Thermal Corrections to Entanglement Entropy

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Certificate

This is to certify that this dissertation entitled Thermal Corrections to Entanglement Entropy towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Saikat Bera** at Indian Institute of Science Education and Research under the supervision of **Prof. Sunil Mukhi**, Professor and Chair, Department of Physics , during the academic year 2016-2017.

Prof. Sunil Mukhi

Committee:

Prof. Sunil Mukhi

Prof. Nabamita Banerjee

This thesis is dedicated to my family and teachers.

Declaration

I hereby declare that the matter embodied in the report entitled Thermal Corrections to Entanglement Entropy are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, under the supervision of **Prof. Sunil Mukhi** and the same has not been submitted elsewhere for any other degree.

Saikat Bera

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Abstract

This thesis deals with the study of quantum entanglement across spatial boundaries in conformal-invariant quantum field theories (2d CFT) in two dimensions. In 2d CFT, Rényi entropy and entanglement entropy has been evaluated successfully using the replica method for non-compact systems at zero and non-zero temperatures and for compact systems at zero temperature (Ref.[2],[4]). However, for a compact system at finite temperature, an exact evaluation of the entropy measure using the replica trick has not been successfully performed. In order to gain more insights into the problem, calculations have been done by approximating the system to a compact system at zero temperature and expanding about it to obtain the leading order correction term to the entropy measure (Ref.[6]). In this thesis we propose a method to calculate the second correction term and eventually a general approach for calculating higher order correction terms to Rényi entropy and entanglement entropy at low temperature expansion for a spatially compactified 1d system.

Contents

Introduction

Symmetries play a crucial role in characterising and hence understanding of the different natural phenomena. Although a system might not be exactly solvable, studying its underlying symmetries may lead to great insights on the problem even without solving for it's dynamics. Additionaly, violation of symmetries is in itself of great interest and symmetry breaking has been used to describe many physical phenomena like ferromagnetism and superconductivity.

Among the symmetries of a physical system, the most widely studied are translational invariance and rotational invariance. Scale invariance is a symmetry which is not often encountered in physical systems. This however becomes extremely important when studying statistical systems at a critical point. Statistical systems like the Ising model, at criticality are the best examples of systems have scaling symmetry. When a statistical system is in a state far from the critical point, the particles of the system interact with each other upto a characteristic length scale called the correlation length. Correlation length is a characteristic property of the phase the system is present in. Now, at the critical point, the correlation length becomes infinite i.e. all particles in the system can interact with all other particles freely. Thus, the system can no longer be characterized by the correlation length and hence correlation functions scale as a power law rather than exponential. So, the behaviour of the system at the critical point is described by a scale invariant theory. It has been proved by Polchinski (Ref.[1]) that in $d = 2$ spacetime dimensions, scale invariance implies conformal invariance. Therefore, these systems have an effective description in terms of Conformal Field Theory (CFT). Thus, CFT is essential in studying the properties of statistical systems at criticality.

Quantum entanglement is a phenomenon in which two or more particles interact with each other irrespective of the separation between them and the quantum state of one particle cannot be described independently of the others. Entropy is a measure of the number of microstates of a system which are in thermal equilibrium given its macrostate (macroscopic thermodynamic properties). Rényi entropy and Entanglement entropy are two widely used measures of entropy. These can further be used to study the fluctuations in thermal equilibrium which implies that thermodynamic quantities at temperatures close to the equilibrium temperature can be determined by studying the behaviour of those fluctuatuions. Thus, Rényi entropy and entanglement entropy are imperative to the study of statistical systems, especially near criticality.

In this project we study the Rényi entropy and entanglement entropy at finite temperature, and investigate the calculation of correction terms in a low temperature expansion.

Chapter 1

Conformal Field Theory

This chapter contains a brief review of Conformal Field Theory and definitions of many objects such as correlation functions, primary fields, radial quantization, operator product expansion and vertex operators, which are crucial for the understanding of all the subsequent work done in the project.

1.1 Conformal Transformations in d Euclidean Dimension

A general conformal transformation in d euclidean dimension is an invertible map $x \to x'$ such that the metric tensor is left invariant upto a scale

$$g'_{\mu\nu}\left(x'\right) = \Lambda\left(x\right) g_{\mu\nu}\left(x\right) \tag{1.1.1}$$

A conformal transformation is angle preserving i.e. it does not affect the angle between two arbitrary curves crossing each other at some point. The set of conformal transformations form a group. This group has the Poincaré group as a subgroup corresponding to the special case $\Lambda\left(x\right) \equiv 1$. A general conformal transformation includes translation, rotation, dialation and special conformal transformation (SCT). Thus, the finite conformal transformations are given by:

$$\begin{aligned} x'^{\mu} &= x^{\mu} + a^{\mu} && \text{(Translation)} \\ x'^{\mu} &= M^{\mu}_{\nu} x^{\nu} && \text{(Rotation)} \\ x'^{\mu} &= \lambda x^{\mu} && \text{(Dilation)} \\ x'^{\mu} &= \tfrac{x^{\mu} - b^{\mu} x^2}{1 - 2b.x + b^2 x^2} && \text{(SCT)} \end{aligned} \tag{1.1.2}$$

The corresponding generators of these transformations are given by:

$$\begin{aligned} P_{\mu} &= -i\partial_{\mu} && \text{(Translation)} \\ L_{\mu\nu} &= i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) && \text{(Rotation)} \\ D &= -ix^{\mu}\partial_{\mu} && \text{(Dilation)} \\ K_{\mu} &= -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu}) && \text{(SCT)} \end{aligned} \tag{1.1.3}$$

The commutation rules obeyed by these generators define the conformal algebra. One finds that the conformal group in d dimension is isomorphic to $SO(d+1,1)$.

The transformation law for a spinless field $\phi(x)$ under a finite conformal transformation $x \to x'$ can be derived by using the above generators. One finds that the field transforms as:

$$\phi(x) \to \phi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \phi(x) \tag{1.1.4}$$

where, $|\partial x'/\partial x|$ is the Jacobian of the transformation and Δ is called the scaling dimension of the field. A field transforming according to the above transformation law is called a 'quasi-primary' field.

1.2 Correlation Functions

Henceforth we work in Euclidean CFT, obtained by continuing the time to imaginary values.

Let us now examine the consequences of conformal invariance on two-, three- and four-point correlation functions of quasi-primary fields. Henceforth it will be understood that all correlation functions involve time-ordered products of fields. Now the two-point correlation function of fields ϕ_1 and ϕ_2 is given by:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \frac{1}{Z} \int [d\phi] \;\; \phi_1 (x_1) \;\; \phi_2 (x_2) \;\; e^{-S[\phi]} \tag{1.2.1}$$

where, Z is the partion function and $S[\phi]$ is the action of the theory. Thus, the above equation represents a functional integral over the set of all independent fields in the theory.

Now translation and rotation invariance implies that $\langle \phi_1 (x_1) \phi_2 (x_2) \rangle$ is a function of the distance between the points, i.e.

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \; h (|x_1 - x_2|) \tag{1.2.2}$$

Now, from eq.(1.1.4) it is seen that invariance of the two-point function under any transformation $x \to x'$ is given by:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \left| \frac{\partial x'}{\partial x} \right|_{x=x_2}^{\Delta_2/d} \langle \phi_1 (x'_1) \phi_2 (x'_2) \rangle \tag{1.2.3}$$

Thus, from eq.(1.2.3) we see that under scale transformation $x \to \lambda x$, the two-point function satisfies the following relation:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \lambda^{\Delta_1 + \Delta_2} \; \langle \phi_1 (\lambda x_1) \phi_2 (\lambda x_2) \rangle \tag{1.2.4}$$

Thus, eq.(1.2.2) in addition with eq.(1.2.4) gives:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \tag{1.2.5}$$

where C_{12} is a constant.

Now, for a special conformal transformation (SCT), from eq.(1.1.2) we have:

$$\left| \frac{\partial x'}{\partial x} \right| = \frac{1}{(1 - 2b.x + b^2 x^2)^d} \tag{1.2.6}$$

which in turn gives the invariance condition as:

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1 + \Delta_2}} \frac{(\gamma_1 \gamma_2)^{(\Delta_1 + \Delta_2)/2}}{\gamma_1^{\Delta_1} \gamma_2^{\Delta_2}} \tag{1.2.7}$$

where, $\gamma_i = (1 - 2b.x_i + b^2 x_i^2)$.

The above condition is satisfied only when $\Delta_1 = \Delta_2$. Therefore, the two-point function of two quasi-primary fields is given by:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \rangle = \begin{cases} \frac{C_{12}}{|x_1 - x_2|^{2\Delta_1}} & \text{if} \quad \Delta_1 = \Delta_2 \\ 0 & \text{if} \quad \Delta_1 \neq \Delta_2 \end{cases} \tag{1.2.8}$$

A similar analysis can be done for three-point functions and four-point functions. For a three-point correlation function, invariance under translation, rotation and dialation leads to the following form:

$$\langle \phi_1 (x_1) \phi_2 (x_2) \phi_3 (x_3) \rangle = \frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c} \tag{1.2.9}$$

where, $x_{ij} = |x_i - x_j|$ and $a + b + c = \Delta_1 + \Delta_2 + \Delta_3$. Invariance of eq.(1.2.9) under SCT implies:

$$\frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c} = \frac{C_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c} \frac{(\gamma_1\gamma_2)^{a/2}(\gamma_2\gamma_3)^{b/2}(\gamma_3\gamma_1)^{c/2}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}\gamma_3^{\Delta_3}} \tag{1.2.10}$$

which leads to the following set of constraints:

$$a + c = 2\Delta_1 \ , \ a + b = 2\Delta_2 \ , \ b + c = 2\Delta_3 \tag{1.2.11}$$

The above set of equations has a unique solution for a, b, c given by:

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 \\ b &= \Delta_2 + \Delta_3 - \Delta_1 \\ c &= \Delta_3 + \Delta_1 - \Delta_2 \end{aligned} \tag{1.2.12}$$

Therefore the three-point function of quasi-primary fields is given by:

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}} \tag{1.2.13}$$

Thus, the three-point functions of quasi-primary fields are determined upto a constant C_{123}.

