Exploring Random Walks on Networks: An Analysis of Extreme Events

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by

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Certificate

This is to certify that this dissertation entitled Exploring Random Walks on Networks: An Analysis of Extreme Eventstowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents work carried out by Pranav M at Indian Institute of Science Education and Research under the supervision of Prof M.S. Santhanam, Professor, Department of Physics, during the academic year 2022-2023.

Santhanam

Prof M.S. Santhanam

Committee:

Prof M.S. Santhanam

Prof Amit Apte

This thesis is dedicated to my Mother, a source of constant strength through some tough times

Declaration

I hereby declare that the matter embodied in the report entitled Exploring Random Walks on Networks: An Analysis of Extreme Events are the results of the work carried out by me at the Department of Data Science, Indian Institute of Science Education and Research, Pune, under the supervision of Prof M.S. Santhanam and the same has not been submitted elsewhere for any other degree.

Pranav M

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I am grateful to all the people in my life with whose support and constant presence I have done anything at all in my life. This project, and my life, in addition to groups of people, would not have been as colorful as it is, without the environments I grew up in, including my family, my school, and especially the free environment at IISER Pune.
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Abstract

With its key role in our modern lives, understanding transportation systems and mobility from scientific frameworks is vital to designing more resilient and robust systems of travel. In contrast to the conventional approach of looking at complex models directly, we promote building up from simpler models that are more interpretable to more realistic models. We also additionally realize the multimodal nature of transportation and transit and the crucial role it plays in future developments. Using the rich analytical vocabulary of network science, we decide to investigate the behavior of simple random walks on multilayer networks. Inspired by previous works in the area, we enquire especially regarding the evolution and propagation of extreme events that are borne out of such dynamics. We further introduce features into the random walk dynamics to track new sources of extreme behavior. We also explore the phenomena of non-markovian random walks so as to enquire about its implications on the occurrence of extreme events. We see the consistent occurrence of extreme events when extended to multilayer networks and also in random walks with rerouting strategies. We uncover sudden transitions in the states of the networks in random walks with cascades and a decreasing influence of adding memory on the occurrence of extreme events. Our study demonstrates the importance of this modeling approach to understand real-world dynamics on networks and opens up several avenues to further explore this and an opportunity to pair it with data-based approaches

Contents

List of Figures

Chapter 1

Introduction

Network Science has become a cornerstone in understanding many complex systems and phenomena of the real world. The ability of networks to represent relations and dynamics has made them a tractable tool for understanding these systems. The History of Network Science is a very interesting one. While theoretical works began as an offshoot of graph theory in the 1950s and were boosted by the works of Paul Erdos and Alfred Renyi, whose network model we will be using here, it was an empirical tool in social science fields such as Sociology by the 1930s. It has now grown to be a much bigger field, having applications ranging from physics to biology and economics.

One of these complex systems is transportation systems. Scientific analysis of transportation systems is important for developing more robust and resilient ones. While networks have been used for some time in understanding transport systems, it is limited to a single type of connection and thus unable to capture the complexity associated with multi-modal transportation systems that exist at different spatial scales. Multilayer networks, equipped with handling different kinds of connections(edges), are therefore more suited for this purpose. The framework represents the various modes of transport, such as roads, railways, and metro, as different layers of the multilayer network [1, 2]

As the complexity of systems increases, modelers face the challenge of balancing simplicity and realism in their models. In this work, we initially use simple models like random walks to understand the critical properties of the network and the transportation infrastructure it represents. We then move towards more sophisticated random walk models to observe more interesting behaviors, such as cascades and walks with memory. The changes to the model are chosen so that they mimic some properties of the transportation system that the modeler is trying to get insights about using the simulations.

We begin with idealized simplistic models of motion on the networks that represent the traffic, using discrete-time random walks. This simple framework does, however, help us understand the critical properties of the network and, thereby, the transportation infrastructure it represents. We move towards more sophisticated random walk models that help us observe more interesting behaviors, such as cascades and walks with memory.

Random walks are limited in understanding transport systems, as they are stochastic models blind to the environment and temporal changes. As a first step, random walks with rerouting strategies will be introduced. These rerouting strategies help walkers avoid congested nodes and thus lead to different dynamics. We will also look at random walks with cascades with random walks, where we will also be concerned with the changing structure of the network. In memory walks again, we will see how dynamical constraints on the motion of walkers can impact how extreme events occur on the network.

The focus remains on extreme events characterized by a deviation from mean behavior. In transportation systems, these remain unfavorable, and in real systems, there is a tendency to minimize such events. This thesis will focus on characterizing different extreme events related to random walks on networks and understanding their basic behavior. And to understand extreme events, we need to have multiple walkers on the network, which we achieve by actualizing large numbers of simultaneous random walkers that follow the same rules on the network. When working with multilayer networks, there will be an emphasis on understanding the dynamics of the separated layers and how the network structure impacts the dynamics and extreme events as a phenomenon.

Chapter 2

Background

2.1 Networks & Models

Network Science, as described earlier, began as an offshoot of graph theory. The field has been hugely successful in describing a lot of real-world systems. This has been largely due to the ability of networks to represent the different classes of objects and relationships that exist in these systems. When we need sometimes to extract network structures from the systems, a lot of times, we are required to choose a suitable model to describe the system. In such cases, different pre-existing network models are chosen to suit the system we are trying to model.

Different network models have been constructed which follow different linking strategies and thus provide the user with different parameters to manipulate. In many cases, these provide useful approximations of real networks associated with complex systems.

Throughout this thesis, the computational construction of networks has been done using the Networkx package.[3]

2.1.1 Erdos-Renyi Network

Random Networks were the next step from the fixed networks of graph theory. They present us with network structures with degree distributions rather than fixed degrees. One of them is the famous Erdos-Renyi Network(citation). One of the notations for Erdos-Renyi (ER) Networks is G(n,p), where n is the number of nodes in the network and p is the probability of an edge between any two given nodes and is the defining parameter. Thus, in an ER network, the probability that a node has a degree d, where d is strictly less than n is given by,

$$f(d) = \binom{n-1}{d}(d)^p(n-d-1)^{1-p}$$

Thus the mean degree is $(n-1)p$ and the standard deviation is $\sqrt{(n-1)p(1-p)}$. These graphs show low clustering as a result of the degree distribution and are not common in social systems.

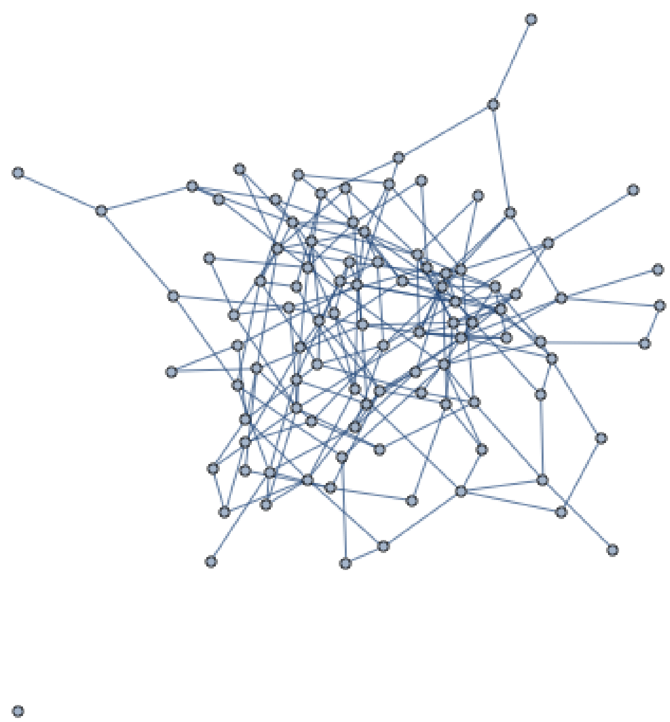

Figure 2.1: Illustration of an Erdos-Renyi Network

2.1.2 Barabasi-Albert Network

Scale-free networks are those in which the degree distribution follows a power law, at least asymptomatically. The probability of a node having a degree k, follows the following rule asymptomatically.

$$P(k) \approx k^{-\gamma}$$

where γ is a parameter that lies between 2 and 3. When the value is 3, this network is called the Barabasi-Albert network. While a lot of real networks follow this pattern, among Transportation networks, it is seen in airways. The underlying process behind scale-free networks is growing nodes and preferential attachment to nodes with higher degrees, which leads to the formation of hubs.

