2D CONFORMAL FIELD THEORY AND FOUR-POINT FUNCTIONS OF THE BABY-MONSTER MODULE

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Certificate

This is to certify that this dissertation entitled 2D CONFORMAL FIELD THEORY AND FOUR-POINT FUNCTIONS OF THE BABY-MONSTER MODULEtowards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by **Girish Lingadahalli Muralidharaa**t Indian Institute of Science Education and Research under the supervision of **Dr.Sunil Mukhi**, Professor, Department of Physics , during the academic year 2016-2017.

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This thesis is dedicated to **My Family**

Declaration

I hereby declare that the matter embodied in the report entitled 2D CONFORMAL FIELD THEORY AND FOUR-POINT FUNCTIONS OF THE BABY-MONSTER MODULE are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research Pune, under the supervision of **Dr.Sunil Mukhi** and the same has not been submitted elsewhere for any other degree.

Girish. L.M.
28/04/17
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Abstract

Field theoretical studies involve the evaluation of four-point Correlator functions of the involved fields. This applies equally to the calculation of Scattering Amplitudes in High Energy Physics or determining the behavior of statistical systems at critical points in Condensed matter Physics. Studying field theory for two dimensional systems with Conformal symmetry becomes important in many physical instances.

In earlier studies, a *dual* to well known *Ising model* theory called *Baby Monster theory* has been identified. From an altogether different approach too, this theory has been developed and is of interest in mathematics of Modular forms. Inspired by this, we would like to understand the Baby Monster theory from a field theorist view point.

In this work we first try to summarize some of the important concepts of Conformal Field theory in two dimensions. Main work is initiated with discussion on some of the different methods that can be used for calculating four-point functions in two dimensional conformal field theory. Eventually, using the properties and similarities arriving due to duality with Ising theory, we calculate the four-point functions of fields in the Baby-Monster theory.

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Chapter 1

Symmetries and Conservation laws

Studying dynamic physical systems is a complicated task. The mathematically rich concept of Symmetry simplifies this task. The knowledge about symmetries has provided both explanations and predictions on the behavior of physical systems. A system has symmetry if it is invariant under certain coordinate transformations. It is said to be continuous symmetry under continuous transformations. Due to Emmy Noether, it is a well established classical result that every continuous symmetry of a classical physical system implies a conserved quantity.

1.1 Infinitesimal transformation and Noether's theorem

In classical field theory, a field configuration obeying equation of motion is said to have symmetry under a transformation if the variation in the action(S) vanishes [1]. Noether's theorem states that "for every continuous symmetry of a field theory corresponds a conserved current and a conserved charge".

Consider an infinitesimal coordinate and field transformation,

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\delta x^{\mu}}{\delta \omega_a} \\ \Phi'(x') &= \Phi(x) + \omega_a \frac{\delta \mathcal{F}}{\delta \omega_a}(x) \end{aligned} \tag{1.1}$$

where $\{\omega_a\}$ is a set of infinitesimal variation parameters kept to first order. The variation in the action is,

$$\delta S = \int d^d x \, \partial_\mu j_a^\mu \, \omega_a \tag{1.2}$$

Where,

$$j_a^\mu = \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \partial_\nu \Phi - \delta_\nu^\mu \mathcal{L} \right\} \frac{\delta x^\nu}{\delta \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \frac{\delta \mathcal{F}}{\delta \omega_a} \tag{1.3}$$

j_a^μ is called the *current* associated with the transformation. Vanishing of the action implies,

$$\partial_\mu j_a^\mu = 0 \tag{1.4}$$

This is the "*existence of a conserved current*". For example, the conserved current of a theory with translational invariance is a *second rank* tensor called the *Energy-Momentum Tensor*, $T^{\mu\nu}$. Consider the quantity,

$$Q_a = \int d^{d-1} x j_a^0 \tag{1.5}$$

This is the conserved charge associated with the transformation. For example, momentum is the conserved charge for translational invariance.

A point to be noted here is that the Noether's theorem is a classical result. The consequence of symmetries on the behaviour of systems in the quantum regime is expressed in terms of *Correlation functions* and *Ward identities.*

1.2 Correlation Functions and Ward Identities

1.2.1 Correlation Function

One of the key problems in field theories is the calculation of *scattering amplitudes* of asymptotic free particles. These amplitudes can be calculated via correlation functions. Correlation functions are a measure on how a system evolves from initial vacuum state to a final vacuum with intermediate stages where the fields interact via creation and annihilation of particles.

Correlation functions are of interest also in study of *phase transitions*. Consider a two dimensional systems of particles with variable spins(σ) placed on a planar lattice. The spin of one particle at site i can influence the spin of another particle at site j. As a result, the statistical average leads to the two-point correlation function,

$$\langle \sigma_i \sigma_j \rangle = \begin{cases} exp\Big(- \dfrac{|i-j|}{\zeta} \Big) & T >> T_C \\ \dfrac{1}{|i-j|^{\alpha}} & T = T_C \end{cases} \tag{1.6}$$

Where the correlation length $\zeta > 0$. The above equation predicts that the correlation between spins at different sites falls off exponentially with distance when the temperature is greater than the *critical temperature(T_C)*. But at T_C, one finds a very different behaviour and is dependent on a quantity α called *critical exponent*. It has been observed that at T_C many systems fall into a set with common critical exponent.
For a general field Φ, n-point correlation is defined as,

$$\langle \Phi(x_1)\Phi(x_2)...\Phi(x_n) \rangle = \langle 0 | \mathcal{T}(\Phi(x_1)\Phi(x_2)...\Phi(x_n)) | 0 \rangle \tag{1.7}$$

Here $\mathcal{T}$ denotes time ordering of the fields and $|0\rangle$ represents the ground state(vacuum) of the theory. Correlation functions can be defined thorough the path integral formalism as (here we have continued to imaginary, which is Euclidean formalism),

$$\langle \Phi(x_1)\Phi(x_2)...\Phi(x_n) \rangle = \frac{\int [d\Phi] \Phi(x_1)\Phi(x_2)...\Phi(x_n) e^{-S[\Phi]}}{\int [d\Phi] e^{-S[\Phi]}} \tag{1.8}$$

Here, $[d\Phi]$ denotes the integration measure. The effect of continuous symmetry transformation leads to the following identity for correlation functions. Consider the transformation,

$$\begin{gathered} x \longrightarrow x' \\ \Phi'(x') = \mathcal{F}(\Phi(x)) \end{gathered} \tag{1.9}$$

As a consequence of the symmetry of the action and the invariance of the functional integration measure in the path integral, the following identity of correlation functions can be established,

$$\langle \Phi(x'_1)\Phi(x'_2)...\Phi(x'_n) \rangle = \langle \mathcal{F}(\Phi(x_1))\mathcal{F}(\Phi(x_2))...\mathcal{F}(\Phi(x_n)) \rangle \tag{1.10}$$

1.2.2 Ward Identities

Information about the symmetry under transformations is encoded in Ward identities which establish a connection between classically conserved currents and correlation functions,

$$\frac{\partial}{\partial x^\mu} \langle j_a^\mu(x)\phi(x_1).....\phi(x_n)\rangle = -i\sum_{i=1}^{n}\delta(x-x_i)\langle \phi(x_1)...G_a\phi(x_i)...\phi(x_n)\rangle \tag{1.11}$$

Here G_a is the *generator* of symmetry transformation. Also,

$$[Q_a, \Phi] = -iG_a\Phi \tag{1.12}$$

In other words, as a consequence of Ward-identities, the conserved charges can be identified as the generators of symmetry transformations.

Chapter 2

Conformal Field Theory

Systems with conformal symmetry are invariant under a class of transformations called conformal transformations. Even though it is not an exact symmetry of nature, it is plays role in systems that are of physical relevance. Some systems with conformal invariance are[2],

1. Free boson/fermion field theory with vanishing mass.

2. String theory, a favored candidate for quantum gravity. Here conformal field theory(CFT) arises as the two dimensional field theory living on the world-volume of string moving on a background space-time.

3. Statistical models in two dimensions at a second order continuous phase transition, called critical points. Like the case described in eq. (1.6), the description of such systems are characterized by critical exponents. An example that we will discuss later in some detail is Ising model, which is a model for two dimensional ferromagnet. Correlation function (1.6) at the critical temperature (T_C) is invariant under conformal transformations. Different primary fields in the CFT correspond to different critical exponentsin the system.

2.1 Conformal Invariance in General Dimension

A conformal transformation is an invertible mapping of the coordinates and leaves the *metric*, $g_{\mu\nu}$, invariant upto a scale,

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x) \tag{2.1}$$

A set of finite transformations that obey the above requirement (for d $\geq$3) form a group called *conformal group*. They are,

Type	Transformation	Generators
Translation	$x'^{\mu} = x^{\mu} + a^{\mu}$	$P_{\mu} = -i\partial_{\mu}$
Dilation	$x'^{\mu} = \lambda x^{\mu}$	$D = -ix^{\mu}\partial_{\mu}$
Lorentz	$x'^{\mu} = M^{\mu}_{\nu} x^{\nu}$	$L_{\mu\nu} = i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$
The special conformal transformation	$x'^{\mu} = \frac{x^{\mu} - b^{\mu}x^2}{1 - 2b.x + b^2x^2}$	$K_{\mu} = -i(2x_{\mu}x^{\nu}\partial_{\nu} - x^2\partial_{\mu})$

Special Conformal Transformation(SCT) is conformal a with scale factor[2],

$$\Lambda(x) = (1 - 2b.x + b^2x^2)^2 \tag{2.2}$$

In effect, SCT is a translation preceded and succeeded by an inversion,

$$\frac{x'^{\mu}}{x'^2} = \frac{x^{\mu}}{x^2} - b^{\mu} \tag{2.3}$$

The *conformal algebra* is the set of commutation rules that the above generators obey among themselves. It can be shown that the conformal algebra (in d dimensions) is *so(d+1,1)* with $\frac{1}{2}(d+2)(d+1)$ parameters.
Conformal invariants are functions that are left unchanged under the action of the conformal group. Simplest invariants called *anharmonic ratios or corss ratios*, can be constructed with a minimum of 4 points. They are given as,

$$\frac{|x_1 - x_2||x_3 - x_4|}{|x_1 - x_3||x_2 - x_4|} \quad \text{and} \quad \frac{|x_1 - x_2||x_3 - x_4|}{|x_2 - x_3||x_1 - x_4|} \tag{2.4}$$

2.1.1 Conformal Invariance in Classical Field Theory

Under an arbitrary coordinate transformation, the change in the action is,

$$\delta S = \frac{1}{d}\int d^d x T^{\mu}_{\mu}\partial_\rho \epsilon^\rho \tag{2.5}$$

Hence *tracelessness* of the energy-momentum tensor implies the invariance of the action under conformal transformations. If the theory has scale invariance, then a symmetric EM tensor can be made traceless.[1] This suggests that full conformal invariance is a consequence of Poincaré invariance and scale invariance.

For the effect of conformal symmetry on classical fields, if we demand that the field $\Phi(x)$ forms an irreducible representation of Lorentz group, then the field transformation generator for dilation will be a multiple of the identity(Schur's lemma), $-i\Delta$(reflects the fact that they are non-Hermitian). Also from commutation relations, the corresponding generator for SCT will vanish. If fields transformation as:

$$\Phi'(x') = \lambda(x)^{\frac{\Delta}{2}}\Phi(x) \tag{2.6}$$

then they are called quasi-primary fields with Δ, called the *scaling dimension*.

2.1.2 Conformal Invariance in Quantum Field Theory

For the study of conformal transformation on quantum fields, correlation functions and Ward identities are important. Covariance under conformal transformations forces the *two-point* correlation function between two quasi-primary fields to be,

$$\langle\phi_1(x_1)\phi_2(x_2)\rangle = \begin{cases} \frac{C_{12}}{|x_1-x_2|^{2\Delta_1}} & if \quad \Delta_1 = \Delta_2 \\ 0 & if \quad \Delta_1 \neq \Delta_2 \end{cases} \tag{2.7}$$

[1]Theory with rotational invariance can be made to have symmetric EM tensor.)

And the *three-point* correlation function takes the form,

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}} \tag{2.8}$$

Where C_{123} is called the structure constant. Relevant substitution of currents and generators into the general Ward identity will produce conformal Ward identities assosiated with the conformal group.

2.2 Two Dimensional CFT

Unlike in other dimensions, the two dimension case has an infinite set of *local* conformal transformations. The global conformal group in two dimensions is the a 6-parameter group so(3,1) but the local conformal invariance gives us more useful information.

