Fibrations Over Topological Groups

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled *Fibrations Over Topological Groups* towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Karthik Suraj Vasisht at Indian Institute of Science Education and Research under the supervision of Steven Spallone, Professor, Department of Mathematics, during the academic year 2018-2023.

Steven Spallone

Committee:

Steven Spallone

Vivek Mohan Mallick

This thesis is dedicated to my Guru

Declaration

I hereby declare that the matter embodied in the report entitled *Fibrations Over Topological Groups* are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of Steven Spallone and the same has not been submitted elsewhere for any other degree.

Karthik Suraj Vasisht

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Abstract

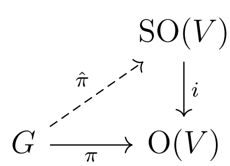
This thesis serves three purposes. First, we define Wu classes of representations and compute them for orthogonal representations of cyclic groups. Next, we provide an exposition to simplicial homotopy theory and discuss some pivotal constructions such as the Kan Loop Functor. We show how to recover a topological group from its classifying space up to homotopy equivalence. Finally, we provide partial results in the characterization of principal fibrations with fiber an Eilenberg-MacLane space.

Contents

Introduction

Consider a finite-dimensional real vector space V with non-degenerate quadratic form Q. The group of isometries of (V, Q) is called the *orthogonal group* of V. The group $\mathrm{O}(V, Q)$ has a subgroup $\mathrm{SO}(V, Q)$ consisting of orientation preserving isometries. We proceed to omit "Q" and write $\mathrm{O}(V)$ instead of $\mathrm{O}(V, Q)$, etc.

Let G be a sufficiently nice topological group and (π, V) a representation of G. We call π an *orthogonal* representation if $\pi(g)$ preserves Q for every $g \in G$. If π maps into $\mathrm{SO}(V)$ we call π *achiral*. The group $\mathrm{SO}(V)$ can be viewed as the subgroup of $\mathrm{O}(V)$ consisting only isometries with determinant 1. Thus, an orthogonal representation is achiral if and only if $\det \pi(g) = 1$ for every $g \in G$.



As another example, consider the universal cover $\mathrm{Spin}(V)$ of $\mathrm{SO}(V)$. An achiral representation is called *spinorial* if it lifts to $\mathrm{Spin}(V)$. One can ask which orthogonal representations are spinorial. The criterion for spinoriality involves the vanishing of certain special cohomology classes called Stiefel-Whitney classes (SWCs). Let BG denote a classifying space of G. Write $w_k(\pi)$ for the k^{th} Stiefel-Whitney class (SWC) associated to π. The class $w_k(\pi)$ is the pullback of the k^{th} universal SWC $w_k \in H^*(BO(n), \mathbb{Z}/2\mathbb{Z})$, i.e., $w_k(\pi) = \pi^*(w_k)$. A classical fact is that π lifts to $\mathrm{Spin}(V)$ if and only if $w_2(\pi) = 0$.

$$\begin{array}{ccc} & & \mathrm{Spin}(V) \\ & \overset{\hat{\pi}}{\nearrow} & \downarrow \rho \\ G & \xrightarrow[\pi]{} & SO(V) \end{array}$$

We may rephrase the earlier discussion in terms of SWCs as well. It is known that $w_1(\pi) = w_1(\det \pi)$, and it follows that the criterion for achirality is precisely the vanishing of $w_1(\pi)$.

In the two examples above, the SWCs w_1 and w_2 are associated to the groups $\mathrm{SO}(V)$ and $\mathrm{Spin}(V)$, respectively, via a certain lifting property. This motivates the following question: Given a k, is there a group G_k and a map $G_k \to \mathrm{O}(V)$ such that π lifts to G_k if and only if $w_k(\pi) = 0$?

Before pursuing this question let us first consider an analogous question in the category of topological spaces.

Definition 0.0.1. Let Y be a topological space and A an abelian group. Let $\alpha \in H^i(Y, A)$. An *α-space* over Y is a pair (Z, p) consisting a space Z and a map $p : Z \to Y$, with the property that a map $f : X \to Y$ lifts to Z if and only if $f^*(\alpha) = 0$.

A natural question is whether α-spaces exist. We prove following existence theorem in Section 4.1:

Theorem 0.0.1. When Y is locally compact, α-spaces exist over Y.

Coming back to our original aim, we are interested in studying the analogs of α-spaces in a category of sufficiently nice groups, those whose underlying spaces are CW complexes.

Definition 0.0.2. A topological group whose underlying space is a CW complex is called a *CW group.*

Definition 0.0.3. Let G be a CW group and A an abelian group. Let BG denote a classifying space for G and let $\alpha \in H^i(BG, A)$. An α-group is a pair (H, p) consisting a CW group H and a map $p : H \to G$, with the property that a group homomorphism $f : K \to G$ lifts to H if and only if $f^*(\alpha) = 0$.

The discussion earlier in this introduction exhibits SO(V) and Spin(V) as a w_1-group over O(V) and a w_2-group over SO(V), respectively. We conjecture that α-groups exist over CW groups. In Section 4.2.2, we present a construction which is a candidate for α-groups.

One concept that appears at multiple places in this thesis is that of a classifying space of a topological group. Classifying spaces appear in multiple exciting areas of mathematical research like group extensions. There are many down to earth examples of classifying spaces: The circle S^1 is a model for a classifying space of $\mathbb{Z}$, and $\mathbb{R}P^\infty$ is a model for a classifying space of C_2. An interesting question is whether one can recover a topological group from its classifying space. In this thesis, we survey how the theory of simplicial sets, and in particular the Kan loop group functor, can be used to answer this question. The Kan loop group functor is also intimately tied to the candidate construction of α-groups over CW groups.

Structure of the Thesis

This thesis is structured around three primary objectives. First, we define Wu classes and examine how to associate Wu classes to representations. We then compute Wu classes for orthogonal complex representations of cyclic groups. This thesis endeavors to provide a comprehensive and accessible introduction to the field of simplicial homotopy theory. Finally, we discuss recent progress made towards proving our conjecture.

This thesis is divided into four chapters. Chapter 1 deals with a few preliminary notions such as Hurewicz and principal fibrations. We also discuss some basics about Clifford algebras and the Pin group. In Chapter 2, we discuss Steenrod Squares as well as Wu classes of representations. We also discuss the Bockstein homomorphism and its relation to the first Steenrod Square operation. The computations of Wu classes discussed earlier may be found in this chapter. Chapter 3 introduces simplicial homotopy theory and describes elementary notions such as simplicial homotopy and homotopy groups. The pivotal constructions discussed include the singular complex of a topological space, the geometric realization of a simplicial set, simplicial classifying complexes, and the Kan loop group functor. Finally, chapter 4 deals with fibrations over both topological spaces and topological groups. In this chapter, we give a proof of Theorem 0.0.1 and the progress towards proving our conjecture.

Original Contribution

This thesis comprises multiple visual examples and explanations to present the theory in a friendly manner. In Section 3.4.1 we provide a pictorial description of the classifying complex of C_2 and show its realization is homeomorphic to $\mathbb{R}P^\infty$. In Example 12 we provide pictorial descriptions of PTCPs which are analogous to the double covers of S^1. The proof of Theorem 4.1.6 and Propositions 4.2.1 and 4.2.2 are original and can be found in Section 4.1 and Subsection 4.2.1 respectively.

Notation

1. The real numbers with the usual topology is denoted $\mathbb{R}$.
2. The additive group of integers is denoted $\mathbb{Z}$.
3. The cyclic group of order n is denoted by both C_n and $\mathbb{Z}/n\mathbb{Z}$.
4. An isomorphism of two groups or rings is denoted $\cong$.
5. The category of topological spaces is denoted **Top**.
6. A continuous function between topological spaces is called a *continuous map* or sometimes just a *map*.
7. Let (X, x_0) be a pointed topological space. The n^{th} homotopy group of X at x_0 is denoted $\pi_n(X, x_0)$.
8. Let X be a topological space and A an abelian group. We denote the k^{th} singular homology group of X with coefficients in A by $H_k(X; A)$.
9. Let X be a topological space and A an abelian group. We denote the k^{th} singular cohomology group by $H^k(X, A)$. The singular cohomology ring with ring operation as the usual cup product is denoted by $H^*(X, A)$.
10. The cup product between $x, y \in H^*(X, A)$ is denoted $x \cup y$.
11. Let G be a CW group and let (π, V) be a finite dimensional orthogonal complex representation of G. We let Θ_π denote the character of π.

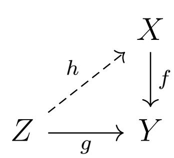
Chapter 1

Preliminaries

This chapter provides an overview of the basic concepts used throughout the thesis. While the material covered here is not new, it forms a basis for constructions that are developed in later chapters. The concepts discussed here can be found in relevant standard texts to which the reader is referred throughout the chapter. On a whole, the discussion on fibrations, homotopy equivalences, and weak homotopy equivalences refers to [Hat02]. More information on the Clifford algebras can be found in [BD85] or [HFH91].

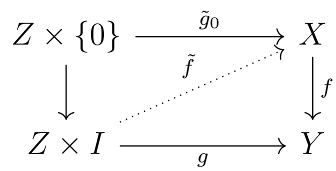
1.1 Fibrations

Definition 1.1.1. Let X, Y and Z be topological spaces and let $f : X \to Y$, $g : Z \to Y$ be maps. A map $h : Z \to X$ is said to be a *lift* of f if $f \circ h = g$.



Definition 1.1.2. Let X, Y and Z be topological spaces. Let $f : X \to Y$ be a map and $g : Z \times I \to Y$ be a homotopy. The map f is said to have the *homotopy lifting property* (HLP) with respect to Z if, given a lift $\tilde{g}_0$ of g_0, there is a homotopy $\tilde{g}_t$ such that $f \circ \tilde{g}_t = g_t$ for all $t \in [0, 1]$.

The space X above is called the *total space* of the fibration f, and Y is called the *base space*. In the definition, we may equivalently say that the homotopy g lifts to a homotopy $\tilde{g} : Z \times I \to X$. This is represented in the following diagram:



Definition 1.1.3. A map that has the HLP for all discs D^n is called a *Serre fibration*. A map that has the HLP for all spaces is called a *Hurewicz fibration*.

It is immediate that any Hurewicz fibration is a Serre fibration.

Example 1. Let X be a topological space. The identity map $\mathrm{id} : X \to X$ is a fibration.

Example 2. Let B, F be topological spaces. The projection $B \times F \to B$ is called the *product fibration*. Fix a point $f_0 \in F$. This map is easily seen to be a fibration since any homotopy $g : Z \times I \to B$ lifts to a homotopy $\tilde{g} : Z \times I \to B \times F$ given by $\tilde{g}(z,t) = (g(z,t), f_0)$.

Example 3. Let X be a topological space. Loosely, a *cover* or *covering space* of X is a topological space $\widetilde{X}$ with a surjection $p : \widetilde{X} \to X$ satisfying: for every $x \in X$, there is an open neighborhood U_x of x such that the fiber over x is a disjoint union of open sets which each map homeomorphically to U_x. Any covering space projection is a fibration since covering spaces have the homotopy lifting property.

Usually in the case of fiber bundles, the fibers over points in the base space are homeomorphic to each other. When it comes to fibrations however, we consider a homotopy-theoretic analog of this:

Proposition 1.1.1 ([Hat02, Proposition 4.61])**.** *Let $p : E \to B$ be a fibration. Let a and b be two points in the same path component of B. Then, the fibres F_a and F_b over a and b respectively are homotopy equivalent.*

Definition 1.1.4. Let $p : E \to B$ be a fibration and let $f : A \to B$ be a map. Define $f^*(E) = \{(a,e) \in A \times E \mid f(a) = p(e)\}$. The *pullback fibration* $p^* : f^*(E) \to A$ is the map taking a pair (a,e) to $f(a)$.

We commonly say that $p^*(E)$ is the pullback of p over f. By construction, the following diagram commutes, where the top map is the projection to the first coordinate of $p^*(E)$:

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ {\scriptstyle p^*}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ A & \xrightarrow[f]{} & B \end{array}$$

Definition 1.1.5. Let $p_1 : E_1 \to B$ and $p : E_2 \to B$ be two fibrations. A fiber-preserving map $f : E_1 \to E_2$ is called a *fiber homotopy equivalence* if there is a fiber-preserving map $g : E_2 \to E_1$ such that the compositions $f \circ g$ and $g \circ f$ are homotopic to the identity via fiber-preserving maps.

Proposition 1.1.2. Let $p : E \to B$ be a fibration and $f_t : A \to B$ a homotopy. Then, $f_0^*(E)$ and $f_1^*(E)$ are fiber homotopy equivalent.

In particular, this implies that a fibration over a contractible base space is fiber homotopy equivalent to a product fibration.

The following proposition about the long exact sequence associated with a fibration is especially important as it allows for a simple calculation of homotopy groups of the base space with respect to the fiber and the total space.

Proposition 1.1.3. Let $p : E \to B$ be a Serre fibration. Choose basepoints $b_0 \in B$ and $x_0 \in p^{-1}(b_0)$. Then, there is a long exact sequence of homotopy groups

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(B, b_0) \to 0.$$

We now discuss an important construction that will play an important role in Chapter 4. Let $p : E \to B$ be a fibration and $f : A \to B$ a map. Let $E_f = \big\{(a, \gamma) \in A \times B^I \mid \gamma(0) = f(a)\big\}$. Define a map $p_f : E_f \to B$ sending every pair (a, γ) to the endpoint $\gamma(1)$. The space E_f is called the *mapping path space* of f. The fiber F_f over a point $b_0 \in B$ is the set of pairs (a, γ) with $\gamma(1) = b_0$. The mapping path space is topologised with the compact-open topology.

Proposition 1.1.4 ([Hat02, Page 407])**.** The map p_f is a fibration.

Fix a point $b_0 \in B$. Let i denote the inclusion of b_0 into B. The mapping path space E_i consists of pairs (b_0, γ), where γ is a path in B starting at b_0. This space can be identified with all paths in B which start at b_0.

Definition 1.1.6. Let B be a topological space. The space of all paths in B which start at a fixed b_0 is called the *path space* of B and is denoted PB.

Since the path space is a special case of the mapping path space, it is topologised with the compact-open topology.

Proposition 1.1.5. Let B be a path-connected topological space. Then, PB is contractible.

Proof. Let b_0 be a distinguished point in B. Define a map $f : PB \to \{b_0\}$ sending a path γ to its starting point $\gamma(0)$. Let $i : \{b_0\} \to PB$ be the map sending b_0 to the constant path at b_0. We leave it to the reader to show that the two maps defined are homotopy inverses of each other. It follows that PB is homotopy equivalent to a point, i.e., PB is contractible. □

Define a map $\pi : PB \to B$ sending a path γ in PB to its endpoint $\gamma(1)$. The map π is a fibration and is called the *path fibration* over B.

Definition 1.1.7. Let X be a path-connected topological space. The *based loop space* at $x_0 \in X$ is defined as the space of loops based at x_0.

Notation 1.1.1. Let B be a topological space with base point b_0. We denote the based loop space of B at b_0 by $\Omega(B, b_0)$. If B is path connected, we simply write ΩB.

Proposition 1.1.6. Let B be a path-connected topological space and let b_0 be a base point in B. The fiber of $\pi : PB \to B$ over b_0 is the loop space of B. Further, the fiber over any other point b_1 in B is homotopy equivalent to ΩB.

Proof. This is an obvious consequence of Proposition 1.1.1. We give another heuristic argument here. Since B is path-connected, there is a path from b_0 to b_1. Any loop based at b_1 can be "moved" along the path joining b_0 and b_1 to give a loop at b_0. Thus the based loop spaces at both points are homotopy equivalent. □

Proposition 1.1.7. Let B be a topological space. For b in B, let l_b denote the constant loop at b. The homotopy groups of B at b_0 can be computed from the homotopy groups of ΩB by

$$\pi_n(B, b) \cong \pi_{n-1}(\Omega B, l_b)$$

Proof. We start with the fiber sequence

$$\Omega B \to PB \xrightarrow{\pi} B.$$

By Proposition 1.1.3, we get a long exact sequence of homotopy groups associated to the above fiber sequence.

$$\cdots \to \pi_n(\Omega B, l_b) \to \pi_n(PB, l_b) \to \pi_n(B, b) \to \pi_{n-1}(\Omega B, l_b) \to \cdots \to \pi_0(B, b) \to 0.$$

Since PB is contractible, $\pi_i(PB, l_b) = 0$ for all $i \geq 0$. The proof follows. □

Let B, X be topological spaces and $f : X \to B$ be a continuous map. Write π for the path fibration over B.

Definition 1.1.8. The pullback fibration $f^*(PB)$ is called a *principal fibration* over X.

1.2 Homotopy Equivalences and Weak Homotopy Equivalences

Let X and Y be topological spaces. Let A be an abelian group. We denote the k^{th} singular homology group of X with coefficients in A by $H_k(X; A)$. Similarly, the k^{th} singular cohomology group of X with ceofficients in A is denoted $H^k(X, A)$.

Definition 1.2.1. X and Y are said to be homotopy equivalent if there maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$.

Proposition 1.2.1. Let $f : (X, x_0) \to (Y, y_0)$ be a homotopy equivalence. Then,

1. The induced map $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is a bijection for $n = 0$ and an isomorphism for $n > 0$.
2. The induced map $f_{**} : H_n(X, A) \to H_n(Y, A)$ is an isomorphism for all $n \geq 0$.
3. The induced map $f^* : H^n(Y, A) \to H^n(X, A)$ is an isomorphism for all $n \geq 0$.

Definition 1.2.2. Let $(X, x_0), (Y, y_0)$ be pointed topological spaces. A map $f : (X, x_0) \to (Y, y_0)$ is said to be a *weak homotopy equivalence* if the induced map on homotopy groups is a bijection for $n = 0$, and an isomorphism for $n > 0$.

Proposition 1.2.2. Let $f : X \to Y$ be a weak homotopy equivalence. For any coefficient group A, the following statements are true.

1. The induced map $f_* : H_n(X, A) \to H_n(Y, A)$ is an isomorphism for all $n \geq 0$.

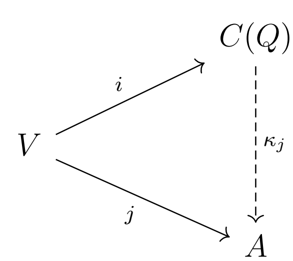
2. The induced map $f^* : H^n(Y, A) \to H^n(X, A)$ is an isomorphism for all $n \geq 0$.

1.3 A Pin Group

Let V be a finite-dimensional complex vector space. Let Q be a non-degenerate quadratic form. Let $O(Q)$ denote the group of linear automorphisms of V that preserve Q. In this section, we describe the construction of a well known double cover of $O(Q)$ called the Pin group, denoted $\text{Pin}(Q)$.

