On the Björling problem for Born-Infeld solitons and the interpolation problem for timelike minimal surfaces

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled "On the Björling problem for Born-Infeld solitons and the interpolation problem for timelike minimal surfaces " towards the partial fulfillment of the BS-MS dual degree program at the Indian Institute of Science Education and Research, Pune represents work carried out by Sreedev M at Indian Institute of Science Education and Research under the supervision of Dr.Rukmini Dey, associate professor, ICTS-TIFR Bangalore, department of mathematics, and Dr. Anisa Chorwadwala, assistant professor, IISER Pune, department of mathematics during the academic year 2022-2023.

Rukmini Dey

Dr. Rukmini Dey and Dr. Anisa Chorwadwala

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Dr. Mainak Poddar

This thesis is dedicated to my little sister

Declaration

I hereby declare that the matter embodied in the report entitled "On the Björling problem for Born-Infeld solitons and the interpolation problem for timelike minimal surfaces " are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rukmini Dey and Dr. Anisa Chorwadwala and the same has not been submitted elsewhere for any other degree.

Sreedev M

Acknowledgments

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I want to express my gratitude to ICTS-TIFR Bangalore for their hospitality and for giving me a chance to meet so many outstanding researchers from all around the world. I would also like to thank Prof. Mainak Poddar and Prof. Anisa Chorwadwala for their love and support. I'm beyond grateful to the maths club coordinators in IISER Pune, Heerak, Prasanna, and Padmapriya for giving me a lot of chances with their events during this period for me to explore a different side of myself. Finally, as an individual, I'm forever indebted to all the wonderful people who I met in IISER Pune, who showed me a lot of beautiful and unique things in life and loved me when they had no reason to!

Abstract

Minimal surfaces are zero mean curvature surfaces that appear in nature as idealized soap films. The minimal surface theory is filled with lots of beautiful geometric results, bridging various mathematical branches such as complex analysis, functional analysis, PDE theory, and so on.

This thesis is a combination of mainly two completed research works and one ongoing research work about zero mean curvature surfaces. The Björling problem and its solution is a well-known result for minimal surfaces in Euclidean three-space. The minimal surface equation is similar to the Born-Infeld equation, which is naturally studied in physics. For the first research work, we ask the question of the Björling problem for Born-Infeld solitons. This begins with the case of locally Born-Infeld soliton surfaces and later moves on to graph-like surfaces. We also present some results about their representation formulae.

The singular Björling problem and its solution for timelike minimal surfaces is another famous result in minimal surface theory. In the second research work, we give different proofs of this theorem using split-harmonic maps. This is motivated by a similar solution of the singular Björling problem for maximal surfaces using harmonic maps. As an application, we study the problem of interpolating a given split-Fourier curve to a point by a timelike minimal surface. This is inspired by an analogous result for maximal surfaces. We also solve the problem of interpolating a given split-Fourier curve to another specified split-Fourier curve by a timelike minimal surface.

The third and ongoing research work is about understanding the geometry behind the interpolation problems of minimal surfaces. Jesse Douglas earlier gave some existence results for interpolation problems of minimal surfaces, based on area. We try to make these results more concrete by studying the relationship between the existence of minimal surfaces interpolating two curves with the distance between them and giving the explicit parametrization of such minimal surfaces.

Contents

Introduction

0.1 Original contribution

In this section, We first list the preprints which were put up on arXiv as a part of this thesis.

0.1.1 List of preprints coming from this thesis

1. Manikoth, Sreedev. On the Bjorling problem for Born-Infeld solitons, 2022. https://arxiv.org/abs/2210.08752.
2. Manikoth, Sreedev. Split-harmonic maps and the interpolation problem for timelike minimal surfaces, 2022. https://arxiv.org/abs/2210.17137.

0.1.2 Ongoing research work

1. Dey. Rukmini, Manikoth. Sreedev, Relative geometry of curves and the interpolation problems of minimal surfaces. (In progress)

The above list contains all the original contributions coming from this thesis. These preprints and ongoing works are described in Chapters 4, 5 and 6. We start the thesis by giving a flavor of minimal surface theory in both Euclidean-three space and the Lorentz-Minkowksi space. The first three chapters are expository, serving this purpose.

The Swedish mathematician Emmanuel Björling asked the following problem for minimal surfaces: Given a real analytic curve α and a real analytic normal vector field n along the curve, can we find a minimal surface X which contains α and its tangent planes along α are specified by n. Björling himself solves this problem in [4] and the solution was later again investigated by Herman Schwarz.

In the fourth chapter, we ask the same Björling problem for a different kind of surfaces called Born-Infeld solitons. We present our solution to the Björling problem of locally Born-Infeld soliton surfaces and also a graph of a function kind of Born-Infeld solitons. We also present a few results about their representation formula given by Barbishov and Chernikov and also a few results using E A Paxton's results about timelike minimal surfaces over compact sets. This is a semi-expository chapter and this led to my first preprint on arXiv, which can be found at [21].

For a specified Jordan curve c in the Euclidean space, the Platue problem asks one to find a minimal surface X with $\partial X = c$(here ∂X denotes the boundary of the surface X). This problem was solved independently in the 1930s by Jesse Douglas(in [13]) and Tibor Rado (In [25])). Douglas in the same period also solved this problem for two given contours(In [14]). From there on various mathematicians were interested in studying different variations of this problem using techniques involving a diverse set of mathematics branches. For instance, we note that in [26], Illia Vekua used the implicit function theorem for Banach spaces to study this problem for curves that are sufficiently close to a plane curve. Similarly, Rukmini Dey, Rahul Kumar Singh, and Pradip Kumar study this problem using the Inverse function theorem for Banach spaces in [9], [10] and using harmonic functions and Fourier analysis in [7]. Mathematicians were also interested in studying this problem for specific examples(in Lorentz-Minkowski space, using maximal surfaces) which can be seen at [16].

In the fifth chapter, we introduce split-harmonic maps, and split-Fourier curves, and using them we solve the singular Björling problem and the interpolation problem for timelike minimal surfaces. We interpolate a split-Fourier curve to a point and another arbitrary split-Fourier curve and give certain algebraic conditions on the split-Fourier coefficients, which if satisfied ensures that the curves can be interpolated. This work was inspired by [7] and searches for similar techniques in split complex numbers. This is a completely original work, which leads to my second preprint which can be seen at [22].

In the sixth chapter, we give a slight introduction to our current and ongoing research work. This is a work in progress, which is titled as in [11].

We also note that all pictures in this thesis are created using Desmos and Geogebra.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we list some basic definitions and formulas which will be used throughout this thesis. Since we are mostly dealing with Euclidean-three space or Lorentz-Minkoswski space, the definitions might look much simpler and shorter than the ones introduced in a typical Riemannian geometry course.

1.2 Basic definitions and formulae

Definition 1.2.1 (Surfaces). *By a C^n surface, we mean a function $X : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ which is C^n.*

From now on, unless specified otherwise we will always assume our surfaces are at least C^2.

Definition 1.2.2 (Regular points). *point $w \in \Omega$ of a surface X is said to be a regular point if its jacobian $DX(w)$ has rank 2. If the rank of the jacobian is ≤ 1 we say, the point w is a singular point.*

Definition 1.2.3 (Tangent space). *We define tangent space of a surface $X : \Omega \to \mathbb{R}^3$ at a regular point $w \in \Omega$, to be the plane, span$\{X_u(w), X_v(w)\}$ in $\mathbb{R}^3$.*

Definition 1.2.4 (Surface normal). *At a regular point $w \in \Omega$ we define, surface normal $N(w)$ of the surface to be,*

$$N(w) = \frac{X_u(w) \times X_v(w)}{|X_u(w) \times X_v(w)|}$$

Definition 1.2.5 (The Gauss and mean curvature formulae). *we fix a regular point w. To simplify the formula, we use the notation N for $N(w)$ and X_u for $X_u(w)$. Then the mean curvature $H(w)$ which will denote by H is given by,*

$$H = \frac{\langle X_v, X_v\rangle\langle N, X_{uu}\rangle + \langle N, X_{vv}\rangle\langle X_u, X_u\rangle - 2\langle X_u, X_v\rangle\langle N, X_{uv}\rangle}{2(\langle X_u, X_u\rangle\langle X_v, X_v\rangle - \langle X_u, X_v\rangle^2)}$$

The Gauss curvature $K(w)$ which we will denote by K is given by,

$$K = \frac{\langle N, X_{uu}\rangle\langle N, X_{vv}\rangle - \langle N, X_{uv}\rangle^2}{\langle X_u, X_u\rangle\langle X_v, X_v\rangle - \langle X_u, X_v\rangle^2}$$

We refer to [12] for the details of these definitions and formulae. Roughly speaking, replacing the Euclidean inner product in the above equations with the Lorentz-Minkowski bilinear form leads to the formulae for mean curvature and Gauss curvature in the Lorentz-Minkowski space. For similar definitions in Lorentz-Minkowski space, we refer to [20].

Other basic theorems from differential geometry such as Weingarten equations and so on, will be referred from [12] whenever required.

Chapter 2

Minimal surfaces in the three-dimensional Euclidean space

2.1 Introduction

In this chapter, we introduce the theory of minimal surfaces in Euclidean-three space. This is an expository chapter that is mostly based on [12]. We start by showing that for regular surfaces of class C^2, the property local area minimizing is equivalent to having zero mean curvature everywhere. Soon we would see that over simply connected domains this is equivalent to saying the surface is the real part of certain holomorphic curves called isotropic curves. Using this correspondence between complex analysis and geometry, we deduce the Weirstrass-Enneper representation and the Björling representation formula of minimal surfaces.

2.2 Locally area minimizing property and mean curvature

Let

$$X : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$$

be a class C^2, regular surface. We introduce a few terms.

Definition 2.2.1 (Variation of a surface)**.** *Fix an $\epsilon_0 > 0$. A variation of X is a family of surfaces,*

$$Z : \overline{\Omega} \times (-\epsilon_0, \epsilon_0) \to \mathbb{R}^3$$

which is of class C^2 with satisfying,

$$Z(w, 0) = X(w)$$

for all $w \in \Omega$.

For a fixed $\epsilon \in (-\epsilon_0, \epsilon_0)$, we can consider the surface $U : \Omega \to \mathbb{R}^3$ defined by, $U(w) = Z(w, \epsilon)$. We denote this surface by $Z(., \epsilon)$. We denote it's area by $A_\Omega(Z(., \epsilon))$.

Definition 2.2.2 (First variation of a surface)**.** *The vector field,*

$$Y(w) = \frac{\partial}{\partial \epsilon} Z(w, \epsilon)|_{\epsilon=0}$$

is defined to be the first variation of Z

Definition 2.2.3 (First variation of area of a surface)**.** *We define the first variation of the area at X, in direction of the vector field Y to be,*

$$\delta_\Omega A(X, Y) = \frac{d}{d\epsilon} A_\Omega(Z(., \epsilon))|_{\epsilon=0}$$

Now we state the main result which relates mean curvature with local area minimizing property.

Theorem 2.2.1. *The first variation of the area in the direction of Y vanishes for all compactly supported vector fields Y on Ω $\iff$ mean curvature of X, H is identically zero.*

Proof. We first prove in the forward direction. Since Z is of class C^2, by Taylor expansion(around 0 for ϵ) we can write,

$$Z(w, \epsilon) = Z(w, 0) + \epsilon Y(w) + \epsilon^2 R(w, \epsilon)$$

Where R is a reminder term satisfying $lim_{\epsilon \to 0} R(w, \epsilon) = O(1)$.

Let N be the surface normal of X. Now using Gauss and Weingarten equations one can calculate the area of $Z(w, \epsilon)$ and find the first variation of area, in terms of mean curvature. We refer to [12] pages 54-56, for more details. We can show,

$$\delta_\Omega A(X, Y) = -2 \int \langle Y, N \rangle H dA$$

for all compactly supported vector fields Y. Since $\langle Y, N \rangle$ can be any arbitrary compactly supported function(for different choices of the compactly supported vector fields Y), by the fundamental theorem of the calculus of variations we conclude that H must be identically zero.

Proof of the backward direction follows from the fact that H being identically zero implies,

$$\delta_\Omega A(X, Y) = -2 \int \langle Y, N \rangle H dA$$

is zero for every vector field Y. □

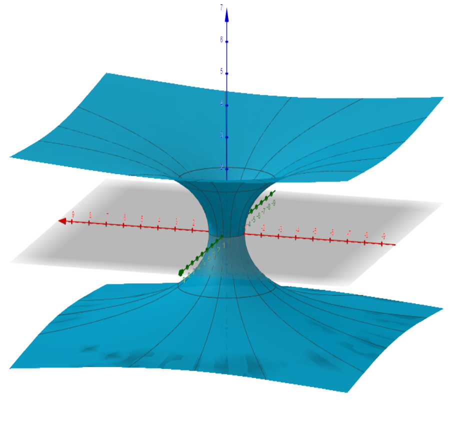
Thus for regular surfaces of class C^2, the local area minimizing property is equivalent to having zero mean curvature. This motivates the following definition.