2.2.1 The Conformal Group in Two Dimensions

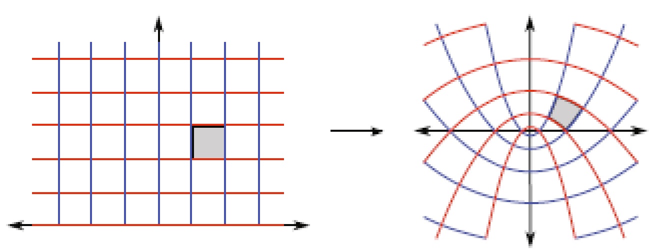

Figure 2.1: Conformal transformation in two dimensions. Image credite:[2]

Local Conformal Group

Consider mapping of the plane $x^\mu \to y^\mu$. Under this mapping, condition for the conformal invariance reads as,

$$\frac{\partial y^2}{\partial x^1} = \frac{\partial y^1}{\partial x^2} \quad \text{and} \quad \frac{\partial y^1}{\partial x^1} = -\frac{\partial y^2}{\partial x^2} \tag{2.9}$$

or as,

$$\frac{\partial y^2}{\partial x^1} = -\frac{\partial y^1}{\partial x^2} \quad \text{and} \quad \frac{\partial y^1}{\partial x^1} = \frac{\partial y^2}{\partial x^2} \tag{2.10}$$

Here, (2.9) is Cauchy-Riemann condition for *Holomorphic functions* and (2.10) is for *Anti-Holomorphic functions*.

If we introduce complex coordinates, $z = x^1 + ix^2$ then the holomorphic condition becomes,

$$\partial_{\bar{z}} w(z, \bar{z}) = 0 \tag{2.11}$$

This represents any holomorphic mapping (and anti-holomorphic mapping) of the complex plane onto itself. Indeed, it is well known that any analytic mapping of the plane preserves angles.

Global Conformal Group

Global conformal mappings, unlike for local ones, are required to be *invertible*. The complete set of all global conformal maps, called the *super confrmal group*, are mappings of the form,

$$f(z) = \tfrac{az+b}{cz+d} \quad with \quad ad - bc = 1 \tag{2.12}$$

It is evident from the structure of the above equation that the global conformal group is isomorphic to complex invertible 2 × 2 unit-determinant matrices, i.e $SL(2, \mathbb{C})$ isomorphic to $so(3, 1)$. This shows that Global conformal group is a 6-parameter(3-complex) group.

Local Conformal Generators

The conformal generators for local conformal transformation defined as,

$$l_n = -z^{n+1}\partial_z \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \tag{2.13}$$

These differential operators obey what is called the *Witt algebra*.

$$\begin{aligned} [\ell_n, \ell_m] &= (n-m)\ell_{n+m} \\ [\bar{\ell}_n, \bar{\ell}_m] &= (n-m)\bar{\ell}_{n+m} \\ [\ell_n, \bar{\ell}_m] &= 0 \end{aligned} \tag{2.14}$$

2.2.2 Correlation Functions and Conformal Ward identity

In CFT_2, fields with spin(S) transform as *primary fields* where we replace the role of scaling dimension for quasi-primary fields by *conformal dimension* and separate the holomorhpic and anti-holomorphic parts as,

$$h = \tfrac{\Delta+S}{2} \quad \bar{h} = \tfrac{\Delta-S}{2} \tag{2.15}$$

Under a conformal mapping, primary fields transform as,

$$\Phi'(w, w') = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \Phi(z, \bar{z}) \tag{2.16}$$

Correlation Functions

With the introduction of holomorphic and anti-holomorphic coordinates, eq. 1.10 becomes,

$$\langle\Phi_1(w_1, \bar{w}_1)\Phi_2(w_2, \bar{w}_2)...\Phi_n(w_n, \bar{w}_n)\rangle = \prod_{i=1}^{n} \left(\frac{dw}{dz}\right)^{-h_i}_{w=w_i} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}_i}_{\bar{w}=\bar{w}_i} \langle\Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)...\Phi_n(z_n, \bar{z}_n)\rangle \tag{2.17}$$

Eq. (2.7) for *two-point* correlation function reads as,

$$\langle\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \tag{2.18}$$

if the conformal dimensions of the two fields are same. Otherwise it is zero.
The *Three-point correlation* is given by,

$$\begin{aligned}\langle\phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = C_{123} &\frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}}. \\ \times &\frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\end{aligned} \tag{2.19}$$

Four-point correlation functions depend on cross ratios, the simplest conformal invariants. In two dimensions, all four points lie on the same plane and all different types of cross ratios are linearly related.

$$\eta = \tfrac{z_{12}z_{34}}{z_{13}z_{24}}, \quad 1 - \eta = \tfrac{z_{14}z_{23}}{z_{13}z_{24}}, \quad \tfrac{\eta}{1-\eta} = \tfrac{z_{12}z_{34}}{z_{14}z_{23}} \tag{2.20}$$

Given three distinct points, z_1, z_2, z_3 it is always possible to find a global conformal transformation that takes them to any other set of three points[1], say: 1, ∞ and 0 respectively. Then $\eta = -z_4$ and the four-point function will depend on this point solely.

Conformal Ward Identity

Ward identities *eq.*(1.11) for translation, rotation and scaling when expressed in complex coordinates can be combined into a single expression with separated holomorphic and anti-holomorphic parts. This is called the *Conformal Ward identity*,

$$\delta_{\epsilon,\bar{\epsilon}} \langle X \rangle = -\frac{1}{2\pi i} \oint_C dz \epsilon(z) \langle T(z) X \rangle + \frac{1}{2\pi i} \oint_C d\bar{z} \epsilon(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle \tag{2.21}$$

Where X is the string of fields $\Phi_1, ...\Phi_n$ and,

$$T = -2\pi T_{zz}; \quad \bar{T} = -2\pi \bar{T}_{\bar{z}\bar{z}} \tag{2.22}$$

The Conformal Ward identity reflects the consequences of local conformal transformations on correlation functions. When the conformal ward identity is applied to global conformal transformations, $SL(2, \mathbb{C})$, the following relations on correlation functions are obtained,

$$\begin{aligned} \Sigma_i \partial_{w_i} \langle X \rangle &= 0 \\ \Sigma_i (w_i \partial_{w_i} + h_i) \langle X \rangle &= 0 \\ \Sigma_i (w_i^2 \partial_{w_i} + 2 w_i h_i) \langle X \rangle &= 0 \end{aligned} \tag{2.23}$$

Chapter 3

The Operator Formalism of CFT

3.1 Virasoro Algebra

3.1.1 Operator Product Expansion

Correlation functions involving two or more fields with coinciding positions have singularities and tend to diverge. *Operator Product Expansion*(OPE) expresses the product of fields or operators at different positions as a sum of single operators, well defined at neighborhood of each position and multiplied by a diverging quantity that embodies the quantum fluctuations at coinciding positions[1]. Ward identity is one useful tool to obtain OPE between two fields. Standard expression for the OPE between EM tensors is,

$$T(z)T(w) \sim \frac{\frac{c}{2}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{3.1}$$

Here C is called the *central charge* which plays an important role in characterizing the short-distance behavior of a CFT. Relation of CFT with Anti de-Sitter(AdS) space in AdS/CFT correspondence is also captured by central charge through *Brown-Henneaux* relation where central charge of the CFT is related to the radius(L) of AdS[3],

$$c = \frac{3L}{2G_N} \tag{3.2}$$

Where, G_N is Newton's gravitational constant.

3.1.2 Radial Quantization

In the Euclidean time formalism we have been using, we can choose the time direction to be radial direction from the origin. This is the essence of radial quantization and the operator formalism of CFT. This is achieved by mapping the initial theory defined on a cylinder onto a complex plane. Let the cylinder be defined $\xi = t + ix$. Mapping onto a plane is the change of coordinates given by,

$$z = e^{\frac{2\pi\xi}{L}} \tag{3.3}$$

'Radial ordering' on the arguments of operators appearing in OPEs should be maintained analogous to the time ordering in correlation functions. With this construction, commutation relations between operators A and B can be obtained from OPE between fields a(z) and b(w) through contour integrals as,

$$[A, B] = \oint_0 dw \oint_w dz a(z) b(z) \qquad with \qquad A = \oint a(z) dz \tag{3.4}$$

3.1.3 Virasoro Algebra

The *mode expansion* of a conformal field, $\Phi(z, \bar{z})$ of dimension $(h, \bar{h})$ is,

$$\begin{aligned} \Phi(z, \bar{z}) &= \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} z^{-m-h} \bar{z}^{-n-\bar{h}} \Phi_{m,n} \\ \Phi_{m,n} &= \frac{1}{2\pi i} \oint dz z^{m+h-1} \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+\bar{h}-1} \Phi(z, \bar{z}) \end{aligned} \tag{3.5}$$

The above mode expansion implies Hermitian conjugation relation,

$$\Phi^{\dagger}_{m,n} = \Phi_{-m,-n} \tag{3.6}$$

Hence the Energy-Momentum tensor can be expanded as,

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \quad ; \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n \tag{3.7}$$

If we apply the commutation relation eq.(3.4) to the conformal ward identity eq.(2.21) and identify the modes of EM tensor as the *quantum generators* of local conformal transformations, we get the *Virasoro algebra*,

$$\boxed{\begin{array}{l}[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [L_n, \bar{L}_m] = 0 \\ [\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}\end{array}} \tag{3.8}$$

Note the appearence of a central extension term, C, relative to the Witt algebra. With this algebra in hand it becomes easy to construct the *Hilbert space* of the CFT with generators acting as ladder operators. Some of the key features of this Hilbert space are,

1. The Vacuum state is represented by $|0\rangle$.

2. The Asymptotic state of a primary field is defined as,

$$|h, \bar{h}\rangle \equiv \Phi(0,0)|0\rangle \tag{3.9}$$

3. The eigenvalue equation for the Hamiltonian, defined as $(L_0 + \bar{L}_0)$, is

$$(L_0 + \bar{L}_0)|h, \bar{h}\rangle = (h + \bar{h})|h, \bar{h}\rangle \tag{3.10}$$

4. Ladder operation on the eigenstate is,

$$\begin{aligned} L_m|h, \bar{h}\rangle &= 0|h, \bar{h}\rangle \\ L_{-m}|h, \bar{h}\rangle &\longrightarrow |h+m, \bar{h}\rangle \end{aligned} \tag{3.11}$$

   $\forall m > 0$ and corresponding equations hold for anti-holomorphic generators. i.e. raising ladder operator $L_{-m}(m > 0)$ on any eigen state takes it to a state of higher dimension by virtue of Virasoro algebra as can be seen below:

$$\begin{aligned} [L_0, L_{-m}]|h\rangle &= mL_{-m}|h\rangle \\ \Rightarrow L_0 L_{-m} - L_{-m}L_0|h\rangle &= mL_{-m}|h\rangle \\ \Rightarrow L_0 L_{-m}|h\rangle &= (mL_{-m} + L_{-m}L_0)|h\rangle \\ \Rightarrow L_0(L_{-m}|h\rangle) &= (m+n)(L_{-m}|h\rangle) \end{aligned} \tag{3.12}$$

3.2 Conformal Families

3.2.1 Descendant Fields

A general state obtained by raising ladder operation similar to above on the *highest-weight state* $|h\rangle$,

$$L_{-k_1}L_{-k_2}...L_{-k_n}|h\rangle \qquad (1 \leq k_1 \leq ... \leq k_n) \tag{3.13}$$

is called a *descendant state* of a primary of the CFT with asymptotic state $|h\rangle$. This descendant is said to be at level N, where, $h' = h + k_1 + k_2 ... + k_n \equiv h + N$. The number of linearly independent states at level N is given by the *partition number*, $p(\mathrm{N})$, of integers given by,

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n \tag{3.14}$$

The collection of a primary and its descendants is called a *conformal family* and the mathematical structure of this 'tower' is called a *Verma module*. The lowest states of the Verma module of a primary of conformal dimension h are given in the table below,

Table . Structure of Verma Module, V(c,h)

level	p(level)	Tower of descendants
0	p(0)=1	$\|h\rangle$
1	p(1)=1	$L_{-1}\|h\rangle$
2	p(2)=2	$L_{-1}^2\|h\rangle$, $L_{-2}\|h\rangle$
3	p(3)=3	$L_{-1}^3\|h\rangle$, $L_{-1}L_{-2}\|h\rangle$, $L_{-3}\|h\rangle$
4	p(4)=5	$L_{-1}^4\|h\rangle$, $L_{-1}^2L_{-2}\|h\rangle$, $L_{-1}L_{-3}\|h\rangle$, $L_{-2}L_{-2}\|h\rangle$, $L_{-4}\|h\rangle$
5	p(5)=7	$L_{-1}^5\|h\rangle$, $L_{-1}^3L_{-2}\|h\rangle$, $L_{-1}^2L_{-3}\|h\rangle$, $L_{-1}L_{-2}^2\|h\rangle$, $L_{-1}L{-}4\|h\rangle$, $L_{-2}L_{-3}\|h\rangle$, $L_{-5}\|h\rangle$

Considering corresponding Verma moudule of anti-holomorphic part, $\bar{V}(c,\bar{h})$, we can represent the Hilbert space of the CFT as,

$$\mathcal{H}(c) = \sum_{h,\bar{h}} V(c,h) \otimes \bar{V}(c,\bar{h}) \tag{3.15}$$

3.2.2 Virasoro Characters

For every Verma module, V(c,h), we associate a generating function called a *character*, defined as,

$$\begin{aligned} \chi(c,h)(\tau) &= Tr(q^{L_0-\frac{c}{24}}) \\ &= q^{h-c/24}\sum_{n=0}^{\infty} p(n)q^n \end{aligned} \tag{3.16}$$

Here, $q \equiv e^{2\pi i\tau}$ and τ is a point on the upper half part of the complex plane and the parameter in *modular invariance of theory.* This requirement on τ implies the convergence of character sequence defined above.
In terms of the *Dedekind eta function*,

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{3.17}$$

The Virasoro characters become,

$$\chi(c,h)(\tau) = \frac{q^{h+\frac{(1-c)}{24}}}{\eta(\tau)} \tag{3.18}$$

This statement is actually not true for the identity $h = 0$ and for other special fields as we will see below.

