Introduction to toric varieties

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled 'Introduction to toric varieties', towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Sidharth S. at Indian Institute of Science Education and Research under the supervision of Dr. Vivek Mohan Mallick, Department of Mathematics, during the academic year 2016-2017.

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Dedicated to life

Declaration

I hereby declare that the matter embodied in the report entitled 'Introduction to toric varieties' are the results of the work carried out by me at the Department of Mathematics, IISER Pune, under the supervision of Dr. Vivek Mohan Mallick, and the same has not been submitted elsewhere for any other degree.

Sidharth S.

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Abstract

This thesis is the product of an introductory graduate level study of algebraic geometric objects called toric varieties with devoloping necessary background in algebraic varieties, category theory, scheme theory and cohomology.

Contents

Introduction

The renowned mathematician M. Atiyah once compared algebra to a deal made by the devil to the mathematician. He could obtain the marvellous machinery that is algebra, which will answer all his querries, if only the mathematician was willing to give up his soul: the geometric intuition [Vak p.203]. Moreover, at another instance Atiyah portrayed this dilemma with an even more striking metaphor. He said, “Should you just be an algebraist or a geometer?” is like asking “Would you rather be deaf or blind?” [Vak p.17] These two quotes perhaps capture the essence of algebraic geometry unlike anything else. We are constantly searching for algebraic machinery with which one can categorise and study geometric spaces. At the same time, we wish to impart the soul of geometry into the abstract algebraic contructs such as ‘schemes’; which are often born out of generalisation of spaces, with a structure which is within the scope of our geometric visualisaton.

In this year long project, my aim was to devolop the necessary algebraic geometry machinery and study special class of algebraic objects called toric varieties, at an introductory graduate level. I also developed some cohomology and category theory background with which I look to further widen my understanding of the properties of toric varieties.

In studying modern algebraic geometry, which is largely due to the works of greats like Grothendieck and Serre amongst many others, I have opted for the method laid out by R. Hartshorne in his classical textbook [Har.]. He introduces the construction and properties of algebraic affine varieties and algebraic projective varieties as polynomial zero solutions for an underlying field, which is closed algebraically. Then he follows it up with unrelated and abstract sheaf theory. The construction, in at most generality of abstract entities called schemes, then stems from sheaf theory with some prerequisites in commutative algebra. Finally he establishes the link between algebraic varieties and schemes. I have found this approach quite intuitive and rewarding. Later on it really helped me while studying deeper concepts regarding schemes such as divisors. After gaining some understanding of scheme theory, I studied cohomological algebra from [Har.] with a special emphasis on cohomology of sheaves of modules. In the second half of the project, I initially studied the construction of toric varieties over the complex field $\mathbb{C}$ with emphasis on combinatorics. My references for this part were [Ful.] and [Cox]. But as I studied them further, the concepts from algebraic varieties and scheme theory started becoming apparent. And as far as this thesis is concerned, I will be presenting the work till an invariant on toric varieties called T-divisor.

In writing the thesis, I have tried to provide all the necessary definitions before defining a key theorem or a concept. However, some topics from ring theory are assumed such as the Krull dimension of a ring, valuation, valuation ring and its quotient field, etc. These can be found in any standard text on commutative algebra such as [AtM.]

Chapter 1

Algebraic Varieties

The study of zero sets of polynomials has always been a major part of mathematics as it has huge significance both in practical and abstract terms. By studying algebraic varieties and their properties, one is essentially trying to have a good understanding of just that.

This chapter, we consider algebraically closed fields and polynomials in finitely many variables over those fields as functions acting on related affine spaces. Let the field k be closed algebraically. The set of all n-tuples of the field elements is referred to as the n-affine space over k which is denoted by $\mathbf{A}_k^n$ (equivalently by $\mathbf{A}^n$ in case k is obvious). Also we represent the ring of all polynomials in n variables over k, $k[x_1, x_2, ..., x_n]$, by $\mathbf{K}$.

1.1 Affine and Projective varieties

1.1.1 Algebraic set

The field k be closed algebraically. Also let $\mathbf{A}^n$ be the n-affine space as defined. Then it is possible to consider $g \in \mathbf{K}$ to be a function from $\mathbf{A}_k^n$ to k. Given by, $g : (k_1, k_2, ..., k_n) \mapsto g(k_1, k_2, ..., k_n)$.

Let $Y \subseteq \mathbf{K}$ be an ideal. Then we can define the zero set of Y in $\mathbf{A}_k^n$, denoted by $Z(Y)$, as $\{P \in \mathbf{A}^n \mid f(P) = 0 \; \forall f \in Y\}$.

A set $V \subseteq \mathbf{A}_k^n$ is defined to be an algebraic set if V is zero set of Y where Y is a set of elements of $\mathbf{K}$.

1.1.2 Zariski topology

A topology was introduced by Zariski on $\mathbf{A}^n$, which is imparted via. defining all the algebraic sets of $\mathbf{A}^n$ to be closed. One can easily verify that it satisfies all the conditions for a topology. In particular, the whole space and the empty set, $\mathbf{A}^n = Z(0)$ and $\phi = Z(\mathbf{K})$ are both closed.

Example 1. *For a field k that is closed algebraically, Zariski topology on $\mathbf{A}^1$ gives the 'finite compliment' topology. i.e. a subset of $\mathbf{A}^1$ is open iff it is the compliment of a finite set.*

1.1.3 Affine variety

In any topology, a reducible closed set Y is a set that is closed which can be written as a union of two distinct subsets of Y such that both the subsets are proper and closed in Y. (The closed sets need not be disjoint) We say a closed set is irreducible when it happens to be not reducible.

Definition 1.1.1. *We define an affine variety (or affine algebraic variety) to be an irreducible closed subset (w.r.t. Zariski topology) of affine-n-space along with the induced subspace (Zariski) topology. A quasi-affine variety is defined to be an open subset of an affine variety along with the induced topology.*

Theorem 1.1.1. *A correspondence which reverses the inclusion relation can be established between the algebraic sets of $\mathbf{A}_k^n$ and the ideals which equal their own radical in the polynomial ring $\mathbf{K}$. Concretely the correspondence can be shown as, an algebraic set $Y \mapsto I(Y)$ and a radical ideal $A \mapsto Z(A)$. Moreover, Y is an algebraic variety that is affine iff the corresponding ideal $I(Y)$ is prime.*

Proof. For the first part, we only need to show that $I(Y)$ is an ideal that equals its radical whenever Y is an algebraic set. Which follows from Nullstellensatz theorem due to Hilbert. [see, AtM 1, P.85]
For the second part, let Y be irreducible. We need its ideal to be prime. Suppose for two

polynomials j,k their product $j.k \in I(Y)$. Then $Y \subseteq Z(jk) = Z(j) \cup Z(k)$. Therefore, $Y = (Y \cap Z(j)) \cup (Y \cap Z(k))$, both of which are closed in Y. Hence without loss of generality $Y = Y \cap Z(j)$. i.e. $Y \subseteq Z(j) \Rightarrow j \in I(Y)$.
Now let A be an ideal, prime in $\mathbf{K}$. Assume $Z(A)$ is reducible. i.e. $Z(A) = Y \cup Y'$ both of which are distinct proper closed subsets of $Z(A)$. Therefore, $A = I(Y) \cap I(Y')$. Since A is prime, without loss of generality $A = I(Y) \Rightarrow Z(A) = Y$. Which is a contradiction. Therefore $Z(A)$ is irreducible.

1.1.4 Graded rings and Projective varieties

Definition 1.1.2. *A ring $\overline{G}$ is a graded ring when $\overline{G} = \oplus_{i \geq 0} G_i$, abelian groups G_i such that $G_k.G_l \subseteq G_{i+j}$ for any k,$l \geq 0$.*
An ideal $\overline{a}$ in a graded ring $\overline{G}$ is referred to as homogeneous when it equals the direct sum $\overline{a} = \oplus_{i \geq 0} \overline{a} \cap G_i$

Example 2. *For field k, closed algebraically, the polynomial ring $\mathbf{K} = k[x_1, x_2, ..., x_n]$ can be made into a graded ring $\overline{\mathbf{K}}$. The gradation is given by considering each K_i as the set of every linear combinations of homogeneous polynomials in degree i in $\mathbf{K}$.*

For field k, closed algebraically, let $\overline{\mathbf{K}}$ be the graded ring as defined above. Suppose h is a polynomial homogeneous in degree i in $\overline{\mathbf{K}}$. Then $h(\lambda a_1, \lambda a_2, ..., \lambda a_n) = \lambda^i h(a_1, a_2, ..., a_n)$, where $(a_1, a_2, ..., a_n) \in k^n$,$\lambda \in k$ and $\lambda \neq 0$.
i.e. it is possible to define zero set $(Z(h))$ over the projective-n-space $\mathbf{P}_k^n$ as we defined it for affine-n-space.

Definition 1.1.3. *A subset $V \subseteq \mathbf{P}_k^n$ is defined to be an algebraic set if $V = Z(Y)$ for some set of homogeneous polynomials Y in $\overline{\mathbf{K}}$. We define Zariski topology on $\mathbf{P}_k^n$ using algebraic sets as in affine case.*
A projective algebraic variety (equivalently, projective variety) can be defined as an irreducible closed subset of $\mathbf{P}_k^n$ along with topology induced from its superset. A quasi-projective variety is a subset, open, of a projective variety with the topology induced .

Similar to the affine case, we can define a correspondence between algebraic sets in $\mathbf{P}_k^n$ and the radical ideals, in which the containment of $K = \oplus_{i>0} K_i$ (the irrelevant maximal ideal) is not entire, in the graded polynomial ring $\overline{\mathbf{K}}$. Concretely, an algebraic set $Y \mapsto I(Y)$ and a radical ideal (which do not contain K) $A \mapsto Z(A)$. Moreover, Y is a projective variety iff $I(Y)$ is an ideal which is prime.

1.2 Morphisms

The affine (also quasi-affine) and projective (also quasi-projective) algebraic varieties all have been defined. The word algebraic variety thus has the scope of all four of these.

1.2.1 Regular functions on affine varieties

The field k be closed algebraically. And $V \subseteq A_k^n$ be an affine variety (Or quasi-affine variety).

