Constraining the space of low energy EFTs

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Certificate

This is to certify that this dissertation entitled Constraining the space of low energy EFTs towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rajat Sharma at Indian Institute of Science Education and Research under the supervision of Diptimoy Ghosh , Assistant Professor, Department of Physics , during the academic year 2022-2023.

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This thesis is dedicated to my mentors, family and friends

Declaration

I hereby declare that the matter embodied in the report entitled Constraining the space of low energy EFTs are the results of the work carried out by me at the Department of Physics, Indian Institute of Science Education and Research, Pune, under the supervision of Diptimoy Ghosh and the same has not been submitted elsewhere for any other degree.

Rajat Sharma

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Abstract

We derive the causality and unitarity constraints on dimension 6 and dimension 8 Gluon field strength operators in the Standard Model Effective Field Theory (SMEFT). We use the 'amplitude analysis' i.e. dispersion relation for $2 \to 2$ scattering in the forward limit, to put bounds on the Wilson coefficients. We show that the dimension 6 operators can exist only in the presence of certain dimension 8 operators. It is interesting that the square of the dimension 6 Wilson coefficients can be constrained in this case even at the tree level. We also successfully rederive all these bounds using the classical causality argument that demands that the speed of fluctuations about any non-trivial background should not exceed the speed of light. We also point out some subtleties in the superluminality analysis regarding whether the low-frequency phase velocity can always be used as the relevant quantity for Causality violation. We also explore Bell inequality violation for $2 \to 2$ scattering in Effective Field Theories (EFTs) of photons, gluons, and gravitons. Using the CGLMP Bell parameter (I_2), we show that, starting from an appropriate initial non-product state, the Bell inequality can always be violated in the final state (i.e.,$I_2 > 2$) at least for some scattering angle. For an initial product state, we demonstrate that abelian gauge theories behave qualitatively differently than non-abelian gauge theories (or Gravity) from the point of view of Bell violation in the final state: in the non-abelian case, Bell violation ($I_2 > 2$) is never possible within the validity of EFTs for weakly coupled UV completions. Interestingly, we also find that, for a maximally entangled initial state, scattering can reduce the degree of entanglement only for CP-violating theories. Thus Bell violation in $2 \to 2$ scattering can, in principle, be used to classify CP conserving vs violating theories.

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Introduction

It is well known that dynamics at very high energy scales or short distances are irrelevant to describe low energy or long distance phsyics, i.e. very different energy scales are 'decoupled' from each other. For example, we don't need to know the fine details of nuclei to understand the properties of electronic energy levels in atoms. It's mainly on this idea that the framework of Effective Field Theory (EFT) is built (see [1, 2] for a review), in which we are agnostic of the UV physics inaccessible to us and construct the Lagrangian for the low energy (IR) theory in terms of some physical cut-off energy scale (Λ). The Lagrangian for the EFT, a priori, must contain all the possible operators consistent with the symmetries of the theory e.g., Lorentz invariance and gauge invariance, and the coefficients of these operators can have arbitrary values. However, it has been shown in recent years that sacred principles like relativistic Causality and Unitarity do impose non-trivial constraints on these coefficients and carve out the allowed parameter space [3, 4]. This is interesting also phenomenologically since it leads to enhanced statistical power for experiments sensitive to these operators because one can incorporate IR consistency bounds into the prior probability distribution.

One of the first attempts in this direction due to [5, 3] (which we refer to as the 'amplitude analysis') exploits well-established fundamental principles like micro-causality (leading to analyticity [6, 7]) and unitarity of the S-matrix to constrain the EFT parameter space. This involves using dispersion relations for $2 \to 2$ scattering amplitudes in the forward limit. The 'amplitude analysis' has successfully given linear positivity bounds on dimension 8 operators in a variety of theories but hasn't had much success with dimension 6 operators[1] containing 4 fields; although, one can derive certain sum rules [9, 10, 11]. The reason for this lack of success is the fact that for such an operator, the scattering am-

[1] See recent developments made to constrain dim 6 operators using S-matrix Bootstrap methods [8].

plitude grows as the Mandelstam variable s at the tree level. However, in this thesis, we show that *it's possible to constrain the square of the coefficient of dimension 6 operators containing **3** gluon fields w.r.t those of dim 8 operators.* Such an operator appears in the SMEFT. Similar positivity bounds for the electroweak gauge bosons were obtained in [12]. A lot of effort has been focused on constraining the parameter space of SMEFT using various methods (see, for example, [12, 13, 14, 15, 16] and the references therein), since it is and will be the main aim of present and future particle physics experiments to measure these coefficients. Also, verifying whether the experimentally measured coefficients satisfy our theoretical constraints allows us to test fundamental properties of the UV theory such as locality and Lorentz invariance up to very high energies through experimental signatures at accessible scales [16].

The other method often employed to put constraints on the Wilson coefficients is based on the classical causality argument [3]. One demands that the propagation of perturbations over any non-trivial background should respect causality i.e. the speed of signal propagation should not be superluminal [3, 4, 17, 18]. It is well known that if the wavefront velocity i.e. infinite frequency limit of the phase velocity is (sub)luminal then causality is preserved. Naively, one might conclude that it is not possible to put constraints on the EFT coefficients since the EFT is valid, by definition, in the low-frequency regime i.e. $\omega/\Lambda \ll 1$. However, one can instead consider signal velocity (for a precise definition, see [19]) which, for non-dispersive mediums (the case of our interest), is equal to the group/phase velocity. Thus, the low frequency group/phase velocity can be directly associated with causality obviating the need to take the high-frequency limit. One can also use analyticity in the form of the Kramers-Kronig relation [20], which, for dissipative backgrounds, demands that the phase velocity cannot decrease with increasing frequency [20, 4]. Therefore, the superluminal phase velocity in the EFT can be associated with causality violation. However, in general, it is not very clear how one can determine the dispersive properties of the background medium, which is essential in the usefulness of the Kramers-Kronig relation. We will discuss this in some more detail in section 2.2.

It is not always necessary that *small* superluminal low energy speed violates causality as the observations detecting causality violation may turn out to be unmeasurable within the valid regime of EFT [18, 21]. Therefore for generic EFTs, particularly gravitational ones, scattering phase shift or time delay is perhaps a better probe to detect causality violations [22, 23]. However, for homogeneous backgrounds (as considered in this paper) signals can be allowed to propagate over large distances, and in that case, even the small superluminality can be detected within the EFT regime. Therefore, using this method one can try to rederive or even hope to improve the bounds on EFT coefficients obtained by the 'amplitude analysis'.

We would like to stress that, a priori, it is unclear if the two methods always provide the same constraints (or equivalently, whether using any one of them is enough to maximally constrain the space of EFTs), as naively, they don't seem to be related at all. One is purely based on the classical causality of wave propagation and another on scattering amplitudes which relies on Unitarity, and Froissart bound in addition to micro-causality (the two methods could be somewhat related since the classical causality analysis secretly might also depend on analyticity in the form of Kramers-Kronig relation [4] and unitarity (for dissipative mediums) however, the connection is unclear, as we discuss in section 2.2).

For the classical causality/superluminality analysis, we also point out some subtleties that arise when mass-like terms exist in the dispersion relation. We show, in the particular case of the chiral Lagrangian, that this may lead to deviation from strict positivity. However, in the case of gluonic operators, we demonstrate the mass-like terms can be removed by choosing particular configurations of non-trivial background and polarization of perturbation. This helps us derive constraints on the Wilson coefficients of gluonic operators by demanding subluminal phase velocity as our measure for causality. We show that *the superluminality analysis for dim 6 and 8 gluonic operators, in a nontrivial way, reproduces all bounds that we obtain from the 'amplitude analysis'*. This is the novel and main result of our work. Finally, we mention a non-relativistic example following [24], where superluminality gives stronger bounds than the amplitude analysis. From the examples given in our work, one can gather that one should use both analyses whenever possible in order to get the maximum amount of information on an IR effective theory[2].

Another quantum phenomenon that has recently caught the attention in the context of EFTs is entanglement. Entanglement is a unique relationship between two or more particles, where their states are correlated in such a way that a measurement performed on one particle instantly influences the other particle, regardless of the spatial separation between them. This correlation poses a challenge to the principle of local realism, which asserts that the properties of a state are determined by its local environment. In 1964 [26], John

[2]For fermions, it is unclear how one can implement the superluminality analysis. However, the 'amplitude analysis' can still be carried out [25].

Bell derived a set of inequalities - commonly known as Bell inequalities - for the correlated expectation values that must be satisfied by a local deterministic theory. However, quantum mechanics predicts that the Bell inequalities can be violated for certain correlated expectation values, which has also been experimentally verified [27, 28, 29, 30]. These experiments have played a critical role in establishing quantum mechanics as a fundamental theory of nature.

Entanglement can have significant implications for our understanding of spacetime and information in quantum field theory (QFT) [31, 32]. However, very little is known and explored about the Bell inequalities in the context of QFT [33]. Recently, the interest has been revived after it has been shown that the Bell inequalities can be violated experimentally by the entangled top-quark pairs produced at the LHC [34, 35, 36, 37]. More work along this line has shown that it is possible to experimentally measure Bell violation for hyperons [38] and gauge bosons from Higgs boson decay [39, 40] as well.

These observations have motivated the study of entanglement in $2 \to 2$ scattering in high energy physics [41, 42, 43], particularly in the context of Effective Field Theory (EFT)[44, 45, 46, 47]. The existence of higher dimensional operators in an EFT can modify the degree of entanglement in the final scattered states, which might act as a possible probe of new physics. If we experimentally observe Bell violation in $2 \to 2$ scattering for a particular initial state, then we can directly constrain the corresponding EFT by quantifying the degree of entanglement in the final scattered state. However, a priori, it is not very clear which initial state can be used to probe the quantum nature of the theory and demand Bell violation.

In this work, we consider the CGLMP Bell parameter (I_2) [48] as the measure of entanglement in the states (to our knowledge, this was used in the context of EFT first in [46]). For local hidden variable theories, $|I_2| \leq 2$, however, this inequality can be violated by quantum theories as shown in section 3.1. Therefore, the CGLMP Bell parameter can be used to distinguish between the local hidden variable theories and the quantum theories. We consider the initial states for $2 \to 2$ scattering such that the CGLMP parameter corresponding to them (I_{2i}) satisfies $|I_{2i}| \leq 2$. In other words, the CGLMP parameter for the initial state can, in principle, be explained by a local hidden variable theory, and it does not call for a quantum mechanical origin. We use this condition with the motivation that we want to probe the quantum nature of a theory only through the scattering process, i.e., whether unitary evolution can increase the degree of entanglement beyond $I_2 = 2$. We then calculate the allowed EFT parameter space for which we can observe Bell violation at some energy (within the validity of the EFT regime) and scattering angle.

In this thesis, we first give a brief overview of the concepts and tools used in the work. We then present the main results obtained during the thesis which are mainly reproduced from [49] and [50]. We finally conclude with summary of our results and future outlook.

Chapter 1

Preliminaries

While numerous concepts from Quantum Field Theory (QFT) are relevant to this thesis, it is impractical to summarize all of them in a single chapter. Nevertheless, this chapter provides a review of some important concepts. For a more comprehensive understanding, the primary sources are [51, 52]:

1.1 S-matrix and Scattering Amplitude

In quantum field theory, particles are treated as excitations of a field that pervades all of space-time. When particles interact, they exchange energy and momentum, which causes their behavior to change. The S-matrix describes this behavior change by relating the system's initial and final states. The S-matrix is defined as

$$\langle f| S |i\rangle_{\text{Heisenberg}} = \langle f;\infty|i;-\infty\rangle_{\text{Schrödinger}} \tag{1.1}$$

In the Schrödinger picture representation, the states evolve in time, however, in the Heisenberg picture, we put all the evolution in an operator, leaving the states alone. When the initial and final states are momentum eigenstates that we evolve from $t = -\infty$ to $t = \infty$, the time evolution operator is given by the S-matrix. The S-matrix is defined assuming that all the interactions that change the state happen in a finite time interval, i.e., the states are free of interaction at asymptotic times, $t = \pm\infty$.

We can explicitly get the form of S-matrix by expanding the initial and final momentum eigenstates in terms of creation ($a_p^\dagger(t)$) and annihilation ($a_p(t)$) operators which create and annihilate particles, respectively, with momentum p at time t. Since the fields can also be written as a sum over creation and annihilation operators, we can express the S-matrix in the form of fields. Under the assumption that all interactions happen in a finite interval, we can write the S-matrix as

$$\begin{aligned}\langle p_3 \cdots p_n|S|p_1 p_2\rangle = \left[i \int d^4x_1 e^{-ip_1x_1} \left(\Box_1 + m^2\right)\right] \cdots \left[i \int d^4x_n e^{ip_nx_n} \left(\Box_n + m^2\right)\right] \\ \times \left\langle \Omega \left| T \left\{\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\cdots\phi\left(x_n\right)\right\}\right| \Omega\right\rangle ,\end{aligned} \tag{1.2}$$

This is the LSZ reduction formula where $|\Omega\rangle$ is the ground state and time-ordering operation $T\{...\}$ indicates that all field operators should be ordered such that those at later times are always on the left of those at earlier times. Now we need to calculate the time-ordered correlation function of fields to get the elements of the S-matrix. By going to the interaction picture, which is an intermediate representation of Schrödinger and Heisenberg picture, the time-ordered correlation function of fields can be written in terms of time-ordered correlation of free fields, free vacuum, and interaction term in the Lagrangian.

$$\left\langle \Omega \left| T \left\{\phi\left(x_1\right)\cdots\phi\left(x_n\right)\right\}\right| \Omega\right\rangle = \frac{\left\langle 0 \left| T \left\{\phi_0\left(x_1\right)\cdots\phi_0\left(x_n\right) e^{i\int d^4x \mathcal{L}_{\text{in}}\left[\phi_0\right]}\right\}\right| 0\right\rangle}{\left\langle 0 \left| T \left\{e^{i\int d^4x \mathcal{L}_{\text{in}}\left[\phi_0\right]}\right\}\right| 0\right\rangle}. \tag{1.3}$$

The expansion of the r.h.s of the above equation gives n-point time-ordered correlation function of free fields. *Wick's theorem* tells us that it is given by a sum over all possible ways in which all the fields in the correlation function can be contracted with each other. For example, $\langle 0|T\left\{\phi_0\left(x_1\right)\phi_0\left(x_2\right)\right\}|0\rangle$ is given by the $D_F(x_1, x_2) = D_{12}$ which is the propagator. These contractions can be represented by Feynman diagrams, where vertices represent points where the correlation function is evaluated, and the lines correspond to the propagators. The denominator in eq$^{\text{n}}$(1.3) cancels all the bubble diagrams, connected subgraphs that do not involve any external point, in the numerator. Moreover, the $\Box + m^2$ in the LSZ reduction formula cancels the propagator connecting the external vertex to an internal vertex which allows us to take external lines to be on-shell one-particle states.

When there are no interactions, the S-matrix is simply given by the identity matrix $\mathbb{1}$, therefore it can be written as

$$S = \mathbb{1} + i\mathcal{T}$$

where $\mathcal{T}$, the transfer matrix, describes the deviation from free theory due to the interactions. Since the momentum should be conserved in any interaction, we can write $\mathcal{T} = (2\pi)^4\delta^4(\sum p_i)\mathcal{M}$. We usually refer to $\mathcal{M}$ when we talk about scattering amplitudes which can be calculated from Feynman diagrams using the Feynman rules.

1.2 Unitarity

Unitarity has a lot of significant implications in Quantum Field Theory. Unitarity in a very simple language means that the probabilities should add up to 1. The condition of unitarity also forces the physical states in the Hilbert space to transform in unitary irreducible representations of the Poincaré group. In Schrödinger picture, the unitarity demands that the probability should be conserved,

$$\langle\Psi; t|\Psi; t\rangle = \langle\Psi; 0|\Psi; 0\rangle \tag{1.4}$$

Since $|\Psi; t\rangle = S|\Psi; 0\rangle$ where S is the time evolution operator, e^{-iHt}, we get S-matrix is a unitary matrix and H is a Hermitian matrix. The unitarity of S-matrix, $S^\dagger S = \mathbb{1}$ has a very important implication known as *Optical theorem*, which relates the scattering amplitude to the cross-section. We know the scattering matrix can be written as,

$$S = \mathbb{1} + i\mathcal{T} \tag{1.5}$$

then unitarity implies that

$$i(\mathcal{T}^\dagger - \mathcal{T}) = \mathcal{T}^\dagger\mathcal{T} \tag{1.6}$$

Also, $\langle f|\mathcal{T}|i\rangle = (2\pi)^4\delta^4(\sum p_i)\mathcal{M}(i \to f)$ then using the completeness relation of the Hilbert space $\mathbb{1} = \sum_X \int d\Pi_X \, |X\rangle \, \langle X|$ we get the generalized optical theorem,

$$\mathcal{M}(i \to f) - \mathcal{M}^*(f \to i) = i\sum_X \int d\Pi_X (2\pi)^4\delta^4(p_i - p_X)\mathcal{M}(i \to X)\mathcal{M}^*(f \to X) \tag{1.7}$$

In perturbation theory, the above statement must hold at each order. However, the l.h.s of the above equation have linear terms of amplitude, whereas the r.h.s of the above equation have amplitude squared terms. This means that the optical theorem relates the amplitude at a given order to that at lower order. Therefore, it can relate the loop contributions to the tree-level contributions.

Now if the final state is the same as the initial state i.e., $|i\rangle = |f\rangle = |A\rangle$, then the above relation reduces to,

$$\mathrm{Im}\mathcal{M}(A \to A) = 2E_{\mathrm{CM}}|\vec{p}_i| \sum_X \sigma(A \to X) \tag{1.8}$$

where we have used the formula for cross-section in the center-of-mass frame. This special case is known as the optical theorem, which states that the imaginary part of the forward scattering amplitude is proportional to the total scattering cross-section.

Chapter 2

Causality and Unitarity Constraints on dimension 6 & 8 Gluonic operators in the SMEFT

This chapter is largely produced from [49], which was one of the original works done during the course of this thesis.
This chapter is mainly organized as follows: In section ??, we derive positivity constraints on dim 6 and dim 8 gluonic operators in SMEFT using the 'amplitude analysis' with an overview of the method first. In section 2.2, we first discuss a few subtleties in the superluminality analysis, followed by a demonstration that all the bounds can also be reproduced by the superluminality analysis.

2.1 Positivity constraints from unitarity and analyticity

In this section, we derive constraints on dimension 6 and dimension 8 Gluonic operators of the SMEFT using dispersion relations for $2 \to 2$ scattering amplitude. Let us first review how the 'amplitude analysis' works to put constraints on the Wilson coefficients of a general EFT.

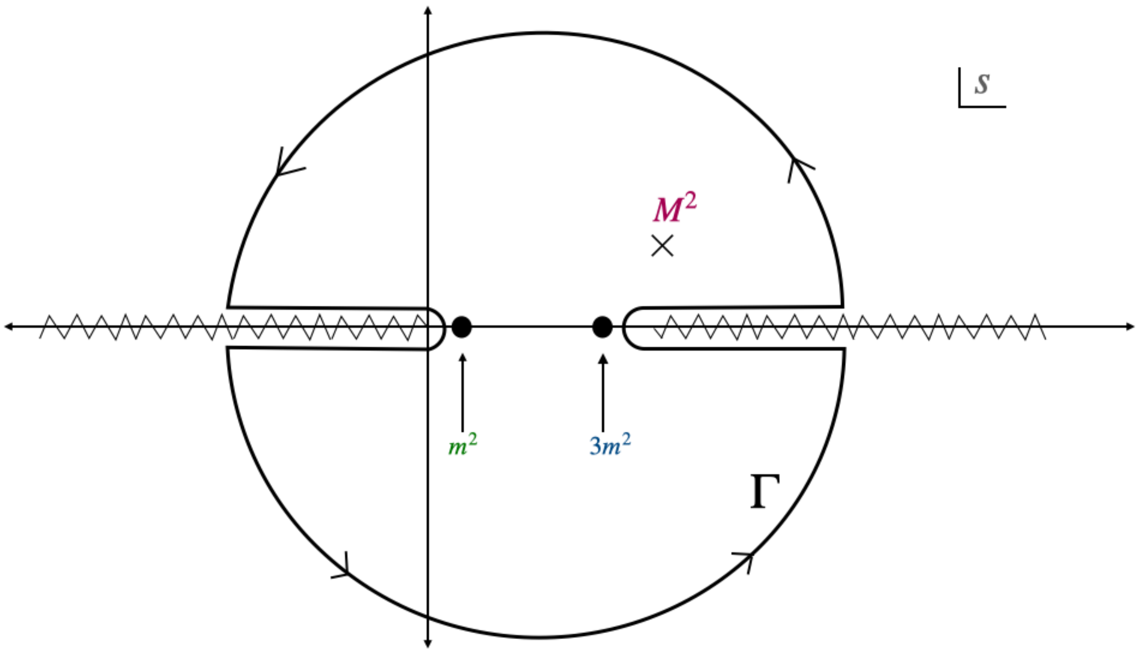

Figure 2.1: Analytic structure of $\mathcal{A}(s)$ in complex s plane. The contour is symmetric about $s = 2m^2$ and go over branch cuts from $-\infty$ to 0 and $4m^2$ to ∞.

