Towards the Plancherel Formula of GL(2) over a p-adic field

A Thesis

submitted to
Indian Institute of Science Education and Research Pune
in partial fulfillment of the requirements for the
BS-MS Dual Degree Programme

by

Papia Bera
20121082

Indian Institute of Science Education and Research Pune
Dr. Homi Bhabha Road,
Pashan, Pune 411008, INDIA.

April, 2017

Supervisor: A. Raghuram

Certificate

This is to certify that this dissertation entitled Towards the Plancherel Formula of GL(2) over a p-adic field towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Papia Bera at Indian Institute of Science Education and Research under the supervision of A. Raghuram, Professor, Department of Mathematics, during the academic year 2016-2017.

30/3/17

A. Raghuram

Committee:

A. Raghuram

Chandrasheel Bhagwat

Declaration

I hereby declare that the matter embodied in the report entitled Towards the Plancherel Formula of GL(2) over a p-adic field are the results of the work carried out by me at the Department of Mathematics, Indian Institute of Science Education and Research, Pune, under the supervision of A. Raghuram and the same has not been submitted elsewhere for any other degree.

Bera
30/3/2017
Papia Bera

Acknowledgments

I have been able to finish this project and all the work behind it only with the help and support of several people. I would like to thank all of them here.

Firstly, I would like to thank my guide, Prof. A. Raghuram. His guidance and direction has been of immense help throughout this project. He encouraged and inspired me to pursue this subject. I am very grateful for all the insights he provided, not just about the project but also on my pursuit of a doctoral research position. He has been very lenient with my many mistakes, and I am very thankful for it.

Next, I would like to thank Chaitanya Ambi, for the many hours of discussion on the project and the insight he provided to the various problems that I had to work out. His talks on Abstract Representation Theory, were very helpful in improving my understanding the problems at hand.

I would like to thank all the faculty and students of the math department, who have always been supportive and encouraging during all my interactions with them. I would especially like to mention my fellow 5th year BS-MS students, Visakh, Ajith, Varun, Dileep, Chris, Omkar and the rest. They provided a lot of moral support on the days that I needed it the most. The many discussions with them always left me with a sense of calm, and helped me get through the most difficult parts.

Finally, I would like to thank my parents and my family, who have always supported me and believed in me and my choices. I would also like to thank my friends at IISER Pune, whose camaraderie has always kept me grounded.

Papia Bera

Abstract

This project aims to write down the Plancherel formula for $GL(2, F)$ where F is a p-adic field. The Plancherel formula for $SL(2, F)$ and $PGL(2, F)$ are known, and we also have a general form of the formula for a real reductive semisimple lie group. Thus, in this project we aim to do the same for $GL(2, F)$. We do not arrive at the final formula, instead we give an approach to the proof of this formula. This project details all the requirements such as, its irreducible representations and their characters, as well as other possible ways to get to the problem. We hope to solve the problem soon.

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Introduction

The Plancherel Theorem states that

Theorem. *For a locally compact abelian group G, the fourier transform that takes functions from $\mathcal{L}^1(G) \bigcap \mathcal{L}^2(G)$ to a subspace of $\mathcal{L}^2(\hat{G})$ can be extended to the whole space, $\mathcal{L}^2(G)$ as an isometry.*

This theorem is an important part of Fourier Analysis, as it is implicitly linked to the fourier inversion theorem. The corresponding formula for the fourier inversion theorem, by abuse of notation, is called the Plancherel formula. For example for the space of real numbers, $\mathbb{R}$, the fourier transform is

$$\hat{f}(t) = \int\limits_{-\infty}^{\infty} f(x)e^{-ixt}dx$$

where for a given $t \in \mathbb{R}$, $\pi(x) = e^{-ixt}$ gives an irreducible representation of $\mathbb{R}$, and the Plancherel formula is

$$f(0) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \hat{f}(t)dt.$$

This is the inverse fourier transform evaluated at 0.

Although, the theorem is only about locally compact abelian groups, the result can be shown for certain non-abelian groups such as $GL(n, F)$, $SL(n, F)$ and $PGL(n, F)$, where F can be real or p-adic. In this project we look at the case of $GL(2, F)$, where F is a p-adic field. The Plancherel formula for such a group G can be written as ,

$$f(e) = \int_{\hat{G}} \Theta_\pi(f) d\pi. \tag{1}$$

where, $f \in \mathcal{C}_c^\infty(G)$, e is the identity of G, and Θ_π is the character distribution of G, that is, Trace of the operator $\pi(f) = \int_G f(g)\pi(g)dg$.

This is essentially the decomposition of the regular representation, whose character distribution is then evaluated at identity.

This project aims to get the Plancherel formula for $GL(2, F)$, where F is a p-adic field. We try to get to this formula by decomposing its regular representation in terms of its irreducible unitary representation. Thus, we study the representation theory for such a group. The classification followed in this project is based on the work done by Bushnell and Henniart in their book "The Local Langlands' Conjecture for $GL(2)$". We parametrize these classifications exactly as given in the book.

Next we look at the character theory for such a group. The character distributions as mentioned in (1), is also studied. The formalisation of these characters is described as given in Jacquet and Langlands' book, "Automorphic Forms on GL(2)".

To prove this formula for $GL(2)$, we study the proof of the formula for $SL(2)$. This is read from the paper by Sally and Shalika, "Plancerel Formula for $SL(2)$ over a local field". The hope is to be able to emulate this proof for the case of $GL(2)$. This is due to the similarities in the classification of their representations. The parametrization of their classification is similar to that seen for $GL(2)$.

Although, there are many similarities between the two groups, they also have some subtle differences which means that the proof for $G = GL(2)$ might not be able to construct similar situations as was possible for $SL(2)$. Thus, we also look at the decomposition of the regular representation of G, in terms of the regular representation of $PGL(2, F)$ and its G-equivariant spaces. We can then look at the Plancherel formula for $PGL(2, F)$ as given in Silberger's book "PGL_2 over the p-adics : its Representations, Spherical Functions, and Fourier Analysis", and try to write down the formula for G.

Thus, we hope to get to the Plancherel formula sometime soon.

Chapter 1

Preliminaries

This chapter is meant to clarify all the basic concepts, terms and definitions which would be repeatedly used in this project. We first have a section on p-adic fields, which briefly discusses the construction, and some important properties of p-adic fields. This leads to a discussion on quadratic extensions of p-adic fields. Finally, we have a section on the definition of direct sum of Hilbert spaces.

1.1 p-adic Fields

Fix a prime number p. We first define a function on $\mathbb{Z}$ such that, $v_p(n) = a$ where, $a+1$ is the smallest integer such that $n \not\equiv 0 (\text{mod } p^{a+1})$ whenever, $n \in \mathbb{Z}$ and $n \neq 0$. Also, define, $v_p(0) = +\infty$. Observe that, for all $m, n \in \mathbb{Z}$,

1. $v_p(m \cdot n) = v_p(m) + v_p(n)$
2. $v_p(m+n) \geq \min\{v_p(m), v_p(n)\}$.

Thus, v_p is a valuation on the ring $\mathbb{Z}$.

Now, consider an absolute value on $\mathbb{Q}$, such that for every $x \in \mathbb{Q}$, $x = \frac{m}{n}$, $m, n \in \mathbb{Z}$,

$$|x|_p = \left|\frac{m}{n}\right|_p = p^{v_p(n) - v_p(m)}.$$

Due to the properties of v_p, we can easily check that $|\cdot|_p$ is an absolute value and therefore defines a metric on $\mathbb{Q}$. It can also be checked that $|\cdot|_p$ is a non-archimedean absolute value. This has some important consequences such as,

1. $\forall x, y \in \mathbb{Q}$, $|x+y|_p \leq \max\{|x|_p, |y|_p\}$
2. $\forall x \in \mathbb{Z}$, $|x|_p \leq 1$

The p-adic field, $\mathbb{Q}_p$ is the metric completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_p$. The ring of integers of such a field is $\mathbb{Z}_p = \{x \in \mathbb{Q}_p \mid |x| \leq 1\}$, the unit disk in $\mathbb{Q}_p$. The space $\mathfrak{p} = \{x \in \mathbb{Q}_p \mid |x| < 1\}$ is the unique prime ideal in the ring $\mathbb{Z}_p$. Here, $\mathfrak{p}$ is a principal ideal, $\mathfrak{p} = \langle p \rangle$. The field $\mathbf{k} = \mathbb{Z}_\mathbf{p}/\mathfrak{p} \cong \mathbb{Z}/\mathbf{p}\mathbb{Z}$, is called the residue field of $\mathbb{Q}_p$. The set of units in the ring of integers is the unit circle, $U = \{x \in \mathbb{Z}_p \mid |x| = 1\}$.