Following a similar analysis for four-point functions lead to their forms being determined upto a multiplicative factor of a function of the anharmonic ratios i.e.

$$\langle \phi_1(x_1)\,\phi_2(x_2)\,\phi_3(x_3)\,\phi_4(x_4)\rangle = h\left(\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{14}x_{23}}\right) \prod_{1\le i<j\le 4} x_{ij}^{\Delta/3-\Delta_1-\Delta_2} \tag{1.2.14}$$

where, $\Delta = \sum_{i=1}^{4} \Delta_i$. This can be seen from the fact that the anharmonic ratios $\frac{x_{12}x_{34}}{x_{13}x_{24}}$ and $\frac{x_{12}x_{34}}{x_{14}x_{23}}$ are invariant under translation, rotation, scaling and SCT. Thus, multiplying the RHS of eq.(1.2.14) with any function of the anharmonic ratios will leave ir invariant under these transformations.

1.3 CFT in $2d$

For systems in $2d$, let the co-ordinates on the plane be (x, τ_x). Now, for a general transformation $x^\mu \to w^\mu(x)$ to be a conformal transformation, the condition $g'_{\mu\nu}(w) \propto g_{\mu\nu}(x)$ is equivalent either to

$$\frac{\partial \tau_w}{\partial x} = \frac{\partial w}{\partial \tau_x} \text{ and } \frac{\partial w}{\partial x} = -\frac{\partial \tau_w}{\partial \tau_x} \tag{1.3.1}$$

or to

$$\frac{\partial \tau_w}{\partial x} = -\frac{\partial w}{\partial \tau_x} \text{ and } \frac{\partial w}{\partial x} = \frac{\partial \tau_w}{\partial \tau_x} \tag{1.3.2}$$

It is observed that eq.(1.3.1) is the Cauchy-Riemann equations for holomorphic functions, while eq.(1.3.2) is the same for antiholomorphic functions. This motivates the use of complex coordinates for $2d$ CFT. The complex coordinates $z, \bar{z}$ are defined according to the following rules:

$$\begin{aligned} z &= x + i\tau & \bar{z} &= x - i\tau \\ \partial_z &= \tfrac{1}{2}\left(\partial_x - i\partial_\tau\right) & \partial_{\bar{z}} &= \tfrac{1}{2}\left(\partial_x + i\partial_\tau\right) \end{aligned} \tag{1.3.3}$$

In terms of the complex coordinate z, the complete set of global conformal transformations (also called projective transformations) is given by the set of invertible maps which map the entire complex plane to itself. These mappings are given by

$$f(z) = \frac{az + b}{cz + d} \tag{1.3.4}$$

where, $a, b, c, d \in \mathbb{C}$ with $ad - bc = 1$ i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$. The set of global conformal transformations is also called the *special conformal group*.

1.3.1 Primary Fields

One of the interesting properties of CFT in two dimension is that the definition of quasi-primary fields is also applicable to fields with non-zero spin. For a given field with scaling dimension Δ and planar spin s, holomorphic conformal dimension 'h' and antiholomorphic conformal dimension '$\bar{h}$' are defined as:

$$h = \tfrac{1}{2}(\Delta + s) \qquad \bar{h} = \tfrac{1}{2}(\Delta - s) \tag{1.3.5}$$

Thus, under the conformal map $z \to w(z)$, $\bar{z} \to \bar{w}(\bar{z})$, a quasi-primary field transforms as:

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{1.3.6}$$

Now, a field in two dimension which transforms according to eq.(1.3.6) under *any* local transformation i.e. holomorphic transformations which are allowed to be singular at 0 and infinity, is called a primary field.

1.3.2 Correlation Functions of Primary Fields

From eq.(1.2.3) and eq.(1.3.6), it is seen that under a conformal transformation, the n-point correlation function of n primary fields with conformal dimensions h_i and $\bar{h}_i$ transforms as:

$$\langle \phi_1 (w_1, \bar{w}_1) \cdots \phi_n (w_n, \bar{w}_n) \rangle = \left[\prod_{1 \leq i \leq n} \left(\frac{dw_i}{dz_i} \right)^{-h_i} \left(\frac{d\bar{w}_i}{d\bar{z}_i} \right)^{-\bar{h}_i} \right] \langle \phi_1 (z_1, \bar{z}_1) \cdots \phi_n (z_n, \bar{z}_n) \rangle \tag{1.3.7}$$

An important point to note in the above equation is that it incorporates the possibility of a field with non-zero spin in the difference $h_i - \bar{h}_i$. Thus the two-point correlation function is given by:

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \rangle = \begin{cases} \frac{C_{12}}{(z_1 - z_2)^{2h} \ (\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} & \text{if } \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} \\ 0 & \text{otherwise} \end{cases} \tag{1.3.8}$$

Similarly, the three-point function is given by:

$$\begin{aligned} \langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \phi_3 (z_3, \bar{z}_3) \rangle = C_{123} & \frac{1}{z_{12}^{h_1 + h_2 - h_3} \bar{z}_{12}^{\bar{h}_1 + \bar{h}_2 - \bar{h}_3}} \\ & \times \frac{1}{z_{23}^{h_2 + h_3 - h_1} \bar{z}_{23}^{\bar{h}_2 + \bar{h}_3 - \bar{h}_1}} \times \frac{1}{z_{31}^{h_3 + h_1 - h_2} \bar{z}_{31}^{\bar{h}_3 + \bar{h}_1 - \bar{h}_2}} \end{aligned} \tag{1.3.9}$$

and the four-point function is given by:

$$\langle \phi_1 (z_1, \bar{z}_1) \phi_2 (z_2, \bar{z}_2) \phi_3 (z_3, \bar{z}_3) \phi_4 (z_4, \bar{z}_4) \rangle = h \left(\zeta, \bar{\zeta} \right) \prod_{1 \leq i < j \leq 4} z_{ij}^{h/3 - h_i - h_j} \ \bar{z}_{ij}^{\bar{h}/3 - \bar{h}_i - \bar{h}_j} \tag{1.3.10}$$

where, $\zeta = \frac{z_{12} z_{34}}{z_{14} z_{32}}$, $h = \sum_{i=1}^{4} h_i$ and $\bar{h} = \sum_{i=1}^{4} \bar{h}_i$.

1.3.3 Radial Quantization

In $2d$ Eucledean formalism, the space and time axes are on equal footing. Thus it is possible to choose a special co-ordinate system and perform quantization along those axes.

One such special quantization is called the radial quantization. In this, the time axis is taken to be radial while the space axis is taken as concentric circles. As we will see, this turns out to be very useful in the study of conformal field theories.

One way to relate this spacetime with the usual notion of spacetime is to consider a system in which the space axis is compactified to length L. The spacetime is then represented by a cylinder with the time axis along the axis of the cylinder (going from -∞ to $+\infty$) and the space axis along the lateral surface of the cylinder (going from 0 to L) with the points $(0, t)$ and (L, t) being identified. This cylinder is also represented by a single complex co-ordinate $\vartheta = t + ix$ (or eqivalently $\vartheta = t - ix$). Now the cylinder can be mapped to the rquired spacetime via the map $z = e^{2\pi\vartheta/L}$, which is equivalent to 'squashing' the cylinder such that the points at $t = -\infty$ is mapped to the origin and the points at $t = +\infty$ is mapped to a circle with infinite radius, on the plane (Fig.1.1).

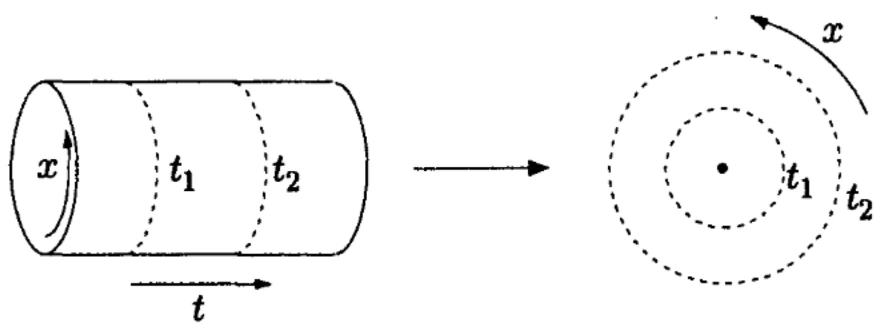

Figure 1.1: Mapping from cylinder to the complex plane

An important point to note is that in the radial quantization picture, the time-ordering of operators in correlators becomes a radial ordering i.e.

$$\mathfrak{R}\phi_1(z)\,\phi_2(w) = \begin{cases} \phi_1(z)\,\phi_2(w) & \text{if } |z| > |w| \\ \phi_2(w)\,\phi_1(z) & \text{if } |w| > |z| \end{cases} \tag{1.3.11}$$

1.3.4 Operator Product Expansion (OPE)

Correlation functions have the typical property of having singularities when the position of two or more fields coincides. This in essence represents the infinite fluctuations of a quantum field taken at a precise location. This is captured by the operator product expansion (OPE). The OPE gives the product of two operators at positions z and w respectively, as a sum of single operators (well defined as $z \to w$) multiplied with a c-number function of $z - w$ (possibly diverging as $z \to w$).

$$O_1(z)\, O_2(w) = \sum_{n=-\infty}^{N} \frac{A_n(w)}{(z-w)^n} \tag{1.3.12}$$

where $A_n(w)$ are non-singular at $w = z$. As an example, the OPE of the energy-momentum tensor with a primary field of conformal dimension $\left(h, \bar{h}\right)$ is given by:

$$\begin{aligned} T(z)\, \phi(w, \bar{w}) &\sim \tfrac{h}{(z-w)^2}\ \phi(w, \bar{w}) + \tfrac{1}{z-w}\ \partial_w \phi(w, \bar{w}) \\ \bar{T}(\bar{z})\, \phi(w, \bar{w}) &\sim \tfrac{\bar{h}}{(\bar{z}-\bar{w})^2}\ \phi(w, \bar{w}) + \tfrac{1}{\bar{z}-\bar{w}}\ \partial_{\bar{w}} \phi(w, \bar{w}) \end{aligned} \tag{1.3.13}$$

The symbol $\sim$ in eq.(1.3.13) means that all the regular terms (non-singular as $z \to w$) in the product of the operators is dropped. This is justified since knowledge of the singular terms alone is often sufficient to fix the entire behaviour of functions using complex analyticity.