2.1.1 Overview

In this section, we give a brief overview, following the discussion in the seminal paper [3], of how the dispersion relations along with well-established principles like micro-causality (leading to analyticity of S-matrix), unitarity, and locality, can help in constraining the low-energy EFT parameters which from the IR side can have arbitrary values, to begin with. We slightly modify the discussion in accordance with our work but the main underlying concepts are the same.

Let us consider $s \leftrightarrow u$ symmetric $2 \to 2$ scattering amplitude, $\mathcal{M}(s,t)$ for a process in which exchanged particles or particles inside a loop are of the same mass m (or massless as would be the case in our work), in the forward limit i.e. initial and final states are exactly same. We define $\mathcal{A}(s) = \mathcal{M}(s,t)|_{t\to 0}$ (forward limit) and take integral around contour as shown in fig. 2.1

$$\mathcal{I} = \oint_C \frac{ds}{2\pi i} \frac{\mathcal{A}(s)}{(s-2m^2)^3} \tag{2.1}$$

where m is the mass of the exchanged particles (or regularized mass for massless exchanged particles), and Λ is the energy cut-off scale of EFT in consideration. It is possible that the theory considered allows for the loops leading to branch cuts starting from $-\infty$ to 0 and

$4m^2$ to $+\infty$. That is why we probe the $s \to 2m^2$ limit instead of $s \to 0$ (which would be okay in case the branch cut doesn't go through or extend up to 0).

Now we evaluate the integral (2.1); since $\lim_{|s|\to\infty} \mathcal{A}(s)/|s|^2 \to 0$ at infinity due to the Froissart bound (more precisely, $\mathcal{A}(s) < s\ ln^2 s$) [53, 54], the integral over the arc at infinity vanishes and we are left just with the integral of discontinuity of $\mathcal{A}(s)$ across the branch cuts,

$$\mathcal{I} = \frac{1}{2\pi i}\int_{cuts} ds \frac{Disc\mathcal{A}(s)}{(s-2m^2)^3}$$

But the integral can also be evaluated in terms of the residues at the poles: at $s = 2m^2$ and $s = m^2,\ 3m^2$ (due to s-channel and u-channel in exchange diagrams). Thus, we get

$$\frac{1}{2}\mathcal{A}''(s=2m^2) + \sum_{s^*=m^2,3m^2} \frac{res\mathcal{A}(s=s^*)}{(s^*-2m^2)^3} = \frac{2}{2\pi i}\int_{s>4m^2} ds \frac{Disc\mathcal{A}(s)}{(s-2m^2)^3} \tag{2.2}$$

In the above equation, we have mapped the integral over the negative branch cut to the positive one using $s \leftrightarrow u$ crossing symmetry. Now, $Disc\mathcal{A}(s) = 2iIm\mathcal{A}(s)$ and from optical theorem (for which initial and final states are required to be identical), we have $Im\mathcal{A}(s) = \sqrt{s(s-4m^2)}\sigma(s)$ where $\sigma(s)$ is the total cross-section of the scattering,

$$\frac{1}{2}\mathcal{A}''(s=2m^2) + \sum_{s^*=m^2,3m^2} \frac{res(\mathcal{A}(s=s^*))}{(s^*-2m^2)^3} = \frac{2}{\pi}\int_{s>4m^2} ds \frac{\sqrt{s(s-4m^2)}\sigma(s)}{(s-2m^2)^3} \tag{2.3}$$

For further analysis, we'll take the $s \leftrightarrow u$ symmetric $\mathcal{A}(s)$ to be of a particular form, which we'll be encountering in further sections.

We concern ourselves with operators only up to dimension 8 in the low-energy EFT and also assume that it contains only 6 and 8 dimension operators in addition to 4-dimensional terms. If the theory allows taking $t \to 0$ i.e. forward limit without causing any divergence problem (which will be the case here) then we can write the $s \leftrightarrow u$ symmetric forward scattering amplitude at tree level as

$$\begin{aligned}\mathcal{A}(s) = \lambda + b\frac{m^2}{\Lambda^2} &+ \frac{1}{\Lambda^4}(c_1 s^2 + c_1 u^2 + c_3 m^4) \\ &+ \frac{1}{\Lambda^4}\left(c_4\frac{s^3}{s-m^2} + c_5\frac{s^2m^2}{s-m^2} + c_6\frac{sm^4}{s-m^2} + c_7\frac{m^6}{s-m^2}\right) \\ &+ \frac{1}{\Lambda^4}\left(c_4\frac{u^3}{u-m^2} + c_5\frac{u^2m^2}{u-m^2} + c_6\frac{um^4}{u-m^2} + c_7\frac{m^6}{u-m^2}\right)\end{aligned} \tag{2.4}$$

Note: If we take the exchange particles to be massless, then it might not be possible to take the forward limit, even with the regularized mass because we also need to put $m \to 0$ at some stage, which is one of the main problems in performing this analysis for EFTs of gravity. Then for the massless case, either the t-channel shouldn't exist or the numerator in the t-channel exchange contribution should converge to 0 faster than t in $t \to 0$ limit avoiding the t-channel pole problem which would be the case for our EFT in consideration.

Putting (2.4) in (2.3) we get,

$$\frac{2}{\Lambda^4}(c_1 + c_4) = \frac{2}{\pi}\int_{s>4m^2} ds \frac{\sqrt{s(s-4m^2)}\sigma(s)}{(s-2m^2)^3} \tag{2.5}$$

The integrand in the r.h.s of the above equation is positive as the cross-section $\sigma(s) > 0$, which then makes the r.h.s manifestly positive. Therefore, the above equation shows that the coefficient of the term which goes as s^2 upon taking $m \to 0$ limit of $\mathcal{A}(s)$ (l.h.s of the above equation) is positive. The same result can be obtained by using a different contour as shown in appendix A.3.

2.1.2 Relative bounds on dim 6 and dim 8 operators

It is clear from the previous section that we cannot put any constraint on the contribution that grows slower than s^2. This is because for the integral to vanish over the arc at infinity, we need minimum $n = 3$ in $\oint_c \frac{\mathcal{A}(s)}{(s-2m^2)^n}$ but then there is no contribution to the residue from terms growing less than s^2, preventing us from constraining them. If we take $n = 2$ and even ignore the contribution from arcs at infinity, then we get the contribution from dim 6 operators towards the residue in eq$^{\mathrm{n}}$(2.5). However, for $n = 2$ the integrand on r.h.s of (2.5) takes form $\sigma(s)/s$ which makes the integral have a non-definitive sign. Thus, the l.h.s also have a non-definitive sign preventing us from constraining the Wilson coefficients using positivity arguments.

This is usually the case for dimension 6 operators containing four fields as their contribution to the tree level amplitude grows as s. However, if the dim 6 operator also gives rise to terms containing only three fields then one indeed gets an s^2 piece through exchange diagrams. The constraint in this case, however, is on the square of the Wilson coefficient (and not on the sign). Also, dimension 8 operators containing four fields give contributions at the same order (s^2/Λ^4) via contact diagrams. Thus, we produce relative bounds on dim 6 and dim 8 operators. Similar bounds for the electroweak gauge bosons were obtained in [12].

Let us now apply the above analysis for dimensions 6 and 8 Gluonic operators in the SMEFT. The independent Gluonic operators for SU(3) as mentioned in [55] and [56, 57] are given in table 2.1. The Lagrangian at dimensions 6 and 8 takes the following form:

X^3		X^4	
$Q_{G^3}^{(1)}$	$f^{abc} G_\mu^{a\nu} G_\nu^{b\rho} G_\rho^{c\mu}$	$Q_{G^4}^{(1)}$	$\left(G^a_{\mu\nu} G^{a\mu\nu}\right)\left(G^b_{\rho\sigma} G^{b\rho\sigma}\right)$
$Q_{G^3}^{(2)}$	$f^{abc} \widetilde{G}_\mu^{a\nu} G_\nu^{b\rho} G_\rho^{c\mu}$	$Q_{G^4}^{(2)}$	$\left(G^a_{\mu\nu} \widetilde{G}^{a\mu\nu}\right)\left(G^b_{\rho\sigma} \widetilde{G}^{b\rho\sigma}\right)$
		$Q_{G^4}^{(3)}$	$\left(G^a_{\mu\nu} G^{b\mu\nu}\right)\left(G^a_{\rho\sigma} G^{b\rho\sigma}\right)$
		$Q_{G^4}^{(4)}$	$\left(G^a_{\mu\nu} \widetilde{G}^{b\mu\nu}\right)\left(G^a_{\rho\sigma} \widetilde{G}^{b\rho\sigma}\right)$
		$Q_{G^4}^{(5)}$	$\left(G^a_{\mu\nu} G^{a\mu\nu}\right)\left(G^b_{\rho\sigma} \widetilde{G}^{b\rho\sigma}\right)$
		$Q_{G^4}^{(6)}$	$\left(G^a_{\mu\nu} G^{b\mu\nu}\right)\left(G^a_{\rho\sigma} \widetilde{G}^{b\rho\sigma}\right)$
		$Q_{G^4}^{(7)}$	$d^{abc} d^{dec}\left(G^a_{\mu\nu} G^{b\mu\nu}\right)\left(G^d_{\rho\sigma} G^{e\rho\sigma}\right)$
		$Q_{G^4}^{(8)}$	$d^{abc} d^{dec}\left(G^a_{\mu\nu} \widetilde{G}^{b\mu\nu}\right)\left(G^d_{\rho\sigma} \widetilde{G}^{e\rho\sigma}\right)$
		$Q_{G^4}^{(9)}$	$d^{abc} d^{dec}\left(G^a_{\mu\nu} G^{b\mu\nu}\right)\left(G^d_{\rho\sigma} \widetilde{G}^{e\rho\sigma}\right)$

Table 2.1: Dimension 6 and 8 gluonic operators in the SMEFT

$$L^{(6)} = \frac{c_6}{\Lambda^2} Q_{G^3}^{(1)} + \frac{c_6'}{\Lambda^2} Q_{G^3}^{(2)}\ ; \quad L^{(8)} = \frac{c_8^{(i)}}{\Lambda^4} Q_{G^4}^{(i)}$$

where c_6, c_6' and $c_8^{(i)}$ are dimensionless Wilson coefficients and Λ is the UV cut-off for the EFT. The above-mentioned dimension 6 operators have parts that contain three fields and three derivatives, e.g., $f^{abc} F_\mu^{a\nu} F_\nu^{b\rho} F_\rho^{c\mu}$ in $Q_{G^3}^{(1)}$. Also, dimension 8 operators have parts containing four fields with four derivatives, e.g.,$\left(F^a_{\mu\nu} F^{b\mu\nu}\right)\left(F^a_{\rho\sigma} F^{b\rho\sigma}\right)$ in $Q_{G^4}^{(3)}$. Both kinds of terms give s^2/Λ^4 growth in the amplitude, the former via an exchange diagram and the latter via a contact diagram. Therefore as discussed above, using the 'amplitude analysis' we can constrain squares of c_6 and c_6' relative to $c_8^{(i)}$.
We calculate scattering amplitude for $gg \to gg$, written explicitly as $|p_1, \epsilon_1, a; p_2, \epsilon_2, b\rangle \to |p_3, \epsilon_3, d; p_4, \epsilon_4, e\rangle$ (where p's are the momenta, ϵ's are polarizations and Latin indices denote color of particles), at tree level getting contribution from Feynman

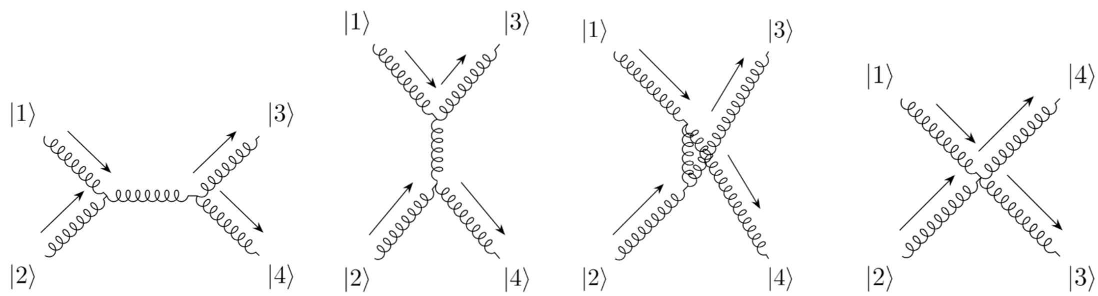

Figure 2.2: Exchange diagrams (first three) get contribution just from dim 6 operators ; contact diagram gets contribution from both dim 6 and dim 8 but the dim 6 contribution is of order s/Λ^2 as explained in the text.

diagrams in fig. 2.2. To relate the imaginary part of amplitude to the cross-section (optical theorem), one needs to take identical initial and final states; therefore, we consider $a = d$ and $b = e$ for our calculations.

Below, we just present those terms that give s^2 contribution to $\mathcal{A}(s)$[1] , for reasons mentioned above.

$$\begin{aligned}
\mathcal{A}(s) &= \mathcal{M}(s,t)|_{t\to 0} \qquad (2.6)\\
&= \Bigg[\Big\{9f^{abc}f^{ab}_{\ \ c}\left(c_6'^2 - c_6^2\right) + 8\delta^{ab}\left(c_8^{(1)} - c_8^{(2)}\right) + 4(1+\delta^{ab})\left(c_8^{(3)} - c_8^{(4)}\right)\\
&\quad + 8d^{abc}d^{ab}_{\ \ c}\left(c_8^{(7)} - c_8^{(8)}\right)\Big\} \times \Big\{|s\epsilon_1\cdot\epsilon_2^* - 2\,\epsilon_1\cdot p_2\epsilon_2^*\cdot p_1|^2 + |s\epsilon_1\cdot\epsilon_2 - 2\,\epsilon_1\cdot p_2\epsilon_2\cdot p_1|^2\Big\}\\
&\quad - 2\Big\{9f^{abc}f^{ab}_{\ \ c}\,c_6'^2 - 8\delta^{ab}c_8^{(2)} - 4(1+\delta^{ab})\,c_8^{(4)} - 8\,d^{abc}d^{ab}_{\ \ c}\,c_8^{(8)}\Big\} \times \Big\{s^2|\epsilon_1|^2|\epsilon_2|^2\Big\}\\
&\quad + 4\Big\{9f^{abc}f^{ab}_{\ \ c}c_6c_6' - 4\delta^{ab}c_8^{(5)} - 2(1+\delta^{ab})c_8^{(6)} - 4d^{abc}d^{ab}_{\ \ c}c_8^{(9)}\Big\}\\
&\qquad \times \varepsilon^{\mu\nu\rho\sigma} Re\Big\{(\epsilon_{1\mu}^*\epsilon_{2\nu}p_{1\rho}p_{2\sigma})(s\epsilon_1\cdot\epsilon_2^* - 2\,\epsilon_1\cdot p_2\epsilon_2^*\cdot p_1)\\
&\qquad\qquad + (\epsilon_{1\mu}^*\epsilon_{2\nu}^*p_{1\rho}p_{2\sigma})(s\epsilon_1\cdot\epsilon_2 - 2\,\epsilon_1\cdot p_2\epsilon_2\cdot p_1)\Big\}\Bigg]
\end{aligned}$$

It should be noted that a and b in the above expression are not contracted, they are colors of incoming and outgoing particles whereas c (color index) and Lorentz indices are contracted. For the t-channel, we get $f^{adc}f^{bec}$ factor in the numerator which is zero for our particular choice of colors, $a = d$ and $b = e$. Therefore, there is no contribution from the t-channel and we are allowed to take the forward limit without any divergence issues due

[1]We took help of the FeynCalc [58, 59, 60] package to verify our calculation.

to the t-channel pole. This isn't possible for massless theories like gravity where one gets s^2/t contribution from the t-channel.

For further analysis we work in the COM frame and we consider general complex polarizations transverse to the momentum,

$$
\begin{aligned}
&p_1 = \{E, 0, 0, E\}\ ,\ p_2 = \{E, 0, 0, -E\}\ \text{(COM frame)} \\
&\epsilon_1 = \epsilon_3 = \{0, \alpha, \beta, 0\}\ ,\ \epsilon_2 = \epsilon_4 = \{0, \gamma, \delta, 0\}
\end{aligned}
$$

Substituting momenta and polarizations in 2.6 we get,

$$
\begin{aligned}
\mathcal{A}(s) =& 4\frac{s^2}{\Lambda^4}\Big[(2\delta^{ab}c_8^{(1)} + (1+\delta^{ab})c_8^{(3)} + 2d^{abc}d^{ab}{}_c c_8^{(7)})(|\alpha\gamma+\beta\delta|^2 + |\alpha\gamma^* + \beta\delta^*|^2) \\
&+ (2\delta^{ab}c_8^{(2)} + (1+\delta^{ab})c_8^{(4)} + 2d^{abc}d^{ab}{}_c c_8^{(8)})(|\alpha^*\delta - \beta^*\gamma|^2 + |\alpha^*\delta^* - \beta^*\gamma^*|^2) \\
&- (2\delta^{ab}c_8^{(5)} + (1+\delta^{ab})c_8^{(6)} + 2d^{abc}d^{ab}{}_c c_8^{(9)}) \\
&\qquad\qquad \times Re\{(\alpha^*\delta - \beta^*\gamma)(\alpha\gamma^* + \beta\delta^*) + (\alpha^*\delta^* - \beta^*\gamma^*)(\alpha\gamma+\beta\delta)\}\Big] \\
&-9\frac{s^2}{\Lambda^4}f^{abc}f^{ab}{}_c\Big[|c_6(\alpha\gamma+\beta\delta) - c_6'(\alpha^*\delta^* - \beta^*\gamma^*)|^2 + |c_6(\alpha\gamma^*+\beta\delta^*) - c_6'(\alpha^*\delta - \beta^*\gamma)|^2\Big]
\end{aligned} \tag{2.7}
$$

where $Re(z)$ denotes the real part of z. From the above expression we can see that contribution from $L^{(6)}$ is always negative which means that if we don't consider dimension 8 gluonic operators in our EFT then from dispersion relation, we'll get

$$
-9\frac{s^2}{\Lambda^4}f^{abc}f^{ab}{}_c\Big[|c_6(\alpha\gamma+\beta\delta) - c_6'(\alpha^*\delta^* - \beta^*\gamma^*)|^2 + |c_6(\alpha\gamma^*+\beta\delta^*) - c_6'(\alpha^*\delta - \beta^*\gamma)|^2\Big] \geq 0
$$

which could be satisfied for arbitrary polarizations only when $c_6 = 0 = c_6'$. It means that dim 6 gluonic operators cannot exist without the presence of higher dimensional operators in the SMEFT. In appendix A.2, we present another example of a dim 6 operator, involving scalar and fermion, with a similar conclusion.
We get different expressions for $\mathcal{A}(s)$ (having s^2 dependence) for different polarizations and colors leading to multiple constraints on linear combinations of coefficients e.g., for $a = d = 1,\ b = e = 2$ and polarizations specified below we get the following expressions:

$\sqrt{2}\epsilon_1$	$\sqrt{2}\epsilon_2$	$\mathcal{A}(s)/(s^2/\Lambda^4)$
$\{0,1,i,0\}$	$\{0,1,-i,0\}$	$(4c_8^{(4)}-9c_6'^2)+(4c_8^{(3)}-9c_6^2)$
$\{0,1,1,0\}$	$\{0,1,-1,0\}$	$2(4c_8^{(4)}-9c_6'^2)$
$\{0,1,1,0\}$	$\{0,1,1,0\}$	$2(4c_8^{(3)}-9c_6^2)$
$\{0,\sqrt{2},0,0\}$	$\{0,1,1,0\}$	$4(c_8^{(4)}+c_8^{(3)}-c_8^{(6)})-9(c_6-c_6')^2$

We can only constrain the $s \leftrightarrow u$ symmetric $2 \to 2$ scattering amplitudes using the amplitude analysis. Therefore, we consider real polarizations or particular combinations of helicity amplitudes, such that the scattering amplitude is $s \leftrightarrow u$ symmetric, to constrain the Wilson coefficients. In table 2.2, we list all the different constraints we get on Wilson coefficients of dim 6 and dim 8 operators from considering different colors and polarizations. Our constraints when considered together are stronger than those presented in [16]. They are also consistent with the expressions obtained for Wilson coefficients in [61] from different UV theories. The inclusion of the dimension 6 operators and the resulting new constraints are the novel results of this section.