We start with m_0 nodes, the links between which are chosen arbitrarily, as long as each node has at least one link. The network develops following two steps:

- Growth: At each time step, we add a new node with $m(m_0)$ links that connect the new node to m nodes already in the network.

- Preferential attachment: The probability (k) that an edge of the new node is connected to node i depends on the degree k_i as $\Pi(k_i) = \frac{k_i}{\sum_j k_j}$ [4]

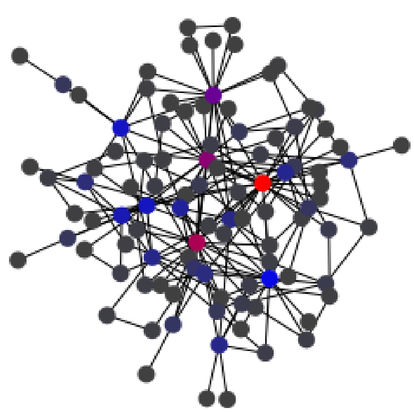

Figure 2.2: Illustration of an Barabasi-Albert Network

2.1.3 Planar Networks

While ER networks and BA networks do not cover the diversity of real or theoretical models of networks, they are instrumental in beginning to understand networks across scales. Transportation networks, especially of Railways and Roadways, resemble neither of these networks but are actually planar networks. Planar networks or planar graphs are those in which the network has a representation where the edges intersect only at vertices(nodes) and do not cross over.

Planar networks tend to have narrow degree distributions skewed towards small degrees, with a small number of nodes with slightly higher degrees. Thus, the abstraction is unable to differentiate between different nodes when it comes to simple random-walk dynamics. The planarity of a network is an important criterion for analyzing and working with networks. Thus, there are algorithms in place for testing the planarity of networks. Real transport and transit network structures can be extracted from open source datasets such as OSMnx [5]

Figure 2.3: A Planar network representing a road network

2.2 Multilayer Networks

Multilayer Networks are born out of a need to represent different categories of relationships within the same structure. In such a network, the nodes on different layers can represent different kinds of objects. The various intra-layer and inter-layer edges can represent different relationships and inherent dynamics. For example, on a social-ecological network, one layer can represent social agents and the edges being communication, whereas the other layer can represent ecological populations and the edges the diffusion or movement of these populations. The interlayer edges could, in this case, indicate extraction dynamics. Similarly, on multimodal transportation systems, one layer could represent a railway network, and the other layer, a road network with vastly different dynamics on each of the layers. [6]

It is important to remember that there are different kinds of multilayer networks and different terminologies associated with them. Multiplex networks are normally associated with structures where the same nodes are represented in all the layers, and the difference is in the respective edge sets. Such cases are commonly present in social networks or communication networks. Multiplex networks form a distinct subset of multilayer networks. In other Multilayer networks, the nodes on different layers are representations of different objects, and thus, the number of nodes can vary from layer to layer. Such kinds of networks are widespread in ecology. In what is called Networks of networks, There are commonly a lot of layers, and even the layers can be multilayered networks onto themselves. The emphasis also is on how the various networks are connected to each other and the emergent interactions between the different networks. Various socio-technical systems can be modeled using Networks of networks.

Multilayer networks can be mathematically represented in various ways. They can be represented by adjacency tensors or supra-adjacency matrices. It's also been possible to extend many of the results and properties of single-layer networks to multilayer networks. In this thesis, we stick to the former method, which is consistent when working with adjacency matrices of single-layer networks as well[7].

Just like in single-layered networks, there are models of multilayer networks too. Although they are less popular, and more work is done using models constructed from real systems.

When running simulations such as random walks on multilayer networks, additional components are involved. One can study the dynamics on each layer separately, study the correlation of phenomena of interest across the layers and enquire about the effect of a parameter related to multilayeredness on the phenomenon of interest[8]. In this thesis, we restrict ourselves to two-layered networks. These networks are constructed by joining two single-layer networks of either the Erdos-Renyi statistics or the Barabasi-Albert Algorithm. We manually select the number of inter-layer edges, and their positions are randomized. The resulting networks follow neither the Erdos-Renyi model nor the Barabasi-Albert Algorithm. Analyzing behaviors on the two layers separately helps us understand if there is any conservation of behavior from the single-layered networks.

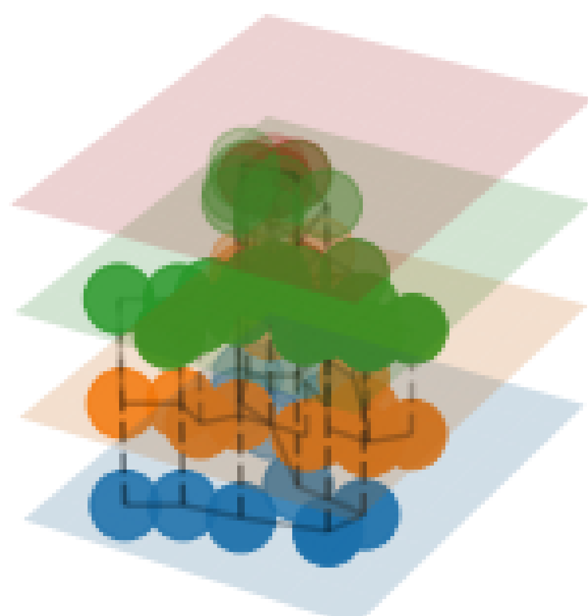

Figure 2.4: A multilayer network for transport system: Transportation Network

2.3 Random Walks

Random walks are one of the most simple yet helpful tools in understanding complex systems. They have been useful in understanding epidemics, forest fires, and other phenomena. With networks, they help us understand their structure and act as a baseline model for understanding more complicated dynamics.

Simple random walks are a class of stochastic processes and are Markovian in nature. This results in behaviors and results that can be analytically calculated using the theory of markovian processes.

Random walks can be conceptualized in various forms. While working with networks, the main two classifications are based on time intervals and biasing. In unbiased discrete-time random walks that we apply here initially, as the name suggests, movement occurs at equidistant time steps, and there is no waiting time, and the jump from the source node to the destination happens without any time delay. Additionally, any of the neighbors of the source node is equiprobable for being the destination. This dynamic is easy to implement but still gives rise to a lot of interesting behavior.

Consider a network G , with corresponding adjacency matrix A. Let the network be unweighted and undirected. Thus, if there is an edge between any two given nodes i and j, both A_{ij} and A_{ji} will be equal to 1. On this network, let there be W non-interacting walkers. The probability that a walker currently at i will move to j is given by A_{ij}/K_i, where K_i is the degree of the node i. We can write down a master equation for the n-step transition probability of a walker starting from node i at time n = 0 to node j at time n as,

$$P_{ij}(n+1) = \sum_k \frac{A_{kj}}{K_k} P_{ik}(n)$$

This leads to a computable stationary distribution.

$$\lim_{n\to\infty} P_{ij}(n) = P_j = \frac{K_j}{2E}$$

On multilayer networks, random walks are especially useful in understanding how the multilayeredness adds to the diversity and structure of the networks. One looks at several metrics that can emerge here to understand the impact of multilayeredness on the dynamics.

