Option Pricing in a Regime Switching Jump Diffusion Model

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Omkar Manjarekar
20121055

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

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Supervisor: Dr. Anindya Goswami

Certificate

This is to certify that this dissertation entitled 'Option Pricing in a Regime Switching Jump Diffusion Model ', towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Omkar Manjarekar at Indian Institute of Science Education and Research under the supervision of Dr. Anindya Goswami, Assistant Professor Department of Mathematics, during the academic year 2016-2017.

Dr. Anindya Goswami

Committee:

Dr. Anindya Goswami

Prof. G. K. Basak

Declaration

I hereby declare that the matter embodied in the report entitled 'Option Pricing in a Regime Switching Jump Diffusion Model ' are the results of the work carried out by me at the Department of Mathematics, IISER Pune, under the supervision of Dr. Anindya Goswami, and the same has not been submitted elsewhere for any other degree.

Omkar Manjarekar

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Abstract

There has been extensive literature available in the theory and practice of option valuation following the pioneering work by Black and Scholes (1973). Contrary to subsequent empirical evidence from the dynamics of financial assets, the Black-Scholes model assumed a constant growth rate r and a constant deterministic volatility coefficient σ. In subsequent studies, to overcome the demerits of B-S-M model, various option valuation models have been proposed and implemented in tune with realistic price dynamics. These include stochastic volatility models, jump-diffusion models, regime-switching models etc. The market in these models is incomplete where a perfect hedge may not be possible by a self-financing portfolio with a pre-determined initial wealth.

In this thesis, we consider a regime-switching jump diffusion model of a financial market, where an observed Euclidean space valued pure jump process drives the values of r and σ. Further, we assume the pure jump process as an age-dependent semi-Markov process. In this, one has an opportunity to incorporate some memory effect of the market. In particular, the knowledge of past stagnancy period can be fed into the option price formula to obtain the price value. We show using Follmer Schweizer decomposition that the option price at time t, satisfies a Cauchy problem involving a linear, parabolic, degenerate and non-local integro-partial differential equation. We study the well-posedness of the Cauchy problem.

Contents

Chapter 1

Preliminaries

I have referred [1], [7], [10], [12] to state following preliminaries

Definition 1.0.1. *Let $(E, \mathcal{E})$ be an Euclidean measurable space. Let $M_P(E)$ be the set of all integer-valued measures on $(E, \mathcal{E})$. We associate $M_P(E)$ with a σ-algebra $\mathcal{M}_P(E)$, which is the smallest σ-algebra on $M_P(E)$ that makes the maps $A : M_P(E) \to \mathbb{N} \cup \{0\}$, $\nu \mapsto \nu(A)$ measurable for all Borel sets A. Let μ be a Radon measure on E. A Poisson random measure with mean measure μ is a measurable function $\wp : (\Omega, \mathcal{F}, P) \to (M_P(E), \mathcal{M}_P(E))$ satisfying the following properties:*

1. *For $A \in \mathcal{E}$ and $k \in \mathbb{N}$,*

$$P[\omega : \wp(\omega)(A) = k] = \begin{cases} e^{-\mu(A)} \frac{(\mu(A))^k}{k!}, & \mu(A) < \infty \\ 0, & \mu(A) = \infty. \end{cases} \tag{1.1}$$

2. *For any $m \in \mathbb{N}$, if $A_1, A_2, \ldots, A_m$ are mutually disjoint sets in $\mathcal{E}$, then $\wp(A_1), \wp(A_2), \ldots, \wp(A_m)$ are independent random variables.*

Theorem 1.0.1. *Let x be of quadratic variation along (τ_n) and F a twice continuously-*

differentiable function on $\mathbb{R}$. Then the Itô formula

$$F(x_t) = F(x_0) + \int_0^t F'(x_{u-})dx_u + \frac{1}{2}\int_{(0,t]} F''(x_{u-})d[x,x]_u + \sum_{u\leq t}[F(x_u) - F(x_{u-}) - F'(x_{u-})\Delta x_u - \frac{1}{2}F''(x_{u-})\Delta x_u^2],$$

holds with

$$\int_0^t F'(x_{s-})dx_s := \lim_{n\longrightarrow\infty} \sum_{t_i(\leq t)\in\tau_n} F'(x_{t_i})(x_{t_{i+1}} - x_{t_i}), \tag{1.2}$$

and the series in (1.2) *is absolutely convergent.*

Definition 1.0.2. *Let $X = (X^0, X^1, \ldots, X^d)$ be a $(d+1)-$dimensional nonnegative semimartingale describing the prices of $d+1$ kinds of assets. We say a strategy $\pi = (\pi^0, \pi^1, \ldots, \pi^d)$ is admissible if $\pi \in L^2(X)$.*

Definition 1.0.3. *An admissible strategy $\pi \in L^2(X)$ is said to be self-financing if the value of the portfolio π, $V^\pi = (V_t^\pi)_{t\geq 0}$ defined by $V_t^\pi = \sum_{i=0}^d \pi_t^i X_t^i$ has a representation $X_t^\pi = X_0^\pi + \int\limits_0^t (\pi_s, dX_s)$.*

Definition 1.0.4. *For each $a \geq 0$ we set*

$$\Pi_a(X) = \{\pi \in SF(X) : X_t^\pi \geq -a, t \in [0,T]\}.$$

where $SF(X)$ is a class of self-financing strategies.

Definition 1.0.5. $\Psi_+ = \Big\{\psi \in L_\infty(\Omega, \mathcal{F}_T, P) : \psi \geq \int\limits_0^T (\pi_s, dX_s)$ *for some strategy* $\pi \in \Pi_+(X)\Big\}$,

where $\Psi_+(X) = \cup_{a\geq 0}\Pi_a(X)$.

Definition 1.0.6. *We say the propery $\overline{NA}_+$(NFLVR) holds if*

$$\overline{\Psi}_+ \cap L_\infty^+(\Omega, \mathcal{F}_T, P) = \{0\}.$$

Definition 1.0.7. *An option is a security(contract) issued by a firm, another financial company giving its buyer the right to buy or sell something of value on specified terms at a fixed instant or during a certain period of time in the future.*

We distinguish options of two kinds: A buyer's option(call option) which gives one the right to buy and seller's option(put option)) which gives one the right to sell.

Definition 1.0.8. *A C_0-semigroup of operators $\mathcal{T}_{t\geq 0}$ on a Banach space V is a map $\mathcal{T} : \mathbb{R}_+ \to BL(V)$, such that*

1. $\mathcal{T}_0 f = f \quad \forall f \in V$,
2. $\mathcal{T}_{t+s} = \mathcal{T}_t \circ \mathcal{T}_s \quad \forall t, s \geq 0$, *and*
3. $\|\mathcal{T}_t f - f\| \to 0$ *as* $t \downarrow 0$, *for all* $f \in V$.

Definition 1.0.9. *Let $\mathcal{T}_{t\geq 0}$ be a C_0-semigroup of operators. The domain of the infinitesimal generator of the semigroup is defined as*

$$D(A) := \left\{ x \in V \mid \lim_{t\to 0} \frac{\mathcal{T}_t x - x}{t} \text{ exists} \right\}$$

and the infinitesimal generator of x is the operator A, defined such that

$$Ax := \lim_{t\to 0} \frac{\mathcal{T}_t x - x}{t}$$

for all $x \in D(A)$.

Theorem 1.0.2. *Let $\mathcal{T}_t$ be a C_0 semigroup and let A be its infintesimal generator. Then*

1. *For $x \in X$,*
$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \mathcal{T}_s x ds = \mathcal{T}_t x. \tag{1.3}$$
2. *For $x \in X$, $\int_0^t \mathcal{T}_s x ds \in D(A)$ and*
$$A\Big(\int_0^t \mathcal{T}_s x ds \Big) = \mathcal{T}_t x - x. \tag{1.4}$$

3. For $x \in D(A)$, $\mathcal{T}_t x \in D(A)$ and

$$\frac{d}{dt}\mathcal{T}_t x = A\mathcal{T}_t x = \mathcal{T}_t A x. \tag{1.5}$$

Theorem 1.0.3. *Consider*

$$\frac{d}{dt}\varphi(t) = A\varphi(t), \quad \text{with} \;\; \varphi(0) = x. \tag{1.6}$$

If A is infinitesimal generator of a C_0 semigroup, $\mathcal{T}_t$, and $x \in D(A)$, then the Cauchy problem has a solution

$$\varphi(t) = \mathcal{T}_t x. \tag{1.7}$$

Definition 1.0.10. *Let A be the infinitesimal generator of a C_0 semigroup $\mathcal{T}_t$. Let $x \in X$ and $f \in L^1\big((0,T);X\big)$. The function $\varphi \in C([0,T];X)$ given by*

$$\varphi(t) = \mathcal{T}_t x + \int_0^t \mathcal{T}_{t-s} f(s)ds, \quad 0 \le t \le T, \tag{1.8}$$

is called the mild solution of the initial value problem

$$\frac{d\varphi}{dt} = A\varphi(t) + f(t), \quad \varphi(0) = x \tag{1.9}$$

on $[0,T]$.

