Functorial Knot Theory

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Functorial Knot Theory towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Visakh Narayanan at The Indian Institute of Science Education and Research, Pune, under the supervision of Dr. Rama Mishra, Associate Professor, Department of Mathematics, during the academic year 2016-2017.

Rama Mishra
29/03/2017

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This thesis is dedicated to nobody

Declaration

I hereby declare that the matter embodied in the report entitled Functorial Knot Theory are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, under the supervision of Dr. Rama Mishra and the same has not been submitted elsewhere for any other degree.

29/03/2017

Visakh Narayanan

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Abstract

The thesis titled “Functorial Knot Theory”, expository in nature, aims to show some recent connections between Knot theory and Category theory. In the first part we discuss how the language of categories and functors effectively enhances the development of new knot invariants such as Jones polynomial and Khovanov homology. The second part demonstrates the surprising application of knot theory in proving some important coherence theorems in category theory.

Contents

Introduction

Knot theory studies topological embeddings of S^1 in $\mathbb{R}^3$, which are called knots. The central problem in knot theory is to classify all knots upto ambient isotopy. As a result knot invariants came into the picture. A knot invariant is a rule of attaching certain kind of mathematical objects to knots in such a way that the attachment is same for equivalent knots. The mathematics of knot theory is the study of different kinds of knot invariants. Modern knot theory involves combinatorial, algebraic and topological invariants.

The introduction of category theory has revolutionized mathematics and the way in which it is percieved. One observation is that, the idea of categories and functors was motivated by the study of certain algebraic invariants of topological spaces like homology. Later many invariants of different objects were discovered to possses such special properties. In the language of categories we call such invariants as functors. Functors could be seen as the central objects of study in category theory. Definition of a category is chosen in a way which makes the definition of a functor easier. This opens up a new way of looking at mathematical objects.

Classical knot theory can be regarded as a study of numerical and polynomial invariants of knots. Among these, the polynomial invariant constucted by Vaughan Jones, which is named after him as the Jones polynomial is the most interesting one, mainly because of its ease of calculation and its beautiful characteristics. There are many different ways of constructing it. Mikhail Khovanov later introduced a graded (co)homology theory for knots, which also has its graded euler characteristic as the Jones polynomial. Because of this we can also regard it as a functorialization of the Jones polynomial. Here the chain groups are graded and hence the complex and corresponding homology are bigraded. It is called as, the

Khovanov homology, after him. Whenever the phrase 'homology' is used it implicitly refers to the functoriality also. Now one can construct a category where objects are links and an arrow between two given links is a compact orientable 2-manifold with boundary such that the boundary is the disjoint union of the two links. Khovanov's construction also includes computation of a chain map between the chain complexes of two links corresponding to any such cobordism between them.

Bringing category theory into knot theory has enhanced the clarity with which the subject was understood. Surprisingly the converse is also being explored. Certain theorems classified as coherence theorems can be proved using knot theoremtic techniques. The goal of this project is to explore most of the topics discussed above in good depth and rigor.

Chapter 1

Basic Knot theory

Definition 1.0.1. *A knot is an ambient isotopy class of an embedding of S^1 in S^3 or $\mathbb{R}^3$.*

That is two such embeddings are considered to be the same knot if they are ambient isotopic in S^3. Isotopy classes of embeddings of finitely many copies of S^1 are called links. It is a common convention to refer to the embedding also as knot (link). From now on the isotopy class and the embedding would be represented by the same word. In light of the classification theorem for 1 manifolds, a link is a compact orientable 1-dimensional submanifold of S^3. In other words, two links K_1 and K_2 are considered equivalent iff there exists and orientation preserving homeomorphism $h : S^3 \to S^3$ which is isotopic to identity (homotopy via homeomorphisms) and $h(K_1) = K_2$. The set of all links will be denoted by $\mathscr{L}$. Though we define knots to be any embeddings of S^1, we usually consider only the ones which are atleast C^1. The Figure 1.1 shows examples of embeddings which are not C^1 which are clearly not "models" of any "real" knot. Such knots are called ***wild knots***. Knots corresponding to embeddings which are atleast C^1 are called *tame knots.*

Remark 1.0.1. *It can be shown that every PL (piecewise linear) embedding is isotopic to a C^1 embedding, and every C^1 is equivalent PL embedding. Hence both these are equivalent knot types.*

For different purposes we use different kind of representations. PL representations usually show up when the combinatorial properties are to be studied.

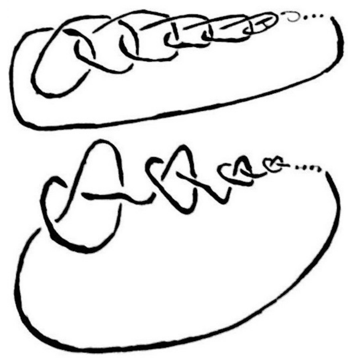


Figure 1.1: wild knots

Figure 1.2: A diagram of trefoil knot.

There are many ways to represent a knot. A projection of a knot to some plane in $\mathbb{R}^3$ is said to be regular, it the only transversal double points as singularities.

Definition 1.0.2. *A regular projection together with a information of over/under passes at every double point is called a knot (link) diagram.*

And it can be shown that every tame knot has a diagram. Hence all tame knots can be studied by considering their diagrams. But more than one diagrams can represent the same knot. Reidemeister solved this problem by introducing a set of three moves on knot diagrams, such that, two diagrams represent equivalent knots if and only if one can be obtained from the other by a finite sequence of Reidemeister's moves. These moves are shown in Figure 1.3.

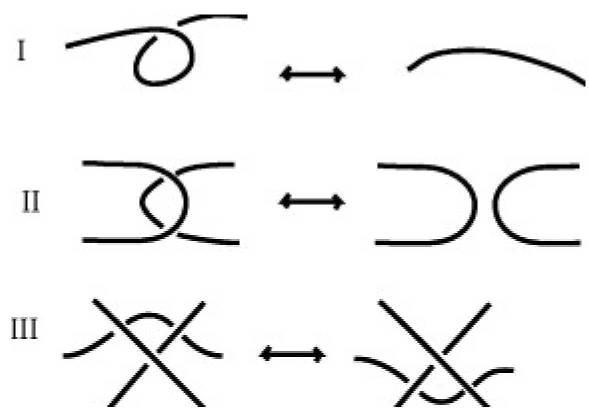


Figure 1.3: Reidemeister moves

Chapter 2

The Jones Polynomial

In 1984, Vaughan Jones discovered a polynomial invariant for knots, which got named after him as the Jones polynomial. It is one of the most celebrated knot invariants. Later several ways to arrive at the polynomial was discovered. The construction originally done by Jones is more algebraic in nature. Louis Kauffman constructed a purely combinatorial model. Here we will present one algebraic and one combinatorial model of the Jones polynomial.

2.1 Via representations of braid groups over Hecke algebras

This section will contain an algebraic construction of the Jones polynomial. We will start from a sequence of groups. Each knot type determines some representation of these. And trace of the representation will correspond to the Jones polynomial.

2.1.1 Braid groups

An embedding of the standard interval I will be refered to as a string connecting its two boundary points. Consider two parallel lines in $3-\textit{space}$, and n equidistant points on each of them. Now consider a disjoint collection of n strings connecting these points

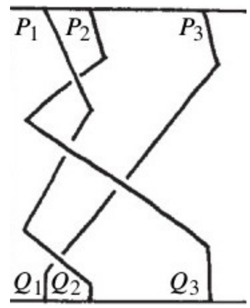


Figure 2.1: Diagram of a 3-braid.

such that the height function in the direction of the lines is monotonic. The coordinate in the perpendicular direction of the lines will be called as height coordinate. This condition ensures that points on one line gets connected only to points on the other line. This whole collection with a choice of one of the lines as initial, is called an $n - braid$.

Two of them will be identified if they are isotopic by a map which respects all these structures. I.e, which sends lines to lines, points to points and strings to strings. Braids are also represented by diagrams which are generic projections of a braid, which has only transversal double points as singularities. Then we have a set of moves defined on diagrams which characterize the isotopy of braids. If we consider the PL version of braids, then the isotopy is characterized by the rule of replacing one side of a triangle with the union of other two, called as Δ-move. The inverse of this operation is also regarded as a Δ-move. In the set of all isotopy classes of n-braids, there is a natural binary operation. Given two classes choose one representative from each and identify the initial line of the last with final line of the first, in such a way that the points are also identified in order. Hence after identification if we remove the line in the middle keeping the points, then this again results in an n-braid. The isotopy class of this braid is defined to be the product of the two classes.

This binary operation has an identity element which is given by the identity braid. Any element multiplied with its mirror image is isotopic to the identity braid. Hence every class is invertible. That is the isotopy classes of n-braids form a group under their natural multiplication. This group is called the braid group, denoted by $\mathscr{B}_n$.

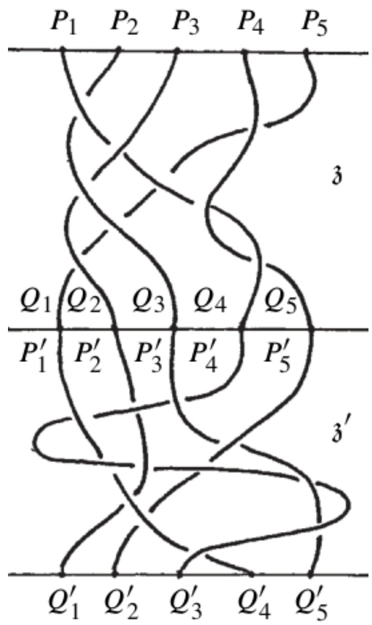


Figure 2.2: Multiplication of braids

It is easy to see that $\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\}$ generate $\mathscr{B}_n$ since given any braid diagram we can always deform it through Δ-moves (in the PL-category) in such a way that there is only one crossing at any given height. And since there are only finitely many crossings, this braid is evidently written as a product of σ_i's. From similar topological arguments it is easily seen

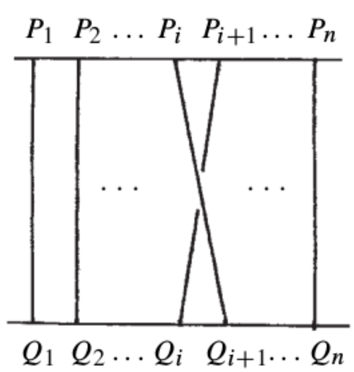


Figure 2.3: The i^{th} generator (σ_i) of $\mathscr{B}_n$

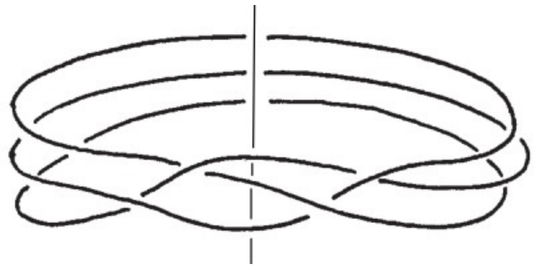


Figure 2.4: Closure of the braid shown in Figure 2.1

that σ_i's satisfy the following relations.

$$\sigma_i\sigma_j = \sigma_j\sigma_i,\ |i-j| \geq 2$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

These two relations determines the group structure. I.e, we have a presentation,

$$\mathscr{B}_n = \langle\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\} \mid R \cup S\rangle$$

$$R = \{\sigma_i\sigma_j\sigma_i^{-1}\sigma_j^{-1} \mid |i-j| \geq 2\}$$

$$S = \{\sigma_i\sigma_{i+1}\sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1}^{-1} \mid 0 < i < n-1\}$$

Note: If $T = \{\sigma_i^2 | \forall 1 \leq i \leq n\}$ then $\langle\{\sigma_1, \sigma_2, \cdots, \sigma_{n-1}\} \mid R \cup S \cup T\rangle$ is a presentation of S_n where each σ_i correspond to the transposition $(i, i+1)$.

