Knots, Links and Spherical braids in Real Projective 3-space

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To my mother

Certificate

Certified that the work incorporated in the thesis entitled Knots, Links and Spherical braids in Real Projective 3-space Submitted by Visakh Narayananwas carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other University or institution.

Rama Mishra

Prof. Rama Mishra

Date: 12/09/2023

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all priciples of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.

Visakh Narayanan

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Abstract

This thesis studies some geometric properties of knots in real projective 3-space. These ideas are borrowed from classical knot theory. Since knots in $\mathbb{R}P^3$ are classified into three disjoint classes: affine, class-0 non-affine, and class-1 knots, it is natural to wonder in which class a given knot belongs. We attempt to answer this question. We provide a structure theorem for these knots which helps in describing their behaviour near the projective plane at infinity. We propose a procedure called *space bending surgery*, on affine knots to produce several examples of knots. Later it is shown that this operation can be altered as a *class changing surgery* on an arbitrary knot in $\mathbb{R}P^3$. We then study the notion of companionship of knots in $\mathbb{R}P^3$ and provide a geometric criterion for a knot to be affine. We also define an invariant called "genus" for knots in $\mathbb{R}P^3$ and show that this genus detects knottedness and gives a criterion for a knot to be affine and of class-1. Using the properties of genus we prove a "non-cancellation" theorem for both the surgery operations mentioned above. We introduce a notion of *cable knot* and show that cable knots with a class 1 companion knot completely characterize genus 1 knots.

In the later part of the thesis, we develop a method for constructing links in $\mathbb{R}P^3$ as plat closures of spherical braids. This method is a generalization of the concept of "plats" in S^3 defined by Joan Birman [**Bir**]. We prove that any link in $\mathbb{R}P^3$ can be constructed in this manner. We introduce a new kind of permutation (called *residual permutation*) associated with a spherical braid in $\mathbb{R}P^3$ and prove that the number of disjoint cycles in the residual permutation of a spherical braid is equal to the number of components of the plat closure link of this braid. This braid representation provides another criterion for detecting affineness. We develop a set of moves on spherical braids in the same spirit as the classical Markov moves on braids. Braids related by a finite sequence of these moves will produce isotopic plat closure links. We conjecture that the converse of this is also true, so we hope to have an analogue of Markov theorem for links in $\mathbb{R}P^3$.

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Chapter 1

Introduction

Three-manifolds may seem harder to understand at first. But as actors and movers in a three-dimensional world, we can learn to imagine them as alternate universes.

-William Thurston, [**Thu**], *1997*

The theory of knots is believed to have started when Lord Kelvin [**Th**] proposed a theory of atoms, where he described the structure of atoms as knots in the fibers of "aether". P.G. Tait made the first table of knots and thereby started the mathematical study of knots[**Tai**]. Classically a knot is described as an embedding of S^1 into S^3 or $\mathbb{R}^3$, which is considered up to ambient isotopy[**Kaw**; **ABDPR**]. This definition has two obvious generalizations. One is that we could look at embeddings of S^1 inside any three-manifold. Or we could consider embeddings of a higher dimensional sphere S^n inside S^{n+2}.

In this thesis, we are interested in the first generalization. We study knot theory in the real projective 3-space, $\mathbb{R}P^3$. So a knot here is a smooth embedding of S^1 in $\mathbb{R}P^3$ and a link is a smooth embedding of disjoint union of copies of S^1 in $\mathbb{R}P^3$. We will mention knots but it would mean knots and links both. As usual, we will identify the embedding by its image. Two knots K_1 and K_2 are said to be ambiently isotopic if there exists an orientation preserving diffeomorphism $\phi : \mathbb{R}P^3 \to \mathbb{R}P^3$ such that $\phi(K_1) = K_2$. Thus the central problem will remain the same: classify knots in $\mathbb{R}P^3$ on the basis of ambient isotopy.

Of late there has been a lot of interest in studying knots in $\mathbb{R}P^3$. This was mainly initiated by Yulia V. Drobotukhina ([**Drob**]). The main idea initially was to generalize tools from classical knot theory which also work for distinguishing knots in $\mathbb{R}P^3$. For example, in [**Drob**] an analogue of the Jones polynomial for links in $\mathbb{R}P^3$ and in [**Huy**] Twisted Alexander polynomial have been developed. In [**Man**] M. Enrico studies the knots in $\mathbb{R}P^3$ by their lifts in S^3. In this thesis, we focus more on geometric properties associated with these knots and define some invariants for knots and links in $\mathbb{R}P^3$. Note that in the case of classical knots, most invariants are defined by representing a knot either by a "diagram" or by a " braid closure". We have tried considering both approaches in the set-up of knots inside $\mathbb{R}P^3$.