Definition 2.2.4 (Minimal surfaces)**.** *We say that a regular surface of class C^2, $X : \Omega \to \mathbb{R}^3$ is a minimal surface if its mean curvature is identically zero.*

Example 1. *The Catenoid,*

$$(x(u, v), y(u, v), z(u, v)) = (\alpha \cosh u \cos v, -\alpha \cosh u \sin v, \alpha u)$$

For a fixed $\alpha \neq 0$ and $-\infty < u < \infty$, $0 \leq v < 2\pi$ is an example of a regular surface with zero mean curvature.



Now given a surface $X(x, y) = (x, y, \psi(x, y))$, which is graph of a function ψ, we calculate its mean curvature and get,

$$H = \frac{(1 + \psi_y^2)\psi_{xx} - 2\psi_x\psi_y\psi_{xy} + (1 + \psi_x^2)\psi_{yy}}{2(1 + \psi_x^2 + \psi_y^2)^{\frac{3}{2}}}$$

This gives us the following result.

Theorem 2.2.2 (minimal surface equation). *A surface $X : \Omega \subset \mathbb{R}^2$, $X(x, y) = (x, y, \psi(x, y))$ which is graph of a function ψ, is a minimal surface if and only if ψ satisfies the equation,*

$$(1 + \psi_y^2)\psi_{xx} - 2\psi_x\psi_y\psi_{xy} + (1 + \psi_x^2)\psi_{yy} = 0.$$

This equation is called the minimal surface equation.

Now we state Bernstein's Theorem. It shows that when $\Omega = R^2$, the only solutions to minimal surface equations are affine linear maps.

Theorem 2.2.3 (Bernstein's Theorem). *If $\psi : \mathbb{R}^2 \to \mathbb{R}^3$ is a solution to the minimal surface equation, then ψ is of the form,*

$$\psi(x, y) = ax + by + c$$

for some real numbers a, b and c.

We refer to [12], pages 65-67 for the proof.

2.3 Relationship with complex analysis; representation formulas

We define conformal parameters.

Definition 2.3.1 (conformal parameters)**.** *a C^2 surface $X : \Omega \to \mathbb{R}^2$ is said to be represented by conformal parameters, if it satisfies the relations,*

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$$

$$\langle X_u, X_v \rangle = 0$$

In fact, given any regular surface, there exists a parameterization where it is represented by conformal parameters. Also, it turns out that in such a parameterization minimal surfaces satisfy

$$\Delta X = 0$$

We refer to [12], pages 74-76 for details.

Using this property, we would generalize the definition of minimal surfaces, to include singularities.

Definition 2.3.2 (Minimal surface definition in general)**.** *A C^2 surface $X : \Omega \to \mathbb{R}^3$, is said to be a minimal surface if it satisfies,*

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$$

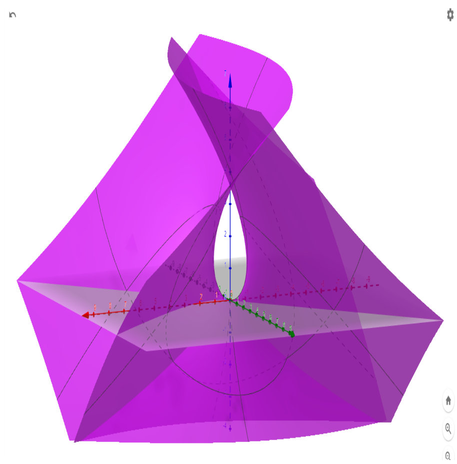
$$\langle X_u, X_v \rangle = 0$$

$$\Delta X = X_{uu} + X_{vv} = 0$$

Example 2. *The Enneper surface,*

$$(x(u,v), y(u,v), z(u,v)) = \left(u - \frac{1}{3}u^3 + uv^2, -u - u^2 v + \frac{1}{3}v^3, u^2 - v^2 \right)$$

with u,v in $\mathbb{R}$ is an example of a minimal surface with singularies.



Now given any minimal surface X on a simply connected domain, we can always find a different minimal surface X^* called the adjoint surface which satisfies,

$$X_u = -X_v^*$$

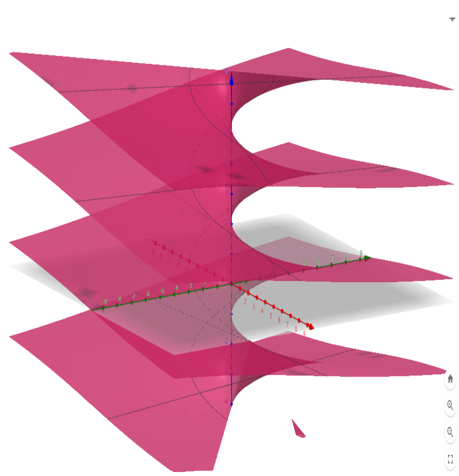
$$X_v = X_u^*$$

We refer to [12] pages 61-64 and page 74 for the details of the proof.

Example 3. *The adjoint surface of the Catenoid(in Example 1) is the Helicoid which is given by,*

$$(x(u,v), y(u,v), z(u,v)) = (\alpha \sinh u \sin v, \alpha \sinh u \cos v, \alpha v)$$

With $-\infty < u < \infty$ *and* $0 \leq v < 2\pi$*.*



We define the isotropic curves in $\mathbb{C}^3$

Definition 2.3.3 (isotropic curves)**.** *A holomorphic curve* $f : \Omega \subset \mathbb{C} \to \mathbb{C}^3$ *is said to be an isotropic curve if its velocity* $f^{'} = (\phi_1, \phi_2, \phi_3)$ *satisfy,*

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$$

We from now on will denote the quantity

$$\phi_1^2 + \phi_2^2 + \phi_3^2$$

by

$$\langle f^{'}, f^{'} \rangle_e.$$

Now we prove the following important theorem

Theorem 2.3.1 (Minimal surfaces and isotropic curves)**.** *A surface* $X : \Omega \to \mathbb{R}^3$ *on a simply connected domain* Ω*, is a minimal surface if and only if there exists an isotropic curve* $f : \Omega \to \mathbb{C}^3$ *with*

$$X = Ref.$$

Proof. We would show the proof in the forward direction, we know that if X is a minimal surface on a simply connected domain, its adjoint surface X^* exists. Then

$$f = X + iX^*$$

is a holomorphic curve as, X^* satisfy Cauchy Riemann equations with X.After doing calculations we get,

$$\langle f^{'}, f^{'} \rangle_e = \langle X_u, X_u \rangle - \langle X_v, X_v \rangle + i\langle X_u, X_v \rangle$$

Now as X is represented by conformal parameters,

$$\langle f^{'}, f^{'} \rangle_e = 0$$

Thus f is an isotropic curve with $X = Ref$. The proof of the backward direction is similar.

□

Now we characterize all the isotropic curves in a simply connected domain Ω, using pairs of meromorphic and holomorphic functions.

Theorem 2.3.2. *Let μ be a holomorphic and ν be a meromorphic function on Ω with $\mu\nu^2$ being holomorphic. Also, we assume μ is not identically zero on Ω. Then all the holomorphic curves f with $f^{'}$ defined by,*

$$f^{'} = (\phi_1, \phi_2, \phi_3) = (\frac{\mu(1-\nu^2)}{2}, \frac{i\mu(1+\nu^2)}{2}, \mu\nu)$$

are isotropic curves. Conversely given any non-constant isotropic curve f on Ω, one can find a pair of meromorphic and holomorphic functions ν, μ with the above properties and $f^{'}$ represented in the above form.

Proof. We first prove in the following direction. As $f^{'}$ is holomorphic, so is f. Also through computations, we can show that,

$$\langle f^{'}, f^{'} \rangle_e = 0.$$

Thus any f defined this way is an isotropic curve.

To prove in the backward direction suppose f is an isotropic curve on Ω. Then $f^{'} = (\phi_1, \phi_2, \phi_3)$ satisfy,

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 0$$

$$(\phi_1 + i\phi_2)(\phi_1 - i\phi_2) + \phi_3^2 = 0$$

If $\phi_1 - i\phi_2 = 0$ is identically, so will be ϕ_3 by the above equation. Then $f^{'}$ will identically zero, implying f is a constant which we assumed to be not the case. Thus $\phi_1^{'} - i\phi_2$ is not identically zero. So we take

$$\nu = \frac{\phi_3}{\phi_1 - i\phi_2}$$

and $\mu = \phi_1 - i\phi_2$ be the pair of meromorphic and holomorphic functions. Now using calculations we can show,

$$(\phi_1, \phi_2, \phi_3) = (\frac{\mu(1-\nu^2)}{2}, \frac{i\mu(1+\nu^2)}{2}, \mu\nu).$$

□

these results lead us to the Weirstrass Enneper representation of minimal surfaces.

Theorem 2.3.3 (Weirstrass-Enneper representation formula)**.** *For every minimal surface $X(w) = (x(w), y(w), z(w))$ defined on a simply connected domain Ω, which is not of the form $X(w) = au + bv + c$ for $a, b, c \in \mathbb{R}$ there is a pair of holomorphic and meromorphic functions, μ v on Ω with the properties $\mu, v \not\equiv 0$ and such that μv^2 is holomorphic on Ω, with*

$$x(w) = x(w_0) + Re \int_{w_0}^{w} \frac{1}{2}\mu(1 - v^2)d\eta$$
$$y(w) = y(w_0) + Re \int_{w_0}^{w} \frac{i}{2}\mu(1 + v^2)d\eta$$
$$z(w) = z(w_0) + Re \int_{w_0}^{w} \mu v d\eta$$

holds for w,w_0 in Ω.

Conversely given a μ, v on a simply connected domain Ω with the above properties, X defines a minimal surface on Ω.

Proof. The proof follows by combining the last two results by noting that minimal surfaces X on a simply connected domain are given by,

$$X = Ref$$

for isotropic curves, f, and isotropic curves have a general form in terms of a pair of meromorphic and holomorphic functions, by theorem 2.3.2. □

Example 4. *For the Enneper surface in Example 1, we have $\mu(z) = 1$, $\nu(z) = z$.*

Now we introduce the Björling problem for minimal surfaces. Given a real analytic curve c and a real analytic unit normal vector field along the curve n, the Bjölring problem asks to find a minimal surface X containing the curve and surface normal along the curve agreeing with the given normal vector field. The following result was given by Herman Schwarz. Here for a surface X, N_X denotes its surface normal.

Theorem 2.3.4 (Solution to the Björling problem)**.** *Given any real analytic curve $c : I \to \mathbb{R}^3$ with $c'(t) \neq 0$ for any t and a real analytic unit normal vector field $n : I \to \mathbb{R}^3$, there exist a minimal surface X with,*

$$X(u, 0) = c(u)$$

$$N_X(u,0) = n(u)$$

This minimal surface $X : \Omega \to \mathbb{R}^3$ is unique with this property. I,e, if we have another minimal surface $Y : \Omega^{'} \to \mathbb{R}^3$ with $Y(u,0) = c(u)$ and $N_Y(u,0) = n(u)$, then $X = Y$ on $\Omega \cap \Omega^{'}$. for a given c, n it is given by the parametrization,

$$X(u,v) = Re\left(c(z) - i\int_{u_0}^{z} n(w) \times dc(w)\right)$$

Where Ω is a domain containing I as $\{(u,0)|u \in I\}$ with analytic extensions of both c and n existing there. In the above parametrization of X, the notations c, and n denote the analytic extensions of the given real analytic curve and normal. Also here $z = u + iv \in \Omega$

Proof. To show existence, we can show using calculations that $X(u,v) = Re\left(c(z) - i\int_{u_0}^{z} n(w) \times dc(w)\right)$ is a minimal surface satisfying

$$X(u,0) = c(u)$$

$$N_X(u,0) = n(u).$$

. The proof of the uniqueness follows from the identity theorem in complex analysis. To show this, suppose we have a minimal surface Y satisfying,

$$Y(u,0) = c(u)$$

$$N_Y(u,0) = n(u)$$

Then from here we note that, $Y_u(u,0) = c^{'}(u)$ and $Y_v(u,0) = N_Y(u,0) \times c^{'}(u,0) = n(u) \times c^{'}(u)$. Let f, g be the isotropic curves with $X = Ref$ and $Y = Reg$. Then these curves satisfy,

$$g^{'}(u,0) = Y_u(u,0) - iY_v(u,0) = c^{'}(u) - in \times c^{'}(u) = f^{'}$$

Now from here, since $g^{'}$ and $f^{'}$ are holomorphic functions and they agree on a set with a limit point(The part of x-axis containing I), by identity theorem we have

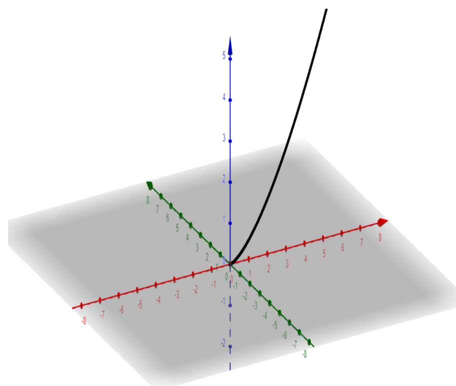
$$g^{'} = f^{'}$$

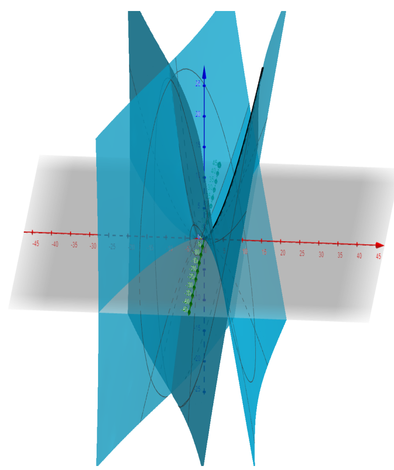
on the intersection of their domains. Thus g and f can differ only a constant. Now using the fact that $Y(u,0) = X(u,0)$ and $N_X(u,0) = N_Y(u,0)$ we conclude that this constant must be zero implying, $X = Y$. □

Example 5. *For the curve,*

$$c(t) = \left(\cosh 2t - 1, 0, \sinh t + \frac{1}{3}\sinh(3t)\right)$$

(This parametrizes the Neil's parabola. The following is a picture of it) and $n(t)$ being it's principal normal, We get the Henneberg surface as the solution to the Björling problem.