2.3.1 Random Walks with Rerouting

We can also study random-walks with additional properties. The reason we do this is to understand more about the structure of the network and to emulate dynamics that will be present in more complicated dynamics present in other systems. Most of these modifications act as constraints in terms of the movement of the individual walkers, whether they are emerging out of the dynamics of that walker, all the walkers, or externally.

The first property we apply to random-walks are the rerouting strategy. In this realization of the algorithm, the walks are no longer simple as some neighbors become inaccessible due to their occupation. A node is no longer a viable destination node if its occupancy is above a set threshold. In our algorithm, we set this threshold to be the same as the analytical extreme event threshold, which in turn is a function of the node degree. The walker, thus, is stuck on the source node for the given time interval, which creates a phenomenon of stuck or non-moving walkers.

2.3.2 Random-Walkers with memory

One of the other modifications is the Random walker with memory. These are walkers that do not visit the last k-visited nodes (random walks with memory) . This leads to constraints in terms of the movement of the walkers. Walkers that have no viable nodes get trapped and are unable to move [9, 10]

Additionally, we could also have walkers with memory k, where the parameter no longer refers to the last k nodes but nodes at the last k time-steps [11] These are called random walkers with decaying memory. In this version of the algorithm, we no longer have walkers getting trapped indefinitely. [12]

2.3.3 Random Walks with node removal

Another modification is random walks with node removal. This version becomes interesting as nodes of the network are removed if and when they encounter extreme events. We look at the evolution of the cascade size that results due to the node removalin the various networks.

[13, 14, 15]

The model is inspired by node percolation models and other network models in epidemiology, where infected patients are no longer susceptible. Cascades have been an important tool in understanding various processes that can be represented on networks such as web attacks. they are also used to understand the structural role of individual nodes on the geometry of the network as a whole. For individual walkers, the algorithm remains the same simple random walks. [16, 17]

2.4 Extreme Events

The scientific definition of extreme events is a crucial aspect of studies that aim to analyze and understand them. In layman's terms, such events are considered anomalous occurrences, but within the dynamics on networks literature, they are defined through density statistics. This involves analyzing the distribution of the density of walkers or agents associated with nodes, which can be obtained through statistical sampling or analytical methods. By applying appropriate thresholds to this distribution, extreme events can be identified. Typically, these thresholds are chosen to be specific orders of magnitude higher than the mean, although they may also be fixed values.

In Transportation and Information Networks, such extreme events are unfavorable as they lead to suboptimal performance of the network as a whole [18, 19]. We can imagine that, in Multimodal transportation systems represented by multilayer networks, there are additional unintended consequences of extreme events, such as congestion, that permeate the particular layer in which they occur. Also, the very multilayeredness of the networks can give rise to additional extreme events. Thus these questions on the existence, distribution, and control of extreme events become crucial practical and engineering questions as we start dealing with real systems[20, 21]

As we deal with different kinds of network dynamics, our conceptualizations of extreme events evolve to fit the model. While in simple random walks, we deal to focus on the overall distribution of extreme events, in other models, we are also interested in the time evolution of congested nodes and stuck walkers that arise due to the extra constraints in place. For example, in random walks with cascades, we track the size of the cascade and also the walkers that are unable to move as they get stuck in isolated nodes.

2.5 Characterizing Extreme Events

We begin with an unbiased discrete random walk on a network. There are no free parameters, and the walkers are restricted to only a neighboring node from the current node. All the neighbors are equally probable to be the destination in this scenario. This walk is executed for a finite amount of time, for finite but multiple numbers of walkers selected according to the experiment. Analytically, occupation probability is the probability that, given a simple random walker, it occupies a given node x of degree K.

$$OP(x) = \frac{K}{2E}$$

This helps us understand the dynamics of random walkers, given that they are non-interacting, the probability that a node has w out of W walkers at any given time step is given by,

$$f(w) = \binom{W}{w} p^w (1-p)^{W-w}$$

it has a mean of $<f> = \frac{WK}{2E}$ and a variance $\sigma^2 = W\frac{K}{2E}(1 - \frac{K}{2E})$ [22]

We characterize extreme events as those instances at which the no.of walkers at a certain node is above a certain threshold. This threshold is given by $<f> + \lambda\sigma$ where λ is a small natural number. Thus, we can also analytically calculate the probability of extreme events for all the different nodes.

$$F(k) = \sum_{j(k)}^{W} f(k)$$

where j is the threshold set for the node of degree k.

This can also be calculated using the closed formula given by

$$F(k) = \sum_{j(k)}^{W} f(k) = I_p(\lfloor q \rfloor + 1, W - \lfloor q \rfloor)$$

where $I_p(x, y)$ is the regularized incomplete beta function [23], where $q = <f> + \lambda\sigma$ is the set threshold. Here we set $\lambda = 4$.

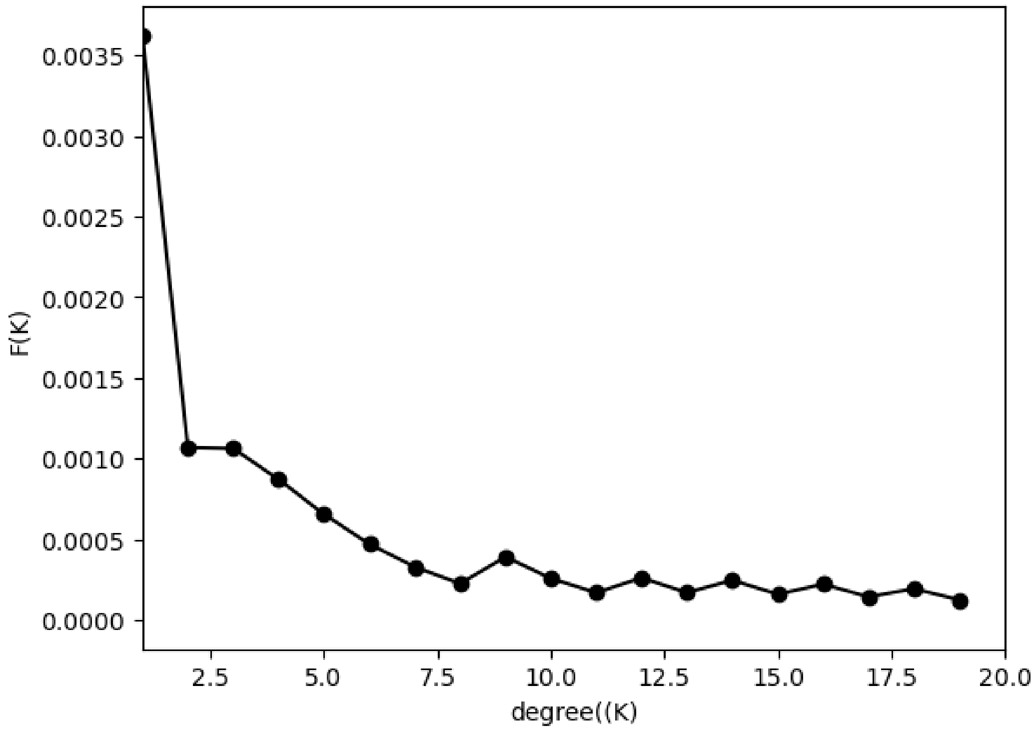

Figure 2.5: Extreme Events on a Barabasi-Albert Network

Understanding this kind of extreme events as well as other varieties that emerge due to the nature of dynamics in each of the setups, in addition to the ones that emerge due to the network structure, will help us move closer to recovering the properties of congestions that are widespread across various technical systems such as transportation, communication, and information networks. [18, 24]

Chapter 3

Implementation

3.0.1 Size of networks and simulations

We work with both Single-layer and double-layered networks. In the former case , we have two setups. In the first three setups for simulations, the size of the single-layer network is 200 nodes. For double-layered networks, present in the first 3 experiments, the size of the network is 400 nodes, 200 in each layer. The inter-layer connections have been manually added to randomized positions. The no.of walkers have been fixed to 400, and for a total of 400 time-steps.