Before we talk about how knots or links behave in $\mathbb{R}P^3$, it is important to understand the topological properties of the ambient space $\mathbb{R}P^3$. Since $\mathbb{R}P^3$ is a quotient space of S^3 by antipodal points being identified, using S^3 as a quotient of a 3 ball B^3, we can represent $\mathbb{R}P^3$ as a quotient of B^3 with its antipodal points identified. This is referred to as the *ball model* to understand $\mathbb{R}P^3$. Also using the mapping cylinder C of the covering map, $h : S^2 \to \mathbb{R}P^2$ we can think of obtaining $\mathbb{R}P^3$ as gluing C with a 3-ball along with their boundary sphere. This representation is referred to as the *mapping cylinder model*. We will see a detailed exposition of both these models in Chapter 2 of this thesis.

The canonical 2 sheeted cover $h : S^3 \to \mathbb{R}P^3$ is an essential tool for developing many ideas towards knot theory in $\mathbb{R}P^3$. We will denote this map by h throughout this thesis. The antipodal map $a : S^3 \to S^3$ is the only non-trivial automorphism of this covering map. As the fundamental group of $\mathbb{R}P^3$ is $\frac{\mathbb{Z}}{2\mathbb{Z}}$, a knot in $\mathbb{R}P^3$ can represent either $\bar{0}$ or $\bar{1}$ in the fundamental group. The knots in $\mathbb{R}P^3$ are referred to as of class-0 or of class-1 accordingly. We have an unknot in both the classes known as the *affine unknot* and the *projective unknot* respectively. All class-0 knots lift to a link, with two components, in S^3. The two components are antipodes of each other. All class-1 knots lift to two paths which together form a knot in S^3. This knot is the antipode of itself. Also $\mathbb{R}^3$ is contained in $\mathbb{R}P^3$ so some knots may lie completely in $\mathbb{R}^3$, they are known as *affine knots*. Affine knots are always of class 0. We will witness that there are non-affine class 0 knots. Thus there are three disjoint families of knots in $\mathbb{R}P^3$ namely affine, non-affine class 0, and class 1. Thus, this itself becomes an important question to investigate - in which family a given knot belongs to? This thesis is mainly addressing this question.

In order to develop a diagram theory we need to know the space where the diagrams are drawn. Since the knots in $\mathbb{R}^3$ are drawn on a plane and in $\mathbb{R}P^3$ it will correspond to a projective plane and hence a link diagram for links in $\mathbb{R}P^3$ will be drawn on a projective plane. Yulia V. Drobotukhina in [**Drob**] provides a complete exposition of this, including a set of Riedemister Moves for these knots. As in classical knot theory, in this situation also a choice of a diagram is not unique. In fact, certain special diagrams are preferred in defining a specific invariant. For instance, some invariants are defined using a representation of knots by an *arc presentation* [**Crom**] or as a *Morse diagram* [**KAt**]. In this thesis, we introduce a new type of tangles known as *residual tangles* and using them show the existence of a specific presentation of link diagrams in $\mathbb{R}P^3$. In classical knot theory, one could generate knots using connected sum [**Rolf**] operation. We realize that this operation is not well defined in the case of knots in $\mathbb{R}P^3$. We define a surgery procedure performed on affine knots to generate any knot in $\mathbb{R}P^3$. We call it a *space bending surgery.* Later, we extend this operation on any knot and obtain a new knot whose class is different from the original knot. We call it a *class changing surgery.* We introduce a notion of companionship of knots in $\mathbb{R}P^3$ and develop some geometric characterization for a knot to be affine.

In the case of classical knot theory, all knots are homologically trivial so they bound a surface called a Seifert surface [**Seif**] of the knot. The minimal genus of all Seifert surfaces of a knot is called the 'genus' of that knot. Unfortunately, all knots in $\mathbb{R}P^3$ cannot bound a surface so we cannot talk of a Seifert surface but we can obtain surfaces containing a knot. We introduce a notion of a *good surface* of a knot k that contains k inside $\mathbb{R}P^3$ and whose lift under the 2 sheeted cover is a connected, orientable surface inside S^3 containing the lift of k. We use this good surface to define a "genus" for a knot in $\mathbb{R}P^3$. We show that it is an invariant for links in $\mathbb{R}P^3$. We have studied several properties of this genus in this thesis that provide characterization for detecting a given knot to be affine or not.

