in [7].

Note that the stock price $P(t)$ at time $t \geq 0$ is given by $P(t) = P(0)\exp\{(r - \frac{\sigma^2}{2})t + \sigma\tilde{W}(t)\}$ Let's denote $(r - \frac{\sigma^2}{2})$ by γ for notational convenience.

For f_t as defined above we have $Q_t = e^{-rt}(c - P(t))^+$. Thus,

$$\begin{aligned}\tilde{E}\left(\sup_{0\leq t<\infty} Q_t\right) &= \tilde{E}\left(\sup_{0\leq t<\infty} e^{-rt}(c - P(t))^+\right)\\ &\leq \tilde{E}\left(\sup_{0\leq t<\infty} 1.c\right) < \infty\end{aligned}$$

This condition is necessary for (3.13) to hold (refer [15] for details).

Define $V_0(x)$, the price of the American perpetual put at time $t = 0$, with the stock price $P(0) = x$ as follows:

$$V_0(x) \triangleq \sup_{\tau \in S_{0,\infty}} \tilde{E}\left(e^{-r\tau}\left(c - x\exp\left\{\gamma\tau + \sigma\tilde{W}(\tau)\right\}\right)^+\right)$$

Since $e^{-rt} \to 0$ as $t \to \infty$ we have for any stopping time τ

$$\begin{aligned}
&\tilde{E}\left(e^{-r\tau}\left(c - x\exp\left\{\gamma\tau + \sigma\tilde{W}(\tau)\right\}\right)^+\right) \\
&= \tilde{E}\left(e^{-r\tau}\left(c - x\exp\left\{\gamma\tau + \sigma\tilde{W}(\tau)\right\}\right)^+ 1_{\tau<\infty} + 0.\limsup_{t\to\infty}\left(c - x\exp\left\{\gamma\tau + \sigma\tilde{W}(\tau)\right\}\right)^+ 1_{\tau=\infty}\right) \\
&= \tilde{E}\left(e^{-r\tau}\left(c - x\exp\left\{\gamma\tau + \sigma\tilde{W}(\tau)\right\}\right)^+ 1_{\tau<\infty}\right).
\end{aligned}$$

Thus, $V_0(x) = \sup_{\tau \in S_{0,\infty}} \tilde{E}\left(e^{-r\tau}\left(c - x\exp\left\{\gamma\tau + \sigma\tilde{W}(\tau)\right\}\right)^+ 1_{\tau<\infty}\right)$.

It can be shown that $V_0(x)$ is convex in x, decreasing on $[0,\infty)$, and $V_0(x) \geq (c-x)^+$ $\forall x \in [0,\infty)$, [7, Lemma 8.2.8]. Define $P^* = \sup\{x \geq 0 : V_0(x) = (c-x)^+\}$. We have,

$$\begin{aligned}
V_0(x) &= (c - x) \quad \text{if} \quad x \leq P^* \\
V_0(x) &> (c - x)^+ \quad \text{if} \quad x > P^*.
\end{aligned}$$

Note that given $P(0) = x$, the optimal stopping time $\rho_0 = \inf\{t \geq 0 : V_0(P(t)) = (c - P(t))^+\}$ can thus be written as $\rho_0 = \inf\{t \geq 0 : P(t) \leq P^*\}$. Since $P(t)$ has continuous paths we can write this as

$$\begin{aligned}
\rho_0 &= \inf\{t \geq 0 : P(t) = P^*\} \quad if \quad x > P^* \\
&= 0 \qquad\qquad\qquad\qquad\quad if \quad x \leq P^*
\end{aligned}$$

To calculate the value of P^* fix $P(0) = x$. Define the function $h(y)$ as follows:

$$h(y) = \tilde{E}\left(e^{-r\alpha_{x,y}}\left(c - x\exp\left\{\gamma\alpha_{x,y} + \sigma\tilde{W}(\alpha_{x,y})\right\}\right)^+ 1_{\alpha_{x,y}<\infty}\right) \tag{3.14}$$

Where $\alpha_{x,y}$ is a stopping time given by $\alpha_{x,y} = \inf\left\{t \geq 0 : x\exp\left\{\gamma t + \sigma\tilde{W}(t)\right\} \leq y\right\}$, or equivalently $\alpha_{x,y} = \inf\left\{t \geq 0 : \left\{\frac{\gamma}{\sigma}t + \tilde{W}(t)\right\} \leq \frac{1}{\sigma}\log\left(\frac{y}{x}\right)\right\}$. If we have $y \geq x$, then $\alpha_{x,y} = 0$.