1.3.5 Vertex Operators

The free boson field ($\phi(z, \bar{z})$) in $2d$ has a very special characteristic that it has logarithmic singularities in its correlator (eq.(1.3.14)), which is a sign of infrared divergences.

$$\langle \phi(z, \bar{z})\, \phi(w, \bar{w}) \rangle \ \sim\ -\left\{ \ln(z-w)\ +\ \ln(\bar{z}-\bar{w}) \right\} \tag{1.3.14}$$

This implies that ϕ in itself is not a conformal field, however exponentials of the free boson field will have power-law correlators due to the logarithmic behaviour of the free boson correlator. This gives us a way to construct infinitely many good conformal fields in the theory, namely the vertex operators (eq.(1.3.15)):

$$\mathcal{V}_\alpha(z,\bar{z}) = :e^{i\alpha\phi(z,\bar{z})}: \tag{1.3.15}$$

The vertex operators are of great importance because each $\mathcal{V}_\alpha(z,\bar{z})$ is a primary field (Ref. [5]) with holomorphic and anti-holomorphic dimension:

$$h_\alpha = \bar{h}_\alpha = \frac{\alpha^2}{2} \tag{1.3.16}$$

The OPE of $\partial\phi$ with $\mathcal{V}_\alpha$ is evaluated by expanding $\mathcal{V}_\alpha$ in it's poynomial form and calculating the OPE of $\partial\phi$ with ϕ thereafter. The common terms are taken out and the resulting answer is re-written in terms of $\mathcal{V}_\alpha$. The result is given by:

$$\partial\phi(z)\,\mathcal{V}_\alpha(w,\bar{w}) \;\sim\; -i\alpha\,\frac{\mathcal{V}_\alpha(w,\bar{w})}{z-w} \tag{1.3.17}$$

The OPE of $\mathcal{V}_\alpha$ with the energy-momentum tensor evaluated in an analogous manner is given by:

$$T(z)\,\mathcal{V}_\alpha(w,\bar{w}) \;\sim\; \frac{\alpha^2}{2}\,\frac{\mathcal{V}_\alpha(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{V}_\alpha(w,\bar{w})}{z-w} \tag{1.3.18}$$

These OPE's will be used for calculation of Rényi entropy and entanglement entropy, as shown in the successive chapters.

Chapter 2

Rényi Entropy and Entanglement Entropy

Quantum entanglement is a phenomenon in which two or more particles interact with each other irrespective of the separation between them and the quantum state of one particle cannot be described independently of the others. Entanglement entropy is a measure of entanglement in a many-body quantum state. Eg. consider a quantum mechanical system in the ground state $|\Psi\rangle$. Assuming non-degeneracy of the ground state, the density matrix for the system is given by $\rho_{tot} = |\Psi\rangle \langle\Psi|$. The von Neumann entropy of the system is given by $S_{tot} = -\ \text{tr}\ \rho_{tot} \log \rho_{tot} = 0$ (since the system is in pure state). Now, if the system is divided into two subsystems A and B, then the total Hilbert space can be written as $\mathcal{H}_{tot} = \mathcal{H}_A \otimes \mathcal{H}_B$. For such a system, the reduced density matrix of the subsystem A is given by $\rho_A = \ \text{tr}_B\ \rho_{tot}$, where the trace is taken only over the Hilbert space $\mathcal{H}_B$.

The entanglement entropy(S_A) of the subsystem A is defined to be the von Neumann entropy of the reduced density matrix ρ_A i.e.

$$S_A = -\ \text{tr}_A\ \rho_A\ \log \rho_A \tag{2.0.1}$$

In addition to the entanglement entropy, there is another measure for entanglement called the Rényi entropy. The Rényi entropy of the subsystem A is given by

$$S_n = \frac{1}{1-n} \log \left(\text{tr } \rho_A^n\right) \tag{2.0.2}$$

with $S_A = \lim_{n\to 1} S_n$.

Note that, if the density matrix is diagonal i.e. $\rho_{tot} = diag(\lambda_1, \ \lambda_2, \ \cdots, \lambda_n)$ then the von Neumann entropy is given by

$$S = -\sum_{i=1}^{n} \lambda_i \ \log\lambda_i \tag{2.0.3}$$

For a density matrix, $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$. Therefore, the RHS of eq.(2.0.3) is always non-negative and hence, the entanglement entropy is *always* non-negative.

The density matrix for a pure state has one of the λ_i equal to one and the rest equal to zero. Now, $\lambda_i \ \log\lambda_i$ is zero for $\lambda_i = 0$ or 1. Therefore, a pure state system has zero entanglement entropy. This is also the minimum value of entanglement entropy. Similarly, the values of RHS of eq.(2.0.3) are bounded above by $\log n$ which is obtained when all λ_i's are equal and equal to $1/n$. Thus, a mixed system is maximally entangled when each subsystem has an equal probability and the maximum entanglement entropy is $\log n$.

2.1 Properties of Entanglement Entropy

For a system at absolute zero temperature, some of the note-worthy properties of entanglement entropy are:

(i) For two subsystems A and B of the system, if B is the complement of A, then $S_A = S_B$. This implies that entanglement entropy is not an extensive quantity. However, this equality does not hold at finite temperature. At non-zero temperature $S_A - S_B = S_{Thermal}$, where $S_{Thermal}$ is the thermal entropy of the entire system.

(ii) If A is further divided into two submanifolds A_1 and A_2, then $S_{A_1} + S_{A_2} \geq S_A$. This property is called subadditivity.

(iii) For three subsystems A, B and C such that they do not intersect each other, the following inequality holds

$$S_{A+B+C} \; + \; S_B \; \leq \; S_{A+B} \; + \; S_{B+C} \tag{2.1.1}$$

This property is called strong subadditivity.

2.2 Entanglement Entropy in 2d CFT

Let a system at zero temperature and in a non-compact space dimension in a 2d CFT be defined on a complex plane with the imaginary axis corresponding to the Euclidean time and the real axis corresponding to the spatial dimension. The subsystem A is defined as the single interval $x \in [u, v]$ at $\tau = 0$ in the flat Euclidean coordinates $(x, \tau) \in \mathbb{R}^2$. Then, the reduced density matrix ρ_A of the subsystem is evaluated using the Euclidean path-integral formalism as shown below(Ref. [2],[4]).

First, the ground state wave function Ψ of the system is obtained by path integrating from $\tau = -\infty$ to $\tau = 0$ in the Euclidean formalism

$$\Psi(\phi_1(x), \tau = 0) = \Psi(\phi_1(x)) = \int_{\phi(\tau=-\infty,x)\equiv 0}^{\phi(\tau=0,x)=\phi_1(x)} [d\phi] \; e^{-S[\phi]} \tag{2.2.1}$$

where, $\phi(\tau, x)$ denotes the fundamental fields of the 2d CFT. The total density matrix ρ is given by the product $\Psi\bar{\Psi}$ where $\bar{\Psi}$ is obtained by path integrating from $\tau = \infty$ to $\tau = 0$. This in turn implies that the density matrix is characterized by the boundary conditions (fields) i.e. $[\rho]_{\phi_1\phi_2} = \Psi(\phi_1(x))\bar{\Psi}(\phi_2(x))$. Thus the system is defined by taking a functional integral over all well behaved functions (those which go to zero as $\tau, x \to \infty$) such that for $\tau \to 0^+$ the functions approach $\phi_2(x)$ and for $\tau \to 0^-$ the functions approach $\phi_1(x)$. The reduced density matrix ρ_A is obtained by integrating ϕ_1 on B assuming $\phi_1(x) = \phi_2(x)$ when $x \in B$. Thus,

$$[\rho_A]_{\phi_2\phi_1} = \frac{1}{Z_1}\int_{\tau=-\infty}^{\tau=\infty}[d\phi]e^{-S[\phi]}\prod_{x\in A}\delta(\phi(+0,x)-\phi_2(x))\cdot\delta(\phi(-0,x)-\phi_1(x)), \qquad (2.2.2)$$

where, Z_1 is the vacuum partition function on $\mathbb{R}^2$. The $1/Z_1$ factor ensures that ρ_A is normalized such that $\mathrm{tr}_A\rho_A = 1$. (Fig. 2.1 (a))

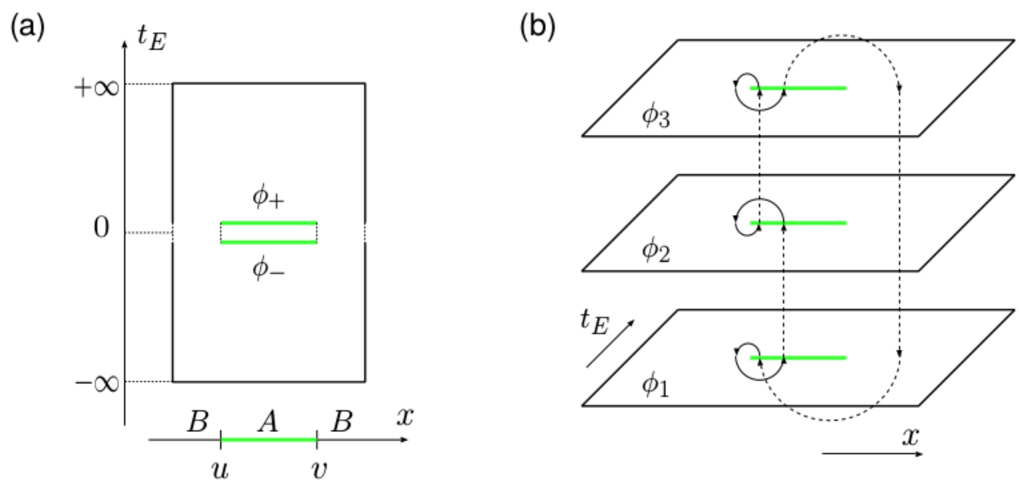

Figure 2.1: (a) The path integral representation of $[\rho_A]_{\phi_+\phi_-}$,
(b) The 3-sheeted Riemann surface corresponding to the calculation of $\mathrm{tr}_A\ \rho_A^3$ Ref.[4]

2.3 Computation of Entanglement Entropy

The computation of entanglement entropy in $2d$ CFT is done using the replica trick, which is explained in the next section (Ref.[4]). The procedure includes evaluating $\mathrm{tr}_A\ \rho_A^n$, then differentiating it's logarithm with respect to n and finally taking the limit $n \to 1$ (where ρ_A is normalized i.e. $\mathrm{tr}_A\ \rho_A = 1$).

$$S_A = -\frac{\partial}{\partial n} \log \mathrm{tr}_A \; \rho_A^n |_{n=1} \tag{2.3.1}$$

To evaluate $\mathrm{tr}_A \; \rho_A^n$, we trace over n copies of ρ_A successively i.e. we compute

$$[\rho_A]_{\phi_1 \phi_2} [\rho_A]_{\phi_2 \phi_3} [\rho_A]_{\phi_3 \phi_4} \cdots [\rho_A]_{\phi_{n-1} \phi_n} [\rho_A]_{\phi_n \phi_1} \tag{2.3.2}$$

where the ϕ_i's are integrated over the functional.