Solutions to Björling problem leads to several interesting geometric results about minimal surfaces. We state three such results. We refer to [12] pages 123-135 for many more such results and their proofs.

Theorem 2.3.5. *Any straight line contained in a minimal surface is the axis of symmetry of that surface. Any plane which intersects a minimal surface perpendicularly is a plane of symmetry of that surface.*

Theorem 2.3.6. *A curve c in a Minimal surface is a geodesic line of curvature of that surface if and only if it is a plane curve.*

Theorem 2.3.7. *Given any regular real analytic curve c, There exists a minimal surface containing it as a geodesic.*

Chapter 3

Zero mean curvature surfaces in the Lorentz-Minkowski space

This chapter is also expository. Here we will introduce the theory of zero mean curvature surfaces in Lorentz-Minkowki space. In the first section, we define Lorentz-Minkowski space and introduce maximal surfaces. Like minimal surfaces, we will see that the geometry of maximal surfaces can also be studied using the tools of complex analysis.

Then in the second section, in a similar spirit, we define split-complex numbers and show how it is related to the geometry of timelike minimal surfaces.

3.1 Zero mean curvature property in the Lorentz Minkowski space

In this section we will be mainly referring to [20], [19], and [2], We start with defining the Lorentz-Minkoskwi space.

Definition 3.1.1. *The Lorentz-Minkowski space,* $\mathbb{L}^3$ *is defined to be, the vector space* R^3 *with the indefinite bilinear form ,*

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3.$$

Each vector $v \in \mathbb{L}^3$ has a casual character.

Definition 3.1.2. *A vector $v \in \mathbb{L}^3$ is called ,*

- *spacelike if it is the zero vector or if it is nonzero and* $\langle v, v\rangle > 0$
- *timelike if* $\langle v, v\rangle < 0$
- *lightlike if it is a nonzero vector and* $\langle v, v\rangle = 0$.

Using the above definition, we define the causal characters of curves.

Definition 3.1.3. *A curve in $\mathbb{L}^3$ is said to be,*

- *spacelike if all the tangent vectors are spacelike*
- *lightlike if all the tangent vectors are lightlike*
- *timelike if all the tangent vectors are timelike*

Similarly, we define casual characters of the planes and surfaces in $\mathbb{L}^3$.

Definition 3.1.4. *A plane in $\mathbb{L}^3$ is said to be,*

- *spacelike if its normal vector is timelike*
- *lightlike if its normal vector is lightlike*
- *timelike if its normal vector is spacelike*

Definition 3.1.5. *A C^2 regular surface $X : \Omega \subset \mathbb{R}^2 \to \mathbb{L}^3$ is said to be,*

- *spacelike if all its tangent planes are spacelike.*
- *lightlike if all its tangent planes are lightlike.*
- *timelike if all its tangent planes are timelike.*

Now we can study the geometry of zero mean curvature surfaces in Lorentz-Minkowski space. We define the two most widely studied zero mean curvature surfaces in the Lorentz-Minkoswski space.

Definition 3.1.6. *A C^2 regular surface $X : \Omega \to \mathbb{L}^3$ is said to be a maximal surface if it is spacelike and has zero mean curvature.*

Definition 3.1.7. *A C^2 regular surface $X : \Omega \to \mathbb{L}^3$ is said to be a timelike minimal surface if it is timelike and has zero mean curvature.*

Example 6. *The $x - y$ plane,*

$$X(u, v) = (u.v.0)$$

for $u, v \in \mathbb{R}$, is an example of a regular spacelike surface with zero mean curvature. Similarly the $x - z$ plane,

$$Y(u, v) = (u, 0, v)$$

for $u, v \in \mathbb{R}$, is an example of a regular timelike surface with zero mean curvature

Using similar techniques as we used to show that minimal surfaces in Euclidean-three space are locally area minimizing, one can show that maximal surfaces are locally area maximizing. Surprisingly timelike minimal surfaces are neither locally area minimizing nor locally area maximizing.

Similar to minimal surfaces in Euclidean-three space, the geometry of maximal surfaces can also be studied using complex analysis. We introduce the most general definition of maximal surfaces, with respect to conformal parameters. Note that here we are using,

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3.$$

Definition 3.1.8. *A C^2 surface $X : \Omega \to \mathbb{L}^3$, from a simply connected domain Ω is said to be a maximal surface if it satisfies,*

$$\Delta X = X_{uu} + X_{vv} = 0$$

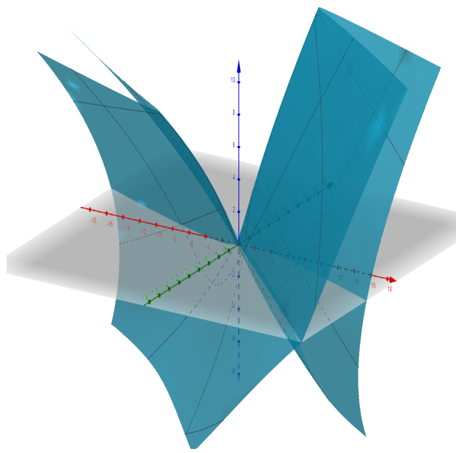
$$\langle X_u, X_v \rangle = 0$$

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$$

Example 7. *The Enneper surface of first kind,*

$$X(u,v) = (u - u^2 v + \frac{1}{3}u^3, -v + u^2 v - \frac{1}{3}v^3, v^2 - u^2)$$

with (u,v) *in* $\mathbb{R}^2 - \{(u,v)|u^2+v^2=1\}$ *is an example of a maximal surface with singularities.*



We refer to [18] page 1085 for more details. Now similar to minimal surfaces in Euclidean-three space, this definition leads to an equivalence of maximal surfaces with real parts of certain holomorphic curves in the complex three-space. This gives rise to the Weirstrass Ennepper representation formula and the solution to the Björling problem for maximal surfaces. Proofs are similar to the case of minimal surfaces in the Euclidean-three space and We refer to [19] and [2] for the details.

Theorem 3.1.1. *For every maximal surface* $X(w) = (x(w), y(w), z(w))$ *defined on a simply connected domain* Ω*, which is not of the form* $X(w) = au + bv + c$ *for* $a, b, c \in \mathbb{R}$ *there is a pair of holomorphic and meromorphic functions,* μ v *on* Ω *with the properties* $\mu, v \not\equiv 0$ *and such that* μv^2 *is holomorphic on* Ω*, with*

$$x(w) = x(w_0) + Re \int_{w_0}^{w} \frac{1}{2}\mu(1+v^2)d\eta$$

$$y(w) = y(w_0) + Re \int_{w_0}^{w} \frac{i}{2}\mu(1-v^2)d\eta$$

$$z(w) = z(w_0) + Re \int_{w_0}^{w} -\mu v d\eta$$

holds for w,w_0 *in* Ω.

Conversely given a μ, v on a simply connected domain Ω with the above properties, X defines a maximal surface on Ω.

Example 8. *For the Enneper surface of the first kind in example-6, we have to take $\mu = 2$ and $\nu = z$.*

Theorem 3.1.2. *Given any real analytic spacelike curve $c : I \to \mathbb{L}^3$ with $c^{'}(t) \neq 0$ for any t and a real analytic timelike unit normal vector field $n : I \to \mathbb{L}^3$, there exist a maximmal surface X with,*

$$X(u, 0) = c(u)$$

$$N_X(u, 0) = n(u)$$

This maximal surface $X : \Omega \to \mathbb{R}^3$ is unique with this property. I,e, if we have another maximal surface $Y : \Omega^{'} \to \mathbb{L}^3$ with $Y(u, 0) = c(u)$ and $N_Y(u, 0) = n(u)$, then $X = Y$ on $\Omega \cap \Omega^{'}$. for a given c, n it is given by the parametrization,

$$X(u, v) = Re\left(c(z) + i\int_{u_0}^{z} n(w) \times dc(w)\right)$$

Where Ω is a domain containing I as $\{(u, 0)|u \in I\}$ with analytic extensions of both c and n existing there. In the above parametrization of X, the notations c, and n denote the analytic extensions of the given real analytic curve and normal. Also here $z = u + iv \in \Omega$

Example 9. *Fix a real number $k > 0$. For the curve being $x - axis$,*

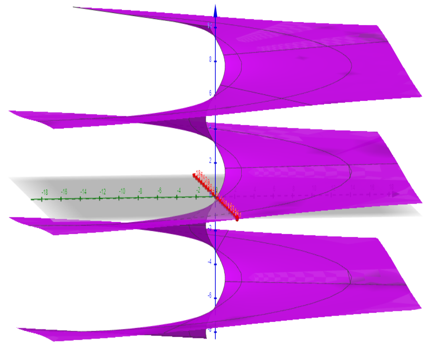
$$c(t) = (t, 0, 0)$$

with $t > \frac{1}{k}$ and the normal being,

$$n(t) = \frac{-1}{\sqrt{k^2t^2 - 1}}(0, 1, kt)$$

We get The helicoid of the first kind as the solution to the Björling problem. It is given by,

$$X(u, v) = \frac{1}{k}(\cosh u \cos v, \cosh u \sin v, v)$$



which is defined on all the points of the $u - v$ plane except at the points with $u = 0$.

Similar to the minimal surfaces in the Euclidean-three space case, the solution to the Björling problem leads to so many interesting geometric results. We state one of them and We refer to [2] for the details.

Theorem 3.1.3. *Consider the following families of maximal surfaces.*

- *spacelike planes*
- *Helicoids of the first kind*
- *Helicoids of the second kind*
- *Cayley's ruled surfaces*

Given any ruled maximal surface, it is congruent to a piece of a member of the above family.

3.2 Split-complex numbers and the geometry of timelike minimal surfaces

In this section, we will be mainly referring to [5] and [18]. When we generalize the definition for timelike minimal surfaces to include singularities, we get the following.

Definition 3.2.1. *A C^2 surface $X : \Omega \to \mathbb{L}^3$ from a simply connected domain Ω is said to be a timelike minimal surface if it satisfies,*

$$X_{uu} - X_{vv} = 0$$

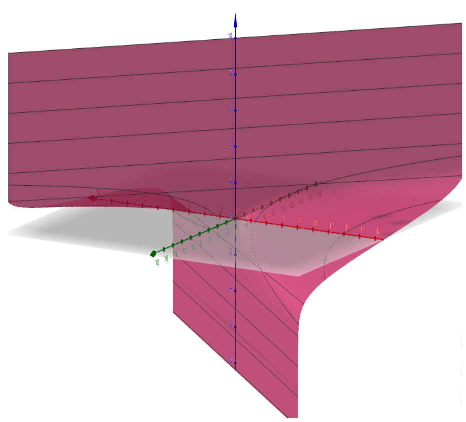
$$\langle X_u, X_v \rangle = 0$$

$$\langle X_u, X_u \rangle = \langle X_v, X_v \rangle$$

Example 10. *The Lorenzian helicoid of the third kind,*

$$X(u, v) = (\sinh u \cosh v, \sinh u \sinh v, v)$$

with u and $v \in \mathbb{R}$ is an example of a timelike minimal surface with singularities.



Unlike the previous situations, here we have the wave equation instead of the Laplace equation. Thus the coordinate functions of timelike minimal surfaces, in conformal parameters are not harmonic. So they are studied using a new kind of number system instead of complex numbers, which are split complex numbers.

Definition 3.2.2. *The ring of split complex numbers is the ring*

$$\mathbb{C}' = \frac{\mathbb{R}[x]}{(x^2 - 1)}$$

We denote the image of x in this ring as, k'. Thus we write,

$$\mathbb{C}' = \{x + k'y | x, y \in \mathbb{R}\}$$

In this ring, one can define analogs of holomorphic maps, which are called split-holomorphic maps.

Definition 3.2.3. *A map $f : \mathbb{C}^{'} \to \mathbb{C}^{'}$ is said to be split-holomorphic if $f(x + k^{'}y) = u + k^{'}v$ satisfy,*

$$u_x = v_y$$

$$u_y = v_x$$

Example 11. *The map $f : \mathbb{C}^{'} \to \mathbb{C}^{'}$ given as $f(x+k^{'}y) = x^2+y^2+2xyk^{'}$ is split-holomorphic.*

We note that the above equations are similar to the Cauchy-Riemann equations. With this number system, one can show that the timelike minimal surfaces are related to certain split-holomorphic curves in the split-complex three-space. Using this, we get the solution to the Björling problem for timelike minimal surfaces.