Colors	$\sqrt{2}\epsilon_1=\{0,1,1,0\}$	$\sqrt{2}\epsilon_1=\{0,1,1,0\}$	$\sqrt{2}\epsilon_1=\{0,1,-1,0\}$	$\sqrt{2}\epsilon_1=\{0,1,1,0\}$
	$\sqrt{2}\epsilon_2=\{0,1,1,0\}$	$\sqrt{2}\epsilon_2=\{0,1,-1,0\}$	$\sqrt{2(A+B)}\epsilon_2=$ $\left\{0,\sqrt{A}+\sqrt{B},\sqrt{A}-\sqrt{B},0\right\}$	$\sqrt{2(A+B)}\epsilon_2=$ $\left\{0,\sqrt{A}+\sqrt{B},\sqrt{B}-\sqrt{A},0\right\}$
$a=1,b=8$	$c_8^{(3)}+\frac{2}{3}c_8^{(7)}>0$	$c_8^{(4)}+\frac{2}{3}c_8^{(8)}>0$	$4\left(c_8^{(3)}+\frac{2}{3}c_8^{(7)}\right)\left(c_8^{(4)}+\frac{2}{3}c_8^{(8)}\right)>\left(c_8^{(6)}+\frac{2}{3}c_8^{(9)}\right)^2$	
$a=1,b=2$	$c_8^{(3)}>\frac{9}{4}c_6^2$	$c_8^{(4)}>\frac{9}{4}c_6'^2$	$4\left(c_8^{(3)}-\frac{9}{4}c_6^2\right)\left(c_8^{(4)}-\frac{9}{4}c_6'^2\right)>\left(c_8^{(6)}-\frac{9}{2}c_6c_6'\right)^2$	
$a=1,b=4$	$c_8^{(3)}+\frac{1}{2}c_8^{(7)}>\frac{9}{16}c_6^2$	$c_8^{(4)}+\frac{1}{2}c_8^{(8)}>\frac{9}{16}c_6'^2$	$4\left(c_8^{(3)}+\frac{1}{2}c_8^{(7)}-\frac{9}{16}c_6^2\right)\left(c_8^{(4)}+\frac{1}{2}c_8^{(8)}-\frac{9}{16}c_6'^2\right)>\left(c_8^{(6)}+\frac{1}{2}c_8^{(9)}-\frac{9}{8}c_6c_6'\right)^2$	
$a=4,b=8$	$c_8^{(3)}+\frac{3}{2}c_8^{(7)}>\frac{27}{16}c_6^2$	$c_8^{(4)}+\frac{3}{2}c_8^{(8)}>\frac{27}{16}c_6'^2$	$4\left(c_8^{(3)}+\frac{3}{2}c_8^{(7)}-\frac{27}{16}c_6^2\right)\left(c_8^{(4)}+\frac{3}{2}c_8^{(8)}-\frac{27}{16}c_6'^2\right)>\left(c_8^{(6)}+\frac{3}{2}c_8^{(9)}-\frac{27}{8}c_6c_6'\right)^2$	
$a=b=1$	$c_8^{(1)}+c_8^{(3)}+\frac{1}{3}c_8^{(7)}>0$	$c_8^{(2)}+c_8^{(4)}+\frac{1}{3}c_8^{(8)}>0$	$4\left(c_8^{(1)}+c_8^{(3)}+\frac{1}{3}c_8^{(7)}\right)\left(c_8^{(2)}+c_8^{(4)}+\frac{1}{3}c_8^{(8)}\right)>\left(c_8^{(5)}+c_8^{(6)}+\frac{1}{3}c_8^{(9)}\right)^2$	
$a=b=4$	$c_8^{(1)}+c_8^{(3)}+c_8^{(7)}>0$	$c_8^{(2)}+c_8^{(4)}+c_8^{(8)}>0$	$4\left(c_8^{(1)}+c_8^{(3)}+c_8^{(7)}\right)\left(c_8^{(2)}+c_8^{(4)}+c_8^{(8)}\right)>\left(c_8^{(5)}+c_8^{(6)}+c_8^{(9)}\right)^2$	

Table 2.2: The table contains the constraints on dim 6 and dim 8 operators' Wilson coefficients obtained using the amplitude analysis. ϵ_1 and ϵ_2 denote the polarizations of particles 1 and 2 respectively. The color of particle 1 is denoted by 'a' and that of particle 2 by 'b'. $A=4(2\delta^{ab}c_8^{(1)}+(1+\delta^{ab})c_8^{(3)}+2d^{abc}d^{ab}{}_c c_8^{(7)})-9f^{abc}f^{ab}{}_c c_6^2$; $B=4(2\delta^{ab}c_8^{(2)}+(1+\delta^{ab})c_8^{(4)}+2d^{abc}d^{ab}{}_c c_8^{(8)})-9f^{abc}f^{ab}{}_c c_6'^2$

For future convenience, we will refer to a particular constraint in table 2.2 using notation $C(i,j)$ where i refers to the row, and j to the column no. of the table; for example, $C(2,1)$ refers to the constraint $c_8^{(3)} > \frac{9}{4}c_6^2$.

The bounds in table 2.2 together significantly reduce the allowed parameter space of the Wilson coefficients. Interestingly, from constraint C(2,1) and C(2,2) we can directly see that for the existence of dim 6 operators we need some specific dim 8 operators to be present. For example, for $Q_{G^3}^{(1)}$ we need $Q_{G^4}^{(3)}$ to exist and for $Q_{G^3}^{(2)}$ we need $Q_{G^4}^{(4)}$. This aspect was appreciated in the case of the electroweak bosons in [12]. More bounds involving dim 8 operators can be obtained by relaxing the elastic forward scattering limit as shown in [62].

2.2 Superluminality

We saw in the previous section that assuming an EFT to have a UV-completion that is Lorentz invariant and unitary one gets positivity constraints on the Wilson coefficients using $2 \to 2$ forward scattering amplitudes. It is well known that one can also reproduce some of these constraints by referring only to IR physics. It turns out that arbitrary values or signs of these coefficients can lead to superluminal propagation of field fluctuations over non-trivial backgrounds [3]. Therefore, instead of working with an S-matrix, we can work with classical wave propagation to derive interesting bounds on the Wilson coefficients by demanding the EFT to be compatible with causality. A priori, it is not absolutely clear whether this analysis would give weaker or stronger bounds compared to the 'amplitude analysis'.

In this section, we first look at the classical causality/subluminality analysis more closely; we investigate how this analysis is modified for massive fields. We then proceed to apply this analysis to the Gluonic operators considered in sec. 2.1.2. Let us first briefly outline how superluminality analysis can be used to put bounds on the Wilson coefficients by demanding that signal velocity cannot be superluminal.

Consider the following Goldstone Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial\pi)^2 + \frac{c_3}{\Lambda^4}(\partial\pi)^4 \tag{2.8}$$

Naively, it may seem that the 8-dimensional operator will not contribute to the free propagation of perturbations as it is not a quadratic term. However, this is only true for trivial backgrounds; for non-trivial backgrounds with non-vanishing derivatives one can get terms quadratic in perturbations,

$$\mathcal{L} \supset \frac{c_3}{\Lambda^4}(\partial\pi_0)^2(\partial\pi)^2 \tag{2.9}$$

where π_0 is the background field. One has to be slightly careful in choosing a background as it might happen that the background itself disallows wave-like (propagating) solutions in which case it would not be possible to run this analysis. Therefore, we usually choose a background whose scale of variation is much larger than that of field fluctuations, allowing for wave-like solutions.

It is also important to point out that given a general dispersion relation, one cannot always directly demand phase or group velocity to be subluminal as it is well known in the literature that they both can be superluminal while remaining in perfect agreement with causality [63, 64, 65]. Instead, one usually demands the wavefront velocity be luminal. However, as already mentioned in sec., one has the so-called Kramers-Kronig relation [66, 20]

$$n(\infty) = n(0) - \frac{2}{\pi}\int_0^\infty \frac{d\omega}{\omega} Im\, n(\omega) \tag{2.10}$$

where $n(\omega)$ denotes the refractive index of the medium. For dissipative mediums, one has $\mathrm{Im}\, n(\omega) > 0$ which implies that the high frequency phase velocity is larger than the low frequency one. And since the wavefront velocity - infinite frequency limit of phase velocity - dictates the speed of information transfer, superluminal phase or group velocity can be related to violation of causality. However, if the medium allows for exhibiting gain (e.g., in the case of a Laser) then one can get $\mathrm{Im}\, n(\omega) < 0$ and in that case, the low-frequency phase velocity cannot be a statement about causality.

The situation gets even more involved if we have a mass-like term in the equation of motion. For example, take the massive Klein-Gordon equation

$$\partial^2\phi + m^2\phi = 0 \tag{2.11}$$

From the above eq$^{\text{n}}$ we get the following dispersion relation and phase velocity,

$$\omega^2 = |\vec{k}|^2 + m^2 \qquad ; \qquad v_p^2 = 1 + \frac{m^2}{|\vec{k}|^2} \tag{2.12}$$

In this case phase velocity is not a good object to consider since $v_p(\infty)$=1 (preserving causality) while being superluminal in the low energy limit (EFT regime). The group velocity on the other hand is (sub)luminal

$$v_g = \frac{|\vec{k}|}{\sqrt{|\vec{k}|^2 + m^2}} \tag{2.13}$$

It was shown in [63] that it is the group velocity which is equal to the signal velocity for Klein-Gordon modes with real mass.

2.2.1 Massive goldstone boson

Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\pi)^2 + \frac{c_3}{\Lambda^4}(\partial\pi)^4 - \frac{1}{2}m^2\pi^2 - J\pi \tag{2.14}$$

where J is what sources the background, with the following EOM

$$\partial^2\pi\left(1 + \frac{4c_3(\partial\pi)^2}{\Lambda^4}\right) + \frac{8c_3}{\Lambda^4}(\partial_\nu\pi)(\partial^\mu\partial^\nu\pi)(\partial_\mu\pi) + m^2\pi + J = 0 \tag{2.15}$$

The linearised EOM for the fluctuations $\xi = \pi - \pi_0$ with $\partial_\mu\pi_0 = C_\mu$(constant) reads

$$\partial^2\xi + \frac{8c_3}{\Lambda^4}C^\mu C^\nu\partial_\mu\partial_\nu\xi - \frac{4c_3C^2}{\Lambda^4}m^2\xi + m^2\xi = 0 \tag{2.16}$$

where we have assumed that the background terms on the l.h.s are canceled by the source J. Taking Fourier transform we get,

$$k_\mu k^\mu = -\frac{8c_3}{\Lambda^4}\left(C.k\right)^2 - \frac{4c_3C^2}{\Lambda^4}m^2 + m^2 \tag{2.17}$$

If the mass term is absent then subluminality demands $c_3 > 0$ as obtained by [3]. As mentioned in the above discussion, for a Klein-Gordon like dispersion relation it is the group velocity that should be (sub)luminal since the group velocity is the signal velocity. It might seem unclear whether this holds true here (since the dispersion relation is more general here, and not exactly Klein-Gordon like). However, with the choice of a purely space-like background, $C_\mu = \big(0, 0, 0, C_{(3)}\big)$, the dispersion relation takes the following form,

$$\omega^2 = ak^2 + m'^2 \tag{2.18}$$

where, $a = 1 - \frac{8c_3 C_{(3)}^2}{\Lambda^4}$ and $m'^2 = m^2 \left(1 + \frac{4c_3 C_{(3)}^2}{\Lambda^4}\right)$. This form is very close to Klein-Gordon since the parameter a is very close to unity; therefore, one can perform the analysis assuming that the group velocity is signal velocity.

We now consider the propagation of perturbations along the z-axis, then the expression for group velocity from eq$^{\text{n}}$(2.17) reads,

$$v_g = \frac{d\omega}{dk} = \frac{k + \frac{8c_3 C_{(3)}}{\Lambda^4}(C_{(0)}\omega - C_{(3)}k)}{\sqrt{k^2 + m^2 - \frac{4c_3 C^2 m^2}{\Lambda^4} - \frac{8c_3}{\Lambda^4}(C_{(0)}\omega - C_{(3)}k)^2 + \frac{8c_3 C_{(0)}}{\Lambda^4}(C_{(0)}\omega - C_{(3)}k)}} \tag{2.19}$$

We choose the background to be varying only in the z-direction i.e. $C_\mu = \big(0, 0, 0, C_{(3)}\big)$, then demanding the group velocity to be subluminal gives

$$\frac{m^2}{k^2}\left(1 + \frac{4c_3 C_{(3)}^2}{\Lambda^4}\right) + \frac{8c_3 C_{(3)}^2}{\Lambda^4} > 0 \tag{2.20}$$

Taking the limit $k \gg m$, we can drop the second term in the parenthesis and get

$$\frac{m^2}{k^2} + 8c_3\varepsilon > 0 \tag{2.21}$$

where, $0 < \varepsilon = \frac{C_{(3)}^2}{\Lambda^4} \ll 1$. Finally, we get

$$c_3 > -\frac{m^2/k^2}{8\varepsilon} \tag{2.22}$$

This is a ratio of two small positive quantities. Therefore, one does not get a strict positivity bound in this case from the superlumianlity analysis, unlike for the massless pions.

However, the amplitude analysis still gives a strict positivity bound on c_3 as it is unaffected by the mass term.

2.2.2 Gluon field strength operators

We now attempt to derive constraints on the Wilson coefficients of the Gluonic operators considered in sec. 2.1.2, using the superluminality analysis. As we will see below, here we will not need to take recourse to the Kramers-Kronig relation because we will be able to choose backgrounds where the dispersion relation takes the simplest form i.e it is non-dispersive:

$$\omega = v|\vec{k}| \tag{2.23}$$

where v is a constant. In this case, phase and group velocities are the same and are equal to the signal velocity. Due to the non-abelian nature of Lagrangian, here the calculations are tedious and are given in Appendix A.1. Here, we just state the main results highlighting the key assumptions that went into the analysis. We'll work with one higher-dimensional operator at a time keeping the calculations and analysis easy to follow.

First, we'll take dim 8 operator $Q^{(1)}_{G^4} = \left(G^a_{\mu\nu}G^{a\mu\nu}\right)\left(G^b_{\rho\sigma}G^{b\rho\sigma}\right)$ in addition to the four dimensional term in the Lagrangian

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{c_8^{(1)}}{\Lambda^4}\left(G^a_{\mu\nu}G^{a\mu\nu}\right)\left(G^b_{\rho\sigma}G^{b\rho\sigma}\right)$$

giving equation of motion,

$$\begin{aligned} &-\partial_\alpha G^{f,\alpha\beta} + \frac{8c_8^{(1)}}{\Lambda^4}\left(2G^a_{\mu\nu}(\partial_\alpha G^{a,\mu\nu})G^{f,\alpha\beta} + G^a_{\mu\nu}G^{a,\mu\nu}\partial_\alpha G^{f,\alpha\beta}\right) \\ =&-g_s G^{a,\beta\nu}A^h_\nu f^{afh} + \frac{8c_8^{(1)}}{\Lambda^4}g_s f^{bfj}G^a_{\mu\nu}G^{a,\mu\nu}G^{b,\beta\sigma}A^j_\sigma \end{aligned} \tag{2.24}$$

We expand $A^{a,\mu} = A^{a,\mu}_0 + h^{a,\mu}$ where we *choose* background $A^{a,\mu}_0$ to be of a particular color 'a' with $\partial^\nu A^{a,\mu} = constant$; such background also solves the equation of motion (2.24). We *look* at the linearised equation of motion for the perturbation $h^{a,\mu}$ of the same color 'a' as that of the background,

$$-\partial_\alpha \partial^\alpha h^{a,\beta} + \frac{32 c_8^{(1)}}{\Lambda^4} F^a_{0\mu\nu} F_0^{a\alpha\beta} \partial_\alpha \partial^\mu h^{a,\nu} = 0 \tag{2.25}$$

where color index 'a' is not contracted, and we'll drop the color index for $F_0^{\mu\nu}$ terms which should be assumed to have color 'a'. Above eq$^{\text{n}}$ when expanded in terms of plane waves gives,

$$k_\mu k^\mu = -\frac{32 c_8^{(1)}}{\Lambda^4} (F_{0\mu\nu} \epsilon^\nu k^\mu)^2 \tag{2.26}$$

When a similar procedure is done considering all 6 and 8 dimensional operators, we get the following equation (color index 'c' is contracted but not 'a'),

$$\begin{aligned} k_\mu k^\mu = -\frac{32}{\Lambda^4} \Bigg\{ & \left(c_8^{(1)} + c_8^{(3)} + d^{aa}{}_c d^{aac} c_8^{(7)}\right) \left(F_{0\mu\nu} \epsilon^\nu k^\mu\right)^2 \\ & + \left(c_8^{(2)} + c_8^{(4)} + d^{aa}{}_c d^{aac} c_8^{(8)}\right) \left(\widetilde{F}_{0\mu\nu} \epsilon^\nu k^\mu\right)^2 \\ & + \left(c_8^{(5)} + c_8^{(6)} + d^{aa}{}_c d^{aac} c_8^{(9)}\right) \left(F_{0\mu\nu} \widetilde{F}_{0\alpha\beta} \epsilon^\nu \epsilon^\beta k^\mu k^\alpha\right) \Bigg\} \end{aligned} \tag{2.27}$$

If we look at the perturbations of the same color as that of the background, then we are effectively considering only the abelian part of our theory. Since there is no analog of dim 6 operators $Q_{G^3}^{(1)}$ and $Q_{G^3}^{(2)}$ in the abelian gauge theory, there is no contribution from dim 6 operators towards the wave propagation. However, we still see signs of the existence of different gluon colors in the form of $d^{aa}{}_c d^{aac}$ factors which give different values for different colors in consideration.

By choosing particular background and polarization for the perturbation, we can get different bounds on Wilson coefficients $c_8^{(i)}$ using dispersion relation (2.27). Consider the z-axis along the direction of propagation of perturbation, perturbation of the same color as that of background let say 1) with polarization, $\epsilon = \{0, 1, 1, 0\}/\sqrt{2}$. *Choose* the background such that only non-zero components of $F^{\mu\nu}$ are $F^{01} = -F^{10}$ and $F^{02} = -F^{20}$ (we have dropped 0 from $F_0^{\mu\nu}$ to avoid confusion with time component), then we get

$$k_\mu k^\mu = -\frac{16}{\Lambda^4}\Bigg\{ \left(c_8^{(1)} + c_8^{(3)} + \frac{1}{3}c_8^{(7)}\right)\omega^2\Big(F_{01}+F_{02}\Big)^2 + \left(c_8^{(2)} + c_8^{(4)} + \frac{1}{3}c_8^{(8)}\right)\left(k^3\right)^2\Big(F_{02}-F_{01}\Big)^2 + \left(c_8^{(5)} + c_8^{(6)} + \frac{1}{3}c_8^{(9)}\right)\omega k^3\left(F_{02}+F_{01}\right)\left(F_{02}-F_{01}\right)\Bigg\} \tag{2.28}$$

Now, if we take $F_{01} = F_{02}$ then for perturbations to be causal we need

$$c_8^{(1)} + c_8^{(3)} + \frac{1}{3}c_8^{(7)} \geq 0 \tag{2.29}$$

which is same as the constraint $C(5,1)$ obtained in sec 2.1.2. Similarly by choosing different polarization and background, explicitly given in the table 2.3, we can reproduce $C(5,2)$ and $C(5,3)$. Also, if instead of color 1 we choose color 4 for both background and perturbations then we get C(6,1), C(6,2) and C(6,3).