Definition 1.0.11. *A two parameter family of bounded linear operators $U(t,s)$, $0 \le s \le t \le T$, on X is called an evolution system if the following two conditions are satisfied:*

1.

$$U(s,s) = I, U(t,r)U(r,s) = U(t,s) \;\; \text{for} \;\; 0 \le s \le t \le T. \tag{1.10}$$

2.

$$(t,s) \longrightarrow U(t,s) \;\; \text{is strongly continuous for} \;\; 0 \le s \le t \le T. \tag{1.11}$$

Remark. If $U(t,s)$ is evolution system associated with $A(t)$, then

$$\frac{\partial}{\partial t}U(t,s) = A(t)U(t,s),$$
$$\frac{\partial}{\partial s}U(t,s) = -U(t,s)A(s).$$

Definition 1.0.12. *Consider*

$$\frac{d\varphi(t)}{dt} = A(t)\varphi(t) + f(t), \quad \varphi(s) = x \tag{1.12}$$

where $f \in L^1\big((0,T);X\big)$ *and there exist an evolution system associated with* $\{A(t)\}_{t\in[0,T]}$. *The function* $\varphi \in C([0,T];X)$ *given by*

$$\varphi(t) = U(t,s)x + \int_s^t U(t,r)f(r)dr$$

is called the mild solution of (1.12).

Remark. Given a C_0 semigroup $\mathcal{T}_t$, defie $U(t,s) = \mathcal{T}_{t-s}$. Let A be the infinitesimal generator of $\mathcal{T}_t$, then $\frac{\partial}{\partial t}\mathcal{T}_{T-t} = -\mathcal{T}_{T-t}A$. To see this, consider for $x \in D(A)$,

$$\lim_{h\longrightarrow 0} \frac{\mathcal{T}(h) - I}{h}\mathcal{T}_{T-t}x = \mathcal{T}_{T-t} \lim_{h\longrightarrow 0} \frac{\mathcal{T}(h) - I}{h}x = \mathcal{T}_{T-t}Ax \text{ exists.}$$

Theorem 1.0.4. *Let* $f : [0,T] \times X \to X$ *be continuous in* t *on* $[0,T]$ *and uniformly Lipschitz continuous on* X. *If* A *is the infinitesimal generator of a* C_0 *semigroup* $\mathcal{T}_t$, $t \geq 0$, *on* X *then for every* $x \in X$ *the initial value problem*

$$\frac{d\varphi(t)}{dt} = A\varphi(t) + f(t,\varphi(t)), \quad t \geq 0 \tag{1.13}$$

$$\varphi(0) = x \tag{1.14}$$

has a unique mild solution $\varphi \in C([0,T] : X)$ *which solves another integral equation as given below*

$$\varphi(t) = \mathcal{T}_t x + \int_0^t \mathcal{T}_{t-s} f(s,\varphi(s))ds.$$

Definition 1.0.13. *Let* $\{X_t\}_{t\geq}$ *be a time-homogeneous markov process in a polish spaces, then the family of operators* $\mathcal{T} \geq 0$, *defined by*

$$\mathcal{T}_t f(x) = E\big(f(X_t)|X_0 = x\big) \quad \forall\ f \text{ continuous and bounded and } x \in S,$$

is a semigroup of operators on $C_b(S)$.

Lemma 1.0.5. *For any nonnegative c,*

$$\sup_{s\in(0,\infty)}\Big(\frac{1+cs}{1+s}\Big)\leq 1+c.$$

Proof.

We write $(0,\infty)=(0,1]\cup(1,\infty)$. We check supremum over $(0,1]$ and $(1,\infty)$ separately. First we note that

$$\sup_{s\in(0,1]}\frac{1+cs}{1+s}\leq\frac{1+c}{1}=1+c. \tag{1.15}$$

Again, since, $0<\frac{1}{s}<1$ for $s\in(1,\infty)$, we have

$$\begin{aligned}\sup_{s\in(1,\infty)}\Big(\frac{1+cs}{1+s}\Big)&=\sup_{s\in(1,\infty)}\Big(\frac{\frac{1}{s}+c}{\frac{1}{s}+1}\Big)\\&\leq\frac{1+c}{0+1}=1+c.\end{aligned}$$

Thus

$$\begin{aligned}\sup_{s\in(0,\infty)}\Big(\frac{1+cs}{1+s}\Big)&=\max\Big(\sup_{s\in(0,1)}\big(\frac{1+cs}{1+s}\big),\sup_{s\in(1,\infty)}\big(\frac{1+cs}{1+s}\big)\Big)\\&\leq 1+c.\end{aligned}$$

□

Chapter 2

Introduction

In 1973, Black, Scholes and Merton considered a mathematical model of asset price to determine price of a European option on the underlying asset. This model, B-S-M model, assumes the dynamics of stock price $\{S_t\}_{t\geq 0}$ follows a geometric Brownian motion, that is,

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \quad S_0 > 0,$$

where $\{W_t\}_{t\geq 0}$ is a standard Wiener process and the drift coefficient, μ, and the volatility coefficient, σ, of the price are taken as positive constants. This model assumes risk-free rates and volatility are constants which is not always true in reality. Because of such limitations, various improved modified versions of this model are being studied and few years later, the regime switching model, a modified version of B-S-M was introduced where μ and σ follow either a Markov or a semi-Markov process. In particular the asset price is given by

$$dS_t = \mu(X_t) S_t dt + \sigma(X_t) S_t dW_t, \quad S_0 > 0,$$

where X_t is a Markov or a semi-Markov process. Later, jump diffusion was introduced to deal with the discontinuity in stock price dynamics which is governed by a Markov or a semi-Markov modulated jump diffusion model as given below

$$dS_t = S_{t-}\Big[\mu(X_t)dt + \sigma(X_t)dW_t + \int_{-\infty}^{\infty} \eta(z) N(dz, dt)\Big], \tag{2.1}$$

where $N(dz, dt)$ is a Poisson random measure with intensity measure $\nu(dz, dt)$, where ν is a finite Borel measure.

If an investor is getting profit without investing anything and any possibility of loss, we say that such market allows an arbitrage which is a sign of lack of equilibrium in the market. So it is important to verify whether a model is arbitrage free under a class of admissible strategies. The equivalence between $\overline{NA}_+$ and the existence of an Equivalent Local Martingale Measure. $\overline{NA}_+$, i.e, The NFLVR property is a refinement of No Arbitrage. No Free Lunch with Vanishing Risk (NFLVR) condition is stronger than No Arbitrage (NA) condition. It has transparent criterion of the absence of arbitrage.

If the market is incomplete, then there are infinitely many equivalent local martingales. So the locally risk minimizing price approach by Föllmer and Schweizer is considered. This kind of work is done in [**?**]. In this approach, one expects to obtain price function of a European option as solution to a system of parabolic Integro-PDEs with appropriate conditions.

The rest of this thesis is arranged in the following manner. In chapter 3, we describe the semi-Markov modulated jump diffusion model and prove that this model has a strong solution. In chapter 4, the NFLVR property of this model is established. The pricing approach is described in chapter 5. In chapter 6, the Cauchy problem is solved.

Chapter 3

Model Description

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space where the filtration satisfies the usual hypothesis and $\chi = \{1, 2, 3, \ldots, k\}$ be a finite set. With some fixed partial ordering $\prec$, consider $\chi_2 := \{(i,j) \in \chi^2 | i \neq j\}$. For each $y > 0$, let $\Lambda_{ij}(y)$ be consecutive right-open and left-closed intervals of length $\lambda_{ij}(y)$ starting from origin. Here $\lambda : \chi_2 \times (0, \infty) \to (0, \infty)$ is a continuously differentiable function with

$$\sup_{y \in (0,\infty)} \sum_{j \neq i} \lambda_{ij}(y) < \infty,$$

$$\lim_{y \to \infty} \Lambda_i(y) = \infty, \text{ where } \Lambda_i(y) := \int_0^y \sum_{j \neq i} \lambda_{ij}(v)\, dv$$

and the diagonal elements are defined as $\lambda_{ii}(y) := -\sum_{j \neq i} \lambda_{ij}(y)$.

We define $h : \chi \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ as

$$h(i, y, z) := \sum_{j \neq i} (j - i) 1_{\Lambda_{ij}(y)}(z)$$

and $g : \chi \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ as

$$g(i, y, z) := y \sum_{j \neq i} 1_{\Lambda_{ij}(y)}(z).$$

Existence-uniqueness of a stong solution to (3.1)-(3.2) is proved in Theorem 2.1.3 of [9].

The solution $X = \{X_t\}_{t\geq 0}$ is shown as a semi-Markov process with instantaneous transition rate λ.