A p-adic field F is a finite field extension of $\mathbb{Q}_p$. The ring of integers, of F is defined in the same manner, $\mathcal{O} = \{x \in F \mid |x| \leq 1\}$. Similarly we have the prime ideal and the unit ring in $\mathcal{O}$, $\mathcal{P} = \{x \in F \mid |x| < 1\}$ and $U_F = \{x \in F \mid |x| = 1\}$. Here, the prime ideal, $\mathcal{P} = \langle \varpi \rangle$ is also a principal ideal, and the element ϖ is called the prime element of F. The residue field, $\mathbf{k}_F = \mathcal{O}/\mathcal{P}$ is also an extension of the residue field of $\mathbb{Q}_p$. If F is an extension of degree n, then, $\mathbf{k}_F = \mathbb{F}_q$ where, $q = p^n$. Here, we can define $U_n = 1 + \mathcal{P}^n$ as the compact open subgroups of U_F, and we get the isomorphism, $U_F/U_n \cong \mathcal{O}/\mathcal{P}^n$.

1.2 Quadratic Extensions of a p-adic field

Hensel's Lemma states that given a p-adic field, F, where $p \neq 2$, and a polynomial whose coefficients lie in the ring of integers $\mathcal{O}$ such that it has a root in the residue field $\mathbf{k}_F$ and the derivative of the polynomial satisfies certain conditions, then the polynomial has a root in $\mathcal{O}$.

Also, any element of $x \in F^\times$ can be written as $x = \varpi^n \cdot y$ where, ϖ is the prime element of F and $y \in U_F$ and n ranges over every integer. That is, $F^\times \cong \mathbb{Z} \times U_F$.

These two properties of F give us that unless $p = 2$, we can say that the order of $F^\times/(F^\times)^2$ is 4. We have that $F^\times/(F^\times)^2 = \{\epsilon, \varpi, \epsilon\varpi, 1\}$, where ϵ is the representative of all square-free units, and ϖ is the prime element, as defined above. Thus, there can be only 4 quadratic extensions possible for the p-adic field, F.

1.3 Direct Integral of Hilbert Spaces

Let (S, μ) be a measure space such that,

1. $S = \bigcup\limits_{i=0} \infty S_i$ where S_i are measurable subsets of S with $\mu(S_i) < \infty$.
2. There exists a function d, such that for A, B, measurable subsets of S, $d(A, B) = \mu(A \triangle B)$, and d satisfies the triangle inequality.
3. $((S, \mu), d)$ forms a seperable pseudo metric space.

Consider a family of Hilbert spaces such that for every $s \in S$, we assign a Hilbert space, H_s to it. A section over such a family of Hilbert spaces is a map, that for each s takes a value in H_s, $v : s \mapsto v_s$ where $v_s \in H_s$. The family of Hilbert spaces , $\{H_s\}_{s\in S}$, is said to be measurable if there exists a set of sections, $\mathfrak{F}$, such that,

1. the map $s \mapsto \langle x(s), y(s)\rangle_{H_s}$ is measurable for all $x, y \in \mathfrak{F}$.
2. If for some section z, the map, $s \mapsto \langle z(s), x(s)\rangle_{H_s}$ is measurable almost everywhere, $\forall x \in \mathfrak{F}$, then $z \in \mathfrak{F}$.
3. $\exists \{x_n\}_{n\in\mathbb{N}}$, a sequence of sections such that, $\overline{\{x_n(s) \mid n \in \mathbb{N}\}} = H_s$ almost everywhere in S.

Given a measurable family of Hilbert spaces, as above, $x \in \mathfrak{F}$ is said to be square integrable if,

$$\int\limits_S \|x(s)\|^2_{H_S} d\mu(s) < \infty.$$

We identify two sections x and y if $\|x(s) - y(s)\|_{H_s} = 0$ almost everywhere on S. With this identification, the set of all square - integrable sections in $\mathfrak{F}$, is called the direct integral of the family of Hilbert spaces, $\{H_s\}_{s\in S}$.

$$\mathcal{H} = \int_S H_s d\mu(s)$$

.

On this space we set, for $x, y \in \mathcal{H}$,

$$\langle x, y\rangle = \int_S \langle x(s), y(s)\rangle_{H_s} d\mu(s).$$

Under this inner product, $\mathcal{H}$ is a pre-Hilbert space.

Chapter 2

Representations of GL(2) over a p-adic field

Let G denote $GL(2, F)$ where F is a p-adic field. The Plancherel formula for G is given by an integration over the unitary representations of G. Thus, in this chapter we give a classification of all the irreducible representations of G as given in the works of Bushnell and Henniart.

Definition. *A smooth representation over this group would be, (π, V) such that for any vector $v \in V$, there exists a compact open subgroup, K in G, which stabilises v.*

Every abstract representation (π, V) of G, has a subspace V^∞ such that π acts as a smooth representation on it. A smooth representation is K-semisimple for every K, a compact open subgroup of G. For G, these are the type of irreducible representations that we must look at.

Consider, $\phi : G \mapsto \mathbb{C}^\times$, a 1-dimensional smooth representation of G or equivalently, a continuous homomorphism from G to $\mathbb{C}^\times$. The commutator group of $GL(2, F)$ being $SL(2, F)$, implies, a map such as ϕ must factor through the determinant map and therefore, every character of G is of the form, $\phi = \chi \circ \det$, where χ is a character of $F^\times$. It can easily be shown that the only possible finite dimensional representations of G are its characters.

To further classify the representations of G, we look at their Jacquet modules. Let, N

be the set of all upper triangular unipotent matrices in G.

$$N = \left\{ \left[\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right], x \in F^{\times} \right\}$$

For an irreducible representation (π, V) of G, we define, $V(N)$ as the subspace of V generated by the vectors $v - \pi(n)v$ for $v \in V$ and $n \in N$. The Jacquet module of V is given by, $V_N = V/V(N)$. All characters of G have a non-trivial Jacquet module.

An irreducible smooth representation (π, V) of G is called cuspidal, if V_N is zero. We shall now look at the classification of representations of G as a cuspidal or a non-cuspidal representation.

2.1 Non-cuspidal representations

Let B be the set of upper triangular matrices in G and T be the set of diagonal matrices in G.

$$B = \left\{ \left[\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right] \mid a, d \in F^{\times}, b \in F \right\}$$

$$T = \left\{ \left[\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right] \mid x, y \in F^{\times} \right\}$$

For any $b \in B$, there exists $t \in T$ and $n \in N$ such that $b = tn$. Consider χ, a character of T. Thus, $\chi = \chi_1 \otimes \chi_2$ where χ_1 and χ_2 are characters of $F^{\times}$. Such a character can be viewed as a representation of B which is trivial on N.

As we discussed above, every character of G is non-cuspidal. To classify the rest, we observe the following proposition.

Proposition 2.1.1. *Let (π, V) be an irreducible smooth representation of G such that, V_N is non-trivial then, π is isomorphic to a G-subspace of $Ind_B^G \chi$, where χ is a character of T.*

Where, the induced representation is of the form,

$$\begin{aligned} \operatorname{Ind}_B^G(\chi) = \{ f : G \mapsto \mathbb{C} \mid f(hg) = \delta_B^{-1/2}(h)\chi(h)f(g),\ h \in B \text{ and } \exists\ K \text{ a compact open} \\ \text{subgroup such that } f(gx) = f(g)\ \forall\ g \in G,\ \forall\ x \in K \} \end{aligned}$$

where, $\delta_b \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix} \right) = \frac{|d|}{|a|}$, is the modular character of B.