3.2.3 Singular Vectors or Null Vectors

Consider a descendant state $|\chi\rangle$ in a verma module V(c,h) such that,

$$L_n|\chi\rangle = 0 \quad (n > 0) \tag{3.19}$$

We need to check only for $n = 1, 2$ cases. Higher n cases just follow from virasoro algebra. Such a state generates a set of descendants inside the given Verma module, which in itself is a module, i.e states in this submodule(V_χ) transform among themselves under any conformal transformations. Such a state $|\chi\rangle$ is called *Singular state or null state.* They have the following properties,

1. They are null vectors, i.e $\langle\chi|\chi\rangle = 0$.
2. Every descendant in V_χ has zero norm.
3. The singular vector and its descendants are orthogonal to the whole Verma module V(c,h).

An irreducible representation of the virasoro algebra is constructed by simply setting the singular vectors to *zero*.
Example: In any CFT, the first state over *identity* is a singular vector.

$$\begin{aligned} L_1[L_{-1}|0\rangle] &\propto L_1|1\rangle = 0 \\ Also, \qquad L_2[L_{-1}|0\rangle] &= 0 \\ \Rightarrow L_n[L_{-1}|0\rangle] = 0 &\quad (n > 0) \end{aligned} \tag{3.20}$$

Thus it is set to zero in any irreducible representation of the virasoro algebra. It follows that the virasoro character for the identity is,

$$\chi(c,0)(\tau) = q^{-\frac{c}{24}}\left(1 + \sum_{n=2}^{\infty} p(n)q^n\right) \tag{3.21}$$

in the absence of null vectors.

Chapter 4

A Minimal Model: Ising Model

4.1 Unitary Minimal Models

Virasoro algebra has a spin-2 current in the form of the EM tensor. However a general CFT, a theory can have spin-1 symmetry via current algebra or Kac-Moody algebra as well as symmetry currents with higher spins > 3.

Minimal Models are a special class of CFTs which have central charge $C < 1$ and whose symmetry algebra is only the Virasoro algebra[1].

A CFT without negative norm states in any of it's Verma module structures is called a *unitary* CFT. A subclass of minimal models are unitary.

Minimal Models are labeled by $\mathcal{M}(p, p')$ with the condition $p = p' + 1$, to satisfy the unitarity condition. The conformal charge and conformal dimensions of the primaries in such a theory are given by,

$$\begin{aligned} h_{r,s} &= \frac{(pr - p's)^2 - (p - p')^2}{4pp'} \qquad 1 \leq r \leq p' - 1, \quad 1 \leq s \leq p - 1 \\ c &= 1 - 6\frac{(p - p')^2}{pp'} \end{aligned} \tag{4.1}$$

It is easily seen from the above constraints on conformal dimensions that a minimal model has a finite number of primaries.

4.2 Operator Algebra and Fusion Rule

The *Operator algebra* consists of the OPE between all the primary fields in a theory. Conformal invariance requires the operator algebra to be,

$$\Phi_1(z,\bar{z})\Phi_2(0,0) = \sum_p \sum_{\{k,\bar{k}\}} C_{12}^{p\{k,\bar{k}\}} z^{h_p-h_1-h_2+K} \bar{z}^{\bar{h}_p-\bar{h}_1-\bar{h}_2+\bar{K}} \Phi_p^{\{k,\bar{k}\}}(0,0) \tag{4.2}$$

where, $K = \sum_i k_i$ and $\{k\}$ is collection of k_i denoting the descendants of primaries. Thus each k_i implies that the primary is acted on by a raising operator L_{k_i}, and $L_{\bar{k}_i}$ for $\bar{k}_i$.

fusion rules gives the information about all the primaries that appear in the operator algebra of two fields. Removing null vectors in the theory leads to constraints and truncation of the operator algebra. The fusion rules for minimal models are as follows,

$$\Phi_{r,s} \times \Phi_{m,n} = \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \quad mod2}}^{k_{max}} \sum_{\substack{l=1+|s-n| \\ l+s+n=1 \quad mod2}}^{l_{max}} \Phi_{k,l} \tag{4.3}$$

where,

$$\begin{aligned} k_{max} &= min(r+m-1, 2p'-1-r-m) \\ l_{max} &= min(s+n-1, 2p'-1-s-n) \end{aligned} \tag{4.4}$$

4.3 Ising Model

Here we will list out some of the properties of Ising Model,

1. $\mathcal{M}(4,3)$ minimal model describes the critical Ising model.

2. Using (4.1), we find that central charge of the Ising model is $c = \frac{1}{2}$.

3. The model has three primaries. They are the identity, Ising spin operator(σ) and the enegy density operator(ϵ), also called the thermal operator.

4. Using (4.1), the conformal dimensions of the primaries are,

$$\begin{aligned} h_{1,1} = h_{2,3} = h_{\mathbb{1}} &= 0 \\ h_{2,1} = h_{1,3} = h_{\epsilon} &= \frac{1}{2} \\ h_{2,2} = h_{1,2} = h_{\sigma} &= \frac{1}{16} \end{aligned} \tag{4.5}$$

and corresponding anti-holomorphic dimensions.

5. The fusion rules (4.3) for $\mathcal{M}(4,3)$ read as,

$$\begin{aligned} \Phi_{1,2} \times \Phi_{1,2} &= \mathbb{1} + \Phi_{2,1} \\ \Phi_{2,1} \times \Phi_{2,1} &= \mathbb{1} \\ \Phi_{2,1} \times \Phi_{1,2} &= \Phi_{1,2} \end{aligned} \tag{4.6}$$

6. The Ising model CFT is equivalent to the unitary *free Majorana fermion* CFT with central charge $c = \frac{1}{2}$ with the identification $\psi\bar{\psi} = \epsilon$ [1].

Four-Point Correlation Function for Free Majorana Fermions

Consider free fermionic field, $\psi\bar{\psi}$ with conformal dimension $h_{\psi} = \frac{1}{2}$. The two-point function is given as,

$$\langle \psi(z_1)\bar{\psi}(\bar{z}_1)\psi(z_2)\bar{\psi}(\bar{z}_2)\rangle = \frac{1}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} \tag{4.7}$$

Consider the four-point function,

$$\langle \psi(z_1)\bar{\psi}(\bar{z}_1)\psi(z_2)\bar{\psi}(\bar{z}_2)\psi(z_3)\bar{\psi}(\bar{z}_3)\psi(z_4)\bar{\psi}(\bar{z}_4)\rangle \tag{4.8}$$

Since the holomorphic and anti-holomorphic parts are independent, this equals,

$$= \Big|\langle \psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle\Big|^2 \tag{4.9}$$

Wick's theorem says that a *2n* point correlation function of *free* fields is the same as all possible products of *two* point correlation functions. In using this theorem, we must take into account that flipping fermionic fields comes with a change of sign. Following this $\left|\langle\psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle\right|^2$ becomes,

$$\left|\langle\psi(z_1)\psi(z_2)\rangle\langle\psi(z_3)\psi(z_4)\rangle - \langle\psi(z_1)\psi(z_3)\rangle\langle\psi(z_2)\psi(z_4)\rangle + \langle\psi(z_1)\psi(z_4)\rangle\langle\psi(z_2)\psi(z_3)\rangle\right|^2 \tag{4.10}$$

Using the result for the two point correlator for each case, we find that

$$\left|\langle\psi(z_1)\psi(z_2)\psi(z_3)\psi(z_4)\rangle\right|^2 = \left|\frac{1}{z_{12}z_{34}} - \frac{1}{z_{14}z_{32}} + \frac{1}{z_{24}z_{31}}\right|^2 \tag{4.11}$$

It will be shown in the next chapter that we get the same four-point function for $\epsilon(z,\bar{z})$ by treating the Ising model as a minimal model. This partially justifies the claim that the free fermion theory is equivalent to the Ising model.

Chapter 5

Methods to Calculate Four-Point Functions

5.1 Conformal Blocks

Consider a general four-point function,

$$\langle \Phi_1(z_1, \bar{z}_1)\Phi_2(z_2, \bar{z}_2)\Phi_3(z_3, \bar{z}_3)\Phi_4(z_4, \bar{z}_4)\rangle \tag{5.1}$$

Under transformation $z_1 \to \infty$, $z_2 \to 1$, $z_3 \to z$ and $z_4 \to 0$ the cross ratio remains invariant. The four-point function depends on these cross ratios. By definition,

$$\lim_{z_4, \bar{z}_4 \to 0} \Phi_4(z_4, \bar{z}_4)|0\rangle = |\Phi_4^{in}\rangle \tag{5.2}$$

similarly,

$$\lim_{z_1, \bar{z}_1 \to 0} \Phi_1(z_1, \bar{z}_1)|0\rangle = |\Phi_1^{in}\rangle \tag{5.3}$$

Structure of Hilbert space in radial quantization implies,

$$\begin{aligned} & [\lim_{z_1, \bar{z}_1 \to 0} \Phi_1(z_1, \bar{z}_1)|0\rangle]^\dagger = [|\Phi_1^{in}\rangle]^\dagger = \langle \Phi_1^{out}| \\ \implies & \qquad \langle 0|[\lim_{z_1, \bar{z}_1 \to 0} \Phi_1(z_1, \bar{z}_1)]^\dagger = \langle \Phi_1^{out}| \end{aligned} \tag{5.4}$$

Also due to *euclidian space* construction, upon Hermitian conjugation the euclidean time must be reversed so that the Minkowski time is left unchanged[1]. Thus hermitian conjugation corresponds to mapping $z \to \frac{1}{\bar{z}}$. For a primary field this implies,

$$[\Phi_1(z_1, \bar{z}_1]^\dagger = \bar{z}_1^{-2h_1} z_1^{-2\bar{h}_1} \Phi_1(\frac{1}{z_1}, \frac{1}{\bar{z}_1}) \tag{5.5}$$

Thus we can write,

$$\langle 0|[\lim_{z_1,\bar{z}_1 \to 0} \Phi_1(z_1, \bar{z}_1)]^\dagger \equiv \langle 0| \lim_{z_1,\bar{z}_1 \to \infty} z_1^{2h_1} \bar{z}_1^{2\bar{h}_1} \Phi_1(z_1, \bar{z}_1) = \langle \Phi_1^{out}| \tag{5.6}$$

This means fields at z_4 and z_1 are asymptotic states. Now we can rewrite the four-point function(for sake of simplicity, let us consider only holomorphic part),

$$\lim_{z_1 \to \infty} z_1^{2h_1} \langle \Phi_1(z_1)\Phi_2(1)\Phi_3(z)\Phi_4(0)\rangle = G_{34}^{21}(z) \tag{5.7}$$

Where,

$$G_{34}^{21}(z) = \langle h_1|\Phi_2(1)\Phi_3(z)|h_4\rangle \tag{5.8}$$

Applying operator algebra(4.2) between $\Phi_3(z)$ and $\Phi_4(0)$,

$$\Phi_3(z)\Phi_4(0) = \sum_p \sum_{\{k\}} C_{34}^{p\{k\}} z^{h_p - h_3 - h_4 + K} \Phi_p^{\{k\}}(0) \tag{5.9}$$

The correlation of descendants are dependent on the correlation of primaries. The exact dependence is given in the next section. But this means that we can write,

$$C_{34}^{p\{k\}} = C_{34}^{p} \beta_{34}^{p\{k\}} \tag{5.10}$$

The operator algebra between $\Phi_3(z)\Phi_4(0)$ can thus be written as,

$$\Phi_3(z)\Phi_4(0) = \sum_p C_{34}^{p} z^{h_p - h_3 - h_4} \Psi_p(z|0) \tag{5.11}$$

Where we have defined,

$$\Psi_p(z|0) = \sum_{\{k\}} \beta_{34}^{p\{k\}} z^K \Phi_p^{\{k\}}(0) \tag{5.12}$$

The function $G^{21}_{34}(z)$ now has the structure,

$$\begin{aligned}
G^{21}_{34}(z) &= \langle h_1|\Phi_2(1)\Phi_3(z)|h_4\rangle \\
&= \langle h_1|\Phi_2(1)\Phi_3(z)\Phi_4(0)|0\rangle \\
&= \langle h_1|\Phi_2(1)\sum_p C^p_{34} z^{h_p-h_3-h_4}\Psi_p(z|0)|0\rangle \\
&= \sum_p C^p_{34}C^p_{21}(C^p_{21})^{-1} z^{h_p-h_3-h_4}\langle h_1|\Phi_2(1)\Psi_p(z|0)|0\rangle \\
&= \sum_p C^p_{34}C^p_{21}\mathcal{F}^{21}_{34}(p|z)
\end{aligned} \tag{5.13}$$

Here we have introduced,

$$\mathcal{F}^{21}_{34}(p|z) = (C^p_{21})^{-1} z^{h_p-h_3-h_4}\langle h_1|\Phi_2(1)\Psi_p(z|0)|0\rangle \tag{5.14}$$

These functions are called the *Conformal Blocks* corresponding to the four-point function. They represent the intermediate conformal families (both primaries and their descendants) that arise during the *scattering* of the four fields.