Theorem 3.2.1. *Given any real analytic spacelike or timelike curve $c : I \to \mathbb{L}^3$ with $c^{'}(t) \neq 0$ for any t and a real analytic timelike or spacelike unit normal vector field $n : I \to \mathbb{L}^3$, there exist a timelike minimal surface X with,*

$$X(u, 0) = c(u)$$

$$N_X(u, 0) = n(u)$$

This timelike minimal surface $X : \Omega \to \mathbb{L}^3$ is unique with this property. I,e, if we have another minimal surface $Y : \Omega^{'} \to \mathbb{L}^3$ with $Y(u, 0) = c(u)$ and $N_Y(u, 0) = n(u)$, then $X = Y$ on $\Omega \cap \Omega^{'}$. for a given c, n it is given by the parametrization,

$$X(u, v) = Re\left(c(z) + k^{'}\int_{u_0}^{z} n(w) \times dc(w)\right)$$

Where Ω is a domain containing I as $\{(u, 0)|u \in I\}$ with analytic extensions of both c and n existing there. In the above parametrization of X, the notations c, and n denote the split-holomorphic extensions of the given real analytic curve and normal. Also here $z = u+k^{'}v \in \Omega$

Note that here all the integrals and real parts are happening over the split-complex numbers. We refer to [5] for the proof.

Example 12. *Fix a real number* $c > 0$. *For the curve being,*

$$c(t) = \frac{1}{c}(\sinh t, 0, 0)$$

and normal being,

$$n(t) = \frac{1}{\cosh t}(0, -\sinh t, 1)$$

One get's the Lorentzian helicoid of the third kind as in example 10 as the solution to the Björling problem.

Like the maximal surface case, the solution to Björling problem for timelike minimal surfaces as well, leads to several interesting geometric results about them. We state one of such results and refer to [5] for more details.

Theorem 3.2.2. *Consider the following families of timelike minimal surfaces.*

- *Lorenzian elliptic catenoids*
- *Lorenzian hyperbolic catenoids*
- *Lorenzian surfaces with spacelike profile curves*
- *Lorenzian parabolic catenoids*

Given any analytic timelike minimal surface in $\mathbb{L}^3$ *which is a surface of revolution, it is congruent to a piece of one of the surfaces in the above families.*

3.3 Born-Infeld solitons

In this section we will be mainly referring to [27], [6], and [21]. We note that,

$$(1 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} - (1 + \psi_x^2)\psi_{yy} = 0$$

is called the Born-Infeld equation. Now we define Born-Infeld soliton functions.

Definition 3.3.1. *(Born-Infeld soliton function) A function* $\psi : \Omega \to \mathbb{R}^2$ *is said to be a Born-Infeld soliton function if it solves the Born-Infeld equation.*

Example 13. *For any C^2 function ϕ we note that,*

$$\psi(x, y) = \phi(x + y)$$

is a Born-Infeld soliton function. For instance,

$$\psi(x, y) = \sin(x + y)$$

is an example of a Born-Infeld soliton function.

In [27] page 617-619, the author presents a way to solve the Born-Infeld equation by hodograph transformations and shows that for suitable functions F and G(which is made precise in [6])

$$x - t = F(r) - \int s^2 G^{'}(s)ds$$

$$x + t = G(s) - \int r^2 F^{'}(r)dr$$

$$\psi(x, t) = \int rF^{'}(r)dr + \int sG^{'}ds$$

is a solution to the Born-Infeld equation. In [6], We also note that in [6], Arka Das shows that Born-Infeld solitons are examples of zero mean curvature surfaces. We will see more about them in the next chapter.

Chapter 4

Solution to the Björling problem of Born-Infeld solitons

4.1 Introduction

In this chapter, we present the solution to the Björling problem for Born-Infeld solitons. This is a completely original work that we put on arXiv as a semi-expository paper, which can be seen at [21]. We give a detailed introduction in the next few paragraphs below.

We note that any non-parametric minimal surface $(x, y, \psi(x, y))$ satisfies the minimal surface equation,

$$(1 + \psi_y^2)\psi_{xx} - 2\psi_x\psi_y\psi_{xy} + (1 + \psi_x^2)\psi_{yy} = 0$$

.

This equation is similar to the Born-Infeld equation,

$$(1 - \psi_y^2)\psi_{xx} + 2\psi_x\psi_y\psi_{xy} - (1 + \psi_x^2)\psi_{yy} = 0$$

This motivates us to ask similar questions about the Born-Infeld solitons. In particular one can ask, can we find an analog of Weierstrass-Ennepper representation formulae for the Born-Infeld solitons? This was answered by Barbishov and Chernikov, which we would shortly see in this article. We can also ask for an analog of the Björling problem. In this chapter, we answer that question. We start with the definition of Born-Infeld soliton general surfaces.

Definition 4.1.1 (Born-Infeld soliton general surfaces)**.** *A surface is said to be a Born-Infeld soliton general surface if it is locally of the form* $(\psi(y, z), y, z)$*,* $(x, \psi(x, z), z)$ *or* $(x, y, \psi(x, y))$ *where* ψ *solves the Born-Infeld equation.*

Example 14. *Later in this chapter, we will show that any timelike minimal surface without singularities is an example of a Born-Infeld soliton general surface.*

in [8], R. Dey and R.K. Singh showed that timelike minimal graphs over $y-z$ plane have Born-Infeld solitons as height functions. We show that the same result holds for timelike minimal graphs over the $x - z$ plane. We also prove that any timelike minimal surface without singularities is locally a graph over $x-z$ or $y-z$ plane. Thus we conclude timelike minimal surfaces without singularities are Born-Infeld soliton general surfaces.

We use the above result to solve Björling problem for Born-Infeld soliton general surfaces. Björling problem for timelike minimal surfaces has already been solved in [5] and [18]. For regular space or timelike curves, one gets a timelike minimal surface without singularities as the solution to the Björling problem. This implies that for regular space or timelike curves, Björling problem for Born-Infeld soliton general surfaces can be solved.

In [24] E. A Paxton showed that any compact subset of a global properly immersed timelike minimal surface is a timelike minimal graph over some timelike plane. We generalize the result in [8] to show timelike minimal graphs over any timelike plane that has the Born-Infeld soliton as a height function. Moreover, we see that for regular real analytic curve c and vector field n which have entire split-holomorphic functions as an analytic extension we get a global properly immersed timelike surface as the solution to the Björling problem. So solving timelike Björling problem for such real analytic strips and looking at their compact subsets is a way to find Born-Infeld solitons. In particular, we note that for real analytic strips of regular real-analytic curves and unit vector fields (c, n) with components as polynomials in t, one can use the above result to find Born-Infeld solitons.

Graph-like Born-Infeld solitons are of special interest to physicists and we try to ask a similar question to them. From [8] we note that spacelike minimal graphs and timelike minimal graphs over $y - z$ plane are Born-Infeld soliton graphs. We use this idea and ask for what kind of real analytic strips (c, n), do we get a spacelike minimal graph or timelike minimal graph as a solution to the Björling problem. We characterize such curves and normals for which graphical solutions can or cannot be found.

Lastly, we go through the representation formulae given by Barbishov and Chernikov. We show that the Barbishov and Chernikov representation formula, like the Weierstrass-Enneper representation, fails at zero Gauss curvature points. In [23] L. McNertney showed that any surface in $\mathbb{L}^3$ which can be expressed as the sum of two lightlike curves with linearly independent velocities is timelike minimal. We see that the Barshiov and Chernikov representation formula also expresses the surface as a sum of two lightlike curves with linearly independent velocities, implying these Born-Infeld solitons are timelike minimal in $\mathbb{L}^3$. Also, normal vector fields of these surfaces parametrized by $r - s$ coordinates are the same.

We note that A. Das also in [6] independently solved the Björling problem for Born-Infeld solitons, $X(\omega(t)) = c(t)$,$N(\omega(t)) = n(t)$ where $\omega(t)$ is a curve in $r - s$ plane determined by c and n. They also shows that Björling problem for Born-Infeld solitons may not have unique solutions.

Our results here are dependent on plenty of earlier work done by several mathematicians. We would have a look at those as we go through them. .

As a summary, we aim to describe three results. We would first prove that any timelike minimal surface without singularities is locally the graph of a Born-Infeld soliton over $y - z$ or $x - z$ plane. This answers the Björling problem for surfaces that are locally Born-Infeld solitons, in the special case when the curve is assumed to be regular. The third section is about some corollaries of E.A Paxton's results ([24]). In the fourth section, we deal with the Björling problem of surfaces that are globally Born-Infeld solitons and present some results. In the last few sections, we study a special class of Born-Infeld solitons, given by Barbishov and Chernikov and we would prove some theorems about them.

Throughout this chapter we will be using the following definition for the Lorentz-Minkowski space, $\mathbb{L}^3$.

Definition 4.1.2. $\mathbb{L}^3$ *is* $\mathbb{R}^3$ *with the metric* $ds^2 = dx^2 + dy^2 - dz^2$

4.2 Regular timelike minimal surfaces and Born-Infeld solitons

We will start with defining Born-Infeld solitons.

Definition 4.2.1 (Born-Infeld soliton)**.** *Let* $\Omega \subset \mathbb{R}^2$ *be an open subset. Let* $(u, v) \in \Omega$*. Now we will denote this subset by* $\Omega_{(u,v)}$*. A map* $\phi\colon \Omega_{(u,v)} \to \mathbb{R}$ *is said to be a Born-Infeld soliton if it solves the Born-Infeld equation in the variables* u, v*.*

First, we will show a lemma about timelike minimal graphs over $x - y$ plane.

Lemma 4.2.1. *Any timelike minimal graph* $X(x, y) = (x, y, \phi(x, y))$ *without singularities is locally a graph of the form* $(x, \psi(x, z), z)$ *or* $(\psi(y, z), y, z)$ *for some Born-Infeld soliton* ψ*.*

Proof. Here the Jacobian of the surface X at a point p looks like this,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \phi_x(p) & \phi_y(p) \end{pmatrix}$$

Thus note that

$$\begin{pmatrix} 1 & 0 \\ \phi_x(p) & \phi_y(p) \end{pmatrix}$$

or

$$\begin{pmatrix} 0 & 1 \\ \phi_x(p) & \phi_y(p) \end{pmatrix}$$

has rank 2 only if their determinants $\phi_x(p)$ or $\phi_y(p)$ is nonzero . In other words, our surface is always locally a graph over $x - z$ or $y - z$ plane if $\phi_x(p)$ or $\phi_y(p)$ is nonzero for any point p. This is always true for a timelike minimal graphs without singularities over $x - y$ plane as they satisfy

$$\phi_x^2(p) + \phi_y^2(p) > 1$$

at all points as the normal of X,

$$N = \frac{(-\phi_x, -\phi_y, -1)}{\sqrt{|\phi_x^2 + \phi_y^2 - 1|}}$$

is spacelike. To show the height function of local $x-z$ or $y-z$ graph is a Born-Infeld soliton, one can compute mean curvature and equate it to zero. This part is similar to R. Dey and R.K. Singh's proof of height functions of timelike minimal graphs without singularities over $y-z$ planes are Born-Infeld solitons(in [8] pages 528 to 530) □

Now we will use this lemma to prove that timelike minimal surfaces without singularities are Born-Infeld soliton general surfaces.

Theorem 4.2.2. *Any timelike minimal surface without singularities is locally a graph of a function over the x-z or y-z plane with their height function being a Born-Infeld soliton.*

Proof. Any regular timelike minimal surface is locally a graph and it is of the form $(x, y, \psi(x, y))$, or $(x, \psi(x, z), z)$ or $(\psi(y, z), y, z)$. By lemma 2.1, we note that the timelike $x-y$ graph without singularities is also locally a $y-z$ or $x-z$ graph. Using the zero-mean curvature condition one can conclude that the height function must be a Born-Infeld soliton. □

We note that alternatively in [20] proposition 3.3(page 75), R. Lopez proved that any timelike minimal surface is locally a graph over $x-z$ or $y-z$ plane by noting that components of the normal are Jacobian of the map from $y-z$,$x-z$ and $x-y$ plane into the image of the surface.

In theorem 3.3 of [18](Page 1091), Y.W Kim, S.E Koh, and S-E Yang proved that if c is a regular spacelike or timelike curve, then there is a timelike minimal surface without singularities solving the Björling problem. This gives us the following result.

Corollary 4.2.3. *If c is a regular real analytic spacelike or timelike curve and n is a real analytic spacelike unit normal vector field, then there exists a Born-Infeld soliton general surface which solves the Björling problem.*

Another corollary of theorem 2.1 is the following.

Corollary 4.2.4. *Let $c : I \to \mathbb{L}^3$ be a regular timelike curve in $\mathbb{L}^3$ such that $c''(t)$ is spacelike for all $t \in I$. Then there exists a Born-Infeld soliton general surface containing c as geodesic(by a geodesic here, we mean a curve such that principal normal agrees with surface normal in $\mathbb{L}^3$).*

Proof. Here since our curve is regular we can give it arc length parametrization which has a constant speed. Now we refer to corollary 3.2, page 489 of [5]. We also note that for a regular curve, we get a timelike minimal surface without singularities as solutions to the Björling problem which is also a Born-Infeld soliton general surface. □

4.3 On compact subsets of timelike minimal surfaces

In this section, we get some corollaries of theorem 1.1 in [24]. We would first prove a lemma.

