Inverse Boundary Value Problems for Polyharmonic Operators in two dimensions

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Certificate

This is to certify that this dissertation entitled “Inverse Boundary Value Problems for Polyharmonic Operators in two dimensions”, towards the partial fulfillment of the Master of Science Degree at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Rajat at Tata Institute of Fundamental Research, Center for Applicable Mathematics, Bengaluru, under my supervision, during the academic year 2022-2023.

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Declaration

I hereby declare that the matter embodied in the report entitled Inverse Boundary Value Problems for Polyharmonic Operators in two dimensions are the results of the work carried out by me at TIFR CAM, Bengaluru, under the supervision of Prof. Venkateswaran P. Krishnan and the same has not been submitted elsewhere for any other degree.

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Abstract

In this thesis, I study an inverse boundary value problem in two dimensions for a polyharmonic operator of the form

$$\mathcal{L} = \partial^m \bar{\partial}^m + \sum_{j,k=0}^{m-1} A_{j,k} \partial^j \bar{\partial}^k, \quad m \geq 2.$$

The inverse problem is whether we can recover uniquely the coefficients $A_{j,k}$ from the set of Cauchy data

$$\mathcal{C}(\mathcal{L}) = \left\{ \left(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}, \partial_\nu^2 u|_{\partial\Omega} \ldots, \partial_\nu^{(2m-1)} u|_{\partial\Omega} \right) : u \in H^{2m}(\Omega), \mathcal{L}u = 0 \right\},$$

where ν is an outer unit normal to $\partial\Omega$.

In joint work with Prof. Krishnan and Dr. Rahul Raju Pattar, I establish that the Cauchy data for a polyharmonic operator uniquely determines all anisotropic perturbations of order at most $m-1$ and several perturbations of orders m to $2m-2$ with some restrictions. This restriction is captured in the following representation of the operator $\mathcal{L}$ as

$$(\partial\bar{\partial})^m + A_{m-1,m-1}(\partial\bar{\partial})^{m-1} + \sum_{l=1}^{m-2} \left(\sum_{j+k=m-l-1} A_{j+l,k+l} \partial^j \bar{\partial}^k \right) (\partial\bar{\partial})^l + \sum_{l=0}^{m-1} \sum_{j+k=l} A_{j,k} \partial^j \bar{\partial}^k.$$

We start this thesis with the Calderón inverse problem, where we followed Prof. Pedro Caro's class lecture notes. This is the first problem that represents many ideas. The proof given here is based on Carleman estimates which is different from the original proof of Sylvester and Uhlmannn. It is the primary technique in partial data inverse problems for constructing CGO solutions. After that, we study the inverse boundary value problem for Schrödinger operator, which will help to discuss our original contribution.

Contents

Chapter 1

What is an inverse problem?

An inverse problem refers to the process of determining the characteristics or properties of an object or system based on indirect measurements or observations. Real-life examples of inverse problems include X-ray, CT, and ultrasound imaging, where measurements are taken of the interior of an object or body to identify healthy and unhealthy parts. The challenge is interpreting the measurements to reconstruct an accurate image of the object of interest.

This section used notes on Introduction to Inverse Problems by Guillaume Bal.

More precisely, An inverse problem (IP) is a problem that involves determining the parameters of a system or object of interest based on the available data or measurements. The first step in solving an IP is to establish a mapping between the parameters and the measurements, which is known as the forward problem or measurement operator (MO), denoted by $\mathfrak{M}$.

The measurement operator (MO) is a mathematical function that maps the parameters of interest, denoted by x, from a functional space $\mathfrak{X}$ to the space of available data, denoted by y, which belongs to another functional space $\mathfrak{Y}$. Solving the inverse problem requires finding point(s) $x \in \mathfrak{X}$ from knowledge of the measurements $y \in \mathfrak{Y}$.

The MO describes our best effort to construct a model for the available data y, which we assume here depends only on the sought parameters x. The choice of $\mathfrak{X}$ describes our best effort to characterize the space where we believe the parameters belong. For reconstructing the object of interest from the data, we need to ask the following questions about MO:

Uniqueness: The first question to ask about the MO is whether it is injective, meaning we have enough data to reconstruct the parameters of interest uniquely. Injectivity means that different parameters in the parameter space cannot produce the same data, which is necessary for a unique reconstruction of the parameters. More precisely, injectivity of MO is:

$$\mathfrak{M}(\mathrm{x}_1) = \mathfrak{M}(\mathrm{x}_2) \implies \mathrm{x}_1 = \mathrm{x}_2 \quad \text{for all } \mathrm{x}_1, \mathrm{x}_2 \in \mathfrak{X}.$$

Then we can uniquely characterize the parameter x from the data y if given in the range of $\mathfrak{M}$.

Stability: After proving $\mathfrak{M}$ is injective, we can talk about an inverse operator $\mathfrak{M}^{-1}$ which is a map from the range of $\mathfrak{M}$ to a unique element in $\mathfrak{X}$. As machines are not perfect, we want to see if we make some measurement error, then what the error will be in the reconstruction of parameters. This is captured by stability estimates which quantify how errors in the available data translate into errors in the reconstruction. The modulus of continuity of the inverse operator $\mathfrak{M}^{-1}$ gives an estimate on the reconstruction error $\|\mathrm{x}_1 - \mathrm{x}_2\|_{\mathfrak{X}}$ based on the error in the data acquired $\|\mathfrak{M}(\mathrm{x}_1) - \mathfrak{M}(\mathrm{x}_2)\|_{\mathfrak{Y}}$.

In this thesis, we will study the uniqueness of MO for the Calderón inverse problem in three and higher dimensions, the inverse problem for the Schrödinger operator in two dimensions, and the inverse problem for the polyharmonic operator in two dimensions.

The idea of the proof for all of these problems follows the same path: Using that both operators have the same measurement operator, we write our integral identity, which involves the object of interest of both, then try to see what type of solutions is needed to show uniqueness. Then, at last, we try to construct these types of solutions.

Chapter 2

Calderón inverse problem

2.1 Introduction

The Calderón problem finds application in electrical impedance tomography (EIT) and optical tomography (OT). EIT is a non-invasive modality(which means no break in the skin is created) that reconstructs tissues' electrical properties (conductivity) from current and voltage measurements on part of boundary. More precisely, let Ω is a bounded open subset of $\mathbb{R}^{\mathrm{n}}$ with some regular boundary and conductivity γ is a bounded function with a positive lower bound on $\overline{\Omega}$. We find that if a voltage potential f is applied on the boundary $\partial\Omega$, then the potential u in Ω solves the Dirichlet problem

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{2.1.1}$$

By the theory of elliptic equations, there is a unique weak solution $u \in H^1(\Omega)$ for any boundary value $f \in H^{1/2}(\partial\Omega)$. It is slightly more regular in the sense the normal derivative on the boundary can be defined in a coherent way as an element of $H^{-1/2}(\partial\Omega)$ by

$$\langle \gamma(\partial_\nu u)|_{\partial\Omega}, g\rangle = \int_\Omega \gamma(x)\nabla u(x) \cdot \nabla v(x)\, dx \tag{2.1.2}$$

where $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega)$. Using that u is a weak solution to (2.1.1), it is straightforward to see that the definition (2.1.2) is independent of the extension v of g.

Note that when $u, v \in C^2(\overline{\Omega})$ and $\gamma \in C^1(\overline{\Omega})$ integration by parts in (2.1.2) gives

$$\langle \gamma(\partial_\nu u)|_{\partial\Omega}, g\rangle = \int_{\partial\Omega} \gamma(x)\partial_\nu u(x) v(x)\, dS(x)$$

where dS is the usual Euclidean surface measure on $\partial\Omega$. Hence the generalized definition (2.1.2) of a normal derivative at the boundary coincides with the classical definition when this makes sense.

The distribution $\gamma(\partial_\nu u)|_{\partial\Omega}$ describes the current flux through the boundary, and it is the natural Neumann data for the equation (2.1.1). Thus we can define the Dirichlet-to-Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by

$$\Lambda_\gamma f = \gamma(\partial_\nu u)\big|_{\partial\Omega}.$$

With $\mathfrak{X} = L^\infty_+(\Omega)$ and $\mathfrak{Y} = \mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$, we define the MO Λ by

$$\Lambda : \mathfrak{X} \ni \gamma \mapsto \Lambda(\gamma) = \Lambda_\gamma \in \mathfrak{Y}.$$

The Calderón problem concerns the inverse of the Λ. There are several important questions besides the injectivity and stability of Λ. We can ask

1. Can we reconstruct γ if we only know the finite number of measurements, i.e., the action of Λ_γ on a finite-dimensional subspace of $H^{1/2}(\partial\Omega)$ is known.

2. Can we reconstruct γ if we only know measurements on some part of the boundary, i.e., for some open set $\Gamma \subset \partial\Omega$ we know $\Lambda_\gamma f|_\Gamma$ for all boundary voltages f.

An interesting fact about the inverse conductivity problem is that the amount of data given in Λ_γ depends on the space's dimension. We can see that the function γ depends on n variables, while the Schwartz kernel of Λ_γ is a function of $2(n-1)$ variables. So, the problem is formally determined in two dimensions, whereas in higher dimensions, it is over-determined. This motivates us somewhat to solve the two-dimensional problem; we have to invoke a different method than the one used for the higher-dimensional problem.

From a physical perspective, it is reasonable to expect that the conductivity at the boundary would be the most readily measurable. In [KV84], Kohn and Vogelius established that, for smooth conductivities, the mapping Λ determines the values of γ and all of its normal derivatives on $\partial\Omega$. It remains a necessary ingredient in many proofs of identifiability in the interior.

We now summarize some uniqueness results for Calderón's problem in the following table:

	n	γ
Sylvester-Uhlmann [SU87]	≥ 3	C^2
Nachman-Sylvester-Uhlmann [NSU88]	≥ 3	$C^{1,1}$
Jerison-Kenig [Cha90]	≥ 3	$W^{2,n/2+}$
Brown [Bro96]	≥ 3	$C^{3/2+}$
Päivärinta-Panchenko-Uhlmann [PPU03]	≥ 3	$W^{3/2,\infty}$
Brown-Torres [Bro03]	≥ 3	$W^{3/2,2n+}$
Haberman-Tataru [HT13]	≥ 3	C^1 , $W^{1,\infty}$ small
Caro-Rogers [CR14]	≥ 3	$W^{1,\infty}$
Nachman [Nac96]	2	C^2
Brown - Uhlmann [BBUU97]	2	C^1
Astala–Päivärinta [AP06]	2	L^∞

The uniqueness problem for $\gamma \in L^\infty(\Omega)$ is open in $n \geq 3$ case. It is fully solved in two dimensions, and the argument uses complex analysis tools. In this chapter, we will prove the uniqueness theorem given by Sylvester-Uhlmann in [SU87].

Theorem 2.1.1. Let Ω be bounded domain with C^2 boundary in $\mathbb{R}^n, n \geq 3$ and let $\gamma_j \in C^2(\overline{\Omega})$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$.