Let $\{T_n\}_{n\geq 1}$ be the increasing sequence of transition times of X. We also define the holding times $\tau_n := T_n - T_{n-1}$ for all $n \geq 1$. Also define $n(t) := \max\{n : T_n \leq t\}$. Hence $T_{n(t)} \leq t \leq T_{n(t)+1}$ and $Y_t = t - T_{n(t)}$.

We define a monotonic increasing non-negative function $F : [0, \infty) \to [0, 1]$ as $F(y|i) := 1 - e^{-\Lambda_i(y)}$ where $\Lambda_i(y)$ is as in page 11. Since $\Lambda_i(y)$ is twice continuously differentiable function of y, thus, $F(y|i)$ is also twice continuously differentiable a.e.. Let $f(y|i) := \frac{d}{dy}F(y|i)$ be the derivative. We also define $p_{ij}(y)$, such that

$$p_{ij}(y) := \Big(\frac{\lambda_{ij}(y)}{-\lambda_{ii}(y)}\Big), \quad j \neq i \tag{3.1}$$

This makes sure that $[p_{ij}(y)]$ is a probability matrix for all y. It can be shown that $F(y|i)$ is the conditional c.d.f of the holding time of X and $p_{ij}(y)$ is the conditional probability that X transits to j given that it is at i for a duration of y.

We consider a market having two types of securities, one is riskless asset called money market account and other one is risky asset called stock. Let $r_t = r(X_t)$ be the spot interest rate and $\{B_t\}_{t\geq 0}$ be the price of one unit of a riskless asset at time t, $B_0 = 1$. Then we have $B_t = e^{\int_0^t r_u du}$. Now let $S = \{S_t\}_{t\geq 0}$ be the price of risky asset which is governed by a semi-Markov modulated jump diffusion model as given below

$$dS_t = S_{t-}\Big[\mu_t dt + \sigma_t dW_t + \int_{-\infty}^{\infty} \eta(z) N(dz, dt)\Big], \tag{3.2}$$

where $S_0 > 0$ is positive, μ_t the drift term, σ_t the volatility coefficient, $\eta : \mathbb{R} \to \mathbb{R}$ is bounded above, continuous and $\eta(z) \geq -1$, W_t is a standard Wiener process and $N(dz, dt)$ is a Poisson random measure with intensity measure $\nu(dz)dt$, where ν is a finite Borel measure. We assume $\mu_t := \mu(X_t)$ and $\sigma := \sigma(X_t) \neq 0$, where X_t is as (3.1)-(3.2). We also assume that W, X and $N(dz, dt)$ are independent and adapted to the filtration $\mathcal{F}_t$.

Theorem 3.0.1. *The SDE* (3.4) *has a strong solution which is given by*

$$S_t = S_0 \exp\Big(\int_0^t (\mu_{u-} - \frac{1}{2}\sigma_{u-}^2) du + \int_0^t \sigma_{u-} dW_u + \int_0^t \int_{\mathbb{R}} \ln(1 + \eta(z_1)) N(dz_1, dt)\Big). \tag{3.3}$$

Proof. We assume that the SDE has a solution, $\{S_t\}_{t\geq 0}$, with the stopping time $\tau = \min\{t \in [0,\infty) \mid S_t \leq 0\}$. By assuming this we first show the uniqueness. Applying Itô Lemma on $\ln S_t$ for $0 \leq t \leq \tau$ we get,

$$
\begin{aligned}
d\ln S_t &= \frac{dS_t^c}{S_{t-}} - \frac{1}{2}\frac{d[S^c]_t}{S_{t-}^2} + \ln S_t - \ln S_{t-} \\
&= \mu_{t-}dt + \sigma_{t-}dW_t - \frac{1}{2}\sigma_{t-}^2 dt + \ln S_t - \ln S_{t-} \\
&= (\mu_{t-} - \frac{1}{2}\sigma_{t-}^2)dt + \sigma_{t-}dW_t + \ln S_t - \ln S_{t-} \\
&= (\mu_{t-} - \frac{1}{2}\sigma_{t-}^2)dt + \sigma_{t-}dW_t + \ln(1 + \int_{\mathbb{R}} \eta(z_1)N(dz_1, dt)) \\
&= (\mu_{t-} - \frac{1}{2}\sigma_{t-}^2)dt + \sigma_{t-}dW_t + \int_{\mathbb{R}} \ln(1 + \eta(z_1))N(dz_1, dt).
\end{aligned}
$$

We integrate both sides from 0 to $t \wedge \tau$ to get

$$
\ln S_{t\wedge\tau} - \ln S_0 = \int_0^{t\wedge\tau} (\mu_{u-} - \frac{1}{2}\sigma_{u-}^2)du + \int_0^{t\wedge\tau} \sigma_{u-}dW_u + \int_0^{t\wedge\tau}\int_{\mathbb{R}} \ln(1 + f(z_1))N(dz_1, dt).
$$

Under the assumptions, $\int_0^t \int_{\mathbb{R}} \ln(1 + \eta(z_1))N(dz_1, dt)$ has finite expectation for any finite stopping time t.

Let $\Omega_1 := \{\omega \in \Omega : \tau(\omega) < \infty\}$, $P(\Omega_1) > 0$. By letting $t \to \infty$ we obtain that $S_{\tau(\omega)-} > 0$. But for $\omega \in \Omega_1$ $S_{\tau(\omega)} \leq 0$. So jump is the only reason for getting $\eta(z) \leq -1$ for some z which is contradiction to the assumption to the assumption on η. Hence $\tau = \infty$ $\mathbb{P}$ $a.s.$ Thus, $S_t > 0$ $\mathbb{P}$ for all $t \in (0,\infty)$.

From above expression, $S = \{S_t\}_{t\geq 0}$ is an adapted and r.c.l.l. process and it is determined uniquely. Hence the uniqueness. □

It is evident that $(S, X, Y) := \{(S_t, X_t, Y_t)\}_{t\geq 0}$ is a strong Markov process. Dynkin's formula states that if A is the infinitesimal generator of (S, X, Y), then $\varphi(S_t, X_t, Y_t) - \varphi(S_0, X_0, Y_0) - \int_0^t A\varphi(S_{u-}, X_{u-}, Y_{u-})du$ is martingale with respect to $\mathcal{F}_t$ for any $\phi \in C_c^\infty$. By denoting the above martingale by $\{M_t\}_{t\geq 0}$, we get

$$\varphi(S_t, X_t, Y_t) = \varphi(S_0, X_0, Y_0) + \int_0^t A\varphi(S_{u-}, X_{u-}, Y_{u-})du + M_t.$$

It is derived in [8] that

$$\begin{aligned} A\varphi(s,i,y) =& \Big[\mu(i)s\frac{\partial}{\partial s} + \frac{1}{2}\sigma^2(i)s^2\frac{\partial^2}{\partial s^2} + \frac{\partial}{\partial y}\varphi(s,i,y)\Big] \\ &+ \Big[\sum_{j\neq i}\lambda_{ij}(y)[\varphi(s,j,0) - \varphi(s,i,y)]\Big] \\ &+ \Big[\int_{\mathbb{R}}(\varphi(s(1+\eta(z)),i,y) - \varphi(s,i,y))\nu(dz)\Big]. \end{aligned} \tag{3.4}$$

In Section 4, we would consider the following PDE with an appropriate terminal condition

$$\begin{aligned} &\frac{\partial\varphi}{\partial t} + A\varphi + (r(i) - \mu(i) + \beta_1(i))s\frac{\partial\varphi}{\partial s} \\ &+ \int_{\mathbb{R}}\big[\varphi(t,s(1+\eta(z)),i,y) - \varphi(t,s,i,y)\big](\beta_2(i) - 1)d\nu = r(i)\varphi(t,s,i,y). \end{aligned} \tag{3.5}$$

where $\beta_1(i)$ and $\beta_2(i)$ do not depend on φ and are yet to be chosen.

Chapter 4

Arbitrage-free model

We say that a market allows an arbitrage, if it enables an investor to get profit without investing anything and any possibility of loss. It is a sign of lack of equilibrium in the market. So we need to check whether this model is arbitrage free (NA) under a reasonably large class of admissible strategies. Sec VII. 2c. Theorem 2 of [12] asserts that the existance of an equivalent local martingale measure (ELMM) implies no free lunch with vanishing risk (NFLVR). In the NFLVR sense arbitrage free market scenario, the nonnegative limit of contingent claims, those can be super replicated by zero capital self financing strategies which ensures portfolio remain bounded below uniformly in t and sample point ω, is essentially zero. So this model to be arbitrage free, it is sufficient to show that there exist an ELMM for this model.

We would need the following lemma for proving Theorem 4.0.3.