For random walks with memory, the size of the network is fixed to a 1000 nodes. It has only been carried out in single-layer networks. The number of simultaneous walkers has been fixed to 1000, and they have all been simulated for 1000 time steps. The averages have been calculated over ensembling over 10 realizations of the experiment.

3.0.2 Initialization of the Random-walk algorithms

Across the models, the initialization has been based on a random seed. The position of any walker at t=0 is chosen at random. The further evolution of the model is dependent on the model chosen.

The walkers are non-interacting. Thus in the first experiment, there is no impact of the dynamics of walkers on each other. In rerouted and random walkers with cascade, there are indirect effects, but the algorithm is set in a way that the effect is not spontaneous but starts only at the next time step. This is achieved by modifying the adjacency matrix(or adjacency list for individual nodes) as a result of the congestions and node deaths that take place. This makes it in practice, even these experiments are non-interacting random walks. In random walks with memory, even though the walks are no longer markovian, and the memory of the previous node is stored for each walker, there is no impact of walkers on each other. The selection of future nodes follows simple random walk dynamics in the first case. In the case of random walks with rerouting, the neighbors which are currently operating above extreme event thresholds become unavailable for the moment.

With random walks with cascades, even though the algorithm for individual walkers remains the same, the network structure and, thus, the adjacency list keep evolving. And in the final case of random walks with memory, the path memory is stored in each walker, and it is unable to visit the last visited k nodes.

3.0.3 Investigation of Extreme Events

While in the simple random walk scenario, the extreme event characterization follows from the pre-existing definition, we have conceptualized novel characterizations of extreme events and other phenomena of interest for the other routing strategies.

In random walks with rerouting, in addition to the distribution of extreme events across degrees, we are also interested in the time series of stuck walkers. Stuck walkers are the walkers who are unable to move as the selected destination node is facing an extreme event. The inspiration for this model comes from congestion scenarios in real transportation systems and is a baseline case for rerouted walks.

In random walks with cascades, inspired by the breakdown of junctions and nodes in road networks facing congestion, nodes go dead. In this situation, these nodes are no longer able to act as destination nodes, and thus this alters the adjacency matrix in itself. There are fewer paths available for the walkers. As the dead nodes do not have the ability to rejuvenate, the evolution of dead nodes is a phenomenon of interest.

Chapter 4

Results & Discussion

The various plots and graphs obtained as a result of running the experiments are laid out here. We will look for trends that are common across the various experiments and the impact of the parameters and the network structure within the experiments to understand patterns that remain the same and the quirks of the models. We begin with the simple random walk model, where we shall investigate the distribution of extreme events. Second, we will look at how extreme events fare in random walks with rerouting strategies in place. The purpose of this is to check whether simple rerouting strategies can significantly alter the distribution of extreme events. In the third section of this chapter, the results from random walks with cascades are presented. The results will help us deduce the impact of node death on random-walk dynamics and also the propagation of extreme events with time in such a network. The results should carry hints in the design of real-world networks. Finally, we explore the impact of memory on extreme events in the final section. The result holds the keys to start appreciating vehicle dynamics on transportation networks.

4.1 Simple Random walks

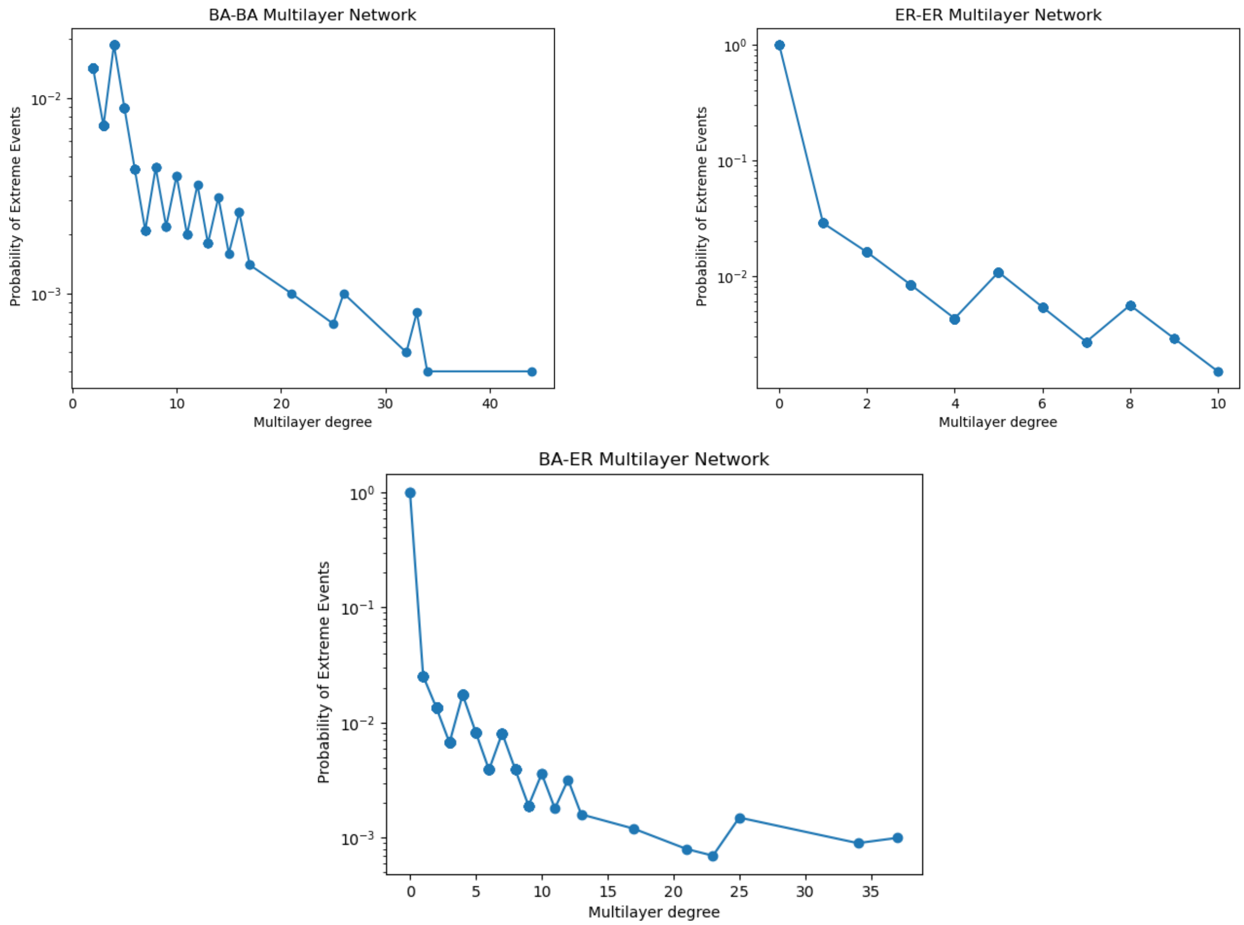

Figure 4.1: Extreme Events on Multilayer Networks

After running the simulation, we find that the trend found in single-layer BA networks, that of lower-degree nodes facing more extreme events as compared to higher-degree nodes, is also the case in Multilayer Networks. This holds true irrespective of the topology of the layers.

We do not observe any new patterns resulting from the multilayeredness of the networks. Given that we only work with two-layered networks with a fixed number of inter-layer links, it remains to see what is the impact of changing these parameters is on the results.

4.2 Random walks with Rerouting

We investigated the behavior of extreme events on single and multilayered networks with rerouting in place.