Now the four-point function can be written as ,

$$\boxed{G^{21}_{34}(z) = \textstyle\sum_p C^p_{34}C^p_{21}\mathcal{F}^{21}_{34}(p|z)\bar{\mathcal{F}}^{21}_{34}(p|\bar{z})} \tag{5.15}$$

The conformal blocks are the constituents of a four-point function that can be determined by conformal invariance. Four-point functions of a rational conformal field theory, a CFT with finite number of primaries, has finite number of conformal blocks due to the truncation of fusion algebra[4]. On the other hand C^p_{34} and C^p_{21} are determined by three point functions and cannot be fixed by conformal invariance.

5.2 Method 1: Using Singular Vectors

5.2.1 Differential Equation for General Correlators

The correlator of descendant fields can be expressed in terms of the correlator of primary fields as(through out we will only deal with the holomorphic part),

$$\langle \Phi_1^{(-r_1,...,-r_k)}(z_1)\Phi_2(z_2)...\rangle = \mathcal{L}_{-r_1}(z_1)...\mathcal{L}_{-r_k}(z_1)\langle \Phi_1(z_1)\Phi_2(z_2)...\rangle \tag{5.16}$$

Where,

$$\mathcal{L}_{-r}(z) = \sum_{i>1}\left\{\frac{(r-1)h_i}{(z_i-z)^r} - \frac{1}{(z_i-z)^{r-1}}\partial_{z_i}\right\} \tag{5.17}$$

Let the Verma module $V(c,h_1)$ of the primary Φ_1 have a singular vector at level n_1,

$$|c, h_1+n_1\rangle = \sum_{Y,|Y|=n_1} \alpha_Y L_{-Y}|c,h_1\rangle \tag{5.18}$$

where Y stands for all linearly independent raising operator combinations to reach level n_1. On setting this singular vector to zero, any correlator function with the singular vector will vanish,

$$\sum_{Y,|Y|=n_1} \alpha_Y \mathcal{L}_{-Y}(z_1)\langle \Phi_1(z_1)\Phi_2(z_2)...\rangle = 0 \tag{5.19}$$

Above equation is the differential equation for calculating correlator functions. This equation can be simplified using the conformal ward identities eq. (2.23)[1],

$$\begin{aligned}\sum_i \partial_{z_i}\langle \Phi_1(z_1)\Phi_2(z_2)...\rangle = 0\\ \sum_i (z_i\partial_{z_i}+h_i)\langle \Phi_1(z_1)\Phi_2(z_2)...\rangle = 0\\ \sum_i (z_i^2\partial_{z_i}+2z_ih_i)\langle \Phi_1(z_1)\Phi_2(z_2)...\rangle = 0\end{aligned} \tag{5.20}$$

The solution to the conformal ward identities is of the form,

$$\langle \Phi_1(z_1)\Phi_2(z_2)...\rangle = \left\{\prod_{i<j} z_{ij}^{\mu_{ij}}\right\}G(z) \tag{5.21}$$

Where G is an arbitrary function of the cross ratio and,

$$\mu_{ij} = \frac{1}{3}\Big(\sum_{k=1}^{4} h_k\Big) - h_i - h_j \tag{5.22}$$

Now let us calculate the four-point correlation functions of fields in the *Ising Model*. It is useful to note the fact that for any minimal model, a singular vector of a Verma module $V_{r,s}$ is present at level rs.

5.2.2 Example-1: All four fields are $\Phi_{2,1}$

$\Phi_{2,1} \equiv \Phi$ correspond to V(c,h)=$V(\frac{1}{2}, \frac{1}{2})$ with singular vector at level-2 and $\mu_{ij} \equiv \mu = -\frac{1}{3}$. For this case, Eq.(5.19) becomes,

$$(\mathcal{L}_{-1}^2 - t\mathcal{L}_{-2})\langle\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle = 0 \tag{5.23}$$

With $t = \frac{p}{p'} = \frac{4}{3}$ and using (5.17) this simplifies to,

$$\left\{\partial_{z_1}^2 - \frac{4}{3}\sum_{i=2,3,4}\left[\frac{h_i}{(z_i - z_1)^2} - \frac{1}{(z_i - z_1)}\partial_{z_i}\right]\right\}\langle\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle = 0 \tag{5.24}$$

Substituting relation eq. (5.21) in the above equation we obtain a differential equation for G(z). Further, as discussed earlier, four-point functions can be made to depend on only one point by changing the differential parameter into, say z, the cross ratio,

$$z = \frac{z_{12}z_{34}}{z_{14}z_{32}} \quad with \quad (1-z) = \frac{z_{24}z_{31}}{z_{14}z_{32}} \tag{5.25}$$

and taking the points $z_1 \to z$, $z_2 \to 0$, $z_3 \to 1$ and $z_4 \to \infty$.
Imposing the above limits, differential operators are modified into,

$$\begin{gathered}\partial_{z_1} = \mu\left[\frac{1}{z} + \frac{1}{z-1}\right] + \partial_z \\ \partial_{z_1}^2 = \mu(\mu-1)\left[\frac{1}{z^2} + \frac{1}{(z-1)^2}\right] + 2\mu^2\left[\frac{1}{z(z-1)}\right] + 2\mu\left[\frac{1}{z} + \frac{1}{z-1}\right]\partial_z + \partial_z^2\end{gathered} \tag{5.26}$$

Making the above changes, we get an *ordinary* differential equation for G(z),

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\partial_z\Big] - \frac{2}{3}\Big[\frac{1}{z^2} + \frac{1}{(z-1)^2}\Big] + \frac{2}{3}\frac{1}{z(z-1)}\right\}G(z) = 0 \tag{5.27}$$

We would like to convert the above equation into the form of a standard *hypergeometric differential equation* form[[1],[5]], whose solutions are given in terms of *hypergeometric functions* given in the table below,

Table. Hypergeometric differential equation(HDE) and it's solutions

$HDE:$	$\Big\{z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\Big\}F(z) = 0$			
$Solutions:$	$F(a,b;c;z) = 1 + \sum_{n=1} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$	and	$z^{1-c}F(a+1-c, b+1-c; 2-c; z)$	
	$where \qquad (x)_n = x(x+1)...(x+n-1)$			

(5.28)

The desired conversion is performed by a substitution for $G(z)$ as,

$$G(z) = [z(1-z)]^{\beta}K(z) \tag{5.29}$$

Substituting this into eq. (5.27) and requiring to get to the standard form(this can be done by asking for 'z' independent coefficients for K(z) in the differential equation), we find two possibilities,

$$\beta = -\frac{2}{3} \quad or \quad 1 \tag{5.30}$$

We can choose either of the βs. Choice of $\beta = -\frac{2}{3}$ and subsequent substitution,

$$G(z) = [z(1-z)]^{-\frac{2}{3}}K(z) \tag{5.31}$$

results in the following hypergeometric differential equation,

$$\left\{z(1-z)\partial_z^2 + \Big(-\frac{2}{3} - \Big(-2-\frac{1}{3}+1\Big)z\Big)\partial_z - \frac{2}{3}\right\}K(z) = 0 \tag{5.32}$$

and corresponding solutions,

$$F\left(-2,-\frac{1}{3};-\frac{2}{3};z\right)=1-z+z^2 \tag{5.33}$$

$$z^{\frac{5}{3}}F\left(-\frac{1}{3},\frac{4}{3};\frac{8}{3};z\right) \tag{5.34}$$

First solution eq.(5.33) substituted in eq. (5.21) gives,

$$\begin{aligned}\langle\Phi(z_1)\Phi(z_2)\Phi(z_3)\Phi(z_4)\rangle &= \frac{G(z)}{(z_{12}z_{23}z_{34}z_{13}z_{24}z_{14})^{1/3}}\\ &= [z(1-z)]^{-\frac{2}{3}}\frac{K(z)}{(z_{12}z_{23}z_{34}z_{13}z_{24}z_{14})^{1/3}}\\ &= \left[\frac{z_{12}z_{34}}{z_{14}z_{32}}\times\frac{z_{24}z_{31}}{z_{14}z_{32}}\right]^{-2/3}\frac{(1-z+z^2)}{(z_{12}z_{23}z_{34}z_{13}z_{24}z_{14})^{1/3}}\\ &\sim \frac{1}{z_{14}z_{32}}\times z^{-1}(1-z+z^2)\end{aligned} \tag{5.35}$$

From *fusion-rules*, we know that,

$$\Phi_{2,1}\otimes\Phi_{2,1}=\mathbb{1} \tag{5.36}$$

The operator algebra implies that the *conformal block* should behave as $z^{-2(\frac{1}{2})+0}=z^{-1}$ to first order. (5.35) shows that solution eq.(5.33) behaves in this way. The second solution (5.34) does not behave according to the requirement and this means it represents a primary that is not a part of the Ising theory. Due to this reason, this block will be excluded from consideration for calculating the four-point function. Including the anti-holomorphic part into the four-point function, the complete result is,

$$\boxed{\begin{aligned}\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle &= \left|\tfrac{1}{z_{14}z_{32}}[z(1-z)]^{-1}(1-z+z^2)\right|^2\\ &= \left|\tfrac{1}{z_{12}z_{34}}-\tfrac{1}{z_{14}z_{32}}+\tfrac{1}{z_{24}z_{31}}\right|^2\end{aligned}} \tag{5.37}$$

Confirming the claim of equivalence with *free Majorana fermionic* theory, we can see here that the above equation is same as the result (4.11).

5.2.3 Example-2: All four fields are $\Phi_{1,2}$

$\Phi_{1,2} \equiv \Phi$ correspond to V(c,h)=$V(\frac{1}{2}, \frac{1}{16})$ with singular vector at level-2 and $\mu_{ij} \equiv \mu = -\frac{1}{24}$. In *Kac-table* for minimal models, $\Phi_{2,1} \rightarrow \Phi_{1,2}$ is achieved by $t \rightarrow 1/t$. Hence for this case, $t = \frac{3}{4}$ in eq. (5.23). Following the same steps in the construction of hypergeometric differential equations, we find that the powers in equation (5.29) are $\beta = -\frac{1}{12}$ *and* $\frac{5}{12}$. *Fusion-rule* for $\Phi_{1,2}$ is,

$$\Phi_{1,2} \otimes \Phi_{1,2} = \mathbb{1} \oplus \Phi_{2,1} \tag{5.38}$$

This implies that there are two conformal blocks. As a result both the solutions to the second order differential equation for G(z) are valid solutions. According to the *operator algebra* (4.2),

$$\Phi(z_1) \times \Phi(z_2) \sim \frac{\mathbb{1}}{z_{12}^{\frac{1}{8}}} + \frac{\Phi_{2,1}}{z_{12}^{-\frac{3}{8}}} \tag{5.39}$$

This implies, the two blocks must have first order behaviors, $z_{12}^{-\frac{1}{8}}$ *and* $z_{12}^{\frac{3}{8}}$. For the choice $\beta = -\frac{1}{12}$, the hypergeometric differential equation obtained is,

$$\left\{ z(1-z)\partial_z^2 + \left(\frac{1}{2} - \left(\frac{1}{4} - \frac{1}{4} + 1\right)z\right)\partial_z + \frac{1}{16}\right\} K(z) = 0 \tag{5.40}$$

Whose solutions are,

$$F(\frac{1}{4}, -\frac{1}{4}; \frac{1}{2}; z) \qquad and \qquad z^{\frac{1}{2}} F(\frac{3}{4}, \frac{1}{4}; \frac{3}{2}; z) \tag{5.41}$$

Here we may mention a hypergeometric identity:

$$F(a - \frac{1}{2}, a; 2a; z) = \left(\frac{1 + \sqrt{1-z}}{2}\right)^{1-2a} \tag{5.42}$$

Making use of the above identity, we can re-write the solutions obtained as,

$$\begin{aligned} F(\frac{1}{4},-\frac{1}{4};\frac{1}{2};z) &= \Big(\frac{1+\sqrt{1-z}}{2}\Big)^{\frac{1}{2}} \\ z^{\frac{1}{2}}F(\frac{3}{4},\frac{1}{4};\frac{3}{2};z) &= \sqrt{2}\Big(1-\sqrt{1-z}\Big)^{\frac{1}{2}} \end{aligned} \tag{5.43}$$