For $y < x$, from the continuity of $\left(\gamma t + \sigma \tilde{W}(t)\right)$ we get, $\alpha_{x,y} = \inf\left\{t \geq 0 : \left\{\frac{\gamma}{\sigma}t + \tilde{W}(t)\right\} = \frac{1}{\sigma}log(\frac{y}{x})\right\}$. We thus have,

$$
\begin{aligned}
h(y) &= \tilde{E}\left(e^{-r\alpha_{x,y}} (c-y)^+ 1_{\alpha_{x,y}<\infty}\right) \\
h(y) &= (c-y)^+ \tilde{E}\left(e^{-r\alpha_{x,y}}\right). \qquad (3.15)
\end{aligned}
$$

The expectation above can be computed, [7, Corollary 7.2.6]. Consider $k, l, m \in [0, \infty)$. For a stopping time $T(l) = \inf\left\{t \geq 0 : kt + \tilde{W}(t) = l\right\}$, the expectation of the exponential random variable $\tilde{E}(e^{-mT(l)})$ is given by $\tilde{E}(e^{-mT(l)}) = \exp\{kl - |l|\sqrt{k^2 + 2m}\}$. Thus, we have

$$
h(y) = \begin{cases} (c-y)^+ & if \quad y > x \\ (c-y)^+ \exp\left\{\frac{\gamma}{\sigma^2}log(\frac{y}{x}) - \frac{|log(\frac{y}{x})|}{\sigma}\sqrt{\frac{\gamma}{\sigma}^2 + 2r}\right\} & if \quad y \in [0,x] \cap [0,c] \\ 0 & if \quad y \in [0,x] \cap [c,\infty) \end{cases} \qquad (3.16)
$$

We see that $h(y)$ attains its maximum when $y \in [0,x] \cap [0,c]$. Substituting $(r - \frac{\sigma^2}{2})$ for γ and simplifying we get, $h(y) = (c-y)\exp\left\{\frac{2r}{\sigma^2} \cdot log(\frac{y}{x})\right\}$. The derivative of $h(y)$ is given by,

$$
h'(y) = \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}} \left(\left(\frac{c-y}{y}\right)\frac{2r}{\sigma^2} - 1\right)
$$

Setting the derivative to zero we see that the maximum is attained at $y = \frac{2cr}{2r+\sigma^2}$. But, P^* also maximises $h(y)$ from the optimality property, and the way $h(y)$ is defined. Thus, we can write $P^* = \frac{2cr}{2r+\sigma^2}$.

The valuation of the American perpetual put as a function of the initial stock price $P_0(x) = x$ is thus given by

$$
V_0(x) = \begin{cases} (c-P^*)(\frac{P^*}{x})^{\frac{2r}{\sigma^2}} & if \quad x \leq P^* \\ (c-x)^+ & if \quad x > P^*. \end{cases} \qquad (3.17)
$$

Mckean [17] showed that the valuation process of the American perpetual put is equivalent to the solution of a particular free boundary problem (see [2] for more details). We look at the pricing of an American perpetual put, assuming it satisfies this free boundary equation, and show that the pricing function is the same as the equation(3.17).

Let $V(p)$ denote the value of the perpetual put as a function of the stock price p, by the conditions presented by McKean it satisfies the following :

$$\frac{1}{2}\sigma^2 p^2 \frac{\mathrm{d}^2 V}{\mathrm{d}p^2} + rp\frac{\mathrm{d}V}{\mathrm{d}p} - rV = 0. \tag{3.18}$$

$$V(\infty) = 0. \tag{3.19}$$

And boundary conditions for the free boundary P^*:

$$V(P^*) = (P^* - p)^+ \tag{3.20}$$

$$\frac{\partial V}{\partial p} \,|_{p=P^*} = -1. \tag{3.21}$$

The differential equation mentioned above is a standard Cauchy-Euler equation whose solution can be obtained by substituting p^r for $V(p)$. We get a general solution of the form

$$V(p) = A_2 p + A_1 p^{-\frac{2r}{\sigma^2}}. \tag{3.22}$$

Due to (3.19) we have $A_2 = 0$, i.e., $V(p) = A_1 p^{-\frac{2r}{\sigma^2}}$. Free boundary conditions give

$$P^* = \frac{2rc}{2r + \sigma^2} \quad \text{and} \quad A_1 = (c - P^*)(P^*)^{\frac{2r}{\sigma^2}}.$$

Which is the same result as the equation (3.17).

Chapter 4

American Option Pricing in a Markov Modulated GBM Market Model

In this chapter, we introduce the market model similar to the one in [5]. In this market model, the parameters of the market model: the interest rate of the bond or equivalently bank interest rate, the appreciation rate of the stock, and the dispersion coefficient or the volatility of the stock due to Brownian motion, are modeled by a finite state Markov chain.

The finite state Markov chain introduced in [5] is assumed to have a stochastic integral representation. This fact is used to arrive at a differential equation analogous to the Black-Scholes equation for option price function. We examine the assumption and look at the precise derivation of the Black-Scholes equation analogue by following the setup introduced in [1]. Lastly we examine the equations for approximate solutions for pricing an American option in regime switching presented in [5].

Original contribution During the literature survey of this paper we found an equation which is imprecise as presented in [5]. We try to present the precise equation and describe the conditions of when the equation holds.