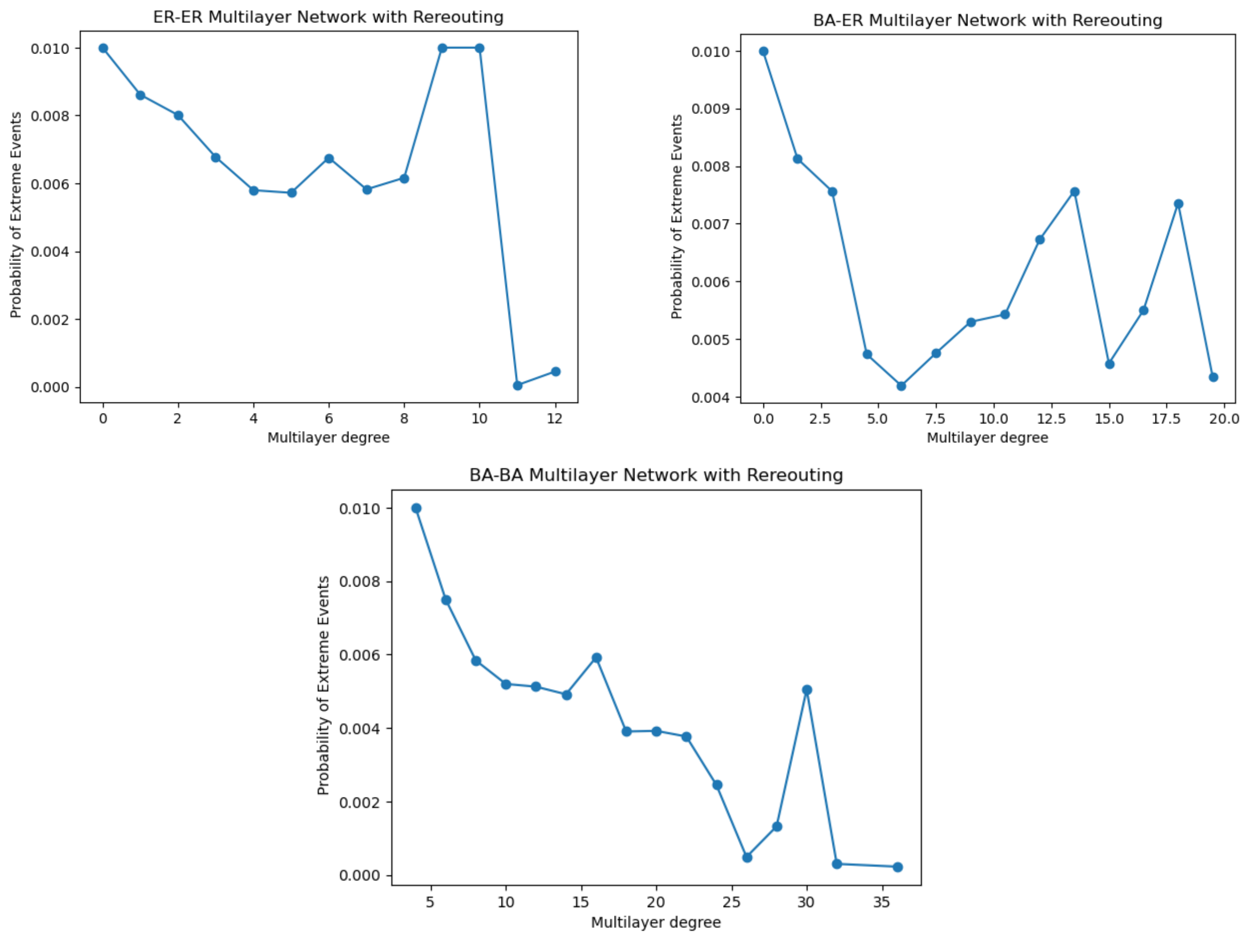

Figure 4.2: Extreme Events on Multilayer Networks with rerouting

We observe here also that there is a decrease in the probability of extreme events with an increase in the node degree. It is interesting that this is consistent across various routing strategies. It is observed, however, that the strategy results in certain cases of the reversal of the decreasing trend across networks. While the reasons are unknown, one speculation would be that rerouting makes nodes with higher degrees adjacent to low-degree nodes more vulnerable to extreme events. Especially in the ER-ER network, where there is lower clustering, this holds more true, as expected. It is also noticeable that the nodes with really

high degrees are not affected by this, as they are linked to more high-degree nodes in BA layers. This indicates that this kind of routing strategy might not imitate those followed in real systems and real traffic.
However, we can investigate other routing strategies that can be used inorder to influence and reduce the occurrence of extreme events. We will be looking at a non-markovian random walk aimed at achieving this goal.

4.3 Random Walks with Node removal

The phenomena of random walks with node removal are particularly interesting to study because of the way in which it modifies the network structure itself. What we will investigate here is the evolving structure itself and how it evolves with time. As there is a propagation of node deaths through the occurrence, we observe the dynamics of this process. The questions include of the rate of the cascading process and its final levels.

We observe that in image 4.3 all the networks, there are abrupt transitions from a low death regime to a high death regime. The rate and timeline of transition vary across networks. We see that thereafter, the process slows down or stops altogether, resulting in constant levels of dead nodes. To understand why this occurs, we will be looking at the dynamics of walkers on the networks and investigate why they do not cause extreme events beyond a point.