We will now consider the behaviour of each conformal block. Substituting each solution K(z) in (5.21), we get conformal blocks,

$$\begin{aligned} \mathcal{F}_1 &= \frac{1}{(z_{14}z_{32})^{\frac{1}{8}}}\frac{1}{[z(1-z)]^{\frac{1}{8}}}\left(\frac{1+\sqrt{1-z}}{2}\right)^{\frac{1}{2}} \\ \mathcal{F}_2 &= \frac{1}{(z_{14}z_{32})^{\frac{1}{8}}}\frac{1}{[z(1-z)]^{\frac{1}{8}}}\left(2(1-\sqrt{1-z})\right)^{\frac{1}{2}} \end{aligned} \tag{5.44}$$

From this we can read off the z dependence of the first conformal block,

$$\begin{aligned} \sim z^{-\frac{1}{8}}F(\frac{1}{4},-\frac{1}{4};\frac{1}{2};z) &= z^{-\frac{1}{8}}\Big(\frac{1+\sqrt{1-z}}{2}\Big)^{\frac{1}{2}} \\ &\sim z^{-\frac{1}{8}}\left(1+1-\frac{1}{2}z\right)^{\frac{1}{2}} \sim z^{-\frac{1}{8}} \end{aligned} \tag{5.45}$$

This shows that $F(\frac{1}{4},-\frac{1}{4};\frac{1}{2};z)$ corresponds to the fusion $\Phi_{(1,2)}\otimes\Phi_{(1,2)} = \mathbb{1}$. The second block,

$$\begin{aligned} z^{-\frac{1}{8}}z^{\frac{1}{2}}F(\frac{3}{4},\frac{1}{4};\frac{3}{2};z) &= z^{-\frac{1}{8}}\sqrt{2}\Big(1-\sqrt{1-z}\Big)^{\frac{1}{2}} \\ &\sim z^{-\frac{1}{8}}\left(1-1+\frac{1}{2}z\right)^{\frac{1}{2}} \sim z^{\frac{3}{8}} \end{aligned} \tag{5.46}$$

$F(\frac{3}{4},\frac{1}{4};\frac{3}{2};z)$ is the conformal block corresponding to the fusion $\Phi_{1,2}\otimes\Phi_{1,2} = \Phi_{2,1}$. Including the corresponding anti-holomorphic part and referring to (5.15), the complete four point function is,

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|z(1-z)|^{\frac{1}{4}}}\Bigg(C^{\mathbb{1}}_{34}C^{\mathbb{1}}_{21}\frac{|1+\sqrt{1-z}|}{2}+ C^{\epsilon}_{21}C^{\epsilon}_{34}2|1-\sqrt{1-z}|\Bigg) \tag{5.47}$$

Here, *structure constants* $C^{\mathbb{1}}_{34} = C^{\mathbb{1}}_{21} \equiv u$ and $C^{\epsilon}_{21} = C^{\epsilon}_{34} \equiv v$. To fix the u and v we study the above equation under the change of variables[5],

$$z = sin^2\theta \tag{5.48}$$

Then the above equation becomes,

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|cos\theta sin\theta|^{\frac{1}{4}}}\Bigg(u^2\frac{|1+cos\theta|}{2}+2v^2|1-cos\theta|\Bigg) \tag{5.49}$$

Under $\theta\to\pi-\theta$, $z = sin^2(\pi-\theta) = sin^2\theta$ and the equation becomes,

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|cos\theta sin\theta|^{\frac{1}{4}}}\Bigg(u^2\frac{|1-cos\theta|}{2}+2v^2|1+cos\theta|\Bigg) \tag{5.50}$$

For the invariance of the four-point function under such a change, it is thus required,

$$\frac{u^2}{2} = 2v^2 \Rightarrow v^2 = \frac{u^2}{4} \tag{5.51}$$

u can be set to unity from the normalization of two point function. Thus the final four-point function after the above substitutions is,

$$\boxed{\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|z(1-z)|^{\frac{1}{4}}}\Bigg(\frac{|1+\sqrt{1-z}|}{2}+\frac{|1-\sqrt{1-z}|}{2}\Bigg)} \tag{5.52}$$

The structure constants can be calculated by sightly different, but much preferred way. For this, exchange the fields at position z_1 and z_2 in the four-point function. This results in the following change in the original cross ratio:

$$z \to (1-z) \tag{5.53}$$

Thus the four-point function,

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|z(1-z)|^{\frac{1}{4}}}\Bigg(C^{\mathbb{1}}_{34}C^{\mathbb{1}}_{21}\frac{|1+\sqrt{1-z}|}{2}+ \\ C^{\epsilon}_{21}C^{\epsilon}_{34}2|1-\sqrt{1-z}|\Bigg) \tag{5.54}$$

under the mentioned swapping of the fields becomes,

$$\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|z(1-z)|^{\frac{1}{4}}}\Bigg(C^{\mathbb{1}}_{34}C^{\mathbb{1}}_{21}\frac{|1+\sqrt{z}|}{2}+ \\ C^{\epsilon}_{21}C^{\epsilon}_{34}2|1-\sqrt{z}|\Bigg) \tag{5.55}$$

But, we know that,

$$|1+\sqrt{1-z}|+|1-\sqrt{1-z}| = |1+\sqrt{z}|+|1-\sqrt{z}| \tag{5.56}$$

as, one can check,

$$\begin{aligned}\Big(|1+\sqrt{1-z}|+|1-\sqrt{1-z}|\Big)^2 &= \Big(|1+\sqrt{z}|+|1-\sqrt{z}|\Big)^2 \\ &= 2+2\sqrt{1-z}\sqrt{1-\bar{z}}+2\sqrt{z}\sqrt{\bar{z}}\end{aligned} \tag{5.57}$$

This results in a similar condition as eq.(5.51),

$$\frac{C^{\mathbb{1}}_{34}C^{\mathbb{1}}_{21}}{2} = 2C^{\epsilon}_{21}C^{\epsilon}_{34} \tag{5.58}$$

and we get back the same result eq. (5.52) for the four-point function.

5.3 Method 2: Through Conformal Blocks' Wronskian

5.3.1 Differential Equation for General Four Point Correlators

As mentioned earlier, rational conformal field theories have a finite number of conformal blocks. By construction, we will build a differential equation whose order is equal to the number of conformal blocks in the theory. The four-point correlation function can be obtained as the solution to this diferential equation in the form of conformal blocks[4]. The crux of this method lies in prior knowledge about the behaviour of conformal blocks under the fusion of involved fields in combinations. This information will be provided through the operator algebra.
As is sufficient for the requirement of our calculations, we consider the fusion of two identical fields($\Phi(z_1)$ and $\Phi(z_2)$)(or a field and it's complex conjugate). As we have discussed earlier, in two dimensional CFT the four-correlator can be made to be function of any one point, say *z₁*. Let there be n conformal blocks, $f_1(z_{12}), f_2(z_{12})....f_n(z_{12})$. Construct *Wronskian* as,

$$W_k = det \begin{bmatrix} f_1 & . & . & f_n \\ \partial_{z_1} f_1 & . & . & \partial_{z_1} f_n \\ . & . & . & . \\ \partial_{z_1}^{k-1} f_1 & . & . & \partial_{z_1}^{k-1} f_n \\ \partial_{z_1}^{k+1} f_1 & . & . & \partial_{z_1}^{k+1} f_n \\ . & . & . & . \\ \partial_{z_1}^{n} f_1 & . & . & \partial_{z_1}^{n} f_n \end{bmatrix} \tag{5.59}$$

In particular, $W_n \equiv W$. W_k is a meromorphic function with possible poles at $\{0, 1, \infty\}$ for the case of four point functions. Consider the equation,

$$W_{n-1} - W_n \frac{W_{n-1}}{W_n} = 0 \tag{5.60}$$

This implies that the determinant of the matrix corresponding to the above equation is *zero*,

$$det \begin{bmatrix} f_1 - \left(\frac{W_{n-1}}{W_n}\right) f_1 & . & . & f_n - \left(\frac{W_{n-1}}{W_n}\right) f_n \\ \partial_{z_1} f_1 - \left(\frac{W_{n-1}}{W_n}\right) \partial_{z_1} f_1 & . & . & \partial_{z_1} f_n - \left(\frac{W_{n-1}}{W_n}\right) \partial_{z_1} f_n \\ . & . & . & . \\ . & . & . & . \\ \partial_{z_1}^{n-2} f_1 - \left(\frac{W_{n-1}}{W_n}\right) \partial_{z_1}^{n-2} f_1 & . & . & \partial_{z_1}^{n-2} f_n - \left(\frac{W_{n-1}}{W_n}\right) \partial_{z_1}^{n-2} f_n \\ \partial_{z_1}^{n} f_1 - \left(\frac{W_{n-1}}{W_n}\right) \partial_{z_1}^{n-1} f_1 & . & . & \partial_{z_1}^{n} f_n - \left(\frac{W_{n-1}}{W_n}\right) \partial_{z_1}^{n-1} f_n \end{bmatrix} = 0 \tag{5.61}$$

We can use two properties of determinants here. Firstly, if we multiply a row of a matrix with a number, the determinant is also multiplied by that number. Secondly, determinant of sum of matrices is same as the sum of determinants of the matrices if the i^{th} row of both matrices are different and all other rows are identical. Then the equation is simplified to,

$$det \begin{bmatrix} f_1 & . & . & f_n \\ \partial_{z_1} f_1 & . & . & \partial_{z_1} f_n \\ . & . & . & . \\ . & . & . & . \\ \partial_{z_1}^{n-2} f_1 & . & . & \partial_{z_1}^{n-2} f_n \\ \partial_{z_1}^{n} f_1 - \frac{W_{n-1}}{W_n} \partial_{z_1}^{n-1} f_1 & . & . & \partial_{z_1}^{n} f_n - \frac{W_{n-1}}{W_n} \partial_{z_1}^{n-1} f_n \end{bmatrix} = 0 \tag{5.62}$$

For the determinant to vanish, the last row of the matrix should be a linear combination of the rest of the rows. This gives us a set of n differential equations each of order n[4],

$$\partial_{z_1}^{n} f_i - \frac{W_{n-1}}{W_n} \partial_{z_1}^{n-1} f_i + \sum_{k=0}^{n-2} \Psi_k \partial_{z_1}^{k} f_i = 0 \tag{5.63}$$

Where we have,

$$\begin{aligned} \Psi_{n-1} &\equiv -\frac{W_{n-1}}{W} = -\frac{W'}{W} \\ \Psi_k &= (-1)^{n-k} \frac{W_k}{W} \end{aligned} \tag{5.64}$$

This is the same as considering n independent solutions of any one nth order differential equation. Hence, the conformal blocks f_i's are independent solutions to an n-th order differntial equation in variable z_1,

$$\partial^n_{z_1} f + \sum_{k=0}^{n-1} \Psi_k \partial^k_{z_1} f = 0 \tag{5.65}$$

Henceforth our discussion will focus only on four-point correlation functions where all four-

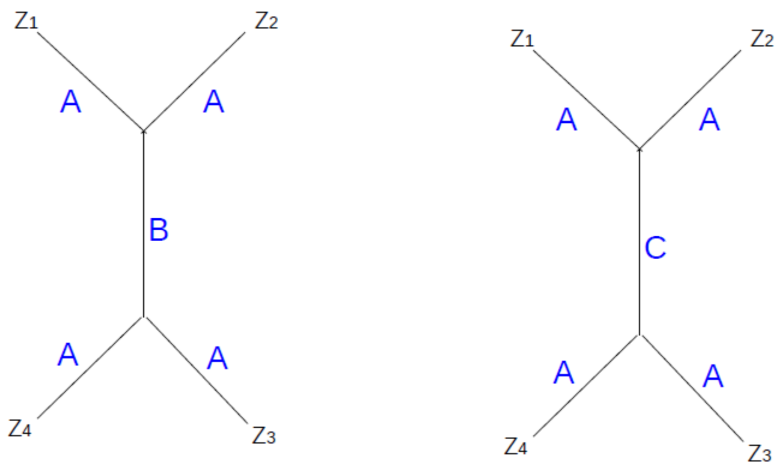

Figure 5.1: Conformal Blocks corresponding to fusion rule (5.66)

fields are *identical*, Hermitian and the fusion-rule dictates the presence of *maximum two* conformal blocks. Let the four fields be denoted by '**A**'(conformal dimension h_A) and fusion-rule,

$$A \otimes A = B \oplus C \tag{5.66}$$

Here '**B**'(h_B) and '**C**'(h_C) are fields in the theory into which the '**A**'s can fuse. Then the conformal blocks can be represented through the diagrams 5.1. Let's name these conformal blocks as f_1 and f_2 respectively. From operator algebra, the behaviour of these blocks as z_1 approaches z_2, z_3 and z_4 are calculated and are tabulated below,

Block	$z_1 \to z_2$	$z_1 \to z_3$	$z_1 \to z_4$
f_1	$z_{12}^{-2h_A+h_B}$	$z_{13}^{-2h_A+h_B}$	$z_{14}^{-2h_A+h_B}$
f_2	$z_{12}^{-2h_A+h_C}$	$z_{13}^{-2h_A+h_C}$	$z_{14}^{-2h_A+h_C}$