4.1 Markov modulated GBM model

4.1.1 The market model

Consider a complete probability space $(\Omega, \mathscr{F}, P)$. Let $X = \{X_t\}_{t\geq 0}$ be an irreducible finite state Markov chain. The state space, without loss of generality, can be assumed to be $\mathcal{X} = \{e_1, e_2 \ldots e_n\}$, where $e_i \in \mathbb{R}^n$ is the ith column of an n dimensional identity matrix. The transition rates for the chain are given as follows:

$$P(X_{t+\delta t} = e_j \mid X_t = e_i) = \lambda_{ij}\delta t + o(\delta t). \tag{4.1}$$

Here, for $i \neq j$, $\lambda_{ij} \geq 0$, and $\lambda_{ii} = -\sum_{j\notin\{i\}} \lambda_{ij}$. The rate matrix is denoted by $A = (\lambda_{ij})$.

We consider two underlying assets of the market, a bond or equivalently a bank account, and a stock. We suppose the instantaneous interest rate of the bank account $r = \{r_t\}_{t\geq 0}$, the appreciation rate of the stock $\mu = \{\mu_t\}_{t\geq 0}$, and the volatility of the stock $\sigma = \{\sigma_t\}_{t\geq 0}$, depend on the state of the X, which represents the states of the economy. We assume there exist vectors $\mu_0 = (\mu_1, \ldots, \mu_n)'$, $\sigma_0 = (\sigma_1, \ldots, \sigma_n)'$ and $r_0 = (r_1, \ldots, r_N)' \in \mathbb{R}^n$ such that

$$\begin{aligned} r_t &= r(X_t) = \langle r, X_t \rangle, \\ \mu_t &= \mu(X_t) = \langle \mu, X_t \rangle, \\ \sigma_t &= \sigma(X_t) = \langle \sigma, X_t \rangle \end{aligned}$$

give the value of the parameters at time t.

Let $B = \{B_t\}_{t\geq 0}$ and $S = \{S_t\}_{t\geq 0}$ denote the money in the bank account and the price of the stock respectively. Then we have the following equations which govern the dynamics of the prices

$$B_t = \exp\left(\int_0^t r(X_u)du\right) \tag{4.2}$$

$$dS_t = S_t\left(\mu(X_t)dt + \sigma(X_t)dW_t\right), \quad S_0 > 0. \tag{4.3}$$

Here $W = \{W_t\}_{t\geq 0}$ is a standard Brownian motion process independent of $X = \{X_t\}_{t\geq 0}$. Let $\mathscr{F}_t = \sigma\left(S_u, X_u, u \leq t\right)$. We can assume $\{\mathscr{F}_t\}$ is right continuous without loss of generality. We will use this filtration henceforth.

4.2 Stochastic integral representation of the Markov chain & BSM equation

Consider a Polish space $\mathcal{S}$. Let $\mathcal{B}(\mathcal{S})$ denote its Borel σ-field on $\mathcal{S}$, $\mathcal{M}(\mathcal{S})$ be the set of all non-negative integer valued σ-finite measures on $\mathcal{B}(\mathcal{S})$. Let $\mathcal{M}_\sigma(\mathcal{S})$ be the smallest σ-field on $\mathcal{M}(\mathcal{S})$ such that $\forall\ B \in \mathcal{B}(\mathcal{S})$, the maps $f_B : \mathcal{M}(\mathcal{S}) \to \mathbb{N}\bigcup\{\infty\}$ defined by $f_B(\mu) := \mu(B)$ are measurable. We equip $\mathcal{M}(\mathcal{S})$ with the σ algebra $\mathcal{M}_\sigma(\mathcal{S})$. A measurable map $\psi : (\Omega, \mathscr{F}) \to (\mathcal{M}(\mathcal{S}), \mathcal{M}_\sigma(\mathcal{S}))$ is called a random point measure on $\mathcal{S}$ with intensity $E\psi$, if it exists. Such ψ is a Poisson random measure with intensity measure $\bar{\psi}$ if (i) $\psi(B_1)$ and $\psi(B_2)$ are independent provide B_1 and B_2 are disjoint; (ii) $\psi(B)$ follows Poisson distribution with rate $\bar{\psi}(B)$.

For $i \neq j \in \{1, 2 \ldots n\}$, let Λ_{ij} denote consecutive (with respect to the lexicographic ordering on $\{1, 2 \ldots n\} \times \{1, 2 \ldots n\}$) left closed right open intervals of the real line with the length of Λ_{ij} being equal to λ_{ij}. We embed $\mathcal{X}$ in $\mathbb{R}$ by identifying e_i with $i \in \mathbb{R}^n$. We then define a function $u : \mathcal{X} \times \mathbb{R} \to \mathbb{R}$ by

$$u(i, z) := \begin{cases} j - i & \text{if } z \in \Lambda_{ij} \\ 0 & \text{otherwise.} \end{cases}$$

Then X_t can be written as

$$X_t = X_0 + \int_0^t \int_{\mathbb{R}} u\left(X_{u-}, z\right) \psi(du, dz). \tag{4.4}$$

Here $\psi(dt, dz)$ is a Poisson random measure with values in $\mathcal{M}\left(\mathbb{R}_+ \times \mathbb{R}\right)$ and intensity measure equal to the Lebesgue measure $dtdz$. The integration is done on the interval $(0, t]$. $\psi(dt, dz), X_0, W$ and S_0, defined on $(\Omega, \mathscr{F}, P)$ are independent.