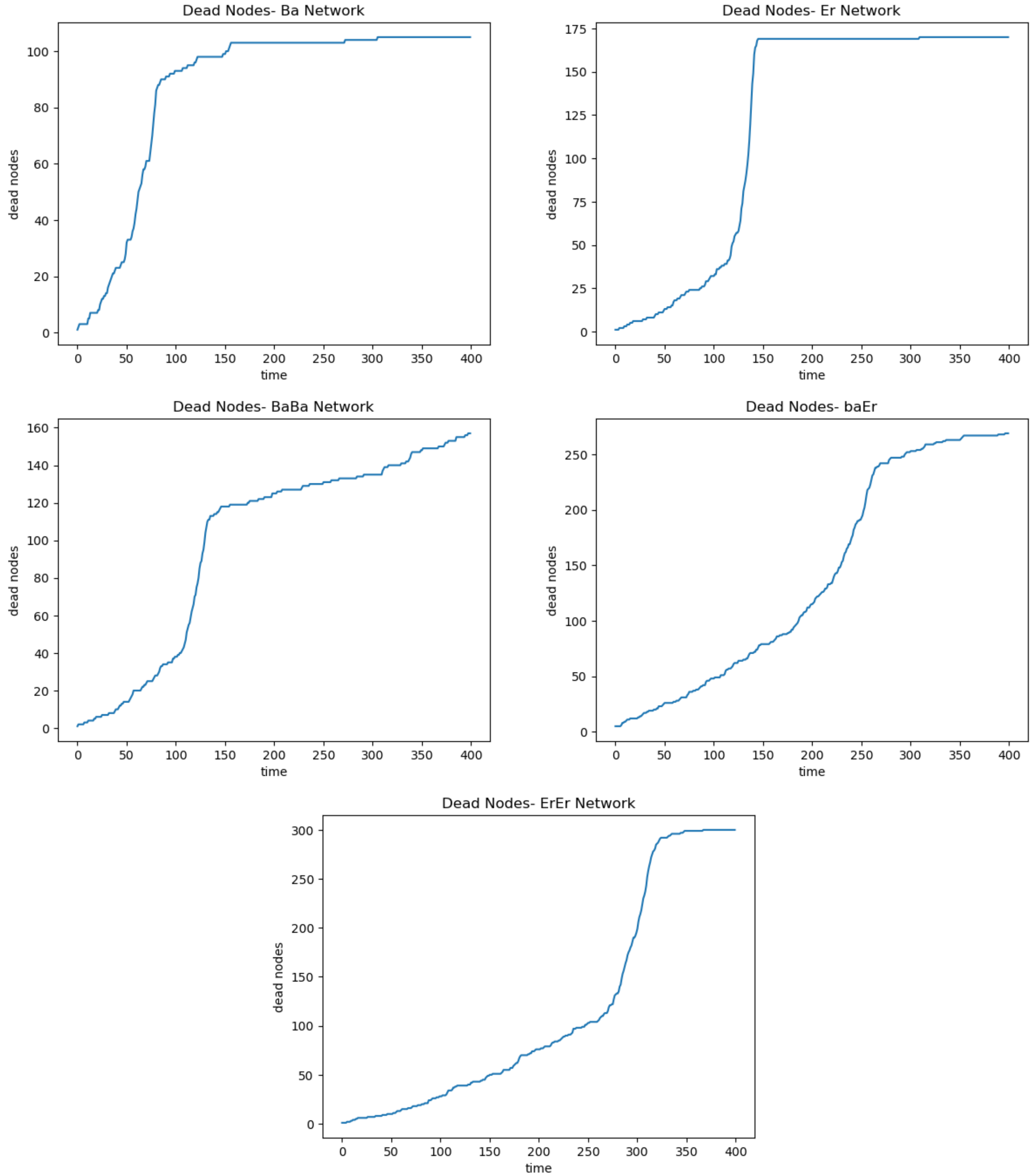

Figure 4.3: Dynamics of Cascade size(no. of dead nodes) on networks

We understand from these plots that networks are vulnerable to such cascades in random-walk dynamics, and we need to further enquire about the cause of such transitions. This result can be compared and is consistent with other cascading behaviors on networks associated with epidemics and deterministic motion.

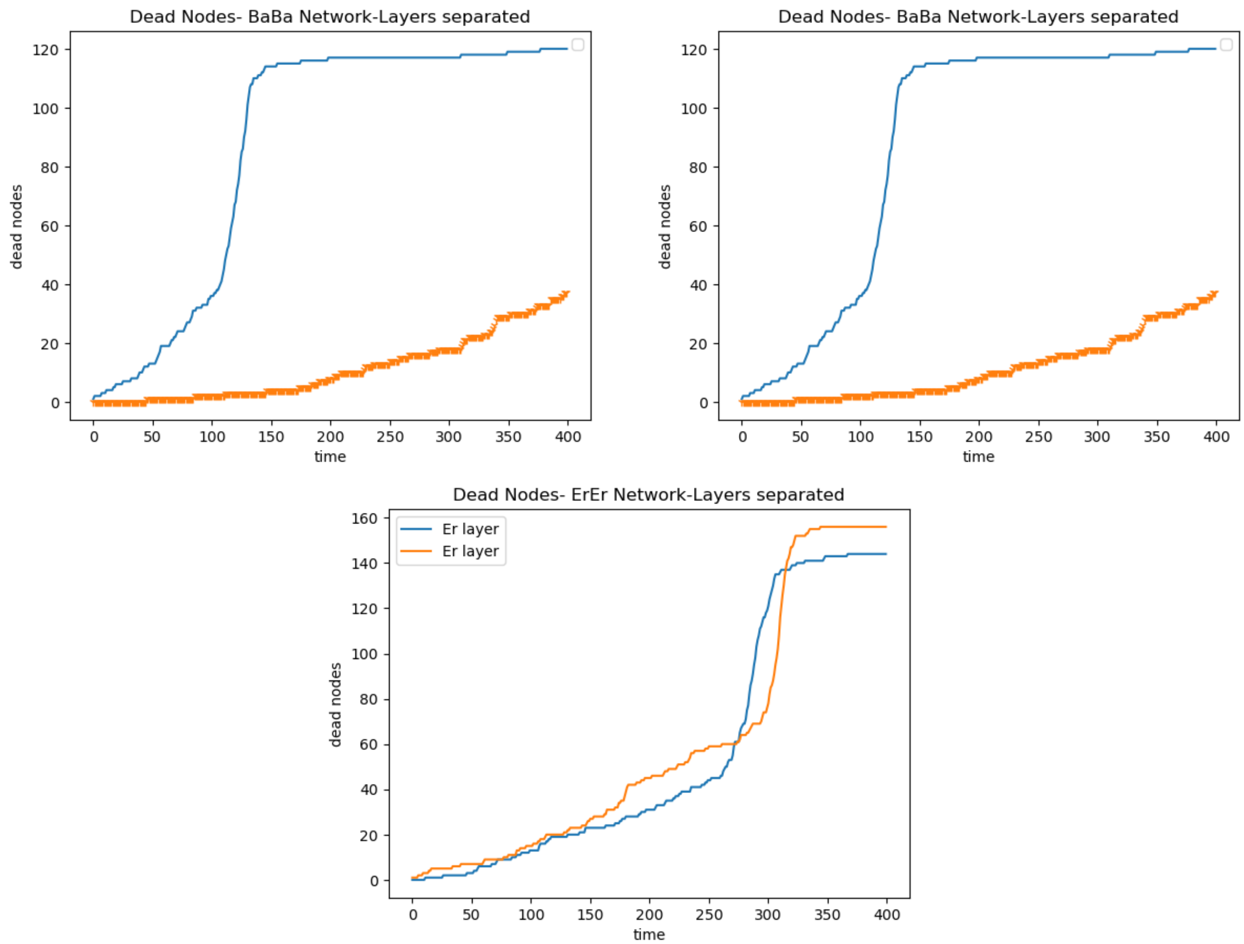

Figure 4.4: Layer separated dynamics- number of dead nodes with cascades

We observe in 4.4 that the dynamics on the individual layers are not identical, and the rate of cascading and timeline of cascades are separate across the layers with one layer cascading earlier and faster than the second one. We can further enquire about the role of the strength of inter-layer coupling on the dynamics. Running correlation analysis, we had observed that the dynamics on the layers are uncorrelated.

The dynamics of stuck walkers is another metric that gets investigated. Since there is a percolation of nodes, the walkers that enter isolated nodes become stuck there as all the neighboring nodes face extreme events.