(5.67)

It can be shown by the change of variables $z_1 \to \frac{1}{w_1}$ and limit $w_1 \to 0$ that the conformal blocks behave as $z_1^{-2h_A}$ as $z_1 \to \infty$. The table for behaviour of the *Wronskian* $W \equiv W_2$,

$$W_2 = det\begin{bmatrix} f_1 & f_2 \\ \partial f_1 & \partial f_2 \end{bmatrix} \tag{5.68}$$

Wronskian	$z_1 \to z_2$	$z_1 \to z_3$	$z_1 \to z_4$
W_2	$z_{12}^{-4h_A+h_B+h_C-1}$	$z_{13}^{-4h_A+h_B+h_C-1}$	$z_{14}^{-4h_A+h_B+h_C-1}$

(5.69)

The behaviour of the Wronskian when $z_1 \to \infty$ is zero if we consider just the first order behaviour of the conformal blocks. Thus we should consider the behaviour of one of the blocks to next order. This shows that the behaviour of the Wronskian is $z_1^{-4h_A-2}$ as $z_1 \to \infty$. The behavour of the conformal blocks as $z_1 \to \infty$ is related to the behaviours as z_1 approaches z_2, z_3 and z_4 (this is due to a theorem by Riemann which states that a holomorphic function on a sphere has equal number of poles and number of zeros. i.e $\#poles = \#zeros$):

$$\boxed{-12h_A + 3h_B + 3h_C - 3 = -4h_A - 2} \tag{5.70}$$

Now we can write the Wronskian as,

$$W = Cz_{12}^{-4h_A+h_B+h_C-1} z_{13}^{-4h_A+h_B+h_C-1} z_{14}^{-4h_A+h_B+h_C-1} \tag{5.71}$$

Where C is a parameter independent of z_1.
As there are two conformal blocks, the differential equation we wish to solve is,

$$\left(\partial_{z_1}^2 + \Psi_1 \partial_{z_1} + \Psi_0\right)G(z) = 0 \tag{5.72}$$

where G(z) is as defined in (5.21) and z is the *cross ratio.* We know from the definition,

$$\Psi_1 = -\frac{\partial_{z_1} W}{W} = (4h_A - h_B - h_C + 1)\left[z_{12}^{-1} + z_{13}^{-1} + z_{14}^{-1}\right] \tag{5.73}$$

Ψ_0 is fixed considering the following properties[4],

- Due to the power law behaviour of blocks, ψ_0 can have at most double pole near z_2, z_3 and z_4.
- The coefficients of the double poles are fixed by the known behaviors of the conformal

blocks.

- Ψ_0 does not have any poles as $z_i \to z_j (i, j = 2, 3, 4)$.

- In the limit $z_2 \to z_4$, function $z_{24}^{-2h_A} z_{13}^{-2h_A}$ is a solution to the differential equation.

These criteria fix,

$$\begin{aligned}\Psi_0 = \Bigg[&\big(-2h_A + h_B\big)\big(-2h_A + h_C\big)\left(\frac{1}{z_{12}^2} + \frac{1}{z_{13}^2} + \frac{1}{z_{14}^2}\right) - \\ &2\big(-2h_A + h_B\big)\big(-2h_A + h_C\big)\left(\frac{1}{z_{12}z_{13}} + \frac{1}{z_{12}z_{14}} + \frac{1}{z_{13}z_{14}}\right)\Bigg]\end{aligned} \tag{5.74}$$

Substituting the above results in the differential equation (5.72) we get,

$$\begin{aligned}&\Bigg\{\partial_{z_1}^2 + (4h_A - h_B - h_C + 1)\big[z_{12}^{-1} + z_{13}^{-1} + z_{14}^{-1}\big]\partial_{z_1} + \\ &\Bigg[\big(-2h_A + h_B\big)\big(-2h_A + h_C\big) \times \left(\frac{1}{z_{12}^2} + \frac{1}{z_{13}^2} + \frac{1}{z_{14}^2} - \frac{2}{z_{12}z_{13}} - \frac{2}{z_{12}z_{14}} - \frac{2}{z_{13}z_{14}}\right)\Bigg]\Bigg\} G(z) = 0\end{aligned} \tag{5.75}$$

Further, four-point function can be made to depend on only one point by changing the differential parameter into, say z, the cross ratio, and taking the points $z_1 \to z$, $z_2 \to 0$, $z_3 \to 1$ and $z_4 \to \infty$. Using relations (5.26) for differential operators, the differential equation is reduced to,

$$\begin{aligned}&\Bigg\{\partial_z^2 + 2\mu\left[\frac{1}{z} + \frac{1}{z-1}\right]\partial_z + \mu(\mu - 1)\left[\frac{1}{z^2} + \frac{1}{(z-1)^2}\right] + \frac{2\mu^2}{z(z-1)} \\ &(4h_A - h_B - h_C + 1)\left[\frac{1}{z} + \frac{1}{z-1}\right]\left(\mu\left[\frac{1}{z} + \frac{1}{z-1}\right] + \partial_z\right) + \\ &\Bigg[\big(-2h_A + h_B\big)\big(-2h_A + h_C\big) \times \left(\frac{1}{z^2} + \frac{1}{(1-z)^2} - \frac{2}{z(z-1)}\right)\Bigg]\Bigg\} G(z) = 0\end{aligned} \tag{5.76}$$

In the following subsections, we will use the current method to reproduce the four-point correlation functions of the *Ising theory* and verify the results of previous section with method-1.

5.3.2 Example-1: All the fields are $\Phi_{2,1}$

$\Phi_{2,1} \equiv \Phi$ corresponds to V(c,h)=$V(\frac{1}{2}, \frac{1}{2})$. From the fusion-rule, we know that,

$$\Phi_{2,1} \otimes \Phi_{2,1} = \mathbb{1} \tag{5.77}$$

Thus we need one conformal block corresponding to identity. But for following the current method, we need a second order differential equation and Wronskian behaviours from two conformal blocks. For this we modify the fusion rule to,

$$\Phi_{2,1} \otimes \Phi_{2,1} = \mathbb{1} \oplus \Phi_{3,1} \tag{5.78}$$

The above result is obtained by modifying the fusion rule eq.(4.3). As the increment in summation is under *mod2* changing (k_{max}) and (l_{max}) to ($k_{max}+2$) and (l_{max}) respectively, we get,

$$\Phi_{r,s} \times \Phi_{m,n} = \sum_{\substack{k=1+|r-m| \\ k+r+m=1 \quad mod2}}^{k_{max}+2} \quad \sum_{\substack{l=1+|s-n| \\ l+s+n=1 \quad mod2}}^{l_{max}} \Phi_{k,l} \tag{5.79}$$

This particular modification will enable us to expand and access the primaries which are not present in the physical theory. $\Phi_{3,1}$ does not lie inside the actual set of primaries for the Ising theory. But here we will include it's corresponding conformal block in deriving the differential equation for the four-point correlators. This consideration will eventually be dropped in the set of solutions obtained.
For $\Phi_{r,s} = \Phi_{3,1}$, conformal dimension is,

$$h_{r,s} = h_{3,1} = \frac{(pr - p's)^2 - (p-p')^2}{4pp'} = \frac{5}{3} \tag{5.80}$$

Now we are in a position to build the differential equation with,

$$\begin{array}{ll} h_A = \frac{1}{2} & h_B = 0 \\ h_C = \frac{5}{3} & \mu_{ij} \equiv \mu = -\frac{1}{3} \end{array} \tag{5.81}$$

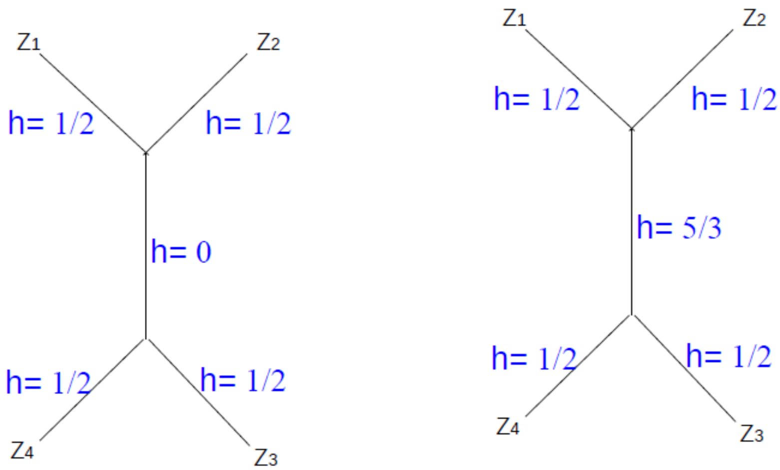

Figure 5.2: Conformal Blocks corresponding to fusion rule (5.78)

Making the above substitutions in equation (5.76), we get,

$$\begin{aligned}\Bigg\{\partial_z^2 + \left(-\frac{2}{3}\right)\left[\frac{1}{z}+\frac{1}{z-1}\right]\partial_z + \left(\frac{4}{9}\right)\left[\frac{1}{z^2}+\frac{1}{(z-1)^2}\right] + \frac{2}{9}\frac{1}{z(z-1)} \\ \left(\frac{4}{3}\right)\left[\frac{1}{z}+\frac{1}{z-1}\right]\left(-\frac{1}{3}\left[\frac{1}{z}+\frac{1}{z-1}\right]+\partial_z\right)+ \\ \left[-(1)\left(\frac{2}{3}\right)\times\left(\frac{1}{z^2}+\frac{1}{(1-z)^2}-\frac{2}{z(z-1)}\right)\right]\Bigg\}G(z)=0\end{aligned} \tag{5.82}$$

which is simplified to,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\partial_z\Big] - \frac{2}{3}\Big[\frac{1}{z^2}+\frac{1}{(z-1)^2}\Big] + \frac{2}{3}\frac{1}{z(z-1)}\right\}G(z)=0 \tag{5.83}$$

Trying to convert this equation to standard *hypergeometric differential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta}K(z) \tag{5.84}$$

we find two possibilities,

$$\beta = -\frac{2}{3} \quad or \quad 1 \tag{5.85}$$

We can choose either of the βs. Here let's choose $\beta = -\frac{2}{3}$. Hence the substitution,

$$G(z) = [z(1-z)]^{-\frac{2}{3}} K(z) \tag{5.86}$$

results in the following differential equation,

$$\left\{ z(1-z)\partial_z^2 + \left(-\frac{2}{3} - \left(-2 - \frac{1}{3} + 1 \right) z \right) \partial_z - \frac{2}{3} \right\} K(z) = 0 \tag{5.87}$$

and corresponding solutions,

$$F\left(-2, -\frac{1}{3}; -\frac{2}{3}; z \right) = 1 - z + z^2 \tag{5.88}$$

$$z^{\frac{5}{3}} F\left(-\frac{1}{3}, \frac{4}{3}; \frac{8}{3}; z \right) \tag{5.89}$$

These two solutions are in agreement with the ones derived from previous method, (5.33) and (5.34). As promised earlier, we retain the *Conformal block* $F\left(-2, -\frac{1}{3}; -\frac{2}{3}; z \right)$ which behaves as $z^{-2(1/2)+0} = z^{-1}$ to first order and the other one is removed. Following the similar steps of method-1, we arrive at the complete four-point function,

$$\boxed{\begin{aligned} \langle \Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4) \rangle &= \left| \tfrac{1}{z_{14}z_{32}} [z(1-z)]^{-1}(1-z+z^2) \right|^2 \\ &= \left| \tfrac{1}{z_{12}z_{34}} - \tfrac{1}{z_{14}z_{32}} + \tfrac{1}{z_{24}z_{31}} \right|^2 \end{aligned}} \tag{5.90}$$

5.3.3 Example-2: All the fields are $\Phi_{1,2}$

$\Phi_{1,2} \equiv \Phi$ corresponds to V(c,h)=$V(\frac{1}{2}, \frac{1}{16})$ and its fusion rule,

$$\Phi_{1,2} \otimes \Phi_{1,2} = \mathbb{1} \oplus \Phi_{2,1} \tag{5.91}$$

We know,

$$\begin{array}{ll} h_A = \frac{1}{16} & h_B = 0 \\ h_C = \frac{1}{2} & \mu_{ij} \equiv \mu = -\frac{1}{24} \end{array} \tag{5.92}$$