2.2 Integral Identity

Proposition 2.2.1 (Weak Integral Identity)**.** If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then

$$\int_\Omega (\gamma_1 - \gamma_2)\nabla u_1 \cdot \nabla u_2 = 0 \tag{2.2.1}$$

for any $u_j \in H^1(\Omega)$ solving $\operatorname{div}(\gamma_j \nabla u_j) = 0$.

Proof. Let v_2 in $H^1(\Omega)$ such that $\operatorname{div}(\gamma_2 \nabla v_2) = 0$ and $v_2|_{\partial\Omega} = u_1|_{\partial\Omega}$. Then $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies

$$\int_\Omega \gamma_1 \nabla u_1 \cdot \nabla u_2 = \int_\Omega \gamma_2 \nabla v_2 \cdot \nabla u_2 = \int_\Omega \gamma_2 \nabla u_2 \cdot \nabla v_2 = \int_\Omega \gamma_2 \nabla u_2 \cdot \nabla u_1.$$

This is exactly as in (2.2.1). $\square$

Lemma 2.2.2 (Reduction to Schrödinger equation)**.**

$$\mathrm{div}(\gamma\nabla u) = 0 \xLeftrightarrow{v=\gamma^{1/2}u} (-\Delta + q)v = 0$$

where $q = \gamma^{-1/2}\Delta\gamma^{1/2}$.

Proof. We can write easily $\mathrm{div}(\gamma\nabla u) = 0$ as

$$-\Delta u - \nabla \log\gamma \cdot \nabla u = 0.$$

We want to replace the first-order term $\nabla\log\gamma\cdot\nabla u$ with a zero-order potential term qu. We can use a Liouville transformation because the coefficient $\nabla\log\gamma$ is a gradient. Indeed, for any ϕ we have

$$e^{-\phi}\circ(-\Delta)\circ e^{\phi} = -\Delta - 2\nabla\phi\cdot\nabla - |\nabla\phi|^2 - \Delta\phi$$

Setting $\phi = \frac{1}{2}\log\gamma$, we obtain

$$\begin{aligned} -\mathrm{div}(\gamma\nabla u) &= -\Delta u - \nabla\log\gamma\cdot\nabla u \\ &= -\gamma^{-1/2}\Delta(\gamma^{1/2}u) + \left(\frac{1}{4}|\nabla\log\gamma|^2 + \frac{1}{2}\Delta\log\gamma\right)u. \end{aligned}$$

It implies

$$-\gamma^{1/2}\mathrm{div}(\gamma\nabla u) = -\Delta(\gamma^{1/2}u) + q(\gamma^{1/2}u),$$

where

$$q = \frac{1}{4}|\nabla\log\gamma|^2 + \frac{1}{2}\Delta\log\gamma,$$

and we find that the function $v = \gamma^{1/2}u$ satisfies the time-independent Schrödinger equation

$$(-\Delta + q)v = 0.$$

By setting $u = 1$ we may observe that $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. □

Proposition 2.2.3. If $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ then

$$\int_\Omega (q_1 - q_2)v_1v_2 = 0$$

for all $v_j \in H^1(\Omega)$ solving $(-\Delta + q_j)v_j = 0$ with $q_j = \gamma_j^{-1/2}\Delta\gamma_j^{1/2}$.

Proof.

$$
\begin{aligned}
\int_\Omega (q_1 - q_2) v_1 v_2 &\overset{-\Delta v_j + q_j v_j = 0}{=} \int_\Omega [\Delta v_1 v_2 - v_1 \Delta v_2] = \int_{\partial\Omega} [\partial_\nu v_1 v_2 - v_1 \partial_\nu v_2] \\
&= \int_{\partial\Omega} \left[(\gamma_1^{1/2} \partial_\nu u_1 + u_1 \partial_\nu \gamma_1^{1/2}) \gamma_2^{1/2} u_2 - \gamma_1^{1/2} u_1 (\gamma_2^{1/2} \partial_\nu u_2 + u_2 \partial_\nu \gamma_2^{1/2}) \right] \\
&= \int_{\partial\Omega} \left[\frac{\gamma_2^{1/2}}{\gamma_1^{1/2}} \gamma_1 \partial_\nu u_1 u_2 + \gamma_2^{1/2} \partial_\nu \gamma_1^{1/2} u_1 u_2 - \frac{\gamma_1^{1/2}}{\gamma_2^{1/2}} u_1 \gamma_2 \partial_\nu u_2 - \gamma_1^{1/2} \partial_\nu \gamma_2^{1/2} u_1 u_2 \right] \\
&= \int_{\partial\Omega} [\gamma_1 \partial_\nu u_1 u_2 - u_1 \gamma_2 \partial_\nu u_2] \\
&= \int_{\partial\Omega} [\Lambda_{\gamma_1}(u_1|_{\partial\Omega}) u_2 - u_1 \Lambda_{\gamma_2}(u_2|_{\partial\Omega})] \\
&= \int_{\partial\Omega} (\Lambda_{\gamma_1} - \Lambda_{\gamma_2})\, (u_1|_{\partial\Omega})\, u_2 = 0.
\end{aligned}
$$

□

2.3 Uniqueness

Now we try to show $\int_\Omega (q_1 - q_2) v_1 v_2 = 0$ for all $v_j \in H^1(\Omega)$ satisfying $(-\Delta + q_j) v_j = 0$ in Ω implies $q_1 = q_2$. Before that let's try to do $\int_\Omega (q_1 - q_2) \vartheta_1 \vartheta_2 = 0$ for all $\vartheta_j \in H^1(\Omega)$ satisfying $\Delta \vartheta_j = 0$ implies $q_1 = q_2$. If we able to find for all $\xi \in \mathbb{R}^{\mathrm{n}}$ harmonic functions ϑ_j s.t. $\vartheta_1 \vartheta_2 = e^{-ix\cdot\xi}$, then using Fourier inversion we are done. Calderon observed that

$$
\begin{aligned}
\Delta(e^{x\cdot\zeta}) = 0 &\iff \zeta \cdot \zeta = 0 \\
&\iff |\mathrm{Re}\zeta| = |\mathrm{Im}\zeta| \text{ and } \mathrm{Re}\zeta \cdot \mathrm{Im}\zeta = 0
\end{aligned}
$$

for $\zeta \in \mathbb{C}^n$. Choose $\xi \in \mathbb{R}^{\mathrm{n}}$ and $\zeta_1, \zeta_2 \in \mathbb{C}^n$ such that

$$
\begin{aligned}
\zeta_j \cdot \zeta_j &= 0 \\
\zeta_1 + \zeta_2 &= -i\xi.
\end{aligned}
$$

For example, we can take

$$
\begin{aligned}
\zeta_1 &= \frac{1}{2}(|\xi|\theta - i\xi) \\
\zeta_2 &= \frac{1}{2}(-|\xi|\theta - i\xi)
\end{aligned}
$$

where $\theta \perp \xi$ and $|\theta| = 1$ which will done this special case. As $(-\Delta + q)$ is a perturbation of Laplacian, we can try

$$v_j = e^{x\cdot\zeta_j}(1 + \omega_j(x, \zeta_j)),$$

in new integral identity.

$$\begin{aligned} 0 &= \int_\Omega (q_1 - q_2)e^{x\cdot(\zeta_1+\zeta_2)}(1 + \omega_1(x, \zeta_1))(1 + \omega_2(x, \zeta_2)) \\ &= \int_\Omega (q_1 - q_2)e^{x\cdot(\zeta_1+\zeta_2)} + \int_\Omega (q_1 - q_2)e^{x\cdot(\zeta_1+\zeta_2)} (\omega_1(x, \zeta_1) + \omega_2(x, \zeta_2) + \omega_1(x, \zeta_1)\omega_2(x, \zeta_2)) \end{aligned}$$

For any $\xi \in \mathbb{R}^{\mathrm{n}}$, if we are able to construct family of solutions $v_j \in H^1(\Omega)$ such that

$$\begin{aligned} &\zeta_j \cdot \zeta_j = 0 \\ &\zeta_1 + \zeta_2 = -i\xi \\ &\|\omega_j\|_{L^2(\Omega)} \to 0 \quad \text{as } |\zeta_j| \to \infty. \end{aligned}$$

then it gives $q_1 = q_2$. Here, we need $n \geq 3$ which can be easily seen by characterizing $\{\zeta \cdot \zeta = 0\}$ in two dimensions. For example, in $n \geq 3$ we can take

$$\begin{aligned} \zeta_1 &= \tau\theta + i\left(\sqrt{\left(\tau^2 - \frac{|\xi|^2}{4}\right)}\omega - \frac{\xi}{2}\right), \\ \zeta_2 &= -\tau\theta - i\left(\sqrt{\left(\tau^2 - \frac{|\xi|^2}{4}\right)}\omega + \frac{\xi}{2}\right). \end{aligned}$$

where $\tau > \frac{|\xi|}{2}$ and $\theta, \omega \in S^{n-1}$ satisfying $\theta \cdot \omega = \omega \cdot \xi = \xi \cdot \theta = 0$. Before going to show that $q_1 = q_2$ we need to be sure that $q_1 = q_2 \implies \gamma_1 = \gamma_2$. A priori, this is not enough to show that $\gamma_1 = \gamma_2$. In particular, multiplying each γ_j by a constant will not change the q_j. Thus we need also determination of γ_j at the boundary. We can see that $q_1 = q_2$ implies

$$\begin{aligned} 0 &= \gamma_1^{-1/2}\Delta\gamma_1^{1/2} - \gamma_2^{-1/2}\Delta\gamma_2^{1/2} \\ &= \gamma_2^{1/2}\Delta\gamma_1^{1/2} - \gamma_1^{1/2}\Delta\gamma_2^{1/2} \\ &= \gamma_2^{1/2}\mathrm{div}(\nabla\gamma_1^{1/2}) - \gamma_1^{1/2}\mathrm{div}(\nabla\gamma_2^{1/2}) \\ &= \mathrm{div}(\gamma_2^{1/2}\nabla\gamma_1^{1/2}) - \nabla\gamma_1^{1/2} \cdot \nabla\gamma_2^{1/2} - \mathrm{div}(\gamma_1^{1/2}\nabla\gamma_2^{1/2}) + \nabla\gamma_2^{1/2} \cdot \nabla\gamma_1^{1/2} \\ &= \mathrm{div}(\gamma_2^{1/2}\nabla\gamma_1^{1/2} - \gamma_1^{1/2}\nabla\gamma_2^{1/2}) \\ &= \mathrm{div}((\gamma_1\gamma_2)^{1/2}\nabla(\log\gamma_1 - \log\gamma_2)) \end{aligned}$$

As $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$ implies that $\log\gamma_1 - \log\gamma_2$ vanish on $\partial\Omega$, this implies that $\gamma_1 = \gamma_2$ in Ω. So,

$$\boxed{\begin{matrix} q_1 = q_2 \\ \gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega} \end{matrix} \implies \gamma_1 = \gamma_2.}$$