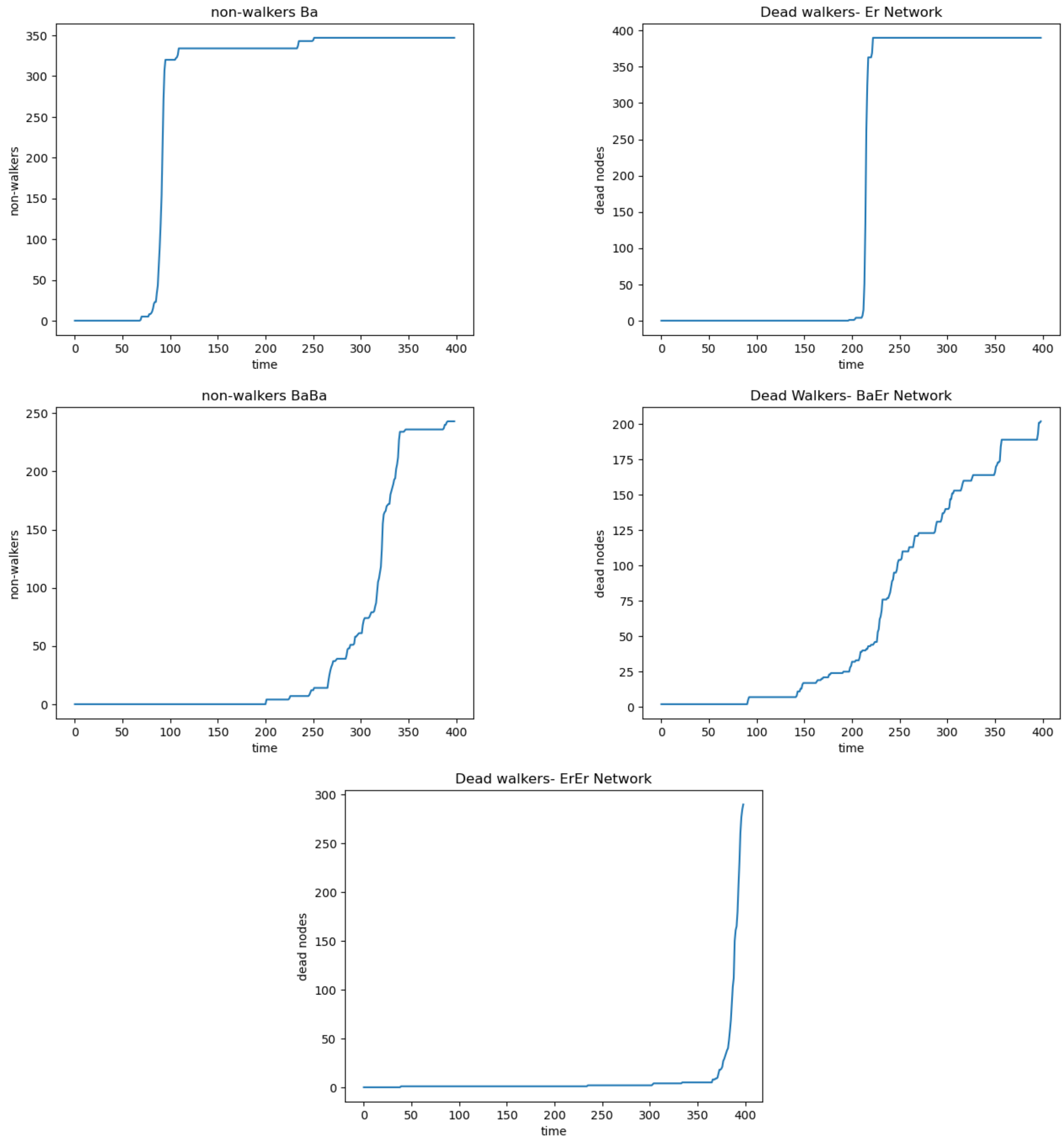

Figure 4.5: The dynamics of stuck walkers on the network when cascades occurs

We observe corresponding to the dynamics of the cascade, there exist abrupt changes in the number of stuck walkers as well. This shows that as there is an increase in the number of isolated walkers with time, a lower number of walkers are free and moving on the network. Due to this, the nodes are less prone to further extreme events. Since the increase is not gradual here, we can conclude that the saturation of cascade size also occurs rather abruptly.

We observe similar behavior on the individual layers in image 4.6, as expected, which can explain the consistency of similar behavior of cascades on the individual layers. We need to understand further the dynamics and evolution of cascades on networks resulting from random-walks in detail to verify this hypothesis.

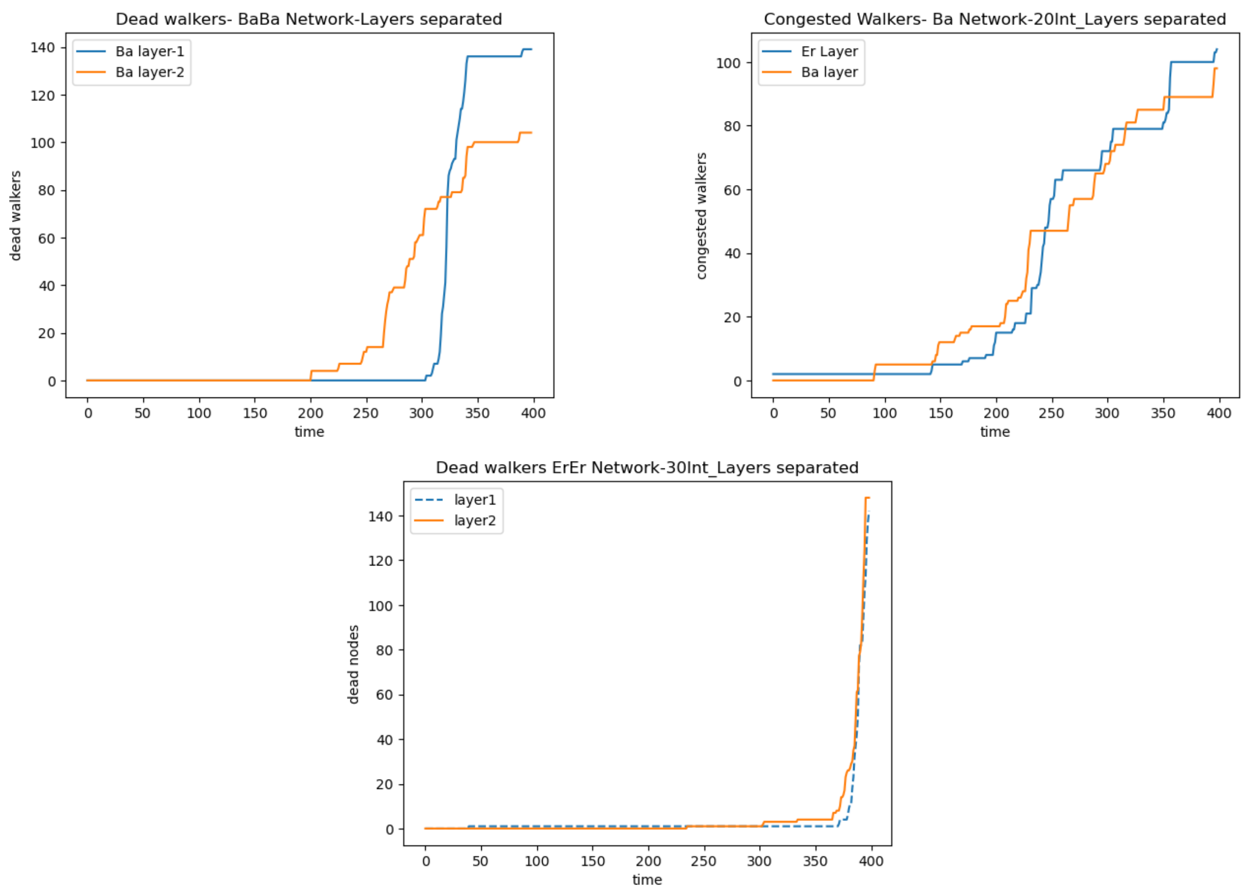

Figure 4.6: Layer-separated dynamics- number of stuck walkers'

4.4 Random Walks with Memory

Memory walks are the only non-markovian dynamics we investigate in the thesis. We look at how the memory parameter k impacts the occurrence and distribution of extreme events.

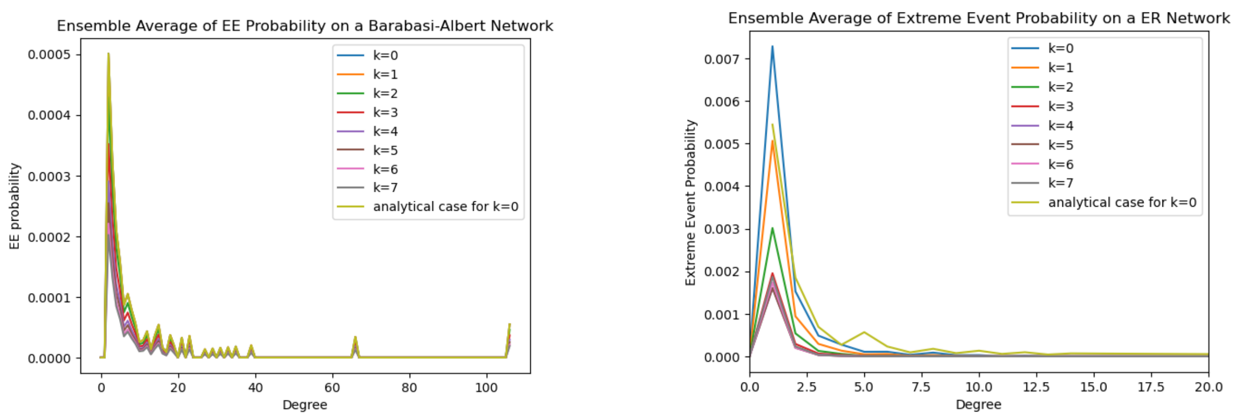

Figure 4.7: Extreme Events on random walks with memory on networks

We observe that as we increase the memory of the walkers, the probability of extreme events decreases for all degrees. This is true for both ER and BA networks. We plot the trend in the peaks of extreme events for further illustration.