Given any braid the i^{th} point on one line can be connected to the i^{th} point on the other line by an new (unknotted) string which doesn't intertwine between any other strings under consideration. If we connect all the points in this fashion, it gives a link. The link obtained is said to be the *closure* of the corresponding braid. For example this operation on the identity n-braid will result in the unlink with n components.

Alexander proved that every link is isotopic to the closure of some braid. Thus braids also represent links. But different braids might represent the same link type. So how to determine whether two braids represent isotopic links? This question was answered by Markov in 1936 through his classic theorem. If we conjugate an n-braid with another n-braid then clearly both of them will have the same closure. Now given any n-braid attaching a trivial string

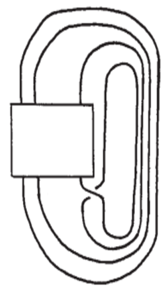


Figure 2.5: Closure after performing a Markov move

at the end and multiplying with either σ_n or σ_n^{-1} will give an $n+1$-braid. This defines a unary operation on the set of all braids which is called as a Markov move. Some times for compactness of writing conjugation will also be refered to as a Markov move.

Figure 2.5 depicts the idea of a Markov move. The empty rectangle denotes a general braid. Clearly by performing the first Riedemeister move on this link, the closure of the initial braid can be obtained. Two braids β_1 and β_2 are said to be Markov equivalent iff there exist a finite sequence $\beta_1 = \alpha_1, \alpha_2, \cdots, \alpha_n = \beta_2$ of braids such that for each i, α_{i+1} can be obtained from α_i by either conjugation with some braid or a Markov move. This defines an equivalence relation in the set of all braids. Markov also proved that the equivalence classes under this relation exactly determines class of all braids with isotopic closures. That is,

Theorem 2.1.1 (Markov)**.** *The closures of two braids are of the link type if and only if they are Markov equivalent.*

Hence if we construct an invariant for braids under Markov moves, then it is equivalent to constructing a link invariant. Vaughan Jones originally constructed some representations of braid groups over certain matrix algerbras such that the trace (upto some scaling) is invariant under Markov moves. Adrian Ocneanu observed that matrix algebra used in Jones' construction is a quotient of some Hecke algebra and the trace function can be extended to this bigger algebra. Thereby, he could come up with a model which is easier to study. Here this invariant will be presented in the language of *skein theory*.

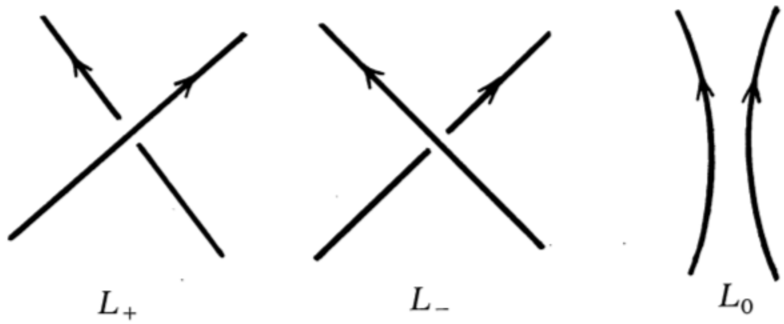


Figure 2.6: Skein related diagrams

2.2 Jones polynomial as a skein invariant

Three oriented link diagrams L_+, L_- and L_0 are said to be skein related iff they look the same outside a ball around a point on the plane and inside the ball they look as shown in Figure 2.6.

Let R be a commutative ring with 1. Consider three units a_+, a_- and a_0 in R. A map $\psi : \mathscr{L} \to R$ is said to be a ***skein invariant*** with skein coefficients a_+, a_- and a_0 iff the following relations hold.

1. $\psi(\bigcirc) = 1$
2. For every skein related diagrams L_+, L_- and L_0:

$$a_+\psi(L_+) + a_-\psi(L_-) + a_0\psi(L_0) = 0$$

The goal of this section is to construct the Jones polynomial as a skein invariant. Besides the construction of Jones polynomial, it will also be a quick glimpse to the theory of skein invariants.

Theorem 2.2.1 (Uniqueness)**.** *A skein invariant $p : \mathscr{L} \to R$ is uniquely determined by its coefficients. I.e, given an ordered 3-tuple of units in R there is atmost one skein invariant with these as coefficients.*

Figure 2.7:

Proof: Suppose $p : \mathscr{L} \to R$ is a skein invariant with coefficients a_+, a_- and a_0. Then by definition $p(\bigcirc) = 1$. Figure 2.7 shows the simplest skein triplets L_+, L_- and L_0. Both L_+ and L_- are isotopic to the unknot $\bigcirc$. Hence p maps both of them to 1. Also L_0 is $\bigcirc^2$, the unlink with 2 components. Then clearly,

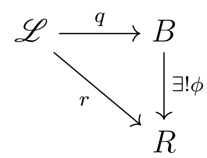
$$p(\bigcirc^2) = -(\frac{a_+ + a_-}{a_0}).$$

By the same trick it is easily shown that,

$$p(\bigcirc^r) = (-(\frac{a_+ + a_-}{a_0}))^{r-1}.$$

Hence the coefficients completely determine the invariant on all the unlinks. Consider an oriented link diagram and a point on one of its components. Start moving through the diagram following the orientation and change all the crossings so that newly visited crossings are always entered through the lower passing. After this inversion of crossings clearly starting from the point we can glue the boundary of a disk on to this component. That is it is an unknot. By this argument it is easy to see that, given any link diagram D, there is a choice of crossing changes so that the diagram changes to an unlink. And each of these transitions taken one by one passes through several stages of skein related diagrams. And the final stage is an unlink. Since p is determined on unlinks, by induction of the same kind as above the invariant can be calculated for the initial diagram by tracing these steps back. (The complexity of this process depends on the number of components and crossings of D. Hence writing a general expression is cumbersome.) Hence p is completely determined by its coefficients and we are done!

The theorem we just proved has very strong implications and it will prove to be inevitable in the rest of the theory. Let $B = \mathbb{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$ be the ring of Laurent polynomials in three variables over $\mathbb{Z}$. Suppose there exist a skein invariant $q : \mathscr{L} \to B$ with coefficients x, y and z. Let $r : \mathscr{L} \to R$ be any skein invariant with coefficients a_+, a_- and a_0. Then there is a unique natural map $\phi : B \to R$ given by $x \mapsto a_+$, $y \mapsto a_-$ and $z \mapsto a_0$ which makes the following diagram commutative:



Hence such a q, if it exists, can be treated as a "universal" member in the family of all skein invariants. Once we construct this invariant then it is trivial to construct any other skein invariant if the required coefficients are known. Note that here we are strongly using the uniqueness theorem proved above. There are many ways to construct this invariant. The following is one of them. For the remaining discussion A will denote the ring $\mathbb{Z}[l, l^{-1}, m, m^-1]$ of Laurent polynomials in two variables l and m (for Lickorish and Millet) over $\mathbb{Z}$.

Suppose there exist a skein invariant $p : \mathscr{L} \to A$ with coefficients l, l^{-1} and m respectively. Let K be a link diagram. It can be shown that if $l^a m^b$ is any monomial which appears in the polynomial $p(K)(l, m)$, then $a \equiv b(mod 2)$. This is easily proved by observing it on unlinks and then proceeding by induction. Thus there exist integers $i := \frac{a-b}{2}$, $j := -\frac{a+b}{2}$ and $k := b$. Let $q(K)(x, y, z)$ be the unique polynomial obtained by replacing each monomial $l^a m^b$ in $p(K)$ by $x^i y^j z^k$. Now $q(K) \in B$ is a homgeneous polynomial of degree 0. Now q defines a map $\mathscr{L} \to B$ by composing with p. Note that for any link K,

$$q(K)(x, y, z) = p(K)((\frac{x}{y})^{\frac{1}{2}}, z(xy)^{-\frac{1}{2}}).$$

Since p is skein invariant, $q(\bigcirc) = 1$. Let L_+, L_- and L_0 be three skein related diagrams. By definition of p as a skein invariant we have:

$$lp(L_+) + l^{-1}p(L_-) + mp(L_0) = 0$$

$$\implies (\frac{x}{y})^{\frac{1}{2}} q(L_+) + (\frac{y}{x})^{\frac{1}{2}} q(L_-) + z(xy)^{-\frac{1}{2}} q(L_0) = 0$$

multiplying both sides by $(xy)^{\frac{1}{2}}$ we have,

$$xq(L_+) + yq(L_-) + zq(L_0) = 0.$$

Again by the uniqueness theorem, q that we constructed out of p is the skein invariant described above. Hence constructing such a p does the job. From now on, p will be regarded as the **"universal skein invariant"**.

2.3 Construction of the universal skein invariant

Let F be a field and $q \in F$ be any element. Then the n^{th} **Hecke algebra** over F associated to q denoted by $H(n, q)$ can be defined as the unital F-algebra generated by $T_1, T_2, \cdots, T_{n-1}$ with the relations:

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \forall i$$
$$T_i T_j = T_j T_i,\ |i - j| > 1$$
$$T_i^2 = (q - 1)T_i + q, \forall i.$$

From the presentation of S_n discussed earlier, $H(n, 1)$ is isomorphic to the group algebra of S_n. It can be shown that $H(n, q)$ is an $n!$-dimensional vector space over F. Considering this sometimes $H(n, q)$ is said to be the q-deformation of the group algebra of S_n. If q is clear from the context, for simplicity, $H(n, q)$ will sometimes be written as H_n. Fix a $q \in F$. The third relation guarantees that each of T_i's are invertible. Hence multiplication by T_i is a vector space automorphism of H_n. By abusing notation this automorphism will also be denoted as T_i. In this notation $T_i \in Aut(H_n)$. From the definition, the inclusion $H_n \hookrightarrow H_{n+1}$ must be clear. And hence H_{n+1} is an (H_n, H_n)-bimodule.