Definition 1.2.1. *A function g from V to the base field k, at a point $p \in V$, is said to be regular if $g = m/n$ on U; m,$n \in k[x_1, x_2, ..., x_n]$ and n is nowhere zero on U, a neighborhood U open and containing p, such that $U \subseteq V$.*

For a given function $h : V \to k$, if such a neighborhood and such polynomials exist for all points of the variety V, we say h regular on variety V.

1.2.2 Regular functions on projective varieties

For a field k, closed algebraically, and $V \subseteq \mathbf{P}_k^n$, a quasi-projective or projective variety,

Definition 1.2.2. *A function $g : V \to k$ is said to be regular at a point $p \in V$ if $g = m/n$ on U; m,n are homogeneous polynomials of same degree in $k[x_1, x_2, ..., x_n]$ and n is nowhere zero on a neighborhood U open and containing p, $U \subseteq V$.*

For a given function $h : V \to k$, if such a neighborhood and such polynomials exist for all points of the variety V, we say h regular on variety V.

1.2.3 Morphism between algebraic varieties

For a field k, closed algebraically, and X, Y, algebraic varieties over k.

Definition 1.2.3. *We define a morphism between X and Y as a function $\phi : X \to Y$ continuous such that for all open sets $U \subseteq Y$ and all regular functions $g : U \to k$, $f.\phi : \phi^{-1}(U) \to k$ is regular.*

Remark 1.2.1. *Observe that the morphism between two varieties is in a way dependent on the locally defined regular functions on the varieties. It is insightful to compare this with the morphisms between manifolds which depend on their locally defined structure functions. In particular an isomorphism between two varieties is necessarily bicontinuous and bijective. But a bicontinuous and bijective function need not be a morphism of varieties.*

Example 3. *$\lambda : \mathbf{A}^1 \to Z(x^3 - y^2) \subseteq \mathbf{A}^2$ given by $t \mapsto (t^2, t^3)$ is bicontinuous and bijective. But not an isomorphism of varieties.*

1.3 Rational maps between algebraic varieties

1.3.1 Open subsets of algebraic varieties

Proposition 1.3.1. *Any subset of an algebraic variety V is dense in V if the subset is open.*

Proof. Let Y be an algebraic variety. Suppose an open subset $U \subseteq Y$ is not dense in Y. Which means there exists atleast one open $U' \subseteq Y$ for which $U \cap U' = \phi$. Therefore the union of their compliments $U'^c \cup U^c = Y$. Since U'^c and U^c are both closed in Y and are necessarily distinct, it is in contradiction to the definition of Y as irreducibile. Hence U is dense in Y.

1.3.2 Rational maps

For two algebraic varieties X and Y.

Definition 1.3.1. *We define a rational map $\varphi : X \to Y$ as a collection $\{(U, \varphi_U)\}$, U open in X with morphism $\varphi_U : U \to Y$, modulo the equivalence relation $(V, \varphi_V) \sim (U, \varphi_U)$ if φ_V and φ_U agree where U and V intersect.*

We call a rational map $\varphi : X \to Y$ as dominant if there exists an equivalence class (U, φ_U) such that the image of φ_U is dense in Y.
Observe that composition of dominant rational maps are possible which will allow us to define the category with varieties as objects and rational, dominant maps as morphisms.

1.3.3 Birational equivalence

Let X and Y be two algebraic varieties.

Definition 1.3.2. *A birational equivalence (or simply birationality) between X and Y is rational morphisms $\psi : X \to Y$, $\varphi : Y \to X$; their compositions give identity rational maps on respective varieties.*

Chapter 2

Some category theory

Category theory can be understood as a language which has the potential to analyse seemingly unrelated branches of mathematics from an objective point of view and show the similarities between their underlying structures. The heart of category theory, perhaps, is the concept of universal properties. Defining an object by demanding a universal property is slightly different from the conventional definitions, in the sense that there can be many objects in the same category that satisfy the property. However, between any two such objects, there will always exist a unique isomorphism. This key aspect can be summarised by the phrase 'universal properties determine an object unique upto unique isomorphism'.

In this section the basic definitions regarding categories are assumed. For a category $\mathscr{C}$ the objects in a category will be denoted by $\mathrm{Obj}(\mathscr{C})$ and for any A, $B \in \mathrm{obj}(\mathscr{C})$ the morphisms from A to B will be denoted by $\mathrm{Mor}(A,B)$. We will concern ourselves only with those categories where Obj. and Mor. are sets. Such categories are called small categories

2.1 Limits and Adjoints

In order to carry out operations of analytical flavour in categories, one requires the notion of limit. Universal properties corresponding to two types of limits have been shown. i.e. inverse limit and colimit. Surprisingly in this framework, many special objects in various categories turn out to be limits of particular objects in those respective categories. e.g:

product, coproduct, etc.

2.1.1 Inverse limit

Let ζ be any category with σ_i, σ_j, ... $\in$ obj(ζ). For a small category $\mathscr{I}$ with i, j, ... $\in$ obj($\mathscr{I}$) and a functor $E : \mathscr{I} \longrightarrow \zeta$ such that $E(i) = \sigma_i \; \forall \; i$ then we call E, a diagram indexed by $\mathscr{I}$ in ζ. Where $\mathscr{I}$ is the index category.

Definition 2.1.1. *The inverse limit of a diagram σ_i, σ_j, ...indexed by $\mathscr{I}$ is an object $\varprojlim$ $\in$ obj(ζ) with morphisms $\rho_i : \varprojlim \longrightarrow \sigma_i$, $\forall \; i$ satisfying the below universal property.*

For every morphism $x : i \longrightarrow j$ in $\mathscr{I}$, the diagram

$$\begin{array}{ccc} \varprojlim & \xrightarrow{\rho_i} & \sigma_i \\ & \searrow^{\rho_j} & \downarrow{\scriptstyle E(x)} \\ & & \sigma_j \end{array}$$

commutes.

Example 4. *(Insightful) Let ζ be the category of sets with morphism denoted by inclusion, and let a partially ordered set be the index category. Then the inverse limit of the indexed diagram of sets is the intersection of all the indexed sets.*

2.1.2 Colimit

Definition 2.1.2. *The colimit of a diagram σ_i, σ_j, ...indexed by $\mathscr{I}$ is an object $\varinjlim$ $\in$ obj(ζ) with morphisms $\lambda_i : \sigma_i \longrightarrow \varinjlim$, $\forall \; i$ satisfying the following universal property.*

For every morphism $y : i \longrightarrow j$ in $\mathscr{I}$, the diagram

$$\begin{array}{ccc} \varinjlim & \xleftarrow{\lambda_i} & \sigma_i \\ & \nwarrow^{\lambda_j} & \downarrow{\scriptstyle E(y)} \\ & & \sigma_j \end{array}$$

commutes.

Example 5. *Let ζ be the category of sets with morphism denoted by inclusion, and the index category be a partially ordered set. Then the colimit of the indexed diagram of sets is the disjoint union of all the indexed sets.*

2.1.3 Adjoint Functors

In a way, similar to how universal properties determine an object unique upto unique isomorphism in a category, adjointness determines functors between two categories upto some specifics.
Let $\mathscr{G}$ and $\mathscr{H}$ be two categories. $A : \mathscr{G} \longrightarrow \mathscr{H}$ and $B : \mathscr{H} \longrightarrow \mathscr{G}$ be two covariant functors.

Definition 2.1.3. *A and B are called an adjoint pair if $\forall\ g \in obj(\mathscr{G})$ and $h \in obj(\mathscr{H})$, $\exists$ a bijection $\lambda_{gh} : Mor(A(g),\ h) \longrightarrow Mor(g,\ B(h))$, such that for every morphism $\gamma : g \to g'$ in $\mathscr{G}$*

$$\begin{array}{ccc} Mor(A(g'), h) & \xrightarrow{A\gamma^*} & Mor(A(g), h) \\ \lambda_{g'h}\big\downarrow & & \big\downarrow\lambda_{gh} \\ Mor(g', B(h)) & \xrightarrow{\gamma^*} & Mor(g, B(h)) \end{array}$$

Commutes. Where $A\gamma^*$ and γ^* are induced by $A(\gamma)$ and γ respectively. (Also similar commutative diagram should exist for every morphism $\tau : h \to h'$ in $\mathscr{H}$).

2.2 Abelian categories

Abelian categories are a subclass of categories with special properties. They constitute few of the most important and useful categories. Before defining abelian categories, one needs to specify few universal properties.

Zero object : In a category, A zero object is both initial and final object, i.e. there exists precisely one morphism from the zero object to every other object in the category and there exists precisely one morphism to the zero object from every other object in the category.

Kernel : Let ζ be a category with the zero object. Suppose f is a morphism between two objects A and B of the category. $f : A \longrightarrow B$. Then the kernel of f is the inverse limit

(Denoted by K) of the following diagram.

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

(Remember that inverse limit K guarantees a unique morphism k from K to A. Often we refer to this morphism as the kernel rather than the object K)

Cokernel : Let ζ be a category with the zero object. Suppose g is a morphism between two objects D and C of the category. $g : D \longrightarrow C$. Then the cokernel of g is the colimit (Denoted by K') of the following diagram.

$$\begin{array}{ccc} D & \xrightarrow{g} & C \\ \downarrow & & \\ 0 & & \end{array}$$

(Similarly, colimit K' guarantees a unique morphism k' from C to K'. Often we refer to this morphism as the cokernel rather than the object K')

Monomorphism : A morphism $f : A \longrightarrow B$ is a monomorphism, if $\forall$ morphisms σ_1, $\sigma_2 : X \longrightarrow A$ such that compositions $f \cdot \sigma_1 = f \cdot \sigma_2$, then $\sigma_1 = \sigma_2$.

Epimorphism : A morphism $g : C \longrightarrow D$ is an epimorphism, if $\forall$ morphisms λ_1, $\lambda_2 : D \longrightarrow Y$ such that compositions $\lambda_1 \cdot g = \lambda_2 \cdot g$, then $\lambda_1 = \lambda_2$.