It turns out the Markov modulated market model presented in (4.2)-(4.4) is not complete. Because of the incompleteness there will be claims that are not attainable, and thus perfect hedging of a claim might not be possible. One resorts to a risk-minimising option pricing in such a case. The details of this are presented in [1].

Here we do not go into the details of the incompleteness, but rather assume the measure P is itself a risk neutral measure.

Under this assumption the price of a European call option with strike price K and expiration time T at time t is given by the following. (Check Buffington [5] for details.)

$$C(t,T,s,x) = E\left[\exp\left(-\int_t^T r(X_v)dv\right)(S_T - K)^+ \mid S_t = s, X_t = x\right]. \tag{4.5}$$

Define $V(t,s,x)$ as follows:

$$V(t,s,X_t) = \exp\left(-\int_0^t r(X_v)dv\right) C(t,T,s,X_t) \tag{4.6}$$

$$\implies V(t,S_t,X_t) = \exp\left(-\int_0^t r(X_v)dv\right) C(t,T,S_t,X_t) \tag{4.7}$$

$$= E\left[\exp\left(-\int_0^T r(X_v)dv\right)(S_T - K)^+ \mid S_t, X_t\right] \tag{4.8}$$

$$= E\left[\exp\left(-\int_0^T r(X_v)dv\right)(S_T - K)^+ \mid \mathscr{F}_t\right] \tag{4.9}$$

using the Markovity of (S,X) w.r.t. $\{\mathscr{F}_t\}$. Consequently $\{V(t,S_t,X_t)\}$ is a $\mathscr{F}_t$-martingale. Let $\boldsymbol{V}(t,s)$ be defined as follows:

$$\boldsymbol{V}(t,s) = \left(V\left(t,s,e_1\right),\ldots,V\left(t,s,e_2\right)\right),$$

so that $V\left(t,S_t,X_t\right) = \langle \boldsymbol{V}\left(t,S_t\right), X_t\rangle$. Applying Ito's formula on $V(t,S_t,X_t)$ we get from (4.4):

$$\begin{aligned} &dV\left(t,S_t,X_t\right)\\ =&\frac{\partial}{\partial t}V\left(t,S_t,X_{t-}\right)dt + \int_{\mathbb{R}}\left\{V\left(t,S_t,X_{t-}+u\left(X_{t-},z\right)\right) - V\left(t,S_t,X_{t-}\right)\right\}\psi(dt,dz)\\ &+\frac{\partial}{\partial s}V\left(t,S_t,X_{t-}\right)dS_t + \frac{1}{2}\frac{\partial^2}{\partial s^2}V\left(t,S_t,X_{t-}\right)d\langle S,S\rangle_t \end{aligned} \tag{4.10}$$

$$\begin{aligned} =&\frac{\partial}{\partial t}V\left(t,S_t,X_{t-}\right)dt + \mu\left(X_{t-}\right)S_t\frac{\partial}{\partial s}V\left(t,S_t,X_{t-}\right)dt\\ &+\frac{1}{2}\sigma^2\left(X_{t-}\right)S_t^2\frac{\partial^2}{\partial s^2}V\left(t,S_t,X_{t-}\right)dt + \sum_{j\in\mathcal{X}}V\left(t,S_t,j\right)\lambda_{X_{t-},j}dt + d\tilde{M}_t \end{aligned} \tag{4.11}$$

where $\tilde{M}_t$ is a martingale given by

$$\begin{aligned} \tilde{M}_t = \tilde{M}_0 \quad & + \int_0^t S_v \sigma\left(X_{v-}\right) \tfrac{\partial}{\partial s} V\left(v, S_v, X_{v-}\right) dW_v \\ & + \int_0^t \int_{\mathbb{R}} \left\{ V\left(v, S_v, X_{v-} + u\left(X_{v-}, z\right)\right) - V\left(v, S_v, X_{v-}\right) \right\} \hat{\psi}(dv, dz) \end{aligned} \tag{4.12}$$

where $\hat{\psi}(dt, dz) := \psi(dt, dz) - dtdz$ is the compensated Poisson random measure.