Consider $H_n \oplus (H_n \otimes_{H_{n-1}} H_n)$ as an (H_n, H_n) bimodule. Every element of this module has a form $a + \sum_j b_j \otimes c_j$ where a, b_j's and c_j's are elements of H_n. Then there is a bimodule map

$$\varphi : H_n \oplus (H_n \otimes_{H_{n-1}} H_n) \to H_{n+1}$$

defined by

$$a + \sum_j b_j \otimes c_j \mapsto a + \sum_j b_j T_n c_j.$$

If $u \in H_{n-1}$ then we know that $bu \otimes c = b \otimes uc$. So for φ to be a well defined map these two should have the same image. Since $u \in H_{n-1}$, it is expressed as product (sum of products) of $T_1, T_2, \cdots, T_{n-2}$. By definition T_n commutes with all of these and hence $uT_n = T_n u$. And thus φ is a well defined map. Infact this indicates the naturality behind replacing the tensor symbol $\otimes_{H_{n-1}}$ with T_n. With some simple calculations [**3**] we can show that:

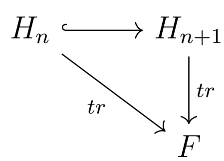
Theorem 2.3.1 (Structure theorem)**.** *φ is an isomorphism of bimodules. I.e,*

$$H_{n+1} \cong H_n \oplus (H_n \otimes_{H_{n-1}} H_n).$$

This theorem has many implications. It shows how Hecke algebras of different order are related. Thereby it provides a way to use induction on the sequence of algebras. In the following this fact is used extensively in proving existence of a function.

Theorem 2.3.2. *Let $z \in F$ be any element. Then for every n there exists a map $tr : H_n \to F$ which satisfy:*

1. *The following diagram commutes:*



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2. *$tr(1) = 1$.*

3. *tr is F-linear and $tr(ab) = tr(ba),\ \forall a, b \in H_n$.*

4. *If $a, b \in H_n$ then $tr(aT_n b) = z tr(ab)$.*

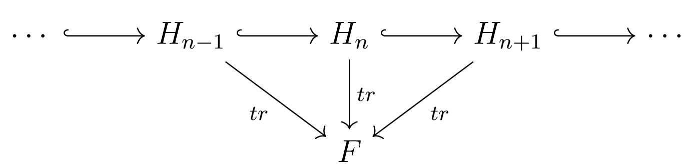
This map on H_n will be called the *trace* corresponding to z.
Proof: Note that $H_1 = F$. The map $tr : H_1 \to F$ is defined to be the identity map of F.

Clearly it satisfies the 2 and 3. Now this function can be extended to higher algebras by induction, in a way such that all the properties are satisfied. Suppose tr is defined on H_n. By the structure theorem, every element in H_{n+1} can be written as $a + \sum_j b_j T_n c_j$ for a, b_j and c_j are in H_n for each j. Now define $tr : H_{n+1} \to F$ by:

$$tr(a + \sum_j b_j T_n c_j) = tr(a) + \sum_j ztr(b_j c_j).$$

Clearly tr satisfies $1, 2$ and 4. Also it is F-linear. It remains to show that $tr(ab) = tr(ba)$ for every $a, b \in H_{n+1}$. This can be proved case by case by considering whether a or b can be expressed as a product (sum of products) containing T_n or not[**3**]. Hence $tr : H_{n+1} \to F$ satisfies all the required properties and the proof is complete.

Thus there exists a $tr : H_n \to F$ for each n. From 1 the following diagram commutes:



Hence this induces a unique trace map on the direct limit of H_n's. As discussed earlier each T_i in H_n is invertible and correspond to an automorphism of H_n. If the third axiom of H_n is not considered, then clearly the algebra generated, say G_n would be isomorphic to the group algebra of $\mathscr{B}_n$. There is a canonical representation of $\mathscr{B}_n$ on G_n. After quotienting G_n with the third relation H_n is obtained. Composing the canonical representation with this quotient map yields a representation of $\mathscr{B}_n$ over H_n. Hence the natural group homomorphism:

$$\rho : \mathscr{B}_n \to Aut(H_n)$$

$$\sigma_i \mapsto T_i$$

is a representation of the n-braid group on the n^{th} Hecke algebra. Note that this doesn't depend on the underlying field. Since the image of ρ can be identified as a subset of H_n in a natural way, ρ can be composed with the trace map. Hence each braid is assosiated to an element in F, called its trace. This is an invariant for braids under isotopy. But our goal is invariance under Markov equivalence. It turns out that for a clever choice of a field F this works out. For this it is convinient to set the following notations.

Let $F' = \mathbb{C}(q, z)$ be the field of all rational functions in two variables q and z with coefficients in $\mathbb{C}$. Let $w = 1 - q + z$ and define:

$$F = F'(\sqrt{\frac{q}{zw}})$$

. Let H_n denote the n^{th} Hecke algebra $H(n, q)$ corresponding to $q \in F$ and tr denote the trace assosiated to $z \in F$. Let ρ be the representation of $\mathscr{B}_n$ on H_n. Let $e : \mathscr{B}_n \to \mathbb{Z}$ be the group homomorphism $\sigma_i \mapsto 1$. Let $\mathscr{B} = \amalg_n \mathscr{B}_n$ and let $n : \mathscr{B} \to \mathbb{N}$ the unique function such that $n(\mathscr{B}_n) = \{n\}$. Now for a braid α define:

$$V_\alpha(q, z) = (\frac{1}{z})^{\frac{n(\alpha)+e(\alpha)-1}{2}} (\frac{q}{w})^{\frac{n(\alpha)-e(\alpha)-1}{2}} tr(\rho(\alpha))$$

This defines a map $V : \mathscr{B} \to F$ defined by $\alpha \mapsto V_\alpha(q, z)$. The variables q and z in the paranthesis is to indicate that this quantity is an expression written in terms of these variables.

Theorem 2.3.3. *The map $V : \mathscr{B} \to F$ is an invariant of Markov equivalence classes.*

Proof: It is enough to varify invariance under conjugation and Markov moves. Suppose $\alpha, \gamma \in \mathscr{B}_n$. Let $\beta = \gamma\alpha\gamma^-1$. From definition $n(\alpha) = n(\beta) = n$ and $e(\alpha) = e(\beta)$ since $\mathbb{Z}$ is abelian.

$$\begin{aligned} tr(\rho(\beta)) &= tr(\rho(\gamma)\rho(\alpha)\rho(\gamma^{-1})) \\ &= tr(\rho(\gamma^{-1})\rho(\gamma)\rho(\alpha)) \\ &= tr(\rho(\alpha)) \end{aligned}$$

And hence $V_\alpha(q, z) = V_\beta(q, z)$ and V is invariant under conjugation.

By abusing notation let α denote the image of the above α under the canonical injection $\mathscr{B}_n \hookrightarrow \mathscr{B}_{n+1}$. Then define $\delta := \alpha\sigma_n$ and $\delta' := \alpha\sigma_n^{-1}$. Clearly δ and δ' are obtained by performing Markov moves on $\alpha \in \mathscr{B}_n$. Note that $n(\alpha) = n$ and thus $n(\delta) = n(\delta') = n + 1$. Also $e(\delta) = e + 1$ and $e(\delta') = e - 1$ where $e = e(\alpha)$.

$$\begin{aligned} tr(\rho(\delta)) &= tr(\rho(\alpha)\rho(\sigma_n)) \\ &= tr(\rho(\alpha)T_n) \\ &= ztr(\rho(\alpha)) \end{aligned}$$

$$\begin{aligned} \implies V_\delta(q,z) &= (\frac{1}{z})^{\frac{n(\delta)+e(\delta)-1}{2}}(\frac{q}{w})^{\frac{n(\delta)-e(\delta)-1}{2}}tr(\rho(\delta)) \\ &= (\frac{1}{z})^{\frac{n+1+e+1-1}{2}}(\frac{q}{w})^{\frac{n+1-e-1-1}{2}}ztr(\rho(\alpha)) \\ &= \frac{1}{z}(\frac{1}{z})^{\frac{n+e-1}{2}}(\frac{q}{w})^{\frac{n-e-1}{2}}ztr(\rho(\alpha)) \\ &= (\frac{1}{z})^{\frac{n+e-1}{2}}(\frac{q}{w})^{\frac{n-e-1}{2}}tr(\rho(\alpha)) \\ &= V_\alpha(q,z) \end{aligned}$$

Also observe that $T_n^{-1} = \frac{1}{q}(T_n + 1 - q)$. Thus:

$$\begin{aligned} tr(\rho(\delta')) &= tr(\rho(\alpha)\rho(\sigma_n^{-1})) \\ &= tr(\rho(\alpha)T_n^{-1}) \\ &= tr(\rho(\alpha)\frac{1}{q}(T_n + 1 - q)) \\ &= \frac{1}{q}[ztr(\rho(\alpha)) + (1-q)tr(\rho(\alpha))] \\ &= \frac{1}{q}[(1-q+z)tr(\rho(\alpha))] \\ &= \frac{w}{q}tr(\rho(\alpha)) \end{aligned}$$

$$\begin{aligned} \implies V_{\delta'}(q,z) &= (\frac{1}{z})^{\frac{n(\delta')+e(\delta')-1}{2}}(\frac{q}{w})^{\frac{n(\delta')-e(\delta')-1}{2}}tr(\rho(\delta')) \\ &= \frac{q}{w}(\frac{1}{z})^{\frac{n+e-1}{2}}(\frac{q}{w})^{\frac{n-e-1}{2}}\frac{w}{q}tr(\rho(\alpha)) \\ &= (\frac{1}{z})^{\frac{n+e-1}{2}}(\frac{q}{w})^{\frac{n-e-1}{2}}tr(\rho(\alpha)) \\ &= V_\alpha(q,z) \end{aligned}$$

This shows that V is invariant under all the Markov moves and hence it is invariant of Markov equivalence classes. That is V does not depend on the braid but only on the isotopy class of its closure link. Thus we have constructed a link invariant.

Theorem 2.3.4. *Given any link K, there exists a unique Laurent polynomial $p_K(l,m) \in A$ such that*

$$p_K(i(\frac{z}{w})^{\frac{1}{2}}, i(q^{-\frac{1}{2}} - q^{\frac{1}{2}})) = V_\alpha(q,z)$$

whenever α is any braid whose closure is isotopic to K. Morover $K \mapsto p_K(l,m)$ is a skein invariant with coefficients $l, l^{-1}, m \in A$.

Proof: Note that p_K is well defined since $V_\alpha(q,z)$ doesn't depend on the choice of α by the previous theorem. It is easy to see that whenever L_+, L_- and L_0 are three skein related diagrams, there exist $\gamma, \beta \in \mathscr{B}_n$ such that they are isotopic to closures of the braids $\alpha_+ := \gamma\sigma_k\beta$, $\alpha_- := \gamma\sigma_k^{-1}\beta$ and $\alpha_0 = \gamma\beta$ respectively. For further discussions it is convenient to define the map W by:

$$W_\alpha = (\frac{1}{z})^{\frac{n(\alpha)+e(\alpha)-1}{2}}(\frac{q}{w})^{\frac{n(\alpha)-e(\alpha)-1}{2}}\rho(\alpha)$$

which assosiates an element of H_n to a braid $\alpha \in \mathscr{B}_n$ such that $tr(W_\alpha) = V_\alpha(q,z)$. For proving this theorem we need the following:

Lemma 2.3.5 (Skein invariance lemma)**.** *If α_+, α_- and α_0 as as defined above and $l = i(\frac{z}{w})^{\frac{1}{2}}$ and $m = i(q^{-\frac{1}{2}} - q^{\frac{1}{2}})$ then:*

$$lW_{\alpha_+} + l^{-1}W_{\alpha_-} + mW_{\alpha_0} = 0$$

The lemma can be proved by straight forward calculations by substituting all the symbols in the L.H.S. by their definitions. Hence the proof may be skipped. By taking trace on both sides of the equation in the lemma, we obtain:

$$lV_{\alpha_+} + l^{-1}V_{\alpha_-} + mV_{\alpha_0} = 0$$

$$\implies lp_{L_+} + l^{-1}p_{L_-} + mp_{L_0} = 0$$

If ι is the identity 1-braid (with closure as $\bigcirc$), then it is trivial to check that $V_\iota(q,z) = 1$. As a consequence we have $p_\bigcirc(l,m) = 1$. Thus $K \mapsto p_K(l,m)$ is a skein invariant.