Thus to classify the non-cuspidal representations, we must look at which characters of T give us an irreducible smooth representation of G when induced. This gives the following Irreducibilty Criterion :

Theorem 2.1.2. *Let* $\chi = \chi_1 \otimes \chi_2$ *be a character of* T*, and* $(\pi, V) = Ind_B^G \chi$*, then* π *is reducible if and only if* $\chi_1 \chi_2^{-1}$ *is either of the characters* $x \mapsto \|x\|^{\pm 1}$

These irreducible induced representations are called the principal series representations of G.

The above theorem does not mention the uniqueness of these induced representations. Therefore, we must look at the conditions under which they are distinct. Let χ and ξ be two characters of T. By Frobenius Reciprocity, we can say that

$$\mathrm{Hom}_G(\mathrm{Ind}_B^G \chi, \mathrm{Ind}_B^G \xi) \cong \mathrm{Hom}_T((\mathrm{Ind}_B^G \chi)_N, \xi)$$

also,

$$0 \longrightarrow \chi^w \longrightarrow (\mathrm{Ind}_B^G \chi)_N \longrightarrow \chi \longrightarrow 0$$

and thus, $\mathrm{Ind}_B^G \chi$ and $\mathrm{Ind}_B^G \xi$ are distinct if and only if $\xi \neq \chi$ or χ^w, where $w = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and χ^w is defined as the character $\chi^w(g) = \chi(wgw^{-1})$.

Thus we have a distinct classification for the principal series representations.

Following this, we must also look at the G-subspaces of the induced representations which are reducible. Observe that δ_B as a character of B is trivial on N, therefore, we can also look at it as a character of T. We find that

$$0 \longrightarrow \mathbb{1}_G \longrightarrow \mathrm{Ind}_B^G \delta_B^{1/2} \longrightarrow \mathrm{St}_G \longrightarrow 0$$

The irreducible G-quotient of $\mathrm{Ind}_B^G \delta_B^{1/2}$ is called the Steinberg representation . From the above short exact sequence we can see that any $\mathrm{Ind}_B^G \phi \delta_B^{1/2}$, where ϕ is a character of $F^\times$, has an irreducible G-quotient, $\phi.\mathrm{St}_G$. The Steinberg representation and its twists are called the special representations.

From the definition of δ_B and the Irreducibility Criterion, we can see that the above representations exhaust all the possible reducible representations of the form $\mathrm{Ind}_B^G \chi$ such that χ is a character of T. Thus, we have classified all non-cuspidal representations of G.

2.2 Cuspidal Representations

Consider the following rings in G.

Definition 2.2.1. *A chain order $\mathfrak{U}$ is a ring with an integer, $e_{\mathfrak{U}}$, associated with it such that, there exists $g \in G$ for which*

$$g\mathfrak{U}g^{-1} = \begin{cases} \mathfrak{M} = \left[\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{smallmatrix}\right] & \text{and } e_{\mathfrak{U}} = 1 \\ \mathfrak{J} = \left[\begin{smallmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{P} & \mathcal{O} \end{smallmatrix}\right] & \text{and } e_{\mathfrak{U}} = 2 \end{cases}$$

For such a $\mathfrak{U}$, we write its Jacobson radical as $\mathfrak{P} = \text{rad}\,\mathfrak{U}$. For every $\mathfrak{U}$, $\mathfrak{P} = \Pi\mathfrak{U} = \mathfrak{U}\Pi$, for some Π, the prime element of $\mathfrak{U}$. For example,

$$\text{rad}\,\mathfrak{M} = \varpi\mathfrak{M}, \quad \text{rad}\,\mathfrak{J} = \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right)\mathfrak{J}$$

where, ϖ is a prime element of F. We define, $\mathfrak{P}^n = \Pi^n\mathfrak{U} = \mathfrak{U}\Pi^n$ for any $n \geq 0$. The units of the ring are defined as ,

$$U_{\mathfrak{U}}^0 = U_{\mathfrak{U}} = \mathfrak{U}^\times$$

$$U_{\mathfrak{U}}^n = 1 + \mathfrak{P}^n,\ n \geq 1$$

For example, $U_{\mathfrak{M}} = GL(2, \mathcal{O})$ and $U_{\mathfrak{J}} = I$, where I is the Iwahori subgroup.

$$I = \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \text{ such that } a, d \in U_F, c \in \mathcal{P} \text{ and } d \in \mathcal{O}\}$$

Fix a character, $\psi \in \hat{F}$, such that $\psi \neq 1$ and $\psi|_{\mathcal{P}} = \mathbb{1}$ but $\psi|_{\mathcal{O}} \neq \mathbb{1}$, that is ψ is of level 1. We can define ψ_A as a character of $A = M(2, F)$ such that, $\psi_A(x) = \psi(\text{tr}_A x)$ where tr_A denotes the trace map. Similarly, for any $a \in A$ we can also define the character, $a\psi_A(x) = \psi_A(ax) = \psi(\text{tr}_A(ax))$. The map $a \mapsto a\psi_A$ gives an isomorphism $A \cong \hat{A}$.

With such a fixed character, we now have the following isomorphism.

Proposition 2.2.1. *Let $\mathfrak{U}$ be a chain order with radical $\mathfrak{P}$. Let n be any integer, then*

$$\mathfrak{P}^{-n}/\mathfrak{P}^{1-n} \longrightarrow (\widehat{U_{\mathfrak{U}}^n/U_{\mathfrak{U}}^{n+1}})$$

$$a + \mathfrak{P}^{1-n} \mapsto \psi_{A,a}|_{U_{\mathfrak{U}}^n}$$

is an isomorphism, where $\psi_{A,a}(x) = \psi_A(a(x-1))$ *and* $(\widehat{U_{\mathfrak{U}}^n/U_{\mathfrak{U}}^{n+1}})$ *is the set of all characters of* $U_{\mathfrak{U}}^n$ *which are trivial on* $U_{\mathfrak{U}}^{n+1}$*.*

This isomorphism leads to the definition of a stratum.

Definition 2.2.2. *A stratum is a triple* $(\mathfrak{U}, n, a)$ *such that* $\mathfrak{U}$ *is a chain order with radical* $\mathfrak{P}$*, n is an integer and* $a \in \mathfrak{P}^{-n}$*.*

This strata is equivalent to identifying a character, $\psi_{A,a}$ of $U_{\mathfrak{U}}^n$ which is trivial on $U_{\mathfrak{U}}^{n+1}$ corresponding to the coset $a + \mathfrak{P}^{1-n}$ according to the isomorphism in the above proposition (2.2.1). Thus, we say that two strata, $(\mathfrak{U}, n, a_1)$ and $(\mathfrak{U}, n, a_2)$ are equivalent if and only if $a_1 \cong a_2 (\text{mod } \mathfrak{P}^{1-n})$.

Next we distinguish the fundamental strata.

Definition 2.2.3. *Let* $(\mathfrak{U}, n, a)$ *be a stratum and radical of* $\mathfrak{U}$ *is* $\mathfrak{P}$*. It is called fundamental if the coset* $a + \mathfrak{P}^{1-n}$ *contains no nilpotent elements of A. Equivalently, there exists an integer* $r \geq 1$ *such that* $a^r \in \mathfrak{P}^{1-rn}$*.*

Let (π, V) be an irreducible smooth representation of G. A stratum, $(\mathfrak{U}, n, a)$ such that $n \geq 1$ is said to be contained in π, if π contains the corresponding character $\psi_{A,a}$ of $U_{\mathfrak{U}}^n$. Distinguishing the fundamental strata was important as except for certain conditions on π, an irreducible smooth representation will always contain a fundamental strata. The exceptions lie at the point when π contains the trivial character of $U_{\mathfrak{M}}^1$. In which case we can easily classify the possible cuspidal representations as c-$\text{Ind}_{ZK}^G \Lambda$, where $K = GL(2, \mathcal{O})$, Z is the center of G, and Λ is a representation of ZK which contains λ, a representation of K inflated from an irreducible cuspidal representation of $GL(2, k)$, k being the residue field of F.

