Foundations and Applications of Finite Strength Quantum Measurement and Weak Value

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by

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Certificate

This is to certify that this dissertation entitled "Foundations and Applications of Finite Strength Quantum Measurement and Weak Value" towards the partial fulfillment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Varad R. Pande at Bose Institute, Kolkata under the supervision of Dipankar Home, Senior Professor, CAPSS, Bose Institute, Kolkata during the academic year 2016-17.

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Declaration

I hereby declare that the matter embodied in the report entitled "Foundations and Applications of Finite Strength Quantum Measurement and Weak Value" are the results of the work carried out by me at the Department of Physics, Bose Institute, Kolkata under the supervision of Prof. Dipankar Home and the same has not been submitted elsewhere for any other degree.

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Abstract

Quantum measurement is analyzed from a conceptual point of view by comparing weak and strong interaction regimes using von-Neumann's measurement model and explicit conditions for weakness have been derived for both weak value and expectation value. The effect of a weak interaction up to strength of orders 1 and 2 without post-selection on the system and pointer states is studied. This engenders a simple interpretation of the imaginary part of the weak value. A general mathematical formalism of weak measurement within which previous results fit is derived. Effect of multiple degrees of freedom and correlations among these within the pointer states in the context of weak measurement is reviewed. Weak measurement is studied as a tool to create entanglement between previously uncorrelated quantum states. A mathematical formalism irrespective of the pointer state used is derived under which one can obtain the joint weak value of incompatible observables as well as second order of the weak values. Using a bipartite correlated resource state, a protocol for determination of weak value corresponding to a weak interaction done by one party by another spatially separated party is conceptualized and mathematically demonstrated.

Contents

Chapter 1

Prologue

While embarking on an investigative journey about any matter in science or otherwise, one should first know – whether in the social[1], subjective logical [2] or a need-based [3] context – the necessity to understand it. This thesis is conceived owing to the superposition of all the three contexts for weak quantum measurement in particular, and quantum measurement, in general. All work in this thesis is original except for section 3.6 which is a review and reproduction of an old result and the postulates of quantum mechanics below.

Among the central features of quantum mechanics like entanglement [27], non-classical correlations [18], unitary or non-unitary evolution of the wave-function or the density matrix, non-locality [4], error-disturbance [20] or fundamental fluctuations [26] uncertainty principles etc. perhaps quantum measurement is the most crucial. It is important not just due to the technical intricacies of the procedure itself, but also (maybe, more so) due to the features endowed to quantum mechanics due to it. A lack of understanding, whether philosophical or operational, of quantum mechanical measurement will and most likely does, resist us from any reasonable success in the unification of physics. Without going further into the reasons of this due to a relative lack of maturity in my thoughts about it, I will restrict this document to quantum measurement and within that, to weak quantum measurement.

Quantum mechanics has seen an empirical development through the aggregation of results of various different experiments. Therefore, almost all of the counter intuitive features of quantum mechanics are in some way or another connected to quantum measurement. All features of quantum non-locality like EPR paradox [9], entanglement, steering [33] are related to measurement. Definitions of other non-classical correlations like quantum discord are centrally dependent on measurement [21]. Mixedness of quantum states and therefore, the partial tracing operation, is related to measurement. According to the Copenhagen interpretation of quantum mechanics, postulate 3 provides for the measurement of quantum systems.

1.1 Postulates of the Quantum

In order to motivate the investigations in this thesis, it is pertinent to begin with the postulates of quantum mechanics [19]:

[1]through other people's work/interest

[2]due to a perceived lack of a sound framework in which all features of that matter fit

[3]due to fruitful technological applications

Postulate 1: Every isolated physical system is defined on a separable, complex vector space on which an inner product is defined (a Hilbert space). The state of the physical system is uniquely and completely represented by a state vector of unit norm in this complex vector space.

The above postulate when extended to continuous variable quantum states provides for the wave-function $\psi(x,t)$ defined in position and time. This wave-function represents a vector in an infinite dimensional Hilbert space. In the above, the unit norm provides for the so called normalization of the quantum state. As an example, the quantum information processing researcher is interested (mostly) in the finite dimensional quantum state which can be defined as qubits (a two-level quantum system), qutrits (a three-level quantum system) and so on. On the other hand, a quantum chemist is interested in the state which is defined continuously in space and time represented by the wave-function. The second postulate provides for the evolution (change) of the above defined quantum state with time.

Postulate 2: A *closed* quantum state evolves from time t to t' according to a *unitary* evolution $U(t,t')$ such that

$$|\psi'\rangle = U(t,t')\,|\psi\rangle \tag{1.1}$$

where $|\psi'\rangle$ is the state of the quantum system at time t' and $|\psi\rangle$ is its state at time t.

The above unitary U can be defined using the time dependent Schrödinger equation:

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi\rangle = H\,|\psi\rangle$$

where H is a Hamiltonian which completely determines the dynamics of the quantum state with time. The third postulate provides for the trickiest and perhaps the most important (because it is operational) aspect of quantum mechanics. After many experimental observations of measurements and their outcomes, quantum measurement is defined by postulate 3:

Postulate 3: Quantum mechanical measurements are defined via the action of the Hermitian operators (which correspond to physical observables) that act on the Hilbert space on which the given quantum state is defined. The outcome of these actions is termed an eigenvalue, which is a real number.

One can immediately see that the extent of correspondence between the notion of the quantum state and the quantum observable is such that one may wonder which of the two is more integral to the existing reality. The postulates of quantum mechanics do not answer whether the state came first or its Hermitian operator (observable). Now, let me define the essentials:
Quantum State: A vector in a finite or infinite dimensional complex vector space on which an inner product is defined. By virtue of its construction, this quantum state can be expanded in terms of its *eigenfunctions* which form the complete orthonormal basis for the given Hilbert space. In principle, there exists an infinite number of possible complete orthonormal basis sets that define a given Hilbert space. Operationally, in quantum measurement, however, a natural basis arises which pertains to the outcomes of an experiment.

Hermitian Operator: A Hermitian operator is the mathematical equivalent of the operational fact of projective measurement. Such an operator *acts* on the given quantum state and renders a particular outcome which is one of its eigenvalues. This action reduces the state to one of its eigenstates (which are part of the basis). Therefore, any subsequent measurement using the same observable will keep giving the same outcome.

Note that nothing stated till this point actually provides for a way an observer will know of the outcome of a measurement on a quantum system that he performed. Such considerations require a model of measurement in which the state on which the observer will record the outcome is called the *pointer* and the state on which the measurement is performed is called the *system*. Such a model of measurement was first provided for by Jon vonNeumann in his seminal volume [31]. Since any dynamics of a quantum state with time are defined through unitary evolution (postulate 2), it is only logical for the aforementioned model of measurement to follow the same procedure. Now, the unitary used here is an interaction unitary. This unitary is composed of a Hamiltonian constructed using the observables acting on the system and the pointer Hilbert spaces respectively. Picking up from these postulates of quantum mechanics, I will go on to the next section which expounds on the foundations of finite strength quantum measurement.

Chapter 2

Foundations of Finite Strength Quantum Measurement and Weak Value

2.1 Quantum Measurement Model

I consider von Neumann's model of measurement [31] and provide a critical discussion (which the current literature, including [1,8], lacks) of the ways *expectation value* and the *weak value* (a term I define later) occur for a given strength of measurement interaction and subsequent post-selection. This model of measurement is conceived from a combination of all the three basic postulates of quantum mechanics described above and is therefore indispensable for any operational understanding of the quantum measurement process. Note that the term 'measurement interaction' encompasses the term 'weak interaction' when the strength of interaction is considered up to the first order expansion of the joint unitary evolution (defined later on).

Defining detection Detection entails recording the outcome that can be seen from each pointer. This can be visualized by thinking of pairwise interaction between multiple identically prepared measuring devices (representing the pointer) and the systems (the system that is desired to be measured). Detection would in this case mean that one *looks* at and *records* the outcomes each of the devices would show via a deflection of their pointers after their respective interactions with the systems. It is necessary to keep in mind that this schematic is overly simplistic and is not by itself an accurate representation of the procedure described above/below.

Measurement interaction or coupling Measurement interaction or coupling entails the joint evolution of the system and pointer state under a unitary composed of the interaction Hamiltonian between the system and the pointer states.

Post-Selection A strong measurement of some observable on the given quantum state and selecting one of the outcomes of that measurement. This observable may or may not be a projector with respect to the basis in which the given quantum state is defined.

The quantum state of the *pointer* on which the outcome of the measurement is recorded, is a continuously distributed Gaussian function of momentum/position defined on an infinite dimensional Hilbert space and the *system* state is a discrete variable pure quantum state defined on an n-dimensional Hilbert space. The coupling that occurs between the system and the pointer results from their joint evolution under the following unitary interaction.

$$U = e^{-i\int \hat{H}dt}$$

$\hat{H} = -g(t)\hat{q}\otimes\hat{A}$, where the real eigenvalues a_i of the system state are part of the spectral decomposition of the Hermitian operator $\hat{A}$ in its eigenbasis. The evolved state is

$$e^{i\int g(t)\hat{q}\otimes\hat{A}dt}e^{-q^2/4(\Delta q)^2}\sum_i^n \alpha_i \left|A = a_i\right\rangle = \sum_i^n \alpha_i e^{-(\Delta p)^2(p-a_i)^2/}\left|A = a_i\right\rangle$$

The Fourier transform assists us in deriving the above:

$$e^{ia_i\hat{q}}e^{-(\Delta p)^2p^2} = e^{ia_i\hat{q}}\int dqe^{-q^2/4(\Delta q)^2}e^{-ipq} \tag{2.1}$$

$$= \int dqe^{-q^2/4(\Delta q)^2}e^{-iq(p-a_i)} \tag{2.2}$$

$$= e^{-(\Delta p)^2(p-a_i)^2} \tag{2.3}$$

As can be seen, the system and the pointer are in a joint entangled state [1]. The pointer momentum distribution is divided into Gaussians centered at the respective eigenvalues a_i. This fact is the origin of the *correlation* or *entanglement* between the system and the pointer states after the interaction. The Gaussian pointer distributions that correspond to each eigenvalue a_i are now associated with the respective eigenstates $|a_i\rangle$ of the system, thus causing entanglement between the Gaussian pointer and the system states.

2.2 Strong Measurement - Expectation Value

In the above, if $\Delta p << \min_n |a_i - a_j|$ (where $i, j \in \{1, n\}$), it is ensured that the Gaussians that are centered at (or, whose peaks lie at) different $a_i's$ are well separated from each other. In other words, the different peaks centered at the respective eigenvalues are well-resolved. A projective measurement on the system state will destroy its superposition and reduce it to one of its eigenstates $|A = a_k\rangle$, corresponding to the eigenvalue a_k. Now, due to the aforementioned correlation, the corresponding pointer state will be a Gaussian peaked at a_k. Hence, an observer looking at the pointer will see or *detect* the value a_k. In this case, the final pointer wave-function, for pointer detection of one of the particles that corresponds to the system eigenstate (which occurs due to the projection), $|A = a_k\rangle$ of the large ensemble is

$$\phi(p) = e^{-(\Delta p)^2(p-a_k)^2}$$

The above state and therefore the eigenvalue a_k will approximately occur $|\alpha_k|^2$ times when one goes on to build the complete measurement statistics by performing a large number of measurements on an ensemble os quantum systems prepared in the given joint state. This is because, according to the Copenhagen interpretation, $|\alpha_k|^2$ represents the probability of occurrence of the outcome $|A = a_k\rangle$ with eigenvalue a_k, where α_k is the probability amplitude. I use the standard way to calculate the expectation value of $\hat{A}$,

[1]For pure states, 'non-separability' of the wave-function is equivalent to 'entanglement' and these terms will be used alternately depending on the context but carry the same meaning for pure quantum states

$\langle\psi_i|\, A\, |\psi_i\rangle$ by sandwiching the measurement interaction between initial and final system states which are identical (and I have ignored the post-selected system state - which is identical to the pre-selected one) and are written in the eigen-basis of $\hat{A}$, (this can always be done for a complete orthonormal basis) that is:

$$\begin{aligned}
&\left\langle \sum_{j=1}^{n} \alpha_j^* \psi_j^\dagger \right| e^{i\hat{A}\otimes\hat{q}} \left| \sum_{i=1}^{n} \alpha_i \psi_i \right\rangle e^{-q^2/4(\Delta q)^2} \\
&= \left\langle \sum_{j=1}^{n} \alpha_j^* \psi_j^\dagger \right| e^{ia_i\hat{q}} \left| \sum_{i=1}^{n} \alpha_i \psi_i \right\rangle e^{-q^2/4(\Delta q)^2} \\
&= \sum_{j,i=1}^{n} \alpha_j^* \alpha_i \delta_{ij} e^{ia_i\hat{q}} e^{-q^2/4(\Delta q)^2} \\
&= \sum_{i=1}^{n} |\alpha_i|^2 e^{ia_i\hat{q}} e^{-q^2/4(\Delta q)^2} \\
&= \sum_{i=1}^{n} |\alpha_i|^2 e^{-(\Delta p)^2 (p-a_i)^2}
\end{aligned} \tag{2.4}$$

In the above, the interaction strength g has been considered to be 1, such that, $\int g(t)dt = 1$, and the Fourier transform 2.1 has been used. Also, $|A = a_i\rangle \equiv \Psi_i$ and $|A = a_j\rangle \equiv \Psi_j$. On doing a large number of measurements, one detects the eigenvalue a_i with the probability of $|\alpha_i|^2$ and eventually gets the expectation value of $\langle\hat{A}\rangle = \sum_{i=1}^{n} |\alpha_i|^2 a_i$

2.3 Weak Measurement - Expectation Value

It is instructive to compare the above expression (2.4) with the one given below:

$$\exp\left\{ ig \sum_{i=1}^{n} |\alpha_i|^2 a_i \hat{q} \right\} \quad e^{-q^2/4(\Delta q)^2} \tag{2.5}$$

Through this expression, one will be able to argue that the peak of final Gaussian pointer wave-function can be approximated to lie at the expectation value. By using the Fourier transform technique described earlier (2.1), the expectation value appears as the shift of the peak of the Gaussian pointer in the momentum space:

$$e^{-(\Delta p)(p-g\langle\hat{A}\rangle)^2}$$

In the above expression, note that the expectation value of $\hat{A} = \sum_{i=1}^{n} |\alpha_i|^2 a_i$ appears in the exponential of the interaction. The above expression signifies that the post-interaction pointer state (without projective measurement(post-selection) and detection) can be approximated by a Gaussian distribution in its momentum representation which is centered around the expectation value of system observable, $\langle\hat{A}\rangle_w$. To actually decipher (I have avoided the use of the word 'measure' due to the risk of it being taken out of context) this expectation value with desired accuracy, however, one will have to do measurements over a large number of particles and build up the statistics as done before when the interaction was strong. The main consequence of doing a weak interaction is that a *single* detection of the pointer will not result in any *discernible* outcome. A single outcome will neither be an expectation value (obviously), nor an eigenvalue. It will be a fuzzy value will a finite uncertainty which can be owed to the weakness of the interaction prior to the detection. In the case of strong interaction, however, each detection will yield a *definite* outcome which will be one of the eigenvalues of the observable $\hat{A}$ acting on the system state $|\psi\rangle$. As explained in the section above, the result of ensemble measurement is the build up of the measurement statistics that depend on the probability with which each eigen-state of $\hat{A}$ is present in the eigenstate decomposition of the system state.

2.4 Weakness and Strength

In order to justify the observations made in the section above, it is imperative that the expressions 2.4 and 2.5 are equal to each other in the 'weak limit'. This is subject to the satisfaction of certain approximations. To get an understanding of these approximations, it is necessary to Taylor expand the exponential (and ignore the pointer state distribution for the moment) in both cases. Case 1 (2.4) :

$$\begin{aligned} &\sum_{i=1}^{n} |\alpha_i|^2 e^{iga_i\hat{q}} \\ &= \sum_{i=1}^{n} |\alpha_i|^2 (1 + iga_i\hat{q} - g^2 a_i^2 \hat{q}^2 - ig^3 a_i^3 \hat{q}^3 + ...) \\ &= (\sum_{i=1}^{n} |\alpha_i|^2 + ig\sum_{i=1}^{n} |\alpha_i|^2 a_i\hat{q} - g^2 \sum_{i=1}^{n} |\alpha_i|^2 a_i^2 \hat{q}^2 - ig^3 \sum_{i=1}^{n} |\alpha_i|^2 a_i^3 \hat{q}^3 + ...) \\ &= (1 + ig\langle\hat{A}\rangle\hat{q} - g^2\langle\hat{A}^2\rangle\hat{q}^2 - ig^3\langle\hat{A}^3\rangle\hat{q}^3 + ...) \end{aligned}$$

Here, $\langle\hat{A}^k\rangle$ are the higher orders of the expectation value of $\hat{A}$. Now, lets consider case 2 (2.5) by expanding the exponential (again ignoring the pointer state distribution):

$$\begin{aligned} &\exp\left\{ig\sum_{i=1}^{n} |\alpha_i|^2 a_i\hat{q}\right\} \\ &= 1 + ig\sum_{i=1}^{n} |\alpha_i|^2 a_i\hat{q} - g^2(\sum_{i=1}^{n} |\alpha_i|^2 a_i\hat{q})^2 - ig^3(\sum_{i=1}^{n} |\alpha_i|^2 a_i\hat{q})^3 + ... \\ &= 1 + ig\langle\hat{A}\rangle\hat{q} - g^2\langle\hat{A}\rangle^2\hat{q}^2 - ig^3\langle\hat{A}\rangle^3\hat{q}^3 + ... \end{aligned}$$

By looking at the similarities and differences between the final expressions for cases 1 and 2, one can derive the conditions under which these two expressions are equal. These general conditions can be defined as *conditions for weakness*. For the two expressions to be equal, it is necessary and sufficient that the terms of orders 2 and higher be small enough (compared to the first two terms) to be ignored. For case 1 (2.4), we have:

$$g^k\langle\hat{A}^k\rangle q^k << \max[1, g\langle\hat{A}\rangle q] \quad \forall \quad k \geq 2$$

For case 2 (2.5), we have:

$$g^k\langle\hat{A}\rangle^k q^k << \max[1, g\langle\hat{A}\rangle q] \quad \forall \quad k \geq 2$$