We have seen that in the case of the classical knot theory a braid representation of knots is a key component to introduce a plethora of quantum invariants and in particular polynomial invariants. One would like to find a similar connection of knots and links in $\mathbb{R}P^3$ with braids. Joan Birman introduced the notion of the braid group of an arbitrary manifold in [**Bir**]. The n-string braid group of any manifold is described as the group of "motions" of n special points in it. The standard Artin's braid group [**Bir**] will appear as the braid group of the plane in this definition. The braid group of the 2-sphere is of particular importance for our purposes. We will refer to the elements of this group as "spherical braids" and the elements of Artin's braid group as "classical braids". Birman introduced the concept of *plats* in S^3 [**Bir**]. It provides a different "closure" of braids than the standard closure in [**Alex**]. Then it is proven that *every classical link is isotopic to a plat.* Using the representation of $\mathbb{R}P^3$ as in the mapping cylinder model, we use spherical braids with even strands, place them on the mapping cylinder and define a closure operation that we call *plat closure* (Chapter 5, Section 5.1), to obtain links in $\mathbb{R}P^3$. We prove that every link in $\mathbb{R}P^3$ is ambient isotopic to the plat closure of some spherical braid. A beautiful feature of a classical braid is the permutation that it defines. The number of disjoint cycles in the permutation of the classical braid is equal to the number of components in the closure link. We introduce a new permutation called *residual permutation* associated to an even strand spherical braid in $\mathbb{R}P^3$ (Section 5.2) and show that the number of disjoint cycles in this permutation counts the number of components in the plat closure link. We also introduce some moves on spherical braids in such a way that under these moves the plat closures are ambient isotopic. We hope to prove that these moves are complete to capture the ambient isotopy of plat closure links in $\mathbb{R}P^3$. Thus we hope to prove a Markov Theorem. Once this is established, using the representations of braid groups of S^2 we may be able to obtain many invariants for knots and links in $\mathbb{R}P^3$.

This thesis is organized as follows: It contains six chapters including the Introduction as Chapter 1. Chapter 2 contains all the material and known results that are used in proving the main results. We have divided this chapter into three sections. In Section 2.1 we discuss the topology and geometry of $\mathbb{R}P^3$. We include all the basic definitions required to introduce knots and links in $\mathbb{R}P^3$. Section 2.2 discusses braids in any manifold and spherical braids in particular. We split this into two subsections. In Section 2.2.1 we present spherical braids in $\mathbb{R}P^3$ while Section 2.2.2 discusses the braid groups of S^2 in detail including a convenient presentation that is used in our main results in Chapter 5. In Section 2.3 we provide a quick account of Reidemeister type moves on diagrams in $\mathbb{R}P^3$ which were introduced in [**Drob**]. Chapter 3 has two main sections. Section 3.1 discusses three surgery operations, introduced by us, that can be performed on knots in $\mathbb{R}P^3$. In Section 3.1.1 we define what we mean by a projectivization of an affine knot. This helps in converting an affine knot into a class-1 knot. We define residual tangles in Section 3.1.2. In Section 3.1.3, we develop a surgery operation, which we call as "*space bending surgery*" which is performed on an affine link to produce a link in $\mathbb{R}P^3$. Since an affine link in $\mathbb{R}P^3$ corresponds to a classical link in $\mathbb{R}^3$, we may regard this as a surgery being done on a classical link. Associated to this procedure, we get a complete classification for links in $\mathbb{R}P^3$. This results in the following theorem which we call a "Structure theorem" for knots in $\mathbb{R}P^3$.

Theorem (3.1.1. (The structure theorem))**.** *Any knot in $\mathbb{R}P^3$ can be obtained from an affine knot by a space bending surgery.*

In Section 3.1.4 we provide a more general procedure called the "class changing surgery". It can be performed on an arbitrary link in $\mathbb{R}P^3$. This procedure can be repeated arbitrarily many times. Each time we perform it, the homology class of the knot strictly changes.

In Section 3.2 we generalize the notion of companionship of knots from classical knot theory to knots in $\mathbb{R}P^3$. It is known that the classical unknot is a companion of every knot in S^3. As a parallel to this, we have the following theorem.

Theorem (3.2.1.)**.** *The projective unknot is a companion to every knot in $\mathbb{R}P^3$.*

Then we define a property called "projective inessentiality", (see Definition 3.2.3). Using this we provide a geometric characterization for a knot to be affine through the following theorem.

Theorem (3.2.2.)**.** *A knot is affine if and only if it is projectively inessential.*

In Chapter 4, we study the association of a class of surfaces to a given link in $\mathbb{R}P^3$, which we call as "good surfaces". We prove that

Theorem (4.0.1)**.** *Every link in $\mathbb{R}P^3$ has a good surface.*

Refer to Definition 4.0.1 and Definition 4.0.2 where we define the *genus* of a knot in $\mathbb{R}^3$. The philosophy we adopt here is that "the complexity of a knot should also reflect in the complexity of the surfaces on which it can be embedded". We analyze many of the properties of this genus. For instance, we have the following.

Corollary (4.0.3.)**.** *The genus of every knot in $\mathbb{R}P^3$ is less than or equal to 3.*

Recall that the genus defined for classical knots using Seifert surfaces detects knottedness in S^3, see [**Rolf**]. Similarly, we prove that the genus defined using good surfaces detects knottedness in $\mathbb{R}P^3$.