Next, we try to classify the fundamental strata and try to distinguish an irreducible representation in terms of which type fundamental strata it contains.

Consider the strata $(\mathfrak{U}, n, a)$ such that $e_{\mathfrak{U}} = 1$. Then, $a \in \mathfrak{P}^{-n}$ and $\mathfrak{P}^{-n} = \varpi^{-n}\mathfrak{U}$, since $e_{\mathfrak{U}} = 1$ and $\mathfrak{P}$ is a G-conjugate of the radical of $\mathfrak{M}$. Therefore, there exists some, $a_0 \in \mathfrak{U}$ such that $a = \varpi^{-n} a_0$. We define $f_a(t)$ as the characteristic polynomial of the image of a_0 in $\mathfrak{U}/\mathfrak{P} = M(2, k)$ where k is the residue field of F. This characteristic polynomial remains the same for $a_1 \cong a_2 (\text{mod } \mathfrak{P}^{1-n})$.

Using the above, we classify the fundamental strata into the following:

Definition 2.2.4. *Let $(\mathfrak{U}, n, a)$ be a fundamental strata,*

1. *if $e_{\mathfrak{U}} = 2$ and n is odd, then the strata is ramified simple*
2. *if $e_{\mathfrak{U}} = 1$ and $f_a(t)$ is irreducible in $k[t]$, then the strata is unramified simple.*
3. *if $e_{\mathfrak{U}} = 1$ and $f_a(t)$ has distinct roots in k, then the strata is split*
4. *if $e_{\mathfrak{U}} = 1$ and $f_a(t)$ has a repeated root in $k^\times$, then the strata is essentially scalar.*

It can be shown that an irreducible smooth cuspidal representation either contains only a simple strata or is a twist of a representation that contains a simple strata. Whereas, the principal series representations either contain only the fundamental split strata or are a twist of one which does.

Thus, to classify the cuspidal representations we look at simple strata.

For both the unramified and ramified simple strata $(\mathfrak{U}, n, a)$, $E = F[a]$ is a quadratic extension. Consider, $J_a = E^\times U_{\mathfrak{U}}^{[(n+1)/2]}$. This is a subgroup of G which is compact modulo Z and is also the maximal subgroup that normalises $\psi_{A,a}$, the character of $U_{\mathfrak{U}}^n$ associated with the given strata.

Theorem 2.2.2. *For a given simple strata $(\mathfrak{U}, n, a)$, let Λ be an irreducible representation of J_a such that $\Lambda|_{U_{\mathfrak{U}}^{[n/2]+1}}$ contains $\psi_{A,a}$. Then, the representation,*

$$\pi_\Lambda = c\text{-}Ind_{J_a}^{G} \Lambda$$

is irreducible and cuspidal.

Here, the compact induction of Λ from J_a to G is defined as,

$$\begin{aligned} \text{c-Ind}_{J_a}^{G}(\Lambda) = \{ f : G \mapsto \mathbb{C} \mid f(hg) = \Lambda(h) f(g)\ h \in J\ \text{ and } \exists\ K \text{ a compact open subgroup} \\ \text{such that } f(gx) = f(g)\ \forall\ g \in G,\ \forall\ x \in K \text{ and } \mathrm{Supp} f \text{ is compact in } H\backslash G \}. \end{aligned}$$

The above theorem, describes an irreducible cuspidal representation which can be identified by three parameters - a simple strata $(\mathfrak{U}, n, a)$, its associated subgroup, J_a, and a representation of J_a which contains the character associated with the given strata. Using this, we define objects called cuspidal types in order to try and parametrize every irreducible cuspidal representation of G.

Definition 2.2.5. *A cuspidal type is a triple* $(\mathfrak{U}, J, \Lambda)$ *where* $\mathfrak{U}$ *is a chain order,* J *a subgroup of* G *which is compact modulo* Z *and* Λ *is an irreducible representation of* J *such that one of the following happens:*

1. $\mathfrak{U} \cong \mathfrak{M}$*,* $J = ZK = ZU_{\mathfrak{M}}$ *and* $\Lambda|_{U_{\mathfrak{M}}}$ *is inflated from an irreducible cuspidal representation of the group* $GL(2, \boldsymbol{k})$
2. $(\mathfrak{U}, n, a)$ *is a simple stratum,* $J = J_a$ *as defined above and* Λ *is an irreducible representation of* J *such that* $\Lambda|_{U_{\mathfrak{U}}}^{[n/2]+1}$ *contains* $\psi_{A,a}$
3. *There exists* $(\mathfrak{U}, J, \Lambda_0)$ *such that it satisfies either of the above two conditions and* $\Lambda \cong \Lambda_0 \otimes \chi \circ det$ *for some character* χ *of* $F^{\times}$

Each cuspidal type, $(\mathfrak{U}, n, a)$ gives an irreducible cuspidal representation, $\pi_\Lambda = \text{c-nd}_J^G \Lambda$.

Theorem 2.2.3. *The map between the conjugacy classes of cuspidal types and the equivalence classes of irreducible cuspidal representations of* G*.*

$$(\mathfrak{U}, J, \Lambda) \longrightarrow \textit{c-Ind}_J^G \Lambda$$

is a bijection.

Thus, all the cuspidal representations of G are classified by their cuspidal types. These are also called the supercuspidal representations of G.

Chapter 3

Characters of Irreducible Representations of $GL(2, F)$

The Plancherel formula for G will be of the form (1). Here the integrand is called the character distribution of an irreducible unitary representation (π, V) of G. As we shall see in this chapter, the character distribution can be determined by a function over G which is dependent only on π. By abuse of notation, we call this function the character of π. We study this integrand and the correspoding character in this chapter.

3.1 The Character as a Locally Integrable Function

Let (π, V) be a smooth representation of G and K be some compact open subgroup of G. The subspace $V^K = \{v \in V \mid \pi(k)v = v \ \forall k \in K\}$, is the set of all K-fixed vectors in V.

Definition. *A smooth representation (π, V) of G is said to be admissible if for every compact open subgroup K, the subspace V^K is of finite dimension.*

Using the classification of representations in the previous chapter, we can prove that every irreducible representation of G is admissible. Given such an irreducible representation

(π, V) of G and a function $f \in \mathcal{C}_c^\infty(G)$, consider the operator,

$$Q_\pi = \pi(f) = \int_G f(g)\pi(g)dg.$$

The representation π being admissible implies that $\pi(f)$ is a finite rank operator. Therefore, $Tr(\pi(f))$, the trace of $\pi(f)$ exists and is a continuous functional on $\mathcal{C}_c^\infty(G)$. There exists a locally integrable function $\chi_\pi(g)$ defined almost everywhere on G, such that,

$$Tr(\pi(f)) = \int_G f(g)\chi_\pi(g)dg.$$

This is a result of Godemont and Harishchandra for irreducible unitary representations of semisimple Lie groups. For the purposes of this project, we shall prove this result for G, as given in the works of Jacquet and Langlands, *Automorphic Forms on GL(2)*. For both the cuspidal and non-cuspidal representations, we shall prove that such a χ_π exists by giving an explicit form for the function.

3.2 Character of Non-cuspidal Representations

We have classified all the non-cuspidal representations of G into three different types - the characters of G, the principle series representations and the special representations. The principal series representations are the induced representations of characters of T, inflated to B, which satisfy certain conditions. That is, the principal series can be written as $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ where μ_1 and μ_2 are characters of $F^\times$. The special representations are twists of the Steinberg representation, which is the G-quotient of $\mathrm{Ind}_B^G \delta_B^{1/2}$,

$$0 \longrightarrow \mathbb{1}_G \longrightarrow \mathrm{Ind}_B^G \delta_B^{1/2} \longrightarrow \mathrm{St}_G \longrightarrow 0.$$

Therefore the character of St_G can be written as $\chi_{\mathrm{St}_G} = \chi_{\mathrm{Ind}_B^G \delta_B^{1/2}} - 1$.