We can write the integral version of (4.11) as

$$\begin{aligned} & V(t, S_t, X_t) - V(0, S_0, X_0) - \tilde{M}_t + \tilde{M}_0 \\ & \qquad = \int_0^t \Big(\frac{\partial}{\partial t} V\left(t, S_t, X_{t-}\right) + \mu\left(X_{t-}\right) S_t \frac{\partial}{\partial s} V\left(t, S_t, X_{t-}\right) \\ & \qquad\qquad + \frac{1}{2} \sigma^2\left(X_{t-}\right) S_t^2 \frac{\partial^2}{\partial s^2} V\left(t, S_t, X_{t-}\right) + \langle \boldsymbol{V}(t, S_t), A^* X_{t-} \rangle \Big) dt. \end{aligned} \tag{4.13}$$

Notice that the LHS of above equation is a martingale, and the RHS is a bounded variation process. A martingale with bounded variation is a constant martingale. For the integral of RHS to be constant the integrand needs to be zero, as the integrand is non-negative the way it is defined. This imples,

$$\begin{aligned} \frac{\partial}{\partial t} V\left(t, S_t, X_{t-}\right) + \mu\left(X_{t-}\right) S_t \frac{\partial}{\partial s} V\left(t, S_t, X_{t-}\right) + \frac{1}{2} \sigma^2\left(X_{t-}\right) S_t^2 \frac{\partial^2}{\partial s^2} V\left(t, S_t, X_{t-}\right) \\ + \langle \boldsymbol{V}(t, S_t), A^* X_{t-} \rangle = 0. \end{aligned} \tag{4.14}$$

We see from (4.6) that $C(t, T, S_t, X_t)$ also follows equation (4.14). The boundary condition for C is $C(T, T, s, X) = (s - K)^+$. Write

$$C_i(t, s) \triangleq C(t, T, s, e_i) \tag{4.15}$$

$$\boldsymbol{C}(t, s) = (C_1(t, s), C_2(t, s), \ldots, C_n(t, s)) \tag{4.16}$$

then we can see $\boldsymbol{C}(t, s)$ satisfies the coupled Black-Scholes equations:

$$r_i C_i(t, s) + \frac{\partial C_i(t, s)}{\partial t} + \mu_i s \frac{\partial C_i(t, s)}{\partial s} + \frac{1}{2} \sigma_i^2 s^2 \frac{\partial^2 C_i(t, s)}{\partial s^2} + \langle \boldsymbol{C}(t, s), A^* e_i \rangle = 0, \quad 1 \leq i \leq n. \tag{4.17}$$

4.3 System of equations for the pricing of American options

Now we consider an American put option. We assume from here onward that number of states of regime switching $n = 2$. Let the transition rate matrix be given by

$$A = \begin{pmatrix} a_{11} & -a_{11} \\ -a_{22} & a_{22} \end{pmatrix}. \tag{4.18}$$

The price of an American option with strike price K and expiration time T at time t is given by

$$J(t, T, s, x) = \sup_{\tau \in S_{t,T}} E\left[\exp\left(-\int_t^\tau r_u du\right)(K - S_\tau) \mid S_t = s, X_t = x\right] \tag{4.19}$$

where $(K - S_\tau)$ is the payoff from exercise strategy with stopping time τ. Let $\boldsymbol{J}(t, s)$ be defined as follows:

$$\boldsymbol{J}(t, s) = (J(t, T, s, e_1), J(t, T, s, e_2)) \triangleq (J_1(t, s), J_2(t, s)). \tag{4.20}$$

If there is no regime switching this problem boils down to the McKean problem [17]. They present a 'continuation region', which for each $t \in [0, T]$ is an interval of the form $[S_t^*, \infty)$ [7]. In the continuation region the option price satisfies the BSM equations, and outside of this region the price of the option is equal to the payoff $(K - S_t)$. The optimal exercise strategy is defined as the first time the option price is equal to the payoff. Similar 'continuation region' and 'stopping boundary' can be defined for the regime switching case also(refer to [5] for details).

Let for $i = 1, 2$

$$\mathcal{C}^i = \left\{(s, t) \in R^+ \times [0, T] : J(t, T, s, e_i) > (K - s)^+\right\}$$
$$\mathcal{S}^i = \left\{(s, t) \in R^+ \times [0, T] : J(t, T, s, e_i) = (K - s)^+\right\}$$

denote the continuation region and stopping region respectively with $X_t = e_i$. For each $t \in [0, T]$ we get an interval $[s^*(e_i, t), \infty)$ of the continuation region. Depending upon the values of μ_i, σ_i and r_i either of the $s^*(e_i, t)$ can be smaller. We assume, without loss of generality $s^*(e_2,t) \geq s^*(e_1,t) \quad \forall t \in [0,T]$.