2.2.1 Additive category

Definition 2.2.1. *A category is called additive when the following three properties are satisfied.*

- *For every objects A, B in the category, Mor(A, B) is an abelian group. [Hence denoted by Hom(A, B)]*
- *For every objects A, B in the category, product of A and B exists. (Therefore finite product exists by induction)*
- *Zero object exists.*

2.2.2 Abelian category

Definition 2.2.2. *An abelian category is an additive category $\mathscr{A}$ with the following three properties.*
- *If f is a morphism in $\mathscr{A}$, the kernel as well as the cokernel exist for f.*
- *If f is a monomorphism in $\mathscr{A}$, the cokernel of f has f as its kernel.*
- *If f is an epimorphism in $\mathscr{A}$, the kernel of f has f as its cokernel.*

Example 6. 1. *Category of abelian groups.*
2. *Category of modules over a ring.*

2.3 Injective and Projective objects

2.3.1 Chain complexes

Let $\mathscr{C}$ be an abelian category. A chain complex $X^\bullet$ in $\mathscr{C}$ is a collection of objects X^i, $i \in \mathbf{Z}$, in $\mathscr{C}$ along with morphisms $m^i : X^i \to X^{i+1}$ such that composition $m^{i+1} \cdot m^i$ is 0 $\forall i$.
All chain complexes (or simply complexes) $X^\bullet$ are considered to be consisting of infinite elements, by setting X^j to be zero whenever X^j is not defined for some $j > N$ for some integer N.
A **morphism** f between two chain complexes $X^\bullet$ and $Y^\bullet$, defined in the same abelian category, is a set of morphisms $f^i : X^i \to Y^i$ such that the diagram

$$\begin{array}{ccc} X^i & \xrightarrow{m^i} & X^{i+1} \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle f^{i+1}} \\ Y^i & \xrightarrow{n^i} & Y^{i+1} \end{array}$$

commutes $\forall$ i.

2.3.2 Additive and Exact functors

Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories. A covariant functor F from $\mathscr{C}$ to $\mathscr{D}$ is said to be **additive** if for every a,$b \in \mathscr{C}$, the induced map $f : \mathrm{Hom}(a, b) \to \mathrm{Hom}(F(a), F(b))$ is a group

homomorphism. Similarly for $f' : \text{Hom}(a,b) \to \text{Hom}(F'(b), F'(a))$ in case of a contra variant functor F'.
An additive covariant functor F between abelian categories $\mathscr{C}$ and $\mathscr{D}$ is said to be **exact** if for every short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathscr{C}$, the induced sequence

$$0 \to F(A) \to F(B) \to F(C) \to 0$$

is exact in $\mathscr{D}$. Similarly for a contra variant functor, arrows reversed in $\mathscr{D}$.

2.3.3 Injective and Projective objects

Let $\mathscr{C}$ be an abelian category and C be a fixed object in $\mathscr{C}$. Then we can define a contravariant left exact functor $\text{Hom}(\cdot, C)$ from $\mathscr{C}$ to category of abelian groups. Which is given by $A \to \text{Hom}(A, C)$. Similarly we can define a covariant functor $\text{Hom}(C, \cdot)$.

Definition 2.3.1. *An object C of an abelian category $\mathscr{C}$ is said to be injective if the contravariant functor $Hom(\cdot, C)$ is exact.*

Definition 2.3.2. *An object D of an abelian category $\mathscr{C}$ is said to be projective if the covariant functor $Hom(D, \cdot)$ is exact.*

2.3.4 Injective and Projective resolutions

Definition 2.3.3. *Let $\mathscr{C}$ be an abelian category. An object $C \in \mathscr{C}$ is said to have an injective resolution if there exists a chain complex $I^\bullet$ with $i \geq 0$, consisting entirely of injective objects, with a morphism $c : C \to I^0$ such that the complex,*

$$0 \to C \xrightarrow{c} I^0 \to I^1 \to \dots\dots\dots$$

is exact.

Definition 2.3.4. *Let $\mathscr{C}$ be an abelian category. An object $D \in \mathscr{C}$ is said to have a projective resolution if there exists a chain complex $I^\bullet$ with $i \geq 0$, consisting entirely of projective objects, with a morphism $d : I^0 \to D$ such that the complex,*

$$......... \to I^1 \to I^0 \xrightarrow{d} \to D \to 0$$

is exact.

Remark 2.3.1. *Observe that projective and injective objects are nothing but generalisations of projective and injective modules. The corresponding resolutions are also generalisations of sequences of modules.*

Chapter 3

Sheaves

In mathematics when we deal with topological spaces, often one notices that the information about the space as a whole is contained in information regarding open sets. i.e. a global picture of the space can be created from local data. We make use of this concept in defining manifolds and differentiable manifolds where the local information is given by real functions. With the concept of sheaves, we wish to generalise this concept even further.

3.1 Presheaves and sheaves

Definition 3.1.1. *A presheaf $\mathscr{F}$ on a topological space X is (in most generality) defined as the following information.*
a)If $U \subseteq X$ is an open subset, then $\exists$ a set $\mathscr{F}(U)$.
b)If $V \subseteq U$ in X is an inclusion of open sets, then $\exists$ a map $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ (Called the restriction map) subject to the following conditions.
0. $\mathscr{F}(\phi)$ is a one element set.
1. The restriction map $\rho_{UU} : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map for all U.
2. For a 3 chain open sets $U \subseteq V \subseteq W$ in X, $\rho_{WU} = \rho_{VU} \cdot \rho_{WV}$.

In the above definition, if $\mathscr{F}(U)$ is a set with additional structure for all open $U \subseteq X$, i.e., $\mathscr{F}(U)$ is an abelian group/a ring/an R module (for some ring R) etc., for all open U, the presheaf defined is the presheaf of abelian groups/presheaf of rings/presheaf of R modules etc.

respectively. In such cases, the restriction maps ρ_{UV} are defined to be structure preserving maps with respect to the additional structure in the category of $\mathscr{F}(U)$ s. Moreover, whenever the context is clear for an element $s \in \mathscr{F}(U)$ and open $V \subseteq U$, instead of $\rho_{UV}(s)$, we write $s|_V$ (Read, s restricted to V).

Definition 3.1.2. *Let X be a topological space and for any open $U \subseteq X$ let $\{V_i\}$ be any open covering of U in X.*
A presheaf $\mathscr{F}$ on X is a sheaf on X if it satisfies the following two axioms.
(1) *If $s,t \in \mathscr{F}(U)$ are two elements such that $s|_{V_i} = t|_{V_i}$ for all i, then $s = t$ in $\mathscr{F}(U)$.*
(2) *There exists an element $s \in \mathscr{F}(U)$ such that $s|_{V_i} = s_i$ for each i whenever $\exists$ elements $s_i \in \mathscr{F}(V_i)$ for each i, such that for each i,j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$.*

Remark 3.1.1. *The second axiom implies that we can patch up local information on the space to obtain a global information in case of a sheaf which might not be possible in a presheaf. The first axiom implies that the global information that we obtain by patching up local information will be unique in case of a sheaf.*

Example 7. *From definition, it is clear that all sheaves are presheaves.*
Presheaves that are not sheaves
(1) *Constant Presheaf : Let X be a topological space. For every open set $U \subseteq X$, let $\mathscr{F}(U) = \mathbb{Z}$ (Where, $\mathbb{Z}$ is the set of all integers with group structure). This is a presheaf on X. $\mathscr{F}$ won't be a sheaf on X when X has more than one element and has discrete topology.*
(2) *Let $\mathbb{R}$ be the real space with classical toplogy. For every open $U \subseteq \mathbb{R}$, Let $\mathscr{F}(U)$ be the set of all bounded real valued functions on U. Then with the usual restriction, $\mathscr{F}$ forms a presheaf. $\mathscr{F}$ won't be a sheaf as it will fail to satisfy the second axiom.*
Sheaves
(1) *Let $\mathbb{C}$ be the complex space with classical toplogy. For every open $U \subseteq \mathbb{C}$, Let $\mathscr{F}(U)$ be the set of all holomorphic functions on U. Then with the usual restriction, $\mathscr{F}$ forms a sheaf on $\mathbb{C}$.*
(2) *Constant Sheaf : A constant sheaf $\mathscr{Z}$ on X is defined as follows. Assign discrete topology to $\mathbb{Z}$. Let $\mathscr{Z}(U) = \{$ continuous funtions from U to $\mathbb{Z}$ $\}$ Then with the usual restriction maps, $\mathscr{Z}$ becomes a sheaf.*

3.2 Stalks

The 'stalk' of a presheaf at a point as a notion is inspired from the concept of vector bundle in differential geometry. For $\mathscr{F}$ defined on X, a presheaf, and open $U \subseteq X$, owing to the inspiration we denote the elements of $\mathscr{F}(U)$ as sections of $\mathscr{F}$ over U.
Germs of sections : Germs of sections at a point parallels the concept of germs of functions at a point in differential geometry.

Definition 3.2.1. *Let $p \in$ open $U \subseteq X$ be a point of the topological space. Germ of a section $f \in \mathscr{F}(U)$ (denoted by (f, U)) at p is the representative of (f, U) in the equivalence class $\{(f, openV) \mid p \in V,\ f \in \mathscr{F}(V)\}$ modulo $\sim$. Where,*
$(f,V) \sim (g,W)$ if $\exists$ open $X \subseteq V \cap W$ s.t. $f\mid_X = g\mid_X$.

Stalk of a presheaf at a point

Definition 3.2.2. *Let $p \in$ open $U \subseteq X$ be a point of the topological space. Stalk of a presheaf $\mathscr{F}$ at p (denoted by $\mathscr{F}_p$) is the set of all germs of sections over all open sets containing p.*

Stalk of a sheaf is the same as the stalk of the underlying presheaf.

3.2.1 A category theoretical view

Category theory can often provide valuable new insights to familiar concepts. In this section we will explore the concepts we already defined, in the frame work of category theory.

Presheaves and sheaves

Let X be a topological space. The open sets of X form a category $\mathscr{T}op_X$ where the morphisms are given by inclusion.i.e. $\sharp$ Mor $\{V, U\} = 1$ iff $V \subseteq U$ and 0 otherwise.