2.4 Complex Geometric Optics Solutions

In this section, we construct the solution needed in last section. More precisely, we will construct solutions $v \in H^1(\Omega)$ of the form

$$v = e^{x\cdot\zeta}(1 + \omega(x,\zeta))$$

of equation $(-\Delta + q)v = 0$ with $\|\omega_\zeta\|_{L^2(\Omega)} \to 0$ as $|\zeta| \to \infty$. We find that

$$e^{-x\cdot\zeta}(-\Delta + q)e^{x\cdot\zeta} = -\Delta - 2\zeta\cdot\nabla + q.$$

So, the error term satisfies

$$(-\Delta - 2\zeta\cdot\nabla)\omega(x,\zeta) + q(x)\omega(x,\zeta) = -q.$$

We actually try to find for any given $f \in L^2(\Omega)$ and $\zeta \in \mathcal{V}$, function $u \in L^2(\Omega)$ satisfying

$$Lu = (\Delta + 2\zeta\cdot\nabla - q)u = f$$

and $\|u\|_{L^2(\Omega)} \to 0$ as $|\zeta| \to \infty$. Suppose we can somehow show L^T injective for some spaces. Then we can define a map

$$l(L^T v) = \langle f, v\rangle$$

which is a well defined linear functional. Suppose we are also able to show l bounded then by Hahn Banach theorem l extends to a continuous linear functional $\bar{l}$ on some bigger Hilbert space H. Then by Riesz representation theorem, there exist unique u such that $\tilde{l}(\phi) = \langle u, \phi\rangle$. So,

$$\langle u, L^T v\rangle = l(L^T v) = \langle f, v\rangle$$

which implies

$$\langle Lu, v\rangle = \langle f, v\rangle.$$

It's giving us some idea, we want $v \in C_c^\infty(\Omega)$, $H = L^2(\Omega)$, L^T injective and l bounded. We found that for showing these, we only need to show apriori estimate

$$\|v\|_{L^2(\Omega)} \lesssim \|L^T v\|_{L^2(\Omega)}, \quad v \in C_c^\infty(\Omega)$$

Because of term $-2\zeta \cdot \nabla$ present in L^T we can expect apriori estimate (using Poincare inequality) as $|\zeta| \to \infty$

$$|\zeta|\|v\|_{L^2(\Omega)} \lesssim \|L^T v\|_{L^2(\Omega)}, \quad v \in C_c^\infty(\Omega).$$

and it gives also estimate on u, as $|\zeta| \to \infty$

$$\|u\|_{L^2(\Omega)} \lesssim \frac{1}{|\zeta|}\|f\|_{L^2(\Omega)}$$

which implies $\|u\|_{L^2(\Omega)} \to 0$ as $|\zeta| \to \infty$.

Proposition 2.4.1. For any $\zeta \in \mathcal{V}$ we have

$$|\mathrm{Re}\zeta|\|v\|_{L^2(\mathbb{R}^n)} \leq C\|(\Delta - 2\zeta \cdot \nabla)v\|_{L^2(\mathbb{R}^n)}$$

for all $v \in S(\mathbb{R}^n)$ satisfying

$$\mathrm{Supp}(v) \subset \{x \in \mathbb{R}^n : |\mathrm{Re}\zeta \cdot x| \leq R|\mathrm{Re}\zeta|\},$$

for some $R > 0$.

Proof. We find that for all $v \in S(\mathbb{R}^n)$,

$$\begin{aligned}
\|(\Delta - 2\zeta \cdot \nabla)v\|^2_{L^2(\mathbb{R}^n)} &= \|(-|\xi|^2 - 2i\zeta \cdot \xi)\hat{v}(\xi)\|_{L^2(\mathbb{R}^n)} \\
&= \|(-|\xi|^2 - 2i\mathrm{Re}\zeta \cdot \xi + 2\mathrm{Im}\zeta \cdot \xi)\hat{v}(\xi)\|_{L^2(\mathbb{R}^n)} \\
&= \|(-|\xi|^2 + 2\mathrm{Im}\zeta \cdot \xi)\hat{v}(\xi)\|_{L^2(\mathbb{R}^n)} + \|(2\mathrm{Re}\zeta \cdot \xi)\hat{v}(\xi)\|_{L^2(\mathbb{R}^n)} \\
&= \|(\Delta - 2i\mathrm{Im}\zeta \cdot \nabla)v\|^2_{L^2(\mathbb{R}^n)} + \|(-2i\mathrm{Re}\zeta \cdot \nabla)v\|^2_{L^2(\mathbb{R}^n)} \\
&\geq \|(2\mathrm{Re}\zeta \cdot \nabla)v\|^2_{L^2(\mathbb{R}^n)}
\end{aligned}$$

Let Q be orthogonal transformation such that

$$Q(e_1) = \mathrm{Re}(\zeta)/|\mathrm{Re}(\zeta)|$$

Choose $w(y) = v(Q(y))$ then

$$\frac{\partial w}{\partial y_1}(y) = \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}(Q(y))Q_{j1} = \nabla v(Q(y)) \cdot Q(e_1) = \nabla v(Q(y)) \cdot \frac{\mathrm{Re}(\zeta)}{|\mathrm{Re}(\zeta)|}$$

Then

$$\|(\Delta - 2\zeta \cdot \nabla)v\|_{L^2(\mathbb{R}^n)} \geq 2\|\mathrm{Re}\zeta \cdot \nabla v\|_{L^2(\mathbb{R}^n)} = 2\|\mathrm{Re}\zeta \cdot \nabla v(Q(y))\|_{L^2(\mathbb{R}^n)} = 2|\mathrm{Re}\zeta| \left\| \frac{\partial w}{\partial y_1}(y) \right\|_{L^2(\mathbb{R}^n)}.$$

If $\mathrm{Supp}(w) \subset \{(y_1, y'); y_1 \in [-R, R]\}$, then by Poincare inequality we have

$$\|(\Delta - 2\zeta \cdot \nabla)v\|_{L^2(\mathbb{R}^n)} \gtrsim |\mathrm{Re}\zeta| \|v(Q(y))\|_{L^2(\mathbb{R}^n)} = |\mathrm{Re}\zeta| \|v\|_{L^2(\mathbb{R}^n)}.$$

Then

$$y_1 = e_1 \cdot y = Q(e_1) \cdot Q(y) = \frac{1}{|\mathrm{Re}\zeta|} \mathrm{Re}\zeta \cdot Qy \quad \text{for all } y \in \mathbb{R}^n$$

and

$$|y_1| \leq R \iff |\mathrm{Re}\zeta \cdot Qy| \leq R|\mathrm{Re}\zeta|.$$

So,

$$\mathrm{Supp}(v) \subset \{x \in \mathbb{R}^n : |\mathrm{Re}\zeta \cdot x| \leq R|\mathrm{Re}\zeta|\}.$$

□

Corollary 2.4.2. For any $\zeta \in \mathcal{V}$ such that $|\mathrm{Re}\zeta| \geq 2C\|q\|_{L^\infty(\mathbb{R}^n)}$ we have

$$|\zeta| \|v\|_{L^2(\mathbb{R}^n)} \lesssim \|(\Delta - 2\zeta \cdot \nabla - q)v\|_{L^2(\mathbb{R}^n)}$$

for all $v \in S(\mathbb{R}^n)$ satisfying

$$\mathrm{Supp}(v) \subset \{x \in \mathbb{R}^n : |\mathrm{Re}\zeta \cdot x| \leq R|\mathrm{Re}\zeta|\},$$

for some $R > 0$. We realize also that $\mathrm{Supp}(v)$ contains $\{|x| \leq R\}$.

Proof. By using Proposition 2.4.1 and condition on ζ,

$$\begin{aligned} |\mathrm{Re}\zeta| \|v\|_{L^2(\mathbb{R}^n)} &\leq C\|(\Delta - 2\zeta \cdot \nabla - q)v\|_{L^2(\mathbb{R}^n)} + C\|qv\|_{L^2(\mathbb{R}^n)} \\ &\leq C\|(\Delta - 2\zeta \cdot \nabla - q)v\|_{L^2(\mathbb{R}^n)} + \frac{|\mathrm{Re}\zeta|}{2} \|v\|_{L^2(\mathbb{R}^n)} \end{aligned}$$

which easily proves our corollary. $\square$

As we discussed, the proof of following Theorems depend on apriori estimate we just proved. Therefore, following the discussion, we can easily prove the following theorems.

Theorem 2.4.3. Let Ω be a bounded open set and $q \in L^\infty(\Omega)$. Then for all $f \in L^2(\Omega)$ and $\zeta \in \mathcal{V}$ such that $|\mathrm{Re}\zeta| \geq 2C\|q\|_{L^\infty(\Omega)}$ there exist unique $\omega \in L^2(\Omega)$ such that

$$(\Delta + 2\zeta \cdot \nabla - q)\omega = f$$

and

$$\|\omega\|_{L^2(\Omega)} \lesssim \frac{1}{|\zeta|}\|f\|_{L^2(\Omega)}.$$

By observation, we can prove much general result.

Theorem 2.4.4. Let $q \in L^\infty(\Omega)$. For any $\zeta \in \mathcal{V}$ satisfying $|\mathrm{Re}\zeta| \geq C\|q\|_{L^\infty(\Omega)}$, and for any function $a \in H^2(\Omega)$ satisfying

$$\zeta \cdot \nabla a = 0 \quad \text{in } \Omega,$$

the equation $(\Delta - q)u = 0$ in Ω has a solution

$$u(x) = e^{\zeta \cdot x}(a + w(x, \zeta)),$$

where $w_\zeta \in L^2(\Omega)$ satisfies

$$\|\omega\|_{L^2(\Omega)} \lesssim \frac{1}{|\zeta|}\|(\Delta - q)a\|_{L^2(\Omega)}.$$

We have $\Delta u_\zeta \in L^2(\Omega)$. We proved that $w_\zeta \in L^2(\Omega)$ which implies $u_\zeta \in L^2(\Omega)$ but we want $u_\zeta \in H^1(\Omega)$ as discussed in Proposition 2.2.3. In the following proposition, we prove Cacciopoli estimates which give us u_ζ in $H^1_{\mathrm{loc}}(\Omega)$.

Proposition 2.4.5. Let Ω be a bounded domain in $\mathbb{R}^n$. There exists $C > 0$ such that for all $\chi \in C_c^\infty(\Omega)$ and for all $v \in C^\infty(\overline{\Omega})$,

$$\|\chi \nabla v\|_{L^2(\Omega)} \leq C\left(\|\chi\|_{L^\infty(\Omega)}\|\Delta v\|_{L^2(\Omega)} + \|\chi\|_{W^{1,\infty}(\Omega)}\|v\|_{L^2(\Omega)}\right)$$

Proof.

$$\int_\Omega |\chi \nabla v|^2 = \int_\Omega \chi^2 \nabla v \cdot \nabla \overline{v} = -\int_\Omega 2\chi \nabla \chi \cdot \nabla v \overline{v} - \int_\Omega \chi^2 \Delta v \overline{v}$$

$$\begin{aligned}
\Big|\int_\Omega \chi^2 \Delta v \overline{v}\Big| &\leq \|\chi\|^2_{L^\infty(\Omega)} \|\Delta v\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}\|\chi\|^2_{L^\infty(\Omega)} \left(\|\Delta v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right)
\end{aligned}$$

$$\begin{aligned}
\Big|\int_\Omega 2\chi\nabla\chi \cdot \nabla v \overline{v}\Big| &\leq 2\|\chi\nabla v\|_{L^2(\Omega)} \|\nabla\chi v\|_{L^2(\Omega)} \\
&\leq \frac{1}{2}\|\chi\nabla v\|^2_{L^2(\Omega)} + 2\|\nabla\chi\|^2_{L^\infty(\Omega)} \|v\|^2_{L^2(\Omega)}
\end{aligned}$$

$\square$

The proof of Theorem follows by extending the potential $q \in L^\infty(\Omega)$ to bigger domain Ω' by zero. Then doing an analysis of CGO solutions on Ω', we get $u_\zeta \in H^1_{\mathrm{loc}}(\Omega)$ which implies $u_\zeta \in H^1(\Omega)$. Then using sections integral identity and uniqueness, we prove our Theorem.