Definition 1.3.1. A *Clifford algebra* $C(Q)$ is an associative algebra with unit 1, along with a map $i : V \to C(Q)$ satisfying $(i(v))^2 = Q(v) \cdot 1$. Further, $C(Q)$ is also required to satisfy the following universal property:

If A is any associative $\mathbb{C}$-algebra with unit 1, and there is a map $j : V \to A$ satisfying $(j(v))^2 = Q(v) \cdot 1$ for any $v \in V$, then there is a unique homomorphism κ_j such that the following diagram commutes:



The map $i : V \to C(Q)$ above is called the *structure map* of $C(Q)$.

Let $V^{\otimes 0} = \mathbb{C}$ and $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$, i.e., the n-fold tensor product of V. The tensor algebra $T(V)$ is defined as

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n}.$$

Let $I(V)$ denote the ideal generated by elements of the form $v \otimes v - Q(v) \cdot 1$.

Proposition 1.3.1. The quotient algebra $C(Q) := T(V)/I(V)$ is a Clifford algebra. Further the composition $i : V \hookrightarrow T(V) \to T(V)/I(V)$ defines the structure map of $C(Q)$.

Define the anti-involution map $* : C(Q) \to C(Q)$ by

$$(v_1 v_2 \dots v_k)^* = (-1)^k v_k v_{k-1} \dots v_1$$

for $v_i \in V$, where V is seen as a subspace of $T(V)$. There is also a canonical automorphism $\alpha : C(Q) \to C(Q)$ defined by

$$\alpha(v_1 v_2 \dots v_k) = (-1)^k (v_1 v_2 \dots v_k)$$

.

Let $C(Q)^\times$ denote the group of units of $C(Q)$. The subgroup of $C(Q)$ consisting all $x \in C(Q)$ such that $xx^* = 1$ and $xvx^* \in V$ for every $v \in V$ is called the *Pin group* associated to Q and denoted $\mathrm{Pin}(Q)$. Let $O(Q)$ denote the orthogonal group associated to Q, i.e. the subgroup of $\mathrm{Aut}(V)$ comprising all linear transformations preserving Q.

Proposition 1.3.2. The map

$$\rho : \mathrm{Pin}(Q) \to O(Q)$$
$$\rho(x)(v) = \alpha(x) v x^*$$

is a $2-1$ covering of $O(Q)$.

Chapter 2

Wu classes of Representations

In this chapter, we define the notion of Wu classes and explain how to associate Wu classes to orthogonal complex group representations. The first three sections consist primarily of known results. The association of Wu classes to representations is original and mirrors the definition of Stiefel Whitney Classes of representations. The computations of Wu classes for representations of cyclic groups in Section 2.4 are original. In this chapter, we discuss only finite dimensional representations. Further, all vector bundles will be real of finite rank.

2.1 Associated Bundles

Let G be a CW group. Let $p : E \to B$ be a G-bundle. Let (π, V) be a finite degree real representation of G.

Define a relation $\sim$ on $E \times V$ by $(eg, v) \sim (e, \pi(g)v)$. It is an easy check that $\sim$ is an equivalence relation. Let $E[V]$ denote the quotient of $(E \times V)$ by this equivalence relation. The space $E[V]$ is sometimes written $EG \times_G V$. We represent elements of $E[V]$ by $[e, v]$.

Proposition 2.1.1. The map $p_a : E[V] \to B$ sending $[e, v]$ to $p(e)$ is a vector bundle over B with fiber isomorphic to V.

Definition 2.1.1. $E[V]$ as defined above is called the *associated bundle* of (π, V).

$$\begin{array}{ccc} f^*(E) & \longrightarrow & EG \\ \downarrow & & \downarrow{\scriptstyle p_u} \\ B & \xrightarrow{f} & BG \end{array}$$

As a consequence of the above theorem, there is a map $f : B \to BG$, unique up to homotopy, such that $E = f^*(EG)$. Given a representation (π, V) of G we can form the associated bundle $EG[V]$ over BG.

Lemma 2.1.2 ([MS74, Lemma 3.1])**.** Let $f : B_1 \to B_2$ be a map covered by a bundle map $f : E_1 \to E_2$. Then E_1 is isomorphic to to the pullback bundle $f^*(E_2)$.

Proposition 2.1.3. Let G be a CW group. Let $p : E \to B$ be a principal G-bundle and $f : B \to BG$ be a map corresponding to p. For a representation (π, V) of G, let $E[V]$ and $EG[V]$ be the associated bundles of E and EG with respect to π. Then,

$$E[V] = f^*(EG[V])$$

Proof. Let $p : E[V] \to B$ denote the projection of $E[V]$ and $p_u : EG \to BG$ denote the projection for EG. We wish to show the second diagram below commutes:

$$\begin{array}{ccc} E & \longrightarrow & EG \\ \downarrow{\scriptstyle p} & & {\scriptstyle p_u}\downarrow \\ B & \xrightarrow[f]{} & BG \end{array} \qquad\qquad \begin{array}{ccc} E[V] & \longrightarrow & EG[V] \\ \downarrow{\scriptstyle p_a} & & \downarrow{\scriptstyle (p_u)_a} \\ B & \xrightarrow[f]{} & BG \end{array}$$

BG can be viewed as the quotient of EG by the action of G on EG. Thus, $(p_u)_a(xg) = (p_u)_a(x)$ for $x \in EG, g \in G$. Let $[e, v] \in E[V]$. We have

$$\begin{aligned} f \circ p_*([e, v]) &= f(p(e)) \\ &= p_u(f(e)) \\ &= (p_u)_a([f(e), v]) \\ &= (p_u)_a f'([e, v]) \end{aligned}$$

The statement of the proposition follows from Lemma 2.1.2. ∎

2.2 Steenrod Squares

Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. We denote the induced map on cohomology groups for any n and coefficient group A by f^*.

Let A_1 and A_2 be abelian groups. Fix non-negative integers m and n.

Definition 2.2.1. A *cohomology operation* is a natural transformation $\Theta : H^n(-, A_1) \to H^m(-, A_2)$ such that for topological spaces X, Y the following diagram commutes:

$$\begin{array}{ccc} H^n(X, A_1) & \xrightarrow{\Theta_X} & H^m(X, A_2) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ H^n(Y, A_1) & \xrightarrow[\Theta_Y]{} & H^m(Y, A_2) \end{array}$$

Let X be a topological space. For $x \in H^*(X, \mathbb{Z}/2\mathbb{Z})$, let $|x|$ denote the degree of x. Fix a non-negative integer n.

Definition 2.2.2. The i^{th} *Steenrod Square operation*, denoted Sq^i, is a cohomology operation $\mathrm{Sq}^i : H^n(-, \mathbb{Z}/2\mathbb{Z}) \to H^{n+i}(-, \mathbb{Z}/2\mathbb{Z})$ characterised by the following properties:

1. Sq^0 is the identity map
2. $\mathrm{Sq}^i(x) = 0$ for $i > |x|$
3. $\mathrm{Sq}^k(x \cup y) = \sum\limits_{i+j=k} \mathrm{Sq}^i(x) \cup \mathrm{Sq}^j(y)$
4. $\mathrm{Sq}^i(x) = x \cup x$ if $i = |x|$

An additional property of the Steenrod Squares we will be using is that Sq^1 is the *Bockstein homomorphism* associated to the short exact sequence $0 \to \mathbb{Z}_2 \xrightarrow{\iota} \mathbb{Z}_4 \xrightarrow{2} \mathbb{Z}_2 \to 0$. We discuss the Bockstein homomorphism in the next section.

Definition 2.2.3. The *total Steenrod Square* $\mathrm{Sq} : H^*(X, \mathbb{Z}/2\mathbb{Z}) \to H^*(X, \mathbb{Z}/2\mathbb{Z})$ is the formal sum of Steenrod squares:

$$\mathrm{Sq} = \mathrm{Sq}^0 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \dots$$

Proposition 2.2.1. Sq is a ring homomorphism with respect to the cup product on $H^*(X, \mathbb{Z}/2\mathbb{Z})$.

Proof. Sq is an abelian group homomorphism since each Sq^i is. The rest of the proof follows from property (3) above. Let $\alpha, \beta \in H^*(X, \mathbb{Z}/2\mathbb{Z})$. We have:

$$\begin{aligned}
\mathrm{Sq}(\alpha \cup \beta) &= \sum_{k \geq 0} \mathrm{Sq}^k(\alpha \cup \beta) \\
&= \sum_{k \geq 0} \sum_{i+j=k} \mathrm{Sq}^i(\alpha) \cup \mathrm{Sq}^j(\beta) \\
&= (1 + \mathrm{Sq}^1(\alpha) + \mathrm{Sq}^2(\alpha) + \dots)(1 + \mathrm{Sq}^1(\beta) + \mathrm{Sq}^2(\beta) + \dots) \\
&= \mathrm{Sq}(\alpha) \cup \mathrm{Sq}(\beta)
\end{aligned}$$

□

Example 4. In this example we observe how Steenrod Square operations interact with the mod-2 cohomology ring of $\mathbb{R}P^n$. Recall that $H^*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[\alpha]/(\alpha^{n+1})$, where α is a generator of $H^1(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$. Every cohomology class in $H^*(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z})$ can be represented by a polynomial in α, so it suffices to understand the action of Steenrod squares on α. From the conditions in Definition 3.2.2, we get:

$$\begin{aligned}
\mathrm{Sq}^0(\alpha) &= \alpha \\
\mathrm{Sq}^1(\alpha) &= \alpha^2 \\
\mathrm{Sq}^i(\alpha) &= 0 \text{ for } i > 1
\end{aligned}$$

First, we observe that $\mathrm{Sq}(\alpha) = \alpha + \alpha^2$. Thus, $\mathrm{Sq}(\alpha^k) = (\alpha + \alpha^2)^k = \alpha^k(1 + \alpha)^k$. Using the binomial theorem[1], $\mathrm{Sq}(\alpha^k) = \alpha^k(1 + \alpha)^k = \binom{k}{i}\alpha^{i+k}$. Comparing with $\mathrm{Sq}(\alpha^k) = \mathrm{Sq}^0(\alpha^k) + \mathrm{Sq}^1(\alpha^k) + \dots$, we get $\mathrm{Sq}^i(\alpha^k) = \binom{k}{i}\alpha^{i+k}$.

[1]Since we are working with mod-2 cohomology, $\binom{k}{i}$ here refers to the mod-2 binomial coefficient.

2.3 Defining Wu classes

Given a vector bundle $p : E \to B$, we can associate to it a unique sequence of cohomology classes $w_i(E) \in H^*(B, \mathbb{Z}/2\mathbb{Z})$, called Stiefel Whitney classes (SWCs). We now define another sequence of cohomology classes which relate the SWCs and Steenrod Square Operations. All vector bundles dealt with in this section will be finite rank real vector bundles.

Definition 2.3.1. Let X be a topological space and A an abelian group. The *completed cohomology ring* $\widehat{H}^*(X, A)$ is the ring of possibly infinite formal sums of cohomology classes in $H^*(X, A)$. That is,

$$\widehat{H}^*(X, A) = \left\{ \sum_{i \geq 0} x_i \mid x_i \in H^i(X, A) \right\}$$

Definition 2.3.2. The i^{th} *Wu class* of a vector bundle E, denoted $v_i(E)$, is the unique cohomology class defined recursively via the Steenrod Square operations as

$$v_0(E) = 1, \quad v_i(E) = w_i(E) + \mathrm{Sq}^1 v_{i-1}(E) + \cdots + \mathrm{Sq}^i v_0(E) \quad \text{for } i \geq 1.$$

Definition 2.3.3. The *total Wu class* of a bundle $E \to B$ is the formal sum of Wu classes of E and lies in $\widehat{H}^*(X, \mathbb{Z}/2\mathbb{Z})$.

$$v(E) = 1 + v_1(E) + v_2(E) + \ldots$$

Remark 2.3.1. As defined here, Wu classes are defined for any topological space B. For manifolds, one may use Poincare Duality to define Wu classes. If B is a manifold, the k^{th} Wu class of the tangent bundle of B is defined as the unique cohomology class such that for any $x \in H^{n-k}(B, \mathbb{Z}/2\mathbb{Z})$, we have $\mathrm{Sq}^k(x) = x \cup v_k$. Interested readers are referred to section 11 of [MS74].

Let A and B be topological spaces. Let E and F be vector bundles over B. The following properties are satisfied:

1. $v(E \oplus F) = v(E) \cup v(F)$.
2. Let E' be a vector bundle over a space A. If $f : A \to B$ is covered by a bundle map from E' to E, then $v(E') = f^*(v(E))$.

3. Let γ_1^1 denote the tautological line bundle over $\mathbb{R}P^1$. Then, $v_1(\gamma_1^1) \neq 0$.

Theorem 2.3.1 ([MS74, Theorem 11.4])**.** Let E be a vector bundle over a base space B. Let $w(E)$ and $v(E)$ denote the total SWC and Wu class of E. Then,

$$w(E) = \mathrm{Sq}(v(E))$$

Proposition 2.3.2. Let G be a topological group and (π, V) a representation of G. Let E be a principal G-bundle and $E[V]$ its associated bundle. Then

$$w(E[V]) = f^*(w(EG[V]))$$
$$v(E[V]) = f^*(v(EG[V]))$$

Proof. The proof follows from an elementary application of the naturality axioms for SWCs and Wu classes. □

2.4 Wu Classes of Representations

In this section, we describe how to associate Wu classes to representations. Let G be a CW group.

Proposition 2.4.1 ([BD85, Proposition 6.4])**.** Let (π, V) be an orthogonal complex representation of G. Then, there is a unique real representation (π_0, V_0) such that $\pi_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \pi$.

Let $EG[V_0]$ be the associated bundle of EG with respect to the representation π_0.

Definition 2.4.1. The i^{th} SWC and Wu class of (π, V) are defined as

$$w_i(\pi) := w_i(EG[V_0])$$
$$v_i(\pi) := v_i(EG[V_0])$$

2.5 Wu Classes of Cyclic Groups

This section deals with the computation of Wu classes for representations of C_n. The first subsection deals with the case when $n \equiv 2 \mod 4$ and the second deals with $n \equiv 0 \mod 4$.

Notation 2.5.1. Let G be a discrete topological group. We write $H^*_{\mathrm{gp}}(G, A)$ for the group cohomology of G with coefficients in an abelian group A.

Theorem 2.5.1 ([AM94, Section II.3])**.** Let G be a discrete topological group. Let BG denote a classifying space of G. Then,

$$H^*_{\mathrm{gp}}(G, \mathbb{Z}/2\mathbb{Z}) \cong H^*(BG, \mathbb{Z}/2\mathbb{Z}).$$

We denote by C_n the cyclic group of order n. Let g be a generator of C_n. Let "sgn" denote the linear character of order 2, and let $\chi_\bullet$ be the linear character sending g to $e^{\frac{2\pi i}{n}}$. When $n \equiv 0 \mod 4$, let $s = w_1(\mathrm{sgn}), t = w_2(\chi_\bullet \oplus \chi_\bullet^{-1})$. When $n \equiv 2 \mod 4$, let $x = w_1(\mathrm{sgn})$.

Proposition 2.5.2 ([KT67])**.** The group cohomology of C_n is given by

$$H^*_{\mathrm{gp}}(C_n, \mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}[s,t]/(s^2) & \text{if } n \equiv 0 \mod 4 \\ \mathbb{Z}/2\mathbb{Z}[x] & \text{if } n \equiv 2 \mod 4 \end{cases}$$

2.5.1 Computing Wu classes when $n \equiv 2 \mod 4$

From the above proposition, it is clear that every SWC and Wu class of the associated bundle $EG[V_0]$ can be represented as a formal power series in x. We will denote $v(EG[V_0])$ and $w(EG[V_0])$ by the power series $v(x)$ and $w(x)$ respectively.

By Theorem 2.3.1, we have:

$$w(x) = \mathrm{Sq}(v(x))$$

The action of Sq on $v(x)$ is determined by its action on x. Let $h(x) = x + x^2$. Rewriting the above equation we obtain:

$$w(x) = v \circ h(x)$$

We can view the polynomial $h(x)$ as an element of the *formal power series ring* $\mathbb{Z}/2\mathbb{Z}[[x]]$.

Definition 2.5.1. Let k be a field and let $k[[x]]$ denote the formal power series ring in x. Let $f(x) \in k[[x]]$. An element $g(x) \in k[[x]]$ is said to be a *compositional inverse* of $f(x)$ if $f(g(x)) = g(f(x)) = x$, whenever the left hand side is defined. We denote the inverse of an element $f \in \mathbb{Z}/2\mathbb{Z}[[x]]$ with respect to composition by f^{-1}.

Given a formal power series $f(x)$, it is clear that if $f(x)$ admits a compositional inverse, then $f(x)$ cannot have a constant term. The subset of all formal power series with coefficients in a field $\mathbb{F}_p$ that admit inverses with respect to composition is commonly called the *Nottingham group*, denoted $\mathcal{N}(\mathbb{F}_p)$. The elements of this group are of the form $f(x) = x + \sum_{i \geq 2} a_i x^i$, where $a_i \in \mathbb{F}_p$ for $i \geq 2$. The polynomial $h(x)$ is of this form, so we may compute the compositional inverse of $h(x)$, defined earlier.

Proposition 2.5.3. The inverse of $h(x) = x + x^2$ in $\mathbb{Z}/2\mathbb{Z}[[x]]$ is $h^{-1}(x) = \sum_{k=0}^{\infty} x^{2^k}$.

Proof. We have:

$$\begin{aligned} h \circ h^{-1}(x) &= \left(\sum_{k=0}^{\infty} x^{2^k}\right) + \left(\sum_{k=0}^{\infty} x^{2^k}\right)^2 \\ &= \left(\sum_{k=0}^{\infty} x^{2^k}\right) + \left(\sum_{k=1}^{\infty} x^{2^k}\right) \\ &= x. \end{aligned}$$

Similarly,

$$\begin{aligned}
h^{-1} \circ h(x) &= h^{-1}(x + x^2) \\
&= \sum_{k=0}^{\infty} (x + x^2)^{2^k} \\
&= (x + x^2) + (x + x^2)^2 + (x + x^2)^4 + \dots \\
&= (x + x^2) + (x^2 + x^4) + (x^4 + x^8) + \dots \\
&= x.
\end{aligned}$$

This ends the proof. □

The above calculations are summarized in the following proposition.

Proposition 2.5.4. Let C_n be as above and (π, V) be an orthogonal complex representation of C_n. Let $b_\pi = \frac{1}{2}(\deg - \Theta_\pi(g^{n/2})$. Let x denote the generator of the mod-2 group cohomology of C_n. Then $v(\pi) = \left(1 + \sum_{k=0}^{\infty} x^{2^k}\right)^{b_\pi}$).