Definition 3.2.3. *A presheaf of sets on X is a contravariant functor from $\mathscr{T}op_X$ to category of $\mathscr{S}ets$.*

This interpretation sheds light into the 0^{th} axiom in our earlier definition, namely, $\mathscr{F}(\phi)$ is a one element set. Since ϕ is an initial object in the category $\mathscr{T}op_X$, under the contravariant functor $\mathscr{F}$ it will be mapped to a final object in the category $\mathscr{S}ets$ i.e. a one element set.

Definition 3.2.4. *Let X be a topological space and for any open $U \subseteq X$ let $\{V_i\}$ be any open covering of U in X. A presheaf $\mathscr{F}$ of sets on X is a sheaf if the following equalizer diagram is exact.*

$$\bullet \to \mathscr{F}(U) \to \Pi\ \mathscr{F}(V_i) \rightrightarrows \Pi\ \mathscr{F}(V_i \cap V_j)$$

Observe that the injectivity of the exact sequence corresponds to the first axiom (axiom of uniqueness) and the fact that $\mathscr{F}(U)$ is the equalizer corresponds to the second axiom (axiom of gluability) of our earlier definition.

Definition 3.2.5. *Let $p \in X$ be a point of the topological space. Stalk of a presheaf $\mathscr{F}$ at p (denoted by $\mathscr{F}_p$) is the direct limit of $\mathscr{F}(U)$ for all open U such that $p \in U$, via. the restriction maps.*

3.2.2 The stalk space

This section intends to make the inspiration from differential geometry precise.

Let $\mathscr{F}$ be a presheaf defined on a topological space X. We can define the stalk space of $\mathscr{F}$ over X as a set, to be the disjoint union $\amalg_{p \in X} \mathscr{F}_p$. Denote the set by Spe($\mathscr{F}$). Consider the natural projection map $\pi : \text{Spe}(\mathscr{F}) \to X$. i.e. $\pi(s) = p$ if $s \in \mathscr{F}_p$.
Consider open $U \subseteq X$. Let $s \in \mathscr{F}(U)$ be an element. We define a function $\tilde{s} : U \to \text{Spe}(\mathscr{F})$ as $\tilde{s}(p) = s_p$. Observe that $\pi.\tilde{s} = id_U$. i.e. $\tilde{s}$ is a section over U for every $s \in \mathscr{F}(U)$. Hence the name sections.
Now we make Spe($\mathscr{F}$) into a topological space by giving it the weakest topology such that $\tilde{s}(U)$ is open for every open $U \in X$ and every $s \in \mathscr{F}(U)$. This is known as the stalk space of $\mathscr{F}$ (Or Espace etale of $\mathscr{F}$).

3.3 Morphism of sheaves

Let $\mathscr{F}$ and $\mathscr{G}$ be two presheaves defined on a topological space X. A morphism θ between $\mathscr{F}$ and $\mathscr{G}$ consists of morphisms $\theta(U)$ between $\mathscr{F}(U)$ and $\mathscr{G}(U)$ for every open $U \subseteq X$, such that whenever there is an inclusion of open sets $V \subseteq U$, the following diagram commutes.

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\theta(U)} & \mathscr{G}(U) \\ \rho_{UV}\downarrow & & \downarrow\rho'_{UV} \\ \mathscr{F}(V) & \xrightarrow{\theta(V)} & \mathscr{G}(V) \end{array}$$

Where, ρ_{UV} and ρ'_{UV} are the restriction maps.
A morphism between sheaves is the morphism between the underlying presheaves.
Therefore, presheaves/sheaves of sets/abelian groups/rings etc., over a topological space forms categories correspondingly.

3.3.1 Presheaf kernel, cokernel and image

Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism between presheaves on some topological space X.

Definition 3.3.1. *Presheaf kernel of ϕ is defined as a presheaf on X which takes open $U \subseteq X$ to kernel of the morphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$.*

Definition 3.3.2. *Presheaf cokernel of ϕ is defined as a presheaf on X which takes open $U \subseteq X$ to cokernel of the morphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$.*

Definition 3.3.3. *Presheaf image of ϕ is defined as a presheaf on X which takes open $U \subseteq X$ to image of the morphism $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$.*

Observe that presheaf kernel and presheaf cokernel satisfy the universal property of kernels and cokernel in the category of presheaves.

3.3.2 Direct image sheaf

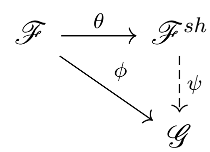
For topological spaces X and Y consider a continuous map $g : X \to Y$.

Definition 3.3.4. *For any sheaf $\mathscr{F}$ on X, we define a direct image sheaf $g_*\mathscr{F}$ on open sets of Y as $g_*\mathscr{F}(U) = \mathscr{F}(g^{-1}(U))$.*

3.4 Sheafification of a presheaf

3.4.1 Universal property

Definition 3.4.1. *Let $\mathscr{F}$ be a presheaf defined on a topological space X. We define the sheaf associated with the presheaf $\mathscr{F}$ (denoted by $\mathscr{F}^{sh}$) to be a sheaf on X along with a presheaf homomorphism $\theta : \mathscr{F} \to \mathscr{F}^{sh}$ such that whenever there exists a presheaf $\mathscr{G}$ on X and a morphism $\phi : \mathscr{F} \to \mathscr{G}$, then $\exists$ a unique morphism $\psi : \mathscr{F}^{sh} \to \mathscr{G}$ such that the below diagram commutes.*



Since this is a universal property, if a sheaf associated with a presheaf exists, then it is unique upto unique isomorphism. Therefore, we are justified in calling it 'the' sheaf associated with a presheaf. Moreover, if the presheaf under consideration is already a sheaf, then the sheaf associated with it will be the same sheaf (unique upto unique isomorphism). Because, the sheaf along with the identity morphism will satisfy the universal property.

3.4.2 Construction

Here we show in a constructive manner that for every presheaf, there exists the sheaf associated with it. We refer to this procedure as 'sheafification' of a presheaf.

Definition 3.4.2. *Let $\mathscr{F}$ be a presheaf defined on a topological space X. Let Spe($\mathscr{F}$) be the stalk space of $\mathscr{F}$. We define $\mathscr{F}^{sh}$ as follows. For every open $U \subseteq X$, Define $\mathscr{F}^{sh}(U) = \{s : U \to Spe(\mathscr{F}) \mid$ for every $p \in U$, $s(p) \in \mathscr{F}_p$ and $\exists$ a neighbourhood $p \in V \subseteq U$ and a section $\mathbf{t} \in \mathscr{F}(V)$ such that for every $q \in V$, $s(q) = \mathbf{t}_q\}$*

Let π be the natural projection map from Spe($\mathscr{F}$) $\to X$.
Observe that $\pi \cdot s = id_U$ for every open $U \subseteq X$ and functions s defined above. Which implies that s correspond to "sections" over U and by construction they are continuous. While defining Spe($\mathscr{F}$) for a sheaf $\mathscr{F}$, we observed that elements of $\mathscr{F}(U)$ define continuous

"sections" $\tilde{s}$ over U. Here, we reverse the analogy and define $F^{sh}(U)$ to be the set of all continuous "sections" from U to Spe($\mathscr{F}$). Hence by construction, it is evident that $F^{sh}(U)$ is a sheaf with the usual restriction of functions. The morphism $\theta : \mathscr{F} \to \mathscr{F}^{sh}$ is given as,
$\theta(U) : \mathscr{F}(U) \to \mathscr{F}^{sh}(U)$
$\theta(U)(s) = \bar{s}$ where,
$\bar{s} : U \to \text{Spe}(\mathscr{F})$ such that,
$\bar{s}(p) = s_p$

3.4.3 Inverse image sheaf

For topological spaces X and Y consider a continuous map $g : X \to Y$.
For $\mathscr{F}$, a sheaf on Y we can define a presheaf $(g^{-1}\mathscr{F})_{pre}$ on the open sets of X as, $(g^{-1}\mathscr{F})_{pre}(V)$ is the colimit of $\mathscr{F}(W)$ over all open $W \subseteq Y$ such that $g(V) \subseteq W$.

Definition 3.4.3. *For $\mathscr{F}$, a sheaf defined on Y, we can define its inverse image sheaf $g^{-1}\mathscr{F}$ on X as the sheaf associated with the presheaf $(g^{-1}\mathscr{F})_{pre}$.*

Chapter 4

Schemes

4.1 Affine and projective schemes; morphisms

4.1.1 Spectrum of a ring

Let A be a commutative ring with identity. We define spectrum of the ring A, Spec(A) as a set to be the set of all prime ideals of A. We can attribute a general version of Zariski topology to Spec(A) given as follows, If I is any ideal in A, define $V(I) :=$ set of all prime ideals in A that contain I. By definition, $V(I) \subseteq \text{Spec}(A)$ for all ideals I in A. We define Zariski topology on Spec(A) by taking all such $V(I)$ s to be closed sets for all ideals I in A.

Remark 4.1.1. *It is easy to verify that this is a topology. In particular, $V(A) = \phi$ and $V(0)$ $=$ Spec(A) are both closed.*

Structure sheaf Let Spec(A) be defined as a topological space as above. We define a sheaf of rings $\mathscr{O}_{Spec(A)}$, called the structure sheaf, on Spec(A) as follows. For an open set U $\subseteq$ Spec(A), define $\mathscr{O}_{Spec(A)}(U) :=$ Set of all functions $s : U \to \amalg_{p \in U} A_p$ such that $s(p) \in A_p$ for every $p \in U$ and there exists an open neighborhood V containing p and contained in U such that $s(q) = a/g$ for all $q \in V$ and for a fixed $a, g \in A$ and g does not belong to the prime ideal q for all q in V.

Definition 4.1.1. *For a ring A, the spectrum of a ring is the topological space Spec(A) along* *with the sheaf of rings $\mathscr{O}_{Spec(A)}$ (structure sheaf). We denote it by (Spec(A), $\mathscr{O}_{Spec(A)}$).*

Remark 4.1.2. *Let A be a ring and (Spec(A), $\mathscr{O}$) be its spectrum.*
- $\mathscr{O}(Spec(A)) \simeq A$
- *For any q in Spec(A), the stalk* $\mathscr{O}_q \simeq A_q$

It is insightful to compare this scenario with the case of an affine variety. Elements of A for Spec(A) is analogous to the functions (polynomials of coordinate ring) on the affine variety. For a point p of Spec(A) the elements of the prime ideal p is analogous to functions that vanish on a point p in an affine variety. For any open set U of Spec(A), the ring $\mathscr{O}(U)$ is analogous to the ring of all regular functions on U for an open set U of an affine variety.