If one considers the $non-trivial$ assumption that $1 > g\langle\hat{A}\rangle\hat{q}$ which is indeed valid in many experimental scenarios (the interaction strength g is controllable), we have for case

1

$$g^k \langle \hat{A}^k \rangle q^k << g\langle \hat{A}\rangle q \quad \forall \quad k \geq 2$$
$$\implies g^{k-1}\frac{\langle \hat{A}^k \rangle}{\langle \hat{A}\rangle} q^{k-1} << 1 \quad \forall \quad k \geq 2$$

and for case 2

$$g^k \langle \hat{A}\rangle^k q^k << g\langle \hat{A}\rangle q \quad \forall \quad k \geq 2$$
$$\implies g^{k-1}\langle \hat{A}\rangle^{k-1} q^{k-1} << 1 \quad \forall \quad k \geq 2$$

In both cases above, the q can be replaced by Δq, the position spread of the initial pointer state. This can always be done since $\Delta q > q \ \forall q$. So, we have Case 1:

$$g^{k-1}\frac{\langle \hat{A}^k \rangle}{\langle \hat{A}\rangle} (\Delta q)^{k-1} << 1 \quad \forall \quad k \geq 2$$

and Case 2

$$g^{k-1}\langle \hat{A}\rangle^{k-1} (\Delta q)^{k-1} << 1 \quad \forall \quad k \geq 2$$

The spread in the position distribution of the initial pointer state is inversely proportional to the spread of its momentum distribution; $\Delta p = \frac{1}{2\Delta q}$. Again, we have, case 1

$$g^{k-1}\frac{\langle \hat{A}^k \rangle}{\langle \hat{A}\rangle} \left(\frac{1}{2\Delta p}\right)^{k-1} << 1 \quad \forall \quad k \geq 2 \tag{2.6}$$
$$\implies (g/2)^{k-1}\frac{\langle \hat{A}^k \rangle}{\langle \hat{A}\rangle} << (\Delta p)^{k-1} \quad \forall \quad k \geq 2 \tag{2.7}$$
$$\implies (g/2)\left(\frac{\langle \hat{A}^k \rangle}{\langle \hat{A}\rangle}\right)^{1/k-1} << \Delta p \quad \forall \quad k \geq 2 \tag{2.8}$$

and case 2

$$g^{k-1}\langle \hat{A}\rangle^{k-1} \left(\frac{1}{2\Delta p}\right)^{k-1} << 1 \quad \forall \quad k \geq 2 \tag{2.9}$$
$$\implies (g/2)^{k-1}\langle \hat{A}\rangle^{k-1} << (\Delta p)^{k-1} \quad \forall \quad k \geq 2 \tag{2.10}$$
$$\implies (g/2)\langle \hat{A}\rangle << \Delta p \quad \forall \quad k \geq 2 \tag{2.11}$$

The weakness condition – truncating both (cases 1 and 2) the expansions up to the first order – holds only if the conditions 2.6 and 2.9 are satisfied together.

2.4.1 Results

After a careful observation of the above two approximations for cases 1 and 2, one can see that three independent factors contribute to the ‘weakness’ of measurement:

1. The measurement interaction strength or coupling strength g between system and pointer states.

2. The expectation value of system measurement observable $\langle \hat{A} \rangle$ and its higher orders $\langle \hat{A}^k \rangle$.

3. The initial variance (spread) of the pointer wave-function in the position (momentum) representation, Δq (Δp).

As prescribed by the weakness conditions described above, lets consider $\Delta p >> \max_n a_i$. Recall that this is opposite to the condition considered in the strong measurement section 2.1.1 above. This time the resolution between the separated Gaussians is so bad that one can only make out a single Gaussian that is peaked at the expectation value of $\hat{A}$, $\sum_i^n |\alpha_i|^2 a_i$. This is the regime in which one defines weak measurement. However, a single detection will not give any discernible result since $\Delta p >> \langle \hat{A} \rangle$. Like in the strong case, here also this expectation value can be deciphered/calculated only after detecting a large number of particles in the ensemble. The uncertainty with which the expectation value is obtained gradually goes down with the number of particles N one detects as $\frac{1}{\sqrt{N}}$.

Definition: Weak Value Before proceeding let us first define an entity called the 'weak value' through the spectral and eigenstate decomposition of the system interaction($\hat{A}$) and post-selection observables($\hat{B}$) and the initial system state.

$$\langle \hat{A} \rangle_w \equiv \frac{\langle \psi_f | \hat{A} | \psi_i \rangle}{\langle \psi_f | \psi_i \rangle} = \frac{\sum_{i=1}^n \beta_i^* a_i \alpha_i}{\sum_{i=1}^n \beta_i^* \alpha_i} \tag{2.12}$$

where $|\psi_f\rangle = \left|\hat{B} = b\right\rangle = \sum_{j=1}^n \beta_j |A = a_j\rangle$, $|\psi_i\rangle = \sum_{i=1}^n \alpha_i |A = a_i\rangle$ and $\hat{A} = \sum_{i=1}^n a_i |a_i\rangle \langle a_i|$ owing to the completeness property of a complete orthonormal basis. Also note that this weak value is a complex quantity in general.

2.5 Post-selected Measurement - What Value?

An explicit treatment to the post-selection process is given here by doing the analysis analogous to the above when the pre-selected system state $|\psi_i\rangle$ is post-selected on another state $|\psi_f\rangle$ by doing a projective measurement using an observable $\hat{B}$. Here, the interaction is sandwiched between $|\psi_f\rangle$, the post-selected state and $|\psi_i\rangle$, the preselected state. One might now expect to determine the value [2] $\langle \psi_f | \hat{A} | \psi_i \rangle$ analogous to the expectation value $\langle \psi_i | \hat{A} | \psi_i \rangle$. Lets consider $\psi_f = \left|\hat{B} = b\right\rangle = \sum_{j=1}^n \beta_j |A = a_j\rangle$

$$\left\langle \sum_{j=1}^n \beta_j^* \psi_j^\dagger \middle| e^{ig\hat{A}\otimes\hat{q}} \middle| \sum_{i=1}^n \alpha_i \psi_i \right\rangle e^{-q^2/4(\Delta q)^2} \tag{2.13}$$

$$= \left\langle \sum_{j=1}^n \beta_j^* \psi_j^\dagger \middle| e^{iga_i\hat{q}} \middle| \sum_{i=1}^n \alpha_i \psi_i \right\rangle e^{-q^2/4(\Delta q)^2} \tag{2.14}$$

$$= \sum_{j,i=1}^n \beta_j^* \alpha_i \delta_{ij} e^{iga_i\hat{q}} e^{-q^2/4(\Delta q)^2} \tag{2.15}$$

$$= \sum_{i=1}^n \beta_i^* \alpha_i e^{iga_i\hat{q}} e^{-q^2/4(\Delta q)^2} \tag{2.16}$$

$$= \sum_{i=1}^n \beta_i^* \alpha_i e^{-(\Delta p)^2 (p - ga_i)^2} \tag{2.17}$$

[2] which will later turn out to be the unnormalized weak value.

$|\psi_i\rangle$ and $\hat{B}$ are written in the orthogonal basis of $\hat{A}$. Looking at the above expressions, it is relevant to ask about the measurement statistics that would come out if one does such a measurement on an ensemble. This question is not answered by the current interpretation of quantum mechanics which defines only the modulus squared of the complex coefficients (probabilities corresponding to the probability amplitudes) to be discerned from the statistics of a measurement. This conundrum might also have led to the claim made by Duck, Stevenson and Sudarshan [8] while trying explain the appearance of the weak value at the peak of the single final Gaussian of the pointer state (will come to this in the next sections). From the above analysis it can be seen that the appearance of the weak value at the peak has got nothing to do with "complicated cancellations" between Gaussians with complex coefficients. It is solely a consequence of the weakness conditions analyzed in detail in the next section. To make this point clearer, we will also proceed with the calculation of weakness conditions that are relevant for the definition and justification of the appearance of the weak value.

One can immediately see the part of the weak value $\sum_{i=1}^{n}\beta_i^*\alpha_i$ is identical to the one that is present in product with the exponentials in the final expression of 2.13. Also, note that the weak value can lie far outside the range of eigenvalues of the observable $\hat{A}$ and can be complex in general. Thus, if one *considers* a procedure[3] of statistical buildup of results with 'probabilities' $\sum_{i=1}^{n}\beta_i^*\alpha_i$ corresponding to the outcome a_i, one will achieve the numerator of the weak value $\sum_{i=1}^{n}\beta_i^*a_i\alpha_i$. As explained above this makes no sense because probabilities can only be normalized real numbers.

2.6 Post-Selected Weak Measurement - Weak Value

Now, as was done for the expectation value, it is necessary to compare the final expression of 2.13 with the one given below:

$$\begin{aligned}(\sum_{i=1}^{n}\beta_i^*\alpha_i)\exp\Big\{ig\frac{\sum_{i=1}^{n}\beta_i^*\alpha_i a_i}{\sum_{i=1}^{n}\beta_i^*\alpha_i}\hat{q}\Big\}\quad e^{-q^2/4(\Delta q)^2} && (2.18)\\ =\langle\psi_f|\psi_i\rangle\exp\Big\{ig\langle\hat{A}\rangle_w\hat{q}\Big\}\quad e^{-q^2/4(\Delta q)^2} && (2.19)\\ =\langle\psi_f|\psi_i\rangle\exp\Big\{-(\Delta p)^2\left[p-g\langle\hat{A}\rangle_w\right]^2\Big\} && (2.20)\end{aligned}$$

Where, I have used Fourier transform to the momentum representation 2.1 and the expression of the spectrally decomposed weak value (2.12). Expanding the above case 2 expression, 2.18 (and ignoring the position distribution of the pointer),

$$\langle\psi_f|\psi_i\rangle\exp\Big\{ig\langle\hat{A}\rangle_w\hat{q}\Big\}=\langle\psi_f|\psi_i\rangle\left(1+ig\langle\hat{A}\rangle_w\hat{q}-g^2\langle\hat{A}\rangle_w^2\hat{q}^2-ig^3\langle\hat{A}\rangle_w^3\hat{q}^3+...\right) \qquad (2.21)$$

$$(2.22)$$

Since $\langle\psi_f|\psi_i\rangle=\sum_{i=1}^{n}\beta_i^*\alpha_i$, I can write,

$$\langle\psi_f|\psi_i\rangle\exp\Big\{ig\sum_{i=1}^{n}\beta_i^*\alpha_i a_i\hat{q}\Big\}=\sum_{i=1}^{n}\beta_i^*\alpha_i(1+ig\sum_{i=1}^{n}\beta_i^*\alpha_i a_i\hat{q}-g^2(\sum_{i=1}^{n}\beta_i^*\alpha_i a_i)^2\hat{q}^2-ig^3(\sum_{i=1}^{n}\beta_i^*\alpha_i a_i)^3\hat{q}^3+...) \qquad (2.23)$$

[3]identical to that of the expectation value

Now, let us expand the case 1 expression (ignoring the pointer distribution in position), 2.13,

$$\sum_{i=1}^{n} \beta_i^* \alpha_i e^{iga_i\hat{q}} = \sum_{i=1}^{n} \beta_i^* \alpha_i (1 + iga_i\hat{q} - g^2 a_i^2 \hat{q}^2 - ig^3 a_i^3 \hat{q}^3 + ...) \tag{2.24}$$

$$= (\sum_{i=1}^{n} \beta_i^* \alpha_i + ig \sum_{i=1}^{n} \beta_i^* \alpha_i a_i \hat{q} - g^2 \sum_{i=1}^{n} \beta_i^* \alpha_i a_i^2 \hat{q}^2 - ig^3 \sum_{i=1}^{n} \beta_i^* \alpha_i a_i^3 \hat{q}^3 + ...) \tag{2.25}$$

$$= \langle \psi_f | \psi_i \rangle \left(1 + ig\langle \hat{A} \rangle_w \hat{q} - g^2 \langle \hat{A}^2 \rangle_w \hat{q}^2 - ig^3 \langle \hat{A}^3 \rangle_w \hat{q}^3 + ...\right) \tag{2.26}$$

$$\tag{2.27}$$

where $\langle \psi_f | \psi_i \rangle = \sum_{i=1}^{n} \beta_i^* \alpha_i$.

We now proceed to derive the conditions under which the above expression (2.24) can be approximated to the case 2 (2.23) expression. Analyzing for case 1 first (2.24):

$$g^k \langle \hat{A}^k \rangle_w q^k << g\langle \hat{A} \rangle_w q \quad \forall k \geq 2$$

$$\implies g^{k-1} \frac{\langle \hat{A}^k \rangle_w}{\langle \hat{A} \rangle_w} q^{k-1} << 1 \quad \forall k \geq 2$$

$$\implies (g/2)^{k-1} \frac{\langle \hat{A}^k \rangle_w}{\langle \hat{A} \rangle_w} << \Delta p^{k-1} \quad \forall k \geq 2$$

$$\implies (g/2) \left(\frac{\langle \hat{A}^k \rangle_w}{\langle \hat{A} \rangle_w} \right)^{1/k-1} << \Delta p \quad \forall k \geq 2$$

Now, analyzing for case 2:

$$g^k \langle \hat{A} \rangle_w^k q^k << g\langle \hat{A} \rangle_w q \quad \forall k \geq 2$$

$$\implies g^{k-1} \langle \hat{A} \rangle_w^{k-1} q^{k-1} << 1 \quad \forall k \geq 2$$

$$\implies (g/2)^{k-1} \langle \hat{A} \rangle_w^{k-1} q^{k-1} << \Delta p^{k-1} \quad \forall k \geq 2$$

$$\implies (g/2) \langle \hat{A} \rangle_w << \Delta p \quad \forall k \geq 2$$

AAV used the modulus of the quantities on the LHS to justify their approximations [1].

$$\Delta q << \max_n \frac{| \langle \psi_f | \psi_{in} \rangle |}{| \langle \psi_f | \hat{A}^n | \psi_{in} \rangle |^{1/n}}$$

AAV's approximations which were later corrected (sic.) by DSS [8] are similar to the ones I arrived at when I did the analogous treatment for the expectation value in (2.6 and 2.9). Assuming that $g\langle \hat{A} \rangle_w \hat{q} < 1$, DSS arrived at the following approximations:

$$\Delta q << \frac{1}{\langle \hat{A} \rangle_w}$$

and

$$\Delta q << \min \left| \frac{\langle \psi_f | \hat{A} | \psi_{in} \rangle}{\langle \psi_f | \hat{A}^n | \psi_{in} \rangle} \right|^{1/k-1}$$

These approximations and assumptions do not have a sound justification because we are comparing complex quantities with a potentially non zero imaginary part on the RHS with real quantities on the LHS [4]. Therefore, in order to justify the truncation of the expansion up to first order, we should consider separate approximations for the real and the imaginary parts of the weak value.

Now, I consider the real and imaginary parts separately to justify the approximations: Case 1:

$$1 + ig\,\mathrm{Re}\langle\hat{A}\rangle_w q - g\,\mathrm{Im}\langle\hat{A}\rangle_w q - g^2\,\mathrm{Re}\langle\hat{A}\rangle_w^2 - ig^2\,\mathrm{Im}\langle\hat{A}\rangle_w^2 q^2 - ig^3\,\mathrm{Re}\langle\hat{A}\rangle_w^3 q^3 + g^3\,\mathrm{Im}\langle\hat{A}\rangle_w^3 q^3 + \tag{2.28}$$

$$g^4\,\mathrm{Re}\langle\hat{A}\rangle_w^4 q^4 + ig^4\,\mathrm{Im}\langle\hat{A}\rangle_w^4 q^4 + ig^5\,\mathrm{Re}\langle\hat{A}\rangle_w^5 q^5 - g^5\,\mathrm{Im}\langle\hat{A}\rangle_w^5 q^5 - g^6\,\mathrm{Re}\langle\hat{A}\rangle_w^6 q^6 - ig^6\,\mathrm{Im}\langle\hat{A}\rangle_w^6 q^6 + ... \tag{2.29}$$

Real Part:

$$1 - g\,\mathrm{Im}\langle\hat{A}\rangle_w q - g^2\,\mathrm{Re}\langle\hat{A}\rangle_w^2 q^2 + g^3\,\mathrm{Im}\langle\hat{A}\rangle_w^3 q^3 + g^4\,\mathrm{Re}\langle\hat{A}\rangle_w^4 q^4 - g^5\,\mathrm{Im}\langle\hat{A}\rangle_w^5 q^5 - g^6\,\mathrm{Re}\langle\hat{A}\rangle_w^6 q^6 + \tag{2.30}$$

$$g^7\,\mathrm{Im}\langle\hat{A}\rangle_w^7 q^7 + g^8\,\mathrm{Re}\langle\hat{A}\rangle_w^8 q^8 \tag{2.31}$$

Imaginary Part:

$$g\,\mathrm{Re}\langle\hat{A}\rangle_w q - g^2\,\mathrm{Im}\langle\hat{A}\rangle_w^2 q^2 - g^3\,\mathrm{Re}\langle\hat{A}\rangle_w^3 q^3 + g^4\,\mathrm{Im}\langle\hat{A}\rangle_w^4 q^4 + g^5\,\mathrm{Re}\langle\hat{A}\rangle_w^5 q^5 - g^6\,\mathrm{Im}\langle\hat{A}\rangle_w^6 q^6 - \tag{2.32}$$

$$g^7\,\mathrm{Re}\langle\hat{A}\rangle_w^7 q^7 + g^8\,\mathrm{Im}\langle\hat{A}\rangle_w^8 q^8 + ... \tag{2.33}$$