Now to probe dim 6 operators using superluminality analysis, we need to consider the background and perturbations of different colors. This makes the analysis rather involved. We'll first consider operator $Q_{G^3}^{(1)} = f^{abc}G_\mu^{a\nu}G_\nu^{b\rho}G_\rho^{c\mu}$ for which we have,

$$L = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} + \frac{c_6}{\Lambda^2}f^{abc}G_\mu^{a\nu}G_\nu^{b\rho}G_\rho^{c\mu}$$

giving equation of motion,

$$-\partial_\alpha G^{f,\alpha\beta} + \frac{6c_6}{\Lambda^2}f^{fbc}\partial_\alpha\left(G^{c,\rho\alpha}G^{b,\beta\rho}\right) = -g_s G^{a,\beta\nu}A_\nu^h f^{afh} + \frac{6c_6}{\Lambda^2}g_s f^{abc}\left(f^{afh}G^{b,\nu\rho}G_\rho^{c,\beta}A_\nu^h\right) \tag{2.30}$$

We again expand $A^{a,\mu} = A_0^{a,\mu} + h^{a,\mu}$ where we *choose* background $A_0^{a,\mu}$ to be of particular color 'a' with $\partial^\nu A^{a,\mu} = constant$, and *look* at the linearised equation of motion for perturbation $h^{a,\mu}$ of different color 'f',

$$
\begin{aligned}
&-\partial_\alpha\Big(H^{f,\alpha\beta}+g_sf^{faj}A_0^{a,\alpha}h^{j,\beta}+g_sf^{fia}h^{i,\alpha}A_0^{a,\beta}\Big) \\
&+\frac{6c_6}{\Lambda^2}f^{fba}\partial_\alpha\Big(F_0^{a,\rho\alpha}(H^{b,\beta}{}_\rho+g_sf^{baj}A_0^{a,\beta}h^j_\rho+g_sf^{bia}h^{i,\beta}A^a_{0\rho}) \\
&\qquad\qquad -F_0^{a,\beta\rho}(H^{b,\alpha}_\rho+g_sf^{baj}A^a_{0\rho}h^{j,\alpha}+g_sf^{bia}h^i_\rho A_0^{a,\alpha})\Big) \\
&=-g_s\Big(H^{d,\beta\nu}+g_sf^{daj}A_0^{a,\beta}h^{j,\nu}+g_sf^{dia}h^{i,\beta}A_0^{a,\nu}\Big)A^a_{0\nu}f^{dfa}-g_sF_0^{a,\beta\nu}h^h_\nu f^{afh} \\
&+\frac{6c_6}{\Lambda^2}g_sf^{daa}\Big(f^{dfh}F_0^{a,\nu\rho}F^{a,\beta}_{0\rho}h^h_\nu\Big) \\
&+\frac{6c_6}{\Lambda^2}g_sf^{dba}\Big(f^{dfa}(H^{b,\nu\rho}+g_sf^{baj}A_0^{a,\nu}h^{j,\rho}+g_sf^{bia}h^{i,\nu}A_0^{a,\rho})F^{a,\beta}_{0\rho}A^a_{0\nu}\Big) \\
&+\frac{6c_6}{\Lambda^2}g_sf^{dac}\Big(f^{dfa}F_0^{a,\nu\rho}(H^{c,\beta}_\rho+g_sf^{caj}A^a_{0\rho}h^{j,\beta}+g_sf^{cia}h^i_\rho A_0^{a,\beta})A^a_{0\nu}\Big)
\end{aligned}
\tag{2.31}
$$

In the above equation and the following equations, 'f' and 'a' are not contracted but are free color indices and we'll again drop 'a' from $F_0^{\mu\nu}$ terms which should always be assumed to have color 'a'.

For further analysis, we take WKB approximation in which the scale of variation of background(r) is much larger than that of perturbations (ω^{-1}). We also *choose* the background field to be arbitrarily small, then at leading order in $r\omega$ and A_0 we get,

$$
\begin{aligned}
&-\partial_\alpha\partial^\alpha h^{f,\beta}+2g_sf^{fba}A_0^{a,\alpha}\partial_\alpha h^{b,\beta}-g_sf^{fba}A_{0\rho}\partial^\beta h^{b,\rho} \\
&+\frac{6c_6}{\Lambda^2}f^{fba}\Big(F_{0\rho\alpha}\partial^\alpha\partial^\beta h^{b,\rho}+F_0^{\beta\rho}\partial_\alpha\partial^\alpha h^b_\rho\Big)=0
\end{aligned}
\tag{2.32}
$$

To get the dispersion relation for perturbation of color 'f', we try to write the differential equation just in terms of color 'f'. So we assume a particular solution for perturbation of some different color 'b' which satisfies the EOM,

$$
-\partial^\alpha h^{b,\rho}+2g_sf^{bga}A_0^{a,\alpha}h^{g,\rho}-g_sf^{bga}A_{0\nu}\delta^{\alpha\rho}h^{g,\nu}+\frac{6c_6}{\Lambda^2}f^{bga}\left(F_0^{\sigma\alpha}\partial^\rho h^g_\sigma+F_0^{\rho\sigma}\partial^\alpha h^g_\sigma\right)=0 \tag{2.33}
$$

The above particular solution for other colors 'b' modifies the eq$^{\text{n}}$(2.32) to,

$$
\begin{aligned}
-\partial^\alpha\partial_\alpha h^{f,\beta} =& 2g_s^2 f^{fba} f^{gba}\left(2A_0^\alpha A_{0\alpha} h^{g,\beta} - \frac{3}{2}A_{0\nu}A_0^\beta h^{g,\nu}\right) \\
&+ g\frac{6c_6}{\Lambda^2} f^{fba} f^{gba}\left(5F_0^{\beta\rho}A_0^\alpha \partial_\alpha h^{g,\rho} - A_{0\nu}F_0^{\rho\beta}\partial^\rho h^{g,\nu} + 5F_{0\rho\alpha}A_0^\alpha\partial^\beta h^{g,\rho}\right) \\
&+ 36\frac{c_6^2}{\Lambda^4} f^{fba} f^{gba}\left(F_0^{\beta\rho}F_0^{\sigma\alpha}\partial_\alpha\partial_\rho h_\sigma^g + F_{0\rho}^\beta F_0^{\rho\sigma}\partial^\alpha\partial_\alpha h_\sigma^g + 2F_0^{\rho\alpha}F_0^{\sigma\alpha}\partial^\beta\partial_\rho h_\sigma^g\right)
\end{aligned}
\tag{2.34}
$$

Since we have assumed a particular type of solution for other colors, we want to see how that affects the perturbation of color 'f', so we try to write the differential equation just in terms of perturbation of color 'f' as mentioned before. But it is not possible to replace the mass-like term (the term inside parenthesis in the first line) in the above equation unless we have an explicit solution for perturbations of all color $h^{g,\beta}$. So, for now, we'll assume that we can choose a particular background A_0 such that the mass-like term vanishes. We'll give below an explicit example of background and polarization of the perturbation where this assumption is satisfied.

In the second and third lines of the eq$^{\text{n}}$(2.34), when $g \neq f$ we again substitute h^g in terms of other colors using the solution assumed eq$^{\text{n}}$(2.33). But this gives terms of higher order in A_0 or $\mathcal{O}\left(\frac{1}{\Lambda^4}\right)$ which we can ignore w.r.t the terms where $g = f$. Therefore only terms with $g = f$ survives at $\mathcal{O}\left(\frac{1}{\Lambda^4}\right)$ and leading order in A_0,

$$
\begin{aligned}
-\partial^\alpha\partial_\alpha h^{f,\beta} =& g\frac{6c_6}{\Lambda^2} f^{fba} f^{fba}\left(5F_0^{\beta\rho}A_0^\alpha \partial_\alpha h^{f,\rho} - A_{0\nu}F_0^{\rho\beta}\partial^\rho h^{f,\nu} + 5F_{0\rho\alpha}A_0^\alpha\partial^\beta h^{f,\rho}\right) \\
&+36\frac{c_6^2}{\Lambda^4} f^{fba} f^{fba}\left(F_0^{\beta\rho}F_0^{\sigma\alpha}\partial_\alpha\partial_\rho h_\sigma^f + F_{0\rho}^\beta F_0^{\rho\sigma}\partial^\alpha\partial_\alpha h_\sigma^f + 2F_0^{\rho\alpha}F_0^{\sigma\alpha}\partial^\beta\partial_\rho h_\sigma^f\right)
\end{aligned}
\tag{2.35}
$$

We expand the above equation in terms of waves with transverse polarization and consider the spatial wave vector, $\vec{\kappa}$ to be complex in general. We then get the following dispersion relation,

$$
\omega^2 = |\vec{k}|^2 + 36\frac{c_6^2}{\Lambda^4}(f^{fba})^2\left(F_0^{\beta\rho}k_\rho\epsilon_\beta\right)^2 \tag{2.36}
$$

where $\vec{k}$ denotes the real part of the spatial wave vector. After considering all dim 6 and dim 8 operators we get the following dispersion relation,

$$\begin{aligned}\frac{k_\mu k^\mu}{4} =& \frac{9}{\Lambda^4} f^{afc} f^{af}_{\ \ c}\Big[c_6(F_0^{\mu\nu} k_\mu \epsilon_\nu) - c_6'(\widetilde{F}_0^{\alpha\beta} k_\alpha \epsilon_\beta)\Big]^2 \\ &- \frac{4}{\Lambda^4}\Big[(2\delta^{af} c_8^{(1)} + (1+\delta^{af}) c_8^{(3)} + 2d^{afc} d^{af}_{\ \ c} c_8^{(7)})\,(F_0^{\mu\nu} k_\mu \epsilon_\nu)^2 \\ &\quad + (2\delta^{af} c_8^{(2)} + (1+\delta^{af}) c_8^{(4)} + 2d^{afc} d^{af}_{\ \ c} c_8^{(8)})\left(\widetilde{F}_0^{\alpha\beta} k_\alpha \epsilon_\beta\right)^2 \\ &\quad - (2\delta^{af} c_8^{(5)} + (1+\delta^{af}) c_8^{(6)} + 2d^{afc} d^{af}_{\ \ c} c_8^{(9)})\,(F_0^{\mu\nu} k_\mu \epsilon_\nu)\left(\widetilde{F}_0^{\alpha\beta} k_\alpha \epsilon_\beta\right)\Big]\end{aligned} \tag{2.37}$$

where 'f' denotes the color of perturbation and 'a' of the background. Note that the above dispersion relation is only valid, assuming that mass-like term in the eq$^{\text{n}}$(2.34) vanish. We had a similar situation in the previous section where we had a mass-like term dependent on the background field, which refrained us from getting a constraint on the Wilson coefficients of the theory. But in this case, since it depends on contracted four-vectors, it is possible to make the mass-like term zero along with non-zero derivatives of background by choosing an appropriate A_0^μ.

We now try to get different constraints on Wilson coefficients by considering particular polarization and the background. Consider the perturbation with polarization $\epsilon = \{0,1,1,0\}/\sqrt{2}$ and choose background of the form $A_{0\mu} = E\{\sqrt{2}t, t, -t, 0\}$ where E is some arbitrarily small constant. Under this configuration, mass-like term in eq$^{\text{n}}$(2.34) vanish and we get following non-zero components of $F_{\mu\nu}$, $F_{01} = -F_{10}$ and $F_{02} = -F_{20}$ with $F_{01} = -F_{02}$, which reduces the dispersion relation (2.37) to the following form,

$$k_\mu k^\mu = \frac{72}{\Lambda^4} f^{afc} f^{af}_{\ \ c} (c_6')^2 \left(k^3\right)^2 (F_{02})^2 - \frac{32}{\Lambda^4}\left(c_8^{(4)} + 2d^{afc} d^{af}_{\ \ c} c_8^{(8)}\right)\left(k^3\right)^2 (F_{02})^2 \tag{2.38}$$

and for the perturbation to be causal (dictated by the phase velocity) we require,

$$9 f^{afc} f^{af}_{\ \ c} (c_6')^2 - 4\left(c_8^{(4)} + 2d^{afc} d^{af}_{\ \ c} c_8^{(8)}\right) < 0 \tag{2.39}$$

We can reproduce the $C(1,2), C(2,2), C(3,2)$ and $C(4,2)$ bounds of table 2.2 using the above relation by choosing different colors for perturbations and background. The remaining bounds of table 2.2 can also be reproduced by choosing different background and polarization configurations, details of which have been relegated to the appendix A.1. In the table 2.3, we present all the bounds obtained on dim 6 and 8 gluonic operators using the superluminality analysis by considering different configurations of the background and perturbation.

2.2.3 A Non-relativistic Example: Stronger bound from Superluminality

Since we got similar bounds from both superluminality and amplitude analysis, therefore, one might think that this is always the case. Here, following [24] we present a counter-example where superluminality gives a stronger bound. Consider the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\pi)^2 - \frac{4c_3}{3\Lambda^2}\dot{\pi}^3 + \frac{2c_4}{3\Lambda^4}\dot{\pi}^4 \tag{2.40}$$

This Lagrangian emerges from the EFT of Inflation [67] Lagrangian in a particular limit [24].

The $2 \to 2$ forward scattering amplitude at tree level is given by

$$\mathcal{A}(s) = \left(c_4 - (2c_3)^2\right)\frac{s^2}{\Lambda^4} \tag{2.41}$$

Performing the 'amplitude analysis' gives the following bound

$$c_4 > (2c_3)^2 \tag{2.42}$$

As a side remark, it is important to note that, due to subtleties related to spontaneously broken Lorentz invariance, the derivation of the above bounds in [24] may not be completely rigorous, see [68] for a recent discussion.

Now, let us check what superluminality gives for this Lagrangian. We derive linearised EOM for the fluctuations, $\xi = \pi + \alpha t$, where α is a small quantity.

$$\ddot{\xi} + \frac{8c_3}{\Lambda^2}\alpha\ddot{\xi} + \frac{8c_4}{\Lambda^4}\alpha^2\ddot{\xi} - \partial_i^2\xi = 0 \tag{2.43}$$

Up to $O(\alpha)$ we have

$$\ddot{\xi} + \frac{8c_3}{\Lambda^2}\alpha\ddot{\xi} - \partial_i^2\xi = 0 \tag{2.44}$$

then the phase velocity of the fluctuation is given by

$$v_{phase}^2 = 1 - \frac{8c_3\alpha}{\Lambda^2} \tag{2.45}$$

Since α can have any sign, therefore, the only way to preserve (sub)luminality is by taking $c_3 = 0$.
Now, up to $O(\alpha^2)$ we have

$$\ddot{\xi} + \frac{8c_4\alpha^2}{\Lambda^2}\ddot{\xi} - \partial_i^2\xi = 0 \tag{2.46}$$

The phase velocity is given by

$$v_{phase}^2 = 1 - \frac{8c_4\alpha^2}{\Lambda^4} \tag{2.47}$$

(Sub)luminality demands, $c_4 \geq 0$. Therefore, superluminality in this particular case gives a stronger bound than the 'amplitude analysis'.

Colors	$\sqrt{2}\epsilon = \{0,1,1,0\}$	$\sqrt{2}\epsilon = \{0,1,1,0\}$
	$A_{0\mu} = E\{\sqrt{2}(x+y), x+y, -(x+y), 0\}$	$A_{0\mu} = E\{\sqrt{2}t, t, -t, 0\}$
$a=1, f=8$	$c_8^{(3)} + \frac{2}{3}c_8^{(7)} > 0$	$c_8^{(4)} + \frac{2}{3}c_8^{(8)} > 0$
$a=1, f=2$	$c_8^{(3)} > \frac{9}{4}c_6^2$	$c_8^{(4)} > \frac{9}{4}c_6'^2$
$a=1, f=4$	$c_8^{(3)} + \frac{1}{2}c_8^{(7)} > \frac{9}{16}c_6^2$	$c_8^{(4)} + \frac{1}{2}c_8^{(8)} > \frac{9}{16}c_6'^2$
$a=4, f=8$	$c_8^{(3)} + \frac{3}{2}c_8^{(7)} > \frac{27}{16}c_6^2$	$c_8^{(4)} + \frac{3}{2}c_8^{(8)} > \frac{27}{16}c_6'^2$
$a=f=1$	$c_8^{(1)} + c_8^{(3)} + \frac{1}{3}c_8^{(7)} > 0$	$c_8^{(2)} + c_8^{(4)} + \frac{1}{3}c_8^{(8)} > 0$
$a=f=4$	$c_8^{(1)} + c_8^{(3)} + c_8^{(7)} > 0$	$c_8^{(2)} + c_8^{(4)} + c_8^{(8)} > 0$

Colors	$\sqrt{2}\epsilon = \{0,1,-1,0\}$	$\sqrt{2}\epsilon = \{0,1,1,0\}$
	$A_{0\mu} = E\{\sqrt{2(D+B)}t, (\sqrt{D}+\sqrt{B})t, (\sqrt{D}-\sqrt{B})t, 0\}$	$A_{0\mu} = E\{\sqrt{2(D+B)}t, (\sqrt{D}+\sqrt{B})t, (\sqrt{B}-\sqrt{D})t, 0\}$
$a=1, f=8$	$4\left(c_8^{(3)} + \frac{2}{3}c_8^{(7)}\right)\left(c_8^{(4)} + \frac{2}{3}c_8^{(8)}\right) > \left(c_8^{(6)} + \frac{2}{3}c_8^{(9)}\right)^2$	
$a=1, f=2$	$4\left(c_8^{(3)} - \frac{9}{4}c_6^2\right)\left(c_8^{(4)} - \frac{9}{4}c_6'^2\right) > \left(c_8^{(6)} - \frac{9}{2}c_6c_6'\right)^2$	
$a=1, f=4$	$4\left(c_8^{(3)} + \frac{1}{2}c_8^{(7)} - \frac{9}{16}c_6^2\right)\left(c_8^{(4)} + \frac{1}{2}c_8^{(8)} - \frac{9}{16}c_6'^2\right) > \left(c_8^{(6)} + \frac{1}{2}c_8^{(9)} - \frac{9}{8}c_6c_6'\right)^2$	
$a=4, f=8$	$4\left(c_8^{(3)} + \frac{3}{2}c_8^{(7)} - \frac{27}{16}c_6^2\right)\left(c_8^{(4)} + \frac{3}{2}c_8^{(8)} - \frac{27}{16}c_6'^2\right) > \left(c_8^{(6)} + \frac{3}{2}c_8^{(9)} - \frac{27}{8}c_6c_6'\right)^2$	
$a=f=1$	$4\left(c_8^{(1)} + c_8^{(3)} + \frac{1}{3}c_8^{(7)}\right)\left(c_8^{(2)} + c_8^{(4)} + \frac{1}{3}c_8^{(8)}\right) > \left(c_8^{(5)} + c_8^{(6)} + \frac{1}{3}c_8^{(9)}\right)^2$	
$a=f=4$	$4\left(c_8^{(1)} + c_8^{(3)} + c_8^{(7)}\right)\left(c_8^{(2)} + c_8^{(4)} + c_8^{(8)}\right) > \left(c_8^{(5)} + c_8^{(6)} + c_8^{(9)}\right)^2$	

Table 2.3: The table contains the constraints on dim 6 and dim 8 operators' Wilson coefficients obtained using the superluminality analysis. $A_{0\mu}$ and ϵ represent the background field and polarization of the perturbation, respectively. The color of the background is denoted by 'a' and that of perturbation by 'f'. $D = 4(2\delta^{ab}c_8^{(1)} + (1+\delta^{ab})c_8^{(3)} + 2d^{abc}d^{ab}{}_c c_8^{(7)}) - 9f^{abc}f^{ab}{}_c c_6^2$; $B = 4(2\delta^{ab}c_8^{(2)} + (1+\delta^{ab})c_8^{(4)} + 2d^{abc}d^{ab}{}_c c_8^{(8)}) - 9f^{abc}f^{ab}{}_c c_6'^2$

Chapter 3

Bell violation in $2 \to 2$ scattering in photon, gluon and graviton EFTs

This chapter is largely produced from [50], which was one of the original works done during the course of this thesis. The chapter is mainly organized as follows: In section 3.1, we give a brief overview of the CGLMP Bell parameter and its validity. In section 3.2, we explore the possibility of Bell violation in photons, gluons, and graviton EFTs. Section 3.3 discusses CP violation from the point of view of Bell violation.