Lemma 4.0.1. *Let $Z = \{Z_t; t \in [0,T]\}$ be an adapted process which is defined as follows*

$$Z_t = \exp\{\int_0^t \phi_u dW_u - \frac{1}{2}\int_0^t \phi_u^2 du + \int_0^t \int_{\mathbb{R}} \ln \Gamma(z,u) N(dz,du) - \int_0^t \int_{\mathbb{R}} [\Gamma(z,u) - 1]\nu(dz)du\}, \tag{4.1}$$

$\phi = \{\phi_t; t \in [0,T]\}$ and $\Gamma = \{\Gamma(\cdot,t); t \in [0,T]\}$ are previsible process and Borel previsible process such that $E[\int_0^t \phi_u^2 du] < \infty$ and $\Gamma > 0$, respectively. Then Z is a positive local martingale under P with $Z_0 = 1$.

Proof. From (4.1), it is obvious that $Z > 0$ with $Z_0 = 1$. We derive the following

$$\begin{aligned}\Delta Z_t =& Z_t - Z_{t-}\\ =& \exp\{\int_0^t \phi_u dW_u - \frac{1}{2}\int_0^t \phi_u^2 du + \int_0^t \int_{\mathbb{R}} [\Gamma(z,u)-1]\nu(dz)du\} \exp\{\int_{[0,t]}\int_{\mathbb{R}} \ln\Gamma(z,u)N(dz,du)\}\\ &- \exp\{\int_0^t \phi_u dW_u - \frac{1}{2}\int_0^t \phi_u^2 du + \int_0^t \int_{\mathbb{R}} [\Gamma(z,u)-1]\nu(dz)du\} \exp\{\int_{[0,t)}\int_{\mathbb{R}} \ln\Gamma(z,u)N(dz,du)\}\\ =& Z_{t-}[\int_{\mathbb{R}} \big(\Gamma(z,t)-1\big)N(dz,\{t\})].\end{aligned}$$

We define $y_t := \int_0^t \phi_u dW_u - \frac{1}{2}\int_0^t \phi_u^2 du + \int_0^t \int_{\mathbb{R}} \ln\Gamma(z,u)N(dz,du) - \int_0^t \int_{\mathbb{R}} \big(\Gamma(z,u)-1\big)\nu(dz)du$ and apply Itô formula on $Z_t = \exp\{y_t\}$ to get,

$$\begin{aligned}Z_t - Z_0 =& \int_0^t Z_{u-}dy_u + \frac{1}{2}\int_0^t Z_{u-}d[y]_u^c + \sum_{0<u\le t}\Big[Z_u - Z_{u-} - Z_{u-}\int_{\mathbb{R}} \ln\Gamma(z,u)N(dz,\{u\})\Big]\\ Z_t - 1 =& \int_0^t Z_{u-}\phi_u dW_u - \int_0^t Z_{u-}\frac{1}{2}\phi_u^2 du + \int_0^t Z_{u-}\int_{\mathbb{R}} \ln\Gamma(z,u)N(dz,du)\\ &- \int_0^t Z_{u-}\int_{\mathbb{R}} \big(\Gamma(z,u)-1\big)\nu(dz)du] + \frac{1}{2}\int_0^t Z_{u-}\phi_u^2 du\\ &+ \sum_{0<u\le t}\int_{\mathbb{R}} Z_{u-}\Big(\Gamma(z,u) - 1 - \ln\Gamma(z,u)\Big)N(dz,\{u\})\\ Z_t =& 1 + \int_0^t Z_{u-}\phi_u dW_u + \int_0^t Z_{u-}\int_{\mathbb{R}} [\Gamma(z,u)-1]\tilde{N}(dz,du) \qquad (4.2)\end{aligned}$$

where $\tilde{N}(dz,du) := N(dz,du) - \nu(dz)du$.

From the last equation we can see that Z is a $P-$local martingale. □

Now we'll state Lemma 3.2 of [6] which will also be useful for proving next theorem.

Lemma 4.0.2. *Let Q be defined on $\mathcal{F}_T$ by $\frac{dQ}{dP} = Z_T$. Then the process $\tilde{W}_t := W_t - \int_0^t \phi_u du$ is a Weiner process under Q and*

$$\int_0^t \int_{\mathbb{R}} \Big[\Gamma(z,u) - 1\Big]\Big(N(dz,du) - \Gamma(z,u)\nu(dz)du\Big)$$

is a $Q-$martingale with respect to its natural filtration which implies that the compensator *measure of $N(dz,dt)$ under Q is given by $\tilde{\nu}(dz,dt) := \Gamma(z,t)\nu(dz)dt$.*

Lemma 4.0.2 implies that $\tilde{M}(dz,du) := N(dz,du) - \tilde{\nu}(dz,du)$ is the compensated Poisson random measure with respect to the measure Q.

We apply Itô formula on $S_t^* = \frac{S_t}{B_t} = \exp\{-\int_0^t r_u du\}S_t$, where S_t^* is the discounted stock price to obtain

$$\begin{aligned}
dS_t^* &= \exp\Big\{-\int_0^t r_u du\Big\}dS_t - S_{t-}\exp\Big\{-\int_0^t r_u du\Big\}r_{t-}dt \\
&= \exp\Big\{-\int_0^t r_u du\Big\}\big[dS_t - S_{t-}r_{t-}dt\big] \\
&= \exp\Big\{-\int_0^t r_u du\Big\}\Big[S_{t-}\Big(\mu_{t-}dt + \sigma_{t-}dW_t + \int_{\mathbb{R}} \eta(z)N(dz,dt)\Big) - S_{t-}r_{t-}dt\Big] \\
&= \big[\mu_{t-} - r_{t-}\big]S_{t-}^*dt + \sigma_{t-}S_{t-}^*dW_t + S_{t-}^*\int_{\mathbb{R}} \eta(z)N(dz,dt).
\end{aligned}$$

Using Lemma 4.0.2 we can rewrite the above SDE as below

$$\begin{aligned}
dS_t^* &= \big[\mu_{t-} - r_{t-}\big]S_{t-}^*dt + \sigma_{t-}S_{t-}^*\big[d\tilde{W}_t + \phi_{t-}dt\big] + \int_{\mathbb{R}} S_{t-}^*\eta(z)\big[\tilde{M}(dz,dt) + \tilde{\nu}(dz,dt)\big] \\
&= \big[\mu_t - r_t + \sigma_t\phi_t + \int_{\mathbb{R}} \eta(z)\Gamma(z,t)\nu(dz)\big]S_{t-}^*dt + S_{t-}^*\sigma_t d\tilde{W}_t \\
&\quad + \int_{\mathbb{R}} S_{t-}^*\eta(z)\tilde{M}(dz,dt). \qquad (4.3)
\end{aligned}$$

where $\tilde{M}(dz,dt) = N(dz,dt) - \tilde{\nu}(dz,dt) = N(dz,dt) - \Gamma(z,t)\nu(dz,dt)$.

We wish to choose Γ and ϕ such that the discounted price is a martingale under Q. It is possible only when the drift term in (4.3) is zero. Thus we have

$$\mu_t - r_t + \sigma_t\phi_t + \int_{\mathbb{R}} \eta(z)\Gamma(z,t)\nu(dz) = 0. \qquad (4.4)$$

Hence we get one equation with two unknowns, i.e. ϕ_t and $\Gamma(z,t)$. Hence (4.4) leads to many different possibilities corresponding to the pair (ϕ,Γ). We would like to select one which satisfies an additional relation such that (4.2) can be represented as

$$dZ_t = \Psi_{t-} Z_{t-}(\sigma_{t-} dW_t + \int_{\mathbb{R}} \eta(z)\tilde{N}(dz, dt)). \tag{4.5}$$

Now comparing (4.2) and (4.5), we get

$$\phi_t = \Psi_t \sigma(X_t)$$

and

$$\Psi_t \eta(z) = \Gamma(z, t) - 1.$$

Now by substituting above in (4.4), we get

$$\begin{aligned} \Psi_t \sigma_t^2 &= r_t - \mu_t - \int_{\mathbb{R}} \eta(z)(1 + \Psi_t \eta(z))\nu(dz) \\ &= r_t - \mu_t - \int_{\mathbb{R}} \eta(z)\nu(dz) - \int_{\mathbb{R}} \eta^2(z)\Psi_t \nu(dz). \end{aligned}$$

Taking Ψ_t terms together,

$$\Psi_t[\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)] = r_t - \mu_t - \int_{\mathbb{R}} \eta(z)\nu(dz).$$

Therefore Ψ_t can be written as

$$\Psi_t = \frac{r_t - \mu_t - \int_{\mathbb{R}} \eta(z)\nu(dz)}{\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)},$$

which suggests

$$\left.\begin{aligned} \Gamma(z, t) &= \frac{r_t - \mu_t - \int_{\mathbb{R}} \eta(z)\nu(dz)}{\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)}\eta(z) + 1 \\ \phi_t &= \frac{r_t - \mu_t - \int_{\mathbb{R}} \eta(z)\nu(dz)}{\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)}\sigma_t. \end{aligned}\right\} \tag{4.6}$$

In view of conditions in Lemma 4.0.1 also need $\Gamma(z, t) > 0$. Thus, in view of (4.6), we require following condition which we assume to be satisfied by the model parameters.