Proof.

$$\begin{aligned}
v(x) &= w \circ h^{-1}(x) \\
&= w\left(\sum_{k=0}^{\infty} x^{2^k}\right) \\
&= \left(1 + \sum_{k=0}^{\infty} x^{2^k}\right)^{b_\pi}.
\end{aligned}$$

□

2.5.2 Computing Wu Classes when $n \equiv 0 \mod 4$

In this section, we assume $n \equiv 0 \mod 4$. Recall from Proposition 2.5.2 that the group cohomology ring of C_n for $n \equiv 0 \mod 4$ is given by

$$H^*(C_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[s, t]/(s^2).$$

Computing the action of the Steenrod Squares on s we get

$$\begin{aligned}\mathrm{Sq}(s) &= \mathrm{Sq}^0(s) + \mathrm{Sq}^1(s) + \dots \\ &= s + s^2 \\ &= s\end{aligned}$$

since $|s| = 1$ and $s^2 = 0$ in $H^*(C_n, \mathbb{Z}/2\mathbb{Z})$.

$$\begin{aligned}\mathrm{Sq}(t) &= \mathrm{Sq}^0(t) + \mathrm{Sq}^1(t) + \mathrm{Sq}^2(t) \\ &= t + \mathrm{Sq}^1(t) + t^2\end{aligned}$$

The $\mathrm{Sq}^1(t)$ term can be computed via the Bockstein homomorphism which we discuss now.

For a topological space X, let $C_n(X)$ denote the free abelian group on the singular n-simplices of X. Let

$$0 \to G \to H \to K \to 0$$

be an exact sequence of abelian groups. Applying the $\mathrm{Hom}(C_n(X), -)$ functor to this sequence we obtain a sequence

$$0 \to \mathrm{Hom}(C_n(X), G) \to \mathrm{Hom}(C_n(X), H) \to \mathrm{Hom}(C_n(X), K) \to 0$$

which is exact since $C_n(X)$ is projective.

This induces a long exact sequence of cohomology groups:

$$0 \to H^0(X, G) \to H^0(X, H) \to H^0(X, K) \xrightarrow{\beta_0} H^1(X, G) \to \dots$$

Definition 2.5.2. The boundary homomorphisms $\beta_n : H^n(X, K) \to H^{n+1}(X, G)$ are called *Bockstein homomorphisms.*

Proposition 2.5.5. Let X be a topological space and let A be an abelian group. Fix a positive integer n. Then, $\mathrm{Sq}^1 : H^n(X, \mathbb{Z}/2\mathbb{Z}) \to H^{n+1}(X, \mathbb{Z}/2\mathbb{Z})$ is precisely the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$.

Proof. The statement can be found as property 7 in section 4L of [Hat02]. The proof of the proposition can be found as the proof of Theorem 4L.12. □

The Bockstein homomorphisms are connecting homomorphisms in the singular cohomology of a space X. When $X = BG$, where G is discrete, we may substitute the singular cohomology of BG with the group cohomology of G. We are interested in the case when $G = C_n$. Fortunately, the group cohomology of C_n is fairly simple. Consider the following projective resolution of $\mathbb{Z}$:

$$\cdots \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{N} \mathbb{Z}G \xrightarrow{\sigma-1} \mathbb{Z}G \xrightarrow{\text{aug}} \mathbb{Z} \to 0.$$

The above resolution is called *standard* or *bar* resolution. We denote the group cohomology of C_n with respect to the above resolution by $H^*(C_n, A)$.

Theorem 2.5.6 ([Ser79])**.** Let $G = C_n$ and g be a generator of C_n. Let $N = \sum_{i=0}^{n-1} g^i$. The group cohomology of C_n with coefficients in a G-module A is given by:

$$H^k(C_n, A) = \begin{cases} A^G & \text{if } k = 0 \\ A^G/NA & \text{if } k \text{ is even and } k > 0 \\ {}_NA/(\sigma-1)A & \text{if } k \text{ is odd} \end{cases}$$

Corollary 2.5.7. The group cohomology of C_n with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is given by

$$H^k(C_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \text{ for } n \geq 0$$

Definition 2.5.3. Let G be a discrete group. The *Tate cohomology groups* of G with coefficients in an abelian group A are given by:

$$\begin{aligned} \hat{H}^n(G, A) &= H^n_{gp}(G, A) \quad \text{if } n \geq 1. \\ \hat{H}^0(G, A) &= A^G/NA \\ \hat{H}^{-1}(G, A) &= {}_NA/I_GA \\ \hat{H}^{-n}(G, A) &= H_{n-1}(G, A) \quad \text{if } n \geq 2. \end{aligned}$$

By Proposition 6 in Chapter 4 of [Ser79], the three cohomology groups defined above agree for all non-negative k. We are now in a position to compute the Bockstein homomorphism β_2. Let $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z}$ denote the inclusion map, sending $1 \in \mathbb{Z}/2\mathbb{Z}$ to $2 \in \mathbb{Z}/4\mathbb{Z}$. Define a map $\alpha : \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ that sends 1 to 1. The maps defined fit into the following short exact sequence:

$$0 \to \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/2\mathbb{Z} \to 0$$

The associated long exact sequence of cohomology groups for computing the Bockstein homomorphism is:

$$\cdots \to H^2(BG, \mathbb{Z}/2\mathbb{Z}) \to H^2(BG, \mathbb{Z}/4\mathbb{Z}) \xrightarrow{a} H^2(BG, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta_2} H^3(BG, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{b} H^3(BG, \mathbb{Z}/4\mathbb{Z}) \to \cdots$$

Since $G = C_n$, Theorem 2.5.1 allows us to rewrite the above long exact sequence in terms of the group cohomology of C_n.

$$\cdots \to H^2_{gp}(G, \mathbb{Z}/2\mathbb{Z}) \to H^2_{gp}(G, \mathbb{Z}/4\mathbb{Z}) \xrightarrow{a} H^2_{gp}(G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\beta_2} H^3_{gp}(G, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{b} H^3_{gp}(G, \mathbb{Z}/4\mathbb{Z}) \to \cdots$$

Using Theorem 2.5.6 and the fact that our coefficient ring is a trivial C_n module we have:

$$H^k_{gp}(C_n, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$
$$H^k_{gp}(C_n, \mathbb{Z}/4\mathbb{Z}) = \mathbb{Z}/4\mathbb{Z}$$

We may rewrite the above long exact sequence as follows:

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\text{inj}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{surj}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\text{inj}} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{surj}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \ldots$$

The natures of the homomorphisms (injective, surjective, or the 0 map) above are completely determined by the fact the sequence is exact. Since the map $0 \to \mathbb{Z}/2\mathbb{Z}$ is injective, the successive maps are surjective, then trivial (0 map) and the pattern repeats. The result of this analysis is the following proposition.

Proposition 2.5.8. The Steenrod Square operation $\mathrm{Sq}^1 : H^2(BC_n, \mathbb{Z}/2\mathbb{Z}) \to H^3(BC_n, \mathbb{Z}/2\mathbb{Z})$ is 0-map.

It follows from the above proposition that $\mathrm{Sq}(t) = t + t^2$. Now, any member of $H^*_{gp}(C_n, \mathbb{Z}/2\mathbb{Z})$ can be represented as a polynomial in 2 variables, s and t. We write $w(s,t)$ for the power series representing $w(\pi)$ and $v(s,t)$ for the power series representing $v(\pi)$. By Theorem 2.3.1, we have

$$w(s,t) = \mathrm{Sq}(v(s,t))$$

Let $h(s,t) = (s, t + t^2)$. We may write $\mathrm{Sq}(v(s,t)) = v \circ h(s,t)$. The inverse of h lies in the formal power series ring $\mathbb{Z}/2\mathbb{Z}[[s]]/(s^2) \times \mathbb{Z}/2\mathbb{Z}[[t]]$.

Proposition 2.5.9. The compositional inverse of $h(s,t) = (s, t + t^2)$ in $\mathbb{Z}/2\mathbb{Z}[[s]]/(s^2) \times \mathbb{Z}/2\mathbb{Z}[[t]]$ is $g(s,t) = (s, \sum_{i \geq 0} t^{2^i})$.

Proof. We have,

$$\begin{aligned} h \circ g(s,t) &= f(s, \sum_{i \geq 0} t^{2^i}) \\ &= (s, \sum_{i \geq 0} t^{2^i} + (\sum_{i \geq 0} t^{2^i})^2) \\ &= (s, \sum_{i \geq 0} t^{2^i} + \sum_{i \geq 1} t^{2^i}) \\ &= (s,t) \end{aligned}$$

$$\begin{aligned} g \circ h(s,t) &= g(s, t + t^2) \\ &= (s, \sum_{i \geq 0} (t + t^2)^{2^i}) \\ &= (s, t + t^2 + t^2 + t^4 + t^4 + t^8 + \dots) \\ &= (s,t) \end{aligned}$$

This ends the proof. $\square$

Let $\delta_\pi = 1$ when $\det \pi = 1$ and $\delta_\pi = 0$ when $\det \pi \neq 1$. We note the following regarding SWCs for representations of C_n for $n \equiv 0 \mod 4$. Let $b_\pi = \frac{1}{4}(\deg\pi - \Theta_\pi(g^n/2))$. (See [MS22].)

$$w(\pi) = (1 + \delta_\pi s)(1 + t)^{b_\pi}.$$

Rewriting the above as a polynomial in s, t we have

$$w(s,t) = (1 + \delta_\pi s)(1 + t)^{b_\pi}.$$

Theorem 2.5.10. Let (π, V) be an orthogonal complex representation of C_n. Let $b_\pi = \frac{1}{4}(\deg\pi - \Theta_\pi(g^n/2))$.

1. If $n \equiv 2 \mod 4$, then
$$v(\pi) = (1 + x)^{2b_\pi}$$

2. If $n \equiv 0 \mod 4$, and $\det \pi = 1$, then
$$v(\pi) = \left(1 + \sum_{i \geq 0} t^{2^i}\right)^{b_\pi}$$

3. If $n \equiv 0 \mod 4$, and $\det \pi \neq 1$, then
$$v(\pi) = (1 + s)\left(1 + \sum_{i \geq 0} t^{2^i}\right)^{b_\pi}$$

Proof. The first statement is Proposition 2.5.4. The proof of the second statement is very similar to that of Proposition 2.5.4 as well. For the third statement, we simply compose the compositional inverse of $h(s,t) = (s, t + t^2)$ with the corresponding SWC, i.e.,

$$
\begin{aligned}
v(s,t) &= (w \circ h^{-1})(s,t) \\
&= w(s, \sum_{i \geq 0} t^{2^i}) \\
&= (1+s)\left(1 + \sum_{i\geq 0} t^{2^i}\right)^{b_\pi}
\end{aligned}
$$

□

2.5.3 Computation for Infinite Cyclic Groups

Let (π, V) be an orthogonal complex representation of $\mathbb{Z}$. The unit circle S^1 is a model for a classifying space $B\mathbb{Z}$, and the covering map $p : \mathbb{R} \to \mathbb{Z}$ is a model for the universal principal $\mathbb{Z}$-bundle over S^1. Since $\mathbb{Z}$ is discrete, we have $H^*_{gp}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong H^*(S^1, \mathbb{Z}/2\mathbb{Z})$.

The mod 2 cohomology of S^1 vanishes above degree 1. Let x denote the generator of $H^1(S^1, \mathbb{Z}/2\mathbb{Z})$. From the axiomatic definitions of Wu classes and SWCs and Theorem 2.3.1, we see that $w_0(\pi) = v_0(\pi)$ and $w_1(\pi) = v_1(\pi)$. Thus we have,

$$
v(\pi) = \begin{cases} 1 & \text{if } \det \pi = 1 \\ 1 + x & \text{if } \det \pi \neq 1 \end{cases}
$$

Chapter 3

Simplicial Homotopy Theory

This chapter is primarily a literature review. We have reorganised the information to make it a coherent introduction to simplicial homotopy theory. We start by defining simplicial sets, followed by some examples. The discussion on geometric realization and singular complexes is followed by an introduction to principal fibrations and principal twisted cartesian product. Example 12 provides a visual representation and analysis of a PTCP of two simplicial sets with different twisting functions. Next, we explore the construction of simplicial classifying complexes. Section 3.4.1 analyses the simplicial classifying complex for cC_2. Finally, we discuss Eilenberg-MacLane complexes. We have expanded the proofs of certain propositions to make the material more accessible as a whole.

3.1 Simplicial Sets

Definition 3.1.1. A *simplicial set* K is a graded set indexed by $\mathbb{N} \cup \{0\}$ along with maps $\partial_i : K_n \to K_{n-1}$ (face maps) and $s_i : K_n \to K_{n+1}$ (degeneracy maps), $0 \leq i \leq n$ which satisfy the following conditions:

1. $\partial_i \partial_j = \partial_{j-1} \partial_i$ if $i < j$
2. $s_i s_j = s_{j+1} s_i$ if $i \leq j$
3. $\partial_i s_j = s_{j-1} \partial_i$ if $i < j$

4. $\partial_i s_i = \partial_{i+1} s_i = \mathrm{id}_{K_n}$

5. $\partial_i s_j = s_j \partial_{i-1}$ if $i \geq j + 1$

The members of K_n are called n-simplices. An n-simplex is said to be *degenerate* if it can be written as $s_i y$ for some $(n-1)$-simplex y. A simplex that is not degenerate is said to be *non-degenerate*.

The notion of simplicial sets may be generalised to any category $\mathcal{C}$. Let Δ^* denote the *simplex category* whose objects are totally ordered sets $\{0, 1, 2, \ldots, n\}$ and morphisms are order-preserving functions between them.

Notation 3.1.1. The totally ordered sets $\{1, 2, \ldots, n\}$ will be denoted $[n]$ for each $n \geq 0$.

Definition 3.1.2. A *simplicial object* F in $\mathcal{C}$ is a contravariant functor $F : \Delta^* \to \mathcal{C}$.

Define morphisms $\delta_i : [n-1] \to [n]$ and $\sigma_i : [n+1] \to [n]$ in Δ^* by:

$$\delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$$

The maps are called δ_i and σ_i are called *coface* and *codegeneracy* maps respectively. The face maps of a simplicial object $F : \Delta^* \to \mathcal{C}$ are precisely $\partial_i := F(\delta_i)$ and $s_i := F(\sigma_i)$ respectively, for $0 \leq i \leq n$. This gives us another definition of a simplicial set.

Definition 3.1.3. A *simplicial set* K is a contravariant functor $K : \Delta^* \to \mathbf{Set}$.

Definition 3.1.4. A *simplicial group* G is a graded group $\{G_n\}_{n \geq 0}$ with face and degeneracy maps that satisfy the properties in Definition 3.1.1. Equivalently, a simplicial group may be viewed as a contravariant functor $G : \Delta^* \to \mathbf{Grp}$.

Notation 3.1.2. For the rest of this thesis, all face and degeneracy maps will be denoted by ∂_i and s_i respectively, unless explicitly defined otherwise.

Example 5. For a category $\mathcal{C}$ and $X \in Ob(\mathcal{C})$, there is a contravariant functor $\text{Hom}(-, X) : \mathcal{C} \to \textbf{Set}$ taking an object Y to $\text{Hom}(Y, X)$ and a morphism $f : Y \to Z$ to $\text{Hom}(f) : \text{Hom}(Z, X) \to \text{Hom}(Y, X)$ defined by precomposition. Taking $\mathcal{C}$ as Δ^* we obtain a functor $\Delta[n] : \Delta^* \to \textbf{Set}$. This functor takes an object $[m]$ in Δ^* to $\text{Hom}([m], [n])$. Given a morphism $\phi : [m] \to [p]$, the functor $\Delta[n]$ takes $f : [p] \to [n]$ to $\Delta[n](\phi)(f) = f \circ \phi$. It can easily be checked that this functor defines a simplicial set with face and degeneracy maps $\Delta[n](\delta_i)$ and $\Delta[n](\sigma_i)$ respectively.

Definition 3.1.5. A simplicial set K is said to have the *extension property* if for every $(n+1)$-set of n-simplices $x_0, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ satisfying $\partial_i x_j = \partial_{j-1} x_i$ for $i \leq j$ and $i \neq k$, there is an $(n+1)$-simplex y such that $\partial_i y = x_i$ for all $i \neq k$.

Definition 3.1.6. A simplicial that satisfies the extension condition is called a *Kan complex*.

Kan complexes are of particular importance as we will soon see that the homotopy theory of Kan complexes is equivalent (in some suitable sense) to the homotopy theory of CW complexes. The majority of simplicial sets we will see in this thesis will be Kan complexes. However, looking at a few examples and non-examples of Kan complexes before preceding is useful.

Example 6. Let X be a set. Define a simplicial set cX by setting $cX_n = X$ for all $n \geq 0$ and all face and degeneracy maps as the "identity" map, i.e. the map taking $x \in cX_n$ to $x \in cX_{n+1}$. The conditions in Definition 3.1.1 are trivially satisfied. We call this simplicial set the *constant simplicial set* of X. For any set X, cX is a Kan complex. Suppose $x_0, x_1, \ldots, \hat{x_k}, \ldots x_{n+1} \in cX_n$ form a compatible system of n-simplices. Then, we have $\partial_i x_j = \partial_{j-1} x_i$ for all $i, j \neq k$. Since all face maps are identity, we see that two simplices x_i and x_j belong to the same compatible system if and only if $x_i = x_j$. Hence, compatible systems of simplices in cX consist of copies of one simplex. For any compatible system $x, x, x, \ldots, \hat{x_k}, \ldots x$, $x \in cX_{n+1}$ is a simplex that satisfies $\partial_i x = x_i = x$ for all $i \neq k$, and hence, cX is a Kan complex.

Example 7. Consider the simplicial set Z with $Z_n = \mathbb{Z}$ and face and degeneracy maps as constant maps to the identity element 0. Any $(n+1)$-set of n-simplices will form a compatible system since $\partial_i x_j = \partial_{j-1} x_i = 0$ for any $x_i, x_j \in Z_n$. Thus, there is no $y \in Z_{n+1}$ such that $\partial_i y = x_i$ for $x_i \neq 0$ and that Z is *not* a Kan complex.

Example 8. Let S be the simplicial set with $S_0 = \{v_0, v_1\}$, $S_1 = \{s_0 v_0, s_0 v_1, e_0, e_1\}$ and S_n generated by degeneracies of S_0 and S_1 for $n \geq 2$. Define degeneracy maps from S_1 to S_0 as follows:

$$\partial_0 e_0 = v_1 \qquad \partial_1 e_0 = v_0$$
$$\partial_0 e_1 = v_0 \qquad \partial_1 e_1 = v_1$$

The face and degeneracy maps for higher dimension simplices can be computed from the properties in Definition 3.1.1. We leave it as an exercise to the reader to show that S is *not* a Kan complex. See Figure 3.1 below for a pictorial representation of S.