3.1 CGLMP Bell parameter

We will first briefly overview the CGLMP inequality and the corresponding Bell parameter for qubits following [48]. Let us suppose there are two parties, Alice (A) and Bob (B), with a qubit state each. Alice can carry out two possible measurements, A_1 or A_2, and Bob can carry out two possible measurements, B_1 or B_2. Each measurement can have only two possible outcomes: $A_1, A_2, B_1, B_2 = 0, 1$. For a local hidden variable theory, the qubit system can be described by 16 probabilities c_{jklm} where (j, k) are Alice's local variables and (l, m) are Bob's local variables. The pair (j, k) represents that measurement A_1 has outcome j and measurement A_2 has outcome k; and (l, m) represents that the measurement B_1 has outcome l and measurement B_2 has outcome m. Since c_{jklm} are probabilities, they are positive ($c_{jklm} \geq 0$) and sum to one $\left(\sum_{jklm} c_{jklm} = 1\right)$. The joint probabilities

$P(A_1 = j, B_2 = m)$ then take the following form $P(A_1 = j, B_2 = m) = \sum_{kl} c_{jklm}$ and similarly for $P(A_1 = j, B_1 = l)$, $P(A_2 = k, B_1 = l)$ and $P(A_2 = k, B_2 = m)$. Using the joint probabilities, we define the probability $P(A_a = B_b + k)$ that the measurements A_a and B_b have outcomes that differ by k modulo 2 :

$$P(A_a = B_b + k) \equiv \sum_{j=0}^{1} P(A_a = j, B_b = j + k \bmod 2) \tag{3.1}$$

The CGLMP inequality[1] is a combination of the above probabilities defined as:

$$\begin{aligned} I_2 = &+ [P(A_1 = B_1) + P(B_1 = A_2 + 1) + P(A_2 = B_2) + P(B_2 = A_1)] \\ &- [P(A_1 = B_1 - 1) + P(B_1 = A_2) + P(A_2 = B_2 - 1) + P(B_2 = A_1 - 1)] \end{aligned} \tag{3.2}$$

For a local hidden variable theory, we can have any three probabilities with a $+$ sign in the above expression satisfied along with one with a $-$ sign (or vice versa). Therefore, for such theories $-2 \leq I_2 \leq 2$. However, for a quantum mechanical theory, I_2 can be greater than 2, as shown below.
Consider the following normalized quantum state of entangled qubits (in $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$ basis),

$$|\Psi\rangle = \frac{1}{\sqrt{\sum_{m=0}^{1} |\mu_m|^2}} \sum_{m=0}^{1} \mu_m |m\rangle_A \otimes |m\rangle_B \tag{3.3}$$

The measurements by Alice and Bob are carried out in three steps [48, 70]. First, a variable phase, $e^{i\phi_a(m)}$ for Alice and $e^{i\varphi_b(m)}$ for Bob, which depends on the measurement being carried out is given to each state $|m\rangle$ using phase shifters which are at the disposal of the observer. Thus the state becomes

$$|\Psi\rangle = \frac{1}{\sqrt{\sum_{m=0}^{1} |\mu_m|^2}} \sum_{m=0}^{1} \mu_m e^{i\phi_a(m)} e^{i\varphi_b(m)} |m\rangle_A \otimes |m\rangle_B \tag{3.4}$$

where $\phi_1(m) = \pi\alpha_1 m$, $\phi_2(m) = \pi\alpha_2 m$, $\varphi_1(m) = \pi\beta_1 m$ and $\varphi_2(m) = \pi\beta_2 m$ with $\alpha_1 = 0$, $\alpha_2 = 1/2$, $\beta_1 = 1/4$ and $\beta_2 = -1/4$. These are the optimal measurement settings for which one gets the maximum value of I_2 for an entangled quantum state in Schmidt basis [48].

Then each party carries out a discrete Fourier transform to get the state to the following

[1]For the qubit case, the CGLMP inequality is equivalent to the CHSH inequality [69].

form,

$$|\Psi\rangle = \frac{1}{2\sqrt{\sum_{m=0}^{1}|\mu_m|^2}} \sum_{m,k,l=0}^{1} \mu_m \exp\left[i\Big(\phi_a(m) + \varphi_b(m) + \pi m(k-l)\Big)\right] |k\rangle_A \otimes |l\rangle_B \quad (3.5)$$

The final step is for Alice to measure the projection of the state along the k basis and for Bob to measure along the l basis. Thus the joint probabilities are:

$$P(A_a = k, B_b = l) = \frac{1}{4\sum_{m=0}^{1}|\mu_m|^2} \left|\sum_{m=0}^{1} \mu_m \exp\left[i\pi m\Big(\alpha_a + k + \beta_b - l\Big)\right]\right|^2 \quad (3.6)$$

We can also get the above joint probabilities in one step if we consider the operators A_a and B_b to have the following non-degenerate eigenvectors, respectively [46],

$$|k\rangle_{A,a} = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} X_{j,k}^{(a)} |j\rangle_A, \quad (3.7)$$

$$|l\rangle_{B,b} = \frac{1}{\sqrt{2}} \sum_{j=0}^{1} Y_{j,l}^{(b)} |j\rangle_B, \quad (3.8)$$

where $X_{j,k}^{(a)} = \exp\left(i\pi j\left(k + \alpha_a\right)\right), Y_{j,l}^{(b)} = \exp\left(i\pi j\left(-l + \beta_b\right)\right)$ with α_a and β_b defined as before. Then we get the following joint probabilities,

$$P\left(A_a = k, B_b = l\right) = \langle\Psi| \Big(|k\rangle_{A,a} \otimes |l\rangle_{B,b} \quad {}_{A,a}\langle l| \otimes_{B,b} \langle k|\Big) |\Psi\rangle \quad (3.9)$$

which when evaluated is same as eq^n(3.6).
Using eq$^{\mathrm{n}}$(3.6) and eq$^{\mathrm{n}}$(3.1), we can calculate I_2 for generic μ_m, which is given by

$$I_2 = \frac{2\sqrt{2}\left(\mu_0\mu_1^* + \mu_1\mu_0^*\right)}{\sqrt{\sum_{m=0}^{1}|\mu_m|^2}} \quad (3.10)$$

When I_2 is extremized w.r.t μ_m, we get $-2\sqrt{2} \leq I_2 \leq 2\sqrt{2}$. The $I_2 = 2\sqrt{2}$ corresponds to the maximally entangled state, $|\Psi\rangle = \left(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right)/\sqrt{2}$. Thus for a quantum mechanical theory, $|I_2|$ can be greater than 2, whereas $|I_2|$ is always less than or equal to 2 for a local hidden variable theory. In the rest of the paper, we will denote $|m\rangle \otimes |n\rangle$ as $|m, n\rangle$ for convenience.
Note that if we consider $|\Psi\rangle$ to be a generic superposition in $|0,0\rangle$, $|0,1\rangle$, $|1,0\rangle$ and $|1,1\rangle$

basis,

$$|\Psi\rangle = \sum_{m,n=0}^{1} \mu_{m,n} |m,n\rangle \tag{3.11}$$

then we get the following expression for I_2,

$$I_2 = \frac{2\sqrt{2}\left(\mu_{0,0}\mu_{1,1}^{*} + \mu_{1,1}\mu_{0,0}^{*}\right)}{\sqrt{\sum_{m,n=0}^{1} |\mu_{m,n}|^2}} \tag{3.12}$$

However, now we get $I_2 = 0$ for certain maximally entangled states, $|\Psi\rangle = (|0,1\rangle + |1,0\rangle)/\sqrt{2}$ and $|\Psi\rangle = (|0,0\rangle + i\,|1,1\rangle)/\sqrt{2}$. Therefore, I_2 is a good order parameter for entanglement only if one considers $|0,0\rangle$ and $|1,1\rangle$ as the basis and real coefficients μ_0 and μ_1. For this purpose, we go to $|0,0\rangle$ and $|1,1\rangle$ basis before calculating I_2 and only consider CP-conserving theories as they have real amplitudes.

3.2 Bell inequality in $2 \to 2$ scattering

Since Bell inequalities can be used to distinguish between local hidden variable theories and quantum theories, we try to probe the quantum nature of different EFTs using $2 \to 2$ scattering and Bell inequalities. For this purpose, we relate the CGLMP Bell parameter I_2 to the $2 \to 2$ scattering helicity amplitudes, which can be easily calculated.

Consider $|p_1, h_1, m_1; p_2, h_2, m_2\rangle$ and $|p_3, h_3, m_3; p_4, h_4, m_4\rangle$ to be the basis for initial and final states, respectively, for the scattering process. Here, h_i represents the helicity of the particles, which can take values $h = +1, -1$ for photons and gluons, and $h = +2, -2$ for gravitons; and m_i represents the other quantum numbers, like color for gluons. We sum over the other quantum numbers for final states, m_3 and m_4, so that the final states are entangled only in Hilbert space spanned by helicity states. It would also be interesting to study the entanglement in other quantum numbers, however, one has to consider the appropriate CGLMP Bell parameter, I_d [48].

Now, the final states can be represented as $|p_3, h_3; p_4, h_4\rangle$. We denote the helicity scattering amplitudes as $\mathcal{M}_{h_1,h_2}^{h_3,h_4}(m_1, m_2, s, t, u)$ where we take initial states to be incoming and final states to be outgoing and s,t,u are the usual Mandelstam variables. In the rest of the paper, we use 1 in place of $h = +1, +2$ and 0 in place of $h = -1, -2$, for convenience.

Consider the initial state to be a generic superposition in the $|0,0\rangle$ and $|1,1\rangle$ helicity basis, $|\psi\rangle_i = \sum_{i=0,1} \lambda_i |i, m_1; i, m_2\rangle$. We have suppressed p_i here and in the rest of the paper to make the notation less clumsy. Any state can be written in this basis by the Schmidt decomposition theorem, as also shown in the appendix. We consider the initial states such that the CGLMP parameter corresponding to them (I_{2i}) satisfies $|I_{2i}| \leq 2$ as we want to probe quantum nature of the theory only through the scattering process. We then identify $\mu_{m,n} = \sum_{i=0,1} \lambda_i \mathcal{M}_{i,i}^{m,n}$ in eq$^{\rm n}$(3.11) i.e. we take $|\Psi\rangle$ to be the final state of our scattering process. We convert the final state to $|0,0\rangle$ and $|1,1\rangle$ basis by the Schmidt decomposition method and calculate I_{2f} for different theories and initial states using eq$^{\rm n}$(3.10).
From now on, we will use the parameterization $\lambda_0 \to \cos\theta$ and $\lambda_1 \to \sin\theta$ for convenience.

We consider the scattering of identical particles and assume parity symmetry. Then for photons, gluons, and gravitons scattering (cases that we will be considering), there are total 16 helicity scattering amplitudes for each set of additional quantum numbers m_1 and m_2 (they represent colors for gluons and do not exist for photons and gravitons). Note that we sum over m_3 and m_4 while calculating amplitudes, as already mentioned. However, due to parity symmetry, there are only five distinct scattering amplitudes in the COM frame [71], denoted as

$$\Phi_1(s,t,u) \equiv \mathcal{M}_{1,1}^{1,1}(s,t,u), \quad \Phi_2(s,t,u) \equiv \mathcal{M}_{1,1}^{0,0}(s,t,u), \quad \Phi_3(s,t,u) \equiv \mathcal{M}_{1,0}^{1,0}(s,t,u) \tag{3.13}$$

$$\Phi_4(s,t,u) \equiv \mathcal{M}_{1,0}^{0,1}(s,t,u), \quad \Phi_5(s,t,u) \equiv \mathcal{M}_{1,1}^{1,0}(s,t,u) \tag{3.14}$$

where we have suppressed the m_i dependence of helicity amplitudes. If we don't have quantum numbers other than h_i's i.e. m_i's do not exist, as is the case for photons and gravitons, then due to crossing symmetry, Φ_3 and Φ_4 can be related to Φ_1 as,

$$\Phi_3(s,t,u) = \Phi_1(u,t,s), \quad \Phi_4(s,t,u) = \Phi_1(t,s,u),$$

This leads to only three independent helicity amplitudes, Φ_1, Φ_2 and Φ_5. All the helicity scattering amplitudes for CP conserving theories can be denoted as,

$$\begin{pmatrix} \mathcal{M}_{1,0}^{1,0} & \mathcal{M}_{1,0}^{1,1} & \mathcal{M}_{0,0}^{1,0} & \mathcal{M}_{0,0}^{1,1} \\ \mathcal{M}_{1,0}^{0,0} & M_{1,0}^{0,1} & M_{0,0}^{0,0} & M_{0,0}^{0,1} \\ M_{1,1}^{1,0} & \mathcal{M}_{1,1}^{1,1} & \mathcal{M}_{0,1}^{1,0} & \mathcal{M}_{0,1}^{1,1} \\ \mathcal{M}_{1,1}^{0,0} & \mathcal{M}_{1,1}^{0,1} & \mathcal{M}_{0,1}^{0,0} & \mathcal{M}_{0,1}^{0,1} \end{pmatrix} = \begin{pmatrix} \Phi_3 & \Phi_5 & \Phi_5 & \Phi_2 \\ \Phi_5 & \Phi_4 & \Phi_1 & \Phi_5 \\ \Phi_5 & \Phi_1 & \Phi_4 & \Phi_5 \\ \Phi_2 & \Phi_5 & \Phi_5 & \Phi_3 \end{pmatrix} \tag{3.15}$$

We work in mostly *minus* signature, $\eta_{\mu\nu} = (+,-,-,-)$ throughout our work.

3.2.1 Euler-Heisenberg

Let's consider the following EFT Lagrangian for photons up to dim 8,

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{\Lambda^4}(F^{\mu\nu}F_{\mu\nu})^2 + \frac{c_2}{\Lambda^4}(F^{\mu\nu}\widetilde{F}_{\mu\nu})^2 \tag{3.16}$$

For the case of photons, we don't have to worry about the other quantum numbers, m_1 and m_2, and the low energy amplitudes Φ_1, Φ_2, and Φ_5 can be written as

$$\begin{aligned} \Phi_1(s,t,u) &= g_2 s^2 \\ \Phi_2(s,t,u) &= f_2\left(s^2+t^2+u^2\right) \\ \Phi_5(s,t,u) &= 0 \end{aligned} \tag{3.17}$$

up to $\mathcal{O}(1/\Lambda^4)$ [71, 72], where $g_2 = 8(c_1+c_2)/\Lambda^4$ and $f_2 = 8(c_1-c_2)/\Lambda^4$.

For a generic initial state, $|\psi\rangle_i = \cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$, we get the following final state after $2\to 2$ scattering,

$$\begin{aligned} |\Psi\rangle = &\cos\theta(\mathcal{M}^{0,0}_{0,0}\,|0,0\rangle + \mathcal{M}^{1,1}_{0,0}\,|1,1\rangle + \mathcal{M}^{0,1}_{0,0}\,|0,1\rangle + \mathcal{M}^{1,0}_{0,0}\,|1,0\rangle) \\ &\sin\theta(\mathcal{M}^{0,0}_{1,1}\,|0,0\rangle + \mathcal{M}^{1,1}_{1,1}\,|1,1\rangle + \mathcal{M}^{0,1}_{1,1}\,|0,1\rangle + \mathcal{M}^{1,0}_{1,1}\,|1,0\rangle) \end{aligned} \tag{3.18}$$

Using eq$^{\rm n}$(3.15) and eq$^{\rm n}$(3.17), the final state can be written as

$$|\Psi\rangle = (\sin\theta\;\Phi_1 + \cos\theta\;\Phi_2)\,|1,1\rangle + (\sin\theta\;\Phi_2 + \sin\theta\;\Phi_1)\,|0,0\rangle \tag{3.19}$$

Since the above final state is already in $|0,0\rangle$ and $|1,1\rangle$ basis, we can directly calculate the corresponding CGLMP Bell parameter I_{2f} using eq$^{\rm n}$(3.10),

$$\begin{aligned} I_{2f} &= \frac{2\sqrt{2}\,(2\Phi_1\Phi_2 + (\Phi_1^2+\Phi_2^2)\sin 2\theta)}{|\Phi_1|^2+|\Phi_2|^2+2\Phi_1\Phi_2\sin 2\theta} \\ &= \frac{2\sqrt{2}\,(\sin 2\theta + 4f^2(1+\chi+\chi^2)^2\sin 2\theta + 4f(1+\chi+\chi^2))}{1+4f^2(1+\chi+\chi^2)^2+4f(1+\chi+\chi^2)\sin 2\theta} \end{aligned} \tag{3.20}$$

where $f = f_2/g_2$ and $-1 \leq \chi = t/s \leq 0$ for physical t and s. Also, $\cos\phi_s = 1 + 2\chi$ where ϕ_s is the scattering angle.

We extremize I_{2f} w.r.t θ and χ with the constraint $|I_{2i}| \leq 2$. We find that one can observe Bell violation, i.e., $|I_{2f}| > 2$, for some scattering angle (or equivalently χ) for all values of f_2 and g_2 except for $f_2 = 0$ or $g_2 = 0$. In other words, for any non-zero values of f_2 and g_2, one can observe Bell violation at some scattering angle if $|\psi\rangle_i$ is chosen appropriately. Since we are interested in exploring the possibility of Bell violation due to the quantum evolution of the initial state dictated by the theory, we take the initial state whose CGLMP Bell parameter (I_{2i}) is less than 2, i.e., it can, in principle, also be described by a local hidden variable theory.

For $f_2 = 0$ or $g_2 = 0$, the maximum value for I_{2f} is equal to 2, with the constraint $|I_{2i}| \leq 2$, which lies on the boundary of Bell inequality. Therefore, for $|c_1| = |c_2|$, there is no Bell violation for any value of χ and $|I_{2i}| \leq 2$.

Product initial state: If we take the product state $|1,1\rangle)$ to be the initial state in $2 \to 2$ scattering (i.e. $\theta = \pi/2$ in $|\psi\rangle_i = \cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$), instead of the general state with the constraint $|I_{2i}| \leq 2$, then we get the following I_{2f}

$$I_{2f} = I_2^{1,1} = \frac{8\sqrt{2} f_2 g_2 \left(\chi^2 + \chi + 1\right)}{4\left(\chi^2 + \chi + 1\right)^2 f_2^2 + g_2^2} \tag{3.21}$$

For this particular initial state, we observe Bell violation for some scattering angle, given

$$\frac{\sqrt{2}-1}{2} \leq \left|\frac{f_2}{g_2}\right| \leq \frac{2(\sqrt{2}+1)}{3} \quad \approx \quad 0.2071 \leq \left|\frac{f_2}{g_2}\right| \leq 1.6095 \tag{3.22}$$

Interestingly, the QED 1-loop answer for $\left|\frac{f_2}{g_2}\right| \approx 0.2727$ [73, 74, 75] lies inside the above range. Note that a similar exercise was done in [46] by demanding Bell violation for all scattering angles, which leads to a slightly different range for $|f_2/g_2|$.

For $|f_2/g_2|$ outside the above range (3.22), we don't observe Bell violation for any scattering angle because here we have fixed $\theta = \pi/2$ in the initial state. If we allow θ to vary, then we can observe Bell violation for all non-zero f_2 and g_2, as shown above (eq$^{\text{n}}$(3.20) and the following paragraph).

If one performs an experiment with the product initial state and observes Bell violation at some scattering scale, then the constraints in eq^{n}(3.22) must hold true. However, we do not find any clear *theoretical* motivation to demand Bell violation in $2 \to 2$ scattering with product state as the initial state, as explored by [46]. In fact, on the contrary, we will show that for a product initial state, there is no Bell violation at any scattering angle for EFT of gluons (non-abelian), with a weakly coupled UV completion.

3.2.2 EFT for gluons

Now let's consider the following lagrangian containing only the CP conserving operator for EFT of gluons upto dim 6,

$$L = -\frac{1}{4} G^{a}_{\mu\nu} G^{a\mu\nu} + g^{3} \frac{c_1}{\Lambda^2} f^{abc} G^{a\nu}_{\mu} G^{b\rho}_{\nu} G^{c\mu}_{\rho} \tag{3.23}$$

For the above Lagrangian, we get the following helicity amplitudes up to $\mathcal{O}(1/\Lambda^2)$ for process $|p_1, \epsilon_1, a; p_2, \epsilon_2, b\rangle \to |p_3, \epsilon_3, d; p_4, \epsilon_4, e\rangle$

$$\begin{aligned}
\mathcal{M}^{1,1}_{1,1}(s,t,u,a,b,d,e) &= 2g^2 \frac{s}{tu}(f^{acb} f^{dce} u - s f^{ace} f^{bcd}) \qquad (3.24)\\
\mathcal{M}^{0,0}_{1,1}(s,t,u,a,b,d,e) &= -12 g^4 \frac{c_1}{\Lambda^2} f^{acb} f^{dce} \frac{(t^2+u^2)}{(t-u)} - 12 g^4 \frac{c_1 s}{\Lambda^2} \frac{(t f^{acd} f^{bce} - u f^{ace} f^{bcd})}{(t-u)}\\
\mathcal{M}^{1,0}_{1,0}(s,t,u,a,b,d,e) &= 2g^2 \frac{u}{ts}(f^{ace} f^{dcb} s - u f^{acb} f^{ecd})\\
\mathcal{M}^{0,1}_{1,0}(s,t,u,a,b,d,e) &= 2g^2 \frac{t}{us}(f^{acd} f^{bce} u - t f^{ace} f^{dcb})\\
\mathcal{M}^{1,0}_{1,1}(s,t,u,a,b,d,e) &= 6 g^4 \frac{c_1}{\Lambda^2}(u f^{acd} f^{bce} + t f^{ace} f^{bcd})
\end{aligned}$$

where a, b, c, d, and e represent the color of particles and c is summed over.