Assumption A1:

$$\frac{r(i) - \mu(i) - \int_{\mathbb{R}} \eta(z)\nu(dz)}{\sigma^2(i) + \int_{\mathbb{R}} \eta^2(z)\nu(dz)}\eta(z) > -1.$$

Now we substitute (4.6) in (4.3), drift term becomes zero and we get

$$S_t^* - S_0^* = \int_0^t S_{u-}^* \sigma(X_u) d\tilde{W}_u + \int_0^t \int_{\mathbb{R}} S_{u-}^* \eta(z) \tilde{M}(dz, du).$$

By Lemma 4.0.2, this becomes local martingale under Q. Thus we have proved the following theorem.

Theorem 4.0.3. *Let ϕ_t and $\Gamma(z,t)$ be as in (4.6). Under (A1), the Probability measure Q as defined in Lemma 4.0.2 is an equivalent local martingale measure.*

We reemphasise that the existence of an equivalent local martingale measure implies that this market model has NFLVR property.

Chapter 5

Option pricing

5.1 The Föllmer-Schweizer decomposition

Let ξ_t and ε_t be the number of units invested in assets with prices S_t and B_t respectively at time t. The value of the resulting portfolio at time t is given by

$$V_t = \xi_t S_t + \varepsilon_t B_t.$$

An admissible strategy is defined as a predictable process $\pi = \{\pi_t = (\xi_t, \varepsilon_t), 0 \leq t \leq T\}$ which satisfies the following conditions

(i) $\int_0^T \xi_t^2 d\langle S \rangle_t < \infty$,

(ii) $E(\varepsilon_t^2) < \infty$,

In [3], it is shown that if market is arbitrage free, then existence of an optimal strategy to hedge a contingent claim H with finite variance is equivalent to the existence of Föllmer Schweizer decomposition of $H^* := B_T^{-1} H$, the discounted claim. The F-S decomposition of H^* is given by

$$H^* = H_0 + \int_0^T \xi_t^{H^*} dS_t^* + L_T^{H^*}$$

where $H_0 \in L^2(\Omega, \mathcal{F}_\prime, P)$, $L^{H^*} = \{L_t^{H^*}\}_{0 \leq t \leq T}$ is a square integrable martingale starting with zero and orthogonal to the martingale part of S_t, and $\xi^{H^*} = \{\xi_t^{H^*}\}$ satisfies (i). In [3], it is further asserted that the optimal strategy $\pi = (\xi_t, \varepsilon_t)$ is given by

$$\begin{aligned}\xi_t &:= \xi^{H^*},\\ V_t^* &:= H_0 + \int_0^t \xi_u dS_u^* + L_t^{H^*},\\ \varepsilon_t &:= V_t^* - \xi_t S_t^*,\end{aligned}$$

and $B_t V_t^*$ amounts to the locally risk minimizing price of the claim H at time t. Hence Föllmer Schweizer decomposition is important for the pricing and hedging problems in an incomplete market.

5.2 Derivation of F-S decomposition

In order to price a call option, we take $H = (S_T - K)^+$. We now consider the Cauchy problem given by (3.8) with a terminal condition $\varphi(T, s, i, y) = (s - K)^+$.

For the time being, we assume that the Cauchy problem has a unique classical solution and we denote that by φ. Define $\varphi_t := \varphi(t, S_t, X_t, Y_t)$ and we are looking for ξ_t which is predictable, such that $L = \{L_t\}_{t\geq 0}$ with $L_t := \int_0^t \left[d(\frac{\varphi_t}{B_t}) - \xi_t dS_t^*\right]$ becomes square integrable $P-$martingale and orthogonal to $\overline{M} := \left\{ \int_0^t \sigma_u S_u^* dW_u + \int_0^t \int_{\mathbb{R}} S_u^* \eta(z) \tilde{N}(du, dz) \right\}_{t\geq 0}$, the martingale part of S^*.

$$\begin{aligned}dL_t =& d(\frac{\varphi_t}{B_t}) - \xi_t dS_t^*\\ =& (S_t^* \sigma_t \frac{\partial \varphi}{\partial s} - \xi_t \sigma_t S_t^*) dW_t + \Big[(\mu_t - r_t - \beta_1(X_t)) S_t^* \frac{\partial \varphi}{\partial s} - \big(\mu_t - r_t\big) S_t^* \xi_t\\ &- \Big(\frac{\beta_2(X_t) - 1}{B_t}\Big) \int_{\mathbb{R}} \big(\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)\big) \nu(dz)\\ &- \xi_t S_t^* \int_{\mathbb{R}} \eta(z) \nu(dz)\Big] dt + \int_{\mathbb{R}} \Big(\frac{\varphi(t, S_t(1 + \eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)}{B_t}\\ &- \xi_t S_t^* \eta(z)\Big) \tilde{N}(dz, dt) + \frac{1}{B_t} d\widehat{M}_t.\end{aligned} \tag{5.1}$$

where $\widehat{M}_t = \int_0^t \int_{\mathbb{R}} \big(\varphi(t, S_t, X_t + h(X_t, Y_t, z), Y_t - g(X_t, Y_t, z)) - \varphi(t, S_t, X_t, Y_t)\big) \wp(dt, dz)$. The martingale part of L is equal to

$$\int_0^t \Big[(S_t^*\sigma_t\frac{\partial\varphi}{\partial s} - \xi_t\sigma_t S_t^*)dW_t +$$
$$\int_{\mathbb{R}} \Big(\frac{\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)}{B_t} - \xi_t S_t^*\eta(z)\Big)\tilde{N}(dz, dt) + \frac{1}{B_t}d\widehat{M}_t\Big].$$

Now we are looking for a $\xi := \{\xi_t\}_{t\geq 0}$ such that L becomes orthogonal to $\overline{M}$, i.e., $\langle L, \overline{M}\rangle_t = 0\ \forall\ t$, where $\langle , \rangle$ denotes conditional quadratic covariation.

We note that

$$\begin{aligned} d[L, \overline{M}]_t =& S_t^{*2}\sigma_t^2\big(\frac{\partial\varphi}{\partial s} - \xi_t\big)dt \\ &+ S_t^*\int_{\mathbb{R}}\Big(\frac{\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)}{B_t}\eta(z) - \xi_t S_t^*\eta^2(z)\Big)\tilde{N}(dz, dt). \end{aligned}$$

Hence

$$\begin{aligned} d\langle L, \overline{M}\rangle_t =& S_t^{*2}\sigma_t^2\big(\frac{\partial\varphi}{\partial s} - \xi_t\big)dt \\ &+ S_t^*\int_{\mathbb{R}}\Big(\frac{\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)}{B_t}\eta(z) - \xi_t S_t^*\eta^2(z)\Big)\nu(dz)dt. \end{aligned}$$

Thus $\langle L, M\rangle_t = 0$, if

$$\begin{aligned} &S_t^{*2}\sigma_t^2\big(\frac{\partial\varphi}{\partial s}\big) + S_t^*\int_{\mathbb{R}}\Big(\frac{\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)}{B_t}\Big)\eta(z)\nu(dz) \\ &= S_t^{*2}\sigma_t^2\xi_t + S_t^{*2}\xi_t\int_{\mathbb{R}}\eta^2\nu(dz) \quad \text{holds.} \end{aligned}$$

Therefore, $\langle L, M\rangle_t = 0\ \forall\ t$ if ξ is chosen as

$$\xi_t = \frac{S_t^*\sigma_t^2\big(\frac{\partial\varphi}{\partial s}\big) + \int_{\mathbb{R}}\Big(\frac{\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)}{B_t}\Big)\eta(z)\nu(dz)}{S_t^*\big(\sigma_t^2 + \int_{\mathbb{R}}\eta^2\nu(dz)\big)} \qquad \forall t \in [0, T]. \tag{5.2}$$

Thus we have proved that the above choice of ξ makes martingale part of L orthogonal to $\overline{M}$, irrespective of the choice of $\beta_1(\cdot)$ and $\beta_2(\cdot)$. However, we have not yet established existence of a particular pair $(\beta_1(\cdot), \beta_2(\cdot))$ for which L is a square integrable martingale. Since W, $\tilde{N}$

and $\widehat{M}$ are martingales, to ensure that L is a local martingale, dt term in (5.1) should be zero, i.e.,

$$
\begin{aligned}
&(\mu_t - r_t - \beta_1(X_t))S_t^* \frac{\partial \varphi}{\partial s} - (\mu_t - r_t)S_t^* \xi_t \\
&- \int_{\mathbb{R}} (\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)) \frac{\beta_2(X_t) - 1}{B_t} \nu(dz) - \xi_t S_t^* \int_{\mathbb{R}} \eta(z)\nu(dz) = 0.
\end{aligned}
$$