If the stock price $S_t > s^*(e_2,t)$, then for both states e_1, e_2, (S_t,t) lies in the continuation region. Thus, $J = (J_1(t,S_t), J_2(t,S_t))$ satisfies the pair of Black-Scholes equations:

$$-r_1 J_1 + \frac{\partial J_1}{\partial t} + \mu_1 S_t \frac{\partial J_1}{\partial s} + \frac{1}{2}\sigma_1^2 S_t^2 \frac{\partial^2 J_1}{\partial s^2} + \langle J, A^* e_1 \rangle = 0 \tag{4.21}$$

$$-r_2 J_2 + \frac{\partial J_2}{\partial t} + \mu_2 S_t \frac{\partial J_2}{\partial s} + \frac{1}{2}\sigma_2^2 S_t^2 \frac{\partial^2 J_2}{\partial s^2} + \langle J, A^* e_2 \rangle = 0. \tag{4.22}$$

For $S_t \leq S^*(e_1,t)$, for both states (S_t,t) lies in the stopping region. There

$$J_1(t,S_t) = J_2(t,S_t) = J(t,T,S_t,e_i) = K - S_t. \tag{4.23}$$

For S_t in the transition region $s^*(e_1,t) \leq S_t \leq s^*(e_2,t)$ we have,

$$J_2(t,S_t) = J(t,T,S_t,e_2) = (K - S_t) \tag{4.24}$$

and $J_1(t,S_t) = J(t,T,S_t,e_1)$ satisfying,

$$-r_1 J_1 + \frac{\partial J_1}{\partial t} + \mu_1 S_t \frac{\partial J_1}{\partial s} + \frac{1}{2}\sigma_1^2 {S_t}^2 \frac{\partial^2 J_1}{\partial s^2} + a_{11} J_1 - a_{11}(K - S_t) = 0. \tag{4.25}$$

4.4 Approximate solution for the price of an American put option in the common continuation region.

In this section we will present part of an approximate solution for a finite horizon option, as presented in [5]. We show that the equation in [5] that is equivalent to the equation (4.30) is imprecise, and present scenarios where it holds. We will present the correct equation, and show that the further derivation of the approximate solution as presented in [5] remains the same. Thus, we will not repeat the derivation of the solution, but just state the results.

In the common continuation region $S_t > s^*(e_2,t)$, the price function of the American option satisfies equations (4.21) and (4.22). Additionally, the stopping region gives boundary conditions as follows:

$$J\left(t,T,s^{*}\left(e_{2},t\right),e_{2}\right)=\left(K-s^{*}\left(e_{2},t\right)\right)\qquad and \tag{4.26}$$

$$\frac{\partial J_{2}}{\partial s}\left(t,T,s^{*}\left(e_{2},t\right),e_{2}\right)=-1. \tag{4.27}$$

The second condition comes from the 'smooth pasting' condition. (For deatils check [7] and [17]).

Since the early exercise feature of an American option confers extra rights to the owner, an early exercise premium is paid to the option -writer. We define early exercise premium as ϕ as follows:

$$\phi(t,T,s,x)=J(t,T,s,x)-C(t,T,s,x). \tag{4.28}$$

This quantity is always positive because we can see from equations (4.19) and (4.5) that the price of an American option is greater than the price of a European option with the same parameters.
We define $\boldsymbol{\phi}(t,s)$ as follows:

$$\boldsymbol{\phi}(t,s)=(\phi(t,T,s,e_{1}),\phi(t,T,s,e_{2}))\triangleq(\phi_{1}(t,s),\phi_{2}(t,s)). \tag{4.29}$$

We make the approximation of assuming that $\phi(t,T,s,e_i)$ can be presented in a variables separated format as $\phi_i(t,s)=h_i(s)y_i(t)$. Since, both $J(t,T,s,x)$ and $C(t,T,s,x)$ satisfy the black Scholes equation in the continuation region, so does $\phi(t,T,s,x)$. Thus, we have

$$-r_{i}h_{i}y_{i}+h_{i}\frac{\partial y_{i}}{\partial t}+\mu_{i}s\frac{\partial h_{i}}{\partial s}y_{i}+\frac{1}{2}\sigma_{i}^{2}s^{2}\frac{\partial^{2}h_{i}}{\partial s^{2}}y_{i}+\langle\boldsymbol{\phi},A^{*}e_{i}\rangle=0. \tag{4.30}$$

If we assume the following, we can eliminate the partial derivative with respect to time t.

$$y_{i}(t)=E\left[1-\exp\left(-\int_{t}^{T}r_{u}du\right)\mid X_{t}=e_{i}\right]\implies\frac{\partial y_{i}}{\partial t}=r_{i}\left(y_{i}(t)-1\right). \tag{4.31}$$

Thus equation 4.30 can be written as,

$$-r_{i}h_{i}+\mu_{i}s\frac{\partial h_{i}}{\partial s}y_{i}+\frac{1}{2}\sigma_{i}^{2}s^{2}\frac{\partial^{2}h_{i}}{\partial s^{2}}y_{i}+\sum_{j=1}^{2}\phi_{j}A_{ji}^{*}=0. \tag{4.32}$$