Thus, by looking at characters of representations of the form $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$ where μ_1 and μ_2 are characters of $F^\times$, that is $\chi_{\mathrm{Ind}_B^G \mu_1 \otimes \mu_2} = \chi_{\mu_1, \mu_2}$, we can describe the characters of all non-cuspidal representations.

A regular element of the group G is any element with distinct eigenvalues. Let $\tilde{G}$ be the set of all regular elements of G and let H be the set of all elements of G which have eigen values that do not lie in F. These are the two subsets of G for which we shall define the values of the characters. Both these subsets are open in G and $\tilde{G}$ is dense in G

A Cartan subgroup is the centralizer of a regular element in G. These subgroups are either the subgroup T, the set of all diagonal elements in G, or an isomorphism of the multiplicative group of a quadratic extension E over F. Observe that $\tilde{G}$ is the union of all G-conjugates of every Cartan subgroup in G. Also, $\tilde{G} - H$ is the set of all G-conjugates of T.

Proposition 3.2.1. *Let μ_1, μ_2 be characters of $F^\times$. We define, χ_{μ_1,μ_2} to be the function which is 0 on $\tilde{G} \bigcap H$, undefined on $G - \tilde{G}$ and equal to*

$$\{\mu_1(a)\mu_2(b) + \mu_1(b)\mu_2(a)\} \left| \frac{ab}{(a-b)^2} \right|^{1/2}$$

on an element which lies in some G-conjugate of T, that is on the space $\tilde{G} - H$, with eigen values a and b. Then, χ_{μ_1,μ_2} is continuous on $\tilde{G}$. If $\pi = Ind_B^G \mu_1 \otimes \mu_2$ then for every $f \in \mathcal{C}_c^\infty(G)$,

$$Tr(\pi(f)) = \int_G \chi_{\mu_1,\mu_2}(g) f(g) dg.$$

Proof. Consider the integral,

$$\int_G \chi_{\mu_1,\mu_2}(g) f(g) dg. \tag{3.1}$$

Since χ_{μ_1,μ_2} is 0 outside T, we can show that this is the same as the integral,

$$\frac{1}{2} \int_T \left| \frac{(a-b)^2}{ab} \right| \left\{ \int_{T\backslash G} \chi_{\mu_1,\mu_2}(g^{-1}\alpha g) dg \right\} d\alpha$$

where, the measure dg is the quotient of the measure on G by that on T. Observe that, χ_{μ_1,μ_2} depends only on eigen values and therefore is a class function. Also, $\alpha = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$ is conjugate to $\beta = \left[\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right]$. Thus, we can expand and simplify the above integral to say that (3.1) is equal to,

$$\int_{T} \mu_1(a)\mu_2(b) \left|\frac{(a-b)^2}{ab}\right|^{1/2} \left\{ \int_{T\backslash G} f\left(g^{-1}\alpha g\right) dg \right\} d\alpha.$$

The integral over $T\backslash G$ can be further written as an integral over $GL(2,\mathcal{O}) \times N$,

$$\int_{T} \mu_1(a)\mu_2(b) \left|\frac{a}{b}\right|^{1/2} \left\{ \int_{GL(2,\mathcal{O})\times N} f(k^{-1}\alpha nk)dk\ dn \right\} d\alpha. \tag{3.2}$$

Observe that $G = B \cdot GL(2,\mathcal{O})$. A locally constant function ϕ in the vector space $\mathrm{Ind}_B^G(\mu_1 \otimes \mu_2)$, is determined by its values on $K = GL(2,\mathcal{O})$, since $\phi(xg) = \mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{1/2}\phi(g)$ for every $g \in G$ and $x = \left[\begin{smallmatrix} a & k \\ 0 & b \end{smallmatrix}\right] \in B$. Such a function, ϕ can also be regarded as a function over the compact open subgroup K. Therefore, the following can be seen as an integral operator with kernel $F(k_1, k_2)$, on the space of functions over K.

$$\pi(f)\phi(k_1) = \int_{G} \phi(k_1 g)f(g)dg = \int_{K} \phi(k_2) \left\{ \int_{T\times N} f(k_1^{-1}\alpha n k_2)\mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{1/2} d\alpha\ dn \right\} dk_2$$

$$F(k_1, k_2) = \int_{T\times N} f(k_1^{-1}\alpha n k_2)\mu_1(a)\mu_2(b)\left|\frac{a}{b}\right|^{1/2} d\alpha\ dn$$

It is easy to check that the range of such an operator also lies in the space $\mathrm{Ind}_B^G \mu_1 \otimes \mu_2$. Thus, it is enough to calculate the trace of the integral operator to get $T(r\pi(f))$. The trace of the integral operator is given by,

$$\int_{K} F(k,k)dk$$

which, when expanded, is the same as (3.2)

Thus, we have that for a principal series representation $\pi = \mathrm{Ind}_B^G \mu_1 \otimes \mu_2$, the character is defined on $\tilde{G}$

$$\chi_{\mu_1,\mu_2}(g) = \begin{cases} \{\mu_1(a)\mu_2(b) + \mu_1(b)\mu_2(b)\} \left|\frac{ab}{(a-b)^2}\right|^{1/2} & \text{when } g \in \tilde{G} - H \\ 0 & \text{when } g \in \tilde{G} \bigcap H \end{cases}$$

Similarly for the Steinberg representation, χ_{St_G} is defined on $\tilde{G}$ as follows,

$$\chi_{\mathrm{St}_G} = \begin{cases} \frac{\|a\|+\|b\|}{\|a-b\|} - 1 & \text{when } g \in \tilde{G} - H \\ -1 & \text{when } g \in \tilde{G} \bigcap H \end{cases}$$

All the characters of the special representations can be written as twists of the above character.

3.3 Characters of Cuspidal Representations

Let (π, V) be a cuspidal representation. We can always define a Hermitian form on V. Consider some $\check{v} \in \check{V}$, the smooth dual of V, such that $\check{v} \neq 0$, then we can define the integral

$$(v_1, v_2) = \int_G \langle \pi(g)v_1, \check{v}\rangle \langle v_2, \check{\pi}(g)\check{v}\rangle dg.$$

This forms a Hermitian form on V. We shall use this to show the existence of χ_π in the following proposition.

Proposition 3.3.1. *There exists a locally integrable function, χ_π which is defined and continuous on $\tilde{G} \bigcup H$ such that*

$$Tr(\pi(f)) = \int_G \chi_\pi(g) f(g) dg.$$

Proof. Let ω_π be the central character of π. If $|\omega_\pi| \neq 1$, then there exists some ϕ, a character of $F^\times$ such that $\acute{\pi} = \phi \otimes \pi$ has a central character $|\omega_{\acute{\pi}}| = 1$, that is $\acute{\pi}$ is unitary. Then, if both characters exist, $\chi_{\acute{\pi}}(g) = \phi(\det(g))\chi_\pi$. Therefore, we only need to look at unitary cuspidal representations. Let

$$Q = \pi(f) = \int\limits_G f(h)\pi(h)dh.$$

Consider some orthonormal basis of V, $\{v_i\}$. Then we can define,

$$Q_{ij} = (Qv_i, v_j).$$

π being an admissible representation, Q is a finite rank operator, therefore, only finitely many of the Q_{ij}'s are non-zero.