Lemma 4.3.1. *Let $X : \Omega \to \mathbb{R}^3$ be a timelike minimal surface which is a smooth graph over a timelike plane P. Choose a orthonormal basis $\{b_2, b_3\}$ with respect to $\langle , \rangle_{\mathbb{L}^3}$ for the plane P, with b_2 a spacelike vector ,b_3 a timelike vector . Also, let $b_1 = N$ be the unit spacelike surface normal of the timelike plane P.Then $\{b_1, b_2, b_3\}$ forms a orthonormal basis of $\mathbb{L}^3$. For any $x_2b_2 + x_3b_3$ in P, we can consider*

$$\psi(x_2, x_3) = \langle X, N\rangle_{\mathbb{L}^3}.$$

Then such a ψ satisfies the Born-Infeld equation in variables x_2, x_3 and thus is a Born-infield soliton.

Proof. Since X is a smooth graph over plane P we know that the projection map,

$$\pi : X(\Omega) \to R^2$$

with $\pi(p)$ being the projection of the point $X(p)$ onto the plane P=span$\{b_2, b_3\}$ is a diffeomorphism. Let $\Sigma \subset P$ be the image of this map. Using

$$\phi = \pi^{-1} : \Sigma \to X(\Omega)$$

we get a graph $X(x_2, x_3) = \psi(x_2, x_3)N + x_2b_2 + x_3b_3$ with $\psi(x_2, x_3) = \langle X(x_2, x_3), N\rangle_L$. Here $X(x_2, x_3) : P \to \mathbb{R}^3$ is $X \circ \phi$. To show that such a map $\psi(x_2, x_3)$ satisfies the Born-infield equation, one can compute the mean curvature in this new parametrization and equate it to zero. □

Now we state the main result of this section.

Theorem 4.3.2. *If (c, n) is a smooth real analytic strip with the properties that,*

- *c, a regular curve*
- *c and n have entire split-holomorphic functions as an analytic extension*

then any compact subset of the timelike minimal surface solving this Björling problem is a timelike minimal graph over some timelike plane with height function a Born-Infeld soliton.

Proof. We first note that since c and n have analytic extensions which are entire functions, Ω can be taken to be $\mathbb{R}^2$. Since c is regular, the solution to Björling problem is a regular surface(we refer to theorem 3.3 of [18]). Thus we have a smooth X: $\mathbb{R}^2 \to \mathbb{R}^3$ which is a properly immersed surface, solving this Björling problem. Now by theorem 1.1 (page 3036) of [24] any compact subset of this surface is a timelike graph over some timelike plane. Using lemma 3.1, we conclude that the height function is a Born-Infeld soliton. □

In fact for any $M > 0$, when restricted to a diamond D_M=$\{(u, v)||u| + |v| \leq M\}$ We would still get a timelike graph over some timelike plane. This was used in the proof of theorem 1.1 (page 3036) of [24].

Now we state a special case of Theorem 3.2.

Corollary 4.3.3. *If (c, n) is a smooth real analytic strip with the properties that,*

- *c, a regular curve*
- *$c(t)$ and $n(t)$ have components as polynomials in t with real coefficients.*

for any $M > 0$, the solution to the time like Björling problem when restricted to a diamond D_M=$\{(u, v)||u| + |v| \leq M\}$ is a smooth graph over some timelike plane with height function being a Born-Infeld soliton.

4.4 Björling problem for graph-like Born-Infeld solitons

We first define graphical Born-Infeld solitons.

Definition 4.4.1 (Born-Infeld soliton surface). *A surface X is said to be a Born-Infeld soliton surface if it is of the form $X(y,z) = (\psi(y,z), y, z)$ for some Born-Infeld soliton.*

Example 15. *The surface $X : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$ given by,*

$$X(y,z) = (\tan(y+z), y, z)$$

Is a Born-Infeld soliton surface.

Let us recall the definition of the positive quasi-definite matrix from [17].

Definition 4.4.2. *A matrix J is said to be positive quasidefinite if*

$$A = \frac{J + J^T}{2}$$

is positive definite.

We asked for what kind of real-analytic strips (c, n) the Björling problem gives a graphical solution. We characterized a set of real-analytic strips (c, n) for which one gets a time-like minimal graph as a solution to the Bjorling problem. Note that in the following result, we are using split-complex analysis instead of complex analysis. Here $z = x + k'y$ with $k'^2 = 1$. We refer to [5] for more about split-complex analysis.

Theorem 4.4.1. *Given a real analytic curve $c(t) = (c_1(t), c_2(t), c_3(t))$ which is timelike or spacelike in L^3 and a real analytic spacelike unit normal $n = (n_1(t), n_2(t), n_3(t))$ let,*

$$J_{(c,n)}(t) = \begin{vmatrix} \frac{c_{2_u}(t)}{2} & \frac{c_{2_v}(t)}{2} + (n(t) \times c'(t))_2 \\ \frac{c_{3_u}(t)}{2} & \frac{c_{3_v}(t)}{2} + (n(t) \times c'(t))_3 \end{vmatrix}$$

be a real-valued function defined on I.

Fix a real analytic strip (c, n) *and let* $\Omega_{(c,n)}$ *be a domain where analytic extension of both* c *and* n *exists.*

1) $t \to J_{(c,n)}(t)$ *has a zero in* $I \implies$ *there does not exist a solution for Björling problem of time like Born-Infeld soliton surfaces without singularities.*

2) If $\Omega_{(c,n)}$ *is convex, and if*

$$J_{(c,n)}(z) = \begin{pmatrix} \frac{c_{2_u}}{2} + (Im(n(z) \times c^{'}(z))_2 & \frac{c_{2_v}}{2} + (Re(n(z) \times c^{'}(z))_2 \\ \frac{c_{3_u}}{2} + (Im(n(z) \times c^{'}(z))_3 & \frac{c_{3_v}}{2} + (Re(n(z) \times c^{'}(z))_3) \end{pmatrix}$$

has a non-vanishing determinant and is positive quasi definite for all z *in* $\Omega_{(c,n)}$*, then there is a timelike Born-Infeld soliton surface without singularities in* $\mathbb{L}^3$ *as a solution to the björling problem and it is given by*

$$X(z) = Re\{c(z) + k' \int_{t_0}^{z} n(w) \times c^{'}(w)\, dw\}$$

if c *is timelike and,*

$$X(w) = Re\{c(w) + k' \int_{t_0}^{w} n(\zeta) \times c^{'}(\zeta)\, d\zeta\}$$

When c *is spacelike(Here* $w = k^{'}z$*).*

Proof. We present the proof for the case when c is timelike. When c is spacelike, the proof is similar.

The main idea of the proof is to use the implicit function theorem to understand when the timelike minimal surface solution to the Björling problem becomes a graph over the y-z plane.

For Björling problem for timelike minimal surfaces, we know the solution is

$$X(z) = Re\{c(z) + k' \int_{t_0}^{z} n(w) \times c^{'}(w)\, dw\}$$

(We refer to [5], page 485, theorem 3.1).

Let

$$F(z) = c(z) + k' \int_{t_0}^{z} n(w) \times c^{'}(w)\, dw$$

.

Then

$$X(z) = \frac{F(z) + \overline{F(z)}}{2}$$

$$\frac{\partial X}{\partial z} = \frac{1}{2}\frac{\partial F}{\partial z} = \frac{1}{2}(\frac{\partial c}{\partial z} + k'(n(z) \times c^{'}(z))).$$

We note that for split complex numbers,

$$\frac{\partial X}{\partial z} = \frac{1}{2}\left(\frac{\partial X}{\partial u} + k'\frac{\partial X}{\partial v}\right).$$

Thus

$$\frac{\partial X}{\partial u} = Re(2\frac{\partial X}{\partial z}) = \frac{1}{2}\frac{\partial c}{\partial u} + Im(n(z) \times c^{'}(z))$$

$$\frac{\partial X}{\partial v} = Im(2\frac{\partial X}{\partial z}) = \frac{1}{2}\frac{\partial c}{\partial v} + Re(n(z) \times c^{'}(z))$$

$$\begin{pmatrix} X_u & X_v \end{pmatrix} = \begin{pmatrix} \frac{c_u}{2} + Im(n(z) \times c^{'}(z) & \frac{c_v}{2} + Re(n(z) \times c^{'}(z) \end{pmatrix}.$$

Which implies,

$$\begin{pmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{pmatrix} = \begin{pmatrix} \frac{c_{1_u}}{2} + (Im(n(z) \times c^{'}(z))_1 & \frac{c_{1_v}}{2} + (Re(n(z) \times c^{'}(z))_1 \\ \frac{c_{2_u}}{2} + (Im(n(z) \times c^{'}(z))_2 & \frac{c_{2_v}}{2} + (Re(n(z) \times c^{'}(z))_2 \\ \frac{c_{3_u}}{2} + (Im(n(z) \times c^{'}(z))_3 & \frac{c_{3_v}}{2} + (Re(n(z) \times c^{'}(z))_3) \end{pmatrix}.$$

Thus

$$\begin{pmatrix} y_u & y_v \\ z_u & z_v \end{pmatrix} = \begin{pmatrix} \frac{c_{2_u}}{2} + (Im(n(z) \times c^{'}(z))_2 & \frac{c_{2_v}}{2} + (Re(n(z) \times c^{'}(z))_2 \\ \frac{c_{3_u}}{2} + (Im(n(z) \times c^{'}(z))_3 & \frac{c_{3_v}}{2} + (Re(n(z) \times c^{'}(z))_3) \end{pmatrix}.$$

Let

$$J_{(c,n)}(z) = \begin{vmatrix} \frac{c_{2_u}}{2} + (Im(n(z) \times c^{'}(z))_2 & \frac{c_{2_v}}{2} + (Re(n(z) \times c^{'}(z))_2 \\ \frac{c_{3_u}}{2} + (Im(n(z) \times c^{'}(z))_3 & \frac{c_{3_v}}{2} + (Re(n(z) \times c^{'}(z))_3) \end{vmatrix}.$$

Thus whenever $J_{(c,n)}(z)$ is nonvanishing, by implicit function theorem one can represent the surface as locally a graph over $y - z$ plane. The real-valued function $t \to J_{(c,n)}(t)$ mentioned in theorem

$$J_{(c,n)}(t) = \begin{vmatrix} \frac{c_{2_u}(t)}{2} & \frac{c_{2_v}(t)}{2} + (n(t) \times c^{'}(t))_2 \\ \frac{c_{3_u}(t)}{2} & \frac{c_{3_v}(t)}{2} + (n(t) \times c^{'}(t))_3 \end{vmatrix}$$

is the restriction of this map $z \to J_{(c,n)}(z)$ to I. Using continuity arguments one can conclude that $J_{(c,n)}(z)$ non-vanishing in a some allowed domain $\Omega_{(c,n)}$ is equivalent to $J_{(c,n)}(t)$ non-vanishing on I. This completes the proof of the first part of the theorem.

Now if $\psi : (u, v) \to (y(u, v), z(u, v))$ is injective as well, it would become a diffeomorphism. In [17] theorem 6 (Page 88) D. Gale and H.Nikaido shows that if Ω is a convex domain and if $\psi : \Omega \to \mathbb{R}^2$ has a positive quasi definite Jacobian at all points, then ψ is injective. The second condition ensures this. □

Now we would just state when can we get a spacelike minimal graph or a spacelike Born-Infeld soliton surface without singularities as the solution to the Bjorling problem. The proof is similar, except one has to use complex numbers and complex analysis.

Theorem 4.4.2. *Given a real analytic curve $c(t) = (c_1(t), c_2(t), c_3(t))$ in L^3 and a real analytic timelike unit normal $n = (n_1(t), n_2(t), n_3(t))$ let,*

$$J_{(c,n)}(t) = \begin{vmatrix} \frac{c_{2_u}(t)}{2} & \frac{c_{2_v}(t)}{2} - (n(t) \times c^{'}(t))_2 \\ \frac{c_{3_u}(t)}{2} & \frac{c_{3_v}(t)}{2} - (n(t) \times c^{'}(t))_3 \end{vmatrix}$$

be a real-valued function defined on I.

Fix a real analytic strip (c, n) and let $\Omega_{(c,n)}$ be a domain where analytic extension of both c and n exists.

1) $t \to J_{(c,n)}(t)$ has a zero in $I \implies$ there does not exist a solution for Björling problem *of space like Born-Infeld soliton surfaces without singularities.*

2) If $\Omega_{(c,n)}$ is convex, and if

$$J_{(c,n)}(z) = \begin{pmatrix} \frac{c_{2u}}{2} + (Im(n(z) \times c^{'}(z))_2 & \frac{c_{2v}}{2} - (Re(n(z) \times c^{'}(z))_2 \\ \frac{c_{3u}}{2} + (Im(n(z) \times c^{'}(z))_3 & \frac{c_{3v}}{2} - (Re(n(z) \times c^{'}(z))_3) \end{pmatrix}$$

has a non-vanishing determinant and is positive quasi definite for all z in $\Omega_{(c,n)}$ then there is a spacelike Born-Infeld general surface, without singularities in $\mathbb{L}^3$ as a solution to the Björling problem and it is given by

$$X(z) = Re\{c(z) + i \int_{t_0}^{z} n(w) \times c^{'}(w)\, dw\}$$

We note that A. Das in [6] show that the above positive-quasi definite condition can be replaced with J(c,n)(t) being a P-matrix.