2.4 Replica Method

The computation of eq.(2.3.2) is done using the path integral formalism on a branched surface which is called the replica method. First, consider the system (complex plane) with a cut along the subsystem A (the interval $[u, v]$ at $\tau = 0$). Now, take n such planes. Let the defining CFT on the i^{th} plane be given by the field ϕ_i. The n sheets are then joined along the interval in the following manner : the $\tau \to 0^-$ side of the i^{th} sheet is joined to the $\tau \to 0^+$ side of the $i + 1^{th}$ sheet (Fig. 2.1 (b)), and the $\tau \to 0^-$ side of the n^{th} sheet is joined to the $\tau \to 0^+$ side of the 1^{st} sheet. Thus, the sheets are joined in such a manner that for a point in the i^{th} sheet, an anti-clockwise rotation by 2π about u will take the point to $i + 1^{th}$ sheet and an anti-clockwise rotation by 2π about v will take the point to $i - 1^{th}$ sheet i.e.

$$\phi_i \left(e^{i2\pi} \left(w - u\right)\right) = \phi_{i+1} \left(w - u\right) \qquad \phi_i \left(e^{i2\pi} \left(w - v\right)\right) = \phi_{i-1} \left(w - v\right) \tag{2.4.1}$$

Now, $\mathrm{tr}_A \; \rho_A^n$ is given by the path integral on this n sheeted Riemann surface.

$$\mathrm{tr}_A \rho_A^n \;\; = \;\; \frac{1}{(Z_1)^n} \int_{(\tau,x) \in \mathcal{R}_n} [d\phi] e^{-S(\phi)} \;\; \equiv \;\; \frac{Z_n}{(Z_1)^n} \tag{2.4.2}$$

Alternatively, the boundary conditions (eq.(2.4.1)) can be interpreted as insertions of twist operators σ_i^+ at u and σ_i^- at v, on the i^{th}-sheet. Then,

$$\mathrm{tr}_A \rho_A^n \;=\; \prod_{i=1}^{n} \langle \sigma_i^+(u) \sigma_i^-(v) \rangle \tag{2.4.3}$$

It is important to note that in order to compute the two-point function $\langle \sigma_i^+(u)\sigma_i^-(v)\rangle$, the ultra-violet (UV) divergence needs to be regularized. This is done by introducing a UV cutoff parameter 'a' to the theory. To determine $\langle \sigma_i^+(u)\sigma_i^-(v)\rangle$, we use the uniformization map:

$$z = \left(\frac{w-u}{w-v}\right)^{1/n} \tag{2.4.4}$$

This mapping maps the n sheeted Riemann surface to a complex plane, with each of the i^{th} sheet being mapped to the corresponding i^{th} sector in the plane. Then, we study the tranformation of the energy-momentum tensor under this map. The tranformation law for energy-momentum tensor under a coordinate transformation $z \to w$ is:

$$T'(w) = \left(\frac{dz}{dw}\right)^2 T(z) + \frac{c}{12}\{z;w\} \tag{2.4.5}$$

where, c is the central charge of the theory and $\{z;w\}$ is the Schwarzian derivative given by:

$$\{z;w\} = \frac{(d^3z/dw^3)}{(dz/dw)} - \frac{3}{2}\left(\frac{(d^2z/dw^2)}{(dz/dw)}\right)^2 \tag{2.4.6}$$

Since z is a coordinate on a complex plane, $\langle T(z)\rangle$ is zero by translational and rotational invariance. Hence,

$$\langle T(w)\rangle_{\mathcal{R}_n} = \frac{c}{24}\left(1 - \frac{1}{n^2}\right)\frac{(v-u)^2}{(w-u)^2\ (w-v)^2} \tag{2.4.7}$$

The right hand side (RHS) of eq.(2.4.7) can be interpreted as $\langle T(w)\ \sigma^+(u)\ \sigma^-(v)\rangle$, wehre $T(w)$ is the energy-momentum tensor of each of those n sheets seperated, with each sheet having a twist operator σ^+ at u and σ^- at v. On comparing eq.(2.4.7) as lim $w \to u$ with the OPE of energy-momentum tensor with a primary field (eq.(1.3.13)), we see that the conformal dimension of σ^+ is

$$h_{\sigma^+} = \bar{h}_{\sigma^+} = \frac{c}{24}\left(1 - \frac{1}{n^2}\right) \tag{2.4.8}$$

Similarly, comparing eq.(2.4.7) as lim $w \to v$ with the OPE of energy-momentum tensor with a primary field (eq.(1.3.13)), we get the conformal dimension of σ^- as

$$h_{\sigma^-} = \bar{h}_{\sigma^-} = \frac{c}{24}\left(1 - \frac{1}{n^2}\right) \tag{2.4.9}$$

Thus, contribution of each sheet to $\mathrm{tr}_A\, \rho_A^n$ is

$$\langle \sigma_i^+(u)\sigma_i^-(v)\rangle = \left(\frac{u-v}{a}\right)^{-\left(2h_{\sigma^+}+2\bar{h}_{\sigma^+}\right)} = \left(\frac{u-v}{a}\right)^{-\frac{c}{6}\left(1-\frac{1}{n^2}\right)} \tag{2.4.10}$$

where, a is the UV cutoff. Hence, $\mathrm{tr}_A\, \rho_A^n$ is given by

$$\mathrm{tr}_A\, \rho_A^n = \langle \sigma_i^+(u)\sigma_i^-(v)\rangle^n = \left(\frac{u-v}{a}\right)^{-\frac{c}{6}\left(n-\frac{1}{n}\right)} \tag{2.4.11}$$

From eq.(2.0.2) and eq.(2.4.11) we find the Rényi entropy

$$S_n \sim \frac{c}{6}\left(1 + \frac{1}{n}\right) \log\left(\frac{|u-v|}{a}\right) \tag{2.4.12}$$

Now, the entanglement entropy is given by

$$S_A = \lim_{n\to 1} S_n = \frac{c}{3} \log\left(\frac{|u-v|}{a}\right) \tag{2.4.13}$$

Therefore, the Rényi entropy and entanglement entropy in an infinitely long system at zero temperature are given by

$$S_n = \frac{c}{6}\,\left(1+\frac{1}{n}\right)\,\log\frac{l}{a} \tag{2.4.14}$$

$$S_A = \frac{c}{3}\,\log\frac{l}{a}, \tag{2.4.15}$$

respectively, where, c is the central charge of the CFT and l is the length of the interval ($l = |v-u|$).

2.4.1 Entropy in an Infinitely long system at Finite Temperature

A similar calculation is performed to obtain the Rényi entropy and entanglement entropy of an interval $[r,s]$ in an infinitely long system at finite temperature ($T=\beta^{-1}$). Here, the uniformization map is taken as:

$$z = \left(\frac{e^{\frac{2\pi\theta}{\beta}} - e^{\frac{2\pi r}{\beta}}}{e^{\frac{2\pi\theta}{\beta}} - e^{\frac{2\pi s}{\beta}}}\right)^{1/n} \tag{2.4.16}$$

The original system can be viewed as a sequence of n cylinders of infinite length, with each cylinder having its axis along the spatial direction, having an interval $[r,s]$ and the circumference of the cylinder as the compactified time direction of length β ($=\mathrm{T}^{-1}$). The cylinders are joined to each other along the cut such that the $\tau\to 0^-$ side of the i^{th} cylinder is joined to the $\tau\to 0^+$ side of the $i+1^{th}$ cylinder and the $\tau\to 0^-$ side of the n^{th} cylinder is joined to the $\tau\to 0^+$ side of the 1^{st} cylinder. The mapping $\theta\to w$ given by $w=e^{2\pi\theta/\beta}$ maps the i^{th} cylinder to the i^{th} sheet of the n sheeted Riemann surface ($\mathcal{R}_n$) with the end points of the interval at $u=e^{2\pi r/\beta}$ and $v=e^{2\pi s/\beta}$. Furthermore, the mapping $w\to z$ (eq.(2.4.4)) then maps $\mathcal{R}_n$ to the complex plane $\mathcal{C}$.

Performing the subsequent steps of calculation, the Rényi entropy and entanglement entropy of a single interval in an infinitely long system at finite temperature ($T=\beta^{-1}$) are computed to be

$$S_n = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{2.4.17}$$

and

$$S_A = \frac{c}{3} \log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi l}{\beta}\right)\right) \tag{2.4.18}$$

respectively.

2.4.2 Entropy in a System Compactified on a Circle of circumference L at Zero Temperature

To obtain the Rényi entropy and entanglement entropy of an interval $[r, s]$ in a system compactified on a circle of circumference L at zero temperature, a similar calculation is done with the mapping $\theta \to w$ given by $w = e^{i2\pi\theta/L}$ followed by the mapping $w \to z$ (eq.(2.4.4)). On performing the calculation, the Rényi entropy and entanglement entropy are obtained to be

$$S_n = \frac{c}{6} \left(1 + \frac{1}{n}\right) \log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right) \tag{2.4.19}$$

and

$$S_A = \frac{c}{3} \log\left(\frac{L}{\pi a}\sin\left(\frac{\pi l}{L}\right)\right) \tag{2.4.20}$$

respectively.