That follows if we have

$$
\begin{aligned}
&\xi_t(\mu_t - r_t)S_t^* + S_t^* \int_{\mathbb{R}} \eta(z)\nu(dz) \\
&= (\mu_t - r_t - \beta_1(X_t))S_t^* \frac{\partial \varphi}{\partial s} - \int_{\mathbb{R}} (\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)) \frac{\beta_2(X_t) - 1}{B_t} \nu(dz).
\end{aligned}
\tag{5.3}
$$

Using the expression of ξ_t as in (5.2), the above can be rewritten as

$$
\begin{aligned}
&S_t^{*2}\sigma_t^2 \Big(\mu_t - r_t + \int_{\mathbb{R}} \eta(z)\nu(dz)\Big) \frac{\partial \varphi}{\partial s} \\
&+ \frac{S_t^*}{B_t} \Big(\mu_t - r_t + \int_{\mathbb{R}} \eta(z)\nu(dz)\Big) \int_{\mathbb{R}} (\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t))\eta(z)\nu(dz) \\
= &\, S_t^{*2}(\mu_t - r_t - \beta_1(X_t)) \Big(\sigma_t^2 + \int_{\mathbb{R}} \eta^2 \nu(dz)\Big) \frac{\partial \varphi}{\partial s} \\
&- \frac{S_t^*}{B_t}(\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)) \int_{\mathbb{R}} (\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t))(\beta_2(X_t) - 1)\nu(dz).
\end{aligned}
$$

We rearrange the terms to get

$$
\begin{aligned}
&S_t^{*2}\Big[\sigma_t^2 \Big(\mu_t - r_t + \int_{\mathbb{R}} \eta(z)\nu(dz)\Big) - (\mu_t - r_t) \int_{\mathbb{R}} \eta^2(z)\nu(dz) \\
&\quad + \beta_1(X_t)(\int_{\mathbb{R}} \eta^2(z)\nu(dz) + \sigma_t^2) - (\mu_t - r_t)\sigma_t^2\Big] \frac{\partial \varphi}{\partial s}(t, S_t, X_t, Y_t) \\
= &\quad -\frac{S_t^*}{B_t} \int_{\mathbb{R}} \Big[\Big(\mu_t - r_t + \int_{\mathbb{R}} \eta(z)\nu(dz)\Big)\eta(z) + \Big(\sigma_t^2 + \int_{\mathbb{R}} \eta^2(z)\Big)(\beta_2(X_t) - 1)\Big] \\
&\quad \times \Big(\varphi(t, S_t(1+\eta(z)), X_t, Y_t) - \varphi(t, S_t, X_t, Y_t)\Big)\nu(dz).
\end{aligned}
$$

The above identity holds true irrespective of the function φ if β_1 and β_2 are such that both sides are zero. A direct calculation shows that such β_1 and β_2 exist and are given by

$$
\begin{aligned}
\beta_1(i) &= \frac{\big(\overline{\mu}(i) - r(i) - \int_{\mathbb{R}} \eta(z)\nu(dz)\big) \int_{\mathbb{R}} \eta^2 \nu(dz) - \sigma(i)^2 \int_{\mathbb{R}} \eta(z)\nu(dz)}{\sigma(i)^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)} \\
&= \frac{\big(\overline{\mu}(i) - r(i)\big) \int_{\mathbb{R}} \eta^2(z)\nu(dz) - \int_{\mathbb{R}} \eta(z)\nu(dz)(\sigma(i)^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz))}{\sigma(i)^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)} \\
&= \frac{\big(\overline{\mu}(i) - r(i)\big) \int_{\mathbb{R}} \eta^2(z)\nu(dz)}{\sigma(i)^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)} - \int_{\mathbb{R}} \eta(z)\nu(dz),
\end{aligned}
$$

and

$$
\beta_2(i) = 1 - \frac{\big(\mu(i) - r(i) + \int_{\mathbb{R}} \eta(z)\nu(dz)\big)\eta(z))}{\sigma(i)^2 + \int_{\mathbb{R}} \eta^2(z)\nu(dz)}.
$$

We put these values of β_1 and β_2 in (5.3) to get ξ which is adapted, such that L becomes $P-$martingale and orthogonal to the martingale part of S^*. □

Chapter 6

Pricing equation

Consider the initial boundary value problem (3.8) with terminal condition $\varphi(T)(s,i,y) = K(s)$ where T is the maturity time, K is Lipschitz continuous in s. In this chapter, we aim to establish the existence and uniqueness of a classical solution of the Cauchy problem. To this end, we would first rewrite the Cauchy problem in a manner, suitable for applying general theory of abstract Cauchy problems. Then we establish the existence and uniqueness of the continuous mild solution in Theorem 6.0.2. For this, we now introduce another SDE

$$d\widehat{S}_t = \widehat{S}_t\Big(r(X_t) + \beta_1(X_t)dt + \sigma(X_t)dW_t\Big),$$

where $\{W_t\}_{t\geq 0}$ is the Brownian motion and $\{X_t\}_{t\geq 0}$ is as in (3.1)-(3.2). In the similar line of Theorem 3.0.1, with considerably less effort one can show that the above SDE has a strong positive continuous solution and the solution $\widehat{S} := \{\widehat{S}_t\}_{t\geq 0}$ along with X and Y jointly is strong markov. We call the generator of $\{(\widehat{S}_t, X_t, Y_t)\}_{t\geq 0}$ by $\widehat{A}$ and it is given by

$$\widehat{A}\varphi(s,i,y) = \frac{\partial\varphi}{\partial y}(s,i,y) + \big(r(i)+\beta_1(i)\big)\frac{\partial\varphi}{\partial s} + \frac{1}{2}s^2\sigma^2(i)\frac{\partial^2\varphi}{\partial s^2} + \sum_{j\neq i}\lambda_{ij}(y)\big(\varphi(s,j,0) - \varphi(s,i,y)\big).$$

We can rewrite (3.8) by substituting the expression of A as

$$\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial y} + \big(r(i) + \beta_1(i)\big) s \frac{\partial \varphi}{\partial s} + \frac{1}{2} s^2 \sigma^2(i) \frac{\partial^2 \varphi}{\partial s^2} + \sum_{j \neq i} \lambda_{ij}(y) \big(\varphi(t,s,j,0) - \varphi(t,s,i,y)\big)$$
$$+ \int_{\mathbb{R}} \big[\varphi(t, s(1+\eta(z)), i, y) - \varphi(t,s,i,y)\big] (\beta_2(i)) d\nu = r(i)\varphi.$$

Hence, using the expression of $\widehat{A}$ in the above equation, we have the following Cauchy problem

$$\left.\begin{aligned} \frac{\partial \varphi}{\partial t} + \widehat{A}\varphi + B\varphi = R\varphi \\ \varphi(T) = K \end{aligned}\right\}. \tag{6.1}$$

where $B\varphi(t,s,i,y) := \int_{\mathbb{R}} \big[\varphi(t, s(1+\eta(z)), i, y) - \varphi(t,s,i,y)\big] (\beta_2(i))(dz)$ and $R\varphi(t,s,i,y) = r(i)\varphi(t,s,i,y)$.

A typical expression of K , as in the case of call option takes the form $K(s,i,y) = (s - K_1)^+$, where K_1 is the strike price. Here K need not be in $D(\widehat{A})$. So the classical solution is not assured. We define

$$V := \{\varphi : (0,\infty) \times \chi \times (0,T) \to \mathbb{R} | \sup_{s,i,y} \frac{|\varphi(s,i,y)|}{1+s} < \infty\},$$

clearly $(V, \|\cdot\|_V)$ is a normed linear space where $\|\varphi\|_V := \sup_{s,i,y} |\frac{\varphi(s,i,y)}{1+s}|$. Furthermore, $(V, \|\cdot\|_V)$ is a Banach space consisting continuous functions with at most linear growth.

We define $\overline{\beta_2} := \max_i |\beta_2(i)|$ and using Lemma 1.0.5 with $c = 1 + \eta(z) > 0$ for each z,

$$
\begin{aligned}
\|B\varphi\|_V &= \sup_{s,i,y \in (0,\infty)\times\chi\times(0,T)} \Big|\beta_2(i) \int\limits_{\mathbb{R}} \frac{\varphi(t, s(1+\eta(z)), i, y) - \varphi(t, s, i, y)}{1+s} \nu(dz)\Big| \\
&\le \sup_{s,i,y} \Big[\Big|\beta_2(i)\Big| \int\limits_{\mathbb{R}} \frac{\Big(1 + s\big(1+\eta(z)\big)\Big)}{1+s} \Big|\frac{\varphi\big(t, s(1+\eta(z)), i, y\big)}{1+s\big(1+\eta(z)\big)}\Big| + \Big|\frac{\varphi(t, s, i, y)}{1+s}\Big| \nu(dz)\Big] \\
&\le \overline{\beta_2} \sup_{y,s\in[(0,\infty)]} \Big[\int\limits_{\mathbb{R}} \Big[\big(2+\eta(z)\big)\|\varphi\|_V + \|\varphi\|_V\Big]\nu(dz)\Big] \\
&= \overline{\beta_2} \sup_y \|\varphi\|_V \Big[\int\limits_{\mathbb{R}} \big(3+\eta(z)\big)\nu(dz)\Big] \\
&= \overline{\beta_2}\|\varphi\|_V \Big(3\nu(\mathbb{R}) + \int \eta d\nu\Big) < \infty,
\end{aligned} \tag{6.2}
$$

since ν is a finite measure and η is a bounded function. Hence $\|B\|_V \le \overline{\beta_2}\Big(3\nu(\mathbb{R}) + \int \eta d\nu\Big)$. Thus B is a bounded linear map.