Example 16. *For the curve*

$$c(t) = (sin(2t), t, t)$$

and the normal vectorfield

$$n(t) = \frac{1}{\sqrt{1 + 2\sin^2(2t)}}(1, -\sin 2t. \sin 2t)$$

We can find a solution to the Björling problem of Born-Infeld soliton surfaces. It is given by,

$$X(y, z) = (\sin(y + z), y, z)$$

Where $c : [0, 2\pi) \to \mathbb{R}^3$, $n : [0, 2\pi) \to \mathbb{R}^3$ and $X : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}^3$.

4.5 On solutions given by Barbishov and Chernikov

We refer to pages 617 to 619 of [27] for the representation formula of Born-Infeld soliton surfaces given by Barbishov and Chernikov. We show that it holds at any non-zero Gauss curvature point.

Theorem 4.5.1. *For any timelike Born Infeld soliton surface $(\psi(y, z), y, z)$ without singu-*

larities, for any non-zero Gauss curvature point, there is an open neighborhood with two C^2 functions F and G such that the surface can be represented there as,

$$y - z = F(r) - \int s^2 G^{'}(s) ds$$

$$y + z = G(s) - \int r^2 F^{'}(r) dr$$

$$\psi(y,z) = \int r F^{'}(r) dr + \int s G^{'}(s) ds.$$

Conversely, any graph-like surface $(\psi(y,z), y, z)$ represented this way is a Born-Infeld soliton surface.

Proof. We refer to [27], pages 617-619(Section 17.15) for the proof of the above representation formulae. Note the proof starts with the assumption

$$\psi_y^2 - \psi_z^2 + 1 > 0.$$

This implies the surface is timelike. Also Note that in the proof to go from step 17.89 to 17.90, there was an interchange of the roles of dependent and independent variables. For this, we want the map

$$\psi : (\xi, \eta) \to (u, v)$$

to be a diffeomorphism. So we need the Jacobian of this map ψ to be nonzero. This implies, at such points p

$$(\psi_{yy}\psi_{zz} - \psi_{yz}^2)(p) \neq 0$$

.

The above condition is equivalent to ψ being a local-diffeomorphism. Note that Gauss curvature of $(\psi(y,z), y, z)$ is given by,

$$K(p) = \frac{\psi_{yy}\psi_{zz} - \psi_{yz}^2}{(\psi_y^2 - \psi_z^2 + 1)^2}.$$

Thus if there are no singularities,

$$(\psi_{yy}\psi_{zz} - \psi_{yz}^2)(p) \neq 0$$

is equivalent to saying that Gauss curvature $K(p)$ is nonzero. $\square$

Now we state a result about their surface normal.

Theorem 4.5.2. *The surface normal $N(r,s)$ (in $\mathbb{L}^3$) of all Born-Infeld soliton surfaces described by the representation formula of Barbishov and Chernikov are the same.*

Proof. The proof follows by computation.

$$X_r = \left(rF^{'}(r),\ \ \frac{F^{'}(r)(1-r^2)}{2},\ \ \frac{-F^{'}(r)(1+r^2)}{2}\right).$$

$$X_s = \left(sG^{'}(s),\ \ \frac{G^{'}(s)(1-s^2)}{2},\ \ \frac{G^{'}(s)(1+s^2)}{2}\right).$$

$$N(r,s) = \frac{X_r \times X_s}{|X_r \times X_s|} = \left(\frac{r+s}{1+rs},\ \ \frac{r-s}{1+rs},\ \ \frac{rs-1}{1+rs}\right).$$

Thus $N(r,s)$ is independent of F and G. $\square$

Now we give a geometric interpretation of the above formula.

Theorem 4.5.3. *Graphical surfaces $(\psi(y,z),y,z)$ with $\psi(y,z)$,y,z as described by the representation formula of Barbishov and Chernikov can be written as,*

$$X(r,s) = \frac{\psi(r)+\phi(s)}{2}$$

with

$$\psi(r) = \left(2\int rF^{'}(r)dr,\ \ F(r) - \int r^2F^{'}(r)dr,\ \ -F(r) - \int r^2F^{'}(r)dr\right),$$

$$\phi(s) = \left(2\int sG^{'}(s)ds,\ \ G(s) - \int s^2G^{'}(s)ds,\ \ G(s) + \int s^2G^{'}(s)ds\right),$$

such that ψ, ϕ lightlike curves in $\mathbb{L}^3$ with $\psi^{'}(r)$ and $\phi^{'}(s)$ are linearly independent for all values of r and s. This implies these surfaces are timelike minimal.

Proof. It follows from computation that

$$X(r,s) = \frac{\psi(r) + \phi(s)}{2}$$

and ψ, ϕ are lightlike curves. To show $\psi^{'}(r)$ and $\phi^{'}(s)$ are linearly independent for all values of r and s, note that $\psi^{'}(r) = 2X_r$ and $\phi^{'}(s) = 2X_s$.

$$X_r = \left(rF^{'}(r), \quad \frac{F^{'}(r)(1-r^2)}{2}, \quad \frac{-F^{'}(r)(1+r^2)}{2} \right).$$

$$X_s = \left(sG^{'}(s), \quad \frac{G^{'}(s)(1-s^2)}{2}, \quad \frac{G^{'}(s)(1+s^2)}{2} \right).$$

Since $X(r,s)$ is a given to be the graph of a function, which is a regular surface, X_r and X_s are linearly independent for all values of r and s.

Fact 2.2 in [1](Page 541) confirms these surfaces are timelike minimal. □

Chapter 5

Solution to the interpolation problem of timelike minimal surfaces using split-harmonic maps

5.1 Introduction

[1] In this chapter we present a different solution to the singular Björling problem of timelike minimal surfaces, using split-harmonic maps. We also solve the interpolation problem of timelike minimal surfaces for a class of curves called split-Fourier curves. This is a completely original work, which we put up on arXiv, which can be seen at [22]. We give a detailed introduction in the next few paragraphs.

Singular Björling problem for timelike minimal surfaces asks: given a lightlike curve $\gamma : (a, b) \to \mathbb{L}^3$ and a lightlike vector field $L : (a, b) \to \mathbb{L}^3$ can we find a timelike minimal surface X such that $X(u, 0) = \gamma(u)$ and $X_v(u, 0) = L(u)$. In [18], Y.W Kim, S.E Koh, and S-E Yang solve this using results involving the wave equation. We also note that in [7], R.Dey, P.Kumar, and R.K.Singh solves singular Björling problem for maximal surfaces when $\gamma : \mathbb{S}^1 \to \mathbb{L}^3$, $L : \mathbb{S}^1 \to \mathbb{L}^3$ are lightlike using a representation formula of maximal surfaces involving harmonic maps. This motivates us to solve the singular Björling problem

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for timelike minimal surfaces using a representation formula involving split-harmonic maps. We define split-harmonic maps as follows,

Definition 5.1.1. *A map $f\colon \Omega \subset \mathbb{C}' \to \mathbb{R}$ is said to be split-harmonic if,*

$$f_{xx} - f_{yy} = 0.$$

A map $F\colon \Omega \subset \mathbb{C}' \to \mathbb{C}'$ is said to be split-harmonic if each of its component functions is split-harmonic.

Example 17. *The map $f : \mathbb{C}' \to \mathbb{R}$ given by,*

$$f(x + k'y) = x^2 + y^2$$

is a split-harmonic map.

Throughout this chapter, we will be using the following definition for $\mathbb{L}^3$.

Definition 5.1.2. $\mathbb{L}^3$ *is* $\mathbb{R}^3$ *with the metric* $ds^2 = -dx^2 + dy^2 + dz^2$.

We define split-Fourier curves.

We note that for split-exponential map, $e^{k'\theta} = \cosh\theta + k' \sinh\theta$. We refer to [3], page 11 for more details.

In this chapter, we solve singular Björling problem for timelike minimal surfaces when $\gamma : \mathbb{H}^1 \to \mathbb{L}^3$, $L : \mathbb{H}^1 \to \mathbb{L}^3$ are lightlike. Here $\mathbb{H}^1$ denotes the set $\{x + k'y \in \mathbb{C}' | x > 0, x^2 - y^2 = 1\}$. $\mathbb{H}^1$ is the right branch of the unit hyperbola $x^2 - y^2 = 1$.

Definition 5.1.3. *A curve $\gamma : \mathbb{H}^1 \to \mathbb{L}^3$ is said to be a split-Fourier curve if it has a finite series expansion(I.e, only finitely many terms in the infinite series being nonzero) of the form, $\gamma(\theta) = (\gamma_1 + k'\gamma_2, \gamma_3) = (\Sigma_{-\infty}^{\infty} c_n e^{k'n\theta}, \Sigma_{-\infty}^{\infty} d_n e^{k'n\theta})$ with c_n, d_n being split-complex numbers and $\gamma_3(\theta) = \Sigma_{-\infty}^{\infty} d_n e^{k'n\theta}$ being a real-valued function.*

Example 18. *The curve $\gamma : [0, 1] \to \mathbb{R}^3$ given by,*

$$\gamma(\theta) = (\cosh\theta, \sinh\theta, \sinh\theta) = (e^{k'\theta}, \frac{e^{k'\theta} - e^{-k'\theta}}{2k'})$$

is a split-Fourier curve.

We first develop a representation formula for timelike minimal surfaces involving split-harmonic maps. Then we use this to solve the singular Björling problem. In pursuit of this, we show some results in split-complex analysis which are analogs of corresponding ones in complex analysis. As an application, we study the problem of interpolating a given spacelike or timelike split-Fourier curve to a point p by a timelike minimal surface. This is analogous to a result in [7].

Next, we take two arbitrary split-Fourier curves and find conditions such that there is a timelike minimal surface interpolating them.

5.2 A representation formula for timelike minimal surfaces

We recall the definition of generalized timelike minimal surfaces. Let $z = x + k^{'}y \in \Omega \subseteq \mathbb{C}^{'}$. Here $k^{'^2} = 1$ and $|z|^2 = y^2 - x^2$. We refer to [5] for more about split-complex numbers. We recall the definitions of $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \overline{z}}$ from [5].

$$\frac{\partial F}{\partial z} = \frac{1}{2}(\frac{\partial F}{\partial x} + k^{'}\frac{\partial F}{\partial y})$$

and

$$\frac{\partial F}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial F}{\partial x} - k^{'}\frac{\partial F}{\partial y}).$$

The following definition is motivated by results in [5]

Definition 5.2.1. *A smooth map $F = (u, v, \omega) : \Omega \to \mathbb{L}^3$ is said to be a generalized timelike minimal surface if it satisfies,*

$$F_{xx} - F_{yy} = 0,$$

$$\langle F_x, F_x\rangle + \langle F_y, F_y\rangle = 0,$$

$$\langle F_x, F_y\rangle = 0$$

and

$$-\left|\frac{\partial u}{\partial z}\right|^2+\left|\frac{\partial v}{\partial z}\right|^2+\left|\frac{\partial \omega}{\partial z}\right|^2 \not\equiv 0 \textit{ on } \Omega.$$

We identify $\mathbb{L}^3$ with $\mathbb{C}^{'} \times \mathbb{R}$. Let,

$$h = u + k^{'}v.$$

Also, we define

$$\phi_1 = \frac{\partial u}{\partial z}, \phi_2 = \frac{\partial v}{\partial z} \textit{ and } \phi_3 = \frac{\partial \omega}{\partial z}.$$

Now we present the main result of this section.

Theorem 5.2.1. *A smooth map $F = (h, w) : \Omega \to \mathbb{L}^3$ is a generalized timelike minimal surface if and only if it is split-harmonic with $\omega_z^2 = h_z\overline{h_{\overline{z}}}$ and $|h_z|$ not identically same as $|h_{\overline{z}}|$.*

Proof. We prove in the forward direction.

Since

$$F_{xx} - F_{yy} = 0,$$

F is split-harmonic. By computations we get,

$$4(-\phi_1^2 + \phi_2^2 + \phi_3^2) = \langle F_x, F_x\rangle + \langle F_y, F_y\rangle + 2k^{'}\langle F_x, F_y\rangle,$$

$$h_z = \phi_1 + k^{'}\phi_2$$

and

$$\overline{h_{\overline{z}}} = \phi_1 - k^{'}\phi_2.$$

This implies,

$$4(\omega_z^2 - h_z\overline{h_{\overline{z}}}) = \langle F_x, F_x\rangle + \langle F_y, F_y\rangle + 2k^{'}\langle F_x, F_y\rangle.$$

Since F is represented by conformal parameters,

$$\omega_z^2 = h_z \overline{h_{\overline{z}}}.$$

Using this, one can show by calculating that

$$-\left|\frac{\partial u}{\partial z}\right|^2 + \left|\frac{\partial v}{\partial z}\right|^2 + \left|\frac{\partial \omega}{\partial z}\right|^2 = \frac{-(|h_z| - |\overline{h_{\overline{z}}}|)^2}{2}.$$

From here we get $|h_z|$ not identically same as $|h_{\overline{z}}|$. Proof of the other direction is similar. □

In particular above representation formula shows that for time-like minimal graphs over $x - y$ plane, h is injective.