Chapter 3

Thermal Corrections to Entanglement Entropy

As seen in the previous chapter, for a system with hilbert space $\mathcal{H}_{tot} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$, where $B = \bar{A}$, a measure of entanglement, the entangement entropy is given by

$$S_E = -\mathrm{tr}_A \left(\rho_A \log \rho_A\right) \tag{3.0.1}$$

where, $\rho_A = \mathrm{tr}_B \rho_{tot}$ is the reduced density matrix for subsystem A. Also the Rényi entropy, given by:

$$S_n = \frac{1}{1-n} \log \ (\mathrm{tr}\ \rho_A^n) \tag{3.0.2}$$

with $S_E = \lim_{n \to 1} S_n$.

For a compactified system at finite temperature, the above described path integral prescription for entropy calculation is not easily realised as the geometry of the spacetime of such a system is a torus (genus-1 surface), and hence, joining n such surfaces along the cut will result in a genus-n surface. Now, the path-integral on a genus-n surface is very hard to compute. However, for temperatures close to zero, the system can be approximated by a thin torus which equivalent to a cylinder of infinite length, since the compactification length along the imaginary time axis β becomes infinite, and hence, the entropy can be calculated by a Taylor series expansion around zero temperature. Such a calculation has been done in Ref. [6] for the leading order correction. Here we will follow this method and attempt to extend it to higher-order corrections.

3.1 Leading Order Thermal Correction

For a system at finite temperature, the density matrix is given by the Boltzmann sum over the states:

$$\rho_{tot} = \frac{1}{\mathrm{tr}\left(e^{-\beta H}\right)} \sum_{|\phi\rangle} |\phi\rangle \langle\phi| \; e^{-\beta E_\phi} \tag{3.1.1}$$

Thus, in the low temperature regime, assuming a non-degenerate CFT, the density matrix can be written as:

$$\rho_{tot} = \frac{|0\rangle \langle 0| \; + \; |\phi\rangle \langle\phi| \, e^{-2\pi E_\phi \beta/L} \; + \; \cdots}{1 \; + \; e^{-2\pi E_\phi \beta/L} \; + \; \cdots} \tag{3.1.2}$$

where, L : length of the spatial compactification (circumference of the cylinder)
$\beta = T^{-1}$
$|0\rangle$: ground state of the CFT
$|\phi\rangle$: first excited state of the CFT with energy eigenvalue E_ϕ

Thus, ρ_A is given by partially tracing ρ_{tot} over the Hilbert space of B. Hence, $\mathrm{tr}_A \; \rho_A^n$ is given by:

$$\mathrm{tr}_A \; \rho_A^n = \; \mathrm{tr}_A \left[(\rho_{A,0} + \rho_{A,\phi} e^{-2\pi E_\phi \beta/L} + \cdots)^n \; (1 \; + \; e^{-2\pi E_\phi \beta/L} + \cdots)^{-n} \right] \tag{3.1.3}$$

where, $\rho_{A,0} = \; \mathrm{tr}_B \; |0\rangle \langle 0|$
$\rho_{A,\phi} = \; \mathrm{tr}_B \; |\phi\rangle \langle\phi|$

Expanding eq.(3.1.3) upto first order in $e^{-2\pi E_\phi \beta/L}$ we get

$$\mathrm{tr}_A\ \rho_A^n \sim\ \mathrm{tr}_A\ \rho_{A,0}^n \left[1\ +\ \left(\frac{\mathrm{tr}_A\left[\rho_{A,\phi}\ \rho_{A,0}^{n-1}\right]}{\mathrm{tr}_A\ \rho_{A,0}^n}\ -\ 1\right) n e^{-2\pi E_\phi \beta/L} + \cdots\right] \tag{3.1.4}$$

Since,

$$\log\ (1+x)\ =\ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{3.1.5}$$

Therefore, the leading order thermal correction to Rényi entropy is given by:

$$\delta S_n\ =\ \frac{1}{1-n}\left(\frac{\mathrm{tr}_A\left[\rho_{A,\phi}\ \rho_{A,0}^{n-1}\right]}{\mathrm{tr}_A\ \rho_{A,0}^n}\ -\ 1\right) n e^{-2\pi E_\phi \beta/L} \tag{3.1.6}$$

Now, the proposal is that the term $\frac{\mathrm{tr}_A\left[\rho_{A,\phi}\ \rho_{A,0}^{n-1}\right]}{\mathrm{tr}_A\ \rho_{A,0}^n}$ is proportional to the two-point function on an n-cylinder (similar to as described in the paragraph after eq.(2.4.16)) with one of the cylinders having field ϕ inserted at its end points (points at infinity) and the rest of the $n-1$ cylinders having the vacuum field (Ref. [6]). Thus

$$\frac{\mathrm{tr}_A\left[\rho_{A,\phi}\ \rho_{A,0}^{n-1}\right]}{\mathrm{tr}_A\ \rho_{A,0}^n} = \xi_1\ \langle\phi\left(\infty\right)\phi\left(-\infty\right)\rangle_n \tag{3.1.7}$$

where, ξ_1 is the constant of proportionality. To determine ξ_1, consider the case $n = 1$. Eq.(3.1.7) becomes

$$\frac{\mathrm{tr}_A\ \rho_{A,\phi}}{\mathrm{tr}_A\ \rho_{A,0}} = \xi_1\ \langle\phi\left(\infty\right)\phi\left(-\infty\right)\rangle_1 \tag{3.1.8}$$

where, $\langle\phi\left(\infty\right)\phi\left(-\infty\right)\rangle_1$ is the two-point function on 1-cylinder having operator ϕ inserted at its end points. Now, $\mathrm{tr}_A\ \rho_{A,\phi} = 1$ (since we took ϕ as a normalized eigen function) and $\mathrm{tr}_A\ \rho_{A,0} = 1$, therefore left hand side (LHS) of eq.(3.1.8) is 1. Thus,

$$\xi_1 = \frac{1}{\langle \phi(\infty)\, \phi(-\infty)\rangle_1} \tag{3.1.9}$$

Hence, $\frac{\mathrm{tr}_A\left[\rho_{A,\phi}\ \rho_{A,0}^{n-1}\right]}{\mathrm{tr}_A\ \rho_{A,0}^{n}}$ is given by

$$\frac{\mathrm{tr}\left[\rho_{A,\phi}\ \rho_{A,0}^{n-1}\right]}{\mathrm{tr}\ \rho_{A,0}^{n}} = \frac{\langle \phi(\infty)\, \phi(-\infty)\rangle_n}{\langle \phi(\infty)\, \phi(-\infty)\rangle_1} \tag{3.1.10}$$

Now, $\langle \phi(\infty)\, \phi(-\infty)\rangle_n$ can be evaluated by first mapping the n-cylinder to the complex plane using the uniformization map

$$z = \left(\frac{e^{i2\pi\theta/L} - e^{i2\pi\theta_2/L}}{e^{i2\pi\theta/L} - e^{i2\pi\theta_1/L}}\right)^{1/n} \tag{3.1.11}$$

where, θ is the coordinate on the n-cylinder, z is the coordinate on the complex plane, L is the circumference of each cylinder and θ_1 and θ_2 mark the end points of the interval A i.e. $|\theta_2 - \theta_1| = l$. Thus, under this map, an operator at $\tau = -\infty$ on the j^{th} cylinder is mapped to the location $z_{j,-\infty} = e^{i2\pi(\theta_2 - \theta_1)/nL + 2\pi ij/n}$ and an operator at $\tau = \infty$ on the j^{th} cylinder is mapped to the location $z_{j,\infty} = e^{2\pi ij/n}$. Now, the two-point function is evaluated on the plane

$$\langle \phi(z_{j,\infty}, \bar{z}_{j,\infty})\, \phi(z_{j,-\infty}, \bar{z}_{j,-\infty})\rangle = \frac{1}{\left|z_{j,\infty} - z_{j,-\infty}\right|^{2h_\phi} \left|\bar{z}_{j,\infty} - \bar{z}_{j,-\infty}\right|^{2\bar{h}_\phi}} \tag{3.1.12}$$

where, h_ϕ and $\bar{h}_\phi$ are the holomorphic and anti-holomorphic conformal dimensions of ϕ respectively. This is mapped to $\langle \phi(\infty)\, \phi(-\infty)\rangle_n$ using the transformation law for primary fields under coordinate transformation

$$\begin{aligned}&\langle \phi(\infty)\, \phi(-\infty)\rangle_n \\ &= \left(\frac{dz_{j,\infty}}{d\theta}\bigg|_{\theta(\tau=\infty)} \frac{dz_{j,-\infty}}{d\theta}\bigg|_{\theta(\tau=-\infty)}\right)^{h_\phi} \left(\frac{d\bar{z}_{j,\infty}}{d\bar{\theta}}\bigg|_{\bar{\theta}(\tau=\infty)} \frac{d\bar{z}_{j,-\infty}}{d\bar{\theta}}\bigg|_{\bar{\theta}(\tau=-\infty)}\right)^{\bar{h}_\phi} \langle \phi(z_{j,\infty}, \bar{z}_{j,\infty})\, \phi(z_{j,-\infty}, \bar{z}_{j,-\infty})\rangle\end{aligned} \tag{3.1.13}$$

i.e.

$$\langle \phi(\infty)\, \phi(-\infty) \rangle_n \;=\; \frac{\left(\left.\frac{dz_{j,\infty}}{d\theta}\right|_{\theta(\tau=\infty)} \left.\frac{dz_{j,-\infty}}{d\theta}\right|_{\theta(\tau=-\infty)} \right)^{h_\phi} \left(\left.\frac{d\bar{z}_{j,\infty}}{d\bar{\theta}}\right|_{\bar{\theta}(\tau=\infty)} \left.\frac{d\bar{z}_{j,-\infty}}{d\bar{\theta}}\right|_{\bar{\theta}(\tau=-\infty)} \right)^{\bar{h}_\phi}}{\left| z_{j,\infty} \;-\; z_{j,-\infty} \right|^{2h_\phi} \; \left| \bar{z}_{j,\infty} \;-\; \bar{z}_{j,-\infty} \right|^{2\bar{h}_\phi}} \tag{3.1.14}$$