Theorem (4.0.4)**.** *A link in $\mathbb{R}P^3$ is an unlink if and only if it has genus 0.*

As a corollary to this, we get a "non-cancellation theorem" for space bending surgery and class changing surgery.

Corollary (4.0.5.). *A Space bending surgery or a class changing surgery performed on a non-trivial link will always yield a non-trivial link.*

Recall that the classical genus provides a non-cancellation for the connected sum of classical knots [**Rolf**]. The corollary above may be compared with this, in the sense that, both space bending surgery and class changing surgery are operations of adding links together and thus are similar to a connected sum operation.

Genus provides a complete criteria for a knot to be class-1 or affine, as in the following theorems.

Theorem (4.0.6). *All non-trivial class-1 knots in $\mathbb{R}P^3$ have genus 1.*

An interesting question arising from this is whether there are other types of knots than class-1 knots with genus 1. We introduce the notion of Cable knots and prove the following

Theorem (4.0.7). *Suppose K is a non-trivial knot in $\mathbb{R}P^3$. Then K has genus 1 if and only if it is isotopic to a cable knot with companion J such that J is a class-1 knot.*

Using this theorem one can show the following.

Theorem (4.0.9). *The genus of any non-trivial affine knot in $\mathbb{R}P^3$ is* 2.

As another application of Theorem 4.0.7, we have the following.

Theorem (4.0.11). *There are infinitely many class-*0 *non-affine knots with genus* 1.

Thus we can construct several examples of class-0 non-affine knots of genus 1 by the recipe shown in Figure 4.3. But we do not completely understand how the genus of all class-0 non-affine knots behave. For example, we have the following.

Conjecture (4.0.12.)**.** *There exist class-0 non-affine knots with genus not equal to 1.*

To support this we provide an example of a knot in Figure 4.5, which we strongly suspect to have genus 3. Hence genus detects affine knots and class-1 knots. But the genus of class-0 non-affine knots is only partially understood. Thereby it reveals that the set of class-0 non-affine knots is a tough family of knots.

In Chapter 5 we introduce a braid theory for knots and links in $\mathbb{R}P^3$. This chapter has four sections. In Section 5.1 we explain the notion of *plats* in $\mathbb{R}P^3$ and following Joan Birman, we define a similar closure for spherical braids in $\mathbb{R}P^3$. We refer to it as "projective plat closure" and we prove that it is a complete scheme of representing links in $\mathbb{R}P^3$, via the following result.

Theorem (5.1.2)**.** *Every link in $\mathbb{R}P^3$ is isotopic to the projective plat closure of some spherical braid.*

In Section 5.2 we define a permutation which we call "residual permutation", associated to a spherical braid in $\mathbb{R}P^3$ and prove the following.

Theorem (5.2.1)**.** *The number of components in the plat closure link of a braid β is equal to the number of disjoint cycles in its residual permutation.*

As a remarkable application of this braid theory, we also get another method of detecting when a knot is affine. This is given by the following theorem.

Theorem (5.1.3)**.** *Let L be link in $\mathbb{R}P^3$. Then L is affine if and only if L is isotopic to the plat closure of a $k(k = 2n)$ braid $\beta = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_l}$ such that for some even integer j, $j, j+n \notin \{i_1, i_2, \ldots, i_l\}$.*

In Section 5.3, we discuss some moves on spherical braids, in the spirit of classical Markov moves. These moves generate an equivalence relation of spherical braids which we call M-equivalence. Then we have the following result.

Proposition (5.3.1)**.** *If β and β' are M-equivalent spherical braids then they have isotopic projective plat closures.*

Thus belonging to the same equivalence class is a sufficient condition for two braids to have isotopic plat closure links. We also conjecture that the converse is also true.

Conjecture (5.3.2)**.** *The projective plat closures of two braids are isotopic only if they are M-equivalent.*

Chapter 6 consists of some concluding remarks. In this chapter, we discuss our thoughts on some of the open questions in this subject. A select bibliography is included after Chapter 6. As this thesis has many new constructions and ideas, it can be well appreciated with the help of examples. We have included many interesting examples and pictures of plat closures in Appendix 1. Some examples of residual permutations of spherical braids are provided in Appendix 2.

Chapter 2

Preliminaries

In this chapter, we discuss some of the basic tools which are used in various parts of the thesis. For the work in this thesis we confine ourselves to the smooth category. That is, every manifold and every function that we are talking about is assumed to be smooth unless stated otherwise.

Definition 2.0.1. An **isotopy** [**Hir**] of a submanifold V of a manifold M is a homotopy,

$$F : V \times I \to M,$$

such that, every $F_t := F\big|_{V \times \{t\}}$ is an embedding and F_0 is the inclusion of V inside M. Notice that each $V \times \{t\}$ may be identified as V in a natural way, so that we may think of each F_t as an embedding of the manifold V inside the manifold M.