The expression will look similar, but there will be a major difference. $\langle\hat{A}\rangle^k$ will now be replaced by $\langle\hat{A}^k\rangle$. Thus, we have the case 2 expansion:

$$1 + ig\,\mathrm{Re}\langle\hat{A}\rangle_w q - g\,\mathrm{Im}\langle\hat{A}\rangle_w q - g^2\,\mathrm{Re}\langle\hat{A}^2\rangle_w - ig^2\,\mathrm{Im}\langle\hat{A}^2\rangle_w q^2 - ig^3\,\mathrm{Re}\langle\hat{A}^3\rangle_w q^3 + g^3\,\mathrm{Im}\langle\hat{A}^3\rangle_w q^3 + \tag{2.34}$$

$$g^4\,\mathrm{Re}\langle\hat{A}^4\rangle_w q^4 + ig^4\,\mathrm{Im}\langle\hat{A}^4\rangle_w q^4 + ig^5\,\mathrm{Re}\langle\hat{A}^5\rangle_w q^5 - g^5\,\mathrm{Im}\langle\hat{A}^5\rangle_w q^5 - g^6\,\mathrm{Re}\langle\hat{A}^6\rangle_w q^6 - ig^6\,\mathrm{Im}\langle\hat{A}^6\rangle_w q^6 + ... \tag{2.35}$$

Real Part:

$$1 - g\,\mathrm{Im}\langle\hat{A}\rangle_w q - g^2\,\mathrm{Re}\langle\hat{A}^2\rangle_w q^2 + g^3\,\mathrm{Im}\langle\hat{A}^3\rangle_w q^3 + g^4\,\mathrm{Re}\langle\hat{A}^4\rangle_w q^4 - g^5\,\mathrm{Im}\langle\hat{A}^5\rangle_w q^5 - g^6\,\mathrm{Re}\langle\hat{A}^6\rangle_w q^6 + \tag{2.36}$$

$$g^7\,\mathrm{Im}\langle\hat{A}^7\rangle_w q^7 + g^8\,\mathrm{Re}\langle\hat{A}^8\rangle_w q^8 \tag{2.37}$$

Imaginary Part:

$$g\,\mathrm{Re}\langle\hat{A}\rangle_w q - g^2\,\mathrm{Im}\langle\hat{A}^2\rangle_w q^2 - g^3\,\mathrm{Re}\langle\hat{A}^3\rangle_w q^3 + g^4\,\mathrm{Im}\langle\hat{A}^4\rangle_w q^4 + g^5\,\mathrm{Re}\langle\hat{A}^5\rangle_w q^5 - g^6\,\mathrm{Im}\langle\hat{A}^6\rangle_w q^6 - \tag{2.38}$$

$$g^7\,\mathrm{Re}\langle\hat{A}^7\rangle_w q^7 + g^8\,\mathrm{Im}\langle\hat{A}^8\rangle_w q^8 + ... \tag{2.39}$$

[4] if the position/momentum distribution is considered real-valued

Assuming that $g\,\mathrm{Im}\langle\hat{A}\rangle_w q < 1$ throughout, the following set of 8 approximations emerges:

$$(g/2)\,\mathrm{Im}\langle\hat{A}\rangle_w << \Delta p \quad \forall k \in \{3,5,7,9,...\}$$

$$(g/2)(\frac{\mathrm{Re}\langle\hat{A}\rangle_w^k}{\mathrm{Im}\langle\hat{A}\rangle_w})^{1/k-1} << \Delta p \quad \forall k \in \{2,4,6,8,...\}$$

$$(g/2)(\frac{\mathrm{Im}\langle\hat{A}\rangle_w^k}{\mathrm{Re}\langle\hat{A}\rangle_w})^{1/k-1} << \Delta p \quad \forall k \in \{2,4,6,8,...\}$$

$$(g/2)\,\mathrm{Re}\langle\hat{A}\rangle_w << \Delta p \quad \forall k \in \{3,5,7,9,...\}$$

$$(g/2)(\frac{\mathrm{Im}\langle\hat{A}^k\rangle_w}{\mathrm{Im}\langle\hat{A}\rangle_w})^{1/k-1} << \Delta p \quad \forall k \in \{3,5,7,9,...\}$$

$$(g/2)(\frac{\mathrm{Re}\langle\hat{A}^k\rangle_w}{\mathrm{Im}\langle\hat{A}\rangle_w})^{1/k-1} << \Delta p \quad \forall k \in \{2,4,6,8,...\}$$

$$(g/2)(\frac{\mathrm{Im}\langle\hat{A}^k\rangle_w}{\mathrm{Re}\langle\hat{A}\rangle_w})^{1/k-1} << \Delta p \quad \forall k \in \{2,4,6,8,...\}$$

$$(g/2)(\frac{\mathrm{Re}\langle\hat{A}^k\rangle_w}{\mathrm{Re}\langle\hat{A}\rangle_w})^{1/k-1} << \Delta p \quad \forall k \in \{3,5,7,9,...\}$$

Each condition in the above set of conditions is necessary. They become sufficient only when the set treated as a whole. Observations similar to those made for the weakness criterion while calculating the expectation value hold here.

1. The measurement interaction or coupling strength g between system and pointer.
2. The weak value of system measurement observable $\langle\hat{A}\rangle_w$ and its higher orders $\langle\hat{A}^k\rangle_w$.
3. The initial spread of the pointer wave-function in the position representation Δq.

2.7 Conundrums From the Past

I will quote and briefly discuss the analyses of some eminent physicists concerning the seminal paper [1] which introduced weak values. Let me begin with the inventors Yakir Aharonov, David Z. Albert and Lev Vaidman:

> We have found that the usual measuring procedure for preselected and post-selected ensembles of quantum systems gives unusual results. Under some natural conditions of weakness of the measurement, its result consistently defines a new kind of value for a quantum variable, which we call the weak value.

I. M. Duck, P. M. Stevenson and E. C. G. Sudarshan [8]:

> The surprising effect pointed out by AAV has been shown to be a consequence of constructive and destructive interference between two complex amplitudes. Although surprising, the effect is in no way paradoxical, and involves nothing outside ordinary quantum mechanics.

Asher Peres [22]

> The experimental results, if correctly interpreted, obey the rules of elementary quantum mechanics.

A. J. Leggett [14]

> It is precisely the notion of "standard measuring procedure" which is at issue between us.

From the above comments it is clear that people had diametrically opposite viewpoints about the proposed weak measurement procedure. Leggett had a problem with regarding the weak measurement process to be a standard measuring procedure. Peres had an issue with the processing of the outcomes of the weak measurement process. In particular, he gave the following example of a pointer state after the weak interaction and post-selection. He considered the measurement of the z-component of the spin of an electron which corresponds to the pauli observable $\hat{\sigma}_z$:

$$\begin{aligned}\langle\psi_f|\exp\left(-iq\sigma_z - q^2/4\Delta^2\right)|\psi_i\rangle &= (1/2)(\mu\langle\psi_f|(1+\sigma_z)|\psi_i\rangle e^{-iq} \\ &\quad +\nu\langle\psi_f|(1-\sigma_z)|\psi_i\rangle e^{iq})\exp\left(-q^2/4\Delta^2\right)\end{aligned} \tag{2.40}$$

Fourier transform of the above state is:

$$\begin{aligned}&(1/2)(\mu\langle\psi_f|(1+\sigma_z)|\psi_i\rangle\exp\left(-\Delta^2(p-1)^2\right) \\ &+\nu\langle\psi_f|(1-\sigma_z)|\psi_i\rangle\exp\left(-\Delta^2(p+1)^2\right))\end{aligned} \tag{2.41}$$

A look at the above expression might give the first-hand impression that there are two peaks at ± 1. However, Peres ignores the fact that the coefficient of both the exponents are complex entities in general which have no place in the statistics of a measurement. This example clearly illustrates the point I made through the derivation of the expression 2.13. Needless to say, in the weak limit the center of the Gaussian lies at the weak value. Thus, it can be concluded that post-selection which gives rise to the above indeterminate expression is the root cause of the confusion. Both AAV and DSS seem to regard this as a normal measurement procedure which I agree with.

2.8 Illumination with a Poser

We began with the measurement of the expectation value of an observable with respect to a given quantum state using vonNeumann's model of measurement. For any practical/experimental measurement process this model and hence all its considerations are imperative since it gives an operational interpretation to the quantum measurement process. The model consists of two actions - (i) Weak interaction between system and pointer, (ii) Projective post-selection of the system state. It should be emphatically pointed out that a quantum measurement process cannot give any information about the interaction or the observable involved in the interaction or the state of the system with which the pointer interacts, unless, projective post-selection is performed.

We then demonstrated the method to measure the expectation value of an observable with respect to a given system state using both strong interaction as well as weak interaction between system and pointer states followed by post-selection on the same state. A strong(projective) measurement done on the system state after a weak interaction does not result in collapse of the pointer state into one of Gaussians which leads to a fuzziness or an intrinsic error/uncertainty in the measured value. This intrinsic error can be attributed to the quantum nature of the pointer. Irrespective of the weakness, the result for both cases is the same provided one does the entire measurement process over a large enough number of identically prepared system states.

Then, we moved to the case where post-selection does not result in a projection on the same state[5] due to the usage of a different observable from the one used in the interaction. Here, when strong interaction between the initial system and pointer states is followed by the post-selection on a different system state, the result does not have a justification or any operational meaning within the Copenhagen interpretation of quantum mechanics, if one follows the statistical buildup procedure analogous to the expectation value. However, if the interaction between the system and the pointer is weak and is followed by the projective post-selection, the resulting value has a real experimental existence. It is safe to say that the weak value (or its higher orders - which will have a similar corresponding set of weakness conditions to satisfy) can be defined only in the weak approximation - the conditions for which have been explicitly derived - within the framework of the measurement model. In summary,

1. The conundrum of the past concerning the weak measurement process have been cleared by isolating the weakness of a measurement process as something that is independent of whether one wants to measure the expectation value or the weak value.

2. The surprising nature of the weak value arises due to a combination of the weakness of measurement and the post-selection on the system state which is different from the pre-selected one.

3. An interpretation to the value of the expression 2.13 is still awaited. This implies that either there is an issue with the measurement model or there is an issue with the Copenhagen interpretation of quantum mechanics.

[5] as that of the preselected one

Chapter 3

A general mathematical formulation of WM with and without post-selection

Weak measurement in the sense AAV proposed operationally consists of two actions. (1) $Weak\ interaction$ between two quantum system s(system and pointer) via a coupling Hamiltonian $\hat{H} = \hat{A} \otimes \hat{p}$. Here, the observable $\hat{A}$ corresponds to the measurement on the system wave-function or density matrix and the observable $\hat{p}$ corresponds to the measurement on the pointer wave-function or density matrix. Such a Hamiltonian is implemented using a unitary interaction of coupling strength g. (2) Projective measurement using another observable $\hat{B}$ that does not commute with $\hat{A}$ and selecting some of the outcomes of that projective measurement which amounts to *post selection*.

$$\hat{U} = e^{-ig(t)\hat{A}\otimes\hat{p}} \tag{3.1}$$

3.1 Weak Interaction - First Order Strength

The function g(t) is a delta like function that peaks for a very small time interval during which the weak interaction occurs. Therefore, $\int g(t)dt = g$. In all that follows, the value of g is considered to be very less, i.e, $g << 1$. However, in general, $g \in (0, 1)$, so that $g^n < g$, where $n \in \mathbb{Z}$. The implication of this 'weak coupling' is to expand the unitary interaction between the system state and the pointer state up to first order in its Taylor expansion. A more detailed analysis of the meaning, causes and consequences of the weakness of interaction strength were provided in previous section. Let the above interaction unitary act on the separable bipartite initial system + pointer state $|\psi_{it}\rangle = |\psi_{is}\rangle |\phi_{ip}\rangle$:

$$\begin{aligned}
\hat{U} |\psi_{is}\rangle |\phi_{ip}\rangle &= e^{-ig(t)\hat{A}\otimes\hat{p}} |\psi_{is}\rangle |\phi_{ip}\rangle \\
&= (1 - ig\hat{A} \otimes \hat{p}) |\psi_{is}\rangle |\phi_{ip}\rangle \\
&= |\psi_{is}\rangle |\phi_{ip}\rangle - ig\hat{A} |\psi_{is}\rangle \otimes \hat{p} |\phi_{ip}\rangle
\end{aligned}$$

Here, $|\psi_{is}\rangle$ represents the initial system state and $|\phi_{ip}\rangle$ represents the initial pointer state. Notice that, after the interaction, the joint state of the system and pointer is no longer separable. The weak interaction has converted it to an *entangled state* $|\psi_{ft}\rangle$. This feature of entanglement creation between two quantum systems due to measurement interaction between their respective quantum states is universally true and was first well-formulated by von Neumann. Such an interaction resulting in entanglement is the simplest and the most basic feature of non-classicality. Now, one will be naturally inclined to investigate the effect of the aforementioned weak measurement interaction on the system and the pointer states individually. We address this issue by considering the partial trace of the joint post-interaction state first with respect to the Hilbert space pertaining to the pointer and then with respect to the Hilbert space pertaining to the system. To follow this prescription, we first construct the density matrix ρ_{ft} corresponding to the joint post-interaction state

$$\begin{aligned}\rho_{ft} = |\psi_{ft}\rangle\langle\psi_{ft}| &= (|\psi_{is}\rangle|\phi_{ip}\rangle - ig\hat{A}|\psi_{is}\rangle\otimes\hat{p}|\phi_{ip}\rangle)(\langle\phi_{ip}|\langle\psi_{is}| + ig\langle\phi_{ip}|\hat{p}\otimes\langle\psi_{is}|\hat{A}) \\ &= (|\psi_{is}\rangle|\phi_{ip}\rangle)\langle\phi_{ip}|\langle\psi_{is}| + ig(|\psi_{is}\rangle|\phi_{ip}\rangle)\langle\phi_{ip}|\hat{p}\otimes\langle\psi_{is}|\hat{A} - ig(\hat{A}|\psi_{is}\rangle\otimes\hat{p}|\phi_{ip}\rangle)\langle\phi_{ip}|\langle\psi_{is}| \\ &= |\psi_{is}\rangle\langle\psi_{is}|\otimes|\phi_{ip}\rangle\langle\phi_{ip}| + ig|\psi_{is}\rangle\langle\psi_{is}|\hat{A}\otimes|\phi_{ip}\rangle\langle\phi_{ip}|\hat{p} - ig\hat{A}|\psi_{is}\rangle\langle\psi_{is}|\otimes\hat{p}|\phi_{ip}\rangle\langle\phi_{ip}|\end{aligned}$$

In the calculations, I have used the Hermiticity of $\hat{p}$ and $\hat{A}$, the norm of the initial system and pointer wave-functions to be 1 and the norm of the final joint state is 1, $\langle\psi_{ft}|\psi_{ft}\rangle = 1$. All operators that henceforth appear in this work correspond to physical observables with real eigenvalues are therefore Hermitian operators. Taking the partial trace of the above obtained joint state over the pointer Hilbert space, one gets the system state $\rho_{fs} = \mathrm{Tr}_p(\rho_{ft})$

$$\rho_{fs} = |\psi_{is}\rangle\langle\psi_{is}|\langle\phi_{ip}|\phi_{ip}\rangle + ig|\psi_{is}\rangle\langle\psi_{is}|\hat{A}\langle\phi_{ip}|\hat{p}|\phi_{ip}\rangle - ig\hat{A}|\psi_{is}\rangle\langle\psi_{is}|\langle\phi_{ip}|\hat{p}|\phi_{ip}\rangle \quad (3.2)$$

$$= |\psi_{is}\rangle\langle\psi_{is}| + ig\langle\phi_{ip}|\hat{p}|\phi_{ip}\rangle(|\psi_{is}\rangle\langle\psi_{is}|\hat{A} - \hat{A}|\psi_{is}\rangle\langle\psi_{is}|) \quad (3.3)$$

$$= \rho_{is} + ig\langle\phi_{ip}|\hat{p}|\phi_{ip}\rangle[\rho_{is}, \hat{A}] \quad (3.4)$$

Here and at every further occurrence, $[.,.]$ represents the commutator between the two arguments and $\{.,.\}$ represents the anti-commutator between the two arguments. Also note that $\mathrm{Tr}\left(\rho_{fs}\rho_{fs}^\dagger\right) = 1$. From the above expression it is clear that the non-disturbance or disturbance caused to the system state evident from the change in its expression after the weak interaction is subject to the vanishing or non-vanishing of the commutator $[\rho_{is}, \hat{A}]$. This commutator is related to the expression of the imaginary part of the weak value which I will come to in a later section. Like the system state, the pointer state can also derived in a similar way by taking a partial trace over the system Hilbert space, $\rho_{fp} = \mathrm{Tr}_s(\rho_{ft})$,

$$\begin{aligned}\rho_{fp} = \langle\psi_{is}|\psi_{is}\rangle|\phi_{ip}\rangle\langle\phi_{ip}| + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle|\phi_{ip}\rangle\langle\phi_{ip}|\hat{p} - ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle\hat{p}|\phi_{ip}\rangle\langle\phi_{ip}| \\ = \rho_{ip} + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle[\rho_{ip}, \hat{p}]\end{aligned}$$

Here, $\mathrm{Tr}\left(\rho_{fp}\rho_{fp}^\dagger\right) = 1$. The above expression is analogous to the expression for the reduced system state. The disturbance is accounted for by the commutator between the density matrix of the initial pointer state and pointer measurement observable $\hat{p}$. If one interchanges the roles of the system and the pointer, the above disturbance is also related to the imaginary part of the *pointer* weak value. Note that in both the cases, the disturbance caused on the system(pointer) is also *weighted* by the *expectation value* of the pointer(system) observable with respect to the initial pointer(system) state. This was a non-operational treatment to the disturbance caused to the system or the pointer states after the unitary weak interaction between them. However, in order to experimentally ascertain the change that took place, it is necessary to determine the expectation value of a pointer observable $\hat{p}'$ using the final joint state $|\psi_{ft}\rangle$.

$$\begin{aligned}\langle\psi_{ft}|\,\hat{p}'\,|\psi_{ft}\rangle &= (\langle\phi_{ip}|\,\langle\psi_{is}| + ig\,\langle\phi_{ip}|\,\hat{p}\otimes\langle\psi_{is}|\,\hat{A} \quad , \quad \hat{p}' \quad , \quad |\psi_{is}\rangle\,|\phi_{ip}\rangle - ig\hat{A}\,|\psi_{is}\rangle\otimes\hat{p}\,|\phi_{ip}\rangle) \\ &= \langle\phi_{ip}|\,\hat{p}'\,|\phi_{ip}\rangle\,\langle\psi_{is}|\psi_{is}\rangle + ig\,\langle\phi_{ip}|\,\hat{p}\hat{p}'\,|\phi_{ip}\rangle\,\langle\psi_{is}|\,\hat{A}\,|\psi_{is}\rangle - ig\,\langle\phi_{ip}|\,\hat{p}'\hat{p}\,|\phi_{ip}\rangle\,\langle\psi_{is}|\,\hat{A}\,|\psi_{is}\rangle \\ &= \langle\phi_{ip}|\,\hat{p}'\,|\phi_{ip}\rangle + ig\,\langle\psi_{is}|\,\hat{A}\,|\psi_{is}\rangle\,\langle\phi_{ip}|\,\hat{p}\hat{p}' - \hat{p}'\hat{p}\,|\phi_{ip}\rangle \\ &= \langle\phi_{ip}|\,\hat{p}'\,|\phi_{ip}\rangle + ig\,\langle\psi_{is}|\,\hat{A}\,|\psi_{is}\rangle\,\langle[\hat{p},\hat{p}']\rangle_{ip}\end{aligned}$$