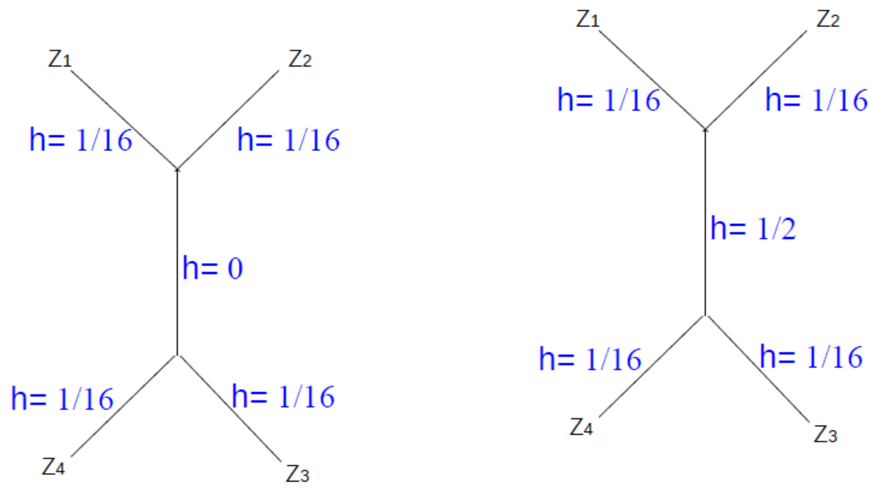

Figure 5.3: Conformal Blocks corresponding to fusion rule (5.91)

Making the above substitutions in equation (5.76), we get,

$$\Bigg\{\partial_z^2 + \left(-\frac{1}{12}\right)\left[\frac{1}{z}+\frac{1}{z-1}\right]\partial_z + \left(\frac{25}{24\times 24}\right)\left[\frac{1}{z^2}+\frac{1}{(z-1)^2}\right] + \frac{2}{24\times 24}\frac{1}{z(z-1)}$$
$$\left(\frac{3}{4}\right)\left[\frac{1}{z}+\frac{1}{z-1}\right]\left(-\frac{1}{24}\left[\frac{1}{z}+\frac{1}{z-1}\right]+\partial_z\right)+ \tag{5.93}$$
$$\left[-\left(\frac{1}{8}\right)\left(\frac{3}{8}\right)\times\left(\frac{1}{z^2}+\frac{1}{(z-1)^2}-\frac{2}{z(z-1)}\right)\right]\Bigg\}G(z)=0$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta}K(z) \tag{5.94}$$

we find two possibilities,

$$\beta = -\frac{1}{12} \quad or \quad \frac{5}{12} \tag{5.95}$$

Choice $\beta = -\frac{1}{12}$ results in the following differential equation,

$$\left\{z(1-z)\partial_z^2 + \left(\frac{1}{2}-\left(\frac{1}{4}-\frac{1}{4}+1\right)z\right)\partial_z + \frac{1}{16}\right\}K(z) = 0 \tag{5.96}$$

Whose solutions are,

$$F(\frac{1}{4},-\frac{1}{4};\frac{1}{2};z) \qquad and \qquad z^{\frac{1}{2}}F(\frac{3}{4},\frac{1}{4};\frac{3}{2};z) \tag{5.97}$$

Using the hypergeometric identity:

$$F(a-\frac{1}{2},a;2a;z)=\Big(\frac{1+\sqrt{1-z}}{2}\Big)^{(1-2a)} \tag{5.98}$$

solutions obtained have the form,

$$\begin{aligned} F(\frac{1}{4},-\frac{1}{4};\frac{1}{2};z) &= \Big(\frac{1+\sqrt{1-z}}{2}\Big)^{\frac{1}{2}} \\ z^{\frac{1}{2}}F(\frac{3}{4},\frac{1}{4};\frac{3}{2};z) &= \sqrt{2}\Big(1-\sqrt{1-z}\Big)^{\frac{1}{2}} \end{aligned} \tag{5.99}$$

These solutions are in agreement with the solutions obtained by Method-1. From this point, following the similar steps of method-1, the complete four-point function with the structure constants,

$$\boxed{\langle\Phi(z_1,\bar{z}_1)\Phi(z_2,\bar{z}_2)\Phi(z_3,\bar{z}_3)\Phi(z_4,\bar{z}_4)\rangle = \frac{1}{|z_{14}z_{32}|^{\frac{1}{4}}}\frac{1}{|z(1-z)|^{\frac{1}{4}}}\left(\frac{|1+\sqrt{1-z}|}{2}+\frac{|1-\sqrt{1-z}|}{2}\right)} \tag{5.100}$$

Chapter 6

Baby-Monster Module

6.1 Coset Theory

Possible 2D CFT candidates can be constructed from established affine CFTs through an algebraic construction called *Coset theory*[1]. This has been generalized in [6] to cosets of meromorphic CFTs. A meromorphic CFT is a self-dual CFT, $\mathcal{H}$ with a single character, i.e. only vacuum primary character($\chi_0^{\mathcal{H}}(\tau)$). Let $\mathcal{D}$ be an affine subtheory of $\mathcal{H}$. It is not required that $\mathcal{H}$ be an affine theory. Then a coset theory can be constructed as,

$$\mathcal{C} = \frac{\mathcal{H}}{\mathcal{D}} \tag{6.1}$$

Let $\mathcal{D}$ be associated with a semi-simple Lie algebra $\mathfrak{h}$ at level k. This allows us to perform a *Sugawara construction* and build the stress-energy tensor of $\mathcal{D}$. Further, the following relation between the central charges of the involved theories holds,

$$C^{\mathcal{C}} = C^{\mathcal{H}} - C^{\mathcal{D}} \tag{6.2}$$

Non-affine theories with central charge $C = 24$ have the character[7],

$$\chi_0^{\mathcal{H}}(\tau) = J(\tau) + \mathcal{N} \qquad \textit{with} \qquad J(\tau) = j(\tau) - 744 = \frac{1}{q} + 196884q + \ldots \tag{6.3}$$

Here $q = e^{2\pi i\tau}$. $j(\tau)$ is known as Klein-j function. It is the unique modular form of weight *zero* with no poles on the upper half plane. $J(\tau)$ as a character corresponds to *Moonshine CFT*. $\mathcal{N}$ is a positive integer which denotes the number of independent states of the theory at level 1. i.e. these states have conformal dimension $h = 1$. Descendants of the vacuum primary at level 1 due to the Virasoro algebra will be removed as a null vector in any irreducible representation of a CFT. Hence, it is possible to have states at level 1 only if the theory has a current algebra, also called a *Kac-Moody algebra*.

As a consequence of this construction, it has been shown in [6] that there is a *bilinear relation* between the characters of the involved theories,

$$\boxed{\chi_0^{\mathcal{H}}(\tau) = \chi_0^{\mathcal{D}}(\tau)\chi_0^{\mathcal{C}}(\tau) + \sum_{i=1}^{p-1} \chi_i^{\mathcal{D}}(\tau)\chi_i^{\mathcal{C}}(\tau)} \tag{6.4}$$

Here we have assumed that $\mathcal{D}$ is a rational CFT with p characters.

6.2 Dual Theories

Classifying *three*-character theories without *Kac-Moody* algebra using the method of *modular invariant differential equations*, *six* interesting solution-theories have been found [8]. As one would predict, $\mathcal{M}_{4,3}$ *Ising model*, a unitary minimal model, is amongst these six possibilities. This is because as we have already seen, Ising model is a three-character theory with one character for each primary. As is the characteristic of minimal models, the Ising model has only a Virasoro algebra, and in particular , no Kac-Moody algebra.

Studying the properties of these six theories, one finds that they are pairwise related by the bilinear relation eq. (6.4) with $\mathcal{H}$ as Monster CFT. Such pairs are called *Duals* to each other. This relation is surprising because none of the dual theories has affine-algebra while traditional coset construction requires that atleast the denominator theory $\mathcal{D}$ have an affine algebra.

For the case of the *Ising model* $\mathcal{M}(4,3)$, with $c = \frac{1}{2}$, its dual denoted $\tilde{\mathcal{M}}(4,3)$ is a theory with $c = \frac{47}{2}$, such that the central charges of the two add to $c = 24$, the central charge of the Moonshine module. This candidate dual $\tilde{\mathcal{M}}(4,3)$ is called the *Baby Monster Module*[[9],[8]]. The Baby Monster has the following properties,(We will denote the primaries of Baby-Monster theory by *tilde* relative corresponding the Ising notation),

1. It is a *three* character theory.

2. Conformal dimensions of the three primaries are 0, $\frac{31}{16}$ and $\frac{3}{2}$.

3. As the Baby Monster is dual to the Ising model, the conformal dimensions of the primaries of the dual theories share the following relations due to (6.4),

$$\begin{aligned} h_0^{(Ising)} &= h_0^{(Babymonster)} = 0 \\ h_{\tilde{\epsilon}}^{(Babymonster)} &= 2 - h_{\epsilon}^{(Ising)} = \frac{31}{16} \\ h_{\tilde{\sigma}}^{(Babymonster)} &= 2 - h_{\sigma}^{(Ising)} = \frac{3}{2} \end{aligned} \tag{6.5}$$

4. The characters of Baby-Monster theory are as follows[[9],[8]],

$$\begin{aligned} \chi_0 &= q^{-\frac{47}{48}}(1 + 96256q^2 + 9646891q^3 + 366845011q^4 + ...) \\ \chi_{\tilde{\epsilon}} &= q^{\frac{23}{24}}(96256 + 10602496q + 420831232q^2 + 9685952512q^3 + ...) \\ \chi_{\tilde{\sigma}} &= q^{\frac{25}{48}}(4371 + 1143745q + 64680601q^2 + 1827005611q^3 + ...) \end{aligned} \tag{6.6}$$

6.3 Verlinde Formula

Fusion rule between two fields can be represented as N_{ijk}. The interpretation of the above notation is that, if two fields i and j can fuse together to form a third field k, then, $N_{ijk} = 1$ (or some integer if there is a multiplicity) and 0 otherwise. With this notation, fusion rules for the Ising model eq.(4.6) becomes,

$$\begin{aligned} N_{\epsilon\epsilon 0} &= 1 \qquad N_{\epsilon\epsilon\sigma} = 1 \\ N_{\sigma\sigma 0} &= 1 \qquad N_{\sigma\sigma\epsilon} = 0 \end{aligned} \tag{6.7}$$

Under modular transformation, $S : \tau \to -\frac{1}{\tau}$, the characters of a CFT transforms as,

$$\chi_i(-\frac{1}{\tau}) = \sum_j M_{ij}\chi_j(\tau) \tag{6.8}$$

Here M is called the *Monodromy Matrix* for the given CFT.

The Verlinde formula is a relation between these Monodromy matrices and the fusion rules for different primaries in the theory[10].

$$N_{ijk} = \sum_{n} \frac{M_{in} M_{jn} M_{kn}}{M_{0n}} \tag{6.9}$$

Here, o corresponds to identity field and n runs over all the primaries fields.

Consider now the bilinear relation (6.4), for the Moonshine module,

$$J(\tau) = \sum_{i} \chi_i \tilde{\chi}_i \tag{6.10}$$

Under $S : \tau \to -\frac{1}{\tau}$, $J(-\frac{1}{\tau}) = J(\tau)$ as it is a zero-weight modular form. This implies,

$$\begin{aligned} J(-\tfrac{1}{\tau}) = J(\tau) &= \sum_i (M\chi)_i (\tilde{M}\tilde{\chi})_i \\ &= \sum_i \sum_j M_{ij}\chi_i \textstyle\sum_k \tilde{M}_{ik}\tilde{\chi}_k \\ &= \sum_{jk} \left(\sum_i \left(M^T\right)_{ji} \tilde{M}_{ik} \right) \chi_j \tilde{\chi}_k \end{aligned} \tag{6.11}$$

Comparing with eq.(6.10), $j = k$ and $M^T \tilde{M} = \mathbb{1}$.

For Ising model, monodromy matrix is[1],

$$M_{(\mathcal{M}_{4,3})} = \frac{1}{2} \begin{pmatrix} 1 & 1 & \sqrt{2} \\ 1 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 \end{pmatrix} \tag{6.12}$$

Here the matrix should be read in the order $(1, \epsilon, \sigma)$. This is an unitary matrix. Hence the transpose is the inverse. This implies, $M^{babymonster} = M^*_{(\mathcal{M}_{4,3})} = M_{(\mathcal{M}_{4,3})}$.

We now have the monodromy matrix for Baby Monster theory. The same matrix, as for Ising model, read in the corresponding order $(1, \tilde{\epsilon}, \tilde{\sigma})$, can now be used to build the fusion rules for Baby Monster theory using Verlinde's formula eq.(6.9). Thus we find the fusion rules for the theory of the Baby Monster,

$$\begin{aligned} N_{\tilde{\epsilon}\tilde{\epsilon}0} &= 1 \qquad & N_{\tilde{\epsilon}\tilde{\epsilon}\tilde{\sigma}} &= 1 \\ N_{\tilde{\sigma}\tilde{\sigma}0} &= 1 \qquad & N_{\tilde{\sigma}\tilde{\sigma}\tilde{\epsilon}} &= 0 \end{aligned} \tag{6.13}$$

6.4 Calculating the Four-Point Functions

For calculating four-point functions of Baby Monster theory we will be using method-2 discussed in the previous chapter. We cannot use method-1 directly due to lack of knowledge about the null vectors in Baby Monster theory. But for using method-2, we only need the fusion rules and operator algebra to obtain the behaviour of Wronskian of conformal blocks and in turn the differential equation to calculate the four-point function.