Now it remains to show that for any K, $p_K(l,m) \in A$, i.e it is a Laurent polynomial with integer coefficients. For this we make use of the theory of skein invariants since p is a skein invariant. As described earlier,

$$p_{\bigcirc^r}(l,m) = \left(\frac{l+l^{-1}}{m}\right)^{r-1}$$

which clearly has integer coefficients. That is the result is true on all unlinks. Hence by induction (of the same kind used in skein theory) it follows that $p_K(l,m)$ has integer coefficients. Hence proof of the theorem is complete.

Hence $p_K(l,m)$ constructed above is the universal skein invariant.

Definition 2.3.1 (Jones polynomial)**.** *The Jones polynomial is the skein invariant* $V : \mathscr{L} \to \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ *with coefficients* $t, -t^{-1}, (t^{\frac{1}{2}} - t^{-\frac{1}{2}})$.

By uniqueness of skein invariants, the Jones polynomial is well defined. As discussed earlier any skein invariant can be obtained from the universal skein invariant. Thus it follows from the theory of skein invariants that, given a link K, its Jones polynomial is given by

$$V_K(t) = p_K(it, i(t^{\frac{1}{2}} - t^{-\frac{1}{2}}))$$

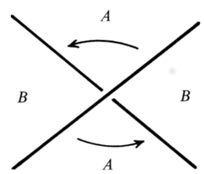


Figure 2.8:

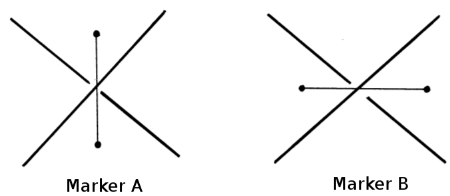


Figure 2.9:

2.4 Kauffman bracket polynomial: A combinatorial approach

Given an unoriented link diagram, which is the image under a generic projection, each of the double points will be isolated. Every double point has an open ball containing it which misses all other double points. This ball seen as a copy of $\mathbb{R}^2$ is divided into four regions by the diagram as shown in Figure 2.8. The regions swiped by the overpass when it is rotated counter-clockwise to match with the underpass is labeled as A and the others are labeled B. At any crossing we have a pair of both the symbols A and B and any one of them can be chosen. The choice is represented by adding markers on crossings as shown in Figure 2.9.

Definition 2.4.1. *A diagram together with a choice of a marker at every double point is called a* ***Kauffman state*** *of the diagram.*

Sometimes Kauffman states are simply refered to as "states". Given any link diagram with n crossings, there are two choices for markers at each crossings. Hence it will have 2^n many states. Once a marker is chosen, the crossing can be "smoothed" out according to the marker as shown in the following figure.

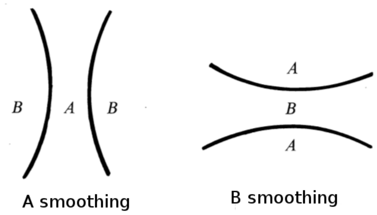


Figure 2.10:

Given any state, after performing smoothings at each crossings, we will obtain a disjoint collection of loops. Thus after smoothings every diagram is a diagram of some unlink. The number of loops in a given state s will be denoted by $|s|$. Let $a(s)$ and $b(s)$ denote the number of A's and B's in s respectively. If L is a link diagram, $S(L)$ will denote the set of all states of L. The symbols A and B can be seen as elements in a ring $\mathbb{Z}[A, A^{-1}, B, B^{-1}, d, d^{-1}]$ where d is another formal symbol which will be given meaning later.

Definition 2.4.2 (Kauffman bracket)**.** *Given an unoriented link diagram L, the Kauffman bracket of L is defined as:*

$$[L] = \sum_{s} A^{a(s)} B^{b(s)} d^{|s|-1}$$

Now $[L]$ is a polynomial in three variables. If B is identified with A^{-1} and d is identified with $-(A^2 + A^{-2})$ then the bracket polynomial can be composed with the quotient map $\mathbb{Z}[A, A^{-1}, B, B^{-1}, d, d^{-1}] \to \mathbb{Z}[A, A^{-1}]$ to get a polynomial in one variable. This Laurent polynomial in A is called *bracket polynomial* and will also be denoted using $[\_]$.

Theorem 2.4.1. *The bracket polynomial $[\_]$ is invariant under Riedemiester move 2 and 3.*

Proof is straight forward application of the definitions. But still it is not invariant under move 1. For resolving this orientation can be made use of. Now consider

$$f_L(A) := A^{-3w(L)}[L]$$

where $w(L)$ denotes the writhe number of L which is the sum of signs of all crossings. The polynomial $f_L(A)$ is said to be the "normalised" bracket polynomial of L.

Theorem 2.4.2. *The normalised bracket polynomial $K \mapsto f_K(A)$ is a skein invariant with coefficients $A^4, -A^4, (A^2 - A^{-2})$ in the ring $\mathbb{Z}[A, A^{-1}]$.*

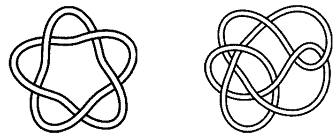


Figure 2.11: 5_1 and 10_{132}

Clearly $w(L)$ is also invariant under Riedemiester moves 2 and 3 and thus $f_L(A)$ will be invariant under move 2 and 3. Hence if it is invariant under move 1 then we are done. It is easily seen that independent of the orientation, any twist in the diagram will always have a sign of -1. Hence while removing it by move 1, the writhe will increase by 1. Substituting the relations between writhe and brackets of the diagram, before and after move 1 and calculating, it follows that f_L is invariant under move 1 also. Hence it is a link invariant. Fact that $f_{\bigcirc}(A) = 1$ is obvious. The skein invariance is also proved by explicitly calculating. Hence the theorem follows.

Consider the map $g : \mathbb{Z}[A, A^{-1}] \to \mathbb{Z}[t^{\frac{1}{4}}, t^{-\frac{1}{4}}]$ given by $A \mapsto t^{\frac{1}{4}}$. Clearly $g \circ f$ is an invariant which satisfy the same skein relation as the Jones polynomial. Hence by uniqueness of skein invariants, $g \circ f$ is the Jones polynomial. That is,

Observation 2.4.3. *The normalised bracket polynomial and Jones polynomial are both the same upto change of variables. Thus the normalised bracket polynomial is another model for Jones polynomial which can be calculated in combinatorial way.*

2.4.1 Some properties of the Jones polynomial

- The polynomial for the mirror image of a link k can be obtained by switching t and t^{-1} in the polynomial for k. Thus it readily tells whether the link is chiral or not.
- The polynomials of any two links determines the polynomial of their disjoint unlinked unions and connected sum. Hence we have some sort of induction techniques on complexity.
- There are many ways to compute it involving algebraic, combinatorial or topological techniques.

But yet it is not known whether it detects the unknot. Also there are some inequivalent couple of knots which share the same Jones polynomial. The pair 5_1 and 10_{132} shown in Figure 2.11 is an example. And also there is **no functoriality** to this invariant, since there is no natural way to talk about an arrow between two polynomials.

Chapter 3

Khovanov Homology

For all discussions of this chapter, R will always denote a fixed commutative ring with 1. The word "module" will always stand for a module over R.

Definition 3.0.3. *Let M be any module over R. A map $d : M \to \mathbb{Z}$ is called a degree function on M iff for each $n \in \mathbb{Z}$ there is a submodule M_n of M with $d(M_n) = \{n\}$ and*

$$M = \bigoplus_{n \in \mathbb{Z}} M_n$$

The module M together with the map d is called a graded module over R. The graded dimension (quantum dimension) of M is the formal series:

$$qdimM := \sum_{n \in \mathbb{Z}} rank\{M_n\} \cdot q^n$$

Note that if M has only finitely many components and each of them are finitely generated, then the graded dimension is a polynomial in an abstract variable q with integer coefficients.

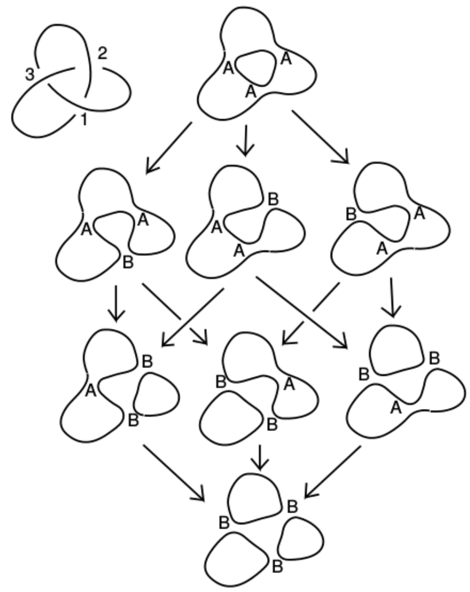


Figure 3.1:

3.1 The cube category and Khovanov complex

Let D be a link diagram with n crossings. Then as described before D will have 2^n Kauffman states. Choose a numbering on the crossings so that each state can be represented as an n-sequence $[X_1 X_2 \cdots X_n]$ where each X_i stands for the symbol A or B. Consider an n dimensional cube whose vertices are the Kauffman states of D with an arrow directed from a state to another iff the first changes to second by the change of exactly one A to B. That is, there is an arrow:

$$[X_1 X_2 \cdots X_i \cdots X_n] \to [X_1 X_2 \cdots \overline{X}_i \cdots X_n]$$

whenever X_i is A and $\overline{X}_i$ is B. Thus the Kauffman states of any link diagram forms a cube. Figure 3.1 shows the cube for a diagram of the trefoil knot.