4.1.2 Morphism of ringed spaces

Let A and B be local rings (A ring in which the non invertible elements form the unique maximal ideal). A ring homomorphism $f : A \to B$ is said to be a local homomorphism of local rings if the pull back of the unique maximal ideal of B by f is exactly the unique maximal ideal of A.

We call a topological space X along with a sheaf of rings $\mathscr{O}_X$ as a ringed space. A ringed space is a locally ringed space if $(\mathscr{O}_X)_p$ is a local ring for every point $p \in X$. Observe that (Spec(A), $\mathscr{O}_{Spec(A)}$) is a locally ringed space

Definition 4.1.2. *Let $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ be spaces that are locally ringed. Between X and Y we define a morphism as $(f, f^\sharp)$ a pair for which, f is continuous $f : X \to Y$ and $f^\sharp$ is a morphism of sheaves $f^\sharp : \mathscr{O}_Y \to f_*\mathscr{O}_X$. [$f_*\mathscr{O}_X$ is the direct image sheaf defined in Ch.3] Such that the morphism induced by $f^\sharp$ at the stalks is a homomorphism of local rings which is local.*

4.1.3 Affine and projective schemes

Definition 4.1.3. *We define an affine scheme as a locally ringed space $(X, \mathscr{O}_X))$ which is isomorphic to spectrum of some ring.(of course, via an isomorphism of locally ringed spaces). A scheme is a ringed space $(Y, \mathscr{O}_Y)$ which is locally affine. i.e. there is an open covering $\{U_i\}$ of Y such that (U_i, $\mathscr{O}_Y|_{U_i}$) is an affine scheme for all i.*

Now we will summarise the construction of an affine projective scheme from a graded ring.

Let $\overline{G} = \oplus_{i\geq 0} G_i$ be a graded ring. We denote by G, the irrelevant maximal ideal ($G = \oplus_{i>0}$ G_i) of $\overline{G}$.

We define Proj($\overline{G}$) as a set to be the set of all homogeneous prime ideals of $\overline{G}$ which do not contain all of G. And define Zariski topology on Proj($\overline{G}$) by defining $V(I) :=$ set of all homogeneous prime ideals in $\overline{G}$ (which do not contain all of G) that contain I for all homogeneous ideal I of $\overline{G}$. We define a structure sheaf and morphisms the same way we defined it for affine schemes.

Definition 4.1.4. *We define a projective scheme as a locally ringed space $(X, \mathscr{O}_X)$ which is isomorphic to Proj. of some graded ring.(of course, via an isomorphism of locally ringed spaces). All projective schemes are locally affine, therefore schemes.*

4.2 Properties of schemes and scheme morphisms

Let X and Y be two schemes. We say Y is a scheme over X if there exists a morphism $Y \to X$. Suppose A is a ring, we say a scheme Y is a scheme over A if there exists a morphism $Y \to \text{Spec}(A)$

4.2.1 Some properties of schemes

Definition 4.2.1. *A scheme $(X, \mathscr{O}_X)$ is said to be integral if $\mathscr{O}_X(U)$ is an integral domain for all $U \subseteq X$, U open.*

Definition 4.2.2. *A scheme $(X, \mathscr{O}_X)$ is said to be locally noetherian if it has an open covering $\{U_i\}$ with $(U_i,\ \mathscr{O}_Y|_{U_i}) \simeq$ Spec(A_i) such that A_i is a noetherian ring for each i. A locally noetherian scheme is noetherian if it is a finite covering.*

Let X and Y be schemes.

Definition 4.2.3. *A morphism $f : X \to Y$ is said to be locally of finite type if there exists an affine open cover $\{U_i\}$ of Y where $U_i \simeq$ Spec(A_i) for some ring A_i such that $f^{-1}(U_i)$ has*

an open cover $\{V_{ij}\}$ where $V_{ij} \simeq Spec(B_{ij})$ for some finitely generated A_i algebra B_{ij} for all i, j.
A locally of finite type morphism is finite type if $f^{-1}(U_i)$ can be covered by finitely many such $\{V_{ij}\}$ for each i.
A finite type morphism is finite if $f^{-1}(U_i) = V_i \simeq Spec(B_i)$ for finitely generated A_i algebra B_i for all i

4.2.2 Separated and proper morphism

Let X be a noetherian scheme and Y be any scheme. Suppose there exists a valuation ring R with quotient field k and suppose there exists morphism of schemes $g_1 : \mathrm{Spec}(k) \to X$ and $g_2 : \mathrm{Spec}(R) \to Y$

Definition 4.2.4. *A morphism of schemes $f : X \to Y$ is said to be separated if there exists at most one morphism $Spec(R) \to X$ such that the following diagram commutes.*

$$\begin{array}{ccc} Spec(k) & \xrightarrow{g_1} & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ Spec(R) & \xrightarrow{g_2} & Y \end{array}$$

A morphism of finite type $f : X \to Y$ is said to be proper if there exists exactly one morphism $Spec(R) \to X$ such that the above diagram commutes.

Remark 4.2.1. *Refer to [Har. p. 96-101] for the definition for separated and proper morphisms for genaral schemes.*

4.2.3 Abstract variety

Theorem 4.2.1. *Consider k, a field closed algebraically. Then the natural functor $\mathscr{F}$ from the category of algebraic varieties over k to the category of schemes over k is full and faithful. Moreover, if V is an algebraic variety over k, then $\mathscr{F}$ maps V to an integral, separated scheme of finite type over k. (meaning morphism from $\mathscr{F}(V) \to spec(k)$ is separated and of finite type).*

Proof. Refer [Har. p. 78,104]

Therefore we define an abstract variety to be a scheme X over an algebraically closed field k such that X is an integral, separated scheme of finite type.

4.3 Sheaf of module; line bundle

4.3.1 Sheaf of module

Let $(X, \mathscr{O}_X)$ be a ringed space.

Definition 4.3.1. *A sheaf $\mathscr{F}$ on X is said to be a sheaf of-$\mathscr{O}_X$-module if for every $U \subseteq X$, U open, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$ module and whenever there is an inclusion of open sets $V \subseteq U \subseteq X$, the restriction morphism $\mathscr{F}(U) \to \mathscr{F}(V)$ is a module homomorphism that respects the ring homomorphism $\mathscr{O}_X(U) \to \mathscr{O}_X(V)$ for all U, V.*

A morphism $\mathscr{F} \to \mathscr{G}$ of sheaves of modules is a morphism of sheaves respecting the module structure.

4.3.2 Locally free sheaves and line bundle

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ and $\mathscr{G}$ be sheaves of-$\mathscr{O}_X$-module on X. The direct sum sheaf $\mathscr{H} = \mathscr{F} \oplus \mathscr{G}$ is defined by $\mathscr{H}(U) = \mathscr{F}(U) \oplus \mathscr{G}(U)$ for every open $U \subseteq X$. Clearly, direct sum sheaf of sheaves of-$\mathscr{O}_X$-module is again a sheaf of-$\mathscr{O}_X$-module.

Definition 4.3.2. *Let $(X, \mathscr{O}_X)$ be a ringed space. A sheaf of-$\mathscr{O}_X$-module $\mathscr{F}$ on X is said to be free if $\mathscr{F} \simeq \oplus^{i \in I} \mathscr{O}_X$ for some indexing set I, where $\mathscr{O}_X$ is taken to be a sheaf of-$\mathscr{O}_X$-module on X.*

Definition 4.3.3. *A sheaf of-$\mathscr{O}_X$-module $\mathscr{F}$ on X is said to be locally free if there exists an open covering $\{U_i\}$ of X such that $\mathscr{F}|_{U_i}$ is a free sheaf of $\mathscr{O}_X|_{U_i}$-module for each i. The rank of $\mathscr{F}$ on an open set is the cardinality of copies of $\mathscr{O}_X|_{U_i}$ in the direct sum.*

It is possible to demonstrate that the rank of a locally free sheaf on a connected topological space X will be equal on all the open subsets of X.[see, Har. p.109] Therefore, owing to the inspiration from the case of a vector bundle over a manifold, we call a locally free sheaf of rank n over a connected topological space X as a vector bundle of rank n over X.

A vector bundle of rank 1 is called a **line bundle**. (or an invertible sheaf).

4.3.3 Sheaf associated to a module

Let A be a commutative ring with identity and M be an A-module. On spec$A$, we define a sheaf $\overline{M}$ called the sheaf associated with the module M as follows.

If $U \subseteq$ spec$A$ is an open set then we define,

$\overline{M}(U) = \{$ Functions $s : U \to \amalg_{p \in U} M_p \mid s(p) \in M_p$, s is locally a fraction of the form $s(p) = m/f$ where $m \in M$ and $f \in A$ $\}$

Remark : Very similar to the way we defined the structure sheaf.

4.3.4 Coherent sheaf

Definition 4.3.4. *A sheaf of-$\mathscr{O}_X$-module $\mathscr{F}$ on a ringed space $(X, \mathscr{O}_X)$ is said to be quasi-coherent if there exists an affine open covering $\{U_i\}$ of X, $U_i \simeq specA_i$ for some ring A_i, such that $\mathscr{F}|_{U_i}$ is isomorphic to the sheaf associated with an A_i module M_i for all i.*
A quasi-coherent sheaf $\mathscr{F}$ is **coherent** *if M_i is a finitely generated A_i module for all i.*

Example 8. *Structure sheaf of any scheme is a coherent sheaf.*

4.4 Divisors

Divisors are certain invariants of schemes. The geometry that is intrinsic to the scheme is closely associated with divisors.

4.4.1 Weil divisor

If A is a noetherian local ring with the unique maximal ideal m, we say A is regular if $dim_K m/m^2 = \dim A$, for the field $K = A/m$.

For a scheme $(X, \mathscr{O}_X)$, if every local ring $(\mathscr{O}_X)_x$ of dimension 1 is regular, we say X is regular in codimension 1.