Consider $u \in V$ such that under the Hermitian form on V, $(u,u) = 1$, that is a unit length vector in V. Now we look at the following equations

$$(\pi(g)^{-1}Q\pi(g)u, u) = (Q\pi(g)u, \pi(g)u) \tag{3.3}$$

$$(Q\pi(g)u, \pi(g)u) = \sum_i (Q\pi(g)u, v_i)(v_i, \pi(g)u) = \sum_j \sum_i (\pi(g)u, v_i)Q_{ji}(v_j, \pi(g)u)$$

Since there are only finitely many Q_{ij}'s which are non-zero, therefore, the above series is a finite sum and we can look at the following integration,

$$\int\limits_{Z\backslash G} (\pi(g)^{-1}Q\pi(g)u, u)dg = \sum_{i,j} Q_{ji} \int\limits_{Z\backslash G} (\pi(g)u, v_j)(v_i, \pi(g)u)dg.$$

π being unitary implies it is also square integrable, therefore the integral on the right converges. We can also apply Schur's Orthogonality relations to get,

$$\frac{1}{d(\pi)} \sum_{i,j} Q_{ij}(v_i, v_j) = \frac{1}{d(\pi)} \sum_i Q_{ii} = \frac{1}{d(\pi)} Tr(\pi(f)).$$

where $d(\pi)$ is the formal degree of π. Also, the expression on the right of (3.3) can be expanded to,

$$\int_G f(h)(\pi(g^{-1}hg)u, u)dh.$$

Therefore, $\frac{1}{d(\pi)}Tr(\pi(f)) = \int\limits_{Z\backslash G} \left\{ \int\limits_G f(h)(\pi(g^{-1}hg)u, u)dh \right\} dg$. The integral over $Z\backslash G$ is the limit of integrals over compact subsets of $Z\backslash G$. Any element $x \in G$ can be written as

$$x = g_1 \left(\begin{smallmatrix} \varpi^p & 0 \\ 0 & \varpi^q \end{smallmatrix} \right) g_2$$

where, g_1 and g_2 are elements of $K = GL(2, \mathcal{O})$ and $p \leq q$. We define, T_r as the set of all elements of G such that $q - p < r$. This is the inverse image of a compact subset $T_r'$of $Z\backslash G$. The above integral is the limit of the integral

$$\int\limits_{T_r'} \left\{ \int\limits_{G} f(h)(\pi(g^{-1}hg)u, u)dh \right\} dg$$

as r approaches infinity. T_r' being compact implies that the above integral converges absolutely. Therefore,

$$\int\limits_{G} f(h) \left\{ \int\limits_{T_r'} (\pi(g^{-1}hg)u, u)dg \right\} dh.$$

Set, $\phi_r(h) = \int\limits_{r'} (\pi(g^{-1}hg)u, u)dg$. The proposition now boils down to showing that the limit $\chi_\pi(h) = d(\pi) \lim\limits_{r \to \infty} \phi_r(h)$ exists.

Since π is an irreducible unitary representation, there exists a conjugate linear map $A : V \longrightarrow \check{V}$ such that $(v_1, v_2) = \langle v_1, Av_2 \rangle$. Therefore, the functions $(\pi(g)u, u)$ are also matrix coefficients. π being cuspidal implies that it is also γ-cuspidal, that is, every matrix coefficient is compactly supported mod Z, and thus, so is $(\pi(g)u, u)$.

Let C be the image of this compact support in $Z\backslash G$. If D is a compact subset of H, we can say that for every $h \in D$ the set $\{g \in G \mid \pi(g^{-1}hg)u, u) \neq 0\}$ has a compact image C' in $Z\backslash G$. Therefore, the integral,

$$\int\limits_{Z\backslash G} (\pi(g^{-1}hg)u, u)dg = \int\limits_{C'} (\pi(g^{-1}hg)u, u)dg$$

is convergent for $h \in D$. If r is large enough then $C' \subset T_r'$ and thus,

$$\phi_r(h) = \int\limits_{Z\backslash G} (\pi(g^{-1}hg)u, u)dg$$

and the sequence $\{\phi_r\}$ converges on H to the limit $(d(\pi))^{-1}\chi_\pi(h)$.

On the rest of $\tilde{G} - H$, which are all the G-conjugates of the diagonal matrices, we can show that for a given compact subset in $\tilde{G} - H$ there exists a constant c and a bound M determined by the compact subset such that, $|\phi_r(h)| \leq cM$ for all r and for all such h in that compact subset. This can be shown by using the fact that, any element of G can be written as a product of an element from B and an element from K, and also h, a conjugate of T, can be written as,

$$h = h_1^{-1} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & \left(1 - \frac{b}{a}\right) x \\ 0 & 1 \end{pmatrix} h_1$$

where $h_1 \in GL(2, \mathcal{O}) = K$. The point h belongs in a compact set if all the values x, a, b and $\left(1 - \frac{b}{a}\right)$ are bounded above and below. Thus we look for the conditions on these bounds for which the argument of the integral can be written as $\pi(g^{-1}hg)u, u)$ where $g \in T_r$ and $h \in T$.

Under these conditions it is easy to show that the integral is bounded. Thus, $\phi_r(h)$ converges over T and therefore over $\tilde{G} \bigcup H$.

The character of an irreducible cuspidal representation (π, V) can be written as,

$$\frac{1}{d(\pi)}\chi_\pi = \int\limits_{Z\backslash G} (\pi(g^{-1}hg)u, u)dg$$

where, $d(\pi)$ is the formal degree of π, u is a vector of unit length corresponding to the Hermitian form on V.

Thus, we have shown that for any irreducible representation (π, V) of G, there exists a function defined almost everywhere on G, such that $Tr(\pi(f)) = \int\limits_G f(g)\chi_\pi(g)dg$. This is called the character of π.

Chapter 4

The Plancherel Formula

In this chapter we try to put down the possible approaches to writing down the Plancherel formula for G. The form of the Plancherel formula that we see in (1) is clearly a decomposition of the regular representation of G, which is then evaluated at the identity. We can try to prove this decomposition by comparing it to the Plancherel formula for $SL(2,F)$. We can also make such a comparison between the Plancherel formula for $SL(2,F)$ and that of $PGL(2,F)$ as well. Therefore, we look at the decompostion of the regular representation of G in terms of the regular representation of G/Z as well.

4.1 A Direct Integral Decompostion

The regular representation of G is the action of the group on $\mathcal{L}^2(G)$, the space of all square-integrable functions over G.

$$\mathcal{L}^2(G) = \left\{ f : G \to \mathbb{C} \mid \int\limits_G |f(g)|^2 dg < \infty \right\}.$$

The Plancherel Formula is essentially a decomposition of $\mathcal{L}^2(G)$ in terms of its irreducible unitary representations. We can also try to obtain this decomposition by first decomposing it in terms of the spaces $\mathcal{L}^2(G/Z,\omega)$. In this section we try to prove that the following map is an isomorphism :

$$\mathcal{L}^2(G/Z,\omega) \longrightarrow \int_{\hat{Z}}^{\oplus} \mathcal{L}^2(G/Z,\omega)d\omega$$

where we look at a direct sum of Hilbert spaces $\mathcal{L}^2(G/Z,\omega)$ over the measure space $\hat{Z}$, the space of all unitary central characters of G.

$$\mathcal{L}^2(G/Z,\omega) = \left\{ f : G \to \mathbb{C} \mid f(gz) = f(g)\omega(z) \ \forall g \in G, z \in Z \text{ and } \int_{G/Z} |f(g)|^2 dg < \infty \right\}$$

Consider for $f \in \mathcal{C}_c^\infty(G)$, the map

$$\Phi : f \mapsto \left(x : \omega \mapsto \left(x(\omega) : g \mapsto \int_Z f(gz)\omega(z)^{-1}dz \right)\right)$$

To show that such a map, when extended to $\mathcal{L}^2(G)$, is an G-invariant isomorphism, we must first check if its well-defined.

For any $f \in \mathcal{C}_c^\infty(G)$ the map gives, $x(\omega)(gz') = \int_Z f(gz'z)\omega(z)^{-1}dz$. On substituting zz' for y, we get $x(\omega)(gz') = \omega(z') \int_Z f(gy)\omega(y)^{-1}dy$, since Z is uni-modular.
Thus, $x(\omega)(gz) = \omega(z)x(\omega)(g)$.