Definition 3.1.1. *By adding exactly the required compositions and identity arrows formally to this cube, a category (small) can be constructed. This category denoted by $\mathscr{C}(D)$ is said to be the cube category assosiated to the diagram D. Each object in a cube category is a disjoint (unlinked) union of circles.*

Let R be a commutative ring with 1. Suppose $F : \mathscr{C}(D) \to R-mod$ is a covariant functor. For each $s \in \mathscr{C}(D)$ let $i(s)$ denote the number of B smoothings in s. And for any $i \in \mathbb{Z}$ define:

$$C^i(D) := \bigoplus_{\{s|i(s)=i\}} F(s)$$

Let $\partial_k : [X_1 \cdots X_k \cdots X_n] \to [X_1 \cdots \overline{X}_k \cdots X_n]$ denote the unique morphism in $\mathscr{C}(D)$ whenever X_k is A (note that the same notation defines different maps for different states). And let $c(s,k)$ denote the number of A smoothings before the index k in a state s. For $s = [X_1 \cdots X_n]$ let $\alpha(s) := \{k | X_k = A\}$. Consider the map:

$$\left(d_s^i := \sum_{k \in \alpha(s)} (-1)^{c(s,k)} F(\partial_k) \right) : F(s) \to C^{i+1}(D).$$

Since $C^i(D)$ is the co-product of $F(s)$'s, each of the maps of the form d_s^i (for every compatible s) together defines a map $d^i : C^i \to C^{i+1}$. Now for $s = [X_1 \cdots X_n]$ if $s^k := [X_1 \cdots \overline{X}_k \cdots X_n]$ for each $k \in \alpha(s)$ then we have,

$$\begin{aligned} d^{i+1}d^i(x) &= d^{i+1}\Big(\sum_{k \in \alpha(s)} (-1)^{c(s,k)} F(\partial_k)(x)\Big) \\ &= \sum_{k \in \alpha(s)} (-1)^{c(s,k)} d_{s^k}^{i+1}(F(\partial_k)(x)) \\ &= \sum_{k \in \alpha(s)} (-1)^{c(s,k)} \sum_{m \in \alpha(s^k)} (-1)^{c(s^k,m)} F(\partial_m \circ \partial_k)(x) \\ &= \sum_{(m,k) \in \alpha(s) \times \alpha(s)} \big[F(\partial_m \circ \partial_k) - F(\partial_k \circ \partial_m)\big](x) \quad \forall x \in F(s) \end{aligned}$$

Hence if $\partial_m \circ \partial_k = \partial_k \circ \partial_m$ in $\mathscr{C}(D)$, then we have $d^{i+1}d^i = 0$ for all $i \in \mathbb{Z}$. Which shows,

$$\cdots \longrightarrow C^{i-1}(D) \xrightarrow{d^{i-1}} C^i(D) \xrightarrow{d^i} C^{i+1}(D) \xrightarrow{d^{i+1}} \cdots$$

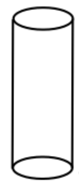


Figure 3.2: Cylinder: no critical point

is a chain complex. The condition $\partial_m \circ \partial_k = \partial_k \circ \partial_m$ can be axiomatically demanded in the definition of the cube category. That is every functor from the cube category of a diagram to a module category "carries" a chain complex and hence a sequence of homology modules.

Note: Thus if there was a category $\mathscr{C}$ where all the cube categories of all link diagrams are subcategories, then a functor on $\mathscr{C}$ will give a homology theory of link diagrams. If it is invariant under Reidemeister moves then its a homology theory of links! Khovanov came up with a functor which gives such a link invariant homology. Also the phrase "homology theory of links" would also encompass the functoriality of this invariant in an appropriate sence.

Definition 3.1.2. *Define **2-cob** to be the category with objects as **finite disjoint union of circles** and a morphism between n circles and m circles is a **smooth compact orientable** 2-**manifold with boundary** such that the boundary is disjoint union of $n + m$ circles.*

Such surfaces are refered to as circle cobordisms. Given a surface Σ_1 from m circles to n circles and another surface Σ_2 from n circles to k circles, clearly we can glue them together at n circles and form a cobordism from m to k circles. This defines a composition operation in this category. And every collection of circles has its identity arrow given by cylinders connecting the corresponding circles. It is easy to see that this surface is indeed the identity of this composition.

Remark 3.1.1. *Also there is a natural **monoidal structure** on this category where tensoring of objects and arrows simply means disjoint union. The empty link (collection of circles) and empty cobordism from empty link to itself will provide identities for the tensoring operations.*

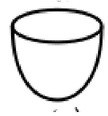


Figure 3.3: Birth of a circle: one critical point of index 0

Figure 3.4: Pair of pants: one critical point of index 1

The infamous **Morse lemma** characterizes the morphisms in this category. On a 2-manifold there are only three different kinds of non-degenerate critical points corresponding to the indices $0, 1$ and 2 (for the sake of brevity we will drop the phrase "non-degenerate" keeping in mind that all kinds of critical points that is refered here will be non-degenerate). For index 1 since the tangent space is a direct sum of two one dimensional orthogonal subspaces, there are two possible surfaces. But they both are topologically equivalent without orientation. And if there are no critical points on a cobordism then it represent planar isotopy of circles and hence it is a union of cylinders. All these surfaces are shown in the Figures 3.2, 3.3, 3.4 and 3.5. Hence any morphism in this category can be constructed using these five cobordisms.

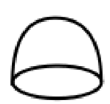


Figure 3.5: Death of a circle: one critical point of index 2

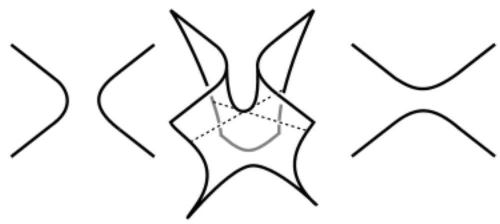


Figure 3.6:

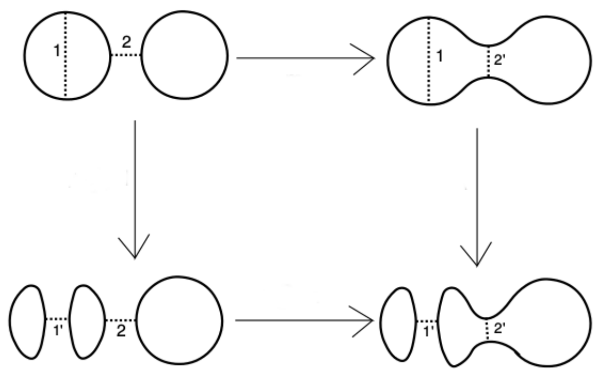


Figure 3.7: The square which is required to be commutative

Observation 3.1.1. *Since each of the states are disjoint collections of circles they are naturally objects in* $2 - cob$*. If a crossing changes from* A *to* B*, then in the states either a circle will split into two or two circles merge and form one circle. This transition is well represented by a cobordism with exactly one critical point of index* 1 *as shown in Figure 3.6. Since the transition does not affect more than two circles, the other circles are connected to there corresponding copies via cylinders. Thus each of the arrows of the form* ∂_k *in* $\mathscr{C}(D)$ *can be represented by a cobordism consisting a pair of pants and some cylinders. Every such arrow correspond to a critical point of index 1. The transitions in which one circle splitting in two and two circles forming one is naturally represented by the two possibilities of critical point of index 1. All this makes* $2 - cob$ *the best place for all the cube categories to live!*

Thus every cube category has an "embedding" in $2 - cob$. It is easily seen that the condition $\partial_m \circ \partial_k = \partial_k \circ \partial_m$ is modeled by the topological equivalence of the corresponding cobordisms. The diagrams in Figures 3.7 and 3.8 illustrates one of the non-trivial cases

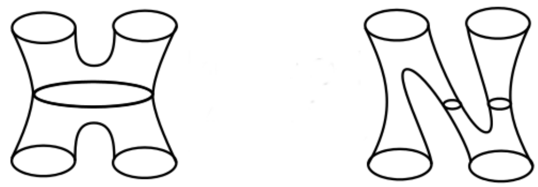


Figure 3.8: Cobordisms representing both the compositions.

of this. Similarly the other cases are also easily verified. As a consequence, if the cube categories are defined as subcategories of $2 - cob$ in the described way, then every functor $F : 2 - cob \to R - mod$ determines a homology theory of link diagrams.

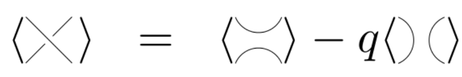
3.2 Definition of Khovanov Homology

This section introduces the co-chain complex corresponding to a link diagram in a form closer to the original algebraic form given by Khovanov. And then we will introduce Viro's interpretation of the complex and enhanced states, which brings in naturality to Khovanov's definition.

Before introducing the construction, it is necessary to modify the definition of Kauffman bracket a bit to suit our purposes. The Kauffman bracket can be defined with a new variable as the unique map $\langle \_ \rangle : \mathscr{L} \to \mathbb{Z}[q, q^{-1}]$ which satisfies:

$$\langle \bigcirc \rangle = q + q^{-1}$$
$$\langle D \amalg \bigcirc \rangle = (q + q^{-1})\langle D \rangle.$$

The inductive relation for the bracket is given as:



This is an unnormalised form of bracket since $\langle \bigcirc \rangle$ is not 1. But it can be normalized by dividing each polynomial by the factor $\langle \bigcirc \rangle = (q + q^{-1})$. Applying the change of variable $A = -q^{-1}$ we will get back the usual bracket.

Let S be a commutative ring with 1 such that there is a monomorphism $i : R \hookrightarrow S$ such that $1 \mapsto 1$. Then S has an obvious R-module structure. Hence S can be tensored with itself over R (in the monoidal category $R - mod$). Also $S \otimes_R S$ is an (S, S)-bimodule. From now on we will drop the subscript R with the tensor symbols where it is clear, for sake of compactness of notation. There are natural isomorphisms $S \to S \otimes R$ and $S \to R \otimes S$. The multiplication in the ring S defines a map $m : S \otimes S \to S$ in an obvious way.

Definition 3.2.1. *S is called a* ***Frobenious algebra*** *over R if there exist an (S, S)-bimodule map $\Delta : S \to S \otimes S$ and an R-module map $\epsilon : S \to R$ such that the following diagram commutes.*

$$\begin{array}{ccccc} R \otimes S & \xleftarrow{\epsilon \otimes 1} & S \otimes S & \xrightarrow{1 \otimes \epsilon} & S \otimes R \\ & \nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow & \\ & & S & & \end{array}$$

The phrase "Frobenious algebra" will refer to the collection $(R, S, m, i, \Delta, \epsilon)$. The maps m, i, Δ, and ϵ are respectively called as the multiplication, unit, co-multiplication and co-unit.

As an example, consider the ring $S = R[x]/(x^2)$ with i as the trivial inclusion. S is generated by the elements 1 and x as a module over R. Define $\Delta : S \to S \otimes S$ by $1 \mapsto 1 \otimes x + x \otimes 1$ and $x \mapsto x \otimes x$, and $\epsilon : S \to R$ by $1 \mapsto 0$ and $x \mapsto 1$. It is easy to verify that these maps indeed make S to a Frobenious algebra over R. This example will be of particular interest to us later on.