Let $f : [0,T]\times V \to V$ be continuous in t and uniformly Lipschitz continuous on V. Theorem 1.0.4 (Theorem 1.2, Chapter 6 of [10]), states that the initial value problem

$$
\left.\begin{aligned}
\frac{\partial\varphi(t)}{\partial t} &= \widehat{A}\varphi + f(t, \varphi(t)), \\
\varphi(0) &= K,
\end{aligned}\right\} \tag{6.3}
$$

has a unique continuous mild solution which solves another integral equation as given below

$$
\varphi(t) = \mathcal{T}_t K + \int\limits_0^t \mathcal{T}_{t-u} f(u, \varphi(u)) du, \tag{6.4}
$$

where $\{\mathcal{T}_t\}_{t\ge 0}$ is the C_0 semigroup generated by $\widehat{A}$.

First we aim to find out expression of mild solution like above for a terminal value problem. To this end we change the direction of time variable. Let $v = T - t$ and define

$$\tilde{\varphi}(v) := \varphi(T-v). \tag{6.5}$$

Thus $\tilde{\varphi}(T) = K$. Then from (6.5) and (6.3)

$$\begin{aligned}\frac{\partial \tilde{\varphi}}{\partial v}(v) = -\frac{\partial \varphi}{\partial t}(T-v) &= -\widehat{A}\varphi(T-v) - f(T-v, \varphi(T-v)) \\ &= -\widehat{A}\tilde{\varphi}(v) - f(T-v, \tilde{\varphi}(v)),\end{aligned}$$

or,

$$\left.\begin{aligned}\frac{\partial \tilde{\varphi}}{\partial v} + \widehat{A}\tilde{\varphi}(v) + f(T-v, \tilde{\varphi}(v)) = 0 \\ \tilde{\varphi}(T) = K.\end{aligned}\right\} \tag{6.6}$$

From (6.5) and (6.4), we know that $\tilde{\varphi}$, the mild solution to (6.6) satisfies,

$$\tilde{\varphi}(v) = \varphi(T-v) = \mathcal{T}_{T-v}K + \int_0^{T-v} \mathcal{T}_{T-v-u} f(u, \tilde{\varphi}(T-u))du.$$

We change variable $\tilde{u} = T-u$, inside the integral to obtain

$$\begin{aligned}\tilde{\varphi}(v) = \varphi(T-v) &= \mathcal{T}_{T-v}K + \int_T^{v} \mathcal{T}_{\tilde{u}-v} f(T-\tilde{u}, \tilde{\varphi}(\tilde{u}))(-1)d\tilde{u} \\ &= \mathcal{T}_{T-v}K + \int_v^{T} \mathcal{T}_{\tilde{u}-v} f(T-\tilde{u}, \tilde{\varphi}(\tilde{u}))d\tilde{u}.\end{aligned}$$

Thus we have proved the following theorem.

Theorem 6.0.1. *The Cauchy problem* $\frac{\partial \varphi}{\partial t} + \widehat{A}\varphi + f(t, \varphi(t)) = 0, \quad \varphi(T) = K$ *has a unique continuous mild solution which solves*

$$\varphi(t) = \mathcal{T}_{T-t}K + \int_t^{T} \mathcal{T}_{u-t} f(u, \varphi(u))du, \tag{6.7}$$

provided f is continuous in t, on $[0,T]$ and uniformly Lipschitz continuous on V.

It is easy to see that the Cauchy problem (6.1) is a special case of the above problem where

$$f(t,\varphi(t)) = (B-R)\varphi(t),$$

where B and R are as in (6.1). Here we note that, using (6.2) since B and R are bounded linear operators, f satisfies the conditions of Theorem 6.0.1.

Thus using (6.7), the mild solution to (6.1) can be written as the solution of the following integral equation

$$\varphi(t) = \mathcal{T}_{T-t}K + \int_t^T \mathcal{T}_{u-t}\big(B-R\big)\varphi(u)du. \tag{6.8}$$

Hence we have proved the following theorem.

Theorem 6.0.2. *The Cauchy problem* (6.1) *has a unique continuous mild solution and that solves* (6.8)*.*

Having proved this, it remains to prove regularity of the mild solution to establish well-posedness. Or in other words the Cauchy problem (6.1) has a classical solution if the following holds.

Theorem 6.0.3. *If $\varphi(t)$ is continuous solution to* (6.8)*, then, $\varphi(t)(s,i,y)$ is C^2 in s for every t,i,y and (for every s,i) in the domain of $D_{t,y}$, where $D_{t,y}\theta$ is defined as*

$$\lim_{h\to 0}\frac{1}{h}\Big(\theta(t+h,s,i,y+h) - \theta(t,s,i,y)\Big)$$

provided the limit exists.

We need the following lemma to prove the above theorem.

Lemma 6.0.4. *For $K:[0,\infty)\to\mathbb{R}$ Lipschitz, $\mathcal{T}_{T-t}K$ is continuously differntiable in t and in $D(\widehat{A})$ for each $t\in[0,T]$.*

Proof. Consider the following Cauchy problem

$$\frac{d\Psi}{dt} + \widehat{A}\Psi = 0, \quad \Psi(T) = K. \tag{6.9}$$

We note that $\mathcal{T}_{T-t}K$ is the unique continuous mild solution of the above Cauchy problem. Since, $\widehat{A}$ is the generator of $\{\widehat{S}_t, X_t, Y_t\}_{t\geq 0}$, for all $s>0, i\in\chi, y\in[0,t]$, using the functions $F, f, p, n(t)$ as in Chapter 3,

$$\begin{aligned}
\mathcal{T}_{T-t}K(s,i,y) =& E[K(\widehat{S}_T)|\widehat{S}_t = s, X_t = i, Y_t = y]\\
=& E\Big[E\big(K(\widehat{S}_T)|\widehat{S}_t, X_t, Y_t, T_{n(t)+1}\big)|\widehat{S}_t = s, X_t = i, Y_t = y\Big]\\
=& P\big(T_{n(t)+1} > T|X_t = i, Y_t = y\big)\times\\
& E\Big[K(\widehat{S}_T)|\widehat{S}_t = s, X_t = i, Y_t = y, T_{n(t)+1} > T\Big]\\
& + \int_0^{T-t} E\Big[K(\widehat{S}_T)|\widehat{S}_t = s, X_t = i, Y_t = y, T_{n(t)+1} = t+v\Big]\times\\
& \frac{f(t - T_{n(t)} + v|i)}{1-F(y|i)}dv\\
=& \frac{1-F(T-T_{n(t)}|i)}{1-F(y|i)}\int_0^\infty K(x)\alpha(x;s,i,T-t)dx\\
& + \int_0^{T-t}\frac{f(y+v|i)}{1-F(y|i)}\sum_{j\neq i} p_{ij}(y+v)\times\\
& \int_0^\infty E\Big[K(\widehat{S}_T, X_T, Y_T)|\widehat{S}_{t+v} = x, Y_{t+v} = 0, X_{t+v} = j, T_{n(t)+1} = t+v\Big]\alpha(x;s,i,v)dxdv
\end{aligned}$$

where $x\mapsto\alpha(x;s,i,v)$ is the probability density function of the lognormal random variable $\ln\mathcal{N}\Big(\ln s + \big(r(i)+\beta(i)-\frac{1}{2}\sigma^2(i)\big)v, \sigma^2(i)v\Big)$. Thus α is in the domain of $\widehat{A}$ and is C^1 in t. Thus, for $y<t$, $\mathcal{T}_{T-t}K(s,i,y)$ satisfies the following integral equation

$$\begin{aligned}
\Psi(t,s,i,y) =& \frac{1-F(T-t+y|i)}{1-F(y|i)}\int_0^\infty K(x)\alpha(x;s,i,T-t)dx\\
& + \int_0^{T-t}\frac{f(y+v|i)}{1-F(y|i)}\sum_j p_{ij}(y+v)\int_0^\infty \Psi(t+v,x,j,0)\alpha(x;s,i,v)dxdv.
\end{aligned}$$

We note that right hand side is C^1 in y as f and p are C^1. Again right hand side is also C^2 in s variable as α is so. The first additive term is also C^1 in t. Finally we observe that on the right hand side $\Psi(t+v,x,j,0)$ is multiplied by C^1 function of v and then integrated from 0 to $T-t$. Therefore the integral on right hand side is C^1 in t as a consequence of integration by parts. Thus left hand side is in $D(\widehat{A})$ and C^1 in t. $\square$

Lemma 6.0.5. *For any $\psi \in V$,*

1. *$\mathcal{T}_{u-t}\psi(s,i,y)$ is C^2 in s and in the domain of $D_{t,y}$, and*

2. *$D_{t,y}\mathcal{T}_{u-t}\psi(s,i,y)$ is continuous.*

Proof. In a similar line of proof of earlier lemma, we can show that $\mathcal{T}_{u-t}\psi(s,i,y)$ satisfies the following equation

$$\begin{aligned}\Psi(t,s,i,y) =& \frac{1-F(u-t+y|i)}{1-F(y|i)}\int_0^\infty \psi(x,i,u-t+y)\alpha(x;s,i,u-t)dx \\ &+ \int_0^{u-t}\frac{f(y+v|i)}{1-F(y|i)}\sum_j p_{ij}(y+v)\int_0^\infty \Psi(t+v,x,j,0)\alpha(x;s,i,v)dxdv. \qquad (6.10)\end{aligned}$$

Part 1.