Definition 2.0.2. An isotopy as above where $V = M$ and each F_t is a diffeomorphism will be called an **ambient isotopy**. The "**ambient isotopy class**" of a submanifold $V \subset M$ is the set of all U such that there exists an ambient isotopy G such that $U = G_1(V)$.

Definition 2.0.3. A **knot** in a manifold M is the ambient isotopy class of the image of an embedding,

$$f : S^1 \to M.$$

Similarly the ambient isotopy class of the image of an embedding,

$$f : S^1 \sqcup S^1 \sqcup ... \sqcup S^1 \to M, \tag{2.1}$$

will be called a **link** in M with n components if n is the number of circles being embedded. As usual, we would not distinguish between the embedding, the image of the embedding, and the isotopy class of the image, when it is clear from the context.

In this thesis, we are interested in knots in $\mathbb{R}P^3$. To be able to include the necessary material required in this work, we divide this chapter in three sections. Section 2.1 focuses on giving an account of the geometric properties of $\mathbb{R}P^3$. Through these properties, we provide a vivid picture of the theory of diagrams for knots in $\mathbb{R}P^3$ proposed in [**Drob**]. After discussing the theory of diagrams, we describe the two different unknots and the three distinct families of links in $\mathbb{R}P^3$. We also give certain simple tests for detecting whether a given link is affine. In Sections 2.2 and 2.3, we introduce the notion of spherical braids and the braid group of S^2 respectively. This is discussed in much more generality by Joan Birman in [**Bir**]. We give a self-contained description which would be sufficient for our purposes.

2.1 The geometry and topology of $\mathbb{R}P^3$

The topology of $\mathbb{R}P^3$ can be understood in many ways. In this thesis, we use the following two topological models frequently to represent $\mathbb{R}P^3$.

Mapping cylinder model. We may think of $\mathbb{R}P^3$ as a compactification of $\mathbb{R}^3$ by adding a projective plane at infinity. Adding an $\mathbb{R}P^2$ at infinity may be understood in the following way. The complement of an open 3-ball in $\mathbb{R}^3$ is homeomorphic to $S^2 \times [0, 1)$. Choose a 3-ball $B \subset \mathbb{R}^3$. Choose a tubular neighbourhood [**Hir**] of ∂B and remove it from $\mathbb{R}^3$. We may identify the remaining two components with B and $S^2 \times [0, 1)$. First we compactify this $S^2 \times [0, 1)$ to $S^2 \times [0, 1]$ by adding a 2-sphere (at infinity) as $S^2 \times \{1\}$. Then we form a quotient of the $S^2 \times [0, 1]$ by identifying each point of the form $(p, 1)$ to the point $(-p, 1)$. Let C be the space thus obtained. Notice that the quotient map takes the sphere $S^2 \times \{1\}$ in $S^2 \times [0, 1]$ and projects it onto a projective plane in C. The boundary of C is a S^2 which is the image of $S^2 \times \{0\}$ under the quotient map. Now by gluing the boundaries of B and C, we obtain a copy of $\mathbb{R}P^3$. Notice that the complement of any open 3-ball in $\mathbb{R}P^3$ is homeomorphic to C. By construction, it must be clear that C is homeomorphic to the mapping cylinder [**Hat1**] of the two sheeted covering $S^2 \to \mathbb{R}P^2$. Thus $\mathbb{R}P^3$ can also be thought of as, joining a closed 3-ball to this mapping cylinder along their boundary sphere. This description of $\mathbb{R}P^3$ is referred as the mapping cylinder model.

Ball model. $\mathbb{R}^3$ is homeomorphic to an open ball. The process we described above, can alternatively be described as follows. Start with a closed 3-ball and identify the diagonally opposite points in its boundary sphere. Thus we obtain $\mathbb{R}P^3$ as a quotient of the 3-ball. A description of $\mathbb{R}P^3$ as the quotient of a 3-ball will be called as a "ball model" for $\mathbb{R}P^3$.

Knots and links in $\mathbb{R}P^3$ are ambient isotopy class of embeddings $f : S^1 \to \mathbb{R}P^3$ and $f : S^1 \sqcup S^1 \sqcup ... \sqcup S^1 \to \mathbb{R}P^3$ respectively as in Definition 2.03 and Definition 2.04.

Notice that,

- Fundamental group of $\mathbb{R}P^3$ is $\frac{\mathbb{Z}}{2\mathbb{Z}}$, so a knot here may represent $\bar{0}$ or $\bar{1}$. We call a knot to be of class 0 or of class 1 accordingly.
- The class of a knot is determined by the mod 2 intersection number of the knot with any projective plane [**Vir**].