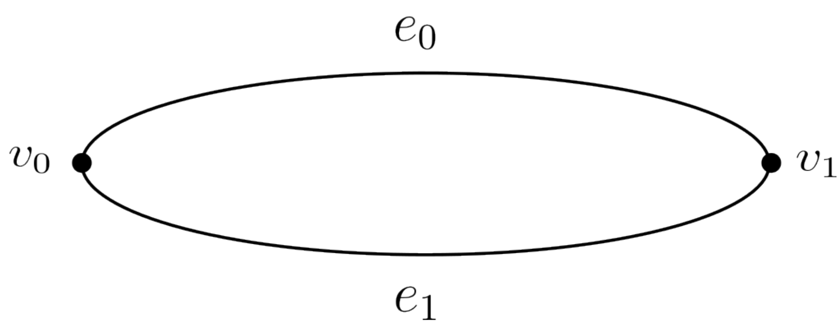


Figure 3.1: A representation of the simplicial set S.

Definition 3.1.7. Let K and L be two simplicial sets. A *simplicial map* $f : K \to L$ is a set of functions $f_n : K_n \to L_n$ such that $\partial_i f_n = f_{n-1}\partial_i$ and $s_i f_n = f_{n+1} s_i$.

$$\begin{array}{ccc} K_n & \xrightarrow{f_n} & L_n \\ {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial_i} \\ K_{n-1} & \xrightarrow{f_{n-1}} & L_{n-1} \end{array} \qquad \begin{array}{ccc} K_n & \xrightarrow{f_n} & L_n \\ {\scriptstyle s_i}\downarrow & & \downarrow{\scriptstyle s_i} \\ K_{n-1} & \xrightarrow{f_{n+1}} & L_{n-1} \end{array}$$

Example 9. Let K, L be simplicial sets and let $l_0 \in L_0$ be a vertex. Define a simplicial map $c_{l_0} : K \to L$ taking any $k \in K_n$ to $s_0^n l_0$. We call this the *constant simplicial map* to l_0. Using properties $(ii), (iv)$, and (iii) in Definition 3.2.2, we can show that f commutes with face and degeneracy maps. For $n = 1$, $\partial_0 f(k) = \partial_0 s_0 l_0 = l_0$ immediately from property (iv). Further, $f(\partial_i k) = l_0$ by the definition of f. For any $k \in K_n$ and $i > 1$, we have more generally:

$$
\begin{aligned}
\partial_i c_{l_0}(k) &= \partial_i s_0^n l_0 \\
&= s_0 \partial_{i-1} s_0^{n-1} l_0 \\
&\vdots \\
&= s_0^{i-1} \partial_1 s_0^{n-i+1} l_0 \\
&= s_0^{n-1} l_0 \\
&= c_{l_0}(\partial_i k)
\end{aligned}
$$

A similar analysis holds for the s_i:

$$
\begin{aligned}
s_i c_{l_0}(k) &= s_i s_0^n l_0 \\
&= s_0 s_{i-1} s_0^{n-1} l_0 \\
&\vdots \\
&= s_0^{n+1} l_0 \\
&= c_{l_0}(s_i l_0)
\end{aligned}
$$

Thus, c_{l_0} is indeed a simplicial map.

3.2 Simplicial Homotopy and Homology Groups

The homotopy groups of a pointed topological space (X, x_0) are instrumental in characterising the weak homotopy type of (X, x_0). In this section, we define analogous constructions for simplicial sets.

Definition 3.2.1. Let K be a simplicial set. Two n-simplices x, x' are said to be *homotopic* if $\partial_i x = \partial_i x'$ for all $0 \leq i \leq n$ and there exists $y \in K_{n+1}$ such that:

(i) $\partial_n y = x$, $\partial_{n+1} y = x'$.

(i) $\partial_i y = s_{n-1}\partial_i y = \partial_i x$.

Proposition 3.2.1 ([May67, Prop. 3.2])**.** If K is a Kan complex, then "two simplices being homotopic" is an equivalence relation.

Let K be a simplicial set and ϕ a distinguished vertex of K. For simplicity, we will denote all degeneracies of ϕ by ϕ as well ϕ . If K is a Kan complex, We call (K, ϕ) a *Kan pair*. Let $\widetilde{K}_n$ be the collection of all n-simplices k that satisfy $\partial_i k = \phi$ for every $0 \leq i \leq n$.

Definition 3.2.2. Let (K, ϕ) be a Kan pair. For $n > 0$, the n^{th} *simplicial homotopy group* is defined as

$$\pi_n(K, \phi) = \widetilde{K}_n / \sim$$

For $n = 0$, define

$$\pi_0(K, \phi) = K_0 / \sim .$$

Let (K, ϕ) be a Kan pair. To define a group structure on $\pi_n(K, \phi)$, let x, y be representatives of $[x], [y] \in \pi_n(K, \phi)$ for a fixed $n > 0$. Consider the ordered collection of n-simplices with $x_{n-1} = x, x_{n+1} = y$ and $x_i = \phi$ for $i < n-1$. These simplices form a compatible system. Since K is Kan, there is a simplex z such that $\partial_i z = x_i$ for $i \neq n$. Define $[x] * [y] = [\partial_n z]$.

Proposition 3.2.2 ([May67, Proposition 4.3])**.** The operation $*$ defines a group operation on $\pi_n(K, \phi)$.

Definition 3.2.3. Two simplicial maps $f, g : K \to L$ are said to be *homotopic* if there exist functions $h_i : K_n \to L_{n+1}$, $0 \leq i \leq n$, such that:

(i) $\partial_0 h_0 = f,\ \partial_{n+1} h_n = g$

(ii) $\partial_i h_j = h_{j-1}\partial_i$, if $i \leq j$

(iii) $\partial_{j+1} h_{j+1} = \partial_{j+1} h_j$

(iv) $\partial_i h_j = h_j \partial_{i-1}$

(v) $s_i h_j = h_{j+1} s_i$ if $i \leq j$

(vi) $s_i h_j = h_j s_{i-1}$ if $i > f$

Notation 3.2.1. We let (0) denote any simplex of $\Delta[1]_0$ and (1) denote any simplex of $\Delta[1]_1$.

Proposition 3.2.3 ([May67, Proposition 6.2])**.** Two simplicial maps $f, g : K \to L$ are homotopic if and only if there exists a simplicial map $F : K \times \Delta[1] \to L$ such that $F(x,(0)) = f(x)$ and $F(x,(1)) = g(x)$ for any simplex x in K.

Definition 3.2.4. Let K, L be simplicial sets. Let $f : K \to L$ be a simplicial map. The map f is said to be a *homotopy equivalence* of simplicial sets if there exists a simplicial map $g : L \to K$ such that the compositions $f \circ g$ and $g \circ f$ are homotopic to the respective identity simplicial maps.

Definition 3.2.5. Let K, L be Kan complexes and $f : K \to L$ a simplicial map between them. The map f is said to be a *simplicial weak homotopy equivalence*, or simply a weak homotopy equivalence in **sSet**, if the following conditions are satisfied.

1. The induced map $f_* : \pi_0(K, \phi) \to \pi_0(L, f(\phi))$ is a bijection.

2. The induced maps $f_* : \pi_n(K, \phi) \to \pi_n(L, f(\phi))$ are isomorphisms for $n > 0$.

Theorem 3.2.4 ([May67, Theorem 12.5])**.** Let $f : K \to L$ be a weak homotopy equivalence of simplicial sets. If K and L are Kan complexes, then f is a homotopy equivalence of simplicial sets.

3.2.1 Simplicial Homology and Cohomology

Let K be a simplicial set. In the categorical perspective, K may be viewed as a contravariant functor $K : \Delta^* \to \mathbf{Set}$. There is a functor $F : \mathbf{Set} \to \mathbf{Ab}$ called the *free* functor taking a set X to the $F(X)$, the free abelian group generated by X. Via composition, we obtain a functor $F^s : \mathbf{sSet} \to \mathbf{sAb}$ taking a simplicial set K to the simplicial abelian group $F^s(K)$. More concretely, $F^s_n(K)$ the abelian group of n-simplices of $F^s(K)$ is precisely the free abelian group generated by K_n. We may give $F^s(K)$ the structure of a chain complex with differential d_n given by

$$d_n = \sum_{i=0}^{n} \partial_i$$

where ∂_i are the degeneracy maps from $F_n^s(K)$ to $F_{n-1}^s(K)$. We denote this chain complex by $C(K)$. Let A be an abelian group. The homology and cohomology of K are defined as the homology and cohomology and $C(K)$.

Example 10. Let X be a topological space and $S_\bullet(X)$ the singular complex associated to X. Then, the homology and cohomology of $S_\bullet(X)$ coincides with the singular homology and cohomology of X. This follows easily from writing out the functors involved as described in the above paragraph.

Example 11. Let K be the simplicial set associated to a simplicial complex $\widetilde{K}$. Then, the homology and cohomology of K are the simplicial homology and cohomology of $\widetilde{K}$. As in the case of the previous example, this follows easily from expanding the definition of homology and cohomology groups.

3.2.2 Singular Complex

Let X be a topological space. We may associate to X a simplicial set $S_\bullet(X)$ called the *singular complex* of X. The set of n-simplices, denoted $S_n(X)$, consists of continuous maps from the standard n-simplex Δ_n to X. For an n-simplex f, the face and degeneracy maps are defined as follows:

$$\partial_i f(t_0, t_1, \ldots, t_{n-1}) = f(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$
$$s_i f(t_0, t_1, \ldots, t_{n+1}) = f(t_0, \ldots, t_i + t_{i+1}, \ldots, t_{n+1})$$

Given any map $g : X \to Y$ of spaces, there is an induced map $S_\bullet(g) : S_\bullet(X) \to S_\bullet(Y)$ by defining $S_\bullet(g)(f) = g \circ f$. One can easily verify from the above facts that $S_\bullet$ defines a functor from the category **Top** of topological spaces to the category **sSet** of simplicial sets.

Theorem 3.2.5. The association of the simplicial set $S_\bullet X$ to a topological space X defines a functor from **Top** to **sSet**.

$$S_\bullet : \mathbf{Top} \to \mathbf{sSet}$$

Proposition 3.2.6 ([May67, Lemma 1.5])**.** For any topological space X, the simplicial set $S_\bullet(X)$ is a Kan complex.

Proposition 3.2.7. Let X, Y be topological spaces. Let $f, g : X \to Y$ be two homotopic maps in **Top**. Then, the induced maps $f_*, g_* : S_\bullet X \to S_\bullet Y$ are homotopic in **sSet**.

Proof. See the discussion before Theorem 16.1 in [May67]. ☐

Proposition 3.2.8 ([May67, Theorem 16.1])**.** Let (X, x_0) be a pointed topological space. Then, $\pi_n(X, x_0) = \pi_n(S_\bullet X, S_\bullet x_0)$.

Proof. See Theorem 16.1 as mentioned and the discussion before it for details. ☐

3.2.3 Geometric Realization

In this section we elaborate on how one associates to a complex K, a topological space $|K|$, called the geometric realization of K. Let $\Delta_n = \{(t_0, t_1, \ldots, t_n) \mid \sum_{i=0}^n t_i = 1, 0 \leq t_0, t_1, ..., t_n \leq 1\} \subset \mathbb{R}^{n+1}$ denote the standard n-simplex with subspace topology. Define face maps $\delta_i : \Delta_{n-1} \to \Delta_n$ and degeneracy maps $\sigma_i : \Delta_{n+1} \to \Delta_n$ as follows:

$$\delta_i(t_0, t_1, \ldots, t_n) = (t_0, t_1, \ldots, 0, t_i \ldots, t_n)$$
$$\sigma_i(t_0, t_1, \ldots, t_n) = (t_0, t_1, \ldots, t_i + t_{i+1}, \ldots, t_n)$$

Give K the discrete topology. Define a topological space $\overline{K} = \bigcup_{\geq 0} K_n \times \Delta_n$ with product topology. Every point in this space is of the form (k_n, x_n) for $k_n \in K_n, x_n \in \Delta_n$. Define relations $\sim_1$ and $\sim_2$ on $\overline{K}$:

$$(\partial_i k_n, x_{n-1}) \sim_1 (k_n, \delta_i x_{n-1})$$
$$(s_i k_n, x_{n+1}) \sim_2 (k_n, \sigma_i x_n).$$

The above relations tell us how to glue/collapse simplices of K with each other. The first relation tells us how to attach the faces of a simplex via face maps. The second relation tells us how to collapse degenerate simplices via the degenerate maps.

Define the quotient space $|K| = \overline{K}/(\sim_1, \sim_2)$. This space is called the **geometric realization** of K. The topology on $|K|$ is the quotient topology.

Notation 3.2.2. Let K be a simplicial set. We denote the equivalence class of a tuple (k_n, u_n) in $|K|$ by $|k_n, u_n|$.

Proposition 3.2.9 ([May67, Thm 14.1])**.** For any simplicial set K, the geometric realization $|K|$ is a CW-complex.

Given a map $f : K \to L$ of simplicial sets, define a map $|f| : |K| \to |L|$ by $|f||k_n, u_n| = |f(k_n), u_n|$. This map is continuous. Given maps $f : K \to L$ and $g : L \to M$, it is easy to see that $|f \circ g| = |f| \circ |g|$. Thus, we have

Proposition 3.2.10. $|-|$ defines a functor from the category of simplicial sets to the category of topological spaces.

$$|-| : \mathbf{sSet} \to \mathbf{Top}$$

Proposition 3.2.11 ([May67, Theorem 14.3])**.** Let K, L be simplicial sets. Suppose that K and L are both countable or that one of K and L is locally finite. Then, $|K| \times |L| \cong |K \times L|$.

Remark 3.2.1. The hypotheses of K and L being countable or locally finite can be removed if we choose to work in the category of **CGHaus** of compactly generated Hausdorff spaces. Given simplicial sets K and L, there is a natural homeomorphism $|X \times Y| \cong |X| \times_{Ke} |Y|$ where $\times_{Ke}$ denotes the *Kelly product*. For a discussion on **CGHaus** the reader is referred to Chapter 5 of [May99]. For a discussion on geometric realizations which live in **CGHaus**, the reader is referred to Chapter I, Section 2 of [GJ99].

Corollary 3.2.12. The geometric realization of a countable simplicial group is a topological group. If we let the geometric realization live in **CGHaus**, then we may remove the countability hypothesis.

Proposition 3.2.13 ([May67, Corollary 14.5])**.** Let K, L be simplicial sets. Let $F : K \times I \to L$ be a simplicial homotopy. Then, $|-|$ induces a homotopy $|F| : |K| \times |I| \to |L|$ between CW complexes.

Corollary 3.2.14. Let K, L be simplicial sets. Let $f, g : K \to L$ be simplicial maps that are homotopic. Then, $|f|$ is homotopic to $|g|$.

Let K be a simplicial set and X a topological space. Define maps $\phi : \mathrm{Hom}_{\mathbf{sSet}}(K, S_\bullet(X)) \to \mathrm{Hom}_{\mathbf{Top}}(T(K), X)$ and $\psi : \mathrm{Hom}(T(K), X) \to \mathrm{Hom}((K, S_\bullet(X))$ by $\phi(f)(|k_n, u_n|) = f(k_n)(u_n)$ and $\psi(g)(k_n)(u_n) = g|k_n, u_n|$. Let K, L be simplicial sets and X, Y be topological spaces. Given a simplicial map $\alpha : K \to L$, there is an induced map $S_\bullet\alpha^* : \mathrm{Hom}_{\mathbf{sSet}}(K, S_\bullet X) \to \mathrm{Hom}_{\mathbf{Top}}(|K|, X)$ defined by precomposing a map $f : K \to S_\bullet X$ with α. Further there is an induced map $|f|^* : \mathrm{Hom}_{\mathbf{sSet}}(L, S_\bullet X) \to \mathrm{Hom}_{\mathbf{Top}}(|L|, X)$ defined by precomposition. Then, the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{sSet}}(K, S_\bullet X) & \xrightarrow{\phi_{K,X}} & \mathrm{Hom}_{\mathbf{Top}}(|K|, X) \\ {\scriptstyle S_\bullet f^*}\downarrow & & \downarrow{\scriptstyle |f|^*} \\ \mathrm{Hom}_{\mathbf{sSet}}(L, S_\bullet X) & \xrightarrow[\phi_{L,X}]{} & \mathrm{Hom}_{\mathbf{Top}}(|L|, X) \end{array}$$

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{sSet}}(K, S_\bullet X) & \xrightarrow{\phi_{K,X}} & \mathrm{Hom}_{\mathbf{Top}}(|K|, X) \\ {\scriptstyle S_\bullet g^*}\downarrow & & \downarrow{\scriptstyle |g|^*} \\ \mathrm{Hom}_{\mathbf{sSet}}(K, S_\bullet Y) & \xrightarrow[\phi_{L,Y}]{} & \mathrm{Hom}_{\mathbf{Top}}(|K|, Y) \end{array}$$

Proposition 3.2.15 ([May67, Proposition 16.2]). The functor $|-| : \mathbf{sSet} \to \mathbf{Top}$ is a left adjoint to the singular complex functor $S_\bullet : \mathbf{Top} \to \mathbf{sSet}$.

To ϕ and ψ we may associate natural transformations $\Phi : TS_\bullet \to 1_{\mathbf{Top}}$ and $\Psi : 1_{\mathbf{sSet}} \to S_\bullet T$ by setting $\Phi(X) = \phi(1_{S(X)})$, $\Psi(K) = \psi(1_{T(K)})$.

Proposition 3.2.16 ([May67, Theorem 16.6]). Let X be a topological space and K a Kan complex.

1. The induced map $\Phi(X)_* : \pi_n(TS_\bullet(X), TS_\bullet(x_0)) \to \pi_n(X, x_0)$ is a bijection for $n = 0$ and an isomorphism for all $n > 1$.

2. The induced map $\Psi(X)_* : \pi_n(S_\bullet T(K), TS_\bullet(x_0)) \to \pi_n(K, \phi)$ is a bijection for $n = 0$ and an isomorphism for all $n > 1$.

The above proposition states that the map $\Phi(X)$ is a weak homotopy equivalence. At this stage we use the following theorem of Whitehead:

Theorem 3.2.17 ([Hat02, Theorem 4.5])**.** If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is a homotopy equivalence.

The above theorem can be generalised to CW complexes that are not connected as well. (See Corollary 3.5.3.10 in [Lur23].)

If X is a CW-complex, then $\Phi(X)$ is a homotopy equivalence. Similarly, if K is a Kan complex, then $\Psi(K)$ becomes a homotopy equivalence. It is now possible to extend the definition of simplicial homotopy groups to arbitrary simplicial sets.

Definition 3.2.6. Let L be a simplicial set and ϕ be a vertex of L. The n^{th} simplicial homotopy group $\pi_n(L, \phi)$ is defined as $\pi_n(S_\bullet(|L|), S\bullet|\phi|)$.

3.3 Kan Fibrations, Principal Fibrations, and PTCPs

In this section, we will define simplicial analogs of fibrations and principal fibrations. Fibrations satisfy the homotopy lifting property with respect to all topological spaces. We expect a similar property to hold for the simplicial analogs.