For $x(\omega)$ to be an element of $\mathcal{L}^2(G/Z,\omega)$, we must also check for square integrability.

$$\int_{G/Z} |x(\omega)(g)|^2 dg = \int_{G/Z} |\int_Z f(gz)\omega(z)^{-1}dz|^2 dg \leq \int_{G/Z} \left(\int_Z |f(gz)\omega(z)^{-1}|dz \right)^2 dg$$

Since, $F \in \mathcal{C}_c^\infty(G)$, and thus compactly supported, therefore there are only finitely many points on $g \in G/Z$ such that $f(gz) \neq 0$ for some $z \in Z$. Also, ω is unitary, therefore,

$$\int_{G/z} |x(\omega)(g)|^2 dg \leq \int_{G/Z} \left(\int_Z |f(gz)|dz \right)^2 dg < \infty$$

Thus, $x(\omega)$ is an element of $\mathcal{L}^2(G/Z,\omega)$.

For x to be a section of the direct integral over $\hat{Z}$,

$$\int_{\hat{Z}} \|x(\omega)\|^2 d\omega < \infty.$$

We first check this for when f is a characteristic function of some compact open subgroup of G. Let $K = GL(2, \mathcal{O})$ and $K_n = 1 + \left[\begin{smallmatrix} \mathcal{P}^n & \mathcal{P}^n \\ \mathcal{P}^n & \mathcal{P}^n \end{smallmatrix}\right]$. Let χ_n be the characteristic function of K_n, that is χ_n take values 1 over K_n and 0 elsewhere. This gives us $x_n = \Phi(\chi_n)$, for some $g \in G$ and some $\omega \in \hat{Z}$

$$x_n(\omega)(g) = \int_Z \chi_n(gz)\omega(z)^{-1} dz$$

We know that, Z being the set of all scalar matrices, $Z \cong F^\times$. Now we consider, ω such that $\omega|_{1+\mathcal{P}^n} \neq \mathbb{1}$. Here, $1 + \mathcal{P}^n = U_n$ is a compact open subgroup of Z. Therefore we can write,

$$\int_Z \chi_n(gz)\omega(z)^{-1} dz = \int_{Z/U_n} \left(\int_{U_n} \chi_n(g\dot{z}u)\omega(\dot{z}u)^{-1} du \right) d\dot{z}$$

Since, $u \in 1 + \mathcal{P}^n \cong U_n$, we have $g\dot{z}u \in K_n \Leftrightarrow g\dot{z} \in K_n$ and χ_n being a characteristic function we get, $\chi_n(g\dot{z}u) = \chi_n(g\dot{z})$.

$$x_n(\omega)(g) = \int_{Z/U_n} \chi_n(g\dot{z})\omega(\dot{z}) \left(\int_{U_n} \omega(u)^{-1} du \right) d\dot{z} = 0$$

Since, $\int_{U_n} \omega(u)^{-1} du = 0$ when $\omega|_{U_n} \neq \mathbb{1}$. Thus, x_n is non-trivial only when $\omega|_{U_n} = \mathbb{1}$.

Next, we look at the space $S = \{\omega \in F^\times \mid \omega|_{U_n} = \mathbb{1}\}$. We know that $Z \cong F^\times \cong \mathbb{Z} \times U_F$. Therefore, $S = \{\widehat{\mathbb{Z} \times (U_F}/U_n)\} \cong S^1 \times (\widehat{U_F/U_n}) \subset \{\hat{Z} = \hat{F^\times}\}$

Since, U_F is compact and U_n is open, thus $(\widehat{U_F/U_n})$ is finite. Also, S^1 is compact, therefore the space S is a compact space in $\hat{Z}$. Thus, the section x is compactly supported over $\hat{Z}$ and the following integral is finite.

$$\int_{\hat{Z}} |x(\omega)|^2 d\omega = \int_{\hat{Z}} \int_{G/Z} |\int_Z f(gz)\omega(z)^{-1} dz|^2 dg \; d\omega < \infty$$

Therefore, we have shown that the map is well-defined for the characteristic functions χ_n.

For any $f \in \mathcal{C}_c^\infty(G)$, one can write f in terms of the characteristic functions defined above, χ_n. Since f is locally constant and compactly supported we can write it as a finite sum, $f = \sum c_i \chi_{U_i}$ where, U_i is a compact open subset of G and χ_{U_i} its characteristic function. Any compact open subset, U_i contains some G translation of K_{m_i} for a sufficiently large m_i, therefore we can write, $f = \sum c_i \chi_{g_i K_{m_i}}$ where, $\chi_{g_i K_{m_i}} = g_i \chi_{K_{m_i}}$ and $\chi_{K_{m_i}} = \chi_{m_i}$, according to our previous notation. Therefore, $f = \sum c_i g_i \chi_{m_i}$

Thus, any function in $\mathcal{C}_c^\infty(G)$ can be written as a linear combination over G translations of χ_n.

The map in question is clearly linear. All we need now is to check for G-invariance. Suppose, $\Phi(f) = x$. Consider the action of some $h \in G$ on f, $h.f(g) = f(gh)$. Then,

$$\int\limits_Z f(ghz)\omega(z)^{-1}dz = x(\omega)(gh) = h.x(\omega)(g) = h.\Phi(f)(\omega)$$

Thus, it is also G-invariant. Therefore, the map is well-defined over $\mathcal{C}_c^\infty(G)$ and since $\mathcal{C}_c^\infty(G)$ is dense in $\mathcal{L}^2(G)$, we can extend the map continuously over the rest of the space.

To show that Φ is an isomorphism, we construct the inverse map. Consider a section $x \in \int_{\hat{Z}}^{\oplus} \mathcal{L}^2(G/Z, \omega)d\omega$. Such an x is said to be compactly supported if there exists a compact subset W of $\hat{Z}$ such that $\|x(\omega)\| = 0$ if $\omega \notin W$. Since $\int_{\hat{Z}} \|x(\omega)\|^2 d\omega < \infty$, thus, we can say that x is a square integrable map from $\hat{Z}$ to $\mathbb{C}$, that is, $x \in \mathcal{L}^2(\hat{Z})$. If we assume x to be compactly supported , then $x \in \mathcal{C}_c^\infty(\hat{Z})$, which is dense in $\mathcal{L}^2(\hat{Z})$. Therefore, we can define a map on $\mathcal{C}_c^\infty(\hat{Z})$ and continuously extend it to the rest of $\mathcal{L}^2(\hat{Z})$.

Consider, $\varphi : x \mapsto f$ such that $f(g) = \int_{\hat{Z}} x(\omega)(g)d\omega$. This when extended to the rest of $\hat{Z}$ and then composed with Φ gives us that $\Phi \circ \varphi = \mathbb{1}_{\hat{Z}}$ and $\varphi \circ \Phi = \mathbb{1}_G$.

Thus, we have that Φ is an isomorphism.

4.2 The Plancherel formula for SL(2)

The unitary representations of $SL(2,F)$ have a similar classification to that of $GL(2,F)$. The finite dimensional representations are all the characters. Then there are the principal series representations which in the case of $SL(2,F)$ can be parametrized by one character of $F^\times$ instead of the pair of characters that we see in $G = GL(2,F)$. There are the special representations which are the twists of the Steinberg representation, which is the same as in G. Finally there are the cuspidal representations parametrized by three objects. A cuspidal representation of $SL(2,F)$ can be written as (h, V, ψ), where $V = F[\sqrt{\theta}]$ is a quadratic extension of F, ψ is a character of C_θ, the set of all units of V, and h (the conductor of ψ) is the smallest integer such that ψ is trivial on $C_\theta^{(h)} = 1 + \mathcal{P}_\theta^h$, where $\mathcal{P}_\theta$ is the prime ideal in V. These parameters are the same as those for the cuspidal representations of G.

The above similarities between the classification of representations of both the groups indicate that the Plancherel formula would also take similar structures in both cases. Therefore, we study the Plancherel formula for $SL(2,F)$ and the proof given for it as explained in the works of Sally and Shalika in order to emulate it for G.