All lines and planes in $\mathbb{R}^3$ get compactified as projective lines and projective planes respectively in $\mathbb{R}P^3$. The complement of any projective plane in $\mathbb{R}P^3$ is diffeomorphic to $\mathbb{R}^3$. The rich geometry admitted by $\mathbb{R}P^3$, similar to the Euclidean space, can be exploited to understand the knots in it. For example, the theory of diagrams for knots in S^3 (see [**Cr-F**]) can be generalized to knots in $\mathbb{R}P^3$ as in [**Vir**]. Since this notion of diagrams will be used throughout the article, we wish to give a quick exposition of this at this point.

Recall how knot diagrams are defined for $\mathbb{R}^3$. Corresponding to any line L in $\mathbb{R}^3$, we can construct a family of lines consisting of all parallel lines to L. Every point in $\mathbb{R}^3$ belongs to exactly one line in this family. Now corresponding to any such family there exists a plane Z passing through the origin, such that, Z intersects each line parallel to L at a unique point. Thus, we can define a projection, $d : \mathbb{R}^3 \to Z$, sending a point of $\mathbb{R}^3$ lying on a line L' parallel to L to the unique point in $L' \cap Z$. Notice that there is a unique line in the family that passes through the origin. We may regard L itself as this line, without loss of generality. Now L and Z are also vector subspaces of the vector space $\mathbb{R}^3$. Thus we get a splitting $\mathbb{R}^3 \approx L \oplus Z$.

It is easy to see that the map d defined above is the same as the projection of the vector space $\mathbb{R}^3$ onto its direct summand Z in the previous splitting. By using this projection map one can have a diagram for a knot K in $\mathbb{R}^3$ on the plane Z. Note that by using different choices of L and Z we have several such projections.

Notice that any family of parallel lines in $\mathbb{R}^3$ meet at a unique point in $\mathbb{R}P^3$ at infinity. Different points in the projective plane at infinity correspond to different families of parallel lines. Consider the ball model of $\mathbb{R}P^3$ presented as a quotient of the closed unit 3-ball $D^3 \subset \mathbb{R}^3$. Choose two diametrically opposite points N and N'. Choose a great circle γ on ∂D^3 disjoint from N and N'. Let $D^2 \subset D^3$, be a closed 2-disk, whose boundary is γ. Notice that since N and N' are diametrically opposite, they lie on distinct hemispheres in $\partial D^3 \setminus \gamma$. Consider a point $p \in D^3 \setminus \{N, N'\}$. It follows from elementary Euclidean geometry that, either there exists a unique circle in $\mathbb{R}^3$ passing through N, N' and p or they are collinear points. If N, N' and p are collinear, let L_p be the unique diametrical line segment from N to N'. It passes through p. Suppose N, N', and p are not collinear. Then as said above, there is a circle, α_p, passing through N, N', and p. It is easy to see that $\alpha_p \setminus \{N, N'\}$ is just two open arcs out of which exactly one contains p. Then let L_p be the arc of α_p which contains p with N and N' added as boundary points. Every circle passing through N and N' has to meet D^2 at least at one point. But notice that for all $p \in D^3 \setminus \{N, N'\}$ we know that L_p intersects D^2 at a unique point. Also if q lies in L_p then L_q is the same as L_p. Then we have a projection $\delta : D^3 \setminus \{N, N'\} \to D^2$, defined by sending every point p to the unique point of $L_p \cap D^2$. It is obvious that it is a continuous projection, with the inverse image of every point $p \in D^2$ as $L_p \setminus \{N, N'\}$. Refer to Figure 2.1.

Under the quotient map $\pi : D^3 \to \mathbb{R}P^3$, N and N' maps to the same point $\overline{N}$ in $\mathbb{R}P^3$. And D^2 maps to a projective plane P. Each of the arcs L_p maps to a projective line, containing $\overline{N}$. Thus δ defines a map from $\mathbb{R}P^3 \setminus \{\overline{N}\} \to P$. We shall refer to this map also as δ. From the context, the reader can deduce which of the projections are being specified. It is easy to see that the points $p \in D^3 \setminus \partial D^3$, the open ball, under δ maps to $D^2 \setminus \partial D^2$ the interior of D^2. We may identify the open ball to $\mathbb{R}^3$, in such a way that the interior of D^2 gets identified to plane Z from the previous example. It is easy to see that there is such an identification where the family of open arcs $L_p \setminus \{N, N'\}$ for all p's in the open ball, maps to a family of parallel lines as before. Hence the projection δ can be thought of as an extension of the projection, $d : \mathbb{R}^3 \to Z$, defined above.