Let E, B be simplicial sets and let $p : E \to B$ be a simplicial map. Let $x_0, x_1, \ldots x_{k-1}, x_{k+1}, \ldots, x_{n+1}$ be a collection of $(n+1)$ n-simplices in E that satisfy the extension condition. p is called a *Kan fibration* if for every $y \in B$ such that $\partial_i y = p(x_i)$, then there exists x in the fiber over y such that $\partial_i x = x$. For $\phi \in B_0$, the *fiber* of p over ϕ is $F := p^{-1}\phi$. The following is a simple consequence of this definition:

Proposition 3.3.1 ([May67, Proposition 7.5])**.** Let B, E be simplicial sets and let $p : E \to B$ be a Kan fibration. If B is a Kan complex, then E is a Kan complex. Also if p is surjective and E is a Kan complex, then B is a Kan complex.

Definition 3.3.1. Let ψ be the simplicial set generated by ϕ. Let $i : (F, \psi) \hookrightarrow (E, \psi)$ be the inclusion map. The sequence

$$(F, \psi) \xrightarrow{i} (E, \psi) \xrightarrow{p} (B, \phi)$$

is called a *fiber sequence*. If E, B are Kan complexes, then the sequence is called a *Kan fiber sequence*.

Theorem 3.3.2 ([May67, Theorem 7.6])**.** Let $(F, \psi) \xrightarrow{i} (E, \psi) \xrightarrow{p} (B, \phi)$ be a Kan fiber sequence. Then the following is an exact sequence:

$$\cdots \to \pi_{n+1}(B, \phi) \to \pi_n(F, \psi) \to \pi_n(E, \psi) \to \pi_n(B, \phi) \to \cdots$$

Finally, we have the covering homotopy property, the simplicial version of the homotopy lifting property.

Proposition 3.3.3 ([May67, Corollary 7.12])**.** Let $p : E \to B$ be a Kan fibration and let K be any simplicial set. Let $\tilde{f} : K \to E$ and $f = p \circ \tilde{f}$. Suppose $F : K \times I \to B$ satisfies $\partial_1 F = f$. Then there exists $\widetilde{F} : K \times I \to E$ such that $p \circ \widetilde{F} = F$ and $\partial_1 \widetilde{F} = \tilde{f}$.

Given two simplicial sets K and L, the *product simplicial set* $K \times L$ is defined by setting $(K \times L)_n = K_n \times L_n$.

Let K be a simplicial set, and G be a simplicial group. We say G acts on K from the left if there is a simplicial map $\phi : G \times K \to K$ such that $\phi(e, k) = k \; \forall \; k \in K_n$ and $\phi(g_1, \phi(g_2, k)) = \phi(g_1 g_2, k)$ for any $g_1, g_2 \in G_n$. Similarly, if there is a simplicial map $\psi : K \times G \to K$ such that $\psi(g_2, \psi(k, g_1)) = \psi(g_1 g_2, k)$, we say G operates on K from the right. Here, e denotes the identity element of G_n.

Definition 3.3.2. Let G be a simplicial group. Let E be a simplicial set with a G action given by a simplicial map ϕ. The simplicial group G is said to act *principally* on K if $\phi(g, k) = k$ only when $g = e$.

Remark 3.3.1. From now, the term G-action will refer to the action of a simplicial group on a simplicial set.

Let K be a simplicial set with a right G-action. Define a simplicial set B by letting $k \sim kg$ for $k \in K$ and $g \in G$. There is a map $p : K \to B$ sending each simplex k to its equivalence class in B. We call p a principal fibration over B with structure group G. More generally, any map that can be represented in this manner is called a principal fibration.

Let F and B be simplicial sets and G a simplicial group that acts on F from the left. Define a simplicial set $E(\tau)$ by $E(\tau)_n = F_n \times B_n$. We add a superscript to the face and degeneracy maps to distinguish between those of the fiber and base space. Define the face and degeneracy maps of $E(\tau)$ as follows:

$$\begin{aligned}
\partial_0(f,b) &= (\tau(b)\partial_0^F(f), \partial_0^B(b)) \\
\partial_i(f,b) &= (\partial_i^F(f), \partial_i^B(b)) \\
s_i(f,b) &= (s_i^F(f), s_i^B(b))
\end{aligned}$$

The function $\tau : B_n \times G$ is called a twisting function. It is not immediately evident that $E(\tau)$ as defined above is indeed a simplicial set. For $E(\tau)$ to be a simplicial set, τ must satisfy certain conditions obtained from forcing the face and degeneracy to satisfy the identities in Definition 3.1.1

For $b \in B_q$,

$$\begin{aligned}
\partial_0\tau(b) &= [\tau(\partial_0 b)]^{-1}\tau(\partial_1 b) \\
\partial_i\tau(b) &= \tau(\partial_{i+1} b),\ i > 0 \\
s_i\tau(b) &= \tau(s_{i+1} b),\ i \geq 0 \\
\tau(s_0 b) &= e_q
\end{aligned}$$

Let $p : E(\tau) \to B$ be the projection map to B. $E(\tau)$ is called a *Twisted Cartesian Product* or TCP.

If $F = G$, the TCP, is called a *principal* TCP as G acts principally on itself. In this case, we may define a right action of G on $E(\tau)$ by $\phi((g,b),h) = (gh,b)$. One can check easily that the quotient of $E(\tau)$ by the G-action gives us a simplicial set isomorphic to B, and $p : E(\tau) \to B$ is a principal fibration.

Example 12. Let $C_2 = \{1, \alpha\}$ denote the cyclic group of order 2. Let cC_2 denote the constant simplicial set of C_2. Let K be the simplicial set in Example 8, which represents a circle with two vertices and two edges:

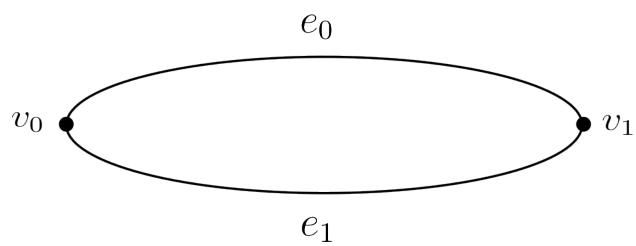


Figure 3.2: Pictorial representation of the geometric realization of K

Let $\tau : K \to cC_2$ be a function taking e_0 to 1 and e_1 to α. Similarly, send any degeneracy of e_0 and e_1 to 1 and α respectively. Define a PTCP $E(\tau) = cC_2 \times_\tau K$. Using the face maps, we try to pictorially depict the set of 0-simplices and 1-simplices of $E(\tau)$.

$$\begin{aligned}
\partial_0(1, e_0) &= (\tau(e_0) \cdot \partial_0 1, \partial_0 e_0) = (1, v_1) \\
\partial_0(1, e_1) &= (\tau(e_1) \cdot \partial_0 1, \partial_0 e_1) = (\alpha, v_0) \\
\partial_0(\alpha, e_0) &= (\tau(e_0) \cdot \partial_0 \alpha, \partial_0 e_0) = (\alpha, v_1) \\
\partial_0(\alpha, e_1) &= (\tau(e_1) \cdot \partial_0 \alpha, \partial_0 e_1) = (1, v_0) \\
\partial_1(1, e_0) &= (\partial_1 1, \partial_1 e_0) = (1, v_0) \\
\partial_1(1, e_1) &= (\partial_1 1, \partial_1 e_1) = (1, v_1) \\
\partial_1(\alpha, e_0) &= (\partial_1 \alpha, \partial_1 e_0) = (\alpha, v_0) \\
\partial_1(\alpha, e_1) &= (\partial_1 \alpha, \partial_1 e_1) = (\alpha, v_1)
\end{aligned}$$

On taking the realization of $E(\tau)$, we get the following diagram:

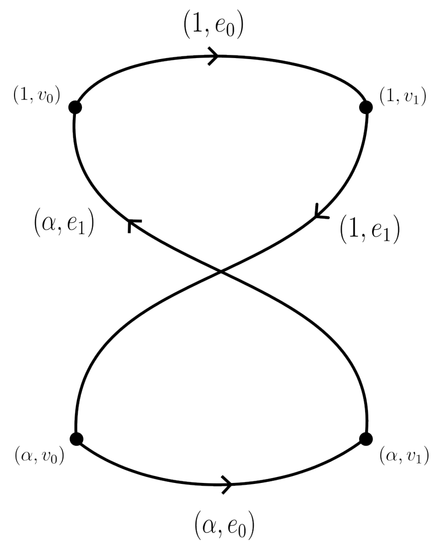


Figure 3.3: PTCP with non-trivial twisting function

Alternatively, let $\tau' : K \to cC_2$ take every n-simplex to the identity elements of $(cC_2)_n$. Define a new PTCP $E(\tau') = cC_2 \times_{\tau'} K$. For $E(\tau')$, we have:

$$
\begin{aligned}
\partial_0(1, e_0) &= (\tau(e_0) \cdot \partial_0 1, \partial_0 e_0) = (1, v_1) \\
\partial_0(1, e_1) &= (\tau(e_1) \cdot \partial_0 1, \partial_0 e_1) = (1, v_0) \\
\partial_0(\alpha, e_0) &= (\tau(e_0) \cdot \partial_0 \alpha, \partial_0 e_0) = (\alpha, v_1) \\
\partial_0(\alpha, e_1) &= (\tau(e_1) \cdot \partial_0 \alpha, \partial_0 e_1) = (\alpha, v_0) \\
\partial_1(1, e_0) &= (\partial_1 1, \partial_1 e_0) = (1, v_0) \\
\partial_1(1, e_1) &= (\partial_1 1, \partial_1 e_1) = (1, v_1) \\
\partial_1(\alpha, e_0) &= (\partial_1 \alpha, \partial_1 e_0) = (\alpha, v_0) \\
\partial_1(\alpha, e_1) &= (\partial_1 \alpha, \partial_1 e_1) = (\alpha, v_1)
\end{aligned}
$$

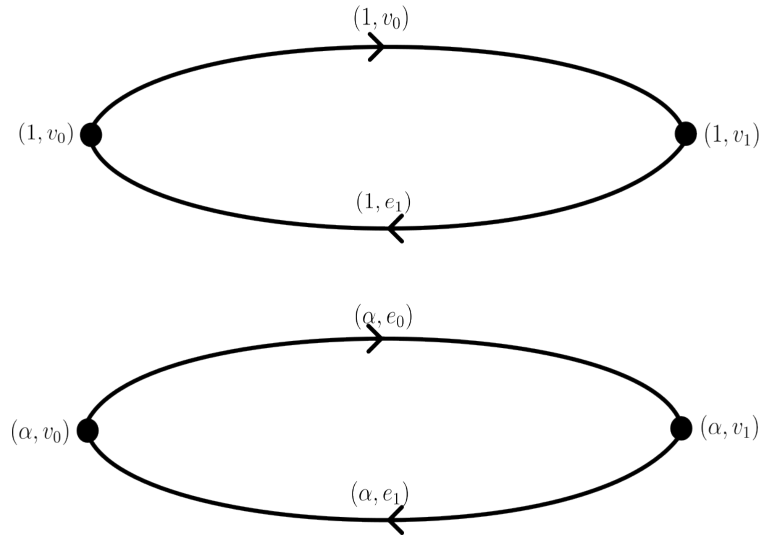


Figure 3.4: PTCP with trivial twisting function

3.4 Simplicial Classifying Complex

Let G be a simplicial group with face maps d_i and degeneracy maps σ_i. Define a simplicial set $\overline{W}(G)$ by setting $(\overline{W}(G))_n = G_{n-1} \times G_{n-2} \times \ldots \times G_0$. For $n = 0$, we set $(\overline{W}G)_0 = \{e\}$. For an element $(g_{n-1}, g_{n-2}, \ldots, g_0)$, define the face and degeneracy maps for each i as:

$$\partial_i(g_{n-1}, \ldots, g_0) = \begin{cases} (g_{n-2}, g_{n-3}, \ldots, g_0) & \text{if } i = 0 \\ (d_{i-1}(g_{n-1}), \ldots, g_{n-i-1} \cdot d_0(g_{n-i}), \ldots, g_0) & \text{if } 0 < i < n \\ (g_{n-1}, g_{n-2}, \ldots, g_1) & \text{if } i > 0 \end{cases}$$

$$s_i(g_{n-1}, \ldots, g_0) = \begin{cases} (e, g_{n-1}, g_{n-2}, \ldots, g_0) & \text{if } i = 0 \\ (\tau_{i-1}(g_{n-1}), \tau_{i-2}(g_{n-2}), \ldots, g_{n-i-1} \cdot \tau_0(g_{n-i}), \ldots g_0) & \text{if } i < n \end{cases}$$

One can verify that the above maps satisfy the conditions in 2.3.2. We call this the *simplicial classifying complex* of the simplicial group G.

Define a simplicial set $W(G)$ by setting $W(G)_n = \overline{W}(G)_{n+1}$, $\partial_i' = \partial_{i+1}$, and $s_i' = s_{i+1}$. The n-simplices of WG are thus tuples in $G_n \times G_{n-1} \times \cdots \times G_0$. There is a natural projection from WG to $\overline{W}G$ sending $(g_n, g_{n-1}, \ldots, g_0)$ to $(g_{n-1}, g_{n-2}, \ldots, g_0)$. This projection can be realised as a principal fibration:

Define a right G-action on WG via the simplicial map

$$\phi : WG \times G \to WG$$
$$((g_n, g_{n-1}, \ldots, g_0), g) \mapsto (g_n \cdot g, g_{n-1}, \ldots, g_0)$$

Level-wise, G_n acts on itself and it follows that the action of G on WG is principal. Define a quotient complex WG/G of WG by identifying $(g_n, g_{n-1}, \ldots, g_0) \in WG$ with $\phi((g_n, g_{n-1}, \ldots, g_0), g)$ for $g \in G$. The projection $p : WG \to \overline{W}G$ is a principal fibration by definition. Define a map $f : \overline{W}G \to WG/G$ sending a tuple $(g_{n-1}, g_{n-2}, \ldots, g_0)$ to the equivalence class $[(e_n, g_{n-1}, \ldots, g_0)] \in WG/G$.

Proposition 3.4.1. The map f as defined above is an isomorphism of simplicial sets.

Proof. Let $(g_{n-1}, \ldots, g_0), (g'_{n-1}, \ldots g'_0) \in \overline{W}G$. If $f((g_{n-1}, \ldots, g_0)) = f((g'_{n-1}, \ldots g'_0))$, then there is a $g_n \in G_n$ such that $(g_n, g_{n-1}, \ldots, g_0) = (g_n, g'_{n-1}, \ldots, g'_0)$. It follows that $g_n = e_n$ and $g'_i = g_i$ for all $0 \leq i \leq n-1$, and hence, f is injective. Suppose $[(g_n, g_{n-1}, \ldots, g_0)] \in WG/G$. It is easy to see that f maps $(g_{n-1}, \ldots, g_0)$ to $[(g_n, g_{n-1}, \ldots, g_0)] \in WG/G$, showing that f is surjective. The verification that f is a simplicial map is left as an exercise. □

It is possible to view the projection $p : WG \to \overline{W}G$ as a PTCP. Define a function $\tau : \overline{W}(G)_n \to G_{n-1}$ by $\tau((g_{n-1}, g_{n-2}, \ldots, g_0)) = g_{n-1}$. It is an easy check that τ is a twisting function. Define a map $\varphi : G \times_\tau \overline{W}G \to \overline{W}G$ sending a tuple $(g, (g_n, g_{n-1}, \ldots, g_0))$ to $(g, g_n, g_{n-1}, \ldots, g_0)$.

Proposition 3.4.2. The map $\varphi : G \times_\tau \overline{W}G \to \overline{W}G$ is an isomorphism of simplicial sets.

Proof. This map is clearly bijective. It remains to show that this map is a simplicial map. We show the case for ∂_0 here.

$$\begin{aligned}\partial_0\varphi(g,(g_{n-1},\dots,g_0)) &= \partial_0(g,g_{n-1},\dots,g_0)\\ &= (g_{n-1}\cdot\partial_0 g,\dots,g_0)\end{aligned}$$

$$\begin{aligned}\varphi(\partial_0(g,g_{n-1},\dots,g_0)) &= \varphi(g_{n-1}\cdot\partial_0 g,(g_{n-2},\dots,g_0))\\ &= (g_{n-1}\cdot\partial_0 g,g_{n-2},\dots,g_0)\end{aligned}$$

The check for other face and degeneracy maps are similar and are left to the reader. □

Proposition 3.4.3 ([May67, Lemma 21.3])**.** For any simplicial group G, the simplicial set $\overline{W}G$ is a Kan complex.

For the remainder of this thesis, elements of $\overline{W}G$ will be tuples $(g_{n-1},\dots,g_0)$.

Proposition 3.4.4. Let G be a simplicial group. Then $\pi_n(WG)=0$ for all $n\geq 0$.

Proof. We base our complex at the 0-simplex e. Let e_n denote the tuple in WG with all entries e. $\widetilde{WG}_1=\{(g_0,g_1)\mid\partial_i'(g_0,g_1)=e\;\forall\, i\}$. Unraveling this, we get $(g_0\cdot g_1)=e$ and $g_1=e$. Thus, $g_0=g_1=e$. A similar process continues. For $\pi_n(WG,e)$, the n^{th} face map forces $(g_1=g_2=\ldots=g_n=e)$. Combining this with any other face map, we get $g_0=e$. So for $n\geq 1$, $\pi_n(WG,e)$ is trivial since the subcomplexes $(\widetilde{WG})_n$ themselves are trivial.

For $n=0$ we inspect when two 0-simplices are homotopic in WG. Let g,g' be two 0-simplices of WG. We want to find $(h,h')\in G\times G$ such that $h\cdot h'=g$ and $h'=g'$. The obvious candidate for this is $(g(g')^{-1},g')$. It follows that any two 0-simplices are homotopic and hence, $\pi_0(WG,e)$ is trivial. □

Proposition 3.4.5. Let cG be the constant simplicial group associated to a discrete group G. Then, $\pi_1(\overline{W}cG)=G$.

Proof. $\overline{W}G$ is obviously connected since $(\overline{W}G)_0=e$. For $n\geq 2$ the argument that $\pi_n(\overline{W}G,e)=e_n$ is similar to that of WG. For $n=1$, $(\widetilde{\overline{W}G})_1=\{g\in G\mid\partial_0 g=\partial_1 g=e\}$.

However, this is true for every g since both face maps from $(\overline{W}G)_1$ are the constant map to e. The result follows. $\square$

Let $\beta : G \to G'$ be a simplicial group homomorphism. Define a map $\beta_* : \overline{W}G \to \overline{W}G'$ by

$$\beta_*(g_{n-1}, \ldots, g_0) = (\beta(g_{n-1}, \ldots, \beta(g_0))$$

The map β_* is the map between $\overline{W}G$ and $\overline{W}G'$ induced from f. This discussion is summarised in the following proposition.