Chapter 3

Inverse problem for Schrödinger operator in two dimensions

3.1 Introduction

Let us now consider the Magnetic Schrödinger operator of the form

$$\mathcal{L}_{A,q} = \sum_{j=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_j} + A_j \right)^2 + q \tag{3.1.1}$$

on domain Ω. Here $A = (A_1, A_2, \ldots, A_n)$ is a magnetic vector potential, and q is a scalar electric potential. If $\vec{A}$ and q are real-valued then $\mathcal{L}_{\vec{A},q}$ is a self-adjoint operator. Thus

$$\mathcal{L}_{\vec{A},q} = -\Delta u - i \operatorname{div}(Au) - iA \cdot \nabla u + |A|^2 u + qu,$$

If we define

$$I_{\vec{A},q}(u,v) = \int_\Omega \left(\nabla u \cdot \nabla \bar{v} + iA \cdot (u \nabla \bar{v} - \bar{v} \nabla u) + (|A|^2 + q) u \bar{v} \right) dx$$

then a function $u \in H^1(\Omega)$ is a weak solution to $\mathcal{L}_{A,q} u = 0$ if

$$I_{\vec{A},q}(u,v) = 0 \quad \text{when} \quad v \in H_0^1(\Omega).$$

Since $\mathcal{L}_{\vec{A},q}u = 0$ can have nontrivial solutions vanishing on $\partial\Omega$, the Dirichlet-to-Neumann map is a multivalued relation. We define it by

$$\Lambda_{A,q} = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}|_{\partial\Omega} + i(A\cdot\nu)u|_{\partial\Omega} \right) : u \in H^1(\Omega), \mathcal{L}_{\vec{A},q}u = 0 \text{ in } \Omega \right\}$$

where

$$\left\langle \frac{\partial u}{\partial \nu}|_{\partial\Omega} + i(A\cdot\nu)u|_{\partial\Omega}, g \right\rangle = I_{\vec{A},q}(u,v).$$

for any $v \in H^1(\Omega)$ with $v|_{\partial\Omega} = g$.

The inverse boundary value problem for (3.1.1) is to recover information of $\vec{A}$ and q from the knowledge of $\Lambda_{\vec{A},q}$.

This problem has a gauge invariance: If ϕ is a smooth function vanishing at $\partial\Omega$, then

$$\Lambda_{\vec{A}+\nabla\phi,q} = \Lambda_{\vec{A},q}. \tag{3.1.2}$$

In fact, we can obtain $e^{-i\phi}\mathcal{L}_{\vec{A},q}(e^{i\phi}u) = \mathcal{L}_{\vec{A}+\nabla\phi,q}u$ which implies $\mathcal{L}_{\vec{A},q}(e^{i\phi}u) = 0$ iff $\mathcal{L}_{\vec{A}+\nabla\phi,q}u = 0$. By using vanishing of ϕ on $\partial\Omega$ we get (3.1.2).

This means we can only hope to recover A up to a gradient term. We are considering A_1 and A_2 as a magnetic vector potentials. Thus the natural problem is to show that

$$\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2} \implies \text{curl}A_1 = \text{curl}A_2 \text{ and } q_1 = q_2.$$

Proposition 3.1.1 (Integral Identity)**.** If $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$ then

$$\int_\Omega \left[i(\vec{A}_1 - \vec{A}_2)\cdot(u_1\nabla\bar{u}_2 - \bar{u}_2\nabla u_1) + (|\vec{A}_1|^2 - |\vec{A}_2|^2 + q_1 - q_2)u_1\bar{u}_2 \right] dx = 0 \tag{3.1.3}$$

for any $u_j \in H^1(\Omega)$ solving of $\mathcal{L}_{\vec{A}_1,q_1}u_1 = 0$ and $\mathcal{L}_{\bar{A}_2,\bar{q}_2}u_2 = 0$.

Proof. If $\Lambda_{A_1,q_1} = \Lambda_{A_2,q_2}$, then there is some v_2 in $H^1(\Omega)$ such that $\mathcal{L}_{A_2,q_2}v_2 = 0$ and

$$\left(u_1|_{\partial\Omega}, \frac{\partial u_1}{\partial\nu}|_{\partial\Omega} + i(A_1\cdot\nu)u_1|_{\partial\Omega} \right) = \left(v_2|_{\partial\Omega}, \frac{\partial v_2}{\partial\nu}|_{\partial\Omega} + i(A_2\cdot\nu)v_2|_{\partial\Omega} \right).$$

Thus

$$I_{A_1,q_1}(u_1,u_2) = I_{A_2,q_2}(v_2,u_2) = \overline{I_{\bar{A}_2,\bar{q}_2}(u_2,v_2)} = \overline{I_{\bar{A}_2,\bar{q}_2(u_2,u_1)}} = I_{A_2,q_2}(u_1,u_2).$$

□

The inverse boundary value problem for the magnetic Schrödinger operator is equivalent to an inverse scattering problem at fixed energy, provided that the coefficients are compactly supported. We now state uniqueness results for the Magnetic Schrödinger operator, which comprises results from inverse scattering theory.

	n	q	A
Khekin - Novikov [NK87]	≥ 3	$e^{-\gamma\langle x\rangle}L^{\infty}$, small	0
Nachman-Sylvester-Uhlmann [NSU88]	≥ 3	$L^{\infty}(\Omega)$	0
Novikov[Nov88]	≥ 3	$L^{\infty}(\Omega)$	0
Jerison-Kenig [Cha90]	≥ 3	$L^{n/2+}$	0
Lavine-Nachman [Nac92, FKS13]	≥ 3	$L^{n/2+}$	0
Novikov [Nov94]	3	$e^{-\gamma\langle x\rangle}L^{\infty}$	0
Sun [Sun93a]	≥ 3	$L^{\infty}(\Omega)$	$W^{2,\infty}(\Omega)$, small
Nakamura-Sun-Uhlmann [NSU95]	≥ 3	$L^{\infty}(\Omega)$	$C^{\infty}(\Omega)$
Eskin-Ralston [ER95]	≥ 3	$e^{-\gamma\langle x\rangle}C^{\infty}$	$e^{-\gamma\lvert x\rvert}C^{\infty}$
Salo [Sal04, Sal06]	≥ 3	$L^{\infty}(\Omega)$	$W^{1,\infty}(\Omega)$
Krupchyk-Uhlmann [KU14]	≥ 3	$L^{\infty}(\Omega)$	$L^{\infty}(\Omega)$
Päivärinta-Salo-Uhlmann [PSU10]	≥ 3	$e^{-\gamma\langle x\rangle}L^{\infty}$	$e^{-\gamma\langle x\rangle}W^{1,\infty}$
Sylvester-Uhlmann [SU86]	2	$W^{2,2}(\Omega)$, small	0
Novikov [Nov86]	2	$e^{-\gamma\langle x\rangle}C^{\infty}$, small	0
Novikov [Nov92]	2	$e^{-\gamma\langle x\rangle}L^{\infty}$, small	0
Sun [Sun93b]	2	$W^{1,\infty}(\Omega)$, small	$W^{3,\infty}(\Omega)$, small
Bukhgeim [Buk08]	2	$W^{1,2+}(\Omega)$	0
Blasten[Blå11]	2	$W^{0+,2+}(\Omega)$	0
Guillarmou-Salo-Tzou [GST11]	2	$e^{-\gamma\lvert x\rvert^2}L^{\infty}\cap C^{1,0+}$	0
Lai [Lai11]	2	$L^{\infty}(\Omega)$	$L^{\infty}(\Omega)$
Imanuvilov-Yamamoto [IY12]	2	$L^{2+}(\Omega)$	0

In this Chapter, we will prove the uniqueness Theorem given by Bukhgeim in [Buk08]. This paper has new techniques in two dimensions that have led too many developments in the study of inverse boundary value problems.

The inverse problem is whether we can uniquely determine the coefficients q from the set of Cauchy data

$$\mathcal{C}_q = \left\{ (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}) : u \in H^1(\Omega), (-\Delta + q)u = 0 \right\}, \tag{3.1.4}$$

where ν is an outer unit normal to $\partial\Omega$. The main Theorem is as follows:

Theorem 3.1.2. If $q_j \in W^{1,p}(\Omega)$, $p > 2$ then $\mathcal{C}_{q_1} = \mathcal{C}_{q_2}$ implies $q_1 = q_2$.

Bukhgeim reduced the problem to a $(\partial, \bar{\partial})$-system. Recall that the Laplace operator is $\Delta = 4\partial\bar{\partial}$ therefore setting $\bar{\partial}u = (-1/4)w$, we can reduce the equation $(-\Delta + q)u = 0$ to the equivalent first order system

$$\begin{pmatrix} (1/4)q & -\partial \\ \bar{\partial} & -1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} = 0,$$

and $\mathcal{C}_q$ determines the boundary value of this system. As we shall see, the Cauchy data at the boundary

$$\mathscr{C}_V = \{ F|_{\partial\Omega} : (D + V)F = 0 \}$$

for first order systems of the form $(D + V)F = 0$ determine any complex-valued matrix potential $V \in W^{1,p}(\Omega)$ with $p > 2$, where

$$D = \begin{pmatrix} 0 & -\partial \\ \bar{\partial} & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} q & 0 \\ 0 & q' \end{pmatrix}. \tag{3.1.5}$$

Our main theorem is the following :

Theorem 3.1.3. Suppose $\mathscr{C}_{V_1} = \mathscr{C}_{V_2}$. Then

1. If $q_j \in W^{1,p}(\Omega)$ and $q'_j \in L^\infty(\Omega)$ then $q_1 = q_2$.
2. If $q_j \in L^\infty(\Omega)$ and $q'_j \in W^{1,p}(\Omega)$ then $q'_1 = q'_2$.