5.3 On split-harmonic maps

In this section, we state some results about split-harmonic maps. We start with the definition of split-holomorphic and split-analytic maps. We refer to [5] for more details.

Definition 5.3.1 (split-holomorphic). *A map $f = u + iv : \Omega \to \mathbb{C}'$ is said to be split-holomorphic if for any $z = x + k'y \in \Omega$, u and v satisfies,*

$$u_x = v_y$$

$$u_y = v_x.$$

We call the above equations as Cauchy-Riemann equations in split-complex analysis.

Definition 5.3.2 (split-analytic). *A map $f = u + iv : \Omega \to \mathbb{C}'$ is said to be split-analytic, if for any $\eta = s + k't \in \Omega$ there is an open ball,*

$$B_R(\eta) = \{x + k'y \in \Omega | \sqrt{(s-x)^2 + (t-y)^2} < R\}$$

In usual subspace topology of Ω in $\mathbb{R}^2$ with $f(z) = \Sigma_0^\infty c_n(z-\eta)^n$, for every $z \in B_R(\eta)$.

Unlike holomorphic maps, not all split-holomorphic maps are analytic. One can look at [3] page 18 for a counterexample. So, we have to impose this extra condition while stating the identity theorem.

Lemma 5.3.1 (Principle of isolated zeros for split-analytic maps)**.** *Suppose f is a split-holomorphic map that is represented by a power series around a zero η in the open ball $B_R(\eta)$. Also, assume f is not identically zero on $B_R(\eta)$. Then there is a $0 < r \leq R$ such that $f(z) \neq 0$ whenever z is in $B_r(\eta) - \{\eta\}$.*

Proof. There exist coefficients c_k such that $f(z) = \Sigma_0^\infty c_k(z-\eta)^k$ in $B_R(\eta)$. Let $n \in \mathbb{N}$ be the smallest number such that $c_n \neq 0$. Since η is a zero of f, $n \geq 1$. Then we have $f(z) = (z-\eta)^n \Sigma_{n+1}^\infty c_k(z-\eta)^k = (z-\eta)^n g(z)$. Here $g(\eta) \neq 0$. Now using continuity of g we note that there is a $0 < r \leq R$ such that $g(z) \neq 0$ whenever z is in $B_r(\eta) - \{\eta\}$. Since $f(z) = (z-\eta)^n g(z)$, this concludes the proof. □

Using this we prove the identity theorem. In the following theorem, we are using the definition of accumulation points following the usual topology of $\mathbb{R}^2$.

Theorem 5.3.2 (Identity theorem for split-analytic maps)**.** *Let $\Omega \subseteq \mathbb{C}'$ be a open and connected domain. Also, let f and g be two split-holomorphic maps that are analytic on Ω. If the set $E = \{z \in \Omega | f(z) = g(z)\}$ contains an accumulation point then $f = g$ on Ω.*

Proof. Consider $h = f - g$. Let $\eta \in E$ be an accumulation point. This implies that for any $r > 0$, $B_r(\eta)$ contains a point $z \neq \eta$ with $h(z) = 0$. Then by lemma 3.1, there is an open ball $B_R(\eta) \subseteq \Omega$ where h is identically zero. Suppose $a \in \Omega \setminus B_R(\eta)$. Since Ω is path connected, there is a path γ with $\gamma(0) = \eta$,$\gamma(1) = a$. Let $t_0 = sup\{t \in [0,1] | h(\gamma(s)) = 0 \forall s \in [0,t]\}$. Note that such a t_0 exists, as this set is non-empty and bounded. Due to continuity of h, we have $h(\gamma(t_0)) = 0$. This imply $\gamma(t_0)$ is a non-isolated zero of h. By lemma 3.1, this implies h must be identically zero in a neighborhood of $\gamma(t_0)$. So unless $t_0 = 1$, we can always find a $\delta > 0$ such that $h(\gamma(t_0 + s)) = 0$ for any $0 < s \leq \delta$. This contradicts the definition of t_0. Thus t_0 must be 1, implying $h(a) = 0$. □

Now we define the hyperbolic annulus.

Definition 5.3.3. *A hyperbolic annulus is a region of the form $D = \{x + k'y | x > 0, a < x^2 - y^2 < b\}$ in the split-complex plane where a and b are two real numbers.*

We note that, unlike circular annulus, the hyperbolic annulus is simply connected. Now we define split-harmonic conjugate maps.

Definition 5.3.4. *For a split-harmonic map* $u : \Omega \subseteq \mathbb{C}' \to \mathbb{R}$*, a map* $v : \Omega \subseteq \mathbb{C}' \to \mathbb{R}$ *is said to be a split-harmonic conjugate of* u *if* v *satisfies,*

$$u_x = v_y$$

$$u_y = v_x.$$

We prove that on a simply connected domain, any split-harmonic map has a split-harmonic conjugate.

Theorem 5.3.3. *Any split-harmonic map* u *on a simply connected domain* Ω *has a split-harmonic conjugate* v*.*

Proof. Let $v(z) = \int_{z_0}^{z} u_y dx + u_x dy$. We note that by Green's theorem, on any closed curve C in the domain,

$$\int_C u_y dx + u_x dy = \int_D (u_{xx} - u_{yy}) dx dy = 0$$

Where D is the region bounded by C. Thus the map v is well defined and it satisfies

$$u_x = v_y$$

$$u_y = v_x.$$

Thus v is a split-harmonic conjugate of u up to a constant.

□

We now state the general form of a split-harmonic map on a simply connected domain.

Theorem 5.3.4 (Representation formula for split-harmonic functions)**.** *Let* Ω *be a simply-connected domain and* $F \colon \Omega \subset \mathbb{C}' \to \mathbb{C}'$ *be a split-harmonic map. Then* F *can be written as*

$$F = h + \overline{g}$$

with h,g being split-holomorphic maps. This representation formula is unique up to an additive constant.

Proof. F being split-harmonic implies,

$$\frac{\partial F}{\partial \overline{z} \partial z} = 0.$$

In particular F_z is split-holomorphic. We fix a point z_0 in Ω. Let $h(z) = \int_{z_0}^{z} F_z dz$. is a well-defined map which is split-holomorphic.We refer to [7] page 16 for proof of well-definiteness of h. Let $g = \overline{F - h}$. We note that

$$g_{\overline{z}} = \frac{\partial \overline{(F-h)}}{\partial \overline{z}} = \overline{F_z - h_z} = 0$$

in Ω.

Thus g is also split-holomorphic. We also have $F = h + \overline{g}$. To show uniqueness up to additive constant, suppose

$$F = h_1 + \overline{g_1} = h_2 + \overline{g_2}$$

then $u = h_1 - h_2 = \overline{g_1 - g_2}$ is both split-holomorphic and anti split-holomorphic(I.e, conjugate of a split-holomorphic map). Using Cauchy-Riemann equations in split-complex analysis, one can show that a map that is both split-holomorphic and anti-split-holomorphic must be a constant. □

The above proof is inspired by a similar result for harmonic maps in [15], page 7.

5.4 The singular Björling problem

In this section, we give a new proof of the singular Björling problem for timelike minimal surfaces.

Let $\mathbb{H}^1$ denotes the set $\{x + k' y \in \mathbb{C}' | x > 0, x^2 - y^2 = 1\}$.

We recall that for split-exponential map, $e^{k'\theta} = \cosh\theta + k' \sinh\theta$. Thus any split-complex number $x + k' y$ with $x > 0$ and $-x < y < x$, can be written as $z = \rho e^{k'\theta}$ for some real numbers θ and ρ. Also any point in $\mathbb{H}^1$ can be written as $e^{k'\theta}$ for some real number θ. We refer to [3], page 11 for more details.

We start with a definition.

Definition 5.4.1. *A map $\alpha : \mathbb{H}^1 \to \mathbb{C}'$ is said to be analytic if for any point η in $\mathbb{H}^1$, there is a neighborhood of it following the usual topology of $\mathbb{R}^2$, where it can be represented by a power series, $\alpha(z) = \Sigma_0^\infty c_n(z-\eta)^n$.*

A curve $\gamma : \mathbb{H}^1 \to \mathbb{L}^3$ is said to be analytic if each of its components is analytic.

Given analytic $\gamma : \mathbb{H}^1 \to \mathbb{L}^3$ and $L : \mathbb{H}^1 \to \mathbb{L}^3$, we define maps g_1 and g_2 on $\mathbb{H}^1$ as follows. Let

$$g_1(e^{k'\theta}) = \Big((L_1 + k'L_2) + k'(\gamma_1 + k'\gamma_2)\Big)\, e^{k'\theta}$$

and

$$g_2(e^{k'\theta}) = \Big((L_1 - k'L_2) + k'(\gamma_1 - k'\gamma_2)\Big)\, e^{k'\theta}.$$

Now we state the main result of this section.

Theorem 5.4.1. *Suppose an analytic lightlike curve $\gamma : \mathbb{H}^1 \to \mathbb{L}^3$, and an analytic lightlike vector field $L : \mathbb{H}^1 \to \mathbb{L}^3$ are given with the properties that,*

- $\langle \gamma', L \rangle = 0$
- *analytic extension $g_1(z)$ of $g_1(e^{k'\theta})$ and $g_2(z)$ of $g_2(e^{k'\theta})$ satisfy $|g_1(z)| \not\equiv |g_2(z)|$.*

Then there is a generalized timelike minimal surface $F = (h, \omega)$ defined on some hyperbolic annulus $A(r,R) = \{x + k'y \in \mathbb{C}' | x > 0, 0 < r < x^2 - y^2 < R\}$, $r < 1 < R$ with singular set atleast $\mathbb{H}^1$ such that,

$$F(e^{k'\theta}) = \gamma(e^{k'\theta}),$$
$$\frac{\partial F}{\partial \rho}|_{e^{k'\theta}} = L(e^{k'\theta})$$

Proof. Since γ and L are analytic on $\mathbb{H}^1$, there is a hyperbolic annulus $A(r,R)$ containing $\mathbb{H}^1$, where analytic extensions of both γ and L exists. We construct a harmonic map h on $A(r,R)$ with,

$$h_\theta = \gamma_1 + k'\gamma_2,$$

$$h_\rho = L_1 + k'L_2.$$

Using,

$$h_z = \frac{1}{2}(h_\rho + k^{'} h_\theta)e^{k^{'}\theta}$$

and

$$h_{\overline{z}} = \frac{1}{2}(h_\rho - k^{'} h_\theta)e^{-k^{'}\theta},$$

We get

$$h_z = \frac{1}{2}(L_1 + k^{'} L_2 + k^{'}(\gamma_1 + k^{'}\gamma_2))e^{k^{'}\theta},$$

$$h_{\overline{z}} = \frac{1}{2}(L_1 + k^{'} L_2 - k^{'}(\gamma_1 + k^{'}\gamma_2))e^{-k^{'}\theta}.$$

Using $dh = h_z dz + h_{\overline{z}} d\overline{z}$ from [3], We fix a point $z_0 \in \mathbb{H}^1$ and define,

$$h(z) = \int_{z_0}^{z} dh = \int_{z_0}^{z} h_z dz + h_{\overline{z}} d\overline{z}.$$

I.e,

$$h(z) \int_{z_0}^{z} (\frac{1}{2}(L_1 + k^{'} L_2 + k^{'}(\gamma_1 + k^{'}\gamma_2))e^{k^{'}\theta})dz + (\frac{1}{2}(L_1 + k^{'} L_2 - k^{'}(\gamma_1 + k^{'}\gamma_2))e^{-k^{'}\theta})d\overline{z}.$$

Where this integral is taken along any path in A(r, R) joining z_0 to z. We refer to [3], page 14 for similar results. By Stoke's theorem, this map is well defined and satisfies

$$h_\theta = \gamma_1 + k^{'}\gamma_2,$$

$$h_\rho = L_1 + k^{'} L_2.$$

Since $h_z = \frac{1}{2}(L_1 + k^{'} L_2 + k^{'}(\gamma_1 + k^{'}\gamma_2))e^{k^{'}\theta}$ is split-analytic, $h_{z\overline{z}} = 0$ and h is split-harmonic. Proof of existence of ω with $\omega_\theta = \gamma_3^{'}$ and $\omega_\rho = L_3$ is similar.

Now to show this (h, ω) satisfies $h_z \overline{h_{\overline{z}}} - \omega_z^2$ we note by computations that,

$$h_z \overline{h_{\overline{z}}}(e^{k^{'}\theta}) = \frac{L_3^2 + \gamma_3^{'2} + 2k^{'} L_3 \gamma_3^{'} e^{2k^{'}\theta}}{4} = \omega_z^2(e^{k^{'}\theta}).$$

Thus we note that the split-holomorphic map $h_z \overline{h_{\overline{z}}} - \omega_z^2$ is zero on $\mathbb{H}^1$. By using theorem

3.2(Identity theorem) we conclude that it is zero on the entire hyperbolic annulus $A(r,R)$.