Proposition 3.4.6. $\overline{W}$ defines a functor from the category of simplicial groups to the category of reduced simplicial sets.

$$\overline{W} : \mathbf{sGrp} \to \mathbf{sSet_0}$$

Proof. The reader is referred to the discussion following Corollary 21.8 in [May67]. $\square$

Notation 3.4.1. Let G be a CW group. Define $BG := |\overline{W}S_\bullet G|$. The space BG will be our model for a classifying space of G. This classifying space comes equipped with a principal fibration $EG := |WS_\bullet G| \to BG$.

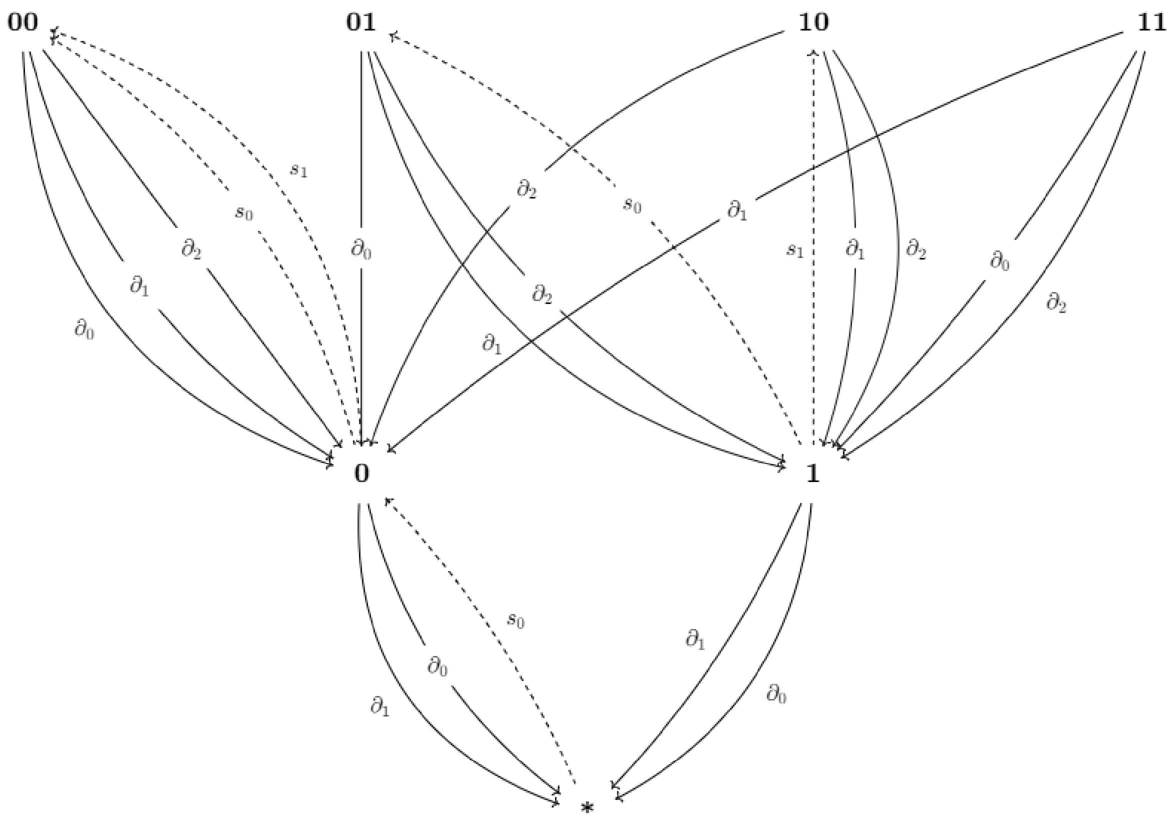
3.4.1 Simplicial Classifying Complex of cC_2

Let C_2 denote the cyclic group of order 2. There is a constant simplicial set denoted cC_2 associated to C_2.

Proposition 3.4.7. Let G be a discrete topological group. Then, $S_\bullet G$ is isomorphic to cG.

Proof. S_nG comprises continuous maps from the standard n-simplex Δ_n and G. The image of a map $f : \Delta_n \to G$ must be a singleton since all connected components of G are singletons. Thus S_nG comprises constant maps to G. Thus S_nG_0 is ismorphic to G_0. One can check that the face and degeneracy maps are simply the identity maps. $\square$

The set of n-simplices of $\overline{W}cC_2$ is $\underbrace{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z}}_{n \text{ times}}$. Each element is a tuple of 0s and 1s, which we will identify (for notational convenience) with binary strings.



The diagram above shows a network of face and degeneracy maps between the 0, 1, and 2 simplices of $\overline{W}cC_2$. Solid arrows denote face maps, which go from up to down. Dotted ones denote degeneracy maps, which go from down to up. The simplices at the far right of the diagram, i.e. the 111...11s, have no dotted arrows ending at them. Recall that $\mathbb{R}P^{\infty}$ is constructed by recursively attaching an n-cell to $\mathbb{R}P^{n-1}$ for each n. We try to relate the existence of only 1 non-degenerate simplex in each level of our network with the construction of $\mathbb{R}P^{\infty}$. We first show that BC_2 has only one cell in each dimension.

Let $n = 0$. The only 0-simplex of $\overline{W}cC_2$ is $*$. It is immediate that $(s_0e, (t, 1-t) \sim (*, \{1\})$, which translates to $(0, (t, 1-t)) \sim (*, \{1\})$, for $0 \leq t \leq 1$. Thus, every element of $\{0\} \times \Delta_1$ is equivalent to $(*, \{1\})$.

Now suppose $n = 1$. The relations for this level are:

$$(\partial_0 0, \{1\}) \sim (0, (0,1)) \Rightarrow (*, \{1\}) \sim (0, (0,1))$$
$$(\partial_1 0, \{1\}) \sim (0, (1,0)) \Rightarrow (*, \{1\}) \sim (0, (1,0))$$
$$(\partial_0 1, \{1\}) \sim (1, (0,1)) \Rightarrow (*, \{1\}) \sim (1, (0,1))$$
$$(\partial_1 1, \{1\}) \sim (1, (1,0)) \Rightarrow (*, \{1\}) \sim (1, (1,0))$$

Figure 3.5: Face relations for $n = 1$

$$(s_0 0, (t_0, t_1, t_2)) \sim (0, (t_0 + t_1, t_2)) \Rightarrow (00, (t_0, t_1, t_2)) \sim (0, (t_0 + t_1, t_2))$$
$$(s_1 0, (t_0, t_1, t_2)) \sim (0, (t_0, t_1 + t_2)) \Rightarrow (00, (t_0, t_1, t_2)) \sim (0, (t_0, t_1 + t_2))$$
$$(s_0 1, (t_0, t_1, t_2)) \sim (1, (t_0 + t_1, t_2)) \Rightarrow (01, (t_0, t_1, t_2)) \sim (1, (t_0 + t_1, t_2))$$
$$(s_1 1, (t_0, t_1, t_2)) \sim (1, (t_0, t_1 + t_2)) \Rightarrow (10, (t_0, t_1, t_2)) \sim (1, (t_0, t_1 + t_2))$$

Figure 3.6: Degeneracy relations for $n = 1$

We can already infer a few things from the above set of relations along with the $n = 0$ case. The first two relations are special cases of (3.5). The third and fourth combined tell us to attach the 1-simplex "1" to e. According to this, the starting and ending points of the 1-simplex "1" are both e, giving us a loop. From the 5th to 8th relations, we gather that every pair in $B_2C_2 \times \Delta_2$ with first entry degenerate can be identified with some pair in $B_1C_2 \times \Delta_1$. As long as there are points in $\{(11)\} \times \Delta_2$ that cannot be identified, (11) will contribute to a 2-cell.

$$(\partial_0(11), (t, 1-t)) \sim (11, (0, t, 1-t)) \Rightarrow (1, (t, 1-t)) \sim (11, (0, t, 1-t))$$
$$(\partial_1(11), (t, 1-t)) \sim (11, (t, 0, 1-t)) \Rightarrow (0, (t, 1-t)) \sim (11, (t, 0, 1-t))$$
$$(\partial_2(11), (t, 1-t)) \sim (11, (t, 1-t, 0)) \Rightarrow (1, (t, 1-t)) \sim (11, (t, 1-t, 0))$$

Figure 3.7: Face relations for $n = 2$

Observing the face maps, we see that there is information on how to attach the edges of (11) to either e or $(1,(t,1-t))$ and the interior does not collapse. Thus (11) is indeed a 2-cell.

Let 1^n denote the string $\underbrace{11...1}_{\text{n times}}$. The above calculations show that there is a canonical representative of each class in BC_2, i.e., the pair $1^n, (t_0, t_1, \ldots, t_n))$

To see why BC_2 is homeomorphic to $\mathbb{R}P^\infty$, we observe how the n-cell is attached to the $(n-1)$-skeleton. From our calculations above, it is clear that the 0-skeleton contains just 1 point $(*, \{1\})$. The 1-skeleton contains e_0 and one 1-cell formed of points of the type $(1,(t,1-t))$ for $t \in [0,1]$, whose boundary is identified with e_0. Now we attach the 2-cell, in 2 steps. Our 2-cell can be identified with Δ_2. The second relation in 3.6 tells us that the line segment formed by points of the form $(11,(t,0,1-t))\Delta_2$ are all identified with the 0-cell. In the first step, we identify all these points, to form a disk whose boundary comprises two 1-cells: one 1-cell with points of the form $(0,t,1-t)$, and one 1-cell with points of the form $(11,(0,t,1-t))$. Let us call this disk $\tilde{\Delta}_2$. According to the first and third relations of (2), we attach the boundary of the $\tilde{\Delta}_2$ onto the 1-cell via the map sending both $(0,t,1-t)$ and $(t,1-t,0)$ to $(1,(t,1-t))$. Recall that one method to construct $\mathbb{R}P^2$ from $\mathbb{R}P^1$ is to attach a disk D_2 by identifying its boundary $\partial D^2 \cong S^1$ to $\mathbb{R}P^1$ via the quotient map $S^1 \to \mathbb{R}P^1$. Identify the boundary of $\widetilde{\Delta}_2$ with S^1 and the 1-skeleton with $\mathbb{R}P^1$. It is easy to see that the resultant map from S^1 to $\mathbb{R}P^1$ is just the quotient map. This similar process is repeated for each $n \geq 2$ indefinitely to give us $\mathbb{R}P^\infty$.

3.5 Eilenberg-MacLane Complexes

For a topological group G, write $B_{\text{Mil}}G$ for its classifying space constructed via the Milnor Construction. For discrete G, the long exact sequence of homotopy groups associated to the fiber sequence $G \to EG \to BG$ entails that BG has only one non-trivial homotopy group $\pi_1(BG) = G$. The classifying space $B_{\text{Mil}}G$ is one instance of a large family of spaces known as Eilenberg-MacLane spaces.

3.5.1 Definition of Eilenberg-MacLane Complexes

Definition 3.5.1. Let G be a group. An *Eilenberg-MacLane space* of type $(G, 1)$ is a path-connected topological space with contractible universal cover and fundamental group isomorphic to G.

The equivalence between Kan complexes and CW complexes described in Subsection 3.2.3 motivates a simplicial analog of Eilenberg MacLane spaces.

Definition 3.5.2. Let K be a simplicial set and let a and b be two n-simplices. K is said to be *minimal* if a and b are homotopic if and only if they are equal.

Definition 3.5.3. Let (K, ϕ) be a Kan pair and let π be a group. K is said to be an *Eilenberg-MacLane complex* of type (π, n) if K is minimal, $\pi_n(K, \phi) = \pi$, and $\pi_i(K, \phi) = 0$ for all $i \neq n$.

Example 13. Let π be a group, and let $c\pi$ denote the constant simplicial group associated to π as defined in Example 6. It is an easy check that $\pi_0(c\pi, e) = \pi$ and $\pi_i(c\pi, e) = 0$ for all $i > 0$. We leave the minimality of $c\pi$ as an exercise. The simplicial set $c\pi$ is a $K(\pi, 0)$.

Example 14. Let $c\pi$ be the constant simplicial group as in the previous example. Consider $\overline{W}(c\pi)$, the simplicial classifying complex of $c\pi$. By the long exact sequence of homotopy groups associated to the principal fibration $W(c\pi) \to \overline{W}(c\pi)$:

$$\cdots \to \pi_1(W(c\pi)) \to \pi_1(\overline{W}(c\pi)) \to \pi_0(c\pi) \to \pi_0(W(c\pi)) \to \pi_0(\overline{W}(c\pi)) \to 0$$

$$\cdots \to \pi_n(c\pi) \to \pi_n(W(c\pi)) \to \pi_n(\overline{W}(c\pi)) \to \pi_{n-1}(c\pi) \to \cdots$$

$\pi_0(\overline{W}(c\pi))$ is trivial since it has only one 0-simplex by definition. The simplicial set $c\pi$ is a $K(\pi, 0)$ as described in the previous example. Further, all the homotopy groups of $W(c\pi)$ are trivial. Using these facts, it follows that $\overline{W}(c\pi)$ is a $K(\pi, 1)$.

We may iterate the process in the previous example to obtain a $K(\pi, n)$ for any $n > 0$. Let π be an abelian group.

Proposition 3.5.1. Let A be a simplicial abelian group. Then $\overline{W}A$ is a simplicial abelian group as well.

Proof. This proof follows from the construction of $\overline{W}A$. The face and degeneracy maps can easily be seen to be abelian group homomorphisms. $\square$

The above proposition implies that the functor $\overline{W}$ can be applied on a simplicial abelian group successively. Using the analysis presented in Example 14, it follows that $\overline{W}^n A$ is a $K(\pi, n)$.

3.5.2 General Construction of a $K(\pi, n)$

We now try to understand how to associate a chain complex to a simplicial abelian group. This seemingly unrelated topic will aid us in describing a general construction for $K(\pi, n)$s. Let G be a simplicial abelian group. Define a map $\delta_n : G_n \to G_{n-1}$ by $\delta_n = \sum_{i=0}^{n} \partial_i$.

Proposition 3.5.2. G with differentials δ_n has the structure of a chain complex.

Proof. We must show that $\delta_{n-1} \circ \delta_n = 0$. Let $g \in G_n$.

$$\delta(\delta(g)) = \sum_{i=0}^{n-1} \sum_{j=0}^{n} g$$

We represent the operation $\delta_{n-1} \circ \delta_n$ as a grid of face maps whose (i, j) entry is $\partial_i \partial_j$

$$\begin{array}{cccccc}
\partial_0\partial_0 & \partial_0\partial_1 & \partial_0\partial_2 & \dots & \partial_0\partial_{n-1} & \partial_0\partial_n \\
\partial_1\partial_0 & \partial_0\partial_i & \partial_0\partial_2 & \dots & \partial_1\partial_{n-1} & \partial_0\partial_n \\
\partial_2\partial_0 & \partial_0\partial_i & \partial_0\partial_2 & \dots & \partial_2\partial_{n-1} & \partial_0\partial_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\partial_{n-2}\partial_0 & \partial_{n-2}\partial_1 & \partial_{n-2}\partial_2 & \dots & \partial_{n-2}\partial_{n-1} & \partial_{n-2}\partial_n \\
\partial_{n-1}\partial_0 & \partial_{n-1}\partial_1 & \partial_{n-1}\partial_2 & \dots & \partial_{n-1}\partial_{n-1} & \partial_{n-1}\partial_n
\end{array}$$

We know that $\partial_i\partial_j = \partial_{j-1}\partial_i$ for $i < j$. Thus $\partial_0\partial_1 = \partial_0\partial_0$, $\partial_0\partial_2 = \partial_1\partial_0$ and so on. Since the sum of indices in this identity shifts down by 1, we get that $(-1)^{i+j}\partial_i\partial_j + (-1)^{i+j-1}\partial_{j-1}\partial_i = 0$. Cancelling out terms for $i = 0$, we get:

$$
\begin{array}{cccccc}
\cancel{\partial_0\partial_0} & \cancel{\partial_0\partial_1} & \cancel{\partial_0\partial_3} & \dots & \cancel{\partial_1\partial_{n-1}} & \cancel{\partial_1\partial_n} \\
\cancel{\partial_2\partial_1} & \partial_1\partial_1 & \partial_0\partial_2 & \dots & \partial_1\partial_{n-1} & \partial_0\partial_n \\
\cancel{\partial_3\partial_1} & \partial_3\partial_2 & \partial_3\partial_3 & \dots & \partial_3\partial_{n-1} & \partial_0\partial_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cancel{\partial_{n-2}\partial_1} & \partial_{n-2}\partial_2 & \partial_{n-2}\partial_3 & \dots & \partial_{n-2}\partial_{n-1} & \partial_{n-2}\partial_n \\
\cancel{\partial_{n-1}\partial_1} & \partial_{n-1}\partial_2 & \partial_{n-1}\partial_3 & \dots & \partial_{n-1}\partial_{n-1} & \partial_{n-1}\partial_n
\end{array}
$$

The highlighted portion represents the terms that may contribute to a nonzero-sum after the first step of cancellation. Continuing this for $i = 1$, we get:

$$
\begin{array}{cccccc}
\cancel{\partial_0\partial_0} & \cancel{\partial_0\partial_1} & \cancel{\partial_0\partial_2} & \dots & \cancel{\partial_0\partial_{n-1}} & \cancel{\partial_0\partial_n} \\
\cancel{\partial_1\partial_0} & \cancel{\partial_1\partial_1} & \cancel{\partial_1\partial_2} & \dots & \cancel{\partial_1\partial_{n-1}} & \cancel{\partial_1\partial_n} \\
\cancel{\partial_2\partial_0} & \cancel{\partial_2\partial_1} & \partial_2\partial_2 & \dots & \partial_2\partial_{n-1} & \partial_2\partial_n \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\cancel{\partial_{n-2}\partial_0} & \cancel{\partial_{n-2}\partial_1} & \partial_{n-2}\partial_2 & \dots & \partial_{n-2}\partial_{n-1} & \partial_{n-2}\partial_n \\
\cancel{\partial_{n-1}\partial_0} & \cancel{\partial_{n-1}\partial_1} & \partial_{n-1}\partial_3 & \dots & \partial_{n-1}\partial_{n-1} & \partial_{n-1}\partial_n
\end{array}
$$

On continuing this process, we observe that all terms get canceled out. This ends the proof. □

Let $D(G)$ denote the chain complex generated by the degenerate simplices of G. That is, $D(G)_n = \sum_{i=0}^{n-1} \mathrm{Im}(s_i : G_{n-1} \to G_n)$. We first conduct a preliminary check that the differential defined for $A(G)$ also defines a differential for $D(G)$.

Proposition 3.5.3. Let G be a simplicial abelian group, and $A(G)$ its associate chain complex with differential δ. Given $x \in D(G)_n$, δx belongs to $D(G)_{n-1}$.