Remark 3.2.1. *As mentioned before, $2 - cob$ is a monoidal category with tensoring induced by disjoint union. Every object in $2 - cob$ is a tensor product of finitely many copies of circle. In a sence, the circle generates the monoid $obj(2 - cob)$. As we have seen before, all cobordisms in this category are generated by just five cobordisms. Hence to define a monoidal functor on $2 - cob$ it is enough to specify the images of the circle and these five morphisms.*

Definition 3.2.2. *Any monoidal functor from $2 - cob \to R - mod$ is called a* ***topological quantum field theory*** *abbreviated as TQFT.*

Let H be a TQFT and let $S = F(S^1)$. Since H is a monoidal functor the identity of $2-cob$ which is the empty disjoint union of circles should be mapped to the identity of $R-mod$ which is trivial module R. Also H sends tensor products to tensor products. Thus H maps n copies of S^1 to the module $S^{\otimes n}$ which is the tensor product of n copies of S. Now the fundamental cobordisms of $2-cob$ corresponds to certain maps under H. Cylinder being identity of S^1 will correspond to the identity map of S. The "birth of a circle" gives a map $i : R \to S$. And the "pair of pants" from two circles to one goes to $m : S \otimes S \to S$ and the other pair of pants goes to $\Delta : S \to S \otimes S$. "Death of a circle" gets mapped to a morphism $\epsilon : S \to R$. Now the above commutative diagram trivially follows because of the topological equivalence of surfaces which correspond to the required composition of maps. Hence every TQFT carries a Frobenious algebra. Now by the same token given a Frobenious algebra S, there is a TQFT which maps the generators of $2-cob$ to corresponding components of S. We just proved:

Theorem 3.2.1. *There is a one to one correspondence between TQFT's and Frobenious algebras.*

Under the light of this theorem, we define:

Definition 3.2.3. *The homology theory given by the TQFT corresponding to the Frobenious algebra $R[x]/(x^2)$ is defined as* ***Khovanov homology****.*

Let $\mathscr{H}$ be the TQFT corresponding to the Frobenious algebra $S := R[x]/(x^2)$. Note that S is isomorphic to the free R-module generated by 1 and x. Each of the Kauffman states of D with k circles are mapped to $S^{\otimes k}$. For an integer i let K_i be the set $\{s|i(s) = i\}$. Each copy of S has two generators 1 and x, thus $rank(\mathscr{H}(s)) = 2^{|s|}$. Hence for the chain group

$$C^i(D) = \bigoplus_{s \in K_i} \mathscr{H}(s)$$

we have,

$$rank\{C^i(D)\} = \sum_{s \in K_i} 2^{|s|}.$$

Now the module S can be given a grading by defining degrees of 1 and x as -1 and 1 respectively. As a consequence, S breaks up into direct sum of two rank-1 submodules S_+

and S_- of degrees -1 and 1 respectively. Every k-fold tensor product of S will break up as direct sum of 2^k rank-1 modules which individually are tensor product of k rank-1 modules which are either a copy of S_+ or S_-. Now the degree of a generator $a_1 \otimes a_2 \otimes \cdots a_n$ (where a_i's are either 1 or x) of such a module can be defined as the sum of degrees of a_i's. This gives a grading on every module in the image of $\mathscr{H}$. That is, each of the chain groups will be graded modules. Hence for any integer j we can define the submodule $C^{i,j}$ as the component of C^i of degree j. It can be shown that the differential maps induced by the morphisms in the cube category also preserves the quantum grading (after certain degree shifts), i.e.

$$d^i\big(C^{i,j}(D)\big) \subset C^{i+1,j}(D)$$

and thus we define $d^{i,j} : C^{i,j}(D) \to C^{i+1,j}(D)$ as the restriction of d^i.

Definition 3.2.4. *The bigraded complex formed by the chain groups $C^{i,j}(D)$ and boundaries $d^{i,j}$ for every integers i, j is defined to be the* ***Khovanov complex*** *of D.*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\cdots \longrightarrow & & C^{i-1,j+1}(D) & \longrightarrow & C^{i,j+1}(D) & \longrightarrow & C^{i+1,j+1}(D) & \longrightarrow & \cdots \\
 & & \bigoplus & & \bigoplus & & \bigoplus & & \\
\cdots \longrightarrow & & C^{i-1,j}(D) & \longrightarrow & C^{i,j}(D) & \longrightarrow & C^{i+1,j}(D) & \longrightarrow & \cdots \\
 & & \bigoplus & & \bigoplus & & \bigoplus & & \\
\cdots \longrightarrow & & C^{i-1,j-1}(D) & \longrightarrow & C^{i,j-1}(D) & \longrightarrow & C^{i+1,j-1}(D) & \longrightarrow & \cdots \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

That is, to the diagram D we have attached a bigraded complex consisting of a chain group $C^{i,j}(D)$, and hence a homology module $H^{i,j}(D)$ for every $i, j \in \mathbb{Z}$. Now it can be proved that the homology is independent of the chosen diagram and depends only on the link represented. Before going into the proof of this, it will be helpful to talk about an interpretation of the definition of the chain complex. This involves certain combinatorial ideas which were due to a mathematician Oleg Viro. These will bring in more clarity and also make the definition resemble the usual definitions of homology in topology.

3.3 Viro's interpretation

Consider the Kauffman states of a diagram D. A state together with a choice of one of the symbols $+$ or $-$ to each circle in it is called an (Viro) enhanced state. Enhanced states were first introduced by a Oleg Viro. Thus each of the Kauffman states s splits into $2^{|s|}$ many enhanced states. From now on, unless specified, "state" will always stand for "enhanced state". For an enhanced state s we define some state evaluations. The notations $i(s), \alpha(s), c(s,k)$ and $|s|$ will be used with their previous meanings. Let n_+ and n_- denote the number of circles labeled $+$ and $-$ respectively. And define $\lambda(s) = n_+ - n_-$ and $j(s) = i(s) + \lambda(s)$. Now we can define $C^{i,j}(D)$ to be the free R module generated over the set of all enhanced states s of D with $i(s) = i$ and $j(s) = j$. Note that $C^{i,j}(D)$ will be trivial except for finitely many values of i and j since there are only finitely many states.
Then the module,

$$C^i(D) = \bigoplus_{j \in \mathbb{Z}} C^{i,j}(D)$$

is a graded R-module (for each i) with the degree of the component $C^{i,j}(D)$ defined to be j. Also graded dimension of each one of such modules is a polynomial in q. These modules naturally look like chain groups of a graded complex with coefficients in R. Inorder to complete this to a **bigraded complex**, it is required to construct graded boundary maps $d^i : C^i(D) \to C^{i+1}(D)$. But then the map should preserve the degree j of each submodule.

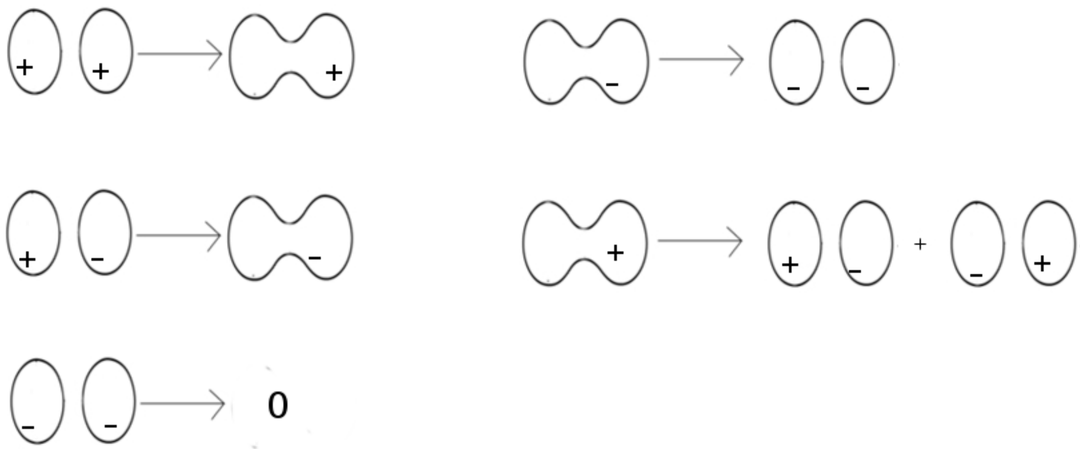


Figure 3.9:

Since the differential will increase the homology grading, i by 1, inorder for quantum grading $j = i + \lambda$ to remain unchanged, λ has to decrease by 1.

Consider the case where two circles merge to form one circle. If both had $+$ on them, then to decrease λ by 1, the only choice is to assign $+$ to the newly formed circle. If one was $+$ and the other was $-$, then the new circle has to be $-$ (note that this doesn't depend on which circle carries a particular symbol). But if both circle were $-$, then none of the signs on the new circle will reduce λ. Hence we declare the image of such states to be 0.

Now consider the case when one circle splits into two. If the circle had $+$, then one of the new circles should be $+$ and the other should be $-$. But there are two different ways to do this since the choice is arbitrary. Hence we define the image to be the sum of both this states. If the circle was $-$, then clearly both the circles in the image have to be $-$. Hence image of every enhanced state under the differential can be easily determined. These are shown in Figure 3.9.

Observation 3.3.1. *Now if merging of circles is compaired as multiplication then the symbols*

+ and − satisfies the same multiplication rule as 1 and x respectively in the ring S. Also splitting of a circle now clearly represents co-multiplication! This shows the naturality of the choice of the Frobenious algebra S. ***The chain groups defined using enhanced states are isomorphic to the chain groups defined by $\mathscr{H}$ in the Khovanov's definition.*** *This follows from a comparison of ranks since they are both free.*

3.4 Invariance under Reidemeister moves

For a chain complex C and an integer k, we denote by $C\{k\}$ the complex obtained by shifting the degree of C by k. Our next aim is to prove that the homology is indeed a link invariant. For this it is enough to prove:

Theorem 3.4.1. *The homology modules $H^{i,j}(D)$ for each $i, j \in \mathbb{Z}$ is invariant under Reidemeister moves on D.*

For the proof of the theorem we will widely use the following lemma from homological algebra.

Lemma 3.4.2. *Let C be a chain complex and C' is a subcomplex of C. If C' has no homology, then for every n, $H_n(C) \cong H_n(C/C')$. If C/C' is also acyclic then for all n, $H_n(C) \cong H_n(C')$.*

Proof: There is an exact sequence of complexes:

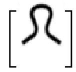
$$0 \longrightarrow C' \longrightarrow C \longrightarrow C/C' \longrightarrow 0$$

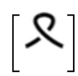
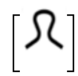
which will give an exact sequence of homology groups:

$$\cdots \longrightarrow H_{n-1}(C/C') \longrightarrow H_n(C') \longrightarrow H_n(C) \longrightarrow H_n(C/C') \longrightarrow H_{n+1}(C') \longrightarrow \cdots$$

if C' is acyclic then by the above exact sequence it follows that $H_n(C) \cong H_n(C/C')$ for all n. Similarly if C/C' is acyclic then $H_n(C) \cong H_n(C')$ for all n.

3.4.1 Invariance under move-I

Let D be a link diagram with a twist . Two different smoothings at this crossing will give two diagrams,  from A smoothing and  from B smoothing. Let $C :=$  ,$A :=$  and $B :=$  represent their corresponding complexes. It is enough to show that C has the same homology as B. Note that in the complex of , the B smoothing which got already applied on  is not counted in the degree. Now by the definition of the complex it follows that:

 $=$  $\oplus$  $\{1\}$.