That is

$$\frac{1}{2}\sigma_1^2 s^2 \frac{\partial^2 h_1}{\partial s^2} + \mu_1 s \frac{\partial h_1}{\partial s} = \frac{(r_1 h_1 - a_{11}(h_1 y_1 - h_2 y_2))}{y_1(t)}. \tag{4.33}$$

$$\frac{1}{2}\sigma_2^2 s^2 \frac{\partial^2 h_2}{\partial s^2} + \mu_2 s \frac{\partial h_2}{\partial s} = \frac{(r_2 h_2 - a_{22}(h_2 y_2 - h_1 y_1))}{y_2(t)}. \tag{4.34}$$

In Appendix B of [5] a solution of the following form is assumed

$$h_1(s) = \rho_1 s^{\eta_1} + \rho_2 s^{\eta_2}, \tag{4.35}$$

$$h_2(s) = \theta_1 s^{\eta_1} + \theta_2 s^{\eta_2}. \tag{4.36}$$

From comparing the coefficients of the s^{η_1} and s^{η_2} four equations are derived. The equivalent equations as we have derived from (4.35), (4.36) and (4.33) are:

$$\frac{1}{2}\sigma_1^2 \rho_1 \eta_1 (\eta_1 - 1) + \mu_1 \rho_1 \eta_1 + a_{11}\rho_1 - a_{11}\theta_1 \frac{y_2}{y_1} - \frac{r_1}{y_1(t)}\rho_1 = 0, \tag{4.37}$$

$$\frac{1}{2}\sigma_1^2 \rho_2 \eta_2 (\eta_2 - 1) + \mu_1 \rho_2 \eta_2 + a_{11}\rho_2 - a_{11}\theta_2 \frac{y_2}{y_1} - \frac{r_1}{y_1(t)}\rho_2 = 0, \tag{4.38}$$

$$\frac{1}{2}\sigma_2^2 \theta_1 \eta_1 (\eta_1 - 1) + \mu_2 \theta_1 \eta_1 - a_{22}\rho_1 \frac{y_1}{y_2} + a_{22}\theta_1 - \frac{r_2}{y_2(t)}\theta_1 = 0, \tag{4.39}$$

$$\frac{1}{2}\sigma_2^2 \theta_2 \eta_2 (\eta_2 - 1) + \mu_2 \theta_2 \eta_2 - a_{22}\rho_2 \frac{y_1}{y_2} + a_{22}\theta_2 - \frac{r_2}{y_2(t)}\theta_2 = 0. \tag{4.40}$$

From equation (4.37) and (4.39) we have,

$$a_{11}\frac{y_2\theta_1}{y_1\rho_1} = \frac{1}{2}\sigma_1^2 \eta_1 (\eta_1 - 1) + \mu_1 \eta_1 + a_{11} - \frac{r_1}{y_1(t)} \tag{4.41}$$

$$a_{22}\frac{y_1\rho_1}{y_2\theta_1} = \frac{1}{2}\sigma_2^2 \eta_1 (\eta_1 - 1) + \mu_2 \eta_1 + a_{22} - \frac{r_2}{y_2(t)}. \tag{4.42}$$

We can thus see η_1 is a solution to the fourth order equation,

$$a_{11}a_{22} = \left(\frac{1}{2}\sigma_1^2 \eta(\eta - 1) + \mu_1 \eta + a_{11} - \frac{r_1}{y_1(t)}\right) \times \left(\frac{1}{2}\sigma_2^2 \eta(\eta - 1) + \mu_2 \eta + a_{22} - \frac{r_2}{y_2(t)}\right). \tag{4.43}$$

We note that, in [5] they arrive at the same equation as (4.43), and thus the further equations follow the result derived in [5]. We will not repeat the same equations. We just state the results from Appendix B of [5].

Proposition 4.1. *The price of the American option in the setting of section 4.3, in the common continuation region is given by*

$$J(t,T,s,e_1) = C(t,T,s,e_1) + \left(\rho_1 s^{\eta_1(t)} + \rho_2 s^{\eta_2(t)}\right) y_1(t) \tag{4.44}$$
$$J(t,T,s,e_2) = C(t,T,s,e_2) + \left(\theta_1 s^{\eta_1(t)} + \theta_2 s^{\eta_2(t)}\right) y_2(t). \tag{4.45}$$

Here $\eta_1(t), \eta_2(t)$ are the two time varying negative roots of the following equation in η. (see [11])

$$F(\eta) = \left(\frac{1}{2}\sigma_1^2\eta(\eta-1) + \mu_1\eta + a_{11} - \frac{r_1}{y_1(t)}\right) \times \left(\frac{1}{2}\sigma_2^2\eta(\eta-1) + \mu_2\eta + a_{22} - \frac{r_2}{y_2(t)}\right) - a_{11}a_{22} = 0.$$