Now, one has the shift in the expectation value of an arbitrary pointer observable after the weak interaction with respect to its expectation value before the weak interaction.

$$\langle\hat{p}'\rangle_{fp} = \langle\hat{p}'\rangle_{ip} + ig\langle\hat{A}\rangle_{is}\langle[\hat{p}',\hat{p}]\rangle_{ip}$$

Like the change in the expression for the state, the change in the expectation value of an arbitrary pointer observable is also weighted by the *expectation value* of the system observable with respect to the initial system state. Similar conclusion can be drawn for an arbitrary system observable's final expectation value.

$$\langle\hat{A}'\rangle_{fs} = \langle\hat{A}'\rangle_{is} + ig\langle\hat{p}\rangle_{ip}\langle[\hat{A}',\hat{A}]\rangle_{is}$$

3.1.1 Results

In the weak limit of the interaction strength when the unitary expansion is restricted to order one,

1. The system state is not disturbed if

$$[\rho_{is},\hat{A}] = 0 \quad \text{or} \quad \langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle = 0$$

2. The pointer state is not disturbed if

$$[\rho_{ip},\hat{p}] = 0 \quad \text{or} \quad \langle\psi_{is}|\,\hat{A}\,|\psi_{is}\rangle = 0$$

3. the expectation value of an arbitrary system observable, $\hat{A}'$ does not change if

$$[\hat{A},\hat{A}'] = 0 \quad \text{or} \quad \langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle = 0$$

4. the expectation value of an arbitrary pointer observable, $\hat{p}'$ does not change if

$$[\hat{p},\hat{p}'] = 0 \quad \text{or} \quad \langle\psi_{is}|\,\hat{A}\,|\psi_{is}\rangle = 0$$

3.2 Measurement back-action and the Weak Value

3.2.1 A brief prelude

Interpretation of the meaning of the real and imaginary parts of the weak value is a topic of much debate. I quote some physicists below:
J. Dressel and A. N. Jordan [7]:

> Specifically, we provide an operational interpretation for the imaginary part of the generalized weak value as the logarithmic directional derivative of the postselection probability along the unitary flow generated by the action of the observable operator.

In simple words, the imaginary part of the weak value determines how the disturbed system state changes in the direction[1] specified by the observable that acts on the system state during the weak interaction.
Yakir Aharonov and Alonso Botero [2]:

> The imaginary part of the complex weak value can be interpreted as a "bias function" for the posterior sampling point.

In this paper, the status of the imaginary part of the weak value is analyzed with the help of statistical techniques by considering the position and momentum Gaussian distributions before the and after the weak measurement. The above interpretation is based mainly on an equation similar to 3.8 which we arrive at later.
In his paper, Aephraim Steinberg [28, 29] states:

> The imaginary part constitutes a shift in the pointer momentum. This latter effect is a reflection of the backaction of a measurement on the particle. It does have physical significance, but, since it does not correspond to a spatial translation of the pointer, should not be thought of as part of the measurement outcome.

Solely because the imaginary part of the weak value represents a shift in the momentum expectation value is not reason enough to state that it reflects measurement backaction. Besides, the imaginary part of the weak value appears in the shift of the spatial measurement as well which can be seen from Equation 3.9.

3.2.2 A Simple Answer

In addition to the individual observations made above, another attribute that summarily applies to all of them is the consideration of a continuous Gaussian pointer. The interpretation I come to valid for any pointer distribution in addition to being the simplest one.

Expressions for the real and the imaginary parts have been explicitly derived in terms of the pre and post-selected system states and the system measurement observable which is part of the interaction unitary [6, 7]. The real part is:

$$\mathrm{Re}\langle A\rangle_w = \frac{\mathrm{Tr}\Big(\hat{\Pi}_f\{\hat{A},\hat{\rho}_{is}\}\Big)}{2\,\mathrm{Tr}\Big(\hat{\Pi}_f\hat{\rho}_{is}\Big)} \tag{3.5}$$

The imaginary part is:

$$\mathrm{Im}\langle A\rangle_w = \frac{\mathrm{Tr}\Big(\hat{\Pi}_f(-i[\hat{A},\hat{\rho}_{is}])\Big)}{2\,\mathrm{Tr}\Big(\hat{\Pi}_f\hat{\rho}_{is}\Big)} \tag{3.6}$$

where Π_f is the projector used for the post-selection on the system state. In the above expression, notice that the commutator $[\hat{A},\hat{\rho}_{is}]$ is identical to the one that determines

[1] of its state space

back-action on the system state 3.2 via a change in its expression. If this commutator is zero, the imaginary part of the weak value is zero. Therefore, if there is no back-action on the system state ($[\hat{A}, \hat{\rho}_{is}] = 0$) because of the weak interaction, the imaginary part of the weak value is zero.

3.3 Weak Interaction - Second Order Strength

Now, we come to the second order treatment of the above weak interaction. While the disturbance in the first order of the interaction strength was subject to the relevant commutation relations, what pattern does the disturbance follow when one expands the interaction upto order 2? The joint final system + pointer state when the interaction is expanded up to order 2:

$$\left|\psi_{ft}^{(2)}\right\rangle = \hat{U}\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle = e^{-ig(t)\hat{A}\otimes\hat{p}}\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle$$
$$= (1 - ig\hat{A}\otimes\hat{p} - g^2\hat{A}^2\otimes\hat{p}^2)\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle$$
$$= \left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle - ig\hat{A}\left|\psi_{is}\right\rangle\otimes\hat{p}\left|\phi_{ip}\right\rangle - g^2\hat{A}^2\left|\psi_{is}\right\rangle\otimes\hat{p}^2\left|\phi_{ip}\right\rangle$$

Constructing the density matrix

$$\rho_{ft}^{(2)} = \left|\psi_{ft}\right\rangle\left\langle\psi_{ft}\right| = (\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle - ig\hat{A}\left|\psi_{is}\right\rangle\otimes\hat{p}\left|\phi_{ip}\right\rangle - g^2\hat{A}^2\left|\psi_{is}\right\rangle\otimes\hat{p}^2\left|\phi_{ip}\right\rangle)(\left\langle\phi_{ip}\right|\left\langle\psi_{is}\right| + ig\left\langle\phi_{ip}\right|\hat{p}\otimes\left\langle\psi_{is}\right|\hat{A} -$$
$$g^2\left\langle\phi_{ip}\right|\hat{p}^2\otimes\left\langle\psi_{is}\right|\hat{A}^2)$$
$$= (\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle)\left\langle\phi_{ip}\right|\left\langle\psi_{is}\right| + ig(\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle)\left\langle\phi_{ip}\right|\hat{p}\otimes\left\langle\psi_{is}\right|\hat{A} - ig(\hat{A}\left|\psi_{is}\right\rangle\otimes\hat{p}\left|\phi_{ip}\right\rangle)\left\langle\phi_{ip}\right|\left\langle\psi_{is}\right| -$$
$$g^2(\hat{A}^2\left|\psi_{is}\right\rangle\otimes\hat{p}^2\left|\phi_{ip}\right\rangle)\left\langle\phi_{ip}\right|\left\langle\psi_{is}\right| + g^2(\hat{A}\left|\psi_{is}\right\rangle\otimes\hat{p}\left|\phi_{ip}\right\rangle)\left\langle\phi_{ip}\right|\hat{p}\otimes\left\langle\psi_{is}\right|\hat{A} -$$
$$g^2(\left|\psi_{is}\right\rangle\left|\phi_{ip}\right\rangle)\left\langle\phi_{ip}\right|\hat{p}^2\otimes\left\langle\psi_{is}\right|\hat{A}^2 =$$
$$\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\otimes\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right| + ig\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\hat{A}\otimes\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right|\hat{p} - ig\hat{A}\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\otimes\hat{p}\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right| -$$
$$g^2\hat{A}^2\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\otimes\hat{p}^2\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right| + g^2\hat{A}\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\hat{A}\otimes\hat{p}\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right|\hat{p} -$$
$$g^2\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\hat{A}^2\otimes\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right|\hat{p}^2$$

On normalizing one has

$$\frac{\rho_{ft}^{(2)}}{\left\langle\psi_{ft}|\psi_{ft}\right\rangle} = \rho_{ft}^{(2)} + g^2\left\langle\psi_{is}\right|\hat{A}^2\left|\psi_{is}\right\rangle\left\langle\phi_{ip}\right|\hat{p}^2\left|\phi_{ip}\right\rangle\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\otimes\left|\phi_{ip}\right\rangle\left\langle\phi_{ip}\right|$$

Taking the partial trace over the pointer Hilbert space, one gets the density matrix of the normalized reduced system state $\rho_{fs}^{(2)}$ after the disturbance is considered up to the second order

$$\rho_{fs}^{(2)} = \left|\psi_{is}\right\rangle\left\langle\psi_{is}\right| + ig\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\hat{A}\left\langle\phi_{ip}\right|\hat{p}\left|\phi_{ip}\right\rangle - ig\hat{A}\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\left\langle\phi_{ip}\right|\hat{p}\left|\phi_{ip}\right\rangle -$$
$$g^2\hat{A}^2\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\left\langle\phi_{ip}\right|\hat{p}^2\left|\phi_{ip}\right\rangle + g^2\hat{A}\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\hat{A}\left\langle\phi_{ip}\right|\hat{p}^2\left|\phi_{ip}\right\rangle -$$
$$g^2\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|\hat{A}^2\left\langle\phi_{ip}\right|\hat{p}^2\left|\phi_{ip}\right\rangle + g^2\left\langle\psi_{is}\right|\hat{A}^2\left|\psi_{is}\right\rangle\left\langle\phi_{ip}\right|\hat{p}^2\left|\phi_{ip}\right\rangle\left|\psi_{is}\right\rangle\left\langle\psi_{is}\right|$$
$$= \rho is + ig\left\langle\phi_{ip}\right|\hat{p}\left|\phi_{ip}\right\rangle[\rho_{is}, \hat{A}] - g^2\left\langle\phi_{ip}\right|\hat{p}^2\left|\phi_{ip}\right\rangle(\{\rho_{is}, \hat{A}^2\} - \hat{A}\rho_{is}\hat{A} - \left\langle\psi_{is}\right|\hat{A}^2\left|\psi_{is}\right\rangle\rho_{is})$$

Similarly, by taking the partial trace over the system Hilbert space, one obtains the normalized reduced pointer state $\rho_{fp}^{(2)}$ when the interaction strength is considered up to the second order

$$\rho_{fp}^{(2)} = |\phi_{ip}\rangle\langle\phi_{ip}| + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle|\phi_{ip}\rangle\langle\phi_{ip}|\hat{p} - ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle\hat{p}|\phi_{ip}\rangle\langle\phi_{ip}| -$$
$$g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle\hat{p}^2|\phi_{ip}\rangle\langle\phi_{ip}| + g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle\hat{p}|\phi_{ip}\rangle\langle\phi_{ip}|\hat{p}-$$
$$g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle|\phi_{ip}\rangle\langle\phi_{ip}|\hat{p}^2 + g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle\langle\phi_{ip}|\hat{p}^2|\phi_{ip}\rangle|\phi_{ip}\rangle\langle\phi_{ip}|$$
$$= \hat{\rho}_{ip} + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle[\hat{\rho_{ip}}, \hat{p}] - g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle(\{\hat{\rho}_{ip}, \hat{p}^2\} - \hat{p}\hat{\rho}_{ip}\hat{p} - \langle\phi_{ip}|\hat{p}^2|\phi_{ip}\rangle\hat{\rho}_{ip})$$

If one expands the interaction unitary up to the second order, the disturbance caused to the(reduced) pointer or the system state is subject to the same commutator that was present in the disturbance caused to the first order. However, in the second order case additional terms are present which cause disturbance even if the first order term comprising the commutator vanishes. Therefore, it can concluded that even if the commutator $[\rho_{is}, \hat{A}]$ ($[\hat{\rho_{ip}}, \hat{p}]$) vanishes, the system (pointer) state gets disturbed when the strength of the weak interaction is considered up to the second order whereas the first order disturbance remains zero. It should be noted that this disturbance due to the second order expansion would be small since only the term containing g^2 survives. Note that all the above expressions are symmetric with respect to the system and the pointer states. That is, if one does not pre-distinguish between the system and the pointer before the weak interaction, one will observe similar changes to the respective disturbed states and to the expectation values of their respective observables. Similarly, one can determine the shift in the expectation value of a pointer observable, $\langle\hat{p}'\rangle_{fp}^{(2)}$ when the interaction strength is considered up to the second order.

$$\left\langle\psi_{ft}^{(2)}\middle|\hat{p}'\middle|\psi_{ft}^{(2)}\right\rangle = (\langle\phi_{ip}|\langle\psi_{is}| + ig\langle\phi_{ip}|\hat{p}\otimes\langle\psi_{is}|\hat{A} - g^2\langle\phi_{ip}|\hat{p}^2\otimes\langle\psi_{is}|\hat{A}^2 \quad , \quad \hat{p}' \quad ,$$
$$|\psi_{is}\rangle|\phi_{ip}\rangle - ig\hat{A}|\psi_{is}\rangle\otimes\hat{p}|\phi_{ip}\rangle - g^2\hat{A}^2|\psi_{is}\rangle\otimes\hat{p}^2|\phi_{ip}\rangle)$$
$$= \langle\phi_{ip}|\hat{p}'|\phi_{ip}\rangle\langle\psi_{is}|\psi_{is}\rangle + ig\langle\phi_{ip}|\hat{p}\hat{p}'|\phi_{ip}\rangle\langle\psi_{is}|\hat{A}|\psi_{is}\rangle - ig\langle\phi_{ip}|\hat{p}'\hat{p}|\phi_{ip}\rangle\langle\psi_{is}|\hat{A}|\psi_{is}\rangle +$$
$$g^2\langle\phi_{ip}|\hat{p}\hat{p}'\hat{p}|\phi_{ip}\rangle\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle - g^2\langle\phi_{ip}|\hat{p}^2\hat{p}'|\phi_{ip}\rangle\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle - g^2\langle\phi_{ip}|\hat{p}'\hat{p}^2|\phi_{ip}\rangle\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle$$
$$= \langle\phi_{ip}|\hat{p}'|\phi_{ip}\rangle + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle\langle\phi_{ip}|\hat{p}\hat{p}' - \hat{p}'\hat{p}|\phi_{ip}\rangle - g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle\langle\phi_{ip}|\hat{p}^2\hat{p}' + \hat{p}'\hat{p}^2 - \hat{p}\hat{p}'\hat{p}|\phi_{ip}\rangle$$
$$= \langle\phi_{ip}|\hat{p}'|\phi_{ip}\rangle + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle\langle[\hat{p}, \hat{p}']\rangle_{ip} - g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle(\langle\{\hat{p}^2, \hat{p}'\}\rangle_{ip} - \langle\hat{p}\hat{p}'\hat{p}\rangle_{ip})$$

On normalization, an extra term comes in

$$\frac{\left\langle\psi_{ft}^{(2)}\middle|\hat{p}'\middle|\psi_{ft}^{(2)}\right\rangle}{\left\langle\psi_{ft}^{(2)}\middle|\psi_{ft}^{(2)}\right\rangle} = \langle\phi_{ip}|\hat{p}'|\phi_{ip}\rangle + ig\langle\psi_{is}|\hat{A}|\psi_{is}\rangle\langle[\hat{p}, \hat{p}']\rangle_{ip} - g^2\langle\psi_{is}|\hat{A}^2|\psi_{is}\rangle(\langle\{\hat{p}^2, \hat{p}'\}\rangle_{ip} - \langle\hat{p}\hat{p}'\hat{p}\rangle_{ip} - \langle\hat{p}^2\rangle_{ip}\rho_{ip})$$

The final expectation value can thus be presented as

$$\langle\hat{p}'\rangle_{fp}^{(2)} = \langle\hat{p}'\rangle_{ip} + ig\langle\hat{A}\rangle_{is}\langle[\hat{p}, \hat{p}']\rangle_{ip} - g^2\langle\hat{A}^2\rangle_{is}(\langle\{\hat{p}^2, \hat{p}'\}\rangle_{ip} - \langle\hat{p}\hat{p}'\hat{p}\rangle_{ip} - \langle\hat{p}^2\rangle_{ip}\rho_{ip})$$

Observations that are similar to those made for the case of the disturbance to the system or the pointer state via a change in their expressions when the interaction strength is considered up to the second order also hold for the shift of the expectation value of an arbitrary system or pointer observable. Note that here the commutator, $[\hat{p}, \hat{p}']$ as well as the anti-commutator, $\{\hat{p}^2, \hat{p}'\}$, in question are different. Analogous expression for the post-weak interaction expectation value would hold for the system observable.

$$\langle\hat{A}'\rangle_{fs}^{(2)} = \langle\hat{A}'\rangle_{is} + ig\langle\hat{p}\rangle_{ip}\langle[\hat{A}, \hat{A}']\rangle_{is} - g^2\langle\hat{p}^2\rangle_{ip}(\langle\{\hat{A}^2, \hat{A}'\}\rangle_{is} - \langle\hat{A}\hat{A}'\hat{A}\rangle_{is} - \langle\hat{A}^2\rangle_{is}\rho_{is})$$

To the first and the second orders in interaction strength, the shift in the expectation value of the arbitrary system(pointer) observable is weighted by expectation values of the first and the second powers of system(pointer) interaction observable respectively. Till now, we have treated only the weak interaction part of a variant of AAV's weak measurement formalism. Now, we come to the part of post-selection which is mainly responsible for the counterintuitive effects of weak measurement.