First we observe that $\frac{\partial\alpha}{\partial s}(x;s,i,v) = \frac{1}{s}O(\ln|x|)\alpha(x;s,i,v)$. Since Ψ is in V (continuous and at most linear growth), using uniform integrability and tightness of

$$v \longmapsto \int_0^\infty \frac{1}{s+\epsilon}|x|^2\alpha(x;s+\epsilon,i,v)dx \quad \text{for } \epsilon \ll 1,$$

we conclude the differentiability of right side of (6.10) with respect to s. In a similar manner existence of partial derivative with respect to s of any higher order can be shown successively.

Part 2.

We next check the applicability of $D_{t,y}$ on the 1st additive term on right of (6.10).

Since F is C^2, it is enough to check for

$$\int_0^\infty \psi(x, i, u - t + y)\alpha(x; s, i, u - t)dx.$$

$D_{t,y}$ of above function is the limit of

$$\frac{1}{\epsilon}\Big[\int_0^\infty \psi\big(x, i, u - t + y\big)\big(\alpha(x; s, i, u - t - \epsilon) - \alpha(x; s, i, u - t)\big)\Big]dx.$$

Due to continuous differentiability of the p.d.f. $\alpha(x; s, i, v)$, on $v > 0$, we can rewrite above as

$$\int_0^\infty \psi\big(x, i, u - t + y\big)\frac{\partial\alpha}{\partial v}\big(x, s, u - t - \epsilon_1)dx \tag{6.11}$$

for some $0 < \epsilon_1(x, s, i, t) < \epsilon$.

Again $\frac{\partial\alpha}{\partial v}\alpha(x; s, i, v) = \alpha(x; s, i, v)O(\ln^2 |x|)$.

As ψ is at most of linear growth with respect to x, there exists c_1 and c_2 such that

$$\big|\psi(x, i, u - t + y)\big| < c_1x + c_2,$$

thus, the modulus of integrand of (6.11) is bounded above by $(c_1x + c_2)\ln^2(|x|)\alpha(x; s, i, v)$ whose integral with respect to x over $[0, \infty)$ is finite.

The finiteness is immediate, since α is p.d.f. of a random variable with finite variance and since $(c_1x + c_2)\ln^2 |x| \leq c_3x^2 + c_4$, for some c_3, c_4. Furthermore, one can also prove that

$$v \longmapsto \int_0^\infty (c_1x + c_2)\ln^2 |x|\alpha(x; s, i, v)dx$$

is right continuous by considering a quotient as above. Hence, (6.11) converges as $\epsilon \to 0$ to $\int_0^\infty \psi(x; i, u - t + y)\frac{\partial\alpha}{\partial v}(x; s, i, u - t)dx$ which is a continuous function.

Thus, the first term is in the domain of $D_{t,y}$ and the image of $D_{t,y}$ is also continuous.

Part 3.

Now we would check if the 2nd term is in the domain of $D_{t,y}$. Although the 2nd term is more involved than the 1st one, but that can also be studied as before. Nevertheless one should be careful that there is a double integral and one of the limits depends on t variable. Furthermore, the variable t appears in the continuous function Ψ not in the form of $t-y$. For all these reasons the analysis of the 2nd term is relatively longer. We call the second term as $\mathcal{B}$ Then $D_{t,y}\mathcal{B}$ is the limit of the following expression

$$\frac{1}{\epsilon}\Big[\int_0^{u-t-\epsilon}\frac{f(y+v+\epsilon|i)}{1-F(y+\epsilon|i)}\sum p_{ij}(y+v+\epsilon)\int_0^{\infty}\Psi(t+v+\epsilon,x,j,0)\alpha(x;s,i,v)dxdv$$
$$-\int_0^{u-t}\frac{f(y+v|i)}{1-F(y|i)}\sum p_{ij}(y+v)\int_0^{\infty}\Psi(t+v,x,j,0)\alpha(x;s,i,v)dxdv\Big].$$

After a suitable substitution, the above expression becomes

$$\int_{\epsilon}^{u-t}\sup p_{ij}(y+v)\int_0^{\infty}\Psi(t+v,x,j,0)\beta_{\epsilon}dxdv$$
$$-\frac{1}{\epsilon}\int_0^{\epsilon}\frac{f(y+v|i}{1-F(y|i)}\sum p_{ij}(y+v)\int_0^{\infty}\Psi(t+v,x,0)\alpha(x;t,s,i,v)dxdv \tag{6.12}$$

where $\beta_{\epsilon}(v,x,s,i,v)=\frac{1}{\epsilon}\Big[\frac{f(y+v|i)}{1-F(y+\epsilon|i)}\alpha(x;s,i,v-\epsilon)-\frac{f(y+v|i)}{1-F(y|i)}\alpha(x;s,i,v)\Big]$. Now due to continuous differentiable of f,α, using Mean value theorem we can rewrite

$$\beta_{\epsilon}=\alpha(x;s,i,v)\Big(-\frac{\partial}{\partial v}\frac{f(y+v+\epsilon-\epsilon_1|i)}{1+F(y+\epsilon|i)}+\frac{\partial}{\partial y}\frac{f(y+v+\epsilon_2|i)}{1-F(y+\epsilon_2|i)}$$
$$+\frac{f(y+v|i)}{1-F(y|i)}\Big(-\alpha_v(x;s,i,v-\epsilon_3)\Big)+\epsilon G_{\epsilon}(v,x,s,i,y)$$

for some, ϵ_1, ϵ_2, ϵ_3 smaller than ϵ, where

$$G_{\epsilon}=\Big(-\frac{\partial}{\partial v}\frac{f(y+v+\epsilon-\epsilon_1|i)}{1-F(y+\epsilon|i)}\Big)\times\Big(\alpha_v(x,s,i,v-\epsilon_3)\Big).$$

We recall that $\alpha_v(x;s,i,v)=\alpha(x,s,i,v)O(\ln^2|x|)$. The expression in (6.12) has two additive terms. For showing convergence of first term, we need to use above expression for applying theorems on convergence of integrals such as dominated convergence theorem, Vitali's convergence theorem, etc. As Ψ is at most of linear growth and continuous, it would be sufficient if we have the following results

(a) $v \longmapsto \int_0^\infty (c_1 x + c_2) \ln^2 |x| \alpha(x; s, i, v) dx$ is bounded and left continuous.

(b) $|x|^2 \alpha(x; s, i, v + \epsilon_2)$ is uniformly integrable and tight with respect to x for $\epsilon_2 \ll 1$.

The result (a) is already established in Part 2. In order to prove (b), we recall here that a family of normal random variables with bounded mean and variance is uniformly integrable and tight. Therefore, (b) follows as here a product of a polynomial and a lognormal density function appears.

Now we can show the convergence of 2nd term of (6.12). Clearly (a) implies boundedness of

$$v \longmapsto \int_0^\infty \Psi(t + v, x, j, 0) \alpha(x, s, i, c) dx,$$

which assures the desired convergence.

Thus, $\mathcal{B}$ is in $D(D_{t,y})$ and hence from Part 2 and Part 3

$$\Psi \in D(D_{t,y}).$$

□

Proof of Theorem 6.0.3. The proof follows from Lemma (6.0.4) and Lemma (6.0.5). □

Chapter 7

Conclusion

In this thesis, we have shown that there exists an equivalent local martingale measure for the model described in chapter 3. This implies that the model is arbitrage free. The locally risk minimizing price approach by Föllmer and Schweizer was considered as market is incomplete. Our main aim was to find price function of a European option. For this, we have established the existence and uniqueness of a classical solution to a system of parabolic Integro-PDEs with appropriate conditions in chapter 6. So we have obtained price function as a solution of this Cauchy problem.

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