θ_1 and θ_2 are as given below:

$$\theta_1 = \left[s_2^*\left(1+\frac{\partial C_2}{\partial s}\right) + \eta_2\left(K - s_2^* - C_2\right)\right]\left[s_2^{*\eta_1} y_2\left(\eta_2 - \eta_1\right)\right]^{-1}$$
$$\theta_2 = \left[s_2^*\left(1+\frac{\partial C_2}{\partial s}\right) + \eta_1\left(K - s_2^* - C_2\right)\right]\left[s_2^{*\eta_2} y_2\left(\eta_1 - \eta_2\right)\right]^{-1}.$$

Here $s_2^* \triangleq s^*(e_2,t)$ is as defined in Section 4.3. $C_2 \triangleq C_2(t,s)$ is as defined in (4.15). *The value of s_2^* is determined by the boundary conditions* (4.26), (4.27) *of smooth pasting and continuity.*

ρ_1 and ρ_2 are as given below:
$\rho_i = \lambda_i^{-1}\theta_i$ where

$$\lambda_i = a_{ii}^{-1}\left(\frac{1}{2}\sigma_i^2\eta_i(t)\left(\eta_i(t)-1\right) + \mu_i\eta_i(t) + a_{ii} - \frac{r_i}{y_i(t)}\right).$$

Bibliography

[1] G. K. Basak, M. K. Ghosh, and A. Goswami, *Risk minimizing option pricing for a class of exotic options in a Markov-modulated market*, Stochastic Analysis and Applications, 29 (2011), pp. 259–281.

[2] A. Bensoussan, *On the theory of option pricing*, Acta Applicandae Mathematica, 2 (1984), pp. 139–158.

[3] J. M. Bismut and B. Skalli, *Temps d'arrêt optimal, théorie générale des processus et processus de markov*, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 39 (1977), pp. 301–313.

[4] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, Journal of Political Economy, 81 (1973), pp. 637–54.

[5] J. Buffington and R. J. R. Elliott, *American options with regime switching*, International Journal of Theoretical and Applied Finance, 05 (2002), pp. 497–514.

[6] G. B. Di Masi, Y. M. Kabanov, and W. J. Runggaldier, *Mean-variance hedging of options on stocks with Markov volatilities*, Theory of Probability & Its Applications, 39 (1995), pp. 172–182.

[7] R. J. Elliott and P. E. Kopp, *Mathematics of Financial Markets*, Springer-Verlag, New-York, 1999.

[8] R. J. R. Elliott, L. Chan, and T. K. Siu, *Option pricing and esscher transform under regime switching*, Annals of Finance, 1 (2005), pp. 423–432.

[9] A. G. Fakeev, *Optimal stopping of a Markov process*, Theory of Probability & Its Applications, 16 (1971), pp. 694–696.

[10] A. Goswami and R. Saini, *Volterra equation for pricing and hedging in a regime switching market*, Cogent Economics & Finance, 2 (2014), pp. 1–11.

[11] X. Guo, *An explicit solution to an optimal stopping problem with regime switching*, Journal of Applied Probability, 38 (2001), pp. 464–481.

[12] J. D. Hamilton, *A new approach to the economic analysis of nonstationary time series and the business cycle*, Econometrica, 57 (1989), pp. 357–384.

[13] J. Harrison and S. R. Pliska, *Martingales and stochastic integrals in the theory of continuous trading*, Stochastic Processes and their Applications, 11 (1981), pp. 215–260.

[14] A. Jobert and L. Rogers, *Option pricing with Markov-modulated dynamics*, SIAM J. Control and Optimization, 44 (2006), pp. 2063–2078.

[15] I. Karatzas, *On the pricing of American options*, Applied Mathematics and Optimization, 17 (1988), pp. 37–60.

[16] I. Karatzas, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, Berlin, 1991.

[17] H. P. J. Mackean, *Appendix : A free boundary problem for the heat equation arising from a problem in mathematical economics*, 1965.

[18] R. C. Merton, *Theory of rational option pricing*, The Bell Journal of Economics and Management Science, 4 (1973), pp. 141–183.

[19] P. V. Moerbeke, *On optimal stopping and free boundary problems*, Advances in Applied Probability, 5 (1973), pp. 33–35.

[20] H. Pham, *Optimal stopping, free boundary, and American option in a jump-diffusion model*, Applied Mathematics and Optimization, 35 (1997), pp. 145–164.

[21] L. C. G. Rogers and D. Williams, *Diffusions, Markov Processes and Martingales*, vol. 2 of Cambridge Mathematical Library, Cambridge University Press, 2 ed., 2000.

[22] X. Su, W. Wang, and K.-S. Hwang, *Risk-minimizing option pricing under a Markov-modulated jump-diffusion model with stochastic volatility*, Statistics & Probability Letters, 82 (2012), p. 1777–1785.

[23] Q. Zhang and X. Guo, *Closed-form solutions for perpetual American put options with regime switching*, SIAM J. Appl. Math., 64 (2004), pp. 2034–2049.

[24] X. L. Zhang, *Numerical analysis of American option pricing in a jump-diffusion model*, Math. Oper. Res., 22 (1997), pp. 668–690.