3.3.1 Results

1. Even if the system (pointer) state is not disturbed when the expansion of the unitary interaction is considered up to the first order, the system (pointer) state is disturbed when the expansion is taken up to the second order.

2. Analogous result holds for the change or the shift in the expectation value of an arbitrary system (pointer) observable.

3.4 After Post-Selection

The discussion in the three subsections above was symmetric with respect to the system and pointer state. Now, we will move to post-selection where the system state will now be projected due to a strong interaction resulting in the disentanglement of the system and pointer states. *Post selection* entails strong von Neumann interaction using another system observable $\hat{B}$ which does not commute with $\hat{A}$. The non-commutation ensures that the state on which the system is post-selected on is different from the state produced as a result of the action of $\hat{A}$ on the pre-selected system state. The expression for the post-selected joint state, $|\phi_{tPS}\rangle$ is

$$|\psi_{tPS}\rangle = (e^{-i|\psi_{fs}\rangle\langle\psi_{fs}|\otimes\hat{\mathbb{I}}})(e^{-ig(t)\hat{A}\otimes\hat{p}})\,|\psi_{is}\rangle\,|\phi_{ip}\rangle$$

The first unitary corresponds to a strong von Neumann interaction implemented using a projector on a subspace of the system Hilbert space and identity on the pointer Hilbert space. Note that the g factor which characterizes the strength of the interaction is 1 here. The second unitary corresponds to the weak interaction which was at the center of discussion in the last section. Proceeding with the computation, one has the final pointer state in the wave-function formulation:

$$\begin{aligned}|\psi_{tPS}\rangle &= (|\psi_{fs}\rangle\langle\psi_{fs}|\otimes\hat{\mathbb{I}})(1-ig\hat{A}\otimes\hat{p})\,|\psi_{is}\rangle\,|\phi_{ip}\rangle\\ &= |\psi_{fs}\rangle\otimes(\langle\psi_{fs}|\psi_{is}\rangle\,(|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle))\end{aligned}$$

While the system state has been projected on $|\psi_{fs}\rangle$ (and will be ignored hereafter) , the pointer state is

$$|\phi_{pPS}\rangle = \langle\psi_{fs}|\psi_{is}\rangle\,(|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle)$$

In the above, the factor $\langle\psi_{fs}|\psi_{is}\rangle$ which depends on the overlap between the *pre* and *post selected* system states indicates the probability of occurrence of this particular pointer state. The factor that weights the shift of this pointer state is defined in literature as the *weak value* [1].

$$\langle\hat{A}\rangle_w = \frac{\langle\psi_{fs}|\,\hat{A}\,|\psi_{is}\rangle}{\langle\psi_{fs}|\psi_{is}\rangle}$$

The denominator in the above expression represents the probability of successful post-selection on $|\psi_{fs}\rangle$ given $|\psi_{is}\rangle$ is the pre-selected state. Looking at the expression of the pointer state after post-selection, one can immediately see that the disturbance caused to the pointer state is now weighted by the weak value. This disturbance is inversely proportional to the probability of post-selection. Lesser the overlap between the pre (corresponding to the weak interaction) and the post-selected (strong von Neumann projected) system state, more is the disturbance caused to the pointer state after the entire procedure. Since the treatment in the above section was done using the density matrix formulation, lets compute the density matrix of the above pointer state.

$$\begin{aligned}
\rho_{pPS} &= \frac{|\phi_{pPS}\rangle\langle\phi_{pPS}|}{\langle\phi_{pPS}|\phi_{pPS}\rangle} \\
&= \frac{|\langle\psi_{fs}|\psi_{is}\rangle|^2(|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle)(\langle\phi_{ip}| + ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p})}{|\langle\psi_{fs}|\psi_{is}\rangle|^2(\langle\phi_{ip}| + ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p})(|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle)} \\
&= \frac{(|\phi_{ip}\rangle\langle\phi_{ip}| - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle\langle\phi_{ip}| + ig\langle\hat{A}\rangle_w^*\,|\phi_{ip}\rangle\langle\phi_{ip}|\,\hat{p})}{(1 + ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle)} \\
&= (|\phi_{ip}\rangle\langle\phi_{ip}| - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle\langle\phi_{ip}| + ig\langle\hat{A}\rangle_w^*\,|\phi_{ip}\rangle\langle\phi_{ip}|\,\hat{p})(1 - ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle \\
&\quad + ig\langle\hat{A}\rangle_w\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle) \\
&= (\rho_{ip} - ig\langle\hat{A}\rangle_w\hat{p}\rho_{ip} + ig\langle\hat{A}\rangle_w^*\rho_{ip}\hat{p} - ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle\,\rho_{ip} + ig\langle\hat{A}\rangle_w\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle\,\rho_{ip}) \\
&= \rho_{ip} + ig\,\mathrm{Re}\langle\hat{A}\rangle_w[\rho_{ip},\hat{p}] - g\,\mathrm{Im}\langle\hat{A}\rangle_w\{\rho_{ip},\hat{p}\} - 2g\,\mathrm{Im}\langle\hat{A}\rangle_w\,\langle\phi_{ip}|\,\hat{p}\,|\phi_{ip}\rangle \qquad (3.7)
\end{aligned}$$

From the above expression it can be inferred that even if the effect of the weak interaction is not present on the post-interaction (without the post-selection) pointer state (because $[\rho_{ip}, \hat{p}] = 0$), the effect of that weak interaction is present in the post-selected pointer state by virtue of the imaginary part of the relevant weak value. This is a purely non-classical effect of strong measurement (post-selection) done on a part of the joint entangled system + pointer state (after the weak interaction).

Now, lets consider the case when neither system nor pointer state was affected after the weak interaction. From the results of section 2.2.1, we know that this may imply $[\rho_{ip}, \hat{p}] = 0$ and $[\rho_{is}, \hat{A}] = 0$. The second identity in turn implies that the imaginary part of the weak value is zero (look at section 2.2.2). From these considerations, it can be inferred that the pointer state after the weak interaction and post-selection remains unaffected.

To demonstrate an operational effect, we will calculate the shift in the expectation value of an arbitrary pointer observable, $\hat{p}'$ with respect to the above pointer state

$$\begin{aligned}
\frac{\langle\phi_{pPS}|\,\hat{p}'\,|\phi_{pPS}\rangle}{\langle\phi_{pPS}|\phi_{pPS}\rangle} &= \frac{(\langle\phi_{ip}| + ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p}\quad,\hat{p}',\quad|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle)}{(\langle\phi_{ip}| + ig\langle\hat{A}\rangle_w^*\,\langle\phi_{ip}|\,\hat{p},\quad|\phi_{ip}\rangle - ig\langle\hat{A}\rangle_w\hat{p}\,|\phi_{ip}\rangle)} \\
&= \frac{\langle\hat{p}'\rangle_{ip} - ig\langle\hat{A}\rangle_w\langle\hat{p}'\hat{p}\rangle_{ip} + ig\langle\hat{A}\rangle_w^*\langle\hat{p}\hat{p}'\rangle_{ip}}{1 - ig\langle\hat{A}\rangle_w\langle\hat{p}\rangle_{ip} + ig\langle\hat{A}\rangle_w^*\langle\hat{p}\rangle_{ip}} \\
&= (\langle\hat{p}'\rangle_{ip} - ig\langle\hat{A}\rangle_w\langle\hat{p}'\hat{p}\rangle_{ip} + ig\langle\hat{A}\rangle_w^*\langle\hat{p}\hat{p}'\rangle_{ip})(1 - 2g\,\mathrm{Im}\langle\hat{A}\rangle_w\langle\hat{p}\rangle_{ip})
\end{aligned}$$

The general expression for the shift in the expectation value of an arbitrary pointer observable, without making any assumptions on the nature of the system or the pointer states and observables, can thus be expressed as

$$\langle\hat{p}'\rangle_{fp} = \langle\hat{p}\rangle_{ip} - 2g\,\mathrm{Im}\langle\hat{A}\rangle_w\langle\hat{p}'\rangle_{ip}\langle\hat{p}\rangle_{ip} - ig\,\mathrm{Re}\langle\hat{A}\rangle_w\langle[\hat{p}',\hat{p}]\rangle_{ip} + g\,\mathrm{Im}\langle\hat{A}\rangle_w\langle\{\hat{p}',\hat{p}\}\rangle$$

If neither the system nor the pointer state is affected after the weak interaction, the effect of that interaction will be present in the expectation value of an arbitrary pointer observable with respect to the pointer state after post-selection, provided $[\hat{p}', \hat{p}] \neq 0$.

3.4.1 Results

1. If neither the system nor the pointer state is affected after the weak interaction, the pointer state may remains unaffected after the post-selection as well.
2. Even if neither the system nor the pointer *state* is disturbed after the weak interaction, the effect of the weak interaction that occurred is present via a change in the *expectation value* of an arbitrary pointer observable with respect to the post-selected pointer state.

3.5 Jozsa - Reloaded

In all the above sections, no restriction was placed with regard to dimensionality, nature of the distribution etc. on the system and pointer states as well as the system and pointer observables involved at any stage of the weak measurement process. Now, lets consider the pointer to be in a continuous variable state which follows a complex valued distribution in position, $\phi(x)$. For the first case, let the observable whose expectation value we are interested in be the conjugate momentum, $\hat{p}_x$.

$$\langle \hat{p}_x \rangle_{fp} = \langle \hat{p}_x \rangle_{ip} + 2g \, \mathrm{Im} \langle \hat{A} \rangle_w \mathrm{var}(\hat{p}_x) \tag{3.8}$$

Instead, if the observable is position, $\hat{x}$, we have,

$$\begin{aligned} \langle \hat{x} \rangle_{fp} &= \langle \hat{x} \rangle_{ip} - 2g \, \mathrm{Im} \langle \hat{A} \rangle_w \langle \hat{x} \rangle_{ip} \langle \hat{p}_x \rangle_{ip} - ig \, \mathrm{Re} \langle \hat{A} \rangle_w \langle [\hat{x}, \hat{p}_x] \rangle + g \, \mathrm{Im} \langle \hat{A} \rangle_w \langle \{\hat{x}, \hat{p}_x\} \rangle \\ &= \langle \hat{x} \rangle_{ip} + \hbar g \, \mathrm{Im} \langle \hat{A} \rangle_w \, \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{var}(x) + g\hbar \, \mathrm{Re} \langle \hat{A} \rangle_w \end{aligned} \tag{3.9}$$

Here, $\mathrm{var}(x) = \langle x^2 \rangle - \langle x \rangle^2$ and we use the Heisenberg equations of motion [12] which prescribe the evolution of an operator under a given Hamiltonian ($H = p^2/2m + V(x)$):

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle x \rangle = \langle [\hat{x}, H] \rangle = i \langle \hat{p}_x \rangle$$

and

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle x^2 \rangle = \langle [\hat{x}^2, H] \rangle = i \langle \hat{x}\hat{p}_x + \hat{p}_x \hat{x} \rangle$$

The expressions for the shift in the expectation values of position as well as momentum derived above match those derived by Jozsa [12].

3.5.1 Results

1. The imaginary part of the weak value appears in a product with the initial position and momentum variance of the pointer wave-function in its position (3.9) and momentum (3.8) shifts respectively.
2. Measuring the expectation value of momentum gives access to only the imaginary part of the weak value while measuring the expectation value of position can potentially give access to both real and imaginary parts of the weak value.

The above results encompass the more primitive results derived in [1, 3, 8, 17].

3.6 Complex Weak Value and the Pointer State Correlations

An important of aspect of weak measurement is that the pointer observable involved in the weak interaction, is, in general, different from the one involved in the post-selection process. Thus, it is relevant to ask how the results obtained above would be modified in the presence of multi-mode pointer states with correlations between the different modes or degrees of freedom. Following [13], we review this problem by considering a two mode correlated pointer state, with complex valued position/momentum distribution in an infinite dimensional Hilbert state (continuous variable):

$$|\phi(p_1, p_2)\rangle = \int \phi(p_1, p_2) |p_1\rangle |p_2\rangle \, dp_1 dp_2$$

where $\phi(p_1, p_2) \neq \phi(p_1)\phi(p_2)$. Here, the weak interaction occurs between the first pointer degree of freedom and the system state whereas the projective measurement interaction occurs between the second degree of freedom and the system state. So, we have $\hat{H}_1 = e^{\sum_k |a_{2k}\rangle\langle a_{2k}| \otimes \hat{q}_2}$ and $\hat{H}_2 = e^{\hat{A}_1 \otimes \hat{q}_1}$. The weak interaction is done using $\hat{H}_1$ followed by the strong interaction using $\hat{H}_2$ and selecting one of the outcomes $|a_{2f}\rangle$. Thus, we have the final pointer state to be:

$$|\phi_f\rangle = \langle a_{2f}|\psi_i\rangle \, (1 + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{q}_1) \, |\phi_i(p_1, p_2 - a_{2f})\rangle$$

where $\langle \hat{A}_1\rangle_w = \frac{\langle a_{2f}|\hat{A}|\psi_i\rangle}{\langle a_{2f}|\psi_i\rangle}$ is the weak value. Calculating the expectation value of an arbitrary pointer observable $\hat{M}$

$$\langle \hat{M}\rangle = \frac{\langle \phi_f| \hat{M} |\phi_f\rangle}{\langle \phi_f|\phi_f\rangle}$$

Numerator:

$$\begin{aligned}
\langle \phi_f| \hat{M} |\phi_f\rangle &= |\langle a_{2f}|\psi_i\rangle|^2 \langle \phi_i| (1 - i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1)\hat{M}(1 + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{q}_1) |\phi_i\rangle \\
&= |\langle a_{2f}|\psi_i\rangle|^2 \langle \phi_i| (1 - i\lambda_1 \langle \hat{A}_1\rangle_w^*)(\hat{M} + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{M}\hat{q}_1) |\phi_i\rangle \\
&= |\langle a_{2f}|\psi_i\rangle|^2 \langle \phi_i| (\hat{M} + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{M}\hat{q}_1 - i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1\hat{M}) |\phi_i\rangle \\
&= |\langle a_{2f}|\psi_i\rangle|^2 (\langle \hat{M}\rangle_i + i\lambda_1 a \langle [\hat{M}, \hat{q}_1]\rangle_i - \lambda_1 b \langle \{\hat{q}_1, \hat{M}\}\rangle_i)
\end{aligned}$$

Where $\langle . \rangle_i$ indicates the expectation value with respect to the initial pointer state. Complex weak value is expanded such that $\langle \hat{A}\rangle_w = a + ib$, where a and b are the real and the imaginary parts of the weak value. Calculating the denominator in 3.6:

$$\begin{aligned}
\langle \phi_f|\phi_f\rangle &= |\langle a_{2f}|\psi_i\rangle|^2 \langle \phi_i| (1 - i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1)(1 + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{q}_1) |\phi_i\rangle \\
&= |\langle a_{2f}|\psi_i\rangle|^2 (1 - 2\lambda_1 b \langle \hat{q}_1\rangle_i)
\end{aligned}$$

Going back to 3.6:

$$\frac{\langle \phi_f| \hat{M} |\phi_f\rangle}{\langle \phi_f|\phi_f\rangle} = (\langle \hat{M}\rangle_i + i\lambda_1 a \langle [\hat{M}, \hat{q}_1]\rangle_i - \lambda_1 b \langle \{\hat{q}_1, \hat{M}\}\rangle_i)(1 - 2\lambda_1 b \langle \hat{q}_1\rangle_i)^{-1}$$

The denominator can be brought to numerator and Taylor expanded up to the first order in the weak limit:

$$\langle\hat{M}\rangle_f = (\langle\hat{M}\rangle_i + i\lambda_1 a\langle[\hat{M},\hat{q}_1]\rangle_i - \lambda_1 b\langle\{\hat{q}_1,\hat{M}\}\rangle_i)(1+2\lambda_1 b\langle\hat{q}_1\rangle_i)$$

Now, let $\hat{M}$ be $\hat{q}_1$. Thus, we have:

$$\begin{aligned}\langle\hat{q}_1\rangle_f &= (\langle\hat{q}_1\rangle_i - 2\lambda_1 b\langle\hat{q}_1^2\rangle_i)(1+2\lambda_1 b\langle\hat{q}_1\rangle_i)\\ &= \langle\hat{q}_1\rangle_i - 2\lambda_1 b\langle\hat{q}_1^2\rangle_i + 2\lambda_1 b\langle\hat{q}_1\rangle_i\langle\hat{q}_1\rangle_i\\ &= \langle\hat{q}_1\rangle_i - 2\lambda_1 b(\langle\hat{q}_1^2\rangle_i + \langle\hat{q}_1\rangle_i\langle\hat{q}_1\rangle_i)\\ &= \langle\hat{q}_1\rangle_i - 2\lambda_1 b\,\delta\hat{q}_1\end{aligned}$$

where $\delta\hat{q}_1 = \langle\hat{q}_1^2\rangle - \langle\hat{q}_1\rangle\langle\hat{q}_1\rangle$. If one considers $\hat{M}=\hat{p}_1$, following a similar calculation and using the canonical commutation relation $[\hat{q},\hat{p}]=i\hbar$, one has

$$\begin{aligned}\langle\hat{p}_1\rangle_f &= (\langle\hat{p}_1\rangle_i - 2\lambda_1 b\langle\hat{p}_1^2\rangle_i)(1+2\lambda_1 b\langle\hat{q}_1\rangle_i)\\ &= \langle\hat{p}_1\rangle + 2\lambda_1 b\langle\hat{p}_1\rangle\langle\hat{q}_1\rangle + \lambda_1 a\hbar - \lambda_1 b\langle\hat{q}_1\hat{p}_1 + \hat{p}_1\hat{q}_1\rangle\end{aligned}$$

Now, using the Heisenberg equations of motion as done earlier under the Hamiltonian $H=(p_1^2+p_2^2)/2m + V(x_1,x_2)$:

$$\begin{aligned}i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{q}_1\rangle &= \langle[\hat{q}_1,H]\rangle\\ &= i\langle\hat{p}_1\rangle\\ i\hbar\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{q}_1^2\rangle &= \langle[\hat{q}_1^2,H]\rangle\\ &= i\langle\hat{q}_1\hat{p}_1+\hat{p}_1\hat{q}_1\rangle\end{aligned}\tag{3.10}$$

we have

$$\langle\hat{p}_1\rangle_f = \langle\hat{p}_1\rangle_i - \hbar\lambda a + \hbar\lambda b\frac{\partial\,\delta(q_1)}{\partial t}$$

The above results are almost identical to Jozsa's except, perhaps, the trivial change of a sign (because the way we expanded the weak interaction unitary). However, if one considers $\hat{M}=\hat{q}_2$, one finds:

$$\begin{aligned}\langle\hat{q}_2\rangle_f &= (\langle\hat{q}_2\rangle_i + i\lambda_1 a\langle[\hat{q}_2,\hat{q}_1]\rangle_i - \lambda_1 b\langle\{\hat{q}_1,\hat{q}_2\}\rangle_i)(1+2\lambda_1 b\langle\hat{q}_1\rangle_i)\\ &= (\langle\hat{q}_2\rangle_i - \lambda_1 b\langle\hat{q}_1\hat{q}_2+\hat{q}_2\hat{q}_1\rangle_i)(1+2\lambda_1 b\langle\hat{q}_1\rangle_i)\\ &= \langle\hat{q}_2\rangle_i + 2\lambda_1 b\langle\hat{q}_2\rangle_i\langle\hat{q}_1\rangle_i - 2\lambda_1 b\langle\hat{q}_1\hat{q}_2\rangle_i\end{aligned}$$

Accounting for the correlation between the pointer degrees of freedom, using $\mathrm{corr}(\hat{q}\hat{q}_2)_i = \langle\hat{q}_1\hat{q}_2\rangle_i - \langle\hat{q}_1\rangle_i\langle\hat{q}_2\rangle_i$ we have

$$\langle\hat{q}_2\rangle_f = \langle\hat{q}_2\rangle_i - 2\lambda b\mathrm{corr}(\hat{q}_1\hat{q}_2)_i$$

Going through similar calculations for $\hat{M}=\hat{p}_2$ and using $\langle\hat{p}_2\rangle_i = \langle\hat{p}_2\rangle_i + a_{2l}$, one finds

$$\langle\hat{p}_2\rangle_f = \langle\hat{p}_2\rangle_{in} + 2\lambda b\mathrm{corr}(q_1,p_2)$$

3.6.1 Results

1. When the projective measurement for post-selection of the system state is done via an interaction with the pointer degree of freedom that is different from the one that is used in the preceding weak interaction, the effect of pointer state correlations are manifested in the shift of the expectation value of the pointer observable corresponding to the strong post-selective interaction.