3.2 Integral Identity

The key of the proof of Theorem 3.1.3 is the construction of families of h-parameterized solutions $U_1 = U_1(x; h)$ and $U_2 = V(x; h)$ with $h > 0$ satisfying $(D + V_1)U_1 = 0$ and

$(D+V_2^*)U_2 = 0$. By assuming $\mathscr{C}_{V_1} = \mathscr{C}_{V_2}$ there exists a solution $\tilde{U}_1$ to $(D+V_2)\tilde{U}_1 = 0$ with $U_1|_{\partial\Omega} = \tilde{U}_1|_{\partial\Omega}$. Therefore, $(D+V_2)(U_1-\tilde{U}_1) = (V_2-V_1)U_1$, using green's formula and vanishing of $U_1-\tilde{U}_1$ on the boundary, we get

$$
\begin{aligned}
0 &= \int_\Omega \langle U_1-\tilde{U}_1, (D+V_2^*)U_2\rangle \\
&= \int_\Omega \langle (D+V_2)(U_1-\tilde{U}_1), U_2\rangle \\
&= \int_\Omega \langle (V_2-V_1)U_1, U_2\rangle.
\end{aligned}
$$

3.3 Complex Geometric Optics Solutions

In this section, we construct CGO solutions solving equation $(D+V)u=0$ of the form

$$
u = \begin{bmatrix} e^{\Phi/h}(a+r_h) \\ e^{\bar{\Phi}/h}(b+\tilde{r}_h) \end{bmatrix} \tag{3.3.1}
$$

where a is holomorphic and b is antiholomorphic function. We find that if error terms r_h and $\tilde{r}_h$ satisfies

$$
\begin{aligned}
r_h + \bar{\partial}^{-1}(e^{-2i\psi/h}q'\tilde{r}_h) &= -\bar{\partial}^{-1}(e^{-2i\psi/h}q'b), \\
\tilde{r}_h - \partial^{-1}(e^{2i\psi/h}qr_h) &= \partial^{-1}(e^{2i\psi/h}qa).
\end{aligned} \tag{3.3.2}
$$

then u given in (3.3.1) solves $(D+V)u=0$. Indeed,

$$
\begin{bmatrix} e^{-\bar{\Phi}/h} & 0 \\ 0 & e^{-\Phi/h} \end{bmatrix} (D+V) \begin{bmatrix} e^{\Phi/h} & 0 \\ 0 & e^{\bar{\Phi}/h} \end{bmatrix} = D + V_h, \tag{3.3.3}
$$

where

$$
V_h = \begin{bmatrix} e^{2i\psi/h}q & 0 \\ 0 & e^{-2i\psi/h}q' \end{bmatrix}.
$$

After substituting u of the form (3.3.1) into the equation $(D+V)u=0$ and, using (3.3.3), we obtain

$$
\begin{aligned}
\bar{\partial}r_h + e^{-2i\psi/h}q'\tilde{r}_h &= -e^{-2i\psi/h}q'b, \\
\partial\tilde{r}_h - e^{2i\psi/h}qr_h &= e^{2i\psi/h}qa.
\end{aligned} \tag{3.3.4}
$$

Since $\bar{\partial}(\bar{\partial}^{-1}f) = f$ and $\partial(\partial^{-1}f) = f$, we can take $r_h, \tilde{r}_h$ as a solution of the system (3.3.2).

<u>For $a = 0$</u> After substituting $a = 0$ in (3.3.2) and solving for r_h respectively we get

$$(I - \mathcal{S}_h)r_h = -\bar{\partial}^{-1}(e^{-2i\psi/h}q'b) \quad \text{with} \quad \mathcal{S}_h(f) = -\bar{\partial}^{-1}(e^{-2i\psi/h}q'\partial^{-1}(e^{2i\psi/h}qf)). \tag{3.3.5}$$

To this end, we estimate the norm of $\mathcal{S}_h$ for which we need the following key estimate from [GT11, Lemma 2.3] and [GT13, Lemma 5.4]. This type of estimate was first given by Bukhgeim in his seminal work [Buk08].

Lemma 3.3.1. Let $r \in (1, \infty)$ and $p > 2$, then there exists $C > 0$ independent of h such that for all $\omega \in W^{1,p}(\Omega)$

$$\|\bar{\partial}^{-1}(e^{-2i\psi/h}\omega)\|_{L^r(\Omega)} \leq Ch^{2/3}\|\omega\|_{W^{1,p}(\Omega)} \quad \text{if } 1 < r < 2 \tag{3.3.6}$$

$$\|\bar{\partial}^{-1}(e^{-2i\psi/h}\omega)\|_{L^r(\Omega)} \leq Ch^{1/r}\|\omega\|_{W^{1,p}(\Omega)} \quad \text{if } 2 \leq r \leq p. \tag{3.3.7}$$

There exist $\epsilon > 0$ and $C > 0$ such that for all $\omega \in W^{1,p}(\Omega)$

$$\|\bar{\partial}^{-1}(e^{-2i\psi/h}\omega)\|_{L^2(\Omega)} \leq Ch^{\frac{1}{2}+\epsilon}\|\omega\|_{W^{1,p}(\Omega)}. \tag{3.3.8}$$

Proof. Let us first prove (3.3.6). Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ which on $\overline{\Omega}$ satisfies

$$\begin{cases} 1 & |z - z_0| > 2\delta \\ 0 & |z - z_0| \leq \delta \end{cases}$$

where $\delta > 0$ is a parameter we will choose later, which can depend on h. Using Minkowski inequality, one can write when $r < 2$

$$\begin{aligned} \left\|\bar{\partial}^{-1}\left((1-\phi)e^{-2i\psi/h}g\right)\right\|_{L^r(\Omega)} &\leq \int_\Omega \left\|\frac{1}{|\cdot - \zeta|}\right\|_{L^r(\Omega)} |(1-\phi(\zeta))g(\zeta)|\, dm(\zeta) \\ &\leq C\|g\|_{L^\infty(\Omega)} \int_\Omega |(1-\phi(\zeta))|\, dm(\zeta) \leq C\delta^2\|g\|_{L^\infty(\Omega)}. \end{aligned} \tag{3.3.9}$$

and we know by Sobolev embedding that $\|g\|_{L^\infty(\Omega)} \leq \|g\|_{W^{1,p}(\Omega)}$. On the support of φ, we observe that since $\varphi = 0$ near z_0, we can use

$$\bar{\partial}^{-1}(e^{-2i\psi/h}\varphi g) = \frac{-h}{2i}\left[e^{-2i\psi/h}\frac{\varphi g}{\bar{\partial}\psi} - \bar{\partial}^{-1}\left(e^{-2i\psi/h}\bar{\partial}\left(\frac{\varphi g}{\bar{\partial}\psi}\right)\right)\right]$$

and the boundedness of $\bar\partial^{-1}$ on L^r to deduce that

$$\begin{aligned}\left\|\bar\partial^{-1}(e^{-2i\psi/h}\varphi g)\right\|_{L^r(\Omega)} &\leq Ch\left(\left\|\frac{\varphi g}{\bar\partial\psi}\right\|_{L^r(\Omega)}+\left\|\frac{(\bar\partial\phi)g}{\bar\partial\psi}\right\|_{L^r(\Omega)}+\left\|\frac{\varphi g}{(\bar\partial\psi)^2}\right\|_{L^r(\Omega)}+\left\|\frac{\phi(\bar\partial g)}{\bar\partial\psi}\right\|_{L^r(\Omega)}\right)\\ &:= Ch(I_1+I_2+I_3+I_4).\end{aligned} \tag{3.3.10}$$

The estimates for I_j are

$$\begin{aligned}I_1 &\leq \left\|\frac{\varphi}{\bar\partial\psi}\right\|_{L^\infty(\Omega)}\|g\|_{L^r(\Omega)}, \qquad I_2 \leq \left\|\frac{\bar\partial\varphi}{\bar\partial\psi}\right\|_{L^r(\Omega)}\|g\|_{L^\infty(\Omega)},\\ I_3 &\leq \left\|\frac{\varphi}{(\bar\partial\psi)^2}\right\|_{L^r(\Omega)}\|g\|_{L^\infty(\Omega)}, \qquad I_4 \leq \left\|\frac{\varphi}{\bar\partial\psi}\right\|_{L^\infty(\Omega)}\|\bar\partial g\|_{L^r(\Omega)},\end{aligned}$$

Using $\bar\partial\psi=\bar z-\bar z_0$ we can find

$$\begin{aligned}\left\|\frac{\varphi}{\bar\partial\psi}\right\|_{L^\infty(\Omega)} &= \left\|\frac{\varphi}{\bar z-\bar z_0}\right\|_{L^\infty(\Omega)} \leq \left\|\frac{1}{|z-z_0|}\right\|_{L^\infty(\Omega\setminus B(z_0,\delta))} = \delta^{-1}\\ \left\|\frac{\bar\partial\varphi}{\bar\partial\psi}\right\|_{L^r(\Omega)} &= \left\|\frac{\bar\partial\varphi}{\bar z-\bar z_0}\right\|_{L^r(\Omega)} \leq C\delta^{-1}\left\|\frac{1}{|z-z_0|}\right\|_{L^r(\{\delta<|z-z_0|<2\delta\})} \leq C\delta^{-1}\left(\int_\delta^{2\delta} s^{1-r}\,ds\right)^{1/r} \leq C\delta^{\frac{2}{r}-2}\\ \left\|\frac{\varphi}{(\bar\partial\psi)^2}\right\|_{L^r(\Omega)} &= \left\|\frac{\varphi}{(\bar z-\bar z_0)^2}\right\|_{L^r(\Omega)} \leq \left\|\frac{1}{|z-z_0|^2}\right\|_{L^r(\Omega\setminus B(z_0,\delta))} \leq C\left(\int_\delta^{1} s^{1-2r}\,ds\right)^{1/r} \leq C\delta^{\frac{2}{r}-2}.\end{aligned}$$

Combining these four estimates with (4.3.10) we obtain

$$\left\|\bar\partial^{-1}(e^{-2i\psi/h}\varphi g)\right\|_{L^r(\Omega)} \leq Ch\|g\|_{W^{1,p}(\Omega)}\left(\delta^{-1}+\delta^{\frac{2}{r}-2}\right)$$

Combining this and (3.3.9) and optimizing by taking $\delta=h^{1/3}$ we see that (3.3.6) is established.