Any link which is disjoint from $\overline{N}$ can be projected to P. It is obvious that one can always assume that any link is disjoint from $\overline{N}$ since if otherwise, there is an isotopy that can be used to remove it from meeting $\overline{N}$. This image of the projection in P can be "drawn" on D^2 by taking its inverse image in D^2 under the quotient map π. Clearly, it will be a collection of arcs in D^2 with boundary points in ∂D^2 forming a set closed under the antipodal map of ∂D^2. As in the classical case, one can always assume the link is in "regular position" and the image in D^2 has only double points as singularities.

Definition 2.1.1. The image of a link in $\mathbb{R}P^3$, in regular position, under δ, with the "over-under" crossing information provided at each double point, is said to be a **diagram** of the link.

Figure 2.2 is an example of a knot diagram. Note that if a knot diagram has $2n$ intersections with the boundary then n mod 2 determines the class of the knot. The knot in Figure 2.2 is a class 0 knot.

Just like the 3-ball, the diametrically opposite points of the closed disk D^2 should be regarded as being identified. In a diagram of a link in $\mathbb{R}P^3$ on a closed 2-disk D^2, any diameter of D^2 is a projective line in the quotient $\mathbb{R}P^2$. Notice that the inverse image of such a line under δ, in D^3 together with N and N' added is a closed 2-disk, say D', transversally intersecting D^2. Refer to Figure 2.3. And in the quotient $\mathbb{R}P^3$, D' becomes a projective plane. Thus diameters of D^2 in a diagram represent projective planes in $\mathbb{R}P^3$.

Proposition 2.1.1. *The complement of any embedded projective plane is diffeomorphic to an open 3-ball.*

Proof: Notice that under the covering map h, any embedded projective plane, P, in $\mathbb{R}P^3$ is covered in S^3 by an embedded 2-sphere S. Then $S^3 \setminus S$ consists of two open 3-balls, say, U_1 and U_2 such that $h|_{U_1} : U_1 \to \mathbb{R}P^3 \setminus P$ is a diffeomorphism. □

Definition 2.1.2. A knot is called an **unknot** if it is isotopic to a knot that has a diagram with no crossings. Similarly, a link is said to be an **unlink** if it is isotopic to a link that has a diagram with no crossings.

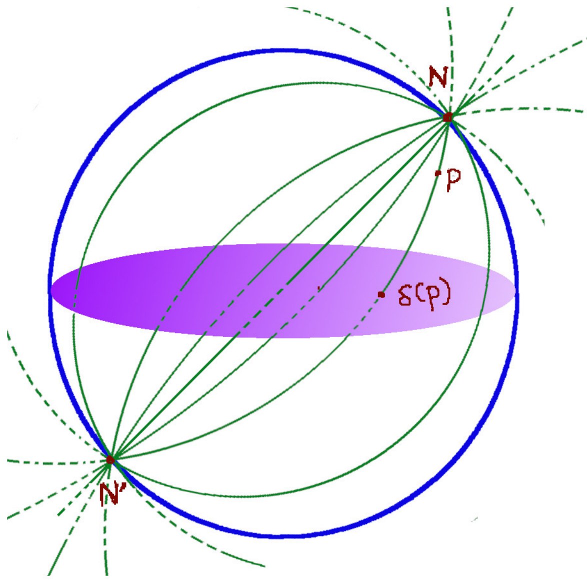

Figure 2.1: The projection map used for making diagrams of knots.

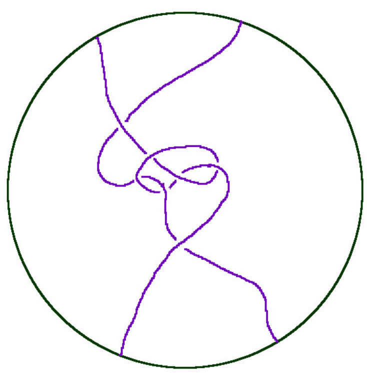

Figure 2.2: An example of a knot diagram in $\mathbb{R}P^3$.