Proof. It suffices to show that the proposition is true for any degenerate simplex. Let $x \in D(G)_n$ be a degenerate simplex. Then $x = s_i y$ for some simplex $y \in G_{n-1}$ and some $0 \leq i \leq n-1$. Using properties 3, 4 and 5 in 3.2.2 we get

$$\begin{aligned}\delta x &= \sum_{j=0}^{n} (-1)^j \partial_j x \\ &= \sum_{i=0}^{n} \partial_j s_i y \\ &= s_{i-1}\left(\sum_{j=0}^{i-1} (-1)^j \partial_j y\right) + s_i \left(\sum_{j=i+2}^{n} \partial_j y\right)\end{aligned}$$

which belongs to $D(G)_{n-1}$. □

Corollary 3.5.4. $D(G)$ with differential δ as defined above is a chain subcomplex of $A(G)$.

Since G comprises abelian groups, $D(G)_n$ is a normal subgroup of $A(G)_n$. Define the *normalised chain complex* associated to G by $A_N(G) := A(G)/D(G)$.

Proposition 3.5.5 ([May67, Corollary 22.3])**.** The map $A(G) \to A_N(G)$ sending any element to its representative in the normalized chain complex is a chain equivalence.

Consider the simplicial set $\Delta[q]$ described in example 5. Using the construction in 3.2.1, we obtain a chain complex $C'_\bullet(\Delta[q])$ with $C_n(\Delta[q]) = F_n^s(\Delta[q]) = F(\mathrm{Hom}_{\Delta^*}([n],[m]))$.

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \mathrm{Hom}([n-1],[m]) & \xrightarrow{\delta_{n-1}} & \mathrm{Hom}([n],[m]) & \xrightarrow{\delta_n} & \mathrm{Hom}([n+1],[m]) & \longrightarrow \cdots \\ & \| & & \| & & \| & \\ & C_{n-1}(\Delta[q]) & & C_n(\Delta[q]) & & C_{n+1}(\Delta[q]) & \end{array}$$

Let $\delta_i : [n-1] \to [n]$ be a coface map and $\sigma_i : [n] \to [n-1]$ be a codegeneracy map in Δ^*. The functor $\mathrm{Hom}_{\Delta^*}([m], -)$ induces maps $\bar{\delta}_i : \Delta[n-1] \to \Delta[n]$ and $\bar{\sigma}_i : \Delta[n] \to \Delta[n-1]$.

Proposition 3.5.6. The maps $\bar{\delta}_i$ and $\bar{\sigma}_i : \Delta[n] \to \Delta[n-1]$ are simplicial maps.

Proof. Let $\delta_j^{(n)}$ denote the j^{th} coface map in Δ^* from $[n-1]$ to $[n]$. The proof is an immediate consequence of the commutativity of the following diagram:

$$\begin{CD}
\mathrm{Hom}([q],[n-1]) @>{(\bar{\delta}_i)_q}>> \mathrm{Hom}([q],[n]) \\
@V{\partial_j}VV @VV{\partial_j}V \\
\mathrm{Hom}([q-1],[n-1]) @>>{(\bar{\delta}_i)_{q-1}}> \mathrm{Hom}([q-1],[n])
\end{CD}$$

For $f \in \mathrm{Hom}([q],[n-1])$,

$$\begin{aligned}
(\partial_j^{(n)}((\bar{\delta}_i)_q)(f)) &= ((\bar{\delta}_i)_q(f)) \circ \delta_j^{(q-1)} \\
&= \delta_i^{(n-1)} \circ f \circ \delta_j^{(q-1)}
\end{aligned}$$

$$\begin{aligned}
(\bar{\delta}_i)_{q-1}(\partial_j^{(n-1)} f) &= \delta_i^{(n-1)} \circ (\partial_j^{(n-1)}(f)) \\
&= \delta_i^{(n-1)} \circ f \circ \delta_j^{(q-1)}
\end{aligned}$$

Showing that $\bar{\delta}_i$ commutes with degeneracy maps is left as an exercise to the reader as is the proof for $\bar{\sigma}_i$ being a simplicial map. □

Let $C_\bullet(\Delta[n])$ denote the normalized chain complex of $C'_\bullet(\Delta[q])$. The maps $\bar{\delta}_i$ and $\bar{\sigma}_i$ defined earlier extend uniquely to maps $\bar{\delta}_i : C_\bullet(\Delta[n]) \to C_\bullet(\Delta[n-1])$ and $\bar{\sigma}_i : C_\bullet(\Delta[n-1]) \to C_\bullet(\Delta[n])$.

Let π be an abelian group. Define the normalized cochain complex $C^\bullet(\Delta[n],\pi)$ by setting $C^\bullet(\Delta[n],\pi) = \mathrm{Hom}_{\mathbb{Z}}(C_\bullet(\Delta[q]),\pi)$. For a fixed n, let $L(\pi, n+1)_m = C^n(\Delta[q],\pi)$. Define maps $\partial_i : L(\pi, n+1)_m \to L(\pi, n+1)_{m-1}$ and $s_i : L(\pi, n+1)_m \to L(\pi, n+1)_{m+1}$:

$$\begin{aligned}
\partial_i u(x) &= u(\partial_i x), \quad u \in L(\pi, n+1)_{m+1},\ x \in C_n(\Delta[q]) \\
s_i v(y) &= v(\sigma_i y), \quad v \in L(\pi, n+1)_m,\ y \in C_n(\Delta[m+1])
\end{aligned}$$

The above face and degeneracy maps make $L(\pi, n+1)$ into a simplicial abelian group. Let $Z^n(\Delta[q], \pi) = \ker \partial_n^*$ be the subgroup of $C_n(\Delta[n], \pi)$ consisting of cocycles. Set

$$\overline{K}(\pi, n)_m = Z^n(\Delta[q], \pi)$$

Proposition 3.5.7. $\overline{K}(\pi, n)$ is a subcomplex of $L(\pi, n+1)$

Proof. It suffices to show that the face and degeneracy maps of $L(\pi, n+1)$ take cocycles to cocycles. That is, if $u \in L(\pi, n+1)_q$ is a cocycle, we need to show that $\delta\partial_i u = 0$ in $C^{n+1}(\Delta[q-1], \pi)$. Consider the following diagram:

$$\begin{array}{ccc} C^n(\Delta[q], \pi) & \xrightarrow{\partial_i^{(n)}} & C^n(\Delta[q-1], \pi) \\ {\scriptstyle \delta^{(q)}}\downarrow & & \downarrow{\scriptstyle \delta^{(q-1)}} \\ C^{n+1}(\Delta[q], \pi) & \xrightarrow[\partial_i^{(n-1)}]{} & C^{n+1}(\Delta[q-1], \pi) \end{array}$$

Proving the commutativity of this diagram is equivalent to proving the proposition. This is because if $u \in C^n(\Delta[q], \pi)$ is a cocycle, then $\partial_i^{(n+1)}(\delta^{(q)} u) = 0$. Using the commutativity of the above diagram, this forces $\delta^{(q-1)}(\partial^{(n)} u) = 0$, showing that $\partial^{(n)} u$ is a cocycle.

To prove commutativity we observe that the following simpler diagram commutes:

$$\begin{array}{ccc} C_n(\Delta[q]) & \xrightarrow{\bar{\delta}^{(n)}} & C_n(\Delta[q-1]) \\ {\scriptstyle \delta_{(q)}}\uparrow & & \uparrow{\scriptstyle \delta_{(q-1)}} \\ C_{n+1}(\Delta[q]) & \xrightarrow[\bar{\delta}^{(n+1)}]{} & C_{n+1}(\Delta[q-1]) \end{array}$$

However, the commutativity of the above diagram follows from Proposition 3.5.6. Functors preserve composition and thus the commutativity of the second diagram implies the commutativity of the first. This completes the proof. □

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^0\big(\Delta[m+1],\pi\big) & \xrightarrow{\partial_0^*} & C^1\big(\Delta[m+1],\pi\big) & \xrightarrow{\partial_1^*} & C^2\big(\Delta[m+1],\pi\big) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \delta^0_{m+1}} & & \Big\downarrow{\scriptstyle \delta^1_{m+1}} & & \Big\downarrow{\scriptstyle \delta^2_{m+1}} & & \\
0 & \longrightarrow & C^0\big(\Delta[q],\pi\big) & \xrightarrow{\partial_0^*} & C^1\big(\Delta[q],\pi\big) & \xrightarrow{\partial_1^*} & C^2\big(\Delta[q],\pi\big) & \longrightarrow & \cdots \\
 & & \Big\downarrow{\scriptstyle \delta^0_{m}} & & \Big\downarrow{\scriptstyle \delta^1_{m}} & & \Big\downarrow{\scriptstyle \delta^2_{m}} & & \\
0 & \longrightarrow & C^0\big(\Delta[m-1],\pi\big) & \xrightarrow{\partial_0^*} & C^1\big(\Delta[m-1],\pi\big) & \xrightarrow{\partial_{1*}} & C^2\big(\Delta[m-1],\pi\big) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

In the figure above, the horizontal rows are the normalized cochain complexes $C^\bullet(\Delta[q],\pi)$ and the vertical columns are the simplicial sets $L(\pi, n+1)$ for $n \geq 0$ with vertical maps as the face maps.

Proposition 3.5.8 ([May67, Theorem 23.9])**.** $\overline{K}(\pi, n)$ as defined above is a $K(\pi, n)$.

Finally, we state the following theorem that relates maps from K into $\overline{K}(\pi, n)$ with cohomology classes of K with coefficients in π.

Theorem 3.5.9 ([May67, Theorem 24.4])**.** There is a bijective correspondence between homotopy classes of simplicial maps $[K, \overline{K}(\pi, n)]$ and $H^n_{\mathbf{SS}}(K, \pi)$.

Chapter 4

Constructions of α-spaces

This chapter deals with the constructions of α-spaces over locally compact topological spaces and CW groups. In the first section, we describe the construction of α-spaces. In the second section, we discuss a possible group structure on α-spaces arising from CW groups. We then present a simplicial construction that leads us to define α-groups. Theorem 4.1.6 is an original result, the proof of which follows easily from known results. The constructions of α-spaces and α-groups are original. The discussions pertaining to Eilenberg-MacLane spaces can be found in [Hat02]. For details on the Kan Loop Group functor, the reader is referred to [May67].

4.1 Fibrations Over Locally Compact Spaces

Let Y be a CW complex and A be a discrete abelian group. Let $K(A,i)$ be an Eilenberg-MacLane space of type (A,i). Let $\alpha \in H^i(Y,A)$. The following is the analog of Theorem 3.5.9 for topological spaces.

Theorem 4.1.1 ([Hat02, Pg. 393])**.** There are natural bijections $T : [X, K(A,n)] \to H^i(X,A)$ for all CW complexes X and all $n > 0$. Such a T has the form $T([f]) = f^*(\alpha_0)$ for a distinguished class $\alpha_0 \in H^n(K(A,n),A)$.

A class α_0 with the property above is called a *fundamental class*.

Corollary 4.1.2. Let $f : X \to K(A,n)$ be a map and let α_0 be a fundamental class in $H^n(K(A,n),A)$. Then, f is nullhomotopic if and only if $f^*(\alpha_0) = 0$

Proof. Suppose f is nullhomotopic. Then, f induces the 0-map on cohomology and $f^*(\alpha_0) = 0$. Conversely, suppose $f^*(\alpha_0) = 0$. We know that a constant map c induces the 0-map on cohomology and $c^*(\alpha_0) = 0$ in $H^n(K(A,n),A)$. By Theorem 4.1.1, f lies in the same homotopy class as the constant map, i.e. f is nullhomotopic. □

We are now in a position to define α-spaces.

Definition 4.1.1. Let Y be a topological space. Let $\alpha \in H^i(Y,A)$. An *α-space* over Y is a pair (Z,p) of a space Z and a map $p : Z \to Y$, with the property that a map $f : X \to Y$ lifts to Z if and only if $f^*(\alpha) = 0$.

The following describes a construction of α-spaces. Let $PK(A,i)$ denote the path space of $K(A,i)$. The space $PK(A,i)$ comes equipped with a principal fibration $\pi : PK(A,i) \to K(A,i)$ as discussed earlier in Section 1.1. Let $f_\alpha : Y \to K(A,i)$ be a representative of the homotopy class of maps in $[Y, K(A,i)]$ corresponding to α as in Theorem 4.1.1. Note that f_α is unique only up to homotopy. We define $Y\langle\alpha\rangle$ to be the total space of the pullback of π along f_α:

$$\begin{array}{ccc} Y\langle\alpha\rangle & \longrightarrow & PK(A,i) \\ \downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f_\alpha} & K(A,i) \end{array}$$

By definition 1.1.8, the map π_α is a principal fibration. It may have come to the reader's attention that the construction of $Y\langle\alpha\rangle$ involves f_α, but the notation is independent of it.

Proposition 4.1.3. Let f_α and g_α be homotopic maps that correspond to α in $H^i(Y,A)$. Then, the α-spaces over Y corresponding to f_α and g_α are homotopy equivalent.

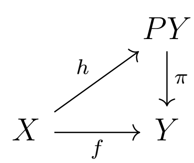
Proof. Note that f_α and g_α are homotopic. The proposition follows from a simple application of Proposition 1.1.2. □

For the rest of this section, we will be interested primarily in locally compact topological spaces. Let X, Y be locally compact topological spaces and let $g : X \to Y$ be a continuous map. Composing with $f_\alpha : Y \to K(A, i)$, we obtain a map from X to $K(A, i)$. By Corollary 4.1.2, $f_\alpha \circ g$ is nullhomotopic if and only if $(f_\alpha \circ g)^*(\alpha_0) = 0$, or equivalently, $g^*(\alpha) = 0$. Now, consider the following proposition.

Proposition 4.1.4 ([Spa66, Chap. 8.2])**.** Let X and Y be locally compact hausdorff spaces. Then a map $f : X \to Y$ is nullhomotopic if and only if it lifts to the path space PY.

Proof. We first show that given a lift to PY, f is nullhomotopic. Let $\tilde{f}$ be a lift to PY. Since PY is contractible, $\tilde{f}$ is nullhomotopic, and hence it is homotopic to a constant map c taking every point in x to the constant path at some $y_0 \in Y$. If $\pi : PY \to Y$ is the path fibration, it follows that the composition $\pi \circ \tilde{f}$ is homotopic to $\pi \circ c$. Since $f = \pi \circ \tilde{f}$, f is homotopic to $\pi \circ c$, which is a constant map, and f is nullhomotopic as required.

Conversely, assume f is nullhomotopic. Then there is a homotopy $H : X \times I \to Y$ such that $H(x, 0) = y_0$ and $H(x, 1) = f(x)$ for some distinguished point y_0. By the exponential correspondence, there is map $h : X \to Y^I$ defined as $h(x)(t) = H(x, t)$. At $t = 0$, we have $h(x)(0) = H(x, 0) = y_0$. Thus, every path in the image of h starts at y_0, and the image of h lies in PY. To complete the proof that h with codomain restricted to PY is the required lift, we must check that the commutative diagram below commutes:

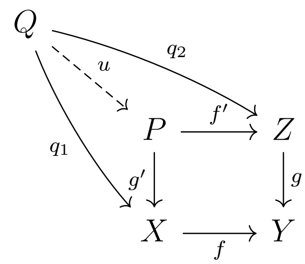


We have $\pi \circ h(x) = h(x)(1) = H(x, 1) = f(x)$ by definition of H. Thus, $h : X \to PY$ is indeed the required lift and this completes the proof. ☐

Remark 4.1.1. For Proposition 4.1.4, [Spa66] assumes that X and Y are locally compact Hausdorff spaces. The exponential correspondence holds for compactly generated spaces as well. (See Chapter 5 of [May99].) Thus, we believe that the given proof can be adapted for compactly generated spaces.

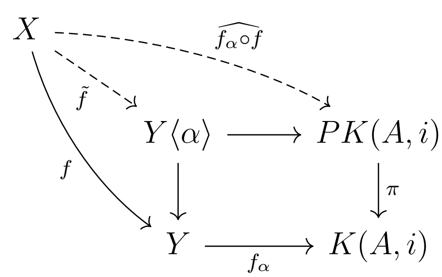
We now deviate a bit and state the universal property of pullbacks.

Proposition 4.1.5. Let P be the pullback of $g : Z \to Y$ along $f : X \to Y$. Let $q_1 : Q \to X$ and $q_2 : Q \to Z$ be maps from a space Q such that $f \circ q_1 = g \circ q_2$. Then there exists a unique map $u : Q \to P$ such that the following diagram commutes:

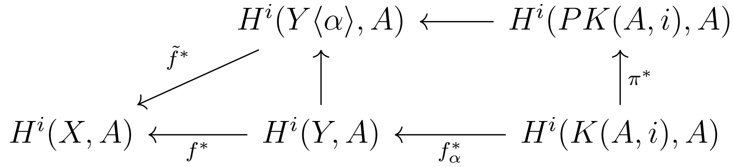


Theorem 4.1.6. Let X be a topological space and $f : X \to Y$ a map. Let $\alpha \in H^i(Y, A)$ be represented by a map f_α to $K(A, i)$. Then, f lifts to a map $\tilde{f} : X \to Y\langle\alpha\rangle$ if and only if $f^*(\alpha) = 0$.

Proof. Consider the following commutative diagram:



Suppose there is a lift $\tilde{f} : X \to Y\langle\alpha\rangle$. We obtain the following induced diagram of degree i cohomology groups:



Since $PK(A, i)$ is contractible, $H^k(PK(A, i)) = 0$. By commutativity of the above diagram, it follows that $f^*(\alpha) = 0$.

Conversely, suppose $f^*(\alpha) = 0$. This is the same as saying $(f^* \circ f_\alpha^*)(\alpha_0) = 0$. By Theorem 4.1.1, $(f^* \circ f_\alpha^*)(\alpha_0) = 0$ if and only if $f_\alpha \circ f$ is nullhomotopic. By prop 4.1.4, $f_\alpha \circ f$ is nullhomotopic if and only if it lifts to the path fibration $\pi : PK(A,i) \to K(A,i)$. The result follows from the universal property of pullbacks. □

The above discussion can be summarised in the following theorem.

Theorem 4.1.7. When Y is locally compact, α-spaces over Y exist.

Proof. The pair $(Y\langle\alpha\rangle, \pi_\alpha)$ is a candidate for an α-space over Y. This pair has the required lifting property, as seen in Theorem 4.1.6. Thus, $(Y\langle\alpha\rangle, \pi_\alpha)$ is indeed an α-space. □

Remark 4.1.2. Although one would expect this lift to be unique at least up to homotopy, this does not appear to be the case. We believe that such a uniqueness property would allow us to treat $Y\langle\alpha\rangle$ as a universal object. However, it is not clear how this can be achieved.

4.2 Fibrations over Topological Groups

In this section, we discuss the analog of the $Y\langle\alpha\rangle$ construction described in the previous section in the context of CW groups. We first define a group structure on $Y\langle\alpha\rangle$ when Y is a topological group. In the next subsection, we describe a simplicial construction of α-spaces.