Let us now focus on the case $2\leq r\leq p$. One can use the boundedness of $\bar\partial^{-1}$ on L^r to obtain

$$\begin{aligned}\left\|\bar\partial^{-1}((1-\varphi)e^{-2i\psi/h}g)\right\|_{L^r(\Omega)} &\leq \left\|(1-\varphi)e^{-2i\psi/h}g\right\|_{L^r(\Omega)}\\ &\leq C\delta^{\frac{2}{r}}\|g\|_{L^\infty(\Omega)}\end{aligned} \tag{3.3.11}$$

Now we can again use the identity

$$\bar{\partial}^{-1}(e^{-2i\psi/h}\varphi g) = \frac{-h}{2i}\left[e^{-2i\psi/h}\frac{\varphi g}{\bar{\partial}\psi} - \bar{\partial}^{-1}\left(e^{-2i\psi/h}\bar{\partial}\left(\frac{\varphi g}{\bar{\partial}\psi}\right)\right)\right]$$

and the boundedness of $\bar{\partial}^{-1}$ on L^r to deduce that for any $r \leq p$, (4.3.10) holds again with all the terms satisfying the same estimates as before, so that

$$\left\|\bar{\partial}^{-1}(e^{-2i\psi/h}\varphi g)\right\|_{L^r(\Omega)} \leq Ch\|g\|_{W^{1,p}(\Omega)}\left(\delta^{-1} + \delta^{\frac{2}{r}-2}\right) \leq Ch\delta^{\frac{2}{r}-2}\|g\|_{W^{1,p}(\Omega)}$$

since now $q \geq 2$. Now by combining the above estimate with (3.3.11) and taking $\delta = h^{1/2}$ we establish (3.3.7). The estimate claimed in (3.3.8) is obtained by interpolating the case $r < 2$ with $r > 2$. □

Lemma 3.3.2. If $q \in L^\infty(\Omega)$ and $q' \in W^{1,p}(\Omega)$, $p > 2$, then $\mathcal{S}_h$ is bounded on $L^r(\Omega)$ for any $1 < r \leq p$ and satisfies $\|\mathcal{S}_h\|_{L^r \to L^r} = O(h^{1/r})$ if $r > 2$ and $\|\mathcal{S}_h\|_{L^2 \to L^2} = O(h^{\frac{1}{2}-\epsilon})$ for any $0 < \epsilon < 1/2$ small.

Proof. Firstly for $2 < r \leq p$ we obtain

$$\begin{aligned}
\|S_h(f)\|_{L^r(\Omega)} &\leq \left\|\bar{\partial}^{-1}\left(e^{-2i\psi/h}q'\partial^{-1}\left(e^{2i\psi/h}qf\right)\right)\right\|_{L^r(\Omega)} \\
&\leq Ch^{1/r}\left\|q'\partial^{-1}\left(e^{2i\psi/h}qf\right)\right\|_{W^{1,r}(\Omega)} \\
&\leq Ch^{1/r}\left\|\partial^{-1}\left(e^{2i\psi/h}qf\right)\right\|_{W^{1,r}(\Omega)} \\
&\leq Ch^{1/r}\left\|e^{2i\psi/h}qf\right\|_{L^r(\Omega)} \\
&\leq Ch^{1/r}\|f\|_{L^r(\Omega)}.
\end{aligned}$$

Further, for $1 < r < 2$,

$$\begin{aligned}
\|S_h(f)\|_{L^r(\Omega)} &\leq \left\|\bar{\partial}^{-1}\left(e^{-2i\psi/h}q'\partial^{-1}\left(e^{2i\psi/h}qf\right)\right)\right\|_{L^r(\Omega)} \\
&\leq C\left\|e^{-2i\psi/h}q'\partial^{-1}\left(e^{2i\psi/h}qf\right)\right\|_{L^r(\Omega)} \\
&\leq C\left\|\partial^{-1}\left(e^{2i\psi/h}qf\right)\right\|_{L^r(\Omega)} \\
&\leq C\left\|e^{2i\psi/h}qf\right\|_{L^r(\Omega)} \\
&\leq C\|f\|_{L^r(\Omega)}.
\end{aligned}$$

For all $\epsilon > 0$ small, interpolating between $r = 1 + \epsilon$ and $r = 2 + \epsilon$, gives the desired result for $r = 2$. $\square$

In view of Lemma 4.3.1, equation (3.3.5) can be solved by using Neumann series by setting (for small $h > 0$)

$$r_h := -\sum_{j=0}^{\infty} \mathcal{S}_h^j \bar{\partial}^{-1}(e^{-2i\psi/h} q' b) \tag{3.3.12}$$

as an element of any $L^r(\Omega)$ for $r \geq 2$. Substituting this expression for r_h into equation (3.3.2) when $a = 0$, we get that

$$\tilde{r}_h = \partial^{-1}(e^{2i\psi/h} q r_h). \tag{3.3.13}$$

We now derive the asymptotics in h for r_h and $\tilde{r}_h$.

Lemma 3.3.3. If $q \in L^\infty(\Omega)$ and $q' \in W^{1,p}(\Omega)$ for some $p > 2$, then there exists $\epsilon > 0$ such that

$$\|r_h\|_{L^2(\Omega)} + \|\tilde{r}_h\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon}).$$

Proof. The statement for r_h is an easy consequence of Lemma 3.3.1 and 4.3.1: indeed

$$\left\|\bar{\partial}^{-1}(e^{-2i\psi/h} q' b)\right\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$$

by Lemma 3.3.1 and $\|\mathrm{S}_h\|_{L^2 \to L^2} = O(h^{\frac{1}{2}-\epsilon})$ thus $\|r_h\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$. The estimate for $\tilde{r}_h$ comes from the fact that $\left\|\partial^{-1}(e^{2i\psi/h} \cdot)\right\|_{L^2 \to L^2} = O(1)$ and (4.3.1). $\square$

The same method can clearly be used by setting $b = 0$ and solving for $\tilde{r}_h$ first. We summarize the results of this section into the following proposition:

Proposition 3.3.4. Let a and b be holomorphic and antiholomorphic functions on Ω. If $q \in L^\infty(\Omega)$ and $q' \in W^{1,p}(\Omega)$ for some $p > 2$, then there exists solutions to $(D+V)F = 0$ in $W^{1,2}(\Omega)$ of the form

$$F_h = \begin{bmatrix} e^{\Phi/h}(r_h) \\ e^{\bar{\Phi}/h}(b + \tilde{r}_h) \end{bmatrix} \tag{3.3.14}$$

where $\|r_h\|_{L^2(\Omega)} + \|\tilde{r}_h\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$ for some $\epsilon > 0$. If conversely $q' \in L^\infty(\Omega)$ and $q \in W^{1,p}(\Omega)$ for some $p > 2$, then there exists solutions to $(D+V)G = 0$ in $W^{1,2}(\Omega)$ of the form

$$G_h = \begin{bmatrix} e^{\Phi/h}(a + r_h) \\ e^{\bar{\Phi}/h}(\tilde{r}_h) \end{bmatrix} \tag{3.3.15}$$

where $\|r_h\|_{L^2(\Omega)} + \|\tilde{r}_h\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$ for some $\epsilon > 0$.

3.4 Uniqueness

In this section, we prove Theorem 3.1.3. Our integral identity, as discussed in Section 3.2, is

$$\int_\Omega \langle (V_2 - V_1)U_1, U_2\rangle = 0, \tag{3.4.1}$$

where $(D + V_1)(U_1) = 0$ and $(D + V_2^*)(U_2) = 0$.

We prove the first part of Theorem 3.1.3. By using Proposition 3.3.4 we consider U_1 and U_2 of the form

$$U_1 = \begin{bmatrix} e^{\Phi/h}(r_h^1) \\ e^{\bar{\Phi}/h}(b + \tilde{r}_h^1) \end{bmatrix} \quad U_2 = \begin{bmatrix} e^{-\Phi/h}(r_h^2) \\ e^{-\bar{\Phi}/h}(b + \tilde{r}_h^2) \end{bmatrix}$$

with $r_h^j, \tilde{r}_h^j$ as constructed in Proposition 3.3.4.

By using U_1 and U_2, the integral identity takes the form

$$0 = \int_\Omega e^{-2i\psi/h}(q_2' - q_1')\left(|b|^2 + \langle b, \tilde{r}_h^2\rangle + \langle \tilde{r}_h^1, \tilde{r}_h^2\rangle + \langle \tilde{r}_h^1, b\rangle\right) + (q_2 - q_1)e^{2i\psi/h}\langle r_h^1, r_h^2\rangle$$

We use the method of stationary phase to obtain that

$$\int_\Omega e^{-2i\psi/h}(q_2' - q_1')|b|^2 = C_{z_0} h e^{2i\psi(z_0)/h}(q_2'(z_0) - q_1'(z_0))|b(z_0)|^2 + o(h). \tag{3.4.2}$$

where $C_{z_0} \neq 0$.

Next, we use the fact that $\|r_h^j\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$, $\|\tilde{r}_h^j\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$, for some $\epsilon > 0$ and obtain the following estimate

$$\int_\Omega (q_2' - q_1')e^{-2i\psi/h}\langle \tilde{r}_h^1, \tilde{r}_h^2\rangle + (q_2 - q_1)e^{2i\psi/h}\langle r_h^1, r_h^2\rangle = O(h^{1+2\epsilon}). \tag{3.4.3}$$

Now, we have the following estimate

$$\int_\Omega e^{-2i\psi/h}(q_2' - q_1')\langle \tilde{r}_h^1, b\rangle$$

$$
\begin{aligned}
&= \int_\Omega e^{-2i\psi/h}(q_2' - q_1')\langle \partial^{-1}(e^{2i\psi/h} q_1 r_h^1), b\rangle \\
&= \int_\Omega \partial^{-1}\left(e^{-2i\psi/h}(q_2' - q_1')\bar{b}\right) e^{2i\psi/h} q_1 r_h^1 \\
&\leq C h^{\frac{1}{2}+\epsilon} \left\|(q_2' - q_1')\bar{b}\right\|_{W^{1,p}} \left\|r_h^1\right\|_{L^2(\Omega)},
\end{aligned}
$$

where we have used Fubini's theorem in the third equality while the last inequality is obtained by applying Lemma 3.3.1. Now, using $\|r_h^1\|_{L^2(\Omega)} = O(h^{1/2+\epsilon})$ to obtain

$$
\int_\Omega e^{-2i\psi/h}(q_2' - q_1')\langle \tilde{r}_h^1, b\rangle = O(h^{1+2\epsilon}). \tag{3.4.4}
$$

Similarly, we obtain

$$
\int_\Omega e^{-2i\psi/h}(q_2' - q_1')\langle b, \tilde{r}_h^2\rangle = O(h^{1+2\epsilon}). \tag{3.4.5}
$$

By matching the asymptotics as $h \to 0$ we get ${q'}_1 = {q'}_2$ since z_0 can be arbitrairly chosen in Ω. Similarly, we can prove the second part.

Chapter 4

Inverse problem for polyharmonic operators in two dimensions

4.1 Introduction

In the case of Magnetic Schrödinger, the principle operator is $-\Delta$, a harmonic operator, and it has all lower-order perturbations. In this section, we will study the operators whose principle operator is $(-\Delta)^m$ and has lower order perturbations.

More precisely, Let

$$\mathcal{P} = (-\Delta)^m + \sum_{|\alpha|<2m} A_\alpha D^\alpha \quad \text{in } \Omega.$$

The inverse problem is whether we can recover the coefficients A_α from the set of Cauchy data

$$\mathcal{C}(\mathcal{P}) = \left\{ \left(u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}, \partial_\nu^2 u|_{\partial\Omega} \ldots, \partial_\nu^{(2m-1)} u|_{\partial\Omega} \right) : u \in H^{2m}(\Omega), \mathcal{P}u = 0 \right\}, \tag{4.1.1}$$

where ν is an outer unit normal to $\partial\Omega$.

In dimensions $n \geq 3$, Krupchyk, Lassas, and Uhlmann [KLU14] established that the Cauchy data for a polyharmonic operator uniquely determine the first-order perturbations. Many authors extended this work; see, for instance, [GK16, BG19, BKS21, BG22]. Till now, the perturbations considered for the polyharmonic operator are of order at most m in $n \geq 3$.