In this section, we define weil divisors only for a noetherian integral separated scheme regular in codimension 1. If X is such a scheme, we say X is a type (+) scheme.

For a scheme X, a generic point of X is a point $x \in X$ such that the closure of x is X. Every non empty closed irreducible subset of a scheme has a unique generic point. [see, Har. p.80]

Definition 4.4.1. *Let X be a type (+) scheme. We say $Y \subset X$ is a prime divisor of X if Y is a closed integral subscheme of codimension* 1.

The free abelian group generated by prime divisors is denoted by Div X. A weil divisor of X is an element of Div X. i.e. If W is a weil divisor of X, then $W = \Sigma_i n_i Y_i$, for some $n_i \in \mathbb{Z}$ where all but finitely many are zero and Y_i are prime divisors.

4.4.2 Divisor class group

Let X be a type (+) scheme and Y be a prime divisor of X. Y has a unique generic point y. The local ring $(\mathscr{O}_X)_y$ is a discrete valuation ring since it is a regular local ring of dimension 1. [See, AtM. p.94] It has as its quotient field, the function field K of X. We denote the corresponding valuation $V_Y : K \to \mathbb{Z}$ as the valuation of Y.

Definition 4.4.2. *Let X be a type (+) scheme and $g \in K^*$ be a non zero function on X. Then we define the divisor of g, $d(g) = \Sigma V_Y(g).Y$ over all prime divisor Y.*

$d(g)$ is a finite sum as all but finitely many $V_Y(g)$ vanish for all g and Y. [see, Har. p.131] Therefore $d(g)$ is a weil divisor for all $g \in K^*$. A weil divisor of this form is called as a principal divisor.

Definition 4.4.3. *The set of all principal divisors form a subgroup of Div X. We call the* *quotient of Div X by the principal divisor subgroup as the divisor class group and denote it by Cl X. This is an invariant for the scheme X.*

4.4.3 Cartier divisor

The Weil divisor though very useful, is limited in its scope. Now we define a divisor for a general scheme.

Let $(X, \mathscr{O}_X)$ be a scheme. Consider an affine open subset $U = specA$ of X. Let $S(U)$ be the set of all elements of $\mathscr{O}_X(U)$ which are not zero divisors in each local ring $(\mathscr{O}_X)_y$, $y \in U$. Now we define a presheaf $\mathscr{H}^{pre}$ given by $\mathscr{H}^{pre}(U)$ is the group $\mathscr{O}_X(U)$ localised at the multiplicatively closed set $S(U)$. Let $\mathscr{H}$ denote the sheaf associated with this presheaf. Let $\mathscr{H}^*$ be the sheaf given by $\mathscr{H}^*(U) =$ multiplicative group of all invertible elements in $\mathscr{H}(U)$. Similarly let $\mathscr{O}^*$ be the sheaf given by $\mathscr{O}^*(U) =$ multiplicative group of all invertible elements in $\mathscr{O}(U)$.

Definition 4.4.4. *A cartier divisor on a scheme $(X, \mathscr{O}_X)$ is defined as a global section of the quotient sheaf $\mathscr{H}^*/\mathscr{O}^*$.*

Definition 4.4.5. *A cartier divisor on a scheme $(X, \mathscr{O}_X)$ is principal if it is in the image of the natural map $\mathscr{H}^*(X) \to \mathscr{H}^*/\mathscr{O}^*(X)$. We define the cartier class group on X, CaCl X as the cartier divisor group quotiented by the principal cartier divisor group.*

Theorem 4.4.1. *When X is an integral separated noetherian scheme with all local rings as U.F.D s, the weil divisor group Div(X)and the cartier divisor group $(\mathscr{H}^*/\mathscr{O}^*)(X)$ are isomorphic. Moreover Cl X and CaCl X are isomorphic.*

Proof. See. [Har. p.140-141]

4.4.4 Picard group

Let $(X, \mathscr{O}_X)$ be a ringed space.

Definition 4.4.6. *The picard group of X, Pic X is defined to be the set of all line bundles, i.e. invertible sheaves (modulo isomorphism) over X w.r.t. $\mathscr{O}_X$. And this equivalence class is given a group structure under tensor product.*

Remark 4.4.1. *If $\mathscr{H}$ and $\mathscr{G}$ are line bundles on X, so is $\mathscr{H} \otimes \mathscr{G}$. Furthermore, for every line bundle $\mathscr{L}$, take its inverse to be the dual sheaf $\mathscr{L}^{-1} = \mathscr{H}om(\mathscr{L}, \mathscr{O}_X)$. $\mathscr{L} \otimes \mathscr{L}^{-1} \simeq \mathscr{H}om(\mathscr{L}, \mathscr{L}) = \mathscr{O}_X$ which is the identity. Hence the group structure.*

Theorem 4.4.2. *On any scheme X, we defined a sheaf $\mathscr{H}$ while defining cartier divisors. There exists an injective homomorphism from $CaCl\ X \to Pic\ X$, where $Pic\ X$ is defined w.r.t. the sheaf $\mathscr{H}$. Furthermore, if X is an integral scheme, then this is an isomorphism of groups.*

Proof. see [Har. p.143-145]

Chapter 5

Cohomology

Homological algebra is a powerfool mathematical tool that allow us to study certain mathematical objects of algebraic flavour in detail. In this project I developed a very basic understanding of cohomological algebra as a prerequisite for studying derived categories on toric varieties later on.

5.1 Definition of sheaf cohomology

5.1.1 Right derived functor

In Chapter 2, we saw chain complexes and injective objects in an abelian category. Suppose $C^\bullet$ is a chain complex in an abelian category $\mathscr{C}$ with morphisms $f^i : C^i \to C^{i+1}$. Then the object at the core of cohomology, the i^{th} cohomological object is defined to be the group $\mathcal{H}^i(C^\bullet) = \text{Ker } f^i / \text{Im } f^{i-1}$.

Next we look at the definition of functors called right derived functors, which we derive from a covariant left exact functor between two abelian categories.

Let $\mathscr{C}$ be an abelian category such that every object of $\mathscr{C}$ has an injective resolution. And let $\mathscr{D}$ be an abelian category. For an object $C \in \mathscr{C}$ we get an injective resolution $C^\bullet$,

$$0 \to C \xrightarrow{c} I^0 \to I^1 \to ....$$

Suppose $F : \mathscr{C} \to \mathscr{D}$ is a covariant left exact functor. Then we define the right derived functors $R^iF : \mathscr{C} \to$ Category of abelian groups, as $R^iF(C) = \mathcal{H}^i(F(C^\bullet))$

Remark 5.1.1. *Observe that right derived functors depend on both the injective resolution and the functor. This is well defined as any two injective resolutions of an object in any abelian category is homotopy equivalent and therefore preserves the i^{th} cohomological object.[Har. p.203,204] Moreover, one can define the same way left derived functors for a right exact covariant functor using projective resolutions.*

5.1.2 Sheaf cohomology

Proposition 5.1.1. *The category of sheaves of modules over a ringed space $(X, \mathscr{O}_X)$ is an abelian category such that every object in the category has an injective resolution.*

Proof. (Sketch) In an abelian category, every object has an injective resolution if every object has an isomorphism to a subobject of some injective object. We make use of a theorem from commutative algebra that for a ring R, every R module has an isomorphism to a submodule of an injective R module. For an $\mathscr{O}_X$-module-sheaf $\mathscr{G}$, we get $(\mathscr{O}_X)_p$ modules $\mathscr{G}_p$ for every point $p \in X$. We define a sheaf of $\mathscr{O}_X$-module $\mathscr{I}$ using this data, such that for every point $p \in X$, $\mathscr{I}_p$ is the injective submodule corresponding to the module $(\mathscr{G})_p$ in the category of $\mathscr{O}_X$-modules. One can show that $\mathscr{I}$ is an injective $\mathscr{O}_X$-module sheaf and there exists an injective morphism of sheaves $\mathscr{G} \to \mathscr{I}$. And the result follows.

Corollary 5.1.2. *The category of sheaves of abelian groups over a topological space is an abelian category such that every object in the category has an injective resolution.*

Proof. Just replace $\mathscr{O}_X$ with the constant sheaf of integers $\mathscr{Z}$ in the above theorem.

Definition 5.1.1. *Let X be a topological space. We can define a functor called the global section functor $\mathcal{G}$ from category of sheaves of abelian groups over X to category of abelian groups as, $\mathcal{G}(\mathscr{F}) = \mathscr{F}(X)$ i.e. the global section of the sheaf $\mathscr{F}$. Since this is a covariant right exact functor, we define the i^{th} cohomology functor $H^i(X, \cdot)$ from the category of sheaves of abelian groups over X to category of abelian groups as the i^{th} right derived functor of $\mathcal{G}$.*

i.e. $H^i(X, \mathscr{F}) = R^i\mathcal{G}(\mathscr{F})$

5.2 A vanishing theorem

The cohomology functors, though very difficult to compute, can contain a lot of information regarding the underlying spaces and sheaves defined on them. As an example consider the following vanishing theorem (due to the works of Grothendieck and Serre) for a noetherian affine scheme and quasi-coherent sheaves defined on it.

Theorem 5.2.1. *Let $(X, \mathscr{O}_X)$ be a noetherian affine scheme such that $X \simeq Spec(R)$, R is a noetherian ring. And $\mathscr{O}_X$-module-sheaf $\mathscr{Q}$ be any quasi-coherent sheaf defined on it. Then the vanishing theorem says $H^i(X, \mathscr{Q})$ vanishes (i.e. equals the zero group)* $\forall\ i > 0$

Proof. (Sketch) We say a sheaf $\mathscr{F}$ on a topological space X is flasque if $\mathscr{F}(V) \to \mathscr{F}(U)$ is surjective $\forall$ open $U \subseteq V \subseteq X$.
The proof proceeds by first showing that the i^{th} cohmology functor vanishes $\forall$ flasque sheaf $\mathscr{F}$ on a topological space X, $\forall i > 0$. Then we show that the sheaf of R injective module is flasque on Spec$R$ for a ring R. The theorem follows.

5.3 Čech cohomology

Since it is nearly impossible to compute cohomology functors using derived functors for sheaves of abelian groups, we introduce a different notion of cohomology that uses an open covering of the underlying space for computing cohomology groups, namely Čech cohomology, which will yield us similar results in certain special cases.