In addition to the above results, further results similar in spirit with some extra terms added can be found for the sequential weak interactions corresponding to two different pointer degrees of freedom and post-selection corresponding to the third degree of freedom [13].

Chapter 4

Applications of Finite Strength Quantum Measurement and Weak Value

After the conceptual and mathematical treatment of weak quantum measurement, we come to some of its applications.

4.1 WM with Qubit Pointer - Creating Transitive Entanglement

The weak measurement field has seen relatively less development when the pointer state is a qubit or defined in a discrete variable state [34]. Steering clear of the foundational issues of a qubit pointer state, especially those that concern the dimensionality of the pointer and the measurement strength, we proceed with the following investigations.

Weak interaction between the system and the pointer states entangles them. Can several such interactions create entanglement between previously uncorrelated states? More precisely, can weak interaction between system and pointer 1 followed by weak interaction between system and pointer 2 followed by post-selection on the system create entanglement between pointer 1 and 2? We proceed with our first analysis by considering a pure, randomly oriented single qubit system state and a pure, separable randomly oriented, 2 qubit pointer state. Sequential weak interactions are performed, with the two different pointer qubits. The initial pointer states are

$$|\psi_{a1}\rangle = \cos\left(\frac{\theta_1}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta_1}{2}\right)|1\rangle \tag{4.1}$$

for first pointer qubit and

$$|\psi_{a2}\rangle = \cos\left(\frac{\theta_2}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta_2}{2}\right)|2\rangle \tag{4.2}$$

for the second pointer qubit. Pointer observables in both the weak interactions are measurements in the randomly oriented Pauli basis in different and arbitrary directions:

$$\hat{P}_1 = \vec{m}.\vec{\sigma} \tag{4.3}$$

where $\vec{m}.\vec{\sigma} = \hat{m}_1\hat{\sigma}_1 + \hat{m}_2\hat{\sigma}_2 + \hat{m}_3\hat{\sigma}_3$ and

$$\hat{P}_2 = \vec{n}.\vec{\sigma} \tag{4.4}$$

where $\vec{n}.\vec{\sigma} = \hat{n}_1\hat{\sigma}_1 + \hat{n}_1\hat{\sigma}_2 + \hat{n}_3\hat{\sigma}_3$ and σ_i are the 2×2 Pauli matrices. Owing to weak coupling, the expansion of the interaction unitaries which couple the first and the second pointer qubits respectively with the system state is considered up to first order: The interaction unitary between the system state and pointer 1 is

$$\hat{U}_1 = e^{-ig\hat{A}_1 \otimes \hat{P}_1 \otimes \mathbb{I}}$$

where $\hat{A}_1$ is the system measurement observable for the first weak interaction. The interaction unitary between the system state and pointer 2 is

$$\hat{U}_2 = e^{-ig\hat{A}_2 \otimes \mathbb{I} \otimes \hat{P}_2}$$

where $\hat{A}_2$ is the system measurement observable for the second weak interaction. Thus, the sequential weak interaction looks like:

$$\begin{aligned}
|\psi_{ft}\rangle &= \hat{U}_2\hat{U}_1 |\psi_{is}\rangle |\psi_{a1}\rangle |\psi_{a2}\rangle \\
&= (1 - ig\hat{A}_2 \otimes \mathbb{I} \otimes \hat{P}_2)(1 - ig\hat{A}_1 \otimes \hat{P}_1 \otimes \mathbb{I}) |\psi_{is}\rangle |\psi_{a1}\rangle |\psi_{a2}\rangle \\
&= |\psi_{is}\rangle |\psi_{a1}\rangle |\psi_{a2}\rangle - ig\hat{A}_2 |\psi_{is}\rangle \otimes \mathbb{I} |\psi_{a1}\rangle \otimes \hat{P}_2 |\psi_{a2}\rangle - ig\hat{A}_1 |\psi_{is}\rangle \otimes \hat{P}_1 |\psi_{a1}\rangle \otimes \mathbb{I} |\psi_{a2}\rangle
\end{aligned}$$

Doing post-selection on the system state $|\psi_{fs}\rangle$, the pointer state becomes:

$$|\psi_{fp}\rangle = \langle |\psi_{fs}\rangle || \psi_{is}\rangle\rangle \left(|\psi_{a1}\rangle |\psi_{a2}\rangle - ig\langle \hat{A}_2 \rangle_w \mathbb{I} |\psi_{a1}\rangle \otimes \hat{P}_2 |\psi_{a2}\rangle - ig\langle \hat{A}_1 \rangle_w \hat{P}_1 |\psi_{a1}\rangle \otimes \mathbb{I} |\psi_{a2}\rangle \right)$$

Plugging in the expressions for the pointer observables during the respective weak interactions and initial pointer states, the joint final state is:

$$\begin{aligned}
|\psi_{ft}\rangle = \langle \psi_{fs}|\psi_{is}\rangle \Bigg(|\psi_{a1}\rangle |\psi_{a2}\rangle - ig_2(A_2)_w |\psi_{a1}\rangle \begin{bmatrix} m_3 & m_1 - im_2 \\ m_1 + im_2 & -m_3 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta_2}{2}\right) \\ e^{(i\phi_2)} \sin\left(\frac{\theta_2}{2}\right) \end{bmatrix} - \\
ig_1(A_1)_w \begin{bmatrix} n_3 & n_1 - in_2 \\ n_1 + in_2 & -n_3 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\theta_1}{2}\right) \\ e^{(i\phi_1)} \sin\left(\frac{\theta_1}{2}\right) \end{bmatrix} |\psi_{a2}\rangle \Bigg)
\end{aligned}$$

From the above state, one can immediately see that the pointer states which were separable before the sequential weak measurement procedure are now entangled (since they are non-separable). It would be beneficial to quantify this correlation using some measure. However, owing to a large number of parameters and limited computational resources, we restrict this analysis to the above expression. Note that the system measurement observables, pre and post selected states are kept flexible and further investigations can be undertaken by considering various cases of these.

4.1.1 Two Qubit Werner State as Pointer

Here, we intend to investigate the change that occurs to the Werner state (which is used as a pointer in this case) by following a sequential weak measurement procedure similar to the one above. Will the amount of the correlations in the Werner state change? The

Werner state is particularly significant because it spans the regimes of separability for $z \in [0, 1/3]$, discord for $z \in [0, 1]$, entanglement for $z \in [1, 1/3]$ [32]:

$$\rho_w = (\frac{1-z}{4})\hat{\mathbb{I}} + z \left|\psi^-\right\rangle \left\langle\psi^-\right| \tag{4.5}$$

where $|\psi^-\rangle\langle\psi^-|$ is the pure, maximally entangled singlet state. Here also, randomly oriented Pauli vectors identical to the ones used in the last section are used as pointer measurement observables for both weak interactions. The final pointer state is given by

$$\rho_{ft} = \mathrm{Tr_s}[\rho_{\mathrm{sf}} \mathrm{U_2 U_1} \rho_{\mathrm{si}} \rho_{\mathrm{w}} \mathrm{U_1^\dagger U_2^\dagger}]$$

where U_1 and U_2 correspond to the sequential weak interactions of 1st and 2nd qubits of the pointer respectively with the system qubit. The above state comes out to be:

$$\rho_{ft} = \mathrm{Tr}[\rho_{\mathrm{sf}}\rho_{\mathrm{si}}][\rho_{\mathrm{w}} - \mathrm{ig_1}\langle\hat{\mathrm{A}}_{\mathrm{1s}}\rangle_{\mathrm{weak}}(\hat{\mathrm{P}}^\dagger_{\mathrm{a1}} \otimes \hat{\mathbb{I}}_{\mathrm{a_2}})\rho_{\mathrm{w}} - \mathrm{ig_2}\langle\hat{\mathrm{A}}_{\mathrm{2s}}\rangle_{\mathrm{weak}}(\hat{\mathbb{I}}_{\mathrm{a1}} \otimes \hat{\mathrm{P}}_{\mathrm{a2}})\rho_{\mathrm{w}} + \mathrm{ig_2}\langle\hat{\mathrm{A}}^\dagger_{\mathrm{2s}}\rangle_{\mathrm{weak}}\rho_{\mathrm{w}}(\hat{\mathbb{I}}_{\mathrm{a1}} \otimes \hat{\mathrm{P}}^\dagger_{\mathrm{a2}}) + ig_1\langle\hat{A}^\dagger_{1s}\rangle_{weak}\rho_w(\hat{\mathbb{I}}_{a2} \otimes \hat{P}^\dagger_{a2})]$$

After analyzing the above state using the positive partial transpose criterion [11, 23], no change was observed in the PPT parameters[1] before the weak measurement and after the weak measurement when the coupling strength for the weak interaction g was considered small.

4.2 Joint Weak Value

Joint expectation value comprises correlations between the two potentially non-local particles on which the two observables are measured. However, it is difficult to measure the joint expectation value of incompatible or non-commuting observables in the laboratory. The reason behind this is the difficulty to engineer a suitable Hamiltonian comprising the observables in a quantum optics architecture [24, 25]. Joint weak value can serve tasks similar to the joint expectation value. In previous works, the joint weak value was calculated for a weak measurement scheme using an engineered Hamiltonian which naturally couples both the momentum dimensions, P_x and P_y of the particle with the incompatible observables $\hat{A}_1$ and $\hat{A}_2$ respectively whose joint weak value we are interested in. In one of these works [25] they successfully obtained the real as well as the imaginary part of the joint weak value. The pointer state used here was a two mode separable Gaussian state. In another work [24] in which the same Hamiltonian was used, in addition to obtaining the joint weak value of incompatible observables, higher orders of the real and the imaginary part of the weak value of single observable was also obtained. The pointer state used in this work was a two dimensional correlated Laguerre-Gauss mode state which is prevalent widely in laser optics. There has been work indicating that the higher orders of the weak value could be beneficial for the estimation of a small parameter (present in the weak interaction) using weak value amplification. Here, we present similar results and more using a Hamiltonian that is easier to implement experimentally because it uses two separate weak interaction corresponding to different degrees of freedom of the pointer. At the outset, in our formalism there is no restriction on the kind of pointer state used. We achieve this by using sequential weak interactions between the system state and the two

[1]done in Mathematica – see supplemental materials

mode pointer state and expand the interaction strength up to the second order. Also, we do this analysis in the Schrödinger picture which simplifies matters here instead of the Heisenberg picture used in the above two papers. Note that in both papers cited above as well, the joint weak value could be obtained only when the weak interaction strength was considered up to the second order with the terms corresponding to the first order dropping out.

4.2.1 General Treatment

We write the total state of the system + pointer and act on it with the sequential weak interaction unitary:

$$|\phi_{ft}\rangle = e^{i\lambda_1 \hat{A}_1 \otimes \hat{q}_1} e^{i\lambda_2 \hat{A}_2 \otimes \hat{q}_2} |\psi_i\rangle |\phi_i(p_1, p_2)\rangle$$

where $\hat{A}_1$ and $\hat{A}_2$ are the observables acting on the system ($|\psi_i\rangle$) Hilbert space during the first and the second weak interactions respectively and $\hat{q}_1$ and $\hat{q}_2$ are those acting on the pointer Hilbert space. $|\phi_i(p_1, p_2)\rangle$ is the two mode initial pointer state and λ_1 and λ_2 are the interaction strengths corresponding to the first and second weak interactions respectively. Expanding both the unitary interactions up to the second order, we have:

$$|\phi_{ft}\rangle = (1 + i\lambda_1 \hat{A}_1 \otimes \hat{q}_1 - \lambda_1^2 \hat{A}_1^2 \otimes \hat{q}_1^2) 1 + i\lambda_2 \hat{A}_2 \otimes \hat{q}_2 - \lambda_2^2 \hat{A}_2^2 \otimes \hat{q}_2^2 |\psi_i\rangle |\phi_i(p_1, p_2)\rangle$$

Doing the post-selection on a system state $|a_l\rangle$, we have the final pointer state:

$$|\phi_f\rangle = \langle a_l|\psi_i\rangle (1 + i\lambda_2 \langle \hat{A}_2\rangle_w \hat{q}_2 - \lambda_2^2 \langle \hat{A}_2^2\rangle_w \hat{q}_2^2 + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{q}_1 - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w \hat{q}_1 \hat{q}_2 - \lambda_1^2 \langle \hat{A}_1^2\rangle_w \hat{q}_1^2)$$

To observe anything, one should calculate the expectation value of an observable with respect to the final pointer state. Thus, we proceed with the calculation of the expectation value of an arbitrary pointer observable $\hat{M}$:

$$\langle \hat{M}\rangle = \frac{\left\langle \phi_f \hat{M} \right| |\phi_f\rangle}{\langle \phi_f|\phi_f\rangle}$$

Calculating the numerator first:

$$\begin{aligned}
\langle \psi_f| \hat{M} |\psi_f\rangle = |\langle a_{3l}|\psi_i\rangle|^2 \langle \phi_i| (1 - i\lambda_2 \langle \hat{A}_2\rangle_w^* \hat{q}_2 - \lambda_2^2 \langle \hat{A}_2^2\rangle_w^* \hat{q}_2^2 - i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1 - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w^* \hat{q}_1 \hat{q}_2 - \\
\lambda_1 \langle \hat{A}_1^2\rangle_w^* \hat{q}_1^2) \hat{M} (1 + i\lambda_2 \langle \hat{A}_2\rangle_w \hat{q}_2 - \lambda_2^2 \langle \hat{A}_2^2\rangle_w \hat{q}_2^2 + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{q}_1 - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w \hat{q}_1 \hat{q}_2 - \lambda_1 \langle \hat{A}_1^2\rangle_w \hat{q}_1^2) \\
= |\langle a_{3l}|\psi_i\rangle|^2 \langle \phi_i| (\hat{M} - i\lambda_2 \langle \hat{A}_2\rangle_w^* \hat{q}_2 \hat{M} - \lambda_2^2 \langle \hat{A}_2^2\rangle_w^* \hat{q}_2^2 \hat{M} - i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1 \hat{M} - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w^* \hat{q}_1 \hat{q}_2 \hat{M} \\
-\lambda_1^2 \langle \hat{A}_1^2\rangle_w^* \hat{q}_1^2 \hat{M} + i\lambda_2 \langle \hat{A}_2\rangle_w \hat{M} \hat{q}_2 + \lambda_2^2 |\langle \hat{A}_2\rangle_w|^2 \hat{q}_2 \hat{M} \hat{q}_2 + \lambda_1\lambda_2 \langle \hat{A}_1\rangle_w^* \langle \hat{A}_2\rangle_w \hat{q}_1 \hat{M} \hat{q}_2 \\
-\lambda_2^2 \langle \hat{A}_2^2\rangle_w \hat{M} \hat{q}_2^2 + i\lambda_1 \langle \hat{A}_1\rangle_w \hat{M} \hat{q}_1 + \lambda_1\lambda_2 \langle \hat{A}_2\rangle_w^* \langle \hat{A}_1\rangle_w \hat{q}_2 \hat{M} \hat{q}_1 + \\
\lambda_1^2 |\langle \hat{A}_1\rangle_w|^2 \hat{q}_1 \hat{M} \hat{q}_1 - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w \hat{M} \hat{q}_1 \hat{q}_2 - \lambda_1^2 \langle \hat{A}_1^2\rangle_w \hat{M} \hat{q}_1^2) |\phi_i\rangle
\end{aligned}$$