We sum over the colors of final states, d and e, which makes the final state entangled only in the helicity basis. Then the amplitudes reduce to the following forms,

$$\Phi_1(s,t,u) = -2g^2\frac{s^2}{tu}\left(\sum_{d,e} f^{ace}f^{bcd}\right) \quad ; \quad \Phi_2(s,t,u) = -12g^4\frac{c_1 s}{\Lambda^2}\left(\sum_{d,e} f^{ace}f^{bcd}\right) \tag{3.25}$$
$$\Phi_3(s,t,u) = -2g^2\frac{u}{t}\left(\sum_{d,e} f^{ace}f^{bcd}\right) \quad ; \quad \Phi_4(s,t,u) = -2g^2\frac{t}{u}\left(\sum_{d,e} f^{ace}f^{bcd}\right)$$
$$\Phi_5(s,t,u) = -6g^4\frac{c_1 s}{\Lambda^2}\left(\sum_{d,e} f^{ace}f^{bcd}\right)$$

For the generic initial state, $|\psi\rangle_i = \cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$, we get the following CGLMP Bell parameter corresponding to the final state (details of which have been relegated to the appendix),

$$I_{2f} = \frac{2\sqrt{2}\,(2\Phi_1\Phi_2 + (\Phi_1^2+\Phi_2^2)\sin 2\theta - 2\Phi_5^2(1+\sin 2\theta))}{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2\sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)} \tag{3.26}$$

For the above Φ_1, Φ_2 and Φ_5 we get the following I_{2f},

$$\begin{aligned} I_{2f} &= 2\sqrt{2}\,\frac{48c_1' uts^2 + (4s^4 + 144c_1'^2u^2t^2)\sin 2\theta - 72c_1'^2u^2t^2(1+\sin 2\theta)}{48c_1' s^2ut\sin 2\theta + 4s^4 + 144c_1'^2u^2t^2 + 72c_1'^2u^2t^2(1+\sin 2\theta)} \\ &= 2\sqrt{2}\,\frac{\sin 2\theta - 12c_1'\chi(1+\chi) + 18c_1'^2\chi^2(1+\chi)^2(\sin 2\theta - 1)}{1 - 12c_1'\chi(1+\chi)\sin 2\theta + 18c_1'^2\chi^2(1+\chi)^2(\sin 2\theta + 3)} \end{aligned} \tag{3.27}$$

where $c_1' = g^2c_1s/\Lambda^2$. We observe Bell violation for some χ and θ (with the constraint $|I_{2i}| \leq 2$) given

$$c_1' \in \left(-\infty, -\frac{2\sqrt{2}}{3(3\sqrt{2}-4)}\right) \cup \left(-\frac{2\sqrt{2}}{3(3\sqrt{2}+4)}, \infty\right) \setminus \{0\} \tag{3.28}$$

Since $c_1' = g^2c_1s/\Lambda^2$ and $s < \Lambda^2$ within the validity of the EFT regime, we finally get

$$c_1 \in \mathbb{R} \setminus 0 \tag{3.29}$$

For any c_1 except 0, we can choose appropriate s so that c_1' lies in the range (3.28).Therefore, for all values of c_1 except 0, Bell inequalities can be violated at some scattering angle for some initial state with $|I_{2i}| \leq 2$.

Product initial state: Now, if we take the product state $|1,1\rangle$ to be the initial state, we get the following I_{2f},

$$I_{2f} = -2\sqrt{2}\frac{12c_1'\chi(1+\chi) + 18c_1'^2\chi^2(1+\chi)^2}{1+54c_1'^2\chi^2(1+\chi)^2} \tag{3.30}$$

We observe Bell violation for some χ if,

$$c_1' < \frac{-4\sqrt{2} + 2\sqrt{2(1+\sqrt{2})}}{3(3-\sqrt{2})} \approx -0.265 \tag{3.31}$$

and since $s < \Lambda^2$ and $g \sim \mathcal{O}(1)$, c_1 has to be at least of $\mathcal{O}(1)$. However, the value of c_1 is expected to be of much smaller order for weakly coupled theories; for example, for weakly coupled UV completion with heavy fermions, one typically gets c_1 of $\mathcal{O}(10^{-4})$ as shown in [73]. Thus, we don't expect to observe Bell violation by the non-abelian gauge theory in $2 \to 2$ scattering for the product state as the initial state.

It is interesting that this is qualitatively different from the abelian case of QED, where it is possible to observe Bell violation for the product initial state. This qualitative difference between abelian and non-abelian gauge theory is mainly due to the contribution from the kinetic term of non-abelian gauge theory towards the MHV amplitude. This contribution doesn't allow the cancellation of energy scale Λ in I_{2f}.

This also shows that Bell violation (for product initial state) cannot be promoted to a principle to constrain EFTs and the QED value satisfying the constraint (3.22) is perhaps just a coincidence.

3.2.3 Bell inequality for RF^2

Now we consider $2 \to 2$ scattering of photons, including the graviton exchange. We use the following curvature conventions for the calculations,

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left[\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right] \tag{3.32}$$

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}, \quad R_{\sigma\nu} = R^\rho_{\sigma\rho\nu}, \tag{3.33}$$

Consider the following EFT action for photon coupled to gravity,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{2}{\kappa^2}R - \frac{1}{4}F_{\mu\nu}^2 + \frac{\hat{\alpha}}{4\Lambda^2}R_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma} + \frac{c_1}{\Lambda^4}(F^{\mu\nu}F_{\mu\nu})^2 + \frac{c_2}{\Lambda^4}(F^{\mu\nu}\widetilde{F}_{\mu\nu})^2\right] \tag{3.34}$$

where $\kappa = 2/M_{\rm pl}$. Taking the gravity to be perturbative i.e. $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, we get the following helicity amplitudes at tree level,

$$\begin{aligned}
\mathcal{M}^{1,1}_{1,1}(s,t,u) &= \Phi_1 = g_2 s^2 + \frac{s\kappa^2}{16}\left[\left(\alpha^2 + \frac{4}{ut}\right)\left(s^2+ut\right) + 6\alpha s\right] \\
\mathcal{M}^{0,0}_{1,1}(s,t,u) &= \Phi_2 = f_2(s^2+t^2+u^2) - \frac{\alpha\kappa^2}{16}\left[2\left(s^2+t^2+u^2\right) - 3\alpha stu\right] \\
\mathcal{M}^{1,0}_{1,0}(s,t,u) &= \Phi_5 = -\frac{\alpha\kappa^2}{16}\left[s^2+t^2+u^2+3\alpha stu\right]
\end{aligned} \tag{3.35}$$

where f_2 and g_2 are same as defined in eq$^{\rm n}$(3.17) and $\alpha = \hat{\alpha}/\Lambda^2$.

For the spinor QED UV completion, $\Lambda = m_e$ (mass of electron) and for $s \ll m_e^2$ we have [76]

$$f_2 = \frac{-e^4}{240\pi^2 m_e^4} \quad ; \quad g_2 = \frac{11e^4}{720\pi^2 m_e^4} \quad ; \quad \alpha = \frac{-e^2}{360\pi^2 m_e^2} \tag{3.36}$$

The I_{2f} is same as eq$^{\rm n}$(3.26) for a generic initial state: $\cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$,

$$I_{2f} = \frac{2\sqrt{2}\left(2\Phi_1\Phi_2 + (\Phi_1^2+\Phi_2^2)\sin 2\theta - 2\Phi_5^2(1+\sin 2\theta)\right)}{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2\sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)} \tag{3.37}$$

After calculating I_{2f} for the above amplitudes and Wilson coefficients, we observe Bell violation for all values of $(e^2M_{pl}^2)/m_e^2$ for some scattering angle and θ.

Product initial state: Taking the product state as the initial state, we observe Bell violation if,

$$\frac{eM_{pl}}{m_e} \geq f(s/M_{\rm pl}) \tag{3.38}$$

where f is some function of $s/M_{\rm pl}$ which increases with decreasing $s/M_{\rm pl}$. For example, $f(s/M_{pl}) \sim 195$ for $s = M_{pl}^2$ and it increases to $f(s/M_{pl}) \sim 1946$ for $s = 0.01M_{pl}^2$.

The above constraint (3.38) is similar to the *Weak Gravity Conjecture* (WGC) ($e/m \geq \mathcal{O}(1)/M_{\rm pl}$) [77] for $f \sim \mathcal{O}(1)$, as noted by [46]. However, within the validity of the EFT regime, $s < \Lambda^2 \ll M_{\rm pl}$, f is of a much higher order than $\mathcal{O}(1)$ and therefore cannot be compared to WGC. If we experimentally observe Bell violation for product initial state, then we get much stronger constraints on the charge-to-mass ratio of fermions coupled to photons than imposed by WGC.

3.2.4 Gravity

Consider the following action for gravity, including the corrections to Einstein's gravity

$$S = \int d^4x \sqrt{-g} \frac{2}{\kappa^2} \left(R + \frac{\hat{\beta}}{3!\Lambda^4} R^3 \right) \tag{3.39}$$

where $R^3 \equiv R^{\mu\nu\kappa\lambda} R_{\kappa\lambda\alpha\gamma} R^{\alpha\gamma}{}_{\mu\nu}$.

Again taking gravity to be perturbative $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, we get the following helicity amplitudes for $2 \to 2$ scattering of gravitons [78]

$$\begin{aligned} \mathcal{M}^{1,1}_{1,1}(s,t,u) &= \Phi_1 = \kappa^2 s \left(\frac{s^2}{ut} + \frac{\beta^2}{16} s^2 ut \right) \\ \mathcal{M}^{0,0}_{1,1}(s,t,u) &= \Phi_2 = \frac{5}{4} \kappa^2 \beta stu \\ \mathcal{M}^{1,0}_{1,0}(s,t,u) &= \Phi_5 = \frac{1}{8} \kappa^2 \beta stu \end{aligned} \tag{3.40}$$

where $\beta = \hat{\beta}/\Lambda^4$. For a generic initial state, $\cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$, we have the following form of I_{2f}

$$I_{2f} = \frac{2\sqrt{2}\left(2\Phi_1\Phi_2 + (\Phi_1^2 + \Phi_2^2)\sin 2\theta - 2\Phi_5^2(1+\sin 2\theta)\right)}{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2 \sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)} \tag{3.41}$$

In this case, as well, we observe Bell violation, i.e., $|I_{2f}| > 2$ for all values of the Wilson coefficient $\hat{\beta}$ for some χ and θ (with the constraint $|I_{2i}| \leq 2$).

Product initial state: For the product initial state, we observe Bell violation at some scattering angle if $1.379 < |\beta| s^2 < 46.417$. Since $s < \Lambda^2$ within the validity of EFT regime, $\hat{\beta}$ must be of a much higher order than $\mathcal{O}(1)$. If one considers $s \sim \Lambda^2$, then $\hat{\beta}$ must be of $\mathcal{O}(1)$ to observe Bell violation; however, then one has to consider higher-dimensional operators as their contributions also become significant. This is similar to the case of non-abelian gauge theory in section (3.2.2).

3.3 CP conserving vs CP violating

In this section, we are not trying to probe the quantum nature of theory but to see if we can differentiate between CP-conserving and CP-violating theories using entanglement and $2 \to 2$ scattering. The CP-violating terms in the Lagrangian of a theory give imaginary contributions to the helicity amplitudes which lead to complex coefficients in the final state. Since the CGLMP Bell parameter is not a good measure of entanglement in states with complex coefficients, we will use another parameter, *concurrence* (Δ), for this purpose. Concurrence is defined as $\Delta = 2\ |\mu_{00}\mu_{11} - \mu_{01}\mu_{10}|$ for a normalized state, $|\Psi\rangle = \mu_{mn}\,|m,n\rangle$. The $\Delta = 1$ corresponds to a maximally entangled state, whereas $\Delta = 0$ corresponds to a product state.

We consider $\theta = -\pi/4$ in the initial state $|\psi\rangle_i = \cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$ i.e. it is maximally entangled. We have the following helicity amplitudes after including CP-violating terms in the theory,

$$\begin{pmatrix} \mathcal{M}_{1,0}^{1,0} & \mathcal{M}_{1,0}^{1,1} & \mathcal{M}_{0,0}^{1,0} & \mathcal{M}_{0,0}^{1,1} \\ \mathcal{M}_{1,0}^{0,0} & \mathcal{M}_{1,0}^{0,1} & \mathcal{M}_{0,0}^{0,0} & \mathcal{M}_{0,0}^{0,1} \\ \mathcal{M}_{1,1}^{1,0} & \mathcal{M}_{1,1}^{1,1} & \mathcal{M}_{0,1}^{1,0} & \mathcal{M}_{0,1}^{1,1} \\ \mathcal{M}_{1,1}^{0,0} & \mathcal{M}_{1,1}^{0,1} & \mathcal{M}_{0,1}^{0,0} & \mathcal{M}_{0,1}^{0,1} \end{pmatrix} = \begin{pmatrix} \Phi_3 & \Phi_5^* & \Phi_5^* & \Phi_2^* \\ \Phi_5 & \Phi_4 & \Phi_1^* & \Phi_5^* \\ \Phi_5 & \Phi_1 & \Phi_4 & \Phi_5^* \\ \Phi_2 & \Phi_5 & \Phi_5 & \Phi_3 \end{pmatrix}$$

Then we get the following final scattered state for the maximally entangled initial state,

$$|\Psi\rangle = (\Phi_1^* - \Phi_2)\,|0,0\rangle + (\Phi_2^* - \Phi_1)\,|1,1\rangle + (\Phi_5^* - \Phi_5)\,(|0,1\rangle + |1,0\rangle) \tag{3.42}$$

which has the concurrence (Δ_f),

$$\Delta_f = \frac{\left||\Phi_1^* - \Phi_2|^2 + (\Phi_5 - \Phi_5^*)^2\right|}{|\Phi_1^* - \Phi_2|^2 + |\Phi_5 - \Phi_5^*|^2} = \frac{\left||\Phi_1^* - \Phi_2|^2 - 4(\mathrm{Im}\Phi_5)^2\right|}{|\Phi_1^* - \Phi_2|^2 + 4(\mathrm{Im}\Phi_5)^2} \tag{3.43}$$

In the case of CP conserving theories, we have real amplitudes; therefore, $\Phi_5 = \Phi_5^*$ and $\Delta = 1$, i.e. the final state is maximally entangled. However, in general, we can have $\Delta < 1$ for the CP-violating theories. Therefore, if one observes a non-maximally entangled final state ($\Delta < 1$) in $2 \to 2$ scattering with $|\psi\rangle_i = (|0,0\rangle - |1,1\rangle)/\sqrt{2}$ as the initial state, then the theory has CP-violating contributions. Note that this statement is true about the full theory since we have used general amplitudes which are not limited to a certain order in Λ.

Chapter 4

Summary

4.1 Summary for causality and unitarity cosntraints

We have derived constraints on dim 6 and dim 8 gluon field strength operators of SMEFT using both the amplitude and superluminality analysis. This significantly reduces the parameter space of the Wilson coefficients. Interestingly, these bounds imply that dim 6 operators can only exist in the presence of certain dim 8 operators.
The amplitude analysis filters out terms growing as even power of s, s^{2n} where $n \geq 1$ ($n = 1$ in our case). It is because of this filtering property that one is not able to put any bounds on the Wilson coefficients of dim 6 operators comprising four fields, e.g. $c\phi^2\partial_\mu\phi\partial^\mu\phi$ as their contribution to the tree level scattering amplitude (forward limit) grow as s. But for dim 6 operators comprising of three fields, like some terms in $Q^{(1)}_{G^3}$ and $Q^{(2)}_{G^3}$, one indeed gets an s^2 dependence at tree level due to exchange diagrams. It is this feature that allowed us to put constraints on the square of the Wilson coefficients of dim 6 gluon field strength operators in SMEFT. We obtained constraints on the magnitude of dim 6 operators' Wilson coefficients in terms of those of dim 8 operators. In appendix A.2, we have given another example of a dim 6 operator (containing 3 fields) whose magnitude can be constrained in terms of dim 8 operators.

In the context of superluminality analysis, we have mentioned the subtleties involving the relation between low-frequency phase velocity and causality. We showed, in the case of chiral Lagrangian, that it is unclear if one gets a strict positivity bound from superluminality when the pion is considered to be massive. The reason for this is that the superluminality analysis takes into account the contribution from operators of all dimensions (unlike the 'amplitude analysis'). As we have argued in our work, the contribution of the dimension 4 operator (other than the kinetic term) to the dispersion relation of the perturbation makes it unclear if one can use the phase velocity to dictate the superluminality of the perturbation. However, in the case of the gluon field strength operators, we managed to get rid of the mass-like terms in the dispersion relation by choosing specific background and polarization of the perturbation. This was possible because the mass-like term entirely depended on four-vector contractions which could be made zero despite having a non-zero field. We showed that interestingly and in a non-trivial way, the superluminality analysis for gluonic operators in the SMEFT reproduces all the bounds obtained from the 'amplitude analysis'.

The above discussion might give the impression that the amplitude analysis always gives similar or stronger bounds than the superluminality analysis. However, this is not always true. In sec. 2.2.3, following [24], we showed an example of a non-relativistic theory (motivated by the EFT of inflation [67]) that the superluminality gives stronger constraints than the 'amplitude analysis' in this case. Thus, we conclude that it is not clear which of the two analyses will give stronger bounds for a particular theory. Hence, ideally, one should perform both analyses in order to obtain the maximum amount of constraints on an IR effective theory.

4.2 Summary for Bell violation in $2 \to 2$ scattering

We explored Bell violation for $2 \to 2$ scattering of photons, gluons and gravitons in the context of EFTs using the CGLMP Bell parameter as the measure of entanglement. We considered the initial state to be entangled in the Hilbert space spanned by the helicity basis, such that the degree of entanglement can be described by a local hidden-variable theory. This condition on the initial state can be described as the relation $|I_{2i}| \leq 2$, where the I_{2i} represents the CGLMP parameter. With this particular choice of the initial state, the Bell inequality for the final state can be violated only due to the quantum nature of the scattering amplitudes, which is dictated by the theory in consideration.

We showed that starting from an appropriate initial state, $2 \to 2$ scattering of photons, gluons, and gravitons could violate the Bell inequality (at least for some scattering angle) for any non-zero value of CP-conserving higher dimensional operators in the corresponding EFTs.

If one considers the initial state to be a product state, which is experimentally easier to prepare, and observes Bell violation at some scattering angle, then the EFT parameter space can be constrained. This was also shown by [46] for the QED case. However, a priori, one can not use this as a principle to constrain the EFTs as we have explicitly shown using the example of the EFT of gluons. In the cases of EFTs for gluons, gravity, and photons including gravity, we observe Bell violation for the initial product state (say $|1,1\rangle$) if the Wilson coefficients of higher dimensional operators are of at least $\mathcal{O}(1)$. In all these cases, the leading operator (4-dim operator) contributes only to the MHV amplitude ($\Phi_1 = \mathcal{M}^{1,1}_{1,1}$), therefore one basis ($|1,1\rangle$) has a significantly higher weight than the other ($|0,0\rangle$) (even after Schmidt decomposition). However, for a significant degree of entanglement in the final state ($I_{2f} > 2$), we need the weights of both bases to be comparable. Thus, the Wilson coefficient must be at least $\mathcal{O}(1)$, so that both the bases have comparable weights. It is interesting that the non-abelian gauge theory (and gravity) is qualitatively different from the abelian gauge theory even from the point of view of Bell violation.

We have also shown that if we consider the initial state in $2 \to 2$ scattering to be a particular maximally entangled state, then we can probe the CP-violating nature of the theory using the degree of entanglement in final states.