6.4.1 All the fields are $\tilde{\Phi}_{1,2}$

We have obtained the fusion rules for Baby Monster theory using Verlinde's formula in the previous section. According to that,

$$\tilde{\Phi}_{1,2} \otimes \tilde{\Phi}_{1,2} = \mathbb{1} \oplus \tilde{\Phi}_{2,1} \tag{6.14}$$

According to the *operator algebra* (4.2),

$$\tilde{\Phi}(z_1) \times \tilde{\Phi}(z_2) \sim \frac{\mathbb{1}}{z_{12}^{\frac{31}{8}}} + \frac{\tilde{\Phi}_{2,1}}{z_{12}^{\frac{31}{8}-\frac{3}{2}}} \tag{6.15}$$

The Fusion algebra only provides the information about the conformal family that appear during the fusion of fields. Hence, it need not be the primary that appears in the fusion, but can also be any of the descendants in that particular conformal family.

Consider the relation eq. (5.70). For the current Baby-Monster case, $h_A = \frac{31}{16}$, $h_B = 0+m$ (m^{th} descendants of the identity) and $h_C = \frac{3}{2} + n$ (n^{th} descendants over the field $\tilde{\Phi}_{2,1}$). Substituting the above values in the relation, we find that,

$$\boxed{m + n = 4} \tag{6.16}$$

We do not know which particular configuration of descendants of the fusion in the conformal blocks is the right one for calculating the four-point function. So in the following, we try to analyze the conformal blocks in each possible scenario individually and see if it gives any insight into what the actual result should be. At this point, it should be noted that the case of first level over identity (case: m=1; n=3) is irrelevant because the Baby monster theory is free of current algebra.

$\boxed{CASE-1: m=4; n=0}$

Set m=4. Then $h_A = \frac{31}{16}$, $h_B = 0+4$ and $h_C = \frac{3}{2}+0$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{221}{144}\Big[\frac{1}{z^2}+\frac{1}{(z-1)^2}\Big] - \frac{643}{144}\frac{1}{z(z-1)}\right\}G(z)=0 \tag{6.17}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta}K(z) \tag{6.18}$$

We find two possibilities,

$$\beta = -\frac{13}{12} \quad or \quad \frac{17}{12} \tag{6.19}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

β	$-\frac{13}{12}$	$\frac{17}{12}$
$Block1$	$F\left(\frac{3}{4},-\frac{19}{4};-\frac{3}{2};z\right)$	$F\left(\frac{1}{4},\frac{23}{4};\frac{7}{2};z\right)$
$Block2$	$z^{\frac{5}{2}}F\left(\frac{13}{4},-\frac{9}{4};\frac{7}{2};z\right)$	$z^{-\frac{5}{2}}F\left(-\frac{9}{4},\frac{13}{4};-\frac{3}{2};z\right)$

(6.20)

$\boxed{CASE-2: m=3; n=1}$

Set m=3. Then $h_A = \frac{31}{16}$, $h_B = 0+3$ and $h_C = \frac{3}{2}+1$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{5}{144}\Big[\frac{1}{z^2}+\frac{1}{(z-1)^2}\Big] - \frac{1075}{144}\frac{1}{z(z-1)}\right\}G(z)=0 \tag{6.21}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta}K(z) \tag{6.22}$$

We find two possibilities,

$$\beta = -\frac{1}{12} \quad or \quad \frac{5}{12} \tag{6.23}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

$$\begin{array}{|c|c|c|} \hline \beta & -\frac{1}{12} & \frac{5}{12} \\ \hline Block1 & F\left(-\frac{11}{4}, \frac{11}{4}; \frac{1}{2}; z\right) & F\left(-\frac{7}{4}, \frac{15}{4}; \frac{3}{2}; z\right) \\ \hline Block2 & z^{\frac{1}{2}} F\left(-\frac{9}{4}, \frac{13}{4}; \frac{3}{2}; z\right) & z^{-\frac{1}{2}} F\left(-\frac{9}{4}, \frac{13}{4}; \frac{1}{2}; z\right) \\ \hline \end{array} \tag{6.24}$$

$\boxed{CASE - 3 : m = 2; n = 2}$

Set m=2. Then $h_A = \frac{31}{16}$, $h_B = 0 + 2$ and $h_C = \frac{3}{2} + 2$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\left[\frac{2z-1}{z(z-1)}\right]\partial_z - \frac{77}{144}\left[\frac{1}{z^2} + \frac{1}{(z-1)^2}\right] - \frac{931}{144}\frac{1}{z(z-1)}\right\}G(z) = 0 \tag{6.25}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta} K(z) \tag{6.26}$$

We find two possibilities,

$$\beta = -\frac{7}{12} \quad or \quad \frac{11}{12} \tag{6.27}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

$$\begin{array}{|c|c|c|} \hline \beta & -\frac{7}{12} & \frac{11}{12} \\ \hline Block1 & F\left(-\frac{15}{4}, \frac{7}{4}; -\frac{1}{2}; z\right) & F\left(-\frac{3}{4}, \frac{19}{4}; \frac{5}{2}; z\right) \\ \hline Block2 & z^{\frac{3}{2}} F\left(-\frac{9}{4}, \frac{13}{4}; \frac{5}{2}; z\right) & z^{-\frac{3}{2}} F\left(-\frac{9}{4}, \frac{13}{4}; -\frac{1}{2}; z\right) \\ \hline \end{array} \tag{6.28}$$

$\boxed{CASE - 4 : m = 0; n = 4}$

Set m=0. Then $h_A = \frac{31}{16}$, $h_B = 0$ and $h_C = \frac{3}{2} + 4$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{1085}{144}\Big[\frac{1}{z^2} + \frac{1}{(z-1)^2}\Big] + \frac{1085}{144}\frac{1}{z(z-1)}\right\}G(z) = 0 \tag{6.29}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta} K(z) \tag{6.30}$$

We find two possibilities,

$$\beta = -\frac{31}{12} \quad or \quad \frac{35}{12} \tag{6.31}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

β	$-\frac{31}{12}$	$\frac{35}{12}$
$Block1$	$F\left(-\frac{31}{4}, -\frac{9}{4}; -\frac{9}{2}; z\right)$	$F\left(\frac{13}{4}, \frac{35}{4}; \frac{13}{2}; z\right)$
$Block2$	$z^{\frac{11}{2}}F\left(\frac{13}{4}, -\frac{9}{4}; \frac{13}{2}; z\right)$	$z^{-\frac{11}{2}}F\left(-\frac{9}{4}, \frac{13}{4}; -\frac{9}{2}; z\right)$

(6.32)

6.4.2 All the fields are $\tilde{\Phi}_{2,1}$

Using the relation eq.(5.70) for $\tilde{\Phi}_{2,1}$ in the particular case: $h_A = \frac{3}{2}$ and $h_B = 0$ implies $h_C = \frac{13}{3}$. As we don't know the configuration of the descendants in the fusion, $h_C = \frac{13}{3}$ will be considered as 4^{th} level descendant of $h_C = \frac{1}{3}$. This is because, increment of levels over identity primary implies equal decrements from field with $h_C = \frac{13}{3}$ so that eq.(5.70) holds.

Thus for the current Baby-Monster case, $h_A = \frac{3}{2}$, $h_B = 0 + m$ (m^{th} descendants of the identity) and $h_C = \frac{1}{3} + n$ (n^{th} descendant) such that $m + n = 4$. Again, the case $m = 1; n = 3$ will not be considered as the theory is free of Kac-Moody algebra.

$\boxed{CASE - 1 : m = 4; n = 0}$

Set m=4. Then $h_A = \frac{3}{2}$, $h_B = 0 + 4$ and $h_C = \frac{1}{3} + 0$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{10}{3}\Big[\frac{1}{z^2} + \frac{1}{(z-1)^2}\Big] + \frac{2}{z(z-1)}\right\}G(z) = 0 \tag{6.33}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^\beta K(z) \tag{6.34}$$

We find two possibilities,

$$\beta = 2 \quad or \quad -\frac{5}{3} \tag{6.35}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

β	2	$-\frac{5}{3}$
*Block*1	$F\left(2, \frac{19}{3}; \frac{14}{3}; z\right)$	$F\left(-1, -\frac{16}{3}; -\frac{8}{3}; z\right)$
*Block*2	$z^{-\frac{11}{3}} F\left(-\frac{5}{3}, \frac{8}{3}; -\frac{8}{3}; z\right)$	$z^{\frac{11}{3}} F\left(\frac{8}{3}, -\frac{5}{3}; \frac{14}{3}; z\right)$

(6.36)

$\boxed{CASE - 2 : m = 3; n = 1}$

Set m=3. Then $h_A = \frac{3}{2}$, $h_B = 0 + 3$ and $h_C = \frac{1}{3} + 1$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{2}{3}\Big[\frac{1}{z^2} + \frac{1}{(z-1)^2}\Big] - \frac{10}{3}\frac{1}{z(z-1)}\right\}G(z) = 0 \tag{6.37}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^\beta K(z) \tag{6.38}$$

We find two possibilities,

$$\beta = 1 \quad or \quad -\frac{2}{3} \tag{6.39}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

β	1	$-\frac{2}{3}$
$Block1$	$F\left(\frac{13}{3}, 0; \frac{8}{3}; z\right)$	$F\left(1, -\frac{10}{3}; -\frac{2}{3}; z\right)$
$Block2$	$z^{-\frac{5}{3}}F\left(\frac{8}{3}, -\frac{5}{3}; -\frac{2}{3}; z\right)$	$z^{\frac{5}{3}}F\left(\frac{8}{3}, -\frac{5}{3}; \frac{8}{3}; z\right)$

(6.40)

$\boxed{CASE-3: m=2; n=2}$

Set m=2. Then $h_A = \frac{3}{2}$, $h_B = 0 + 2$ and $h_C = \frac{1}{3} + 2$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{14}{3}\frac{1}{z(z-1)}\right\}G(z) = 0 \tag{6.41}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^\beta K(z) \tag{6.42}$$

We find two possibilities,

$$\beta = 0 \quad or \quad \frac{1}{3} \tag{6.43}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

β	0	$\frac{1}{3}$
$Block1$	$F\left(-2, \frac{7}{3}; \frac{2}{3}; z\right)$	$F\left(3, -\frac{4}{3}; \frac{4}{3}; z\right)$
$Block2$	$z^{\frac{1}{3}}F\left(-\frac{5}{3}, \frac{8}{3}; \frac{4}{3}; z\right)$	$z^{-\frac{1}{3}}F\left(\frac{8}{3}, -\frac{5}{3}; \frac{2}{3}; z\right)$

(6.44)

$\boxed{CASE-4: m=0; n=4}$

Set m=0. Then $h_A = \frac{3}{2}$, $h_B = 0$ and $h_C = \frac{1}{3} + 4$.

Making the above substitutions in equation (5.76) and simplifying the equation, we get,

$$\left\{\partial_z^2 + \frac{2}{3}\Big[\frac{2z-1}{z(z-1)}\Big]\partial_z - \frac{14}{3}\Big[\frac{1}{z^2} + \frac{1}{(z-1)^2}\Big] + \frac{14}{3}\frac{1}{z(z-1)}\right\}G(z) = 0 \tag{6.45}$$

Trying to convert this equation to standard *hypergeometric dfferential equation* form by the change of function,

$$G(z) = [z(1-z)]^{\beta}K(z) \tag{6.46}$$

We find two possibilities,

$$\beta = -2 \quad or \quad \frac{7}{3} \tag{6.47}$$

After the substitution, we get hypergeometric differential equation and the following solutions:

β	-2	$\frac{7}{3}$
$Block1$	$F\left(-6, -\frac{5}{3}; -\frac{10}{3}; z\right)$	$F\left(7, \frac{8}{3}; \frac{16}{3}; z\right)$
$Block2$	$z^{\frac{13}{3}}F\left(-\frac{5}{3}, \frac{8}{3}; \frac{16}{3}; z\right)$	$z^{-\frac{13}{3}}F\left(\frac{8}{3}, -\frac{5}{3}; -\frac{10}{3}; z\right)$

(6.48)

6.5 Summary and Conclusion

With the calculation of possible candidate-conformal blocks of the four-point function in different descendant-levels scenarios, for both the cases where all the four fields are of conformal dimension $h = \frac{31}{16}$ or $h = \frac{3}{2}$ in the Baby Monster theory, we conclude this work.

We have not singled out the exact conformal-blocks that constitute the full four-point function. But we suspect, the fact that the Baby-Monster theory is 47 copies of the Ising model, might assist in selecting the correct conformal blocks.

Summarizing the work, we have discussed and brought together three different methods [[5],[4],[1]] to calculate four-point functions. Using combinations of these three methods, we have reproduced the four-point functions of Ising theory. Lastly, with the established duality between Ising model and Baby-Monster theory and using the similarity in their fusion rules through Verlinde's formula, we have calculated the possible candidate-conformal blocks for both the cases where all the four fields are of conformal dimension $h = \frac{31}{16}$ or $h = \frac{3}{2}$ in the Baby Monster theory.

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