5.3.1 Defining Čech cohomology

Let I be a well ordered indexing set. Suppose a topological space X has an open cover $\mathcal{V} = \{V_i\}_{i \in I}$.
Consider $\sigma_0 < \sigma_1 < \sigma_2 < \cdots \in I$.

For an element $(\sigma_0, \sigma_1, \cdots, \sigma_n) \in I^n$ we write $V_{\sigma_0.\sigma_1.\cdots\sigma_n} = \bigcap_{0\leq i\leq n} V_{\sigma_i}$
Consider a sheaf of abelian groups $\mathscr{A}$ defined on X.
It is possible to define a chain complex $\check{C}^{\bullet}(\mathcal{V}, \mathscr{A})$ in the category of abelian groups as follows.
For $n \geq 0$

$$\check{C}^n(\mathcal{V}, \mathscr{A}) := \prod_{\sigma_0<\cdots<\sigma_n} \mathscr{A}(V_{\sigma_0\cdots\sigma_n})$$

Observe that an element $g \in \check{C}^n(\mathcal{V}, \mathscr{A})$ is given by components $f_{\sigma_0<\cdots<\sigma_n}$ in $\mathscr{A}(V_{\sigma_0\cdots\sigma_n})$ for each component in the product.
Then we define the boundary map $\delta : \check{C}^{n-1}(\mathcal{V}, \mathscr{A}) \to \check{C}^n(\mathcal{V}, \mathscr{A})$ in terms of components as,

$$(\delta f)_{\sigma_0\cdots\sigma_n} = \sum_{j=0}^{n} (-1)^j f_{\sigma_0\cdots\sigma_n\setminus\ \sigma_j}.$$

Where, $\sigma_0\cdots\sigma_n \setminus\ \sigma_j$ mean σ_j is omitted. One can verify with some algebra that δ^2 is zero. i.e. what we defined is in fact a chain complex.

Definition 5.3.1. *For a topological space X with an open cover $\mathcal{V}$, the n-th Čech cohomological group for a sheaf of abelian groups $\mathscr{A}$ is, denoted by $\check{H}^n(\mathcal{V}, \mathscr{A})$, the n^{th} cohomology group of the complex $\check{C}^{\bullet}(\mathcal{V}, \mathscr{A})$.*

Example 9. *To demonstrate the ease of calculating Čech cohomological group compared to right derived functor, we consider the following example.*
Let a constant sheaf of abelian group $\mathscr{A}$ be defined for some abelian group A on the circle S^1 with classical topology. We consider an open cover for the circle $\mathcal{V} = \{V_0, V_1\}$ where both V_0 and V_1 being respectively the upper open semicircular arc and lower open semicircular arc with their ends overlapping slightly.
We have $\check{C}^0 = \mathscr{A}(V_0) \times \mathscr{A}(V_1) = A \times A$
and $\check{C}^1 = \mathscr{A}(V_0 \cap V_1) = A \times A$ (Since $V_0 \cap V_1$ correspond to two small open sets at the edges of overlap)
$\delta(a, b) = (b - a, b - a)$ and therefore we get $\check{H}^0(\mathcal{V}, \mathscr{A}) = A = \check{H}^1(\mathcal{V}, \mathscr{A})$.

Remark 5.3.1. *One can actually show that if X is a noetherian separated scheme with an affine open cover $\mathcal{V}$ and a quasi-coherent sheaf $\mathscr{A}$ defined on it, the n-th cohomology group with right derived functors is isomorphic to the n-th Čech cohomology group. (Refer [Har. p.222])*

Chapter 6

Toric varieties and fans

6.1 Introduction

The following are some of the key notations that we will be using extensively in this chapter.
Let $V \subseteq \mathbb{C}^n$ be an affine variety.
$I(V) := \{\ f \in \mathbb{C}[x_1,\ x_2,....,x_n] \mid f(x) = 0\ \forall\ x \in V\ \}$
$C[V] := \mathbb{C}[x_1,\ x_2,....,x_n]$ modulo $I(V)$
$C(V) :=$ Field of fractions of $C[V]$
Whenever a polynomial f does not vanish anywhere on $C[V]$,
$C[V]_f := \{\ g/f^l \in C(V) \mid g \in C[V],\ l \geq 0\ \}$
Let A be an ideal in $\mathbb{C}[x_1,\ x_2,....,x_n]$,
$Z(A) := \{\ x \in \mathbb{C}^n \mid f(x) = 0\ \forall\ f \in I\ \}$

To put it very crudely, toric varieties are a special class of varieties which have an algebraic torus embedded in them. The nature of the embedding and the behaviour of the torus within the variety is what makes them fascinating objects of study. In this chapter, we will be looking only at toric varieties that stem from algebraic n-tori over the field $\mathbb{C}$. By an affine algebraic n-torus, we mean the set $(\mathbb{C}^*)^n$. It can be understood as the affine open subset (i.e. compliment of an algebraic set in affine-n-space) $(\mathbb{C}^*)^n = \mathbb{C}^n \setminus Z(x_1.x_2...x_n) \subseteq \mathbb{C}^n$. Furthermore, it is in fact an affine variety with coordinate ring $\mathbb{C}[x_1,\ x_2,....,x_n]_{x_1.x_2...x_n}$ $= \mathbb{C}[x^{\pm 1},\ x^{\pm 2},....,x^{\pm n}]$. We call this the Laurent polynomial ring in n variables. $(\mathbb{C}^*)^n$ has a natural group structure under component-wise multiplication. Hence we define an n-torus

T to be an affine variety isomorphic to $(\mathbb{C}^*)^n$ for some $n \in \mathbb{N}$, such that T inherits a group structure from the isomorphism.

6.2 Definition; construction of affine toric varieties

6.2.1 Characters and one parameter subgroups for a torus

Let T be an n torus.

Definition 6.2.1. *A homomorphism between groups $\chi : T \to \mathbb{C}^*$ is defined to be a character of T for any torus T.*

Example 10. *Let $m = (a_1, ..., a_n) \in \mathbb{Z}^n$. Observe that m gives a character $\chi^m : (\mathbb{C}^*)^n \to \mathbb{C}^*$ given by $\chi^m\ (t_1, ..., t_n) = t_1^{a_1} ... t_n^{a_n}$.*

Definition 6.2.2. *A homomorphism between groups $\lambda : \mathbb{C}^* \to T$ is defined to be one-parameter subgroup for any torus T.*

Example 11. *Let $u = (b_1, ..., b_n) \in \mathbb{Z}^n$. Observe that u gives a character $\lambda^u : \mathbb{C}^* \to (\mathbb{C}^*)^n$ given by $\lambda^u\ (t) = (t_1^{b_1}, ..., t_n^{b_n})$*

Theorem 6.2.1. *χ^m is a character of the torus $(\mathbb{C}^*)^n$ iff $m \in \mathbb{Z}^n$. Thus the characters of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$. Similarly λ^u is a one-parameter subgroups of $(\mathbb{C}^*)^n$ iff $u \in \mathbb{Z}^n$. Thus the one-parameter subgroups of $(\mathbb{C}^*)^n$ form a group isomorphic to $\mathbb{Z}^n$.*

Proof. See [Cox, 1 p.11]

Now consider a torus T along with the isomorphism $\mu : T \to (\mathbb{C}^*)^n$. Its characters and one-parameter subgroups form free abelian groups M and N respectively of rank equal to n. μ induces isomorphisms to $\mathbb{Z}^n$ from M and N. In this arbitrary case, we say $m \in M$ gives the character $\chi^m : T \to \mathbb{C}^*$ and $u \in N$ gives a one-parameter subgroup $\lambda^u : \mathbb{C}^* \to T$

Remark 6.2.1. *Let M and N be the 'character' group and the 'one-parameter subgroup' group for a torus T. We can define a bilinear pairing $\langle , \rangle : M \times N \to \mathbb{Z}$ naturally as follows. For a χ^m corresponding to some $m \in M$ and a λ^u corresponding to some $n \in N$, the composition $\chi^m.\lambda^u : \mathbb{C}^* \to \mathbb{C}^*$ is a character of $\mathbb{C}^*$, which is given by $t \mapsto t^k$ for some $k \in \mathbb{Z}$. Then we define,*

$$\langle\ m\ ,\ u\ \rangle := k$$

Moreover, $N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq T$ *through isomorphism* $u \otimes_{\mathbb{Z}} t \mapsto \lambda^u(t)$*. Hence we represent a torus* T *with its one parameter subgroup* N *as* T_N*.*

6.2.2 Affine toric variety

Definition 6.2.3. *An irreducible algebraic affine variety* V *with an embedding* $T_N \leftrightarrow U$*, where* U *is a Zariski open subset of* V *and* $T_N \simeq (\mathbb{C}^*)^n$ *for some* $n \in \mathbb{N}$*, is defined to be an affine toric variety if there is an algebraic action of* T_N *on* V *which is the extension of the natural group action of* T_N *on itself.*

Example 12. *Consider the affine variety* $Y = Z(\ (x^3 \text{ - } y^2)) \subseteq \mathbb{C}^2$ *which has a cusp at the origin. This is an affine toric variety with torus* C^**, where the embedding is* $t \mapsto (t^2, t^3)$*.*

6.2.3 Affine toric variety from lattice

By a **lattice** of rank n we mean a free abelian group isomorphic to $\mathbb{Z}^n$. For example, a torus T_N has lattices M and N.
Let T_N be a torus along with character lattice M.
We have characters $\chi^{m_i} : T_N \to \mathbb{C}^*$ corresponding to elements of an arbitrary finite subset $\mathscr{M} = \{m_1, m_2, ..., m_s\} \subseteq M$. Then consider the map

$\phi_{\mathscr{M}} : T_N \to \mathbb{C}^s$ defined as $\phi_{\mathscr{M}}\ (t) = (\chi^{m_1}(t),\ \chi^{m_2(t)}, ...,\ \chi^{m_s}(t)) \in \mathbb{C}^s$.