A similar calculation can be done for $\langle \phi(\infty)\, \phi(-\infty) \rangle_1$. Thus, from eq.(3.1.10), eq.(3.1.11) and eq.(3.1.14) we get

$$\frac{\text{tr}\left[\rho_{A,\phi}\; \rho_{A,0}^{n-1}\right]}{\text{tr}\; \rho_{A,0}^{n}} \;=\; \frac{\langle \phi(\infty)\, \phi(-\infty) \rangle_n}{\langle \phi(\infty)\, \phi(-\infty) \rangle_1} \;=\; \frac{1}{n^{2\Delta_\phi}} \left(\frac{\sin\left(\frac{\pi l}{L}\right)}{\sin\left(\frac{\pi l}{nL}\right)} \right)^{2\Delta_\phi} \tag{3.1.15}$$

where, $\Delta_\phi = h_\phi + \bar{h}_\phi$ is the conformal dimension of ϕ. Therefore, the leading order thermal correction in Rényi entropy is given by

$$\delta S_n = \frac{1}{1-n} \left[\frac{1}{n^{2\Delta_\phi - 1}} \left(\frac{\sin\left(\frac{\pi l}{L}\right)}{\sin\left(\frac{\pi l}{nL}\right)} \right)^{2\Delta_\phi} -\; n \right] \; e^{-2\pi E_\phi \beta / L} \tag{3.1.16}$$

Also, the leading order thermal correction in Entanglement entropy is given by

$$\delta S_E \;=\; \lim_{n \to 1}\; \delta S_n \;=\; 2\Delta_\phi \left[1 - \frac{\pi l}{L} \cot\left(\frac{\pi l}{L} \right) \right] \; e^{-2\pi E_\phi \beta / L} \tag{3.1.17}$$

3.2 Higher Order Thermal Corrections

For computing higher-order correction terms we work with a particular theory, the critical Ising model. It is a relatively simple CFT since it has only three primary fields namely the identity field (vacuum), the σ field (having conformal dimension $h_\sigma = \bar{h}_\sigma = 1/16$) and the

ϵ field (having conformal dimension $h_\epsilon = \bar{h}_\epsilon = 1/2$) and the corresponding secondary fields. Thus, the total density matrix is given by

$$\rho_{tot} = \Big[\ |0\rangle\langle 0| + L_{-2}|0\rangle\langle 0| L_{+2} e^{-4\pi\beta/L} + \cdots + |\sigma\rangle\langle\sigma| e^{-2\pi\Delta_\sigma\beta/L} + L_{-1}|\sigma\rangle\langle\sigma| L_{+1}\ e^{-2\pi(\Delta_\sigma+1)\beta/L} + \cdots + |\epsilon\rangle\langle\epsilon| e^{-2\pi\Delta_\epsilon\beta/L} + L_{-1}|\epsilon\rangle\langle\epsilon| L_{+1}\ e^{-2\pi(\Delta_\epsilon+1)\beta/L} + \cdots\Big] \Big/ \Big[1 + e^{-4\pi\beta/L} + \cdots + e^{-2\pi\Delta_\sigma\beta/L} + e^{-2\pi(\Delta_\sigma+1)\beta/L} + \cdots + e^{-2\pi\Delta_\epsilon\beta/L} + e^{-2\pi(\Delta_\epsilon+1)\beta/L} + \cdots\Big] \tag{3.2.1}$$

where, $\Delta_\sigma = h_\sigma + \bar{h}_\sigma = \frac{1}{8}$ and $\Delta_\epsilon = h_\epsilon + \bar{h}_\epsilon = 1$. Note that the contribution to the density matrix is higher from the primary fields than the secondaries. Thus, in a low temperature expansion, the total density matrix can be approximated by considering only the contributions from the primary fields i.e.

$$\rho_{tot} \sim \frac{|0\rangle\langle 0|\ +\ |\sigma\rangle\langle\sigma| e^{-2\pi\Delta_\sigma\beta/L}\ +\ |\epsilon\rangle\langle\epsilon| e^{-2\pi\Delta_\epsilon\beta/L}}{1\ +\ e^{-2\pi\Delta_\sigma\beta/L}\ +\ e^{-2\pi\Delta_\epsilon\beta/L}} \tag{3.2.2}$$

Therefore, the partial density matrix for a subsystem A is given by

$$\rho_A \sim \frac{\rho_{A,0}\ +\ \rho_{A,\sigma} e^{-2\pi\Delta_\sigma\beta/L}\ +\ \rho_{A,\epsilon} e^{-2\pi\Delta_\epsilon\beta/L}}{1\ +\ e^{-2\pi\Delta_\sigma\beta/L}\ +\ e^{-2\pi\Delta_\epsilon\beta/L}} \tag{3.2.3}$$

where $\rho_{A,i} = \mathrm{tr}_B |i\rangle\langle i|$.

$$\Rightarrow \rho_A^n \sim\ (\rho_{A,0} + \rho_{A,\sigma} e^{-2\pi\Delta_\sigma\beta/L} + \rho_{A,\epsilon} e^{-2\pi\Delta_\epsilon\beta/L})^n\ (1 + e^{-2\pi\Delta_\sigma\beta/L} + e^{-2\pi\Delta_\epsilon\beta/L})^{-n} \tag{3.2.4}$$

On expanding eq.(3.2.4) using binomial expansion, one finds an interesting point that for the critical Ising model, the leading order thermal correction is due to the primary σ field i.e. the term $\rho_{A,\sigma}$ as expected. However, the next correction term is given by the term $\rho^2_{A,\sigma}$ rather than $\rho_{A,\epsilon}$. This is so because $2\Delta_\sigma < \Delta_\epsilon$, hence the contribution due to $\rho^2_{A,\sigma}$ is greater than the contribution due to $\rho_{A,\epsilon}$. Thus, the contribution from the primary ϵ field $\rho_{A,\epsilon}$ is comparable to the contribution from $\rho^8_{A,\sigma}$ and hence doesn't come into play till we consider upto the 7^{th} correction term. Therefore, in order to calculate the leading order thermal correction term, working upto order primary σ, we have

$$\rho_A^n \sim \; (1+B)^{-n} \; \Bigg[\; \rho_{A,0}^n + B \; \left(\; \sum_{i=1}^{n} \; \rho_{A,0}^{i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{n-i} \; \right) \\ + B^2 \; \left(\sum_{1\le i<j\le n} \; \rho_{A,0}^{i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{j-i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{n-j} \right) + \cdots \Bigg] \tag{3.2.5}$$

where $B \; = \; e^{-2\pi\Delta_\sigma\beta/L}$. Therefore the first correction term to the Rényi entropy is given by

$$\delta S_n \equiv \frac{1}{1-n} \; \left[\; B \left\{ \frac{\mathrm{tr}_A \; \left(\; \sum_{i=1}^{n} \; \rho_{A,0}^{i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{n-i} \; \right)}{\mathrm{tr}_A \left(\; \rho_{A,0}^n \right)} - n \right\} \right] \\ = \frac{1}{1-n} \left[\frac{1}{n^{2\Delta_\sigma - 1}} \left(\frac{\sin^{2\Delta_\sigma} \left(\frac{\pi l}{L} \right)}{\sin^{2\Delta_\sigma} \left(\frac{\pi l}{nL} \right)} \right) \; - \; n \; \right] \; e^{-2\pi\Delta_\sigma\beta/L} \tag{3.2.6}$$

Also, the first correction term to the Entanglement entropy is given by

$$\delta S_E \; \equiv \; 2\Delta_\sigma \; \left[\; 1 \; - \; \frac{\pi l}{L} \; \cot \left(\frac{\pi l}{L} \right) \; \right] \; e^{-2\pi\Delta_\sigma\beta/L} \tag{3.2.7}$$

From eq.(3.2.4) and eq.(3.1.5) it is seen that for the critical Ising model, the next correction term to the Rényi entropy is given by

$$\delta S_n \equiv \frac{1}{1-n} \; \Bigg[\; B^2 \Bigg\{ \; \frac{\mathrm{tr}_A \; \left(\; \sum_{1\le i<j\le n} \; \rho_{A,0}^{i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{j-i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{n-j} \right)}{\mathrm{tr}_A \; \rho_{A,0}^n} \\ - \; n \; \frac{\mathrm{tr}_A \; \left(\; \sum_{i=1}^{n} \; \rho_{A,0}^{i-1} \; \rho_{A,\sigma} \; \rho_{A,0}^{n-i} \; \right)}{\mathrm{tr}_A \; \rho_{A,0}^n} + \frac{n(n+1)}{2} - n^2 \left(\frac{\mathrm{tr}_A \left[\rho_{A,\sigma} \rho_{A,0}^{n-1} \right]}{\mathrm{tr} \rho_{A,0}^n} \; - \; 1 \right)^2 \Bigg\} \Bigg] \tag{3.2.8}$$

where $B = e^{-2\pi\Delta_\sigma\beta/L}$. Thus, by cyclicity

$$\delta S_n \equiv \frac{1}{1-n} \left[B^2 \left\{ \frac{\mathrm{tr}_A \left(\sum_{1\leq i<j\leq n} \rho_{A,0}^{i-1} \rho_{A,\sigma} \rho_{A,0}^{j-i-1} \rho_{A,\sigma} \rho_{A,0}^{n-j} \right)}{\mathrm{tr}_A \rho_{A,0}^n} \right. \right. \tag{3.2.9}$$

$$\left. \left. + n^2 \frac{\mathrm{tr}_A \left[\rho_{A,\sigma}\rho_{A,0}^{n-1}\right]}{\mathrm{tr}_A \rho_{A,0}^n} + \frac{n(1-n)}{2} - n^2 \left(\frac{\mathrm{tr}_A \left[\rho_{A,\sigma}\rho_{A,0}^{n-1}\right]}{\mathrm{tr}\rho_{A,0}^n} \right)^2 \right\} \right]$$