4.2.1 A Group Structure on α-spaces

Let G be a CW group. Let A be an abelian group. Let $\alpha \in H^i(G, A)$. As a first attempt to construct a CW group $G\langle\alpha\rangle$ and map π_α with the desired properties as in the previous section, one may try to mimic the construction in Section 4.1. Let $f_\alpha : G \to K(A,i)$ be a map corresponding to α.

$$\begin{array}{ccc} G\langle\alpha\rangle & \longrightarrow & PK(A,i) \\ {\scriptstyle\pi_\alpha}\downarrow & & \downarrow{\scriptstyle\pi} \\ G & \xrightarrow{f_\alpha} & K(A,i) \end{array}$$

While this construction above produces a CW complex $G\langle\alpha\rangle$, it is not clear whether $G\langle\alpha\rangle$ admits a group structure with no added hypotheses on G and f_α.

Recall from 3.5.1 that when A is a simplicial abelian group, $\overline{W}A$ is a simplicial abelian group. Using this fact, it is not hard to see that there is a model of $K(A,i)$ as a simplicial abelian group. This was also discussed towards the end of Subsection 3.5.1. Taking the geometric realization of $\overline{K}(A,i)$, we obtain an Eilenberg-MacLane space $K(A,i)$ that is a CW group. For convenience, we write f in place of f_α in the following discussion.

With the added hypotheses that G is locally compact and f is a group homomorphism, $G\langle\alpha\rangle$ admits a group structure. Fix the identity element $e_K \in K(A,i)$ as the base point of $K(A,i)$. The elements of $G\langle\alpha\rangle$ are pairs (g,γ) which satisfy $\gamma(0) = e_K$ and $f(g) = \gamma(1)$. Write $\gamma_{e_K,f(g)} \in PK(A,i)$ for a path starting at e_K and ending at $f(g)$. Note that there may be multiple paths between two fixed points and this notation does not distinguish between such paths. However, this does not pose itself as an issue for our discussion. Let $\gamma^c_{e_K}$ denote the constant path at e_K. Let e_G denote the identity element of G.

Let $(g,\gamma_{e_K,f(g)})$ and $(h,\gamma_{e_K,f(h)})$ be elements of $G\langle\alpha\rangle$. Define an operation on $* : G\langle\alpha\rangle \times G\langle\alpha\rangle \to G\langle\alpha\rangle$ by

$$(g,\gamma_{e_K,f(g)}) * (h,\gamma_{e_K,f(h)}) = (gh,\gamma_{e_K,f(g)}\gamma_{e_K,f(h)})$$

The operation in the second component is the multiplication of paths.

Proposition 4.2.1. The operation $*$ defines a group structure on $G\langle\alpha\rangle$.

Proof. The operation is associative. For if $(g,\gamma_{e_K,f(g)}), (h,\gamma_{e_K,f(h)})$ and $(k,\gamma_{e_K,f(k)})$ are elements of $G\langle\alpha\rangle$,

$$\begin{aligned}
\big((g,\gamma_{e_K,f(g)}) * (h,\gamma_{e_K,f(h)})\big) * (k,\gamma_{e_K,f(k)}) &= (gh,\gamma_{e_K,f(h)}\gamma_{e_K,f(h)}) * (h,\gamma_{e_K,f(h)}) \\
&= (ghk,\gamma_{e_K,f(g)}\gamma_{e_K,f(h)}\gamma_{e_K,f(k)}) \\
&= (g,\gamma_{e_K,f(g)}) * (hk,\gamma_{e_K,f(h)}\gamma_{e_K,f(k)}) \\
&= (g,\gamma_{e_K,f(g)}) * \big((h,\gamma_{e_K,f(h)}) * (k,\gamma_{e_K,f(k)})\big)
\end{aligned}$$

The identity element of $G\langle\alpha\rangle$ with respect to $*$ is $(e_G,\gamma^c_{e_K})$ since both elements individually act as identities. The inverse of an element $(g, \gamma_{e_K, f(g)})$ is $(g^{-1}, \gamma_{e_K, f(g)^{-1}})$. This ends the proof. □

Corollary 4.2.2. Let G be a locally compact group and $G\langle\alpha\rangle$ an α-space with group structure as defined above. Then the projection $\mathrm{pr}_1 : G\langle\alpha\rangle \to G$ is a group homomorphism.

Proof. For $(g, \gamma_{e_K, f(g)}), (h, \gamma_{e_K, f(h)}) \in G\langle\alpha\rangle$,

$$\begin{aligned} \mathrm{pr}_1\big((g, \gamma_{e_K, f(g)}) * (h, \gamma_{e_K, f(h)})\big) &= \mathrm{pr}_1(gh, \gamma_{e_K, f(g)}\gamma_{e_K, f(h)}) \\ &= gh \\ &= \mathrm{pr}_1(g, \gamma_{e_K, f(g)}) * \mathrm{pr}_1(h, \gamma_{e_K, f(h)}). \end{aligned}$$

□

Remark 4.2.1. Although we have constructed an α-space over G which admits a group structure, the machinery of Theorem 4.1.6 does not give us a lift of f_α as a group homomorphism. We have not shown the existence of such a lift but believe it should exist given appropriate hypotheses.

4.2.2 A Simplicial Construction of α-groups

In this subsection, we describe a simplicial construction that is a candidate for an α-group over a given CW group G. We first introduce the *Kan Loop Group functor*, which is central to the construction discussed later.

Definition 4.2.1. A simplicial set K is said to be *0-reduced* if K_0 is a singleton.

Example 15. Let G be a simplicial group. Then, $\overline{W}G$ is a 0-reduced simplicial set.

We may analogously define n-reduced simplicial sets as simplicial sets with K_i singletons for all $i < n$.

Definition 4.2.2. Let K be a simplicial set. A simplicial group $G(K)$ is said to be a *loop group* of K if there exists a PTCP $E(\tau) = G(K) \times_\tau K$ such that $|E(\tau)|$ is contractible.

Example 16. Let π be a group. The loop group of a $K(\pi, n)$ is a $K(\pi, n-1)$.

Loop groups may be defined for arbitrary simplicial sets, but the simplicial sets whose loop groups we are interested in are generally 0-reduced. We now elaborate on an explicit construction of a loop group for 0-reduced simplicial sets. For a reduced simplicial set K, let $F_n(K)$ denote the free group generated by the members of K_n with identity denoted e_n. Define $\mathcal{G}_n(K)$ to be the quotient of $F_n(K)$ by the relation $s_0 k \sim e_n$ for $k \in K_{n-1}$.

Notation 4.2.1. Let $x \in F_n(K)$. We denote the equivalence class of x in $\mathcal{G}_n(K)$ by $[x]$.

One can check that $\mathcal{G}_n(K)$ is a free group with one less generator. Let x be a generator of $\mathcal{G}_n(K)$. Define the face and degeneracy maps of $\mathcal{G}(K)$ on its generators as follows:

$$\begin{aligned} [\partial_0 x]\partial_0[x] &= [\partial_1 x] \\ \partial_i[x] &= [\partial_{i+1} x] \quad \text{if } i > 0 \\ s_i[x] &= [s_{i+1} x] \end{aligned}$$

These maps extend uniquely to group homomorphisms. Define a function $\tau : K \to \mathcal{G}(K)$ sending every simplex in K to its class in $\mathcal{G}(K)$. Define $E(\tau) = \mathcal{G}(K) \times_\tau K$.

Proposition 4.2.3. The function τ is a twisting function.

Proof. The proof is an elementary check of the conditions for a function to be a twisting function. □

Theorem 4.2.4 ([May67, Lemma 26.4, 26.5])**.** The space $|E(\tau)|$ is contractible.

The loop group $\mathcal{G}(K)$ defined above is unique up to homotopy. Let K, L be simplicial sets. Let $f : K \to L$ be a simplicial map. There is an induced map $f_* : \mathcal{G}(K) \to \mathcal{G}(L)$ defined by $f_*(x) = [f(x)]$. Thus, we have the following theorem:

Theorem 4.2.5. The prescription taking a simplicial set K to $\mathcal{G}(K)$ defines a functor from the category of 0-reduced simplicial sets to simplicial groups.

$$\mathcal{G} : \mathbf{sSet_0} \to \mathbf{sGrp}$$

Proposition 4.2.6 ([May67, Theorem 27.1])**.** The loop group functor $\mathcal{G} : \mathbf{sSet_0} \to \mathbf{sGrp}$ is a left adjoint to the simplicial classifying complex functor $\overline{W}$.

Proposition 4.2.7. Let G be a simplicial group and K a 0-reduced simplicial set. Let $\Phi : \mathcal{G}\overline{W} \to 1_{\mathbf{sGrp}}$ and $\Psi : 1_{\mathbf{sSet_0}} \to \overline{W}\mathcal{G}$ be the natural transformations corresponding to the unit-counit adjunction. Then:

1. The map $\Phi(G) : \mathcal{G}\overline{W}(G) \to G$ is a weak homotopy equivalence of simplicial groups.
2. The map $\Psi(K) : K \to \overline{W}\mathcal{G}(K)$ is a weak homotopy equivalence of 0-reduced simplicial sets.

Proof. For the proof, the reader is referred to Section 11 of Kan's paper [Kan58] on the construction of $\mathcal{G}$. □

Before proceeding, we note the following proposition related to the singular complex of a topological group.

Proposition 4.2.8. Let G be a topological group. Then, $S_\bullet G$ is a simplicial group.

Proof. We first show that $S_n(G)$ is a group for every $n \geq 0$. Let Δ^n denote the standard n-simplex. For $f, g \in S_n(G)$, $u \in \Delta^n$, define an operation $*$ by $f * g(u) = f(u)g(u)$, where the multiplication on the right happens in G. It is easy to see that this operation makes $S_n(G)$ a group with inverses defined as $f^{-1}(u) = (f(u))^{-1}$ and identity the map taking everything to the identity of G. It remains to show that the face and degeneracy maps are group homomorphisms. Let $\partial_i : S_n(G) \to S_{n-1}(G)$ be a face map. We have

$$\begin{aligned}
\partial_i(f * g)(t_0, t_1, \ldots, t_{n-1}) &= (f * g)(t_0, t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \\
&= f(t_0, \ldots t_{i-1}, 0, t_i, \ldots, t_{n-1})g(t_0, \ldots t_{i-1}, 0, t_i, \ldots, t_{n-1}) \\
&= (\partial_i f * \partial_i g)(t_0, \ldots t_{i-1}, 0, t_i, \ldots, t_{n-1})
\end{aligned}$$

Similarly, for a degeneracy map $s_i : S_n(G) \to S_{n+1}(G)$, we have

$$
\begin{aligned}
s_i(f * g)(t_0, \dots t_{i-1}, 0, t_i, \dots, t_{n+1}) &= (f * g)(t_0, t_1, \dots t_i + t_{i+1}, \dots, t_{n+1}) \\
&= f(t_0, t_1, \dots t_i + t_{i+1}, \dots, t_{n+1}) g(t_0, t_1, \dots t_i + t_{i+1}, \dots, t_{n+1}) \\
&= s_i(f) * s_i(g)(t_0, \dots t_{i-1}, 0, t_i, \dots, t_{n+1})
\end{aligned}
$$

Thus, $S_\bullet(G)$ is a simplicial group. □

Remark 4.2.2. Given a topological group G, we want a simplicial group whose realization is a topological group that is to G. A natural question that may arise is why not take the constant complex cG. If G is a discrete topological group, then this approach works. $|cG|$ endows G with the discrete topology, and we end up with a CW complex that is homotopy equivalent to G. However, consider the case when G is a connected, path-connected Lie group. Let BG be a classifying space of G. It follows from the long exact sequence of homotopy groups that $\pi_1(BG) = 0$. We expect the same to hold for $|\overline{W}(cG)|$. However, the long exact sequence associated to $cG \to W(cG) \to \overline{W}(cG)$ yields $\pi_1(\overline{W}(cG)) = G$. Since, $|-|$ preserves homotopy groups, $\pi_1(|\overline{W}(cG)|) = G$, which is a contradiction. Thus it is clear that cG does not "capture" the topology of G. The singular complex $S_\bullet G$ however has the property that $|S_\bullet G| \simeq G$ when G is a CW group. Thus $S_\bullet G$ appears to be the appropriate simplicial group that models G.

Since $S_\bullet G$ is a simplicial group, we may consider its simplicial classifying complex $\overline{W}(S_\bullet G)$.

Theorem 4.2.9. Let G be a CW group. There is a map $|\mathcal{G}\overline{W}S_\bullet G| \to G$ which is a homotopy equivalence and continuous group homomorphism.

Proof. Since $S_\bullet G$ is a simplicial group, applying $\overline{W}$ makes sense. $\mathcal{G}$ takes 0-reduced simplicial sets to simplicial groups and $|-|$ takes simplicial groups to topological groups. Thus, the statement of the theorem is plausible.

By Proposition 4.2.7, the map $\Phi(S_\bullet G) : \mathcal{G}\overline{W}S_\bullet G \to S_\bullet G$ is a weak homotopy equivalence of the underlying simplicial sets. The realization $|\Phi(S_\bullet G)|$ is a weak homotopy equivalence. It follows from Theorem 3.2.17 that $|\Phi(S_\bullet G)|$ is a homotopy equivalence since the spaces involved are CW complexes. As $|S_\bullet G|$ is homotopy equivalent to G, we obtain a homotopy equivalence from $|\mathcal{G}\overline{W}S_\bullet G|$ to G.

$$|\mathcal{G}\overline{W}S_\bullet G| \xrightarrow{\simeq} |S_\bullet G| \xrightarrow{\simeq} G$$

Since $|-|$ takes simplicial homomorphisms to group homomorphisms, the homotopy equivalence obtained is a continuous group homomorphism between topological groups. $\square$

Notation 4.2.2. Let $G^\dagger$ denote the topological group $|\mathcal{G}\overline{W}S_\bullet G|$

Let $\alpha \in H^i_{\mathbf{SS}}(\overline{W}S_\bullet G, A)$. Let f_α be a simplicial map from $\overline{W}S_\bullet G$ to $K(A,i)$ be the simplicial map corresponding to α by the one-one correspondence in Theorem 3.5.9. Taking the pullback of p over f_α we have:

$$\begin{array}{ccc}
\overline{W}S_\bullet G_\alpha & \longrightarrow & \mathcal{G}(\overline{K}(A,i)) \times_\tau \overline{K}(A,i) \\
\Big\downarrow{\scriptstyle \pi_\alpha} & & \Big\downarrow{\scriptstyle \pi} \\
\overline{W}S_\bullet G & \xrightarrow[f_\alpha]{} & \overline{K}(A,i)
\end{array}$$

Applying $\mathcal{G}$ to the above diagram, we get:

$$\begin{array}{ccc}
\mathcal{G}\overline{W}S_\bullet G_\alpha & \longrightarrow & \mathcal{G}\big(\mathcal{G}(\overline{K}(A,i)) \times_\tau \overline{K}(A,i)\big) \\
\Big\downarrow{\scriptstyle (\pi_\alpha)_*} & & \Big\downarrow{\scriptstyle \pi_*} \\
\mathcal{G}\overline{W}S_\bullet G & \xrightarrow[(f_\alpha)_*]{} & \mathcal{G}(\overline{K}(A,i))
\end{array}$$

The simplicial set $\mathcal{G}\overline{W}S_\bullet G_\alpha$ is in fact a simplicial group and its realisation is a topological group. Denote its realisation by $G^\dagger_\alpha$. The map p_α is a principal fibration. Further, The induced map $(p_\alpha)_* : \mathcal{G}\overline{W}S_\bullet G_\alpha \to G$ is a simplicial homomorphism. Finally, $|-|$ simplicial homomorphisms to continuous group homomorphisms. Thus, the induced map $|\mathcal{G}(p_\alpha)| : G^\dagger_\alpha \to G^\dagger$ is a continuous homomorphism of CW groups.

We have obtained a continuous group homomorphism $G^\dagger_\alpha \to G^\dagger$ where the underlying topological space $G^\dagger$ is homotopy equivalent to G. As in the previous section, we expect $G^\dagger_\alpha$ to be characterised by a lifting property. We believe that the following holds:

Theorem 4.2.10. Let G be a CW group. Then $G^{\dagger}_{\alpha} \to G^{\dagger}$ is an α-group over $G^{\dagger}$.

Remark 4.2.3. At the moment, the above theorem has not been proven. If we are able to show that any nullhomotopic map to $\overline{K}(A,i)$ lifts to $\mathcal{G}(\overline{K}(A,i)) \times_{\tau} \overline{K}(A,i)$ then we would be able to use the universal property of pullbacks to obtain a lift. This intermediate proposition is also yet to be proven.

4.2.3 Examples of α-spaces over BO(n)

Let $O(n)$ denote the orthogonal group in dimension n and $BO(n)$ its classifying space. Let $K(\mathbb{Z}/2\mathbb{Z}, i)$ denote an Eilenberg-MacLane Space of type $(\mathbb{Z}/2\mathbb{Z}, i)$. We have a path fibration $\pi : PK(\mathbb{Z}/2\mathbb{Z}, i) \to K(\mathbb{Z}/2\mathbb{Z}, i)$. Let g be a map to $K(\mathbb{Z}/2\mathbb{Z}, i)$ corresponding the the i^{th} Wu class v_i. The space $BO(n)\langle v_i \rangle$ is defined as the total space of the pullback of this path fibration via g. Pictorially:

$$\begin{array}{ccc} K(\mathbb{Z}/2\mathbb{Z}, i-1) & \xrightarrow{=} & K(\mathbb{Z}/2\mathbb{Z}, i-1) \\ \downarrow & & \downarrow \\ BO(n)\langle v_i \rangle & \longrightarrow & PK(\mathbb{Z}/2\mathbb{Z}, i) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ BO(n) & \xrightarrow{g} & K(\mathbb{Z}/2\mathbb{Z}, i) \end{array}$$

Definition 4.2.3. A *Wu structure* over M is a lifting of a classifying map $f : M \to BO(n)$ to $BO(n)\langle v_i \rangle$.

$$\begin{array}{ccc} & & BO(n)\langle v_i \rangle \\ & \overset{\tilde{f}}{\nearrow} & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & BO(n) \end{array}$$

Wu structures provide an example of group α-spaces when we take α to be universal Wu classes.

Conclusion and Future Directions

In this thesis, we have computed the Wu classes for representations of C_n for all even n. We have given an introduction to simplicial homotopy theory and surveyed important results in the field. The classifying space construction discussed offers an alternative construction to Milnor's construction and we believe that this construction is a bit simpler. Further, the bar construction can be generalized to any category and is often used to create resolutions of objects. Another useful fact about the classifying complex functor which was discussed was that it exhibits the Kan loop group functor as an adjoint. The adjunction provides a prescription to recover a topological group from its classifying space, up to homotopy equivalence. Finally, we have provided partial results to characterizing and α-spaces. This work is ongoing. The first step forward would be to prove that any nullhomotopic map to a simplicial set lifts to its loop group PTCP.

Simplicial sets are a useful category to do homotopy theory and are useful in the theory of ∞-categories and stable homotopy theory. On the other hand, the theory of loop groups and loop spaces is fascinating in itself. Loop groups provide a rich representation theory which could be an exciting avenue of exploration.

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