Until this work with Prof. Krishnan and Dr. Rahul Raju Pattar, to our knowledge, there is no direct study of the inverse problem for a perturbed polyharmonic operator in two dimensions compared to higher dimensions. Nevertheless, in [IY15, LUW15], the authors have studied the Navier-Stokes equation in two dimensions using the biharmonic operator.

We studied the inverse boundary value problem in two dimensions for a polyharmonic operator of the form:

$$\mathcal{L} = \partial^m \bar{\partial}^m + \sum_{j,k=0}^{m-1} A_{j,k} \partial^j \bar{\partial}^k, \quad m \geq 2. \tag{4.1.2}$$

Note that $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ is identified with $z = x_1 + ix_2 \in \mathbb{C}$ and $\partial = \frac{1}{2}\left(\partial_{x_1} - i\partial_{x_2}\right)$, $\bar{\partial} = \frac{1}{2}\left(\partial_{x_1} + i\partial_{x_2}\right)$.

In this work, we establish that the Cauchy data for a polyharmonic operator in two dimensions uniquely determine all anisotropic perturbations of order at most $m-1$ and several perturbations of orders m to $2m-2$ with some restrictions. This restriction is captured in the following representation of the operator $\mathcal{L}$ as

$$(\partial\bar{\partial})^m + A_{m-1,m-1}(\partial\bar{\partial})^{m-1} + \sum_{l=1}^{m-2}\left(\sum_{j+k=m-l-1} A_{j+l,k+l}\partial^j\bar{\partial}^k\right)(\partial\bar{\partial})^l + \sum_{l=0}^{m-1}\sum_{j+k=l} A_{j,k}\partial^j\bar{\partial}^k.$$

This constraint on the coefficients of orders m to $2m-2$ is required for the techniques employed in this paper to work, mainly to make the equation for the amplitude of complex geometric optics (CGO) solutions to be independent of the coefficients.

Our approach relies on two main techniques - the $\bar{\partial}$-techniques and the method of stationary phase. Bukhgeim first used these techniques in his seminal work [Buk08] to recover the zeroth order perturbation of the Laplacian in two dimensions. This work has led to many developments in studying two-dimensional inverse boundary value problems.

Now, we state the main theorem of this work.

Theorem 4.1.1. Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^2$. Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be two operators of the form (4.1.2) with coefficients $A_{j,k}, \tilde{A}_{j,k} \in W^{j+k+1,p}(\Omega),\ p > 2$, respectively. Assume that

$$\partial_\nu^l A_{j,k} = \partial_\nu^l \tilde{A}_{j,k} \text{ and } A_{0,k} = \tilde{A}_{0,k} \text{ on } \partial\Omega, \quad \text{for } 0 \leq l \leq j-1,\ 0 \leq j,k \leq m-1. \tag{4.1.3}$$

Then $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\tilde{\mathcal{L}})$ implies that $A_{j,k} = \tilde{A}_{j,k}$ on Ω for $0 \leq j,k \leq m-1$.

To prove Theorem 4.1.1, we need a special type of solutions called CGO solutions. The next theorem gives the existence of such solutions in our setting. Let us fix some notational conventions before stating the theorem. Let $\Phi = i(z-z_0)^2$ where $z_0 \in \Omega$ and dS be the surface measure on $\partial\Omega$.

Theorem 4.1.2. Let a be smooth function such that $\bar{\partial}^m a = 0$ in Ω. If $A_{j,k} \in W^{j+k+1,p}(\Omega)$, for some $p > 2$, then for a small $h > 0$ there exist solutions $u \in H^{2m}(\Omega)$ to

$$\mathcal{L}u = \partial^m \bar{\partial}^m u + \sum_{j,k=0}^{m-1} A_{j,k} \partial^j \bar{\partial}^k u = 0, \quad \text{in } \Omega, \tag{4.1.4}$$

of the form

$$u = e^{\Phi/h}(a + r_h), \tag{4.1.5}$$

where the correction term r_h satisfies $\|r_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$ for some $\epsilon > 0$.

4.2 Integral Identity

The key of the proof of Theorem 4.1.1 is the construction of families of h-parameterized solutions $u_1 = u_1(x;h)$ and $v = v(x;h)$ with $h > 0$ satisfying $\mathcal{L}u_1 = 0$ and $\tilde{\mathcal{L}}^* v = 0$. By assuming $\mathcal{C}(\mathcal{L}) = \mathcal{C}(\tilde{\mathcal{L}})$ there exists a solution u_2 to $\tilde{\mathcal{L}} u_2 = 0$ with

$$\left.\begin{aligned} u_2|_{\partial\Omega} &= u_1|_{\partial\Omega}, \\ (\partial_\nu u_2)|_{\partial\Omega} &= (\partial_\nu u_1)|_{\partial\Omega}, \\ \vdots \qquad & \qquad \vdots \\ (\partial_\nu^{(2m-1)} u_2)|_{\partial\Omega} &= (\partial_\nu^{(2m-1)} u_1)|_{\partial\Omega}. \end{aligned}\right\} \tag{4.2.1}$$

Note that

$$\tilde{\mathcal{L}}(u_1 - u_2) = \sum_{j,k=0}^{m-1} (\tilde{A}_{j,k} - A_{j,k}) \partial^j \bar{\partial}^k u_1.$$

Now we use integration by parts and (4.2.1) to obtain the following integral identity

$$\begin{aligned} 0 &= \int_\Omega (u_1 - u_2) \overline{\tilde{\mathcal{L}}^* v} \, dx \\ &= \int_\Omega \tilde{\mathcal{L}}(u_1 - u_2) \bar{v} \, dx \end{aligned}$$

$$= \int_\Omega \left[\sum_{j,k=0}^{m-1} (\tilde{A}_{j,k} - A_{j,k}) \partial^j \bar{\partial}^k u_1 \right] \bar{v}\, dx.$$

4.3 Complex Geometric Optics Solutions

In this section, we prove Theorem 4.1.2 as stated in Introduction. We begin by writing

$$\mathcal{L}u = \partial^m \bar{\partial}^m u + \sum_{j,k=0}^{m-1} A_{j,k} \partial^j \bar{\partial}^k u = 0$$

in the following form

$$\mathcal{L}u = \partial^m \bar{\partial}^m u + \sum_{j,k=0}^{m-1} \partial^j (A'_{j,k} \bar{\partial}^k u) = 0, \tag{4.3.1}$$

where we can define $A'_{j,k} \in W^{j+k+1,p}(\Omega)$ uniquely satisfying

$$A_{j,k} = \sum_{l=j}^{m-1} \binom{l}{j} \partial^{l-j} A'_{l,k}. \tag{4.3.2}$$

Substituting $u = e^{\Phi/h} f$ in (4.3.1), we have

$$\partial^m \left(e^{\Phi/h} \bar{\partial}^m f \right) + \sum_{j,k=0}^{m-1} \partial^j \left(e^{\Phi/h} A'_{j,k} \bar{\partial}^k f \right) = 0.$$

Now, we write $G = e^{\Phi/h} \bar{\partial}^m f$ and the above expression takes the form

$$\begin{aligned} \bar{\partial}^m f &= e^{-\Phi/h} G \\ \partial^m G &= - \sum_{j,k=0}^{m-1} \partial^j \left(e^{\Phi/h} A'_{j,k} \bar{\partial}^k f \right) \end{aligned} \tag{4.3.3}$$

The problem here is that $|e^{\pm\Phi/h}|$ grows too fast when $h \to 0$. This can be solved by choosing $G = e^{\bar{\Phi}/h} g$ to get

$$\bar{\partial}^m f = e^{(\bar{\Phi}-\Phi)/h} g \tag{4.3.4}$$

$$\partial^m g = - \sum_{j,k=0}^{m-1} \partial^j \left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^k f \right) \tag{4.3.5}$$

For (4.3.4), we take the solution

$$f = a + \bar{\partial}^{-m}\left(e^{(\bar{\Phi}-\Phi)/h} g\right), \qquad \text{where } \bar{\partial}^{m} a = 0 \tag{4.3.6}$$

and for (4.3.5) we choose the solution

$$g = -\sum_{j,k=0}^{m-1} \partial^{j-m}\left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k} f\right). \tag{4.3.7}$$

By combining these two, we get an integral equation for g of the form

$$g + \sum_{j,k=0}^{m-1} \partial^{j-m}\left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m}\left(e^{(\bar{\Phi}-\Phi)/h} g\right)\right) = -\sum_{j,k=0}^{m-1} \partial^{j-m}\left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k} a\right). \tag{4.3.8}$$

The above expression for g can be written in the form

$$(I - \mathcal{S}_h) g = w, \tag{4.3.9}$$

where

$$\begin{aligned} \mathcal{S}_h(v) &= -\sum_{j,k=0}^{m-1} \partial^{j-m}\left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m}\left(e^{(\bar{\Phi}-\Phi)/h} v\right)\right) \\ w &= -\sum_{j,k=0}^{m-1} \partial^{j-m}\left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k} a\right). \end{aligned} \tag{4.3.10}$$

The existence of a CGO solution of the form (4.1.5) to the equation (4.1.4) depends on the solvability of (4.3.9). To this end, we estimate the norm of $\mathcal{S}_h$ for which we need the following key estimate from [GT11, Lemma 2.3] and [GT13, Lemma 5.4]. Bukhgeim first gave this type of estimate in his seminal work [Buk08].

Lemma 4.3.1. For any $1 < r \leq p$, the operator $\mathcal{S}_h$ is bounded on $L^r(\Omega)$ and satisfies $\|\mathcal{S}_h\|_{L^r \to L^r} = O(h^{1/r})$ for $r > 2$ and $\|\mathcal{S}_h\|_{L^2 \to L^2} = O(h^{\frac{1}{2}-\epsilon})$ for any $0 < \epsilon < 1/2$ small.