Now, we propose that the term $\frac{\mathrm{tr}_A \left(\rho_{A,0}^{j-1} \rho_{A,\sigma} \rho_{A,0}^{k-j-1} \rho_{A,\sigma} \rho_{A,0}^{n-k} \right)}{\mathrm{tr}_A \rho_{A,0}^n}$ is proportional to the four-point function on an n-cylinder (similar to as described in the paragraph after eq.(2.4.16)) with two of the cylinders (j^{th} and k^{th} cylinder) having field σ inserted at its end points (points at infinity) and the rest of the $n-2$ cylinders having the vacuum field. Thus

$$\frac{\mathrm{tr}_A \left(\rho_{A,0}^{j-1} \rho_{A,\sigma} \rho_{A,0}^{k-j-1} \rho_{A,\sigma} \rho_{A,0}^{n-k} \right)}{\mathrm{tr}_A \rho_{A,0}^n} = \xi_2 \left\langle \sigma_j(\infty)\, \sigma_j(-\infty)\, \sigma_k(\infty)\, \sigma_k(-\infty) \right\rangle_n \tag{3.2.10}$$

where, ξ_2 is the constant of proportionality. To determine ξ_2, consider the case $n = 2$ (since the inequality $1 \leq j < k \leq n$ has no integer solutions for j and k for $n = 1$). Thus,

$$\frac{\mathrm{tr}_A \rho_{A,\sigma}^2}{\mathrm{tr}_A \rho_{A,0}^2} = \xi_2 \left\langle \sigma_1(\infty)\, \sigma_1(-\infty)\, \sigma_2(\infty)\, \sigma_2(-\infty) \right\rangle_2 \tag{3.2.11}$$

where $\langle \sigma_1(\infty)\, \sigma_1(-\infty)\, \sigma_2(\infty)\, \sigma_2(-\infty) \rangle_2$ is the four-point function on 2-cylinder with both the cylinders having field σ at its end point. The four-point functions $\langle \sigma_j(\infty)\, \sigma_j(-\infty)\, \sigma_k(\infty)\, \sigma_k(-\infty) \rangle_n$ and $\langle \sigma_1(\infty)\, \sigma_1(-\infty)\, \sigma_2(\infty)\, \sigma_2(-\infty) \rangle_2$ are evaluated using the uniformization map as described in the previous section. The explicit expression of the term $\mathrm{tr}_A \rho_{A,\sigma}^n / \mathrm{tr}_A \rho_{A,0}^n$ has been evaluated in Ref. [7].

$$\frac{\mathrm{tr}_A \rho_{A,\sigma}^n}{\mathrm{tr}_A \rho_{A,0}^n} = \frac{n^{-2n\left(h_\sigma+\bar{h}_\sigma\right)} \langle \prod_{j=1}^n \sigma\left(z_{j,\infty}, \bar{z}_{j,\infty}\right) \sigma\left(z_{j,-\infty}, \bar{z}_{j,-\infty}\right) \rangle}{\langle \sigma(1)\, \sigma\left(e^{i2\pi(\theta_2 - \theta_1)/L}\right) \rangle^n} \tag{3.2.12}$$

where, z's are new coordinates as defined in the previous section. Thus, $\mathrm{tr}_A \rho_{A,\sigma}^2 / \mathrm{tr}_A \rho_{A,0}^2$

is given by

$$\frac{\mathrm{tr}_A\ \rho_{A,\sigma}^2}{\mathrm{tr}_A\ \rho_{A,0}^2} = \frac{2^{-4\left(h_\sigma+\bar{h}_\sigma\right)}\left\langle \sigma\left(z_{1,\infty},\bar{z}_{1,\infty}\right)\sigma\left(z_{1,-\infty},\bar{z}_{1,-\infty}\right)\sigma\left(z_{2,\infty},\bar{z}_{2,\infty}\right)\sigma\left(z_{2,-\infty},\bar{z}_{2,-\infty}\right)\right\rangle}{\left\langle \sigma\left(1\right)\sigma\left(e^{i2\pi(\theta_2\ -\ \theta_1)/L}\right)\right\rangle^2} \quad (3.2.13)$$

From eq.(3.1.13), eq.(3.2.11) and eq.(3.2.13), the normalization constant ξ_2 is given by

$$\xi_2 = \frac{2^{-4\left(h_\sigma+\bar{h}_\sigma\right)}}{\left\langle \sigma\left(1\right)\sigma\left(e^{i2\pi(\theta_2\ -\ \theta_1)/L}\right)\right\rangle^2}\left[\left(\frac{dz_{1,\infty}}{d\theta}\ \frac{dz_{2,\infty}}{d\theta}\right)\Bigg|_{\theta(\tau=\infty)}\left(\frac{dz_{1,-\infty}}{d\theta}\frac{dz_{2,-\infty}}{d\theta}\right)\Bigg|_{\theta(\tau=-\infty)}\right]^{-h_\sigma}$$
$$\left[\left(\frac{d\bar{z}_{1,\infty}}{d\bar{\theta}}\ \frac{d\bar{z}_{2,\infty}}{d\bar{\theta}}\right)\Bigg|_{\bar{\theta}(\tau=\infty)}\left(\frac{d\bar{z}_{1,-\infty}}{d\bar{\theta}}\ \frac{d\bar{z}_{2,-\infty}}{d\bar{\theta}}\right)\Bigg|_{\bar{\theta}(\tau=-\infty)}\right]^{-\bar{h}_\sigma} \quad (3.2.14)$$

or equivalently

$$\xi_2 = \frac{2^{-4\left(h_\sigma+\bar{h}_\sigma\right)}}{\left\langle \sigma\left(\infty\right)\sigma\left(-\infty\right)\right\rangle_1^2} \quad (3.2.15)$$

where, $\left\langle \sigma\left(\infty\right)\sigma\left(-\infty\right)\right\rangle_1$ is the two point function on 1-cylinder having field σ at its end points. Thus,

$$\frac{\mathrm{tr}_A\ \left(\ \rho_{A,0}^{j-1}\ \rho_{A,\sigma}\ \rho_{A,0}^{k-j-1}\ \ \rho_{A,\sigma}\ \rho_{A,0}^{n-k}\right)}{\mathrm{tr}_A\ \rho_{A,0}^n} = \frac{2^{-4\left(h_\sigma+\bar{h}_\sigma\right)}\ \left\langle \sigma_j\left(\infty\right)\sigma_j\left(-\infty\right)\sigma_k\left(\infty\right)\sigma_k\left(-\infty\right)\right\rangle_n}{\left\langle \sigma\left(\infty\right)\sigma\left(-\infty\right)\right\rangle_1^2} \quad (3.2.16)$$

The four-point function for σ field of the Ising model is given in Ref.[8]. Using the formula presented there in conjunction with the uniformization map

$$z^{(n)} = \left(\frac{e^{i2\pi\theta/L}\ \ -\ \ e^{i\theta_2}}{e^{i2\pi\theta/L}\ \ -\ \ e^{i\theta_1}}\right)^{1/n} \quad (3.2.17)$$

where, $|\theta_2 - \theta_1| = \frac{2\pi l}{L}$, gives the four-point function

$$\langle\sigma\left(z^{(n)}_{j,\infty}, \bar{z}^{(n)}_{j,\infty}\right)\sigma\left(z^{(n)}_{j,-\infty}, \bar{z}^{(n)}_{j,-\infty}\right)\sigma\left(z^{(n)}_{k,\infty}, \bar{z}^{(n)}_{k,\infty}\right)\sigma\left(z^{(n)}_{k,-\infty}, \bar{z}^{(n)}_{k,-\infty}\right)\rangle \tag{3.2.18}$$

$$= \left\{\left[\left(-1+\sqrt{2}\sqrt{\frac{\sin^2\left(\frac{\pi(j-k)}{n}\right)}{\cos\left(\frac{\theta_2-\theta_1}{n}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)}}\right)\left(-1+\sqrt{2}\sqrt{\frac{\sin^2\left(\frac{\pi(j-k)}{n}\right)}{\cos\left(\frac{\bar{\theta}_2-\bar{\theta}_1}{n}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)}}\right)\right]^{1/2}\right.$$

$$\left.+\left[\left(1+\sqrt{2}\sqrt{\frac{\sin^2\left(\frac{\pi(j-k)}{n}\right)}{\cos\left(\frac{\theta_2-\theta_1}{n}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)}}\right)\left(1+\sqrt{2}\sqrt{\frac{\sin^2\left(\frac{\pi(j-k)}{n}\right)}{\cos\left(\frac{\bar{\theta}_2-\bar{\theta}_1}{n}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)}}\right)\right]^{1/2}\right\}$$

$$/\left\{2^{7/4}\left(\frac{\sin^2\left(\frac{\bar{\theta}_2-\bar{\theta}_1}{2n}\right)\sin^2\left(\frac{\theta_2-\theta_1}{2n}\right)e^{i\left(\theta_2-\theta_1-\bar{\theta}_2+\bar{\theta}_1\right)}\sin^4\left(\frac{\pi(j-k)}{n}\right)}{\left(\cos\left(\frac{\theta_2-\theta_1}{n}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)\right)\left(\cos\left(\frac{\bar{\theta}_2-\bar{\theta}_1}{n}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)\right)}\right)^{1/8}\right\}$$

where, $z^{(n)}_{j,\infty} = e^{i(\theta_2-\theta_1)/n\,+\,2\pi ij/n}$, $z^{(n)}_{j,-\infty} = e^{2\pi ij/n}$, $z^{(n)}_{k,\infty} = e^{i(\theta_2-\theta_1)/n\,+\,2\pi ik/n}$ and $z^{(n)}_{k,-\infty} = e^{2\pi ik/n}$. A further simplification yields the result,

$$\langle\sigma\left(z^{(n)}_{j,\infty}, \bar{z}^{(n)}_{j,\infty}\right)\sigma\left(z^{(n)}_{j,-\infty}, \bar{z}^{(n)}_{j,-\infty}\right)\sigma\left(z^{(n)}_{k,\infty}, \bar{z}^{(n)}_{k,\infty}\right)\sigma\left(z^{(n)}_{k,-\infty}, \bar{z}^{(n)}_{k,-\infty}\right)\rangle$$
$$= \left(\frac{\sin\left(\frac{\pi(j-k)}{n}\right)}{\sin\left(\frac{\pi l}{nL}\right)}\right)^{1/2}\frac{1}{\left(2\left(\cos\left(\frac{2\pi l}{nL}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)\right)\right)^{1/4}} \tag{3.2.19}$$