To show $|h_z|$ not identically same as $|h_{\overline{z}}|$, we note that the maps $g_1(e^{k'\theta})$, $g_2(e^{k'\theta})$ agree with the maps h_z ,$\overline{h_{\overline{z}}}$ on $\mathbb{H}^1$. One can show this by calculations. Now by theorem 3.2(identity theorem), thus the maps $g_1(z), g_2(z)$ are same as h_z ,$\overline{h_{\overline{z}}}$. Now the assumption $|g_1(z)|$ is not identically the same as $|g_2(z)|$ ensures the desired result.

To show singular set contains at least $\mathbb{H}^1$, one can compute and prove $(|h_z|-|h_{\overline{z}}|)^2(e^{k'\theta}) = 0$. Here $|h_z|^2 + |h_{\overline{z}}|^2(e^{k'\theta}) = \frac{1}{2}(\gamma_3'^2 - L_3^2) = 2|h_z||h_{\overline{z}}|(e^{k'\theta}) = 2|\omega_z|^2(e^{k'\theta})$. □

5.5 Interpolating a given split-Fourier curve to a point

In this section, we study the problem of interpolating a given spacelike or timelike split-Fourier curve to a point p by a timelike minimal surface. This is similar to a result in [7]. We start with an example, where a curve is interpolated to a point by a timelike minimal surface.

Example 19. *The curve* $c : [0, 2\pi] \to \mathbb{L}^3$ *given by,*

$$c(t) = (\cos t, \sin t.0)$$

is interpolated to the origin, (0,0,0) by the timelike minimal surface, the $x - y$ *plane.I.e, by the surface* $X : \mathbb{R}^2 \to \mathbb{L}^3$ *given by,*

$$X(u, v) = (u, v, 0)$$

In this section, we look for similar instances where a curve gets interpolated to a point. We start by proving that for any analytic split-Fourier curve, split-Fourier coefficients are unique.

Theorem 5.5.1. *For any analytic split-fourier curve* $\gamma(\theta) = (\gamma_1 + k'\gamma_2, \gamma_3) = (\Sigma_{-\infty}^{\infty} c_n e^{k'n\theta}, \Sigma_{-\infty}^{\infty} d_n e^{k'n\theta})$ *, the coefficients* c_n *and* d_n *are unique.*

Proof. Using induction, we can prove that

$$e^{k' n\theta} = \cosh n\theta + k' \sinh n\theta.$$

Thus to show the series $\Sigma_{-\infty}^{\infty} c_n e^{k' n\theta}$ has unique coefficients, it is enough to show its split-real and split-imaginary parts $\Sigma_0^{\infty} a_n \cosh n\theta$ and $\Sigma_0^{\infty} b_n \sinh n\theta$ with a_n and $b_n \in \mathbb{R}$ have unique coefficients. We show that for any function f with a converging series expansion $f(\theta) = \Sigma_0^{\infty} a_n \cosh n\theta$, the coefficients a_n are unique.Proof for $\Sigma_0^{\infty} b_n \sinh n\theta$ is similar.

f is real analytic and it has an analytic extension $f(z) = \Sigma_0^{\infty} a_n \cosh nz$ within a radius of convergence R. Let i denote the complex number with $i^2 = -1$. The point ib is inside this disc for any b with $0 \leq b < R$ and we have $f(ib) = \Sigma_0^{\infty} a_n \cosh nib = \Sigma_0^{\infty} a_n \cos nb$. Let $g(b) = f(ib) = \Sigma_0^{\infty} a_n \cos nb$. Then this is a Fourier expansion of g in $(-R, R)$. By multiplying g with a constant if necessary, we assume that $R > \pi$. Thus the coefficients a_n are Fourier coefficients of g in $[-\pi, \pi]$ and thus they are unique.

□

We solve the problem of interpolating a given spacelike or timelike analytic split-Fourier curve to the point $p = (0, 0, 0)$ using a timelike minimal surface. This is in parallel to a similar result in [7].

Theorem 5.5.2. *A given spacelike or timelike analytic split-Fourier curve*

$$\gamma(\theta) = (\Sigma_{-\infty}^{\infty} c_n e^{k' n\theta}, \Sigma_{-\infty}^{\infty} d_n e^{k' n\theta})$$

can be interpolated as $X(re^{k'\theta})$ *to the point* $p = (0, 0, 0)$ *with* $X(e^{k'\theta}) = p$ *using a timelike minimal surface* X*, if there is an* $r > 0$ *such that* c_n *and* d_n *satisfies,*

$$\Sigma_{n=-\infty}^{\infty} 4n(n-m) \frac{c_n \overline{c}_{n-m} r^{2n-m}}{(r^n - 1)(r^{n-m} - 1)} - \Sigma_{\{(i,j)|i+j=m\}} 4ij \frac{d_i d_j r^k}{(r^i - 1)(r^j - 1)}$$

whenever $m \neq 0, n \in \mathbb{Z}$ *and*

$$\Sigma_{n=-\infty}^{\infty} 4n^2 \left(\frac{\overline{c_n} c_n r^{2n}}{(r^n - 1)^2} + \frac{d_n d_{-n}}{(r^n - 1)(r^{-n} - 1)} \right) = 0$$

Proof. Since γ is analytic, there is a hyperbolic annulus $A(r, R)$ containing $\mathbb{H}^1$ where its analytic extension exists. We construct a timelike minimal surface X with $X(e^{k'\theta}) = p$ and $X(re^{k'\theta}) = \gamma(\theta)$ on this hyperbolic annulus, $A(r, R)$.

We consider split-harmonic maps of the form

$$h(z) = \Sigma_{-\infty}^{\infty} a_n z^n + b_n \frac{1}{\overline{z}^n}.$$

Using $h(e^{k'\theta}) = (0,0)$ we get $a_n = -b_n$. similary using $h(re^{k'\theta}) = \Sigma_0^{\infty} c_n e^{k'n\theta}$, one can compute a_n. From here we get, $a_n = \frac{c_n r^n}{(r^{2n}-1)}$. We note that only finitely many a_n are nonzero, as only finitely many c_n are nonzero(as we assumed split-Fourier curves are only allowed to have finite series expansions, in definition 5.1.3).

Assuming

$$\omega(z) = \Sigma_{-\infty}^{\infty} f_n z^n + g_n \frac{1}{\overline{z}^n}$$

and doing similar computations, we conclude that $f_n = -g_n$ and $f_n = \frac{d_n r^n}{(r^{2n}-1)}$.

Thus the surface $X(z) = (h(z), \omega(z))$ passes through both p and the given curve γ. To show this is timelike minimal, we compute and get

$$h_z \overline{h_{\overline{z}}} - \omega_z^2(e^{k'\theta}) = \frac{1}{4}\left(\Sigma_{-\infty}^{\infty} 2na_n e^{k'n\theta} \Sigma_{-\infty}^{\infty} 2n\overline{a}_n e^{-k'n\theta} - \left(\Sigma_{-\infty}^{\infty} 2nf_n e^{k'n\theta} \right)^2 \right).$$

From here, comparing coefficients of $e^{k'(n-m)\theta}$ when $n \neq m$ and when $n = m$, one gets the conditions given in the theorem. Thus the above quantity vanishing is equivalent to the conditions given in the result. This being zero implies $h_z \overline{h_{\overline{z}}} - \omega_z^2 = 0$ on the entire hyperbolic annulus by identity theorem of split-analytic maps. Also $|h_z|$ is not identically same as $|h_{\overline{z}}|$ as $X(re^{k'\theta})$ is a spacelike or timelike curve. Thus X is a timelike minimal surface interpolating both γ and p. □

s

5.6 Interpolating a split-Fourier curve to another split-Fourier curve

We solve the problem of interpolating a given analytic spacelike or timelike split-Fourier curve to another specified analytic split-Fourier curve.

Theorem 5.6.1. *Given a spacelike or timelike analytic split-Fourier curve*

$$\gamma(\theta) = (\Sigma_{-\infty}^{\infty} c_n e^{k' n\theta}, \Sigma_{-\infty}^{\infty} d_n e^{k' n\theta})$$

and a analytic split-Fourier curve

$$\alpha(\theta) = (\Sigma_{-\infty}^{\infty} l_n e^{k' n\theta}, \Sigma_{-\infty}^{\infty} m_n e^{k' n\theta})$$

, for $r > 0$ let

$$a_n(r) = \frac{r^n c_n - l_n}{r^{2n} - 1}$$

and

$$f_n(r) = \frac{r^n d_n - m_n}{r^{2n} - 1}.$$

γ can be interpolated as $X(re^{k'\theta})$ to α as $X(e^{k'\theta})$ by a timelike minimal surface X if there is an $r > 0$ such that $a_n(r)$ and $f_n(r)$ satisfies,

$$\Sigma_{n=-\infty}^{\infty} 4n(n-m)a_n(r)(\overline{a_{n-m}(r)} - \bar{l}_{n-m}) - \Sigma_{\{(i,j)|i+j=m\}} 4ij f_i(r) f_j(r) = 0,$$

for any $m \neq 0$ and,

$$\Sigma_{n=-\infty}^{\infty} 4n^2 \left(a_n(r)(\overline{a_n(r)} - \bar{l}_n) + f_n(r) f_{-n}(r) \right) = 0.$$

Proof. The proof is similar to that of theorem 5.2. We consider split-harmonic maps of the form,

$$h(z) = \Sigma_{-\infty}^{\infty} a_n z^n + b_n \frac{1}{\bar{z}^n}$$

and

$$\omega(z) = \Sigma_{-\infty}^{\infty} f_n z^n + g_n \frac{1}{\bar{z}^n}.$$

$X(e^{k\theta}) = \alpha$ imply $b_n = l_n - a_n$ and $g_n = m_n - f_n$. Similarly $X(re^{k'\theta}) = \gamma$ implies

$$a_n = \frac{r^n c_n - l_n}{r^{2n} - 1}$$

and

$$f_n = \frac{r^n d_n - m_n}{r^{2n} - 1}$$

. Here we note that only finitely many a_n,b_n are nonzero, as only finitely many c_n,d_n,l_n,m_n are nonzero(as we assumed split-Fourier curves are only allowed to have finite series expansions, in definition 5.1.3).

Thus with these choices of a_n, b_n, f_n, g_n the surface X passes through both α and γ. To show this is timelike minimal we compute and get,

$$h_z\overline{h_{\overline{z}}} - \omega_z^2(e^{k'\theta}) = \frac{1}{4}\left(\Sigma_{-\infty}^{\infty}(2na_n e^{k'n\theta})\Sigma_{-\infty}^{\infty}(2n(\overline{a}_n - \overline{l}_n)e^{-k'n\theta} - \left(\Sigma_{-\infty}^{\infty}2nf_n e^{k'n\theta}\right)^2\right).$$

From here, comparing coefficients of $e^{k'(n-m)\theta}$ when $n \neq m$ and when $n = m$, one gets the conditions given in the theorem. The above quantity vanishing is equivalent to the conditions given in the result. This being zero implies $h_z\overline{h_{\overline{z}}} - \omega_z^2 = 0$ on the entire hyperbolic annulus by identity theorem of split-analytic maps. Also $|h_z|$ is not identically same as $|h_{\overline{z}}|$ as $X(re^{k'\theta})$ is a spacelike or timelike curve. Thus X is a timelike minimal surface interpolating both γ and α. □

Example 20. *The timlike minimal surface $X : \mathbb{C}' \to \mathbb{C}' \times \mathbb{R}$ given by,*

$$X(z) = (\frac{z^4}{4} - \frac{1}{\overline{z}}, 2Rez)$$

Interpolates the split-Fourier curve $c_1 : \mathbb{R} \to \mathbb{R}^3$ to $c_2 : \mathbb{R} \to \mathbb{R}^3$ which are given by,

$$c_1(\theta) = \left(\frac{\cosh 4\theta}{4} - \cosh\theta, \frac{\sinh 4\theta}{4} - \sinh\theta, 2\cosh\theta\right)$$

$$c_2(\theta) = \left(4\cosh 4\theta - \frac{1}{2}\cosh\theta, 4\sinh 4\theta - \frac{1}{2}\sinh\theta, 4\cosh\theta\right).$$

Chapter 6

Conclusion and the ongoing research work

So far in this thesis, we dealt with the Björling problem for Born-Infeld solitons and the interpolation problem for timelike minimal surfaces. We recall that from the last chapter, we got some algebraic conditions about split-Fourier coefficients which ensure interpolation of two given split-Fourier curves by a timelike minimal surface. Right now, given this, we are now exploring,

1. What is the geometric meaning of the algebraic conditions we got?
2. How general is our approach, and what conditions should we impose to interpolate two arbitrary curves?
3. How do geometric parameters associated with given curves such as distance and so on dictate the shape?
4. Given two arbitrary curves on space, how close(to give an explicit number bound) do they have to be for an interpolating minimal or maximal surface to existing and what is the shape(parametrization) of such a surface?

In fact for the interpolation problem of two arbitrary Jordan curves, for minimal surfaces, Jesse Douglas has already given existence results in [14]. We also note that in [9] and in [10], for maximal surfaces, Rukmini Dey, Pradip Kumar, and Rahul Kumar Singh also approach this problem and gave an existence result using the inverse function theorem in Banach manifolds.

But, we, however, are interested in quantifying and coming up with explicit parametrizations and distance bounds for the interpolation of minimal and maximal surfaces. We believe that this can be done using tools of complex analysis and this is a work in progress.

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