That is, for every n, $C^n = A^n \oplus B^{n-1}$, but the boundary maps in the complex of B will be multiplied by and additional $-ve$ sign when it is considered a subcomplex of C. Also  is equivalent to disjoint union of  and a circle. Thus it follows that $A^n = S \otimes B^n$. Hence the multiplication map  $\rightarrow$  gives the diagram:

$$\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^n & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow m & & \downarrow m & & \downarrow m & & \\
\cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^n & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}$$

By definition of the cube category the arrows all squares are commutative. Hence the multiplication map induces an isomorphism. Also all the vertical and horizontal sequences are chain complexes. Hence the above diagram is a bi-complex. By our previous observation it is easily seen that the complex C is the diagonal sum of this complex. From now on for the sake of writing we would just denote this bicomplex as:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^{n-1} & \longrightarrow & A^{n} & \longrightarrow & A^{n+1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \\ \cdots & \longrightarrow & B^{n-1} & \longrightarrow & B^{n} & \longrightarrow & B^{n+1} & \longrightarrow & \cdots \end{array}$$

As discussed before $S = S_+ \oplus S_-$, where $S_+ = \langle 1 \rangle$ and $S_- = \langle x \rangle$. Hence

$$C^n = S_+ \otimes B^n \oplus S_- \otimes B^n \oplus B^{n-1}$$

for each n. Condider the subcomplex, C' defined by

$$\cdots \longrightarrow S_+ \otimes B^{n-1} \oplus B^{n-2} \longrightarrow S_+ \otimes B^n \oplus B^{n-1} \longrightarrow S_+ \otimes B^{n+1} \oplus B^n \longrightarrow \cdots$$

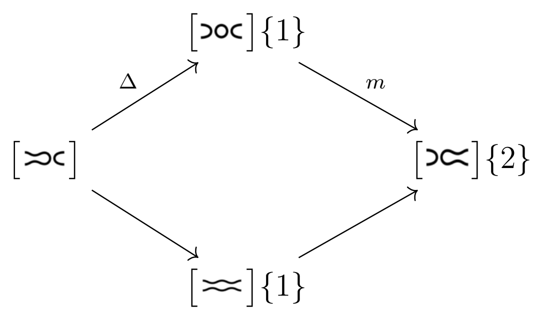
The map m restricted to the submodule $S_+ \otimes B^{n-1}$ is an isomorphism since S_+ generated by the element 1. Its very easy two see that C' is a subcomplex of C and is acyclic. Hence by our previous lemma, the homology of C and C/C' should be isomorphic. The complex C/C' looks like:

$$\cdots \longrightarrow S_- \otimes B^{n-1} \longrightarrow S_- \otimes B^n \longrightarrow S_- \otimes B^{n+1} \longrightarrow \cdots$$

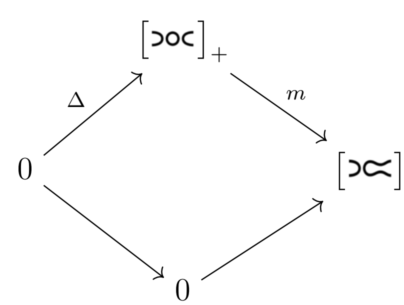
Since $S_- \cong R$ and hence free, the homology of this sequence is isomorphic to the homology of B, by the universal coefficient theorem. Hence homology of C is isomorphic to the homology of B. This proves invariance under move-I.

3.4.2 invariance under move-II

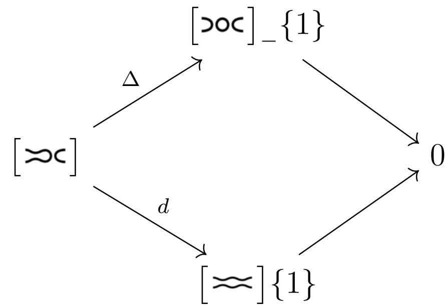
Let D be a diagram with region which looks like ⊃⊂≍. We will prove that if D' is a diagram which resembles D everywhere except this region where it looks like ≍, then D and D' will have the same homology. By the same arguments given in proof of invariance of move-I, we can see that the complex of D is the diagonal sum of the bicomplex:



By the arguments of the previous proof, there is an acyclic bi-complex C' ,



where $[\supset\circ\subset]_+$ denotes the submodule generated by states with + sign on the extra circle. Then by the lemma, homology of C is isomorphic to homology of C/C' which is the complex:



After quotienting with $[\supset\!\circ\!\subset]_+$ it follows that the action of $\Delta : [\asymp\!\subset] \to [\supset\!\circ\!\subset]_-$ is just sending every state of $\asymp\!\subset$ to the corresponding state of $\supset\!\circ\!\subset$ with the extra circle is labeled $-$ sign. Hence Δ is an isomorphism and hence invertible. Consider the map

$$\tau := d\Delta^{-1} : [\supset\!\circ\!\subset]_- \longrightarrow [\asymp]$$

$$C/C' : 0 \longrightarrow [\asymp\!\subset] \xrightarrow{\overline{d}} [\supset\!\circ\!\subset]_-\{1\} \oplus [\asymp]\{1\} \longrightarrow 0$$

For every $n \in \mathbb{Z}$ define $T^n = \left\langle \{ \big(x, \tau(x)\big) | x \in [\supset\!\circ\!\subset]_-^{n-1} \} \right\rangle$ as a submodule of $[\supset\!\circ\!\subset]_-\{1\}^n \oplus [\asymp]\{1\}^n$. Now consider the subcomplex C'' (of the complex C/C') given by:

$$C'' : \ 0 \longrightarrow [\asymp\!\subset] \xrightarrow{\overline{d}} T \longrightarrow 0$$

Since $\overline{d}$ projected to the first component is Δ which is an isomorphism, $ker(\overline{d}) = 0$. For each $x \in [\asymp\!\subset]^n$ we have $d(x) = \tau\Delta(x)$, and hence $\overline{d}(x) = (\Delta(x), \tau\Delta(x)) \in T^n$. Thus we conclude that $im(\overline{d}) = T$. Which means the complex C'' is acyclic. Hence again by the lemma, C has the same homology as:

$$(C/C')/C'' : 0 \longrightarrow \Big([\supset\!\circ\!\subset]_-\{1\} \oplus [\asymp]\{1\} \Big)/T \longrightarrow 0$$

Consider the complex,

$$A : \ 0 \longrightarrow [\asymp]\{1\} \longrightarrow 0$$

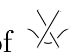
and the inclusion map

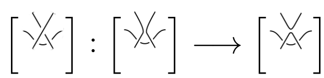
$$i : [\asymp]\{1\} \longrightarrow \Big([\supset\!\circ\!\subset]_{\_}\{1\} \oplus [\asymp]\{1\}\Big)/T$$

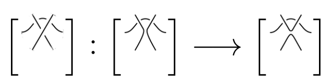
$$x \mapsto \overline{(0, x)}$$

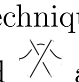
Since $(0, x) \in T$ will mean that $x = 0$, i is injective. Now we know that $\overline{(x, y)} = \overline{(0, y - \tau(x))}$ and hence i is surjective. I.e. i is an isomorphism. Hence the homology of $\supset\!\!\subset$ is isomorphic to the homology of $\asymp$, proving invariance under move-II.

3.4.3 Invariance under move-III

For invariance under third Reidemeister move, we have to show that the homology of  is isomorphic to that of . Note that the complexes of these diagrams are given by:





But  is clearly equivalent to , and  can be obtained from  by a sequence of two Reidemeister moves of type II. Hence all these complexes are isomorphic and by the same techniques used in proof of invariance under move-II, it follows that the homology of  and  are isomorphic. Thus Khovanov homology groups are invariant under all three Reidemeister moves. Hence we have a homology theory of links.

3.5 Euler characteristic

Now for a given link L we will represent its Khovanov bigraded complex by $Kh(L)$. For any link diagram D we have a complex which looks like:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ \cdots & \longrightarrow & C^{i-1,j+1}(D) & \longrightarrow & C^{i,j+1}(D) & \longrightarrow & C^{i+1,j+1}(D) & \longrightarrow & \cdots \\ & & \bigoplus & & \bigoplus & & \bigoplus & & \\ \cdots & \longrightarrow & C^{i-1,j}(D) & \longrightarrow & C^{i,j}(D) & \longrightarrow & C^{i+1,j}(D) & \longrightarrow & \cdots \\ & & \bigoplus & & \bigoplus & & \bigoplus & & \\ \cdots & \longrightarrow & C^{i-1,j-1}(D) & \longrightarrow & C^{i,j-1}(D) & \longrightarrow & C^{i+1,j-1}(D) & \longrightarrow & \cdots \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Given such a bigraded complex, the alternating sum of quantum dimensions of C^i's will be called as the **graded euler characteristic,** χ of the homology. Here we wish to study the graded euler characteristic of Khovanov homology. For the unknot, there is a diagram with just one circle and hence exactly two states. It is easily seen that, $H^{0,1}(\bigcirc) \cong R$ and $H^{0,-1}(\bigcirc) \cong R$ and all the remaining homologies are trivial. Both these are rank 1 over R. Hence the euler characteristic is:

$$\begin{aligned} \chi(\bigcirc) =& \sum_{i,j} (-1)^i q^j rank\{H^{i,j}(D)\} \\ =& q + q^{-1} \end{aligned}$$

But this is the unnormalised bracket of the unknot. To prove the result which this is indicating, we observe that for a diagram D,

$$\chi(D) = \sum_{\text{all enhanced states } e} (-1)^{i(e)} q^{j(e)}$$

Which readily yields $\chi(\bigcirc) = q + q^{-1}$. Also the relation

for the euler characteristic, follows diretly by observing that the enhanced states of the two diagrams obtained after smoothing constitutes the complete family of enhanced states of the diagram with the crossing. And if K is a link diagram, then the enhanced states of $K \sqcup \bigcirc$ will be all states of K together with a $+$ signed circle and all states together with a $-$ signed circle. Thus,

$$\chi(K \sqcup \bigcirc) = q\chi(K) + q^{-1}\chi(K) = (q + q^{-1})\chi(K)$$

Hence by uniqueness of the bracket polynomial, we conclude:

Theorem 3.5.1. *The graded euler characteristic of Khovanov homology is the unnormalised Kauffman bracket polynomial.*

Thus Khovanov homology also carries the bracket polynomial and hence the Jones polynomial. It is proven that unknot is the only link with its homology. Also the 5 crossing and 10 crossing knot which were sharing the same Jones polynomial, have distinct Khovanov homologies. Clearly it is stronger than the Jones polynomial. The following section reveals that Khovanov homology seen in an appropriate way, is **functorial**.

3.6 Functoriality of Khovanov homology

For talking about functoriality of this invariant, first it is required to construct a category of links. This can be done by considering link cobordisms as the morphisms with composition defined by the natural gluing. Khovanov devised a way to construct a chain map between the complexes of two links corresponding to each cobordism between them. We would like to have this chain map to be independent of the embedding of the cobordism, i.e, it should be invariant under ambient isotopy of the cobordism. But in Khovanov's construction, this equivalence of the chain maps corresponding to isotopic surfaces is only achieved upto $-$ve sign. That is the desired functoriality is achieved upto a sign. But for most purposes this is good enough. If the homology is taken over a ring R with $1 = -1$, then it is absolutely functorial. The goal of this section is to give an idea about the functoriality of Khovanov homology by defining chain maps, $Kh(\Sigma)$ corresponding to a link cobordism Σ. For all further discussions a "link cobordism" can be defined as follows:

Definition 3.6.1. *A link cobordism from a link L_0 to another link L_1 is a smooth, compact, orientable 2-manifold with boundary, Σ embedded in $\mathbb{R}^3 \times I$ such that Σ meets the boundary of $\mathbb{R}^3 \times I$ orthogonally and such that,*

$$\partial\Sigma = \Sigma \cap (\mathbb{R}^3 \times \partial I) = L_0 \sqcup L_1.$$

Also $\Sigma \cap (\mathbb{R}^3 \times \{0\})$ should be L_0 and thus L_0 and L_1 will be called source and target of Σ.

The cordinate which is given by the standard interval is usually called time.