In this work, we have explored the possibility of Bell violation by the unitary evolution of qubits for different EFTs using $2 \to 2$ scattering. There is still much to explore on the relationship between entanglement and EFTs. It would be interesting to explore Bell violation by states entangled with respect to quantum numbers other than helicity, like colors for gluons, and if it can restrict the EFT parameter space. It would also be interesting to investigate whether our results hold true even after considering more higher-dimensional operators in the EFT, like dim 8 operators in the EFT of gluons.

Appendix A

Subluminality

A.1 Details of the subluminality analysis for gluons

- For operator $Q_{G^4}^{(1)} = \left(G^a_{\mu\nu}G^{a\mu\nu}\right)\left(G^b_{\rho\sigma}G^{b\rho\sigma}\right)$ we have,

$$L = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \frac{c_8^{(1)}}{\Lambda^4}\left(G^a_{\mu\nu}G^{a\mu\nu}\right)\left(G^b_{\rho\sigma}G^{b\rho\sigma}\right)$$

Applying Euler's Lagrange equation,

$$\partial_\alpha \frac{\partial L}{\partial\left(\partial_\alpha A^f_\beta\right)} - \frac{\partial L}{\partial A^f_\beta} = 0$$

for the above Lagrangian, we get EOM as,

$$\begin{aligned} &-\partial_\alpha G^{f,\alpha\beta} + \frac{8c_8^{(1)}}{\Lambda^4}\left(2G^a_{\mu\nu}(\partial_\alpha G^{a,\mu\nu})G^{f,\alpha\beta} + G^a_{\mu\nu}G^{a,\mu\nu}\partial_\alpha G^{f,\alpha\beta}\right) \\ =&-g_s G^{a,\beta\nu}A^h_\nu f^{afh} + \frac{8c_8^{(1)}}{\Lambda^4}g_s f^{bfj}G^a_{\mu\nu}G^{a,\mu\nu}G^{b,\beta\sigma}A^j_\sigma \end{aligned} \tag{A.1}$$

We expand $A^{a,\mu} = A_0^{a,\mu} + h^{a,\mu}$ where we *choose* background $A_0^{a,\mu}$ of particular color 'a' having $\partial^\nu A^{a,\mu} = constant$, and *look* at the linearised equation of motion for perturbation $h^{a,\mu}$ also of same color 'a', then we get

$$-\partial_\alpha H^{f,\alpha\beta} + \frac{8c_8^{(1)}}{\Lambda^4}\Big(2G^a_{0\mu\nu}\big(\partial_\alpha H^{a,\mu\nu} + g_s f^{a1c} A_0^{1,\mu} h^{c,\nu} + g_s f^{ac1} A_0^{1,\nu} h^{c,\mu}\big) G_0^{f,\alpha\beta} + G^a_{0\mu\nu} G_0^{a,\mu\nu} \partial_\alpha H^{f,\alpha\beta}\Big) = 0 \tag{A.2}$$

where f=a, $G^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_s f^{abc} A^b_{0\mu} A^c_{0\nu}$ and $H^a_{\mu\nu} = \partial_\mu h^a_\nu - \partial_\nu h^a_\mu$
since A^a_0 is non zero only for particular 'a', $G^a_{0\mu\nu} = F^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu}$ and is also non-zero only for color index= 'a'. From now we'll stop writing color index for background for convenience as it is fixed to be 'a'.

$$-\partial_\alpha H^{a,\alpha\beta} + \frac{8c_8^{(1)}}{\Lambda^4}\Big(2F_{0\mu\nu}\partial_\alpha H^{a,\mu\nu} F_0^{\alpha\beta} + F_{0\mu\nu} F_0^{\mu\nu} \partial_\alpha H^{a,\alpha\beta}\Big) = 0 \tag{A.3}$$

Since we are working in Lorentz gauge, $\partial_\alpha h^\alpha = 0$, then writing $H^{a,\mu\nu}$ explicitly

$$-\partial_\alpha \partial^\alpha h^{a,\beta}\left(1 - \frac{8c_8^{(1)}}{\Lambda^4} F_{0\mu\nu} F_0^{\mu\nu}\right) + \frac{32c_8^{(1)}}{\Lambda^4} F_{0\mu\nu} F_0^{\alpha\beta} \partial_\alpha \partial^\mu h^{a,\nu} = 0 \tag{A.4}$$

Rescaling $h^{a,\mu} \rightarrow \dfrac{h^{a,\mu}}{\left(1 - \frac{8c_8^{(1)}}{\Lambda^4} F_{0\mu\nu} F_0^{\mu\nu}\right)}$ and considering terms only up to $\mathcal{O}\left(\dfrac{1}{\Lambda^4}\right)$,

$$-\partial_\alpha \partial^\alpha h^{a,\beta} + \frac{32c_8^{(1)}}{\Lambda^4} F_{0\mu\nu} F_0^{\alpha\beta} \partial_\alpha \partial^\mu h^{a,\nu} = 0 \tag{A.5}$$

Taking the Fourier transform and multiplying the eq$^{\text{n}}$ by normalized polarization of perturbation: ϵ_β,

$$k^2 \cdot \epsilon_\beta \tilde{h}^{a,\beta} = \frac{32c_8^{(1)}}{\Lambda^4} F_{0\mu\nu} F_0^{\alpha\beta} \epsilon_\beta k_\alpha k^\mu \tilde{h}^{a,\nu} \tag{A.6}$$

Also, we can write $\tilde{h}^{a,\nu} = -\epsilon^\nu \, \tilde{h}^{a,\rho} \epsilon_\rho$ considering polarization to be transverse and therefore having components only in spatial direction ($\Rightarrow \epsilon_\nu \epsilon^\nu = -1$)

$$k^2 = -\frac{32c_8^{(1)}}{\Lambda^4} (F_{0\mu\nu} \epsilon^\nu k^\mu)^2 \tag{A.7}$$

Doing the above calculation for other operators gives eq^{n}(2.27).

- For operator $Q_{G^3}^{(1)} = f^{abc} G_\mu^{a\nu} G_\nu^{b\rho} G_\rho^{c\mu}$,

$$L = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \frac{c_6}{\Lambda^2} f^{abc} G_\mu^{a\nu} G_\nu^{b\rho} G_\rho^{c\mu}$$

we get EOM as,

$$-\partial_\alpha G^{f,\alpha\beta} + \frac{6c_6}{\Lambda^2} f^{fbc} \partial_\alpha \left(G^{c,\rho\alpha} G^{b,\beta}{}_\rho \right) = -g_s G^{a,\beta\nu} A^h_\nu f^{afh} + \frac{6c_6}{\Lambda^2} g_s f^{abc} \left(f^{afh} G^{b,\nu\rho} G_\rho^{c,\beta} A^h_\nu \right) \tag{A.8}$$

We again expand $A^{a,\mu} = A_0^{a,\mu} + h^{a,\mu}$ where we *choose* background $A_0^{a,\mu}$ of particular color 'a' with $\partial^\nu A^{a,\mu} = constant$, and *look* at the linearised equation of motion for perturbation $h^{a,\mu}$ of color f.

Then EOM takes the following form for color f,

$$\begin{aligned}
&- \partial_\alpha \left(H^{f,\alpha\beta} + g_s f^{faj} A_0^{a,\alpha} h^{j,\beta} + g_s f^{fia} h^{i,\alpha} A_0^{a,\beta} \right) \\
&+ \frac{6c_6}{\Lambda^2} f^{fba} \partial_\alpha \Big(F_0^{a,\rho\alpha} (H^{b,\beta}{}_\rho + g_s f^{baj} A_0^{a,\beta} h^j_\rho + g_s f^{bia} h^{i,\beta} A^a_{0\rho}) \\
&\qquad\qquad - F_0^{a,\beta\rho} (H_\rho^{b,\alpha} + g_s f^{baj} A^a_{0\rho} h^{j,\alpha} + g_s f^{bia} h^i_\rho A_0^{a,\alpha}) \Big) \\
&= -g_s \left(H^{d,\beta\nu} + g_s f^{daj} A_0^{a,\beta} h^{j,\nu} + g_s f^{dia} h^{i,\beta} A_0^{a,\nu} \right) A^a_{0\nu} f^{dfa} - g_s F_0^{a,\beta\nu} h^h_\nu f^{afh} \\
&+ \frac{6c_6}{\Lambda^2} g_s f^{daa} \left(f^{dfh} F_0^{a,\nu\rho} F_{0\rho}^{a,\beta} h^h_\nu \right) \\
&+ \frac{6c_6}{\Lambda^2} g_s f^{dba} \left(f^{dfa} (H^{b,\nu\rho} + g_s f^{baj} A_0^{a,\nu} h^{j,\rho} + g_s f^{bia} h^{i,\nu} A_0^{a,\rho}) F_{0\rho}^{a,\beta} A^a_{0\nu} \right) \\
&+ \frac{6c_6}{\Lambda^2} g_s f^{dac} \left(f^{dfa} F_0^{a,\nu\rho} (H_\rho^{c,\beta} + g_s f^{caj} A^a_{0\rho} h^{j,\beta} + g_s f^{cia} h^i_\rho A_0^{a,\beta}) A^a_{0\nu} \right)
\end{aligned} \tag{A.9}$$

For this, we work in WKB approximation where the scale of variation of background(r) is much larger than that of perturbations (ω^{-1}), then at order g_s, g_s^2, $1/\Lambda^2$ and leading order of rω in each of them,

$$\begin{aligned}
&-\partial_\alpha\left(H^{f,\alpha\beta}+g_s f^{faj}A_0^{a,\alpha}h^{j,\beta}+g_s f^{fia}h^{i,\alpha}A_0^{a,\beta}\right) \\
&+\frac{6c_6}{\Lambda^2}f^{fba}\partial_\alpha\Big(F_0^{a,\rho\alpha}(H^{b,\beta}{}_\rho+g_s f^{baj}A_0^{a,\beta}h^j_\rho+g_s f^{bia}h^{i,\beta}A^a_{0\rho}) \\
&\qquad\qquad -F_0^{a,\beta\rho}(H^{b,\alpha}_\rho+g_s f^{baj}A^a_{0\rho}h^{j,\alpha}+g_s f^{bia}h^i_\rho A_0^{a,\alpha})\Big) \\
&=-g_s\left(H^{d,\beta\nu}+g_s f^{daj}A_0^{a,\beta}h^{j,\nu}+g_s f^{dia}h^{i,\beta}A_0^{a,\nu}\right)A^a_{0\nu}f^{dfa} \\
&+\frac{6c_6}{\Lambda^2}g_s f^{dba}f^{dfa}H^{b,\nu\rho}F^{a,\beta}_{0\rho}A^a_{0\nu}+\frac{6c_6}{\Lambda^2}g_s f^{dac}f^{dfa}F_0^{a,\nu\rho}H^{c,\beta}_\rho A^a_{0\nu}
\end{aligned} \tag{A.10}$$

In above eq$^{\text{n}}$ since we don't know the relative order of g_s and $1/\Lambda^2$, we ignore only those terms which are definitely of less order than g_s, g_s^2, $1/\Lambda^2$ like g_s^2/Λ^2. Also, in the above equation and all further equations 'f' and 'a' are not contracted but are considered particular color indices and we'll also drop 'a' from background terms.

Lorentz gauge, $\partial_\alpha h^\alpha=0$ implies

$$\begin{aligned}
&-\partial_\alpha\partial_\alpha h^{f,\beta}-2g_s f^{faj}A_0^{a,\alpha}\partial_\alpha h^{j,\beta}+g_s f^{faj}A_{0\nu}\partial^\beta h^{j,\nu} \\
&+g_s^2 f^{dfa}f^{daj}\left(A_0^\beta h^{j,\nu}-h^{j,\beta}A_0^\nu\right)A_0^\nu \\
&+\frac{6c_6}{\Lambda^2}f^{fba}\Big(F_{0\rho\alpha}\partial^\alpha\partial^\beta h^{b,\rho}+F_0^{\beta\rho}\partial_\alpha\partial^\alpha h^b_\rho\Big) \\
&+\frac{6c_6}{\Lambda^2}g_s f^{fba}\Big(f^{baj}A_0^\beta F_0^{\rho\alpha}\partial_\alpha h^j_\rho-f^{baj}F_0^{\rho\alpha}A_{0\rho}\partial_\alpha h^{j,\beta} \\
&\qquad\qquad +f^{baj}A_0^\alpha F_0^{\beta\rho}\partial_\alpha h^f_\rho+f^{baj}H^{j,\nu\rho}F^{\beta}_{0\,\rho}A_{0\nu}+f^{bac}F_0^{\nu\rho}H^{c,\beta}_\rho A_{0\nu}\Big)=0
\end{aligned} \tag{A.11}$$

We can also choose the amplitude ($\max|A_0(x^\mu)|$) of the background to be arbitrary small without affecting other quantities, which would make the terms of order A_0^2 less relevant in comparison to terms of lower order,

$$\begin{aligned}
&-\partial_\alpha\partial_\alpha h^{f,\beta}+2g_s f^{fba}A_0^{a,\alpha}\partial_\alpha h^{b,\beta}-g_s f^{fba}A_{0\rho}\partial^\beta h^{b,\rho} \\
&+\frac{6c_6}{\Lambda^2}f^{fba}\Big(F_{0\rho\alpha}\partial^\alpha\partial^\beta h^{b,\rho}+F_0^{\beta\rho}\partial_\alpha\partial^\alpha h^b_\rho\Big)=0
\end{aligned} \tag{A.12}$$

Now for also some other color 'b' we'll have similar wave equation,

$$
\begin{aligned}
&-\partial^\alpha\partial_\alpha h^{b,\rho}+2g_s f^{bga}A_0^{a,\alpha}\partial_\alpha h^{g,\rho}-g_s f^{bga}A_{0\sigma}\partial^\rho h^{g,\sigma} \\
&+\frac{6c_6}{\Lambda^2}f^{bga}\left(F_0^{\sigma\alpha}\partial_\alpha\partial^\rho h_\sigma^g+F_0^{\rho\sigma}\partial^\alpha\partial_\alpha h_\sigma^g\right)=0
\end{aligned}
\tag{A.13}
$$

We consider the following solution of (A.13),

$$
-\partial^\alpha h^{b,\rho}+2g_s f^{bga}A_0^{a,\alpha}h^{g,\rho}-g_s f^{bga}A_{0\nu}\delta^{\alpha\rho}h^{g,\nu}+\frac{6c_6}{\Lambda^2}f^{bga}\left(F_0^{\sigma\alpha}\partial^\rho h_\sigma^g+F_0^{\rho\sigma}\partial^\alpha h_\sigma^g\right)=0
\tag{A.14}
$$

and substitute in (A.12),

$$
\begin{aligned}
-\partial^\alpha\partial_\alpha h^{f,\beta}=&2g_s^2 f^{fba}f^{gba}\left(2A_0^\alpha A_{0\alpha}h^{g,\beta}-\frac{3}{2}A_{0\nu}A_0^\beta h^{g,\nu}\right) \\
&+g\frac{6c_6}{\Lambda^2}f^{fba}f^{gba}\left(5F_0^{\beta\rho}A_0^\alpha\partial_\alpha h^{g,\rho}-A_{0\nu}F_0^{\rho\beta}\partial^\rho h^{g,\nu}+5F_{0\rho\alpha}A_0^\alpha\partial^\beta h^{g,\rho}\right) \\
&+36\frac{c_6^2}{\Lambda^4}f^{fba}f^{gba}\left(F_0^{\beta\rho}F_0^{\sigma\alpha}\partial_\alpha\partial_\rho h_\sigma^g+F_{0\rho}^\beta F_0^{\rho\sigma}\partial^\alpha\partial_\alpha h_\sigma^g+2F_0^{\rho\alpha}F_0^{\sigma\alpha}\partial^\beta\partial_\rho h_\sigma^g\right)
\end{aligned}
\tag{A.15}
$$

Now we try to get the differential equation just in terms of perturbation of color 'f' assuming that we can choose a particular background A_0 such that mass term vanishes. Then in the above equation in second and third terms of r.h.s, we write h^g in terms of other colors. When $g\neq f$ we get terms of higher order in A_0 or $\mathcal{O}\left(\frac{1}{\Lambda^4}\right)$ from (A.13), therefore only $g=f$ survives at $\mathcal{O}\left(\frac{1}{\Lambda^4}\right)$ and leading order in A_0.

$$
\begin{aligned}
-\partial^\alpha\partial_\alpha h^{f,\beta}=&g\frac{6c_6}{\Lambda^2}f^{fba}f^{fba}\left(5F_0^{\beta\rho}A_0^\alpha\partial_\alpha h^{f,\rho}-A_{0\nu}F_0^{\rho\beta}\partial^\rho h^{f,\nu}+5F_{0\rho\alpha}A_0^\alpha\partial^\beta h^{f,\rho}\right) \\
&+36\frac{c_6^2}{\Lambda^4}f^{fba}f^{fba}\left(F_0^{\beta\rho}F_0^{\sigma\alpha}\partial_\alpha\partial_\rho h_\sigma^f+F_{0\rho}^\beta F_0^{\rho\sigma}\partial^\alpha\partial_\alpha h_\sigma^f+2F_0^{\rho\alpha}F_0^{\sigma\alpha}\partial^\beta\partial_\rho h_\sigma^f\right)
\end{aligned}
$$

Taking the Fourier transform and multiplying by normalized polarization of perturbation color 'f' ϵ_β,

$$
\begin{aligned}
k^\mu k_\mu(\epsilon_\beta\widetilde{h}^{f,\beta})=&ig\frac{6c_6}{\Lambda^2}f^{fba}f^{fba}\epsilon_\beta\left(5F_0^{\beta\rho}A_0^\alpha k_\alpha\widetilde{h}^{f,\rho}-A_{0\nu}F_0^{\rho\beta}k^\rho\widetilde{h}^{f,\nu}+5F_{0\rho\alpha}A_0^\alpha k^\beta\widetilde{h}^{f,\rho}\right) \\
&-36\frac{c_6^2}{\Lambda^4}f^{fba}f^{fba}\epsilon_\beta\left(F_0^{\beta\rho}F_0^{\sigma\alpha}k_\alpha k_\rho\widetilde{h}_\sigma^f+F_{0\rho}^\beta F_0^{\rho\sigma}k^\alpha k_\alpha\widetilde{h}_\sigma^f+2F_0^{\rho\alpha}F_0^{\sigma\alpha}k^\beta k_\rho\widetilde{h}_\sigma^f\right)
\end{aligned}
\tag{A.16}
$$

We consider polarization to be transverse and since we can write $\tilde{h}^{f,\nu} = -\epsilon^{\nu}\,\tilde{h}^{f,\rho}\epsilon_{\rho}$, we get

$$k^{\mu}k_{\mu}\left(1-36\frac{c_6^2}{\Lambda^4}(f^{fba})^2F^{\beta}_{0\rho}F_0^{\rho\sigma}\epsilon_{\beta}\epsilon_{\sigma}\right) = -ig\frac{6c_6}{\Lambda^2}(f^{fba})^2\epsilon_{\beta}\left(5F_0^{\beta\rho}A_0^{\alpha}k_{\alpha}\epsilon^{\rho} - A_{0\nu}F_0^{\rho\beta}k^{\rho}\epsilon^{\nu}\right) \tag{A.17}$$
$$+36\frac{c_6^2}{\Lambda^4}(f^{fba})^2\left(F_0^{\beta\rho}F_0^{\sigma\alpha}k_{\alpha}k_{\rho}\epsilon_{\beta}\epsilon_{\sigma}\right)$$

Considering k^{μ} to be complex in general then for the real part of dispersion relation at leading order we get,

$$k^{\mu}k_{\mu} = 36\frac{c_6^2}{\Lambda^4}(f^{fba})^2\left(F_0^{\beta\rho}k_{\rho}\epsilon_{\beta}\right)^2 \tag{A.18}$$