Computing the denominator:

$$\begin{aligned}
\langle \phi_f|\phi_f\rangle = \langle \phi_i| (1 - i\lambda_2 \langle \hat{A}_2\rangle_w^* \hat{q}_2 - \lambda_2^2 \langle \hat{A}_2^2\rangle_w^* \hat{q}_2^2 - i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1 - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w^* \hat{q}_1 \hat{q}_2 - \lambda_1^2 \langle \hat{A}_1^2\rangle_w^* \hat{q}_1^2 + \\
\lambda_2^2 |\langle \hat{A}_2\rangle_w|^2 \hat{q}_2^2 + \lambda_1\lambda_2 \langle \hat{A}_2\rangle_w^* \langle \hat{A}_1\rangle_w \hat{q}_2 \hat{q}_1 + \lambda_2^2 \langle \hat{A}_2^2\rangle_w^* \hat{q}_2^2 - i\lambda_2 \langle \hat{A}_2\rangle_w^* \hat{q}_2 - \\
i\lambda_1 \langle \hat{A}_1\rangle_w^* \hat{q}_1 + \lambda_1\lambda_2 \langle \hat{A}_1\rangle_w^* \langle \hat{A}_2\rangle_w \hat{q}_1 \hat{q}_2 + \lambda_1^2 |\langle \hat{A}_1\rangle_w|^2 \hat{q}_1^2 - \lambda_1\lambda_2 \langle \hat{A}_1 \hat{A}_2\rangle_w^* \hat{q}_1 \hat{q}_2 - \lambda_1^2 \langle \hat{A}_1^2\rangle_w^* \hat{q}_1^2) |\phi_i\rangle
\end{aligned}$$

The denominator can be brought up and expanded to the second order in λ_1 and λ_2

$$\langle \hat{M}\rangle_f = \langle \phi_f| \hat{M} |\phi_f\rangle \langle \phi_f|\phi_f\rangle^{-1}$$

Thus, we have:

$$\langle\hat{M}\rangle_f = (\langle\hat{M}\rangle_i - \lambda_2 d\langle\{\hat{q}_2,\hat{M}\}\rangle_i - i\lambda_2 c\langle[\hat{q}_2,\hat{M}]\rangle_i - \lambda_2^2 c'\langle\{\hat{q}_2^2,\hat{M}\}\rangle_i + \lambda_2^2 d'\langle[\hat{q}_2^2,\hat{M}]\rangle_i - i\lambda_1 a\langle[\hat{q}_1,\hat{M}]\rangle_i$$
$$-\lambda_1 b\langle\{\hat{q}_1,\hat{M}\}\rangle_i - \lambda_1\lambda_2 x\langle\{\hat{q}_1\hat{q}_2,\hat{M}\}\rangle_i + iy\lambda_1\lambda_2\langle[\hat{q}_1\hat{q}_2,\hat{M}]\rangle_i - \lambda_1^2 a'\langle\{\hat{q}_1^2,\hat{M}\}\rangle_i + ib'\lambda_1^2 d'\langle[\hat{q}_1^2,\hat{M}]\rangle_i +$$
$$\lambda_2^2|\langle\hat{A}_2\rangle_w|^2\langle\hat{q}_2\hat{M}\hat{q}_2\rangle_i + \lambda_1^2|\langle\hat{A}_1\rangle_w|^2\langle\hat{q}_1\hat{M}\hat{q}_1\rangle_i + \lambda_{\lambda 2}(fV + f^*V^\dagger))(1 - 2i\lambda_2 d\langle\hat{q}_2\rangle_i + 2c'\lambda_2^2\langle\hat{q}_2^2\rangle_i$$
$$+ 2b\lambda_1\langle\hat{q}_1\rangle_i + 2x\lambda_1\lambda_2^2\langle\hat{q}_1\hat{q}_2\rangle_i + 2a\lambda_1^2\langle\hat{q}_1^2\rangle_i - \lambda_1\lambda_2 \operatorname{Re}(\langle\hat{A}_2\rangle_w^*\langle\hat{A}_1\rangle_w)\langle\{\hat{q}_2,\hat{q}_1\}\rangle_i +$$
$$i\lambda_1\lambda_2 \operatorname{Im}(\langle\hat{A}_2\rangle_w^*\langle\hat{A}_1\rangle_w)\langle[\hat{q}_1,\hat{q}_2]\rangle_i - \lambda_2^2|\langle\hat{A}_2\rangle_w|^2\langle\hat{q}_2^2\rangle - \lambda_1^2|\langle\hat{A}_1\rangle_w|^2\langle\hat{q}_1^2\rangle -$$
$$4\lambda_2^2 d^2\langle\hat{q}_2^2\rangle + 4\lambda_1^2 b^2\langle\hat{q}_1^2\rangle - 4i\lambda_1\lambda_2 bd\langle\{\hat{q}_1,\hat{q}_2\}\rangle)$$

In the above expression the following replacements have been made by expanding the respective weak values in their real and imaginary components:

$$\langle\hat{A}_1\rangle_w = a + bi$$
$$\langle\hat{A}_2\rangle_w = c + di$$
$$\langle\hat{A}_1^2\rangle_w = a' + b'i$$
$$\langle\hat{A}_2^2\rangle_w = c' + d'i$$
$$\langle\hat{A}_1\hat{A}_2\rangle_w = x + yi$$

as well as $f \equiv \langle\hat{A}_1\rangle_w^*\langle\hat{A}_2\rangle_w$ and $V \equiv \hat{q}_1\hat{M}\hat{q}_2$.

4.2.2 Laguerre-Gauss Mode Pointer State

One need not be perturbed by looking at the above gargantuan expression. Instead, one should think about the pointer states whose specific properties will make a lot of these terms go away. We did this exercise for the Laguerre-Gauss mode pointer state [24]:

$$\phi_{LG} = N[x + isgn(l)y]^{|l|}\exp\left(-\frac{x^2+y^2}{4\sigma^2}\right)$$

The above state is a correlated Gaussian state centered at zero. The state is endowed with orbital angular momentum which is characterized by l. We replace $\hat{M} = \hat{p}_1\hat{p}_2$ and proceed with the calculation of the expectation value of all the relevant commutators and anti-commutators in the above expression using Mathematica[2]. The final expression of $\langle\hat{p}_1\hat{p}_2\rangle_f$ for the Laguerre-Gauss mode pointer state described above when $l = 1$ is

$$\langle\hat{p}_1\hat{p}_2\rangle_f = i\lambda_2 c' l - d'\lambda_2^2 l - i\lambda_1\lambda_2 xl/2 + 2il\lambda_1^2 a' + b'\lambda_1^2 l +$$
$$i\lambda_2^2|\langle\hat{A}_2\rangle_w|^2 l - i\lambda_1^2|\langle\hat{A}_1\rangle_w|^2 l - (3\lambda_1\lambda_2/4)\operatorname{Re}\langle\hat{A}_1\rangle_w^*\langle\hat{A}_2\rangle_w$$

In the above expression, the real part x of the joint weak value $\langle\hat{A}_1\hat{A}_2\rangle_w$ is present, as per our goal. We also see the terms c' and d' which give the full second order weak value $\langle\hat{A}_2^2\rangle_w$ and the terms a' and b' give the full second order weak value $\langle\hat{A}_1^2\rangle_w$. For a choice of $\hat{M} = \hat{p}_1\hat{q}_2$, we will find the imaginary part of the joint weak value, y, after going through similar calculations.

[2]see supplemental materials

4.3 Remote Determination of the Weak Value

The effect of correlations between different degrees of freedom of a multidimensional pointer state have been shown in the foundations section in the context of the complex weak values. Now, we consider a novel scenario exploiting these correlations.

There exist some protocols for transferring the quantum state from one party to another spatially separated party, the most prominent and simplest among which, is, quantum teleportation [5]. However, all these protocols have a drawback. All of them rely on the kind of resource state that is shared between the two parties. Inevitably, the kind of states that can be transferred is also limited. Here, we give a protocol for transferring the real as well as the imaginary parts of the weak value pertaining to a weak interaction carried out by Alice to Bob, who is situated at a distance from Bob. Using the real and the imaginary parts of the weak value the quantum state which was the pre-selected system state in the weak interaction can be found using the methods developed by [15, 16, 30].

4.3.1 Protocol

Alice and Bob share a bipartite correlated quantum state, ρ_{ab} which could be pure or mixed. The state is not separable, $\rho_{ab} neq \rho_a \rho_b$ and is therefore correlated [10]. Alice has an ancilla state ρ_{i1}, unknown to her, whose information she intends to transfer to Bob, who holds the B part of the correlated state. To do so, Alice first performs a weak interaction between her part (A) of the state and the ancilla I_1 using the unitary $U_1 = e^{ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I}}$. In this unitary, the Hamiltonian is designed such that the interaction occurs only between A and I_1 while B remains untouched (identity operation). Thus, after the weak interaction, the state is:

$$\begin{aligned}\rho_{tw} &= U_1 \rho_{i1} \otimes \rho_{ab} U_1^\dagger \\ &= e^{ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I}} \rho_{i1} \otimes \rho_{ab} e^{-ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I}} \\ &= (1 + ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I}) \rho_{i1} \otimes \rho_{ab} (1 - ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I})\end{aligned}$$

Where the coupling unitary interaction is expanded up to the first order in keeping with the weak approximation ($g <<$). Alice then performs the post-selection on the k^{th} eigenstate of ρ_{i1} and on the l^{th} eigenstate of ρ_A using the respective projectors P_{i1}^k and P_a^l.

$$\rho_{tf} = P_{i1}^k P_a^l (1 + ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I}) \rho_{i1} \otimes \rho_{ab} (1 - ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I})$$

Bob's final state is the traced out version of the above state over the parts I_1 and A.

$$\begin{aligned}\rho_b &= \mathrm{Tr}_{i1,a}(\rho_{tf}) \\ &= \mathrm{Tr}_{i1,a}(P_{i1}^k P_a^l (1 + ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I}) \rho_{i1} \otimes \rho_{ab} (1 - ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I})) \\ &= \mathrm{Tr}_{i1,a}(P_{i1}^k P_a^l (\rho_{i1} \otimes \rho_{ab} + ig\hat{I}_1 \rho_{i1} \otimes (\hat{A} \otimes \mathbb{I}) \rho_{ab})(1 - ig\hat{I}_1 \otimes \hat{A} \otimes \mathbb{I})) \\ &= \mathrm{Tr}_{i1,a}((P_{i1}^k \otimes P_a^l \otimes \mathbb{I})(\rho_{i1} \otimes \rho_{ab} - ig\rho_{i1}\hat{I}_1 \otimes \rho_{ab}(\hat{A} \otimes \mathbb{I}) + ig\hat{I}_1 \rho_{i1} \otimes (\hat{A} \otimes \mathbb{I}) \rho_{ab})) \\ &= \mathrm{Tr}_{i1,a}((P_{i1}^k \rho_{i1} \otimes (P_a^l \otimes \mathbb{I}) \rho_{ab} - igP_{i1}^k \rho_{i1} \hat{I}_1 \otimes (P_a^l \otimes \mathbb{I}) \rho_{ab} (\hat{A} \otimes \mathbb{I}) \\ &\quad + igP_{i1}^k \hat{I}_1 \rho_{i1} \otimes (P_a^l \otimes \mathbb{I})(\hat{A} \otimes \mathbb{I}) \rho_{ab}))\end{aligned}$$

Notice that I am using an identity operation on any Hilbert space where there is no action of any operator. Now, I will take the partial trace operation over I_1 and A inside the parenthesis:

$$\rho_b = ((\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\,\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1}\hat{I}_1)\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I})) \\ +ig\,\mathrm{Tr}_{i1}(P_{i1}^k\hat{I}_1\rho_{i1})\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab}))) \\ = \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\,\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1}\hat{I}_1)\otimes\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I}))+ \\ ig\,\mathrm{Tr}_{i1}(P_{i1}^k\hat{I}_1\rho_{i1})\otimes\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab}))) \\ = \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\langle\hat{I}_1\rangle_w^*\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I}))+ \\ ig\langle\hat{I}_1\rangle_w\otimes\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab}))) \tag{4.6}$$

where we have defined the weak value corresponding to the weak interaction performed by Alice between her part A of the shared state and her ancilla, $\hat{I}_1$ whose state needs to be transferred

$$\langle\hat{I}_1\rangle_w \equiv \frac{\mathrm{Tr}_{i1}(P_{i1}^k\hat{I}_1\rho_{i1}))}{\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})}$$

and its conjugate

$$\langle\hat{I}_1\rangle_w^* \equiv \frac{\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1}\hat{I}_1)}{\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})}$$

The above state (4.6) is not normalized. In order to normalize, I first calculate the denominator by taking a complete trace over the state

$$\begin{aligned} \mathrm{Tr}\rho_b &= \mathrm{Tr}(\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\langle\hat{I}_1\rangle_w^*\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I})) \\ &\quad +ig\langle\hat{I}_1\rangle_w\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab})))) \\ &= \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\langle\hat{I}_1\rangle_w^*\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I})) \\ &\quad +ig\langle\hat{I}_1\rangle_w\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab}))) \\ &= \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})((1 - ig\langle\hat{I}_1\rangle_w^*\langle\hat{A}\rangle_{w'}^* \\ &\quad +ig\langle\hat{I}_1\rangle_w\langle\hat{A}\rangle_{w'})) \end{aligned} \tag{4.7}$$

where we have defined the quantity $\langle\hat{A}\rangle_w$ and call it the weak-partial-value. 'Partial' because while the quantum state(density matrix ρ_{ab}) appearing in it is bipartite, the system measurement observable $\hat{A}$ and the post-selection projector P_a^l both act only on the part A of ρ_{ab}, even though the trace operation is performed over the entire expression.

$$\langle\hat{A}\rangle_w = \frac{\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab}\Big)}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}$$

and its conjugate

$$\langle\hat{A}\rangle_w^* = \frac{\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I})\Big)}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}$$

Now, let us normalize the state Bob has

$$\begin{aligned} \rho_b^{(N)} &= \frac{\rho_b}{\mathrm{Tr}\,\rho_b} \\ &= \frac{\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\langle\hat{I}_1\rangle_w^*\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I}))}{(\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})\,\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})((1 - ig\langle\hat{I}_1\rangle_w^*\langle\hat{A}\rangle_{w'}^* + ig\langle\hat{I}_1\rangle_w\otimes\langle\hat{A}\rangle_{w'}))} \\ &\quad \frac{+ig\langle\hat{I}_1\rangle_w\otimes\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab})}{(\mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})\,\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})((1 - ig\langle\hat{I}_1\rangle_w^*\langle\hat{A}\rangle_{w'}^* + ig\langle\hat{I}_1\rangle_w\otimes\langle\hat{A}\rangle_{w'}))} \end{aligned} \tag{4.8}$$

Since g is very small, the denominator can be brought up (to the numerator) and Taylor expanded up to the first order. Thus we have,

$$\begin{aligned}
\rho_b^{(N)} &= \frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) - ig\langle\hat{I}_1\rangle_w^* \,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I}))\\
&+ig\langle\hat{I}_1\rangle_w \otimes \mathrm{Tr}_a((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab}))(1+ig\langle\hat{I}_1\rangle_w^*\langle\hat{A}\rangle_{w'}^* - ig\langle\hat{I}_1\rangle_w\otimes\langle\hat{A}\rangle_{w'})\\
&= \frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}) + ig\langle\hat{I}_1\rangle_w^*\langle\hat{A}\rangle_{w'}^* \,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab})\\
&-ig\langle\hat{I}_1\rangle_w\otimes\langle\hat{A}\rangle_{w'}\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab})\\
&-ig\langle\hat{I}_1\rangle_w^*\,\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})\rho_{ab}(\hat{A}\otimes\mathbb{I})) + ig\langle\hat{I}_1\rangle_w\otimes\mathrm{Tr}_a((P_a^l\otimes\mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_{ab})) \qquad (4.9)
\end{aligned}$$

Whenever Alice gives the signal that post–selection is successful, Bob will find the expectation value of some observable with respect to his state $\rho_b^{(N)}$. Therefore, we have,

$$\begin{aligned}
\langle\hat{B}\rangle_f = \mathrm{Tr}\Big(\hat{B}\rho_b^{(N)}\Big) &= \frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big) + ig\langle\hat{I}_1\rangle_w^*\langle\hat{A}\rangle_{w'}^*\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big)\\
&-ig\langle\hat{I}_1\rangle_w\otimes\langle\hat{A}\rangle_{w'}\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big) - ig\langle\hat{I}_1\rangle_w^*\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}(\hat{A}\otimes\mathbb{I})\Big)\\
&+ig\langle\hat{I}_1\rangle_w\otimes\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})(\hat{A}\otimes\mathbb{I})\rho_{ab}\Big))
\end{aligned}$$

Since the weak value corresponding to Alice's weak interaction with the ancilla can be a complex quantity in general, we decompose it into its real and imaginary parts:

$$\langle\hat{I}_1\rangle_w = \mathrm{Re}\langle\hat{I}_1\rangle_w + i\,\mathrm{Im}\langle\hat{I}_1\rangle_w$$

Therefore, we have,

$$\begin{aligned}
\langle\hat{B}\rangle_f &= \mathrm{Tr}(\hat{B}\rho_b^{(N)})\\
&= \frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}) - ig\mathrm{Re}\langle\hat{I}_1\rangle_w(\langle\hat{A}\rangle_{w'}\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab})\\
&-\langle\hat{A}\rangle_{w'}^*\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}) - \mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})(\hat{A}\otimes\mathbb{I})\rho_{ab}\\
&+\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}(\hat{A}\otimes\mathbb{I})))) + g\mathrm{Im}\langle\hat{I}_1\rangle_w(\langle\hat{A}\rangle_{w'}\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab})\\
&+\langle\hat{A}\rangle_{w'}^*\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}) - \mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})(\hat{A}\otimes\mathbb{I})\rho_{ab}\\
&-\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}(\hat{A}\otimes\mathbb{I})))))
\end{aligned}$$

The above expression can be further simplified if one considers the complex decomposition of the weak-partial-value:

$$\langle\hat{A}\rangle_{w'} = \mathrm{Re}\langle\hat{A}\rangle_{w'} + i\,\mathrm{Im}\langle\hat{A}\rangle_{w'}$$

Hence, we have,

$$\begin{aligned}
\langle\hat{B}\rangle_f &= \frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab})\\
&-ig\,\mathrm{Re}\langle\hat{I}_1\rangle_w(2i\,\mathrm{Im}\langle\hat{A}\rangle_{w'}\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab})\\
&-\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})[(\hat{A}\otimes\mathbb{I}),\rho_{ab}]))\\
&+g\,\mathrm{Im}\langle\hat{I}_1\rangle_w(2\,\mathrm{Re}\langle\hat{A}\rangle_{w'}\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab})\\
&+\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\{(\hat{A}\otimes\mathbb{I}),\rho_{ab}\}))) \qquad (4.10)
\end{aligned}$$

Observing the above expression, one can see two conditions emerging, one that corresponds to the extraction of the real part of the weak value $\langle\hat{I}_1\rangle_w$; another, that corresponds to the extraction of the imaginary part of the weak value $\langle\hat{I}_1\rangle_w$.