For the classification of the cuspidal representations of $SL(2,F)$, we characterize the parameters mentioned above. Assuming F is a p-adic field where $p \neq 2$, there are only three possible quadratic extensions of F, $V = F[\sqrt{\theta}]$ where $\theta = \tau, \epsilon, \epsilon\tau$. Here, τ is a prime element of F and ϵ is a representative of the square free units of F. The quadratic extension V can be seen as a multiplicative subgroup of G. All such possible inclusions that intersect with $SL(2,F)$ are the C_θ's for the corresponding quadratic extensions. These are the compact Cartan subgroups of $SL(2,F)$. The character ψ of C_θ is trivial on C_θ^h. To accommodate the role of h and to limit the possible spaces that C_θ^h can represent, we must modify its definition for different θ's. Here, $C_\epsilon^h = 1 + \mathcal{P}_\epsilon^h$ and $C_\theta^h = 1 + \mathcal{P}_\theta^{2h+1}$, when $\theta = \tau, \epsilon\tau$. The conductor of ψ, h, ranges from 1 to ∞.

Remark 4.2.1. *In the modification of the definition of C_θ^h, we observe that this is also reflected in the classification of cuspidal representations of G. $V = F[\sqrt{\epsilon}]$ being an unramified extension is associated with an unramified simple strata $(\mathfrak{P}, n, a)$, which can have any value for n. Whereas, $V = F[\sqrt{\tau}]$ or $V = F[\sqrt{\epsilon\tau}]$ are ramified extensions which are associated with a ramified simple stratum where n is always odd.*

As seen in the previous chapter, for any irreducible representation (π, V) of $SL(2,F)$ and for some $f \in \mathcal{C}_c^\infty(SL(2,F))$, we can write,

$$\Theta_\pi(f) = Tr(\pi(f)) = \int_G f(g)\theta_\pi(g)dg.$$

For a cuspidal representation, (h, V, ψ) we define, $\Theta_\pi(f) = \Theta_\psi(f)$. For the principal series representation parameterized by a character $\mu \in \hat{F^\times}$, we write this as $\Theta_\mu(f)$ and for the Steinberg representation this is $\Theta_0(f)$

The Plancherel Formula for $SL(2,F)$ is given by the following.

Let $f \in \mathcal{C}_c^\infty(SL(2,F))$, then

$$\frac{2(q^2-1)}{q^2} f(1) = \sum_{\theta=\tau,\epsilon,\tau\epsilon} \sum_{h=1}^{\infty} \sum_{\substack{\psi \in \hat{C}_\theta \\ \text{cond } \psi = h}} \mu(\psi)\Theta_\psi(f) + 2(q-1)\Theta_0(f) + \frac{q^2-1}{q^2} \int_{\mu \in \hat{F^\times}} \frac{1}{|\Gamma(\mu)|^2} \Theta_\mu(f) d\mu \quad (4.1)$$

where $\Gamma(\mu)$ is a function over $\hat{F^\times}$ which along with $d\mu$ gives the Plancherel measure for the principal series representations. Also, $\mu(\psi)$ is the Plancherel measure for the cuspidal representations.

To prove this formula, we first construct the above summation. This summation can be written as a limit of the following distribution,

$$\Lambda_d(f) = \sum_{\theta=\tau,\epsilon,\tau\epsilon} \sum_{h=1}^{d} \sum_{\substack{\psi \in \hat{C}_\theta \\ \text{cond } \psi = h}} \mu(\psi)\Theta_\psi(f) + 2(q-1)\Theta_0(f) + \frac{q^2-1}{q^2} \int_{\mu \in \hat{F^\times}} \frac{1}{|\Gamma(\mu)|^2} \Theta_\mu(f) d\mu \quad (4.2)$$

Thus, what we must prove is that $\lim_{d\to\infty} \Lambda_d(f) = \frac{2(q^2-1)}{q^2} f(1)$ for every $f \in \mathcal{C}_c^\infty(SL(2,F))$.

Consider K_m as defined in the previous section. For this section we redefine K_m as the subgroup of it that lies in $SL(2,F)$ and its characteristic function as χ_n. We can show that,

Lemma 4.2.1. *$\Lambda_d(\chi_m) = \frac{2(q^2-1)}{q^2}$ whenever, $d \geq m$.*

Now consider $f \in \mathcal{C}_c^\infty(SL(2,F))$ and the distribution, $\Lambda(f) = \lim_{d\to\infty} \Lambda_d(f)$. This distribution is supported on the unipotent elements of $SL(2,f)$, that is for any f with support that does not intersect with the unipotent elements, $\Lambda(f) = 0$. Also, for any $g \in G/Z$, $g.\Lambda(f) = \Lambda(g^{-1}.f) = \Lambda(f)$ where, $g^{-1}.f(x) = f(g.x)$ $\forall x \in SL(2,F)$. In other words, Λ is invariant under the adjoint action of G/Z where G and Z are defined as in the previous sections. These properties make the computations easier as, we have the following theorem.

Theorem 4.2.2. *If T is a distribution on $(SL(2,F)-\{1\})$ satisfying the following properties*

1. *T is supported on the unipotent elements of $SL(2,F)$*

2. *T is invariant under the adjoint action of G/Z*

then, T is unique up to scalar multiples.

Now we define a distribution $K(f)$ with the same properties as given in the hypothesis of theorem (4.2.2). To prove (4.1) , the construction of $K(f)$ is such that we have the following corollary.

Corollary 4.2.3. *If T is a distribution on $SL(2,F)$ such that on $SL(2,F)-\{1\}$, T satisfies the conditions given in theorem (4.2.2), then there exists c_1 and c_2 such that for every $f \in \mathcal{C}_c^\infty(SL(2,F))$*

$$T(f) = c_1 K(f) + c_2 f(1)$$

The construction of K is also such that $K(\chi_m - \chi_{m+1}) \neq 0$ whereas, $\Lambda(\chi_m - \chi_{m+1}) = 0$. Therefore, from corollary (4.2.3) and lemma (4.2.1), we have that for Λ, $c_1 = 0$ and $c_2 = \frac{2(q^2-1)}{q^2}$. Therefore,

$$\Lambda(f) = \frac{2(q^2-1)}{q^2} f(1)$$

This proves the Plancherel formula for $SL(2,F)$.

Chapter 5

Conclusion

We have seen the Plancherel Formula as well as its proof for $SL(2,F)$ in the previous chapter. One of the main differences between G and $SL(2,F)$ is that the center of G is $Z \cong F^\times$ is not compact and does not have finite measure, whereas, the center of $SL(2,F)$, given by $Z_S = \{\left[\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right] \mid x = \pm 1\}$, is finite. This difference implies that certain integrals taken over $SL(2,F)$, may not converge over G. Similarly, the Cartan subgroups over $SL(2,F)$ are compact whereas, for G, they are isomprphic to the whole quadratic extension $V = F[\sqrt{\theta}]$ and therefore, do not have finite measure. Due to such subgroups, certain functions over G cannot be constructed in the same way as that for $SL(2,F)$. This may be solved by considering G/Z instead of G. Thus, we look at the direct integral decomposition of $\mathcal{L}^2(G)$ into $\mathcal{L}^2(G/Z,\omega)$, where ω ranges over all the unitary representations of Z.

Next, we must calculate the Plancherel measure for each of the representations classified in the previous chapters. We must attempt to construct the Plancherel formula by constructing distributions similar to Λ and K. We must also check for if Theorem(4.2.2) is valid on G. When all these conditions align, only then we can try to construct a proof for G.

If these conditions are not possible to satisfy then we could try to look at the Plancherel formula for $PGL(2,G)$ as given in Silberger's book "PGL_2 over the p-adics : its Representations, Spherical Functions, and Fourier Analysis". This is essentially the decomposition of $\mathcal{L}^2(G/Z,\omega)$ when ω is trivial. We can try to obtain the Plancherel formula for G by the direct integral decomposition of $\mathcal{L}^2(G)$.

Thus, we can try to follow the strategies formulated in this project to obtain and prove the Plancherel formula for $G = GL(2, F)$, where F is a p-adic field. We hope to get to the formula sometime soon.

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