After considering all dim 6 and dim 8 operators we'll get following dispersion relation,

$$\begin{aligned}\frac{k_{\mu}k^{\mu}}{4} =& \frac{9}{\Lambda^4}f^{afc}f^{af}{}_{c}\Big[c_6(F_0^{\mu\nu}k_{\mu}\epsilon_{\nu}) - c_6'(\widetilde{F}_0^{\alpha\beta}k_{\alpha}\epsilon_{\beta})\Big]^2 \\ &- \frac{4}{\Lambda^4}\Big[(2\delta^{af}c_8^{(1)} + (1+\delta^{af})c_8^{(3)} + 2d^{afc}d^{af}{}_{c}c_8^{(7)})\,(F_0^{\mu\nu}k_{\mu}\epsilon_{\nu})^2 \\ &\quad + (2\delta^{af}c_8^{(2)} + (1+\delta^{af})c_8^{(4)} + 2d^{afc}d^{af}{}_{c}c_8^{(8)})\left(\widetilde{F}_0^{\alpha\beta}k_{\alpha}\epsilon_{\beta}\right)^2 \\ &\quad - (2\delta^{af}c_8^{(5)} + (1+\delta^{af})c_8^{(6)} + 2d^{afc}d^{af}{}_{c}c_8^{(9)})\,(F_0^{\mu\nu}k_{\mu}\epsilon_{\nu})\left(\widetilde{F}_0^{\alpha\beta}k_{\alpha}\epsilon_{\beta}\right)\Big]\end{aligned} \tag{A.19}$$

where 'f' denotes the color of perturbation and 'a' of the background; given the mass term in (A.15) vanish.
Consider the perturbation with polarization $\epsilon = \{0,1,1,0\}/\sqrt{2}$ and choose background of the form $A_{0\mu} = E\{\sqrt{2}(x+y), x+y, -(x+y), 0\}$ where E is some arbitrary small constant. Under this configuration, we have $A_{0\mu}A_0^{\mu} = 0$ and $\epsilon^{\mu}A_{0\mu} = 0$ i.e. mass-like term in eq$^{\text{n}}$(A.15) vanish. For the chosen background, we get the following non-zero components of $F_{\mu\nu}$, $F_{01} = -F_{10}$, $F_{02} = -F_{20}$ and $F_{12} = -F_{21}$ with $F_{01} = F_{02} = F_{12}/\sqrt{2}$, which reduces the dispersion relation (A.19) to the following form,

$$k_{\mu}k^{\mu} = \frac{72}{\Lambda^4}f^{afc}f^{af}{}_{c}(c_6)^2\,(\omega F_{01})^2 - \frac{32}{\Lambda^4}\left(c_8^{(3)} + 2d^{afc}d^{af}{}_{c}c_8^{(7)}\right)(\omega F_{01})^2 \tag{A.20}$$

then by demanding perturbation to be causal, we get

$$9f^{afc}f^{af}{}_{c}(c_6)^2 - 4\left(c_8^{(3)} + 2d^{afc}d^{af}{}_{c}c_8^{(7)}\right) < 0 \tag{A.21}$$

Choosing different combinations of colors for perturbation and background leads to constraints C(1,1), C(2,1), C(3,1), and C(4,1). Now, if we choose the background, $A_{0\mu} = E\{\sqrt{2(D+B)}t, (\sqrt{D}+\sqrt{B})t, (\sqrt{B}-\sqrt{D})t, 0\}$ where,
$D = 4(2\delta^{ab}c_8^{(1)} + (1+\delta^{ab})c_8^{(3)} + 2d^{abc}d^{ab}{}_{c}c_8^{(7)}) - 9f^{abc}f^{ab}{}_{c}c_6^2$;
$B = 4(2\delta^{ab}c_8^{(2)} + (1+\delta^{ab})c_8^{(4)} + 2d^{abc}d^{ab}{}_{c}c_8^{(8)}) - 9f^{abc}f^{ab}{}_{c}c_6'^2$
with perturbation of the same polarization as before, then the subluminality condition gives the following constraint,

$$-2\sqrt{DB} < 4(2\delta^{ab}c_8^{(5)} + (1+\delta^{ab})c_8^{(6)} + 2d^{abc}d^{ab}{}_{c}c_8^{(9)}) - 18f^{abc}f^{ab}{}_{c}c_6c_6' \tag{A.22}$$

Similarly by choosing the background to be
$A_{0\mu} = E\{\sqrt{2(D+B)}t, (\sqrt{D}+\sqrt{B})t, (\sqrt{D}-\sqrt{B})t, 0\}$ along with the polarization $\epsilon = \{0, 1, -1, 0\}/\sqrt{2}$, we get the following constraint

$$2\sqrt{DB} > 4(2\delta^{ab}c_8^{(5)} + (1+\delta^{ab})c_8^{(6)} + 2d^{abc}d^{ab}{}_{c}c_8^{(9)}) - 18f^{abc}f^{ab}{}_{c}c_6c_6' \tag{A.23}$$

Combining the above two constraints we can reproduce C(1,3), C(2,3), C(3,3) and C(4,3).

A.2 An example with Scalar and Fermion

In this appendix, we consider another example of operators of dimensions 6 and 8 which, when subjected to the 'amplitude analysis', get relative bounds on their Wilson coefficients. Consider the following Lagrangian,

$$L^{(6)} = \frac{c_6}{\Lambda^2}\phi\partial_\mu\bar{\Psi}\partial^\mu\Psi \ ; \quad L^{(8)} = i\frac{c_8}{\Lambda^4}(\partial^\mu\partial_\nu\phi\partial_\mu\bar{\Psi}\gamma^\nu\Psi\phi - \partial^\mu\partial_\nu\phi\bar{\Psi}\gamma^\nu\partial_\mu\Psi\phi)$$

where Ψ represents a fermionic field and ϕ is a real scalar field. The second term in $L^{(8)}$ has to be present for it to be hermitian. Note that, the operator $L^{(6)}$ (with ϕ identified as the Higgs doublet, and the normal derivatives replaced by appropriate covariant derivatives) can be written in terms of a linear combination of SMEFT operators of the Warsaw basis up to total derivatives using the EOM (see eq$^{\mathrm{n}}$(6.4) of [55]).

We calculate $2 \rightarrow 2$ scattering amplitude with two scalars and fermions of positive helicities, $\mathcal{M}(\phi f_2^+ \rightarrow \phi f_4^+)$, at tree level up to $\mathcal{O}\left(\frac{1}{\Lambda^4}\right)$. Since the dimension 6 operator considered has **3** fields, we expect to get a contribution scaling like $c_6^2 s^2$ in the amplitude $\mathcal{M}(s,t)$.

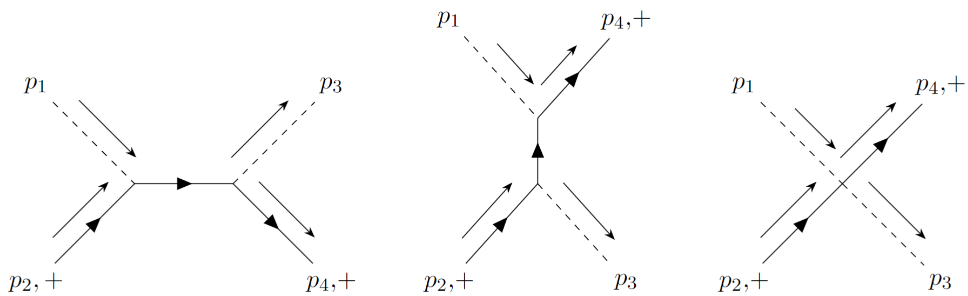

Figure A.1: The first two exchange diagrams represent s and u-channel contributions, and get contribution from $L^{(6)}$. The third contact diagram gets contribution from $L^{(8)}$.

The tree level amplitude gets contribution from the Feynman diagrams in figure A.1 and is given by:

$$\begin{aligned}\mathcal{M}(\phi f_2^+ \phi f_4^+) = &-\frac{c_6^2}{\Lambda^4}\left\{\overline{u}_+(p_4)(\not{p}_1+\not{p}_2)v_-(p_2)\frac{s}{4}+\overline{u}_+(p_4)(\not{p}_2-\not{p}_3)v_-(p_2)\frac{u}{4}\right\} \\ &+\frac{c_8}{\Lambda^4}\Big\{-\overline{u}_+(p_4)\not{p}_1 v_-(p_2)\frac{u}{2}+\overline{u}_+(p_4)\not{p}_1 v_-(p_2)\frac{s}{2} \\ &\qquad +\overline{u}_+(p_4)\not{p}_3 v_-(p_2)\frac{s}{2}-\overline{u}_+(p_4)\not{p}_3 v_-(p_2)\frac{u}{2}\Big\}\end{aligned} \tag{A.24}$$

which using spinor helicity formalism (for detailed introduction check [79]) can be written as:

$$\begin{aligned}\mathcal{M}(\phi f_2^+ \phi f_4^+) = &\frac{c_8}{\Lambda^4}\Big\{\frac{-u}{2}[41]\langle 12\rangle+\frac{s}{2}[41]\langle 12\rangle+\frac{s}{2}[43]\langle 32\rangle-\frac{u}{2}[43]\langle 32\rangle\Big\} \\ &-\frac{c_6^2}{\Lambda^4}\Big\{[41]\langle 12\rangle\frac{s}{4}-[43]\langle 32\rangle\frac{u}{4}\Big\}\end{aligned} \tag{A.25}$$

We now take the forward limit to get $\mathcal{A}(s) = \mathcal{M}(s,t)|_{t\to 0} = \frac{s^2}{\Lambda^4}\left(2c_8 - \frac{c_6^2}{2}\right)$. We don't have to worry about t-channel pole divergence since the t-channel doesn't exist for the process considered. From positivity condition discussed in sec. 2.1.1, we get

$$4c_8 > c_6^2$$

This puts an upper bound on the *magnitude* of c_6 in terms of c_8 similar to what we obtained for the gluonic operators. It also implies that the 6-dimensional operator that we have considered in this example cannot exist on its own, it needs some other operator which gives a positive contribution proportional to s^2 in $\mathcal{A}(s)$ to survive.
One might not have expected to get s^2 dependence from exchange diagrams as there are only two derivatives present in $L^{(6)}$ (unlike the gluonic case which has three derivatives). However, the fermion propagator has $1/p$ dependence instead of the $1/p^2$ dependence for gluons, and more importantly, spinors $\overline{u}$ and v have implicit momentum factors. These momentum factors, in our case, manifest themselves in the form of Mandelstam variables once we take forward limit, leading to s^2 dependence of exchange diagrams.

A.3 The arc variable

In sec 3.2, we derived the constraints by calculating the residue at $\lim_{m^2 \to 0}(s \sim m^2) \to 0$. However, since QCD is confined at low energies it would be preferable to employ a method that circumvents the need to calculate the residue at $s \sim 0$.

To do this, one can define the arc variable [80]

$$a(s) \equiv \int_{\cap_s} \frac{ds'}{\pi i} \frac{\mathcal{M}(s')}{s'^3} \tag{A.26}$$

where $\cap_s$ represents a counterclockwise semicircular path as shown in figure A.2. Also, the Cauchy theorem implies that the integral over the contour $C = \cap_s + \cap_\infty + \cap_{l_1} + \cap_{l_2}$ vanishes. Moreover, due to the Froissart bound, the integral over the arc at infinity i.e. $\cap_\infty$ vanishes. Therefore,

$$a(s) = -\left[\int_{l_1} \frac{ds'}{\pi i} \frac{\mathcal{M}(s')}{s'^3} + \int_{l_2} \frac{ds'}{\pi i} \frac{\mathcal{M}(s')}{s'^3}\right] \tag{A.27}$$

Using crossing symmetry and real analyticity, $\mathcal{M}(s + i\epsilon) = \mathcal{M}^*(-s + i\epsilon)$, we can relate the

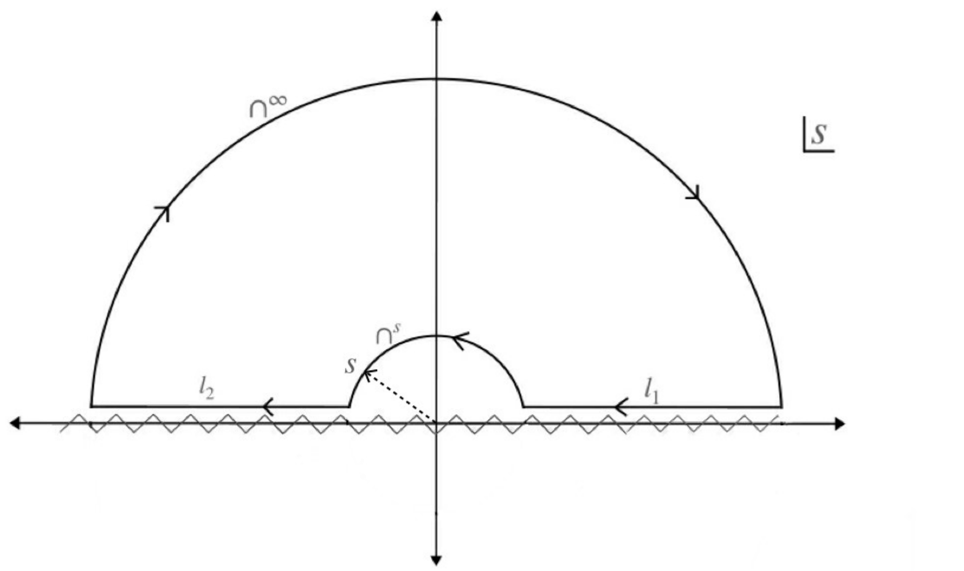

Figure A.2: Contour C in s-complex plane where s represents some energy scale such that $\Lambda_{\rm qcd} < s \ll \Lambda$.

amplitude over l_2 to the amplitude over l_1,

$$a(s) = \int_{s_0}^{\infty} \frac{ds'}{\pi i} \frac{\mathcal{M}(s')}{s'^3} + \int_{-\infty}^{-s_0} \frac{ds'}{\pi i} \frac{\mathcal{M}(s')}{s'^3} \tag{A.28}$$

$$= \int_{s_0}^{\infty} \frac{ds'}{\pi i} \frac{\mathcal{M}(s')}{s'^3} - \int_{s_0}^{\infty} \frac{ds'}{\pi i} \frac{\mathcal{M}^*(s')}{s'^3} = \frac{2}{\pi} \int_{s_0}^{\infty} ds' \frac{\mathrm{Im}\mathcal{M}(s')}{s'^3} \tag{A.29}$$

The optical theorem relates the imaginary part of amplitude to the cross-section, $\mathrm{Im}\mathcal{M}(s') = s'\sigma(s')$,

$$a(s) = \frac{2}{\pi} \int_{s_0}^{\infty} ds' \frac{\sigma(s')}{s'^2} > 0 \tag{A.30}$$

We can systematically compute the arc variable, $a(s)$, as an expansion in s using eq^n(A.26) withing the validity of the EFT regime. For amplitude of the form, $\mathcal{M}(s) = \sum_{n=0} c_{2n} s^{2n}$, which is the case for gluon-gluon scattering, the arc variable is given by the Wilson coefficient, $a(s) = c_2 > 0$ i.e. the coefficient of s^2 in the amplitude is always positive.

Appendix B

Bell violation

B.1 Schmidt decomposition

For a generic initial state, $|\psi\rangle_i = \cos\theta\,|0,0\rangle + \sin\theta\,|1,1\rangle$, we get the following final state after $2 \to 2$ scattering

$$
\begin{aligned}
|\Psi\rangle =& \cos\theta(\mathcal{M}^{0,0}_{0,0}\,|0,0\rangle + \mathcal{M}^{1,1}_{0,0}\,|1,1\rangle + \mathcal{M}^{0,1}_{0,0}\,|0,1\rangle + \mathcal{M}^{1,0}_{0,0}\,|1,0\rangle) \\
& \sin\theta(\mathcal{M}^{0,0}_{1,1}\,|0,0\rangle + \mathcal{M}^{1,1}_{1,1}\,|1,1\rangle + \mathcal{M}^{0,1}_{1,1}\,|0,1\rangle + \mathcal{M}^{1,0}_{1,1}\,|1,0\rangle)
\end{aligned}
\tag{B.1}
$$

For CP conserving theories, we have

$$
\begin{pmatrix}
\mathcal{M}^{1,0}_{1,0} & \mathcal{M}^{1,1}_{1,0} & \mathcal{M}^{1,0}_{0,0} & \mathcal{M}^{1,1}_{0,0} \\
\mathcal{M}^{0,0}_{1,0} & \mathcal{M}^{0,1}_{1,0} & \mathcal{M}^{0,0}_{0,0} & \mathcal{M}^{0,1}_{0,0} \\
\mathcal{M}^{1,0}_{1,1} & \mathcal{M}^{1,1}_{1,1} & \mathcal{M}^{1,0}_{0,1} & \mathcal{M}^{1,1}_{0,1} \\
\mathcal{M}^{0,0}_{1,1} & \mathcal{M}^{0,1}_{1,1} & \mathcal{M}^{0,0}_{0,1} & \mathcal{M}^{0,1}_{0,1}
\end{pmatrix}
=
\begin{pmatrix}
\Phi_3 & \Phi_5 & \Phi_5 & \Phi_2 \\
\Phi_5 & \Phi_4 & \Phi_1 & \Phi_5 \\
\Phi_5 & \Phi_1 & \Phi_4 & \Phi_5 \\
\Phi_2 & \Phi_5 & \Phi_5 & \Phi_3
\end{pmatrix}
$$

Therefore, the normalized final state can be written as

$$
\begin{aligned}
|\Psi\rangle =& \frac{(\sin\theta\Phi_2 + \cos\theta\Phi_1)\,|0,0\rangle + (\sin\theta\Phi_1 + \cos\theta\Phi_2)\,|1,1\rangle + (\sin\theta + \cos\theta)\Phi_5(|0,1\rangle + |1,0\rangle)}{\sqrt{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2\sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)}} \\
=& \mu_{mn}\,|m,n\rangle
\end{aligned}
\tag{B.2}
$$

Now, we will use the Schmidt decomposition theorem [81] to convert the above state to $|0,0\rangle$ and $|1,1\rangle$ basis. The coefficients μ_{mn} can be written in the form of a matrix,

$$\mu_{mn} = \begin{pmatrix} (\sin\theta\Phi_2 + \cos\theta\Phi_1) & (\sin\theta + \cos\theta)\Phi_5 \\ (\sin\theta + \cos\theta)\Phi_5 & (\sin\theta\Phi_1 + \cos\theta\Phi_2) \end{pmatrix} / \sqrt{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2 \sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)}$$

We then diagonalize the above matrix, i.e. find the eigenvalues of the matrix, which are given by the roots of the following equation

$$\begin{aligned} \lambda^2 - & \frac{(\Phi_1+\Phi_2)(\sin\theta+\cos\theta)\lambda}{\sqrt{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2 \sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)}} \\ & + \frac{\Phi_1\Phi_2 + (\Phi_1^2+\Phi_2^2)\sin\theta\cos\theta - \Phi_5^2(1+\sin 2\theta)}{(|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2 \sin 2\theta + 2\Phi_5^2(1+\sin 2\theta))} = 0 \end{aligned} \tag{B.3}$$

These eigenvalues are the new coefficients in $|00\rangle$ and $|11\rangle$ basis. We don't need to explicitly calculate the roots of the above equation as the CGLMP depends just on their product,

$$\begin{aligned} I_{2f} &= 4\sqrt{2}\lambda_1\lambda_2 \\ &= 2\sqrt{2}\frac{2\Phi_1\Phi_2 + (\Phi_1^2+\Phi_2^2)\sin 2\theta - 2\Phi_5^2(1+\sin 2\theta)}{|\Phi_1|^2 + |\Phi_2|^2 + 2\Phi_1\Phi_2 \sin 2\theta + 2\Phi_5^2(1+\sin 2\theta)} \end{aligned} \tag{B.4}$$

One can easily see that in terms of μ_{mn}, we have $\lambda_1\lambda_2 = \mu_{00}\mu_{11} - \mu_{01}\mu_{10}$. Therefore, I_2 can now be directly written as

$$I_{2f} = 4\sqrt{2}\frac{\mu_{00}\mu_{11} - \mu_{01}\mu_{10}}{\sum_{m,n=0}^{1} |\mu_{m,n}|^2} \tag{B.5}$$

for a generic state, $|\Psi\rangle = \sum_{m,n=0}^{1} \mu_{mn} |m,n\rangle$ and real μ_{mn}.

The above result could have also been inferred by directly looking at another parameter quantifying the degree of entanglement, *concurrence* (Δ), which is defined as $\Delta = 2\,|\mu_{00}\mu_{11} - \mu_{01}\mu_{10}|$ for a normalized state. This reduces to $\mu'_{00}\mu'_{11}$ when one goes to $|00\rangle$ and $|11\rangle$ basis as $\mu'_{01} = 0 = \mu'_{10}$. Since the degree of entanglement doesn't depend on the basis, we can infer the relation between the coefficients in different basis as $\mu_{00}\mu_{11} - \mu_{01}\mu_{10} = \mu'_{00}\mu'_{11}$ which is same as we derived from Schmidt decomposition method.

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