4.3.2 Conditions and Pre-conditions

The condition 1 for extracting the real part is:

$$\begin{aligned} \{(\hat{A}\otimes\mathbb{I}),\rho_{ab}\} &= 0 \quad \text{and} \\ [(\hat{A}\otimes\mathbb{I}),\rho_{ab}] &\neq 0 \implies \\ \mathrm{Re}\langle\hat{A}\rangle_{w'} &= 0 \quad \text{and} \\ \mathrm{Im}\langle\hat{A}\rangle_{w'} &\neq 0 \end{aligned} \tag{4.11}$$

Note that here, we have used the expressions for the real and the imaginary parts of the weak value introduced and used in section 2.2.2. The condition 2 for obtaining the imaginary part is the exact reverse of the above condition, that is,

$$\begin{aligned} [(\hat{A}\otimes\mathbb{I}),\rho_{ab}] &= 0 \quad \text{and} \\ \{(\hat{A}\otimes\mathbb{I}),\rho_{ab}\} &\neq 0 \implies \\ \mathrm{Im}\langle\hat{A}\rangle_{w'} &= 0 \quad \text{and} \\ \mathrm{Re}\langle\hat{A}\rangle_{w'} &\neq 0 \end{aligned} \tag{4.12}$$

If condition 1 is satisfied, the expression 4.10 reduces to

$$\frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}) - ig\,\mathrm{Re}\langle\hat{I}_1\rangle_w(2i\mathrm{Im}\langle\hat{A}\rangle_{w'}\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}) \\ -\mathrm{Tr}((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})[(\hat{A}\otimes\mathbb{I}),\rho_{ab}]))) \tag{4.13}$$

thus rendering Bob the real part corresponding to Alice's weak interaction. On the other hand, if condition 2 is satisfied, the expression 4.10 reduces to

$$\frac{1}{\mathrm{Tr}((P_a^l\otimes\mathbb{I})\rho_{ab})}(\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big) + g\,\mathrm{Im}\langle\hat{I}_1\rangle_w(2\,\mathrm{Re}\langle\hat{A}\rangle_{w'}\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big) \\ +\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\{(\hat{A}\otimes\mathbb{I}),\rho_{ab}\}\Big))) \tag{4.14}$$

rendering Bob the imaginary part of the weak interaction performed by Alice. Due to the separate conditions required, Alice and Bob will have to perform two different sets of experiments to obtain the real and the imaginary parts of the weak value. In the first run of experiments they will make their choice for the following entities:

1. The post-selection operators, $\hat{P}_{i1}^k$ and $\hat{P}_a^l$ used by Alice.
2. The joint state ρ_{ab} and the operator $\hat{A}$ such that condition 1 is satisfied
3. $\{(\hat{A}\otimes\mathbb{I}),\rho_{ab}\} = 0$

The above arrangements ensure that Bob already knows the values of the entities $\mathrm{Im}\langle\hat{A}\rangle_{w'}$, $\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big)$ and $\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})[(\hat{A}\otimes\mathbb{I}),\rho_{ab}]\Big)$ so that he can accordingly rescale the expectation value of $\langle\hat{B}\rangle_f$ to obtain the real part of the weak value, $\langle\hat{I}_1\rangle_w$:

$$\mathrm{Re}\langle\hat{I}_1\rangle_w = \frac{i\langle\hat{B}\rangle_f\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})\rho_{ab}\Big) - i\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big)}{2ig\,\mathrm{Im}\langle\hat{A}\rangle_{w'}\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big) - g\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})[(\hat{A}\otimes\mathbb{I}),\rho_{ab}]\Big)}$$

They will agree on similar pre-conditions before the run performed for obtaining the imaginary part of the weak value. In this manner, Bob can obtain the imaginary part of the weak value

$$\mathrm{Im}\langle\hat{I}_1\rangle_w = \frac{i\langle\hat{B}\rangle_f\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})\rho_{ab}\Big) - i\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big)}{g\,\mathrm{Re}\langle\hat{A}\rangle_{w'}\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\rho_{ab}\Big) - g\,\mathrm{Tr}\Big((P_a^l\otimes\mathbb{I})(\mathbb{I}\otimes\hat{B})\{(\hat{A}\otimes\mathbb{I}),\rho_{ab}\}\Big)}$$

After obtaining the real and the imaginary parts of the weak value, Bob can reconstruct the unknown quantum state (using the protocols mentioned above) that was encoded on the ancilla $\hat{I}_1$ which Alice had. Note that this protocol involves a statistical buildup of the results from the measurement of $\hat{B}$ performed by Bob. Thus, each experiment will involve a large number of runs over an ensemble of states for the procedure to be successful.

4.3.3 Classical information transfer

Like every quantum state teleportation protocol, here too classical communication is required to aid the remote reconstruction of the quantum state that is to be transferred. The classical communication occurs by way of Alice informing Bob about the successful post-selection that occurred when she performed the weak measurement (involving the weak interaction and the post-selection) between A and $\hat{I}_1$. Whenever Bob receives the 'yes' signal from Alice, he will know that the post-selection was successful and he will then measure the observable $\hat{B}$ with him. Note that every message about successful post-selection involves 1-bit information transfer. With many runs, he will accumulate the statistics corresponding to the measurement of $\hat{B}$ on his state. Averaging these measurement statistics will give him the expectation value of $\langle\hat{B}\rangle_f$. From this expectation value, he can successfully determine the real and the imaginary parts of the weak value as shown above.

4.3.4 Role of the Correlations

It is relevant to ask where the significance of the correlations between A and B or the non-separability of ρ_{ab} is present in this protocol. One scan of the expression ρ_b in 4.8 and the subsequent normalization reveals that the significance of the non-separability lies in our considerations in the expression 4.6 which led to the definition of the weak-partial-value $\langle\hat{A}\rangle_w$. To show this, lets consider that ρ_{ab} is separable, that is, $\rho_{ab} = \rho_a\rho_b$ and consider the expressions in 4.6 where the partial trace is taken over part A:

$$\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_{ab}) = \mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a\rho_b) = \mathrm{Tr}\Big((P_a^l \otimes \mathbb{I})\rho_a\Big)\rho_b$$

Similar such expressions from 4.6 will end up like this and thus 4.6 will end up as (note that I am replacing ρ_b in 4.6 with ρ_b^{un} to avoid notational confusion with the separated states ρ_a and ρ_b):

$$\begin{aligned}
\rho_b^{un} &= \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a)\rho_b - ig\langle\hat{I}_1\rangle_w^* \,\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a(\hat{A}\otimes\mathbb{I}))\rho_b \\
&\quad + ig\langle\hat{I}_1\rangle_w \otimes \mathrm{Tr}_a((P_a^l \otimes \mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_a)\rho_b)) \\
&= \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})\rho_b((\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a) - ig\langle\hat{I}_1\rangle_w^* \,\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a(\hat{A}\otimes\mathbb{I})) \\
&\quad + ig\langle\hat{I}_1\rangle_w \otimes \mathrm{Tr}_a((P_a^l \otimes \mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_a)))
\end{aligned} \tag{4.15}$$

Notice that the ρ_b can now be taken outside common. The trace of the above unnormalized state is :

$$\begin{aligned}
\mathrm{Tr}\,\rho_b^{un} &= \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})\,\mathrm{Tr}(\rho_b)((\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a) - ig\langle\hat{I}_1\rangle_w^* \,\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a(\hat{A}\otimes\mathbb{I})) \\
&\quad + ig\langle\hat{I}_1\rangle_w \otimes \mathrm{Tr}_a((P_a^l \otimes \mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_a))) \\
&= \mathrm{Tr}_{i1}(P_{i1}^k\rho_{i1})((\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a) - ig\langle\hat{I}_1\rangle_w^* \,\mathrm{Tr}_a((P_a^l \otimes \mathbb{I})\rho_a(\hat{A}\otimes\mathbb{I})) \\
&\quad + ig\langle\hat{I}_1\rangle_w \otimes \mathrm{Tr}_a((P_a^l \otimes \mathbb{I})(\hat{A}\otimes\mathbb{I})\rho_a)))
\end{aligned}$$

Here I have assumed that the initial state ρ_{ab} and its parts ρ_a and ρ_b are normalized so thats $\operatorname{Tr}\rho_b = 1$. Now, normalizing the final Bob's part of the state we have:

$$\begin{aligned}\frac{\rho_b^{un}}{\operatorname{Tr}\rho_b^{un}} &= \frac{\operatorname{Tr}_{i1}(P_{i1}^k\rho_{i1})\rho_b((\operatorname{Tr}_a((P_a^l\otimes\mathbb{I})\rho_a) - ig\langle\hat{I}_1\rangle_w^* \operatorname{Tr}_a((P_a^l\otimes\mathbb{I})\rho_a(\hat{A}\otimes\mathbb{I})) + ...}{\operatorname{Tr}_{i1}(P_{i1}^k\rho_{i1})((\operatorname{Tr}_a((P_a^l\otimes\mathbb{I})\rho_a) - ig\langle\hat{I}_1\rangle_w^* \operatorname{Tr}_a((P_a^l\otimes\mathbb{I})\rho_a(\hat{A}\otimes\mathbb{I})) + ...}\\ &= \rho_b\end{aligned}$$

Everything in the numerator and the denominator cancels out and Bob is left with his initial state ρ_b with no signature of the weak interaction performed by Alice. Thus, it is proved by contradiction that the correlations between A and B or the non-separability of ρ_{ab} is a necessary condition for this protocol to work.

Chapter 5

The Perpetual Epilogue

Weak measurement has been studied from two perspectives. One, from a foundational. Two, from an application-based. The foundational studies involved rigorously studying how the weak value comes about in post-selected quantum measurements using von Neumann's quantum measurement model. We find that the result for the extraction of the expectation value for weak as well as strong interactions between the pointer and the system states is identical. However, the result for the weak value is not. In fact, the result for the weak value cannot be defined in quantum measurement when the interaction preceding the post-selection is strong and the post-selected system state is different from the pre-selected one. Thus, the weak value operationally emerges only in the weak limit. The conditions that are needed to be satisfied for a weak measurement process in the weak limit are derived explicitly in terms of the real and the imaginary parts of the weak value. Even if one in the set of these eight conditons is violated, the weak limit breaks down.

After the conceptual treatment to weak measurement, we move to a mathematical generalization of weak measurement assuming that the condition corresponding to the weak limit is satisfied. Here, we first investigate the earlier unexplored effect of the weak interaction in first as well as second orders of measurement strength on system and pointer states individually. It is found that if the system state is not affected after the weak interaction, the imaginary part of the weak value is zero. This is the simplest interpretation of the imaginary part of the weak value connecting it to measurement backaction. We also calculated the expectation values of the system or pointer observables with respect to the post-interaction system and pointer states. Observations analogous to those made for the expectation value hold here as well. After post-selection is done on the pointer state, the weak value effects a large shift in the expression of the pointer state as well the expectation values of the pointer observables. This shift can be arbitrarily large and is weighted by the real and/or imaginary parts of the weak value as well as the correlations between the different pointer degrees of freedom in case of a multimode pointer state. Several observations connecting the change in the expression of the system or pointer states and the change in expectation values of observables at the post-interaction and post-selection stage to the commutators $[\hat{p}', \hat{p}]$, $[\hat{\rho}_{is}, \hat{A}]$ and $[\hat{\rho}_{ip}, \hat{p}]$ were made. The deeper implications of these entities being zero or non-zero is worth further studies.

Applications of weak measurement are studied where the pointer state is considered to be a qubit. A two qubit separable pointer was successfully converted into a non-separable one by performing sequential weak interactions of each pointer qubit separately with the system qubit and post-selecting on the system qubit. It could be useful to quantify this non-separability in the final pointer state using some measure of quantum correlations. This analysis is also done when the pointer is a two qubit Werner state. The change in the correlation of the Werner state was studied using the PPT criterion for separability of density matrices. No change was observed when the interaction parameter g was considered to be very small.

Obtaining the joint expectation value of incompatible observable is a necessary but difficult task in general. In the light of this, there were schemes to extract joint weak values (which are reported to serve some of the functions of joint expectation value) of non-commuting observables using a specific Hamiltonian and pointer states. We do our analysis using sequential weak interactions considered up to the second order in interaction strength followed by post-selection. Although, the general result has some additional terms in addition to the ones desired, the right choice of the pointer state and the pointer observable whose expectation value is calculated can lead one to the desired answer. We demonstrate this for the two dimensional Laguerre-Gauss mode state and obtain the full second order of the weak value as well as the joint weak value of the system observables involved in the weak interactions. A set of conditions can also be derived for the vanishing or non-vanishing of the commutators and anti-commutators that occur in the expression for the expectation value of an arbitrary observable $\hat{M}$ which enable the extraction of the joint weak value for particular classes of quantum states.

The concept of weak value aids the construction of a protocol for remote determination of the quantum state by transferring the corresponding real and the imaginary parts of the weak value by one party to another spatially separated party. The resource for the protocol is the correlation between the non-separable shared state between Alice and Bob. The state whose information is to be transferred is encoded on the ancilla state carried by Alice. The weak value in its entirety is obtained by Bob through two different sets of the experiments which render its real and imaginary parts separately. From this weak value, Bob can reconstruct the quantum state which was present on Alice's ancilla. As opposed to other protocols, this one does not place any limits on the kind of the state whose information is to be transferred, provided the pre-conditions are satisfied. Significantly, in this protocol the resource state does not dictate the kind of quantum state that whose information can be transferred.

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Chapter 6

Supplemental Materials

PPT Criterion

The positive partial transpose criterion provides a necessary condition for the separability of a density matrix. This criterion provides a necessary and sufficient condition for 2×2 and 3×3 density matrices but only a necessary condition for higher dimensional matrices. If the given density matrix is separable then none of the eigenvalues of its partial transpose are negative. The density matrix of our two qubit Werner state is:

$$\rho_w = \begin{pmatrix} \frac{1-z}{4} & 0 & 0 & 0 \\ 0 & \frac{z+1}{4} & -\frac{z}{2} & 0 \\ 0 & -\frac{z}{2} & \frac{z+1}{4} & 0 \\ 0 & 0 & 0 & \frac{1-z}{4} \end{pmatrix}$$

We are interested in checking the separability between its subsystems(qubits) which are represented by 2×2 matrices. After transposing the first subsystem, we have:

$$\rho_w' = \begin{pmatrix} \frac{1-z}{4} & 0 & 0 & -\frac{z}{2} \\ 0 & \frac{z+1}{4} & 0 & 0 \\ 0 & 0 & \frac{z+1}{4} & 0 \\ -\frac{z}{2} & 0 & 0 & \frac{1-z}{4} \end{pmatrix}$$

The eigenvalues of ρ_w' are:

$$\left\{ \frac{1}{4}(1-3z), \frac{z+1}{4}, \frac{z+1}{4}, \frac{z+1}{4} \right\}$$

In the expression ρ_{ft} of the main text, several combinations[1] of the directional Pauli basis measurements ($m_1 = \{0,1\}$, $m_2 = \{0,1\}$, $m_3 = \{0,1\}$, $n_1 = \{0,1\}$, $n_2 = \{0,1\}$ and $n_3 = \{0,1\}$) were tried for interaction strengths g_1 and g_2 of the order of 10^{-3} to 10^{-6}. However, the eigenvalues of the partially transposed matrix did not show any change.

LG mode expectation values

For the Laguerre Gauss mode pointer state, we have replaced $\hat{M} = \hat{p}_1 \hat{p}_2$ in the expression of $\langle \hat{M} \rangle_f$ of the main text, we get the following terms:

[1] interested reader may contact me for the Mathematica file

- $\langle \{\hat{q}_2, \hat{p}_1\hat{p}_2\} \rangle_i = 0$
- $\langle [\hat{q}_2, \hat{p}_1\hat{p}_2] \rangle_i = 0$
- $\langle \{\hat{q}_2^2, \hat{p}_1\hat{p}_2\} \rangle_i = -i$
- $\langle [\hat{q}_2^2, \hat{p}_1\hat{p}_2] \rangle_i = i$
- $\langle [\hat{q}_1, \hat{p}_1\hat{p}_2] \rangle_i = 0$
- $\langle \{\hat{q}_1, \hat{p}_1\hat{p}_2\} \rangle_i = 0$
- $\langle \{\hat{q}_1\hat{q}_2, \hat{p}_1\hat{p}_2\} \rangle_i = i/2$
- $\langle [\hat{q}_1\hat{q}_2, \hat{p}_1\hat{p}_2] \rangle_i = 0$
- $\langle \{\hat{q}_1^2, \hat{p}_1\hat{p}_2\} \rangle_i = -2i$
- $\langle [\hat{q}_1^2, \hat{p}_1\hat{p}_2] \rangle_i = -i$
- $\langle \hat{p}_1\hat{q}_2\hat{p}_2\hat{q}_2 \rangle_i = i$
- $\langle \hat{p}_2\hat{q}_1\hat{p}_1\hat{q}_1 \rangle_i = -i$
- $\langle \hat{q}_1\hat{p}_1\hat{p}_2\hat{q}_2 \rangle_i = -3/4$
- $\langle \hat{q}_2\hat{p}_2\hat{p}_1\hat{q}_1 \rangle_i = -3/4$

To simplify the above expressions before calculating the relevant entities in Mathematica, I have used the following commutation/anti-commutation identities:

- $\{A, BC\} = \{A, B\}C - B[A, C]$
- $\{AB, C\} = A\{B, C\} - [A, C]B$
- $[AB, C] = A\{B, C\} - \{A, C\}B$
- $[A, BC] = [A, B]C + B[A, C]$

In Mathematica[2] calculations, I have substituted $\hat{p} = i\frac{\partial}{\partial q}$ and computed the respective integrals from $-\infty$ to $+\infty$.

[2]Mathematica notebook available upon request