While we do not yet understand the causes for this, it is important to note that with an increase in memory, there is also an increase in the number of trapped walkers on the network. This may lead to low numbers of walkers being available on the network, resulting in lower probabilities of extreme events on the network.

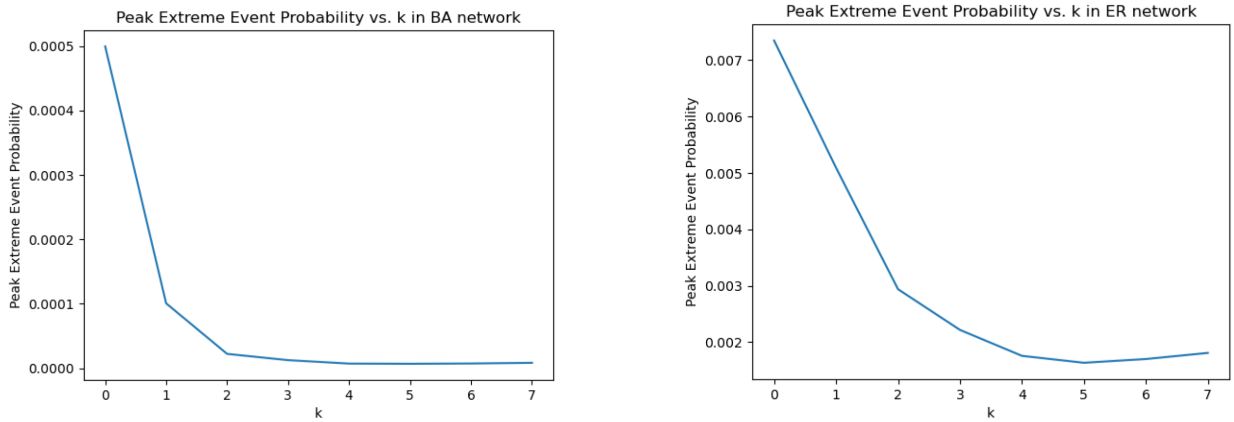

Figure 4.8: Extreme Events peaks on random walks with memory

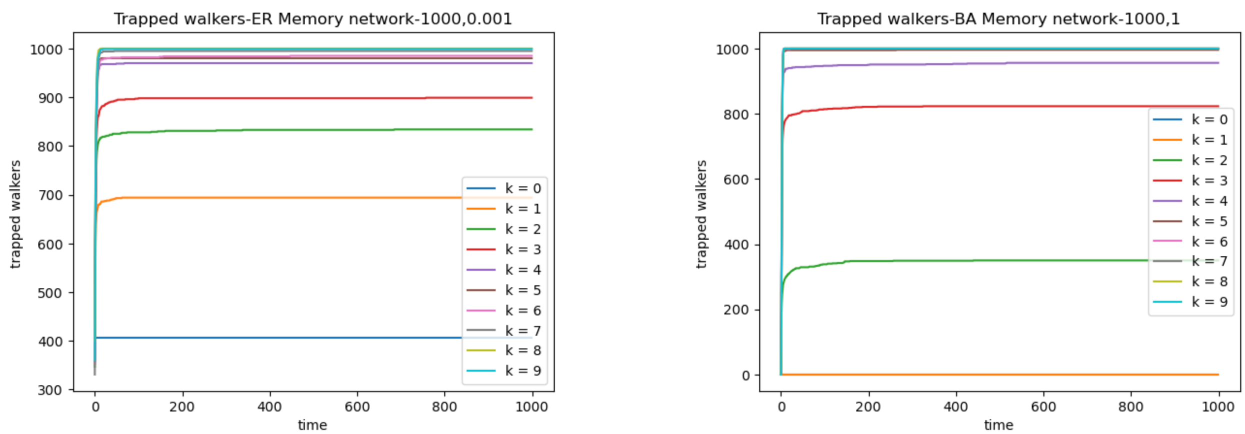

Figure 4.9: Trapped walkers on a random walk with memory

We can see the figure 4.9 that the number of trapped walkers keeps on increasing till it reaches a maximum and that this maximum is higher for higher values of the memory parameter k. This contributes to the reduction of occurrence in the occurrence of extreme events with the increase in k. Thus walkers with higher memory have lower frequencies of extreme events across nodes of varying degrees.

This result can help us in guiding the development of more sophisticated routing strategies as well as in understanding real vehicular dynamics. It is possible to build more sophisticated models of motion up from this idea of non-markovian dynamics.

Chapter 5

Summing up

We have observed a myriad of behaviors shown by various kinds of random walks on networks. We observe the following important patterns:

- smaller degree nodes are more susceptible to extreme events, and this stays true on multilayer networks as well.

- When subjected to a rerouting strategy, the number of stuck walkers keeps on increasing, resulting in more extreme events. This behavior, if present in real systems, indicates the spread of congestion in a system such as transportation networks. We also see that higher-degree nodes are increasingly prone to extreme events in this scenario.

- In random walks with node removal and death, the evolution of the cascade is not continuous but abrupt. The system transitions from a low-death regime to a high-death regime. This trend hints at the presence of critical regions in the network whose death can lead to cascades. We see this is also followed by saturation in dead nodes. This could be due to the lower number of active walkers left on the network, and needs further research for stronger claims to be held.

- In Random walks with memory, extreme events become less prevalent with an increase in the memory of the walkers. However, there is also an increase in the number of trapped walkers with memory, and this dynamic is not very beneficial from point of view of individual walkers.

5.1 Limitations of work and future directions

While the work done during the timespan of this thesis is a launching pad for new directions, especially in to study of extreme events associated with dynamics on multilayer networks, we have been limited in the behaviors we have been able to investigate in detail.

- We have limited scope to networks of BA and ER models. Networks in transportation systems seldom follow this topology. We have also been restrictive in the nature of inter-layer coupling. Work needs to be focused on empirical networks, especially multilayer Networks from Transportation.
- By limiting the dynamics to the regime of random walks, we are following the dynamics of stochastic processes. Real agents on transportation systems follow more deterministic patterns of movement. Random walk dynamics show low levels of self-correlation for dynamics across time. Extracting data from available sources is required to understand more realistic patterns of behavior.
- Discrete-time random walks, as used here, are less suited to study dynamics compared to continuous-time random walks. We also do not directly investigate the dynamics on the edges of the network. We are also concerned with non-interacting walkers, which is not the case with real systems.

Future work in the area of trying to understand Extreme Events must be aimed at being data-based and using empirical and real-time sources in order to understand the phenomenology associated with extreme events and congestion associated with mass dynamics.

5.2 Implications for Transport Systems and other Real Networks

The Thesis has helped us realize the depth and breadth of the work done in understanding transportation systems from the network science background. To design more robust and resilient transportation networks and build optimal multimodal transportation systems, we must continue this work using novel methods from network and data science.

From a theoretical perspective, this thesis is an extension of work on stochastic processes onto multilayer networks and an investigation for properties and behaviors that thus emerge. We also explored the behavior of extreme events in non-markovian dynamics, which is true of real traffic dynamics when we worked with random walks with memory. The work done here can help us understand more about stochastic dynamics on multilayer networks and guide us to recover emergent and interesting properties of multilayer networks.

The work is novel in trying to understand the occurrence of extreme events on random walks with memory and avenues needed to explore for future applications of this investigation. Also, the work has tried the approach of building up from simple random walks to its variations to explore the possibility of modeling real-world dynamics on networks. We also see how cascades resulting from inherent dynamics on the network can further influence future dynamics in the case of stochastic processes. This can be extended to other kinds of network dynamics as well.

We thus conclude the project in hope of more investigations into this approach and more work in trying to build realistic models up from simple models of the dynamics of agents.

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