Now given a finite subset $\mathscr{M} \subseteq M$ of a lattice, we have an embedding $\phi_{\mathscr{M}}$ of the torus T_N into $\mathbb{C}^s$. Define the affine toric variety $Y_{\mathscr{M}}$ associated with $\mathscr{M}$ for the given torus, as the smallest affine variety containing the image of the map $\phi_{\mathscr{M}}$ (Which is the Zariski closure of the image).

6.2.4 Affine toric variety from affine semigroup

An affine semigroup is a semigroup with the following additional properties.

- The binary operation on S is commutative.

- The semigroup is finitely generated, i.e. there is a finite set $\mathscr{A} \subseteq S$ such that $\mathbb{N}\mathscr{A} = \{ \Sigma_{m \in S} \ n_m \chi^m \mid n_m \in \mathbb{N}$ and $n_m = 0$ for all but finitely many $m \} = S$.

- There exists an embedding of the semigroup into some lattice M.

Example 13. $\mathbb{N}^n \subseteq \mathbb{Z}^n$.

Let $S \subseteq M$ be one such affine semigroup. Then $\exists$ a vector space $\mathbb{C}[S]$ generated over $\mathbb{C}$ via. the basis S. Note that the vector space is also an algebra with multiplication being induced due to the structure of semigroup.
i.e. $\mathbb{C}[S] = \{\Sigma_{m \in S} \ c_m \chi^m \mid c_m \in \mathbb{C}$ and $c_m = 0$ for all but finitely many $m\}$ and
$\chi^m . \chi^{m'} = \chi^{m+m'}$

Definition 6.2.4. *Let $S \subseteq M$ for some lattice M be an affine semigroup. Then we define the affine toric variety associated with S as $\sigma_S = Spec(\mathbb{C}[S])$ (where $\mathbb{C}[S]$ is the semigroup algebra associated with S). Moreover the torus of affine toric variety σ_S has character lattice $\mathbb{Z}S$.*

Remark 6.2.2. *One can easily show that $\mathbb{C}[S]$ is an integral domain and a finitely generated $\mathbb{C}$ algebra. Because $\mathbb{C}[S] \subseteq \mathbb{C}[M] = \mathbb{C}[x^{\pm 1}, x^{\pm 2}, ..., x^{\pm n}]$ for $n = dimM$. Therefore $Spec(\mathbb{C}[S])$ is well defined.*

Example 14. *Let $\{e_1, \ . \ . \ . \ , e_n\}$ be a basis of a lattice M, then M is generated by $\mathscr{A} = \{\pm e_1, \ . \ . \ . \ , \pm e_n\}$ as an affine semigroup. Set $x_i = \chi^{e_i}$ Then $\mathbb{C}[M] = \mathbb{C}[x^{\pm 1}, x^{\pm 2}, ..., x^{\pm n}]$*

Remark 6.2.3. *If $S = \mathbb{N}\mathscr{M}$ for a finite set $\mathscr{M} \subseteq M$, then $\sigma_S = Y_{\mathscr{M}}$.*

6.2.5 Affine toric varieties from cones

Let N and M be lattices of same rank with a natural bilinear pairing $\langle , \rangle : M \times N \to \mathbb{Z}$. then we can define a real vector space $V = N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ with its dual space $V^* = M_{\mathbb{R}}$.

Definition 6.2.5. *The set of all points $\sigma = \{ \Sigma \ r_i v_i \mid r_i \geq 0, \ r_i \in \mathbb{R} \}$ is said to be a convex polyhedral cone in V when it is generated by a finite set of vectors $\{v_1, v_2, ..., v_s\}$ in V.
A convex polyhedral cone in V is said to be rational when $v_i \in N$ for every i.*

Let σ be convex rational polyhedral cone in V. The set of all points in V^* given as $\sigma^* = \{u \in V^* \mid \langle u, v \rangle \geq 0 \text{ for all } v \text{ in } \sigma\}$ is defined to be the dual of σ as a cone in V^*.

Definition 6.2.6. *A convex rational polyhedral cone σ in V is strongly convex when σ^* spans V^*.*

Lemma 6.2.2. *Gordon's lemma : If σ is a rational polyhedral cone in V, then $S_\sigma = \sigma^* \cap M$ is a finitely generated semigroup.*

Proof. See [Ful p.12]

Now we define the **affine toric variety associated with the cone** σ as $U_\sigma = \text{Spec}(\mathbb{C}[S_\sigma])$

6.3 A Dictionary

To gain a better understanding of the relations, we can create a 'dictionary' between geometry of affine toric variety, algebra of coordinate ring and combinatorics of affine semigroup.

Affine toric variety V	Coordinate ring $C[V]$	semigroup S
Point	Maximal ideal	Semigroup morphism $S \to \mathbb{C}$
Morphism $V \to V'$	Morphism $C[V'] \to C[V]$	Semigroup morphism $S' \to S$

Remark 6.3.1. *The correspondence between points of affine variety and algebra of coordinate ring is evident from scheme theory. We have already established the derivation of affine toric varieties from semigroups. Now observe that every morphism from $S \to \mathbb{C}$ induces a morphism of algebras $\mathbb{C}[S] \to \mathbb{C}$. Which in turn induces a morphism spec $\mathbb{C} \to$ spec $(\mathbb{C}[S])$. But spec $\mathbb{C}$ is only a point. Therefore a morphism $S \to \mathbb{C}$ correspond to a point in spec$(\mathbb{C}[S])$. To see the other way implication, let x be a point in an affine toric variety. We define a morphism $\overline{x} : S \to \mathbb{C}$ corresponding to x as follows.*
For some $u \in S$, $\overline{x}(u) = \chi^u(x)$, where χ^u is the character corresponding to u.

6.4 Fans and toric varieties

A toric variety is a variety which is locally isomorphic to affine toic varieties.

6.4.1 Fan

A fan Δ in a lattice N is a collection of strongly convex rational polyhedral cones in $N_{\mathbb{R}}$ subjected to the conditions,

• Every face of every cone is again a cone in Δ

• The intersection of any two cones in Δ happens at a common face of both cones.

In this chapter we assume only fans with finitely many cones even though there is no such restriction.

6.4.2 Construction

We have seen that a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ gives an affine toric variety. So every cone in a fan gives one affine toric variety each. We need patching morphisms to glue these affine varieties together to obtain a toric variety. Observe that the intersection of two cones in a fan is a face of both the cones and the face is a cone by definition. The inclusion morphism of this common face to each cone induces a subgroup morphism in the reverse direction and therefore correspond to inclusion morphisms of points according to the dictionary we defined. We use these inclusion morphisms to patch the affine toric varieties to obtain the toric variety.

6.4.3 An example

Let N be isomorphic to $\mathbb{Z}$. Consider the Fan Δ generated by 1 and -1. i.e. $\Delta \simeq \mathbb{R}$.

$$\mathbb{C}[x^{-1}] \hookrightarrow \mathbb{C}[x, x^{-1}] \hookleftarrow \mathbb{C}[x]$$

$$\mathbb{C} \hookleftarrow \mathbb{C}^* \hookrightarrow \mathbb{C}$$

The inclusion map from the common face $\{0\}$ to the cones $\mathbb{R} \geq 0$ and $\mathbb{R} \leq 0$ induces maps between corresponding algebras, providing the patching morphism $a \mapsto a^{-1}$ to glue the affine toric varieties $\mathbb{C}$ (corresponding to the cone generated by 1) and $\mathbb{C}$ (corresponding

to the cone generated by -1) along $\mathbb{C}^*$.
And yields the toric variety $\mathbb{C} \times \mathbb{C}$ modulo ($a \mapsto a^{-1}$ along $\mathbb{C}^*$) $\simeq \mathbb{P}^1$

6.5 Torus invariant divisors

In this section, our aim is to look at divisors that are mapped to themselves under the torus action. i.e. torus invariant divisors. We only look at toric varieties that are generated from a fan.

6.5.1 Fans and orbits

Let Δ be a fan and $T(\Delta)$ be its toric variety.

Theorem 6.5.1. Orbit – cone correspondence theorem *: For a cone σ of dimension k in Δ, the orbit corresponding to it under the action of the torus T_N is $O(\sigma) \simeq (\mathbb{C}^*)^{n-k}$. We define the closure of this orbit in X by $V(\sigma)$. $V(\sigma)$ is a closed subvariety of X.*

Proof. This follows directly from the fact that a k dimensional cone is generated by exactly k many basis vectors and its dual by $n-k$ many basis vectors. Hence the resulting toric variety, which is a closed subvariety of X has an $n-k$ dimensional torus as its dense torus.

6.5.2 Torus invariant divisors

Let Δ be a fan and $T(\Delta)$ be its toric variety.

Recall that subvarieties, integral and closed, which heve codimension 1 generate weil divisors in $T(\Delta)$. Since we are considering torus invariant weil divisors (T-weil divisors), we can invoke orbit-cone correspondence theorem and see that torus invariant subvarieties of codimension 1 in $T(\Delta)$ are given by cones of dimension 1 in Δ. A cone of dimension 1 is a ray edge of the fan. Let $\{\sigma_i\}$ denote the set of all ray edges in Δ. Then a T-weil divisor is of the form

$D_T = \Sigma_i n_i.V(\sigma_i)$, $n_i \in \mathbb{Z}$ and $V(\sigma)$ is the a closed subvariety of X corresponding to the cone σ.

Chapter 7

Conclusion

At this point, algebraic geometry reminds me of the metaphor of 'Akshayapaatra' from 'Mahabharata' which is perhaps true for any branch of science. Similar to how 'Akshayapaatra' is a never ending source of nourishment, there always seem to be a heap of knowledge left to imbibe once I finish learning some concept. I will go to the lengths of sharing my feeling that I barely even scratched the surface of a branch of mathematics that is so immense and fascinating.

While trying to apply the bit of scheme theory and cohomology that I learned on toric varieties, I realised the importance of having to be extremely familiar with these techniques. It was a realisation that learning theorems and doing computaion differs exactly to the degree of knowing something and understanding something. So in the remaining period I will strive to widen my understanding of the already read concepts. Especially cohomological techniques and theory.

If time permits, I would also like to read a bit on triangulated categories and derived categories. The prospect of considering chain complexes as objects in a category and defining a cohomology on complexes of complexes seems fascinating. Intuitively it makes perfect sense, because chain complexes contain more information about the underlying space than the cohomological groups we define.

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