Definition 3.6.2. *A link cobordism Σ is said to be generic iff the projection to the time cordinate is a Morse function on Σ with distinct critical values.*

For a generic cobordism, the intersection with the constant time hyperplanes will be a link in $\mathbb{R}^3$ except for finitely many values of t. For these values the intersection can have a transversal double point or it can be a link together with a point disjoint from it. A generic link cobordism can be projected onto $\mathbb{R}^2 \times I$ in the same manner as links. The images under such projections are called surface diagrams of the cobordism. Again for a generic cobordism, it is possible to construct generic surface diagrams with the only kind of singularities are double points, triple points and whitney umbrella points. Just like Reidemeister moves, the ambient isotopy of the link cobordism can be characterised by surface diagrams under a set of moves introduced by Roseman, named after him as *Roseman moves*.

Definition 3.6.3. *A surface diagram represented as a one parameter family of the diagrams D_t, which are intersections with constant time planes $\mathbb{R}^2 \times \{t\}$, is called a **movie**. Each D_t in a movie is called a still.*

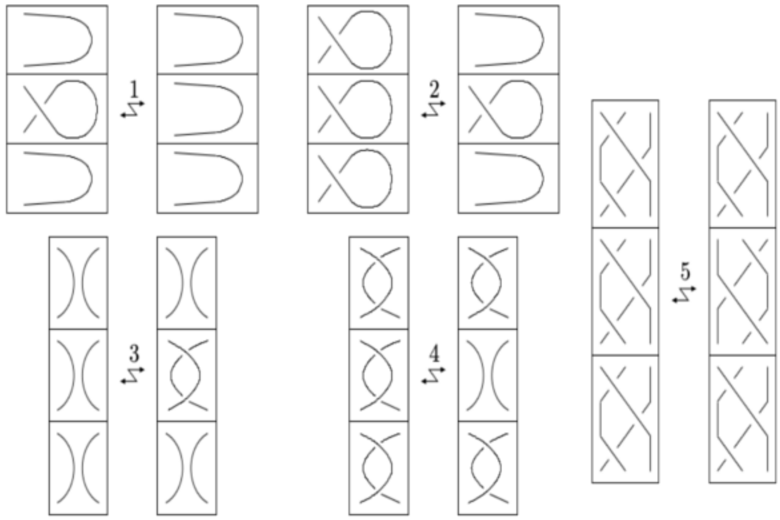


Figure 3.10: Some movie moves (which are Roseman moves)

Carter and Saito came up with a set of moves for movies which characterize the ambient isotopy of the represented cobordism. They are called movie moves. Some of the movie moves are shown in Figure 3.10. As mentioned before, the intersection of the surface diagram with the corresponding plane won't be a link diagram for only finitely many values of t. These will be called critical levels. Also note that, between two critical levels the diagram undergoes planar isotopy. Hence for representing a movie it is enough to consider one still between each pair of consequent critical levels. Some times for more clarity more still are added. If the link diagrams just above and below a critical level are considered, then the transition through this level will either be a Reidemeister move or a Morse modification.

So it is enough to construct maps between complexes of diagrams which differ by a single Reidemeister move or a Morse modification. Since gluing is the composition in the source category, the map corresponding to a surface can be obtained as composition of these maps. The maps for Morse modifications are described as follows:

$$minimum : 1 \mapsto \bigcirc^{+}$$

$$maximum : \begin{cases} \bigcirc^{+} \mapsto & 0 \\ \bigcirc^{-} \mapsto & 1 \end{cases}$$

For a saddle point the maps are completely described by the Frobenious algebra structure on signed circles, which will use the multiplication and co-multiplication operations.

In case of Reidemeister moves, we will use some ideas from the proof of invariance. Suppose D and D' differ by a move-I or II and WLOG assume D has more crossings. Then from the proof of invariance given above, it is clear that the complex of D breaks up as:

$$C(D) = C' \oplus C_0$$

where there is an isomorphism $\varphi : C' \to C(D')$ and C_0 is contractible. Thus we get a map:

$$C(D) = C' \oplus C_0 \xrightarrow{\pi_1} C' \xrightarrow{\varphi} C(D').$$

Where π represents projection map. Now suppose D and D' differ by a move-III. Then from the proof of invariance we know that $C(D) = C^1 \oplus C_0^1$ and $C(D') = C^2 \oplus C_0^2$ where both C_0^1 and C_0^2 are chain contractible and there is an isomorphism $\psi : C^1 \to C^2$. Hence we have the map,

$$C(D) \xrightarrow{\pi} C^1 \xrightarrow{\psi} C^2 \xhookrightarrow{i} C(D')$$

Thus whenever D and D' differ by Reidemeister moves, there is chain map between their complexes $C(D)$ and $C(D')$.

Thus maps corresponding to each of the cobordisms can be computed from there movies. The invariance (upto sign) of these maps are easily checked using movie moves. Thus Kh is a functor! Also Kh is a functorial and it carries the Jones polynomial as its euler characteristic. In this sence, it is a **categorification of the Jones polynomial**.

3.7 Coherence theorems and knot theory

The classsical form of coherence theorem was formulated by Mac Lane [8]. Later much general forms of coherence theorems were discovered. It was a surprising observation that coherence theorems can be stated totally in the language of tangles [8][9]. Once they are characterized by categories of tangles, the study of such categories become very easy. This section is a quick overview of this unexpected application of knot theory in category theory.

Definition 3.7.1. *A monoidal category is a category M together with a functor $\otimes : M \times M \to M$ and an object $I \in M$ so that there exist natural isomorphisms:*

$$\begin{aligned} \rho_A &: A \otimes I \to A \\ \lambda_A &: I \otimes A \to A \\ \alpha_{A,B,C} &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \end{aligned}$$

for every objects A, B and C in M.

It should be noted that the natural isomorphism are also required to satisfy certain commutativity conditions such as the usual "pentagonal identity" of assosiativity (for α). For the sake of brevity such minute details will be neglected in all the coming discussions. A detailed version of these may be found in [9]. A monoidal category can be seen as a generalized version of a monoid where the axioms are equivalences (isomorphisms) instead of equalities. Given a monoidal category M, it is easy to see that, there is a unique 2-category, $\tilde{M}$ with only one object with objects of M as 1-morphisms and arrows in M as 2-morphisms. Composition of 1-morphisms is defined using the tensoring in M. Hence the equality of 1-morphisms are achieved upto isomorphisms. Now in the 2-category, it make sence to choose adjoints (left and right) for objects in M as 1-morphisms. These adjoints denoted by appending the symbol * to the object, comes with unit and counit maps of the form

$$\begin{aligned} h_A &: {}^*A \otimes A \to I \\ e_A &: I \to A \otimes {}^*A \end{aligned}$$

(similar maps η_A and ϵ_A can be defined for right adjoints) are forced to satisy the usual "triangle identities" again upto equivalence. A monoidal category together with a choice of left and right adjoints for every object is said to be an **autonomous category**. In an autonomous category all the natural isomorphism from the monoidal structure and the unit, counit maps for the adjunctions together will be called "structure maps".

Definition 3.7.2. *Given a small category C, there is a unique way to define a category with formal tensor products (with a formal identity I) and adjoints of objects and arrows in C*

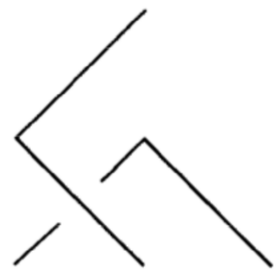


Figure 3.11: A tangle

(with formal structure maps), axiomatically defined to satisy all the required identities for an autonomous category. This category is called the ***free autonomous category*** *generated by C, is denoted by $At(C)$.*

The objects in such a category are just the elements of the free $(I, \otimes, {}^*(), ()^*)$ algebra. The arrows are given by the arrows of C and the formal structure maps. Hence for commutativity of a diagram in such a category is also defined upto equivalence of arrows.

Several categories that arise from knot theory and other branches of mathematics have the structure of a monoidal category. Some of them have extra structure. Some examples of such categories are braided autonomous categories, pivotal monoidal categories and sovereign categories. Each of these will obviously have their "free" versions over a small category C. For a general small category C it is difficult to characterize the arrows in such free categories generated over it. Hence it is difficult to detemine the equivalence of two arrows in such free categories and thus proving commutativity of diagrams becomes hard. Knot theory offers a solution to this problem through coherence theorems. A brief idea of how a coherence theorem will look like is described below.

Definition 3.7.3. *A* ***tangle*** *is a part of a link diagram enclosed in a rectangular region, such that all the intersections of the diagram with the boundary rectangle are transversal and are only on the two horizontal edges.*

For our purposes we will only consider PL-tangles. Figure 3.11 shows an example. Equality of two tangles are defined using isotopies of the plane. Similar to link diagrams these can also be characterized using moves on the diagrams. A usual isotopy is defined by equivalence under all three Reidemeister moves and some additional moves for tangles. Now excluding the first Reidemeister move, we defined "regular" isotopy of tangles. Two tangles with the same number of boundary points on one of the edges of the rectangle can be naturally glued to get another tangle (just like braids). Now for a given small category C and a ring R with 1 we can define (C, R) tangles which are tangles together with labels on them representing objects and arrows of C and elements in the R. The notion of isotopies can be extended to (C, R)-tangles with some more moves for the (C, R) structure.

(C, R)-tangles taken upto some isotopy naturally forms a monoidal category. The objects would be words on the set $ob(C) \times R$ and arrows are (C, R)-tangles with these words as their ends. For different kinds of isotopies of the undelying tangles and for various choices for the ring R, the corresponding monoidal category will have additional structures. All such categories will be refered to as tangle categories. For example, if the tangles are considered upto usual isotopy and the ring is chosen to be $\mathbb{Z}$ then the corresponding category is an autonomous category. Similarly if regular isotopy is considered with the ring $\mathbb{Z}$ then the corresponding will be a braided autonomous category. And for $R = \mathbb{Z}/2\mathbb{Z}$ we have sovereign and pivotal structures.

Coherence theorems assert the existence of an isomorphism between free monoidal categories generated over a small category C and a category of (C, R)-tangles with an appropriate choice of equivalence of tangles and ring R. As an examples:

Theorem 3.7.1. *The category of $(C, \mathbb{Z})$-tangles upto isotopy is isomorphic to the free autonomous category generated over C.*

Theorem 3.7.2. *The category of $(C, \mathbb{Z}/2\mathbb{Z})$-tangles upto regular isotopy is isomorphic to the free sovereign category generated over C.*

As a consequence the equivalence of arrows in free categories over C are characterized by the appropriate isotopy of tangles which are easily verified using the corresponding moves. Hence this simplifies the problems like determining commutativity of diagrams in such categories as discussed earlier. Thus it is a way of encoding the structure of a free category into tangles and then dealing with then using the topological structure of a tangle.

3.8 Conclusion

The Jones polynomial is not the only knot invariant whose categorification is studied. There is Floer homology, graph homology and several other homologies which categorifies different knot polynomials. They arise from representing knots in various forms such as Kauffman states, grid diagrams, graphs etc. As shown in this thesis there is a universal skein invariant which gives all the (skein) knot polynomials by change of variables. Thus a categorification of such an invariant might give rise to a "universal categorification". It would be really interesting to work on. Coherence theorems are really surprising. Encoding all the categorical structure using tangles was such great idea. It is also another interesting field to consider.

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