Using the identity

$$\frac{dz^{(n)}/d\theta}{dz^{(1)}/d\theta} = \frac{1}{n}\frac{z^{(n)}}{z^{(1)}} \tag{3.2.20}$$

we get,

$$\frac{\langle \sigma_j(\infty)\,\sigma_j(-\infty)\,\sigma_k(\infty)\,\sigma_k(-\infty)\rangle_n}{\langle \sigma(\infty)\,\sigma(-\infty)\rangle_1^2}$$
$$= \frac{1}{n^{4\left(h_\sigma+\bar{h}_\sigma\right)}}\left[\frac{z^{(n)}_{j,\infty}}{z^{(1)}_{j,\infty}}\,\frac{z^{(n)}_{j,-\infty}}{z^{(1)}_{j,-\infty}}\,\frac{z^{(n)}_{k,\infty}}{z^{(1)}_{k,\infty}}\,\frac{z^{(n)}_{k,-\infty}}{z^{(1)}_{k,-\infty}}\right]^{h_\sigma}\left[\frac{\bar{z}^{(n)}_{j,\infty}}{\bar{z}^{(1)}_{j,\infty}}\,\frac{\bar{z}^{(n)}_{j,-\infty}}{\bar{z}^{(1)}_{j,-\infty}}\,\frac{\bar{z}^{(n)}_{k,\infty}}{\bar{z}^{(1)}_{k,\infty}}\,\frac{\bar{z}^{(n)}_{k,-\infty}}{\bar{z}^{(1)}_{k,-\infty}}\right]^{\bar{h}_\sigma} \tag{3.2.21}$$
$$\frac{\langle \sigma\left(z^{(n)}_{j,\infty},\bar{z}^{(n)}_{j,\infty}\right)\sigma\left(z^{(n)}_{j,-\infty},\bar{z}^{(n)}_{j,-\infty}\right)\sigma\left(z^{(n)}_{k,\infty},\bar{z}^{(n)}_{k,\infty}\right)\sigma\left(z^{(n)}_{k,-\infty},\bar{z}^{(n)}_{k,-\infty}\right)\rangle}{\langle \sigma\left(z^{(1)}_{j,\infty},\bar{z}^{(1)}_{j,\infty}\right)\sigma\left(z^{(1)}_{j,-\infty},\bar{z}^{(1)}_{j,-\infty}\right)\rangle\,\langle \sigma\left(z^{(1)}_{k,\infty},\bar{z}^{(1)}_{k,\infty}\right)\sigma\left(z^{(1)}_{k,-\infty},\bar{z}^{(1)}_{k,-\infty}\right)\rangle}$$

where, $z^{(1)}_{j,\infty} = z^{(1)}_{k,\infty} = e^{i(\theta_2-\theta_1)} = e^{i2\pi l/L}$ and $z^{(1)}_{j,-\infty} = z^{(1)}_{k,-\infty} = 1$. The two-point functions are evaluated to be

$$\langle \sigma\left(z^{(1)}_{j,\infty},\bar{z}^{(1)}_{j,\infty}\right)\sigma\left(z^{(1)}_{j,-\infty},\bar{z}^{(1)}_{j,-\infty}\right)\rangle = \frac{1}{\left(2\,\sin\left(\frac{\pi l}{L}\right)\right)^{1/4}} = \langle \sigma\left(z^{(1)}_{k,\infty},\bar{z}^{(1)}_{k,\infty}\right)\sigma\left(z^{(1)}_{k,-\infty},\bar{z}^{(1)}_{k,-\infty}\right)\rangle \tag{3.2.22}$$

From eq.(3.2.19) and eq.(3.2.22), we get

$$\frac{\langle \sigma\left(z^{(n)}_{j,\infty},\bar{z}^{(n)}_{j,\infty}\right)\sigma\left(z^{(n)}_{j,-\infty},\bar{z}^{(n)}_{j,-\infty}\right)\sigma\left(z^{(n)}_{k,\infty},\bar{z}^{(n)}_{k,\infty}\right)\sigma\left(z^{(n)}_{k,-\infty},\bar{z}^{(n)}_{k,-\infty}\right)\rangle}{\langle \sigma\left(z^{(1)}_{j,\infty},\bar{z}^{(1)}_{j,\infty}\right)\sigma\left(z^{(1)}_{j,-\infty},\bar{z}^{(1)}_{j,-\infty}\right)\rangle\,\langle \sigma\left(z^{(1)}_{k,\infty},\bar{z}^{(1)}_{k,\infty}\right)\sigma\left(z^{(1)}_{k,-\infty},\bar{z}^{(1)}_{k,-\infty}\right)\rangle}$$
$$= \frac{2^{1/4}\left(\sin\left(\frac{\pi(j-k)}{n}\right)\right)^{1/2}}{\left(\cos\left(\frac{2\pi l}{nL}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)\right)^{1/4}} \tag{3.2.23}$$

Therefore,

$$\frac{\langle \sigma_j(\infty)\,\sigma_j(-\infty)\,\sigma_k(\infty)\,\sigma_k(-\infty)\rangle_n}{\langle \sigma(\infty)\,\sigma(-\infty)\rangle_1^2} = \frac{2^{1/4}}{n^{1/2}}\,\frac{\left(\sin\left(\frac{\pi(j-k)}{n}\right)\right)^{1/2}}{\left(\cos\left(\frac{2\pi l}{nL}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)\right)^{1/4}} \tag{3.2.24}$$

Thus, the second correction term to the Rényi entropy for the critical Ising model is evaluated using eq.(3.2.9), eq.(3.1.15), eq.(3.2.16) and eq.(3.2.24).

$$\delta S_n = \frac{e^{-\pi\beta/2L}}{1-n}\left[\sum_{1\le j<k\le n}\frac{1}{2^{1/4}\ n^{1/2}}\ \frac{\left(\sin\left(\frac{\pi(j-k)}{n}\right)\right)^{1/2}}{\left(\cos\left(\frac{2\pi l}{nL}\right)-\cos\left(\frac{2\pi(j-k)}{n}\right)\right)^{1/4}}\right. \tag{3.2.25}$$

$$\left. +\ n^{7/4}\left(\frac{\sin\left(\frac{\pi l}{L}\right)}{\sin\left(\frac{\pi l}{nL}\right)}\right)^{1/4} + \frac{n\,(1-n)}{2} - n^{3/2}\left(\frac{\sin\left(\frac{\pi l}{L}\right)}{\sin\left(\frac{\pi l}{nL}\right)}\right)^{1/2}\right]$$

Chapter 4

Conclusions

For critical Ising model, it is observed that the calculations required for obtaining the explicit form of the leading order correction to Rényi entropy are the two-point correlation function $\langle \sigma(\infty)\,\sigma(-\infty)\rangle$ and the constant of proportionality ξ_1. Moreover, to calculate ξ_1, the term required to be evaluated is $\frac{\mathrm{tr}_A\ \rho_{A,\sigma}}{\mathrm{tr}_A\ \rho_{A,0}}$.

Similarly, the calculations required for obtaining the explicit form of the second correction term to the Rényi entropy are the four-point correlation function $\langle \sigma_j(\infty)\,\sigma_j(-\infty)\,\sigma_k(\infty)\,\sigma_k(-\infty)\rangle$ (where the σ field is inserted at the end points of the j^{th} and k^{th} cylinders) and the constant of proportionality ξ_2. Also, to calculate ξ_1, the term required to be evaluated is $\frac{\mathrm{tr}_A\ \rho^2_{A,\sigma}}{\mathrm{tr}_A\ \rho^2_{A,0}}$.

This can be further generalised to any i^{th} correction term ($i \leq 7$). The calculations required for obtaining the explicit form of the i^{th} correction term to the Rényi entropy are the $2i$-point correlation function $\langle \sigma_{j_1}(\infty)\,\sigma_{j_1}(-\infty)\,\sigma_{j_2}(\infty)\,\sigma_{j_2}(-\infty)\cdots\sigma_{j_i}(\infty)\,\sigma_{j_i}(-\infty)\rangle$ (where each of the j_a's are the indices of the cylinders, at the end points of which the σ field is inserted) and the constant of proportionality ξ_i (eq.(4.0.1)). All the other quantities involved are calculated inductively i.e. they are already calculated for the $i-1^{th}$ correction term and are hence known.

$$\frac{\mathrm{tr}_A\ \rho^i_{A,\sigma}}{\mathrm{tr}_A\ \rho^i_{A,0}} = \xi_i\ \langle \sigma_1(\infty)\,\sigma_1(-\infty)\,\sigma_2(\infty)\,\sigma_2(-\infty)\cdots\sigma_i(\infty)\,\sigma_i(-\infty)\rangle_i \tag{4.0.1}$$

The $2i$-point functions $\langle \sigma_{j_1}(\infty)\, \sigma_{j_1}(-\infty)\, \sigma_{j_2}(\infty)\, \sigma_{j_2}(-\infty) \cdots \sigma_{j_i}(\infty)\, \sigma_{j_i}(-\infty)\rangle$ and $\langle \sigma_1(\infty)\, \sigma_1(-\infty)\, \sigma_2(\infty)\, \sigma_2(-\infty) \cdots \sigma_i(\infty)\, \sigma_i(-\infty)\rangle_i$ can be evaluated using the uniformization map, given that the $2i$-point function of the theory on a plane is known. The calculation of ξ_i is dependent on the evaluation of the term $\frac{\mathrm{tr}_A\ \rho^i_{A,\sigma}}{\mathrm{tr}_A\ \rho^i_{A,0}}$.

For the 8^{th} correction term, in addition to the σ field, there is also a contribution from the primary field ϵ. Therefore, the correction term will have terms containing two-point, four-point, six-point ... upto sixteen-point function of σ field and also the two-point function of the ϵ field. Thereafter the correction terms will have contributions from the secondaries of both σ and ϵ fields. The 16^{th} onward correction terms will have contributions from the secondaries of all three fields of the theory namely the identity, σ and ϵ fields.

The above done calculations and analyses are applicable to any general CFT provided all the fields present in the theory are known and the calculations are done accordingly, taking into account the conformal dimensions of the fields. It is important to note that fields having lower dimensions will contribute more to the correction of the entropy.

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