Proof. Firstly, for $2 < r \leq p$, we obtain

$$\|S_h(v)\|_{L^r(\Omega)} \leq \sum_{j,k=0}^{m-1} \left\| \partial^{j-m}\left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m}\left(e^{(\bar{\Phi}-\Phi)/h} v\right)\right)\right\|_{L^r(\Omega)}$$

$$
\begin{aligned}
&\le C \sum_{j,k=0}^{m-1} \left\| \partial^{j-m+1} \left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right) \right\|_{L^r(\Omega)} \\
&\le C \sum_{j,k=0}^{m-1} \left\| \partial^{-1} \left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right) \right\|_{L^r(\Omega)} \\
&\le C h^{\frac{1}{r}} \sum_{j,k=0}^{m-1} \left\| A'_{j,k} \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right\|_{W^{1,r}(\Omega)} \\
&\le C h^{\frac{1}{r}} \sum_{k=0}^{m-1} \left\| \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right\|_{W^{1,r}(\Omega)} \\
&\le C h^{\frac{1}{r}} \|v\|_{L^r(\Omega)}.
\end{aligned}
$$

Further, for $1 < r < 2$,

$$
\begin{aligned}
\|S_h(v)\|_{L^r(\Omega)} &\le \sum_{j,k=0}^{m-1} \left\| \partial^{j-m} \left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right) \right\|_{L^r(\Omega)} \\
&\le C \sum_{j,k=0}^{m-1} \left\| \partial^{-1} \left(e^{(\Phi-\bar{\Phi})/h} A'_{j,k} \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right) \right\|_{L^r(\Omega)} \\
&\le C \sum_{j,k=0}^{m-1} \left\| A'_{j,k} \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right\|_{L^r(\Omega)} \\
&\le C \sum_{k=0}^{m-1} \left\| \bar{\partial}^{k-m} \left(e^{(\bar{\Phi}-\Phi)/h} v \right) \right\|_{L^r(\Omega)} \\
&\le C \|v\|_{L^r(\Omega)}.
\end{aligned}
$$

For all $\varepsilon > 0$ small, interpolating between $r = 1 + \varepsilon$ and $r = 2 + \varepsilon$, gives the desired result for $r = 2$. □

Proposition 4.3.2. For all sufficiently small $h > 0$, there exist a solution $g \in H^m(\Omega)$ to the equation

$$(I - \mathcal{S}_h) g = w,$$

where $\mathcal{S}_h$ and w defined in (4.3.10) which satisfies $\|g\|_{L^2} = O(h^{\frac{1}{2}+\epsilon})$.

Proof. Given Lemma 4.3.1, equation (4.3.9) can be solved by using the Neumann series by setting (for small $h > 0$)

$$g = \sum_{j=0}^{\infty} \mathcal{S}_h^j w.$$

as an element of $L^2(\Omega)$. Indeed $\|w\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$ by Lemma 3.3.1 and $\|\mathrm{S}_h\|_{L^2\to L^2} = O(h^{\frac{1}{2}-\epsilon})$ by Lemma 4.3.1 we obtain $\left\|\mathrm{S}_h^j w\right\|_{L^2} = O\left(h^{(\frac{1}{2}-\epsilon)j} h^{\frac{1}{2}+\epsilon}\right)$ which implies $\|g\|_{L^2(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$.

Now that $g \in L^2(\Omega)$ and $A'_{j,k} \in W^{2m+j+k+3,p}(\Omega)$, we use bootstrapping argument on the expression (4.3.8) to conclude that $g \in H^m(\Omega)$. $\square$

Proof of Theorem 4.1.2. Choose g as in Proposition 4.3.2 and let

$$r_h = \bar{\partial}^{-m}\left(e^{(\bar{\Phi}-\Phi)/h} g\right)$$

as observed in (4.3.6). Clearly, $r_h \in H^{2m}(\Omega)$ and $\|r_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}+\varepsilon})$. Then we see that $u = e^{\Phi/h}(a + r_h) \in H^{2m}(\Omega)$ where $\bar{\partial}^m a = 0$ solves $\mathcal{L}u = 0$. This proves Theorem 4.1.2. $\square$

The adjoint operator $\mathcal{L}^*$ has a similar form as the operator $\mathcal{L}$. Hence, by following similar arguments, we can show that the adjoint equation has the same type of CGO solutions as given in Theorem 4.1.2.

Remark 4.3.3. One can write an integral equation for f instead of g by substituting (4.3.7) in (4.3.6), but by following the above procedure, one obtains a coarser estimate, $\|r_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}-\varepsilon})$.

4.4 Uniqueness of Coefficients

In this section, we prove Theorem 4.1.1. Our integral identity, as discussed in Section 4.2 is

$$\sum_{j,k=0}^{m-1} (-1)^j \int_{\Omega} (\tilde{A}'_{j,k} - A'_{j,k}) \bar{\partial}^k u_1 \partial^j \bar{v} = 0, \tag{4.4.1}$$

where $\mathcal{L}(u_1) = 0$ and $\tilde{\mathcal{L}}^*(v) = 0$. By our assumption, we get

$$\sum_{j,k=0}^{m-1} \left((-1)^j \int_{\Omega} (\tilde{A}'_{j,k} - A'{}_{j,k}) \bar{\partial}^k u_1 \partial^j \bar{v}\, dx\right) = 0. \tag{4.4.2}$$

By using Theorem 4.1.2 we consider u_1 and v of the form

$$\begin{aligned} u_1 &= e^{\Phi/h}(a + r_h), \text{ where } \bar{\partial}^m a = 0, \\ v &= e^{-\Phi/h}(b + s_h), \text{ where } \bar{\partial}^m b = 0, \end{aligned} \tag{4.4.3}$$

with r_h and s_h satisfy $\|r_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$ and $\|s_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$ for some $\epsilon > 0$.

By using u_1 and v, the integral identity takes the form

$$\begin{aligned} 0 = &\sum_{j,k=0}^{m-1} (-1)^j \int_\Omega \left[e^{(\Phi-\bar{\Phi})/h} (\tilde{A}'_{j,k} - A'_{j,k}) \bar{\partial}^k a \partial^j \bar{b} \right] \\ &+ \sum_{j,k=0}^{m-1} (-1)^j \int_\Omega \left[e^{(\Phi-\bar{\Phi})/h} (\tilde{A}'_{j,k} - A'_{j,k}) (\bar{\partial}^k a \partial^j \bar{s}_h + \bar{\partial}^k r_h \partial^j \bar{b} + \bar{\partial}^k r_h \partial^j \bar{s}_h) \right] \end{aligned}$$

We use the method of stationary phase to obtain that

$$\begin{aligned} &\sum_{j,k=0}^{m-1} \left((-1)^j \int_\Omega e^{2i\psi/h} (\tilde{A}'_{j,k} - A'_{j,k}) (\bar{\partial}^k a)(\partial^j \bar{b}) \right) \\ &\qquad = \sum_{j,k=0}^{m-1} C_{j,k}(z_0) h e^{2i\psi(z_0)/h} (\tilde{A}'_{j,k}(z_0) - A'_{j,k}(z_0)) \bar{\partial}^k a(z_0) \partial^j \bar{b}(z_0) + o(h). \end{aligned} \tag{4.4.4}$$

where $C_{j,k}(z_0) \neq 0$ for all $0 \leq j,k \leq m-1$.

Next, we use the fact that $\|r_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$, $\|s_h\|_{H^m(\Omega)} = O(h^{\frac{1}{2}+\epsilon})$, for some $\epsilon > 0$ and obtain the following estimate

$$\sum_{j,k=0}^{m-1} (-1)^j \int_\Omega \left[e^{(\Phi-\bar{\Phi})/h} (\tilde{A}'_{j,k} - A'_{j,k}) \bar{\partial}^k r_h \partial^j \bar{s_h} \right] = O(h^{1+2\epsilon}). \tag{4.4.5}$$

Let $\tilde{r}_h$ and $\tilde{s}_h$ be such that

$$\begin{aligned} r_h &= \bar{\partial}^{-m} \left(e^{(\bar{\Phi}-\Phi)/h} \tilde{r}_h \right), \\ s_h &= \bar{\partial}^{-m} \left(e^{(\bar{\Phi}-\Phi)/h} \tilde{s}_h \right) \end{aligned}$$

and they satisfy the equation (4.3.8) in place of g. Using this we have the following estimate

$$\begin{aligned}
&\sum_{j,k=0}^{m-1}(-1)^j\int_\Omega\left[e^{(\Phi-\bar\Phi)/h}(\tilde A'_{j,k}-A'_{j,k})\bar\partial^k r_h\partial^j\bar b\right]\\
&\quad=\sum_{j,k=0}^{m-1}(-1)^j\int_\Omega\left[e^{(\Phi-\bar\Phi)/h}(\tilde A'_{j,k}-A'_{j,k})\bar\partial^{k-m}\left(e^{(\bar\Phi-\Phi)/h}\tilde r_h\right)\partial^j\bar b\right]\\
&\quad=\sum_{j,k=0}^{m-1}(-1)^j\int_\Omega\left[\bar\partial^{k-m}\left(e^{(\Phi-\bar\Phi)/h}(\tilde A'_{j,k}-A'_{j,k})\partial^j\bar b\right)e^{(\bar\Phi-\Phi)/h}\tilde r_h\right]\\
&\quad\le Ch^{\frac12+\epsilon}\sum_{j,k=0}^{m-1}\left\|(\tilde A'_{j,k}-A'_{j,k})\partial^j\bar b\right\|_{W^{1,p}}\|\tilde r_h\|_{L^2},
\end{aligned}$$

where we have used Fubini's theorem in the third equality while the last inequality is obtained by applying Lemma 3.3.1. Now, we apply Proposition 4.3.2 to obtain

$$\sum_{j,k=0}^{m-1}(-1)^j\int_\Omega\left[e^{(\Phi-\bar\Phi)/h}(\tilde A'_{j,k}-A'_{j,k})\bar\partial^k r_h\partial^j\bar b\right]=O(h^{1+2\epsilon}). \tag{4.4.6}$$

Similarly, we obtain

$$\sum_{j,k=0}^{m-1}(-1)^j\int_\Omega\left[e^{(\Phi-\bar\Phi)/h}(\tilde A'_{j,k}-A'_{j,k})\bar\partial^k a\partial^j\bar s_h\right]=O(h^{1+2\epsilon}). \tag{4.4.7}$$

Proof of Theorem 4.1.1. Using the estimates (4.4.4) - (4.4.7) and matching the asymptotics as $h\to 0$, we obtain

$$0=\sum_{j,k=0}^{m-1}(-1)^j(\tilde A'_{j,k}(z_0)-A'_{j,k}(z_0))\bar\partial^k a(z_0)\partial^j\bar b(z_0). \tag{4.4.8}$$

We now show that $A'_{0,0}=\tilde A'_{0,0}$. To this end, let us choose $a=b=1$. With this choice we obtain

$$\tilde A'_{0,0}(z_0)=A'_{0,0}(z_0).$$

Since for any $z_0\in\Omega$ we can choose Φ with a unique critical point at z_0, we have

$$\tilde A'_{0,0}=A'_{0,0}\quad\text{in }\Omega.$$

Next to show that $A'_{0,1} = \tilde{A}'_{0,1}$, we rewrite (4.4.8) by setting the term $\tilde{A}'_{0,0} - A'_{0,0} = 0$. Then, we choose $a = \bar{z}, b = 1$ to obtain

$$A'_{0,1} = \tilde{A}'_{0,1}, \quad \text{in } \Omega.$$

Similarly, one can show $A'_{j,k} = \tilde{A}'_{j,k}$ in an increasing order for $j + k$ by choosing

$$a = \frac{\bar{z}^k}{k!} \text{ and } b = \frac{\bar{z}^j}{j!}$$

and applying the above procedure to obtain

$$\tilde{A}'_{j,k} = A'_{j,k} \text{ in } \Omega \quad \text{for all } \ 0 \leq j, k \leq m - 1.$$

From (4.3.2), we readily obtain that

$$\tilde{A}_{j,k} = A_{j,k} \text{ in } \Omega \quad \text{for all } \ 0 \leq j, k \leq m - 1.$$

This proves Theorem 4.1.1.

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