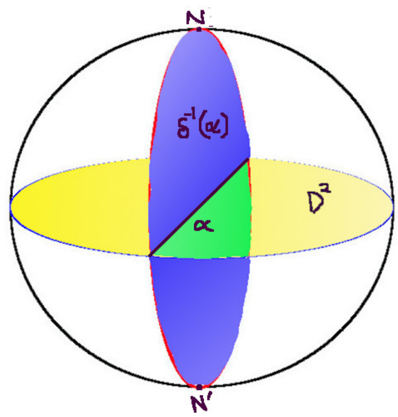

Figure 2.3: Diameters of the target D^2 are images of perpendicular disks

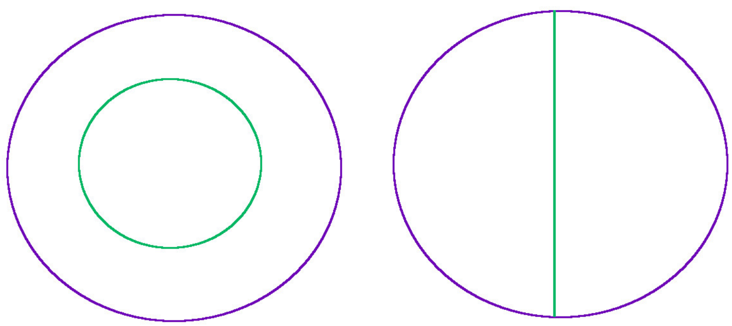

Figure 2.4: Unknots in $\mathbb{R}P^3$

One of the first differences between classical knot theory and projective knot theory is, there are two different candidates which deserve to be regarded as "unknots" in $\mathbb{R}P^3$. We call them the "affine" and "projective" unknots. Refer to Figure 2.4. Notice that any unlink can only have at most one projective unknot as a component.

Definition 2.1.3. A link K is said to be affine, if it is disjoint from some embedded projective plane in $\mathbb{R}P^3$.

Remark 2.1.1. Thus, affine links in $\mathbb{R}P^3$ are contained in an open set homeomorphic to $\mathbb{R}^3$, by Proposition 2.1.1. Hence by the Heine-Borel theorem [**Bor**; **Sim**] we can assume it is contained in a closed 3-ball which is disjoint from a projective plane. Notice that any link in $\mathbb{R}^3$ is compact and hence contained in a closed 3-ball, by Heine-Borel theorem. Now by adding a projective plane at infinity as in the mapping cylinder model, we may obtain an affine link in $\mathbb{R}P^3$. Thus any link K in $\mathbb{R}^3$ can be associated with an affine link $\mathscr{P}(K)$ in

$\mathbb{R}P^3$.

Remark 2.1.2. Since every knot in $\mathbb{R}^3$ is contractible, the affine knots in $\mathbb{R}P^3$ are of class-0. Thus every class-1 knot is non-affine.

Proposition 2.1.2. *Let K be a class-0 knot. If $h^{-1}(K)$ is a link in S^3 with a non-zero linking number between its components, then K is a non-affine knot.*

Proof: Suppose on the contrary that K is affine. Then by definition there exists an embedded projective plane P disjoint from K. The inverse image of any embedded projective plane in $\mathbb{R}P^3$ is a 2-sphere in S^3 whose complement is homeomorphic to two disjoint open 3-balls (refer to Proposition 2.1). Then the two components of $h^{-1}(K)$ lie in the two distinct open balls in S^3 separated by the 2-sphere covering P. Hence, the linking number between the components of $h^{-1}(K)$ must be zero. Which is a contradiction. Hence K must be non-affine. $\square$

The knot shown in Figure 2.2 is an example where the inverse image has a non-zero linking number. Thus we have three distinct families of knots to study in $\mathbb{R}P^3$, **affine, class -0 non-affine, and class -1** knots. One simple criterion for detecting affineness from a diagram is the following.

Remark 2.1.3. If K has a diagram on a closed 2-disk which has a diameter disjoint from the diagram, then K is an affine knot. This is because the inverse image under δ, of any diameter of the disk is a projective plane, say P, with a point missing, and hence K is contained in $\mathbb{R}P^3 \setminus P$. Refer to Figure 2.3.

2.2 Braids

*Space and time are commonly regarded as forms of existence of the real world, matter as its substance. It is in the composite idea of "**motion**" that these three fundamental conceptions enter into intimate relationship.*

-Hermann Weyl, [**Wey**], *1922*

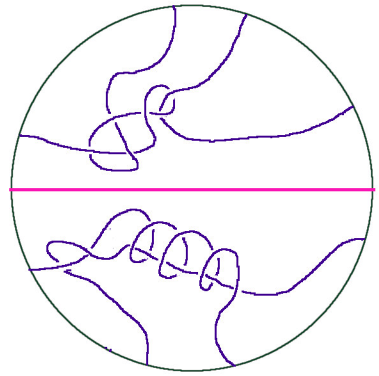

Figure 2.5: Diameter disjoint from the diagram

Albert Einstein introduced the idea of "spacetime" manifolds, through his landmark work [**Ein**] on the theory of relativity. Any manifold with a positive dimension n could be thought of as a $(n-1)+1$ spacetime, where we have $n-1$ dimensions of space and one-dimensional time. Thus, all three manifolds may be interpreted as 2+1-spacetimes. Imagine an abstract universe, which has a two-dimensional space, represented by a surface, say Σ. Suppose the only "matter" in this space is n-particles. We may think of these n particles of this space as n special points on Σ. By providing a concept of time, we may now start making these particles "move". Thus we have a spacetime, $\Sigma \times \mathbb{R}$, where the "worldline" of each particle is an arc. If the particles are not allowed to move back in time, each of the worldlines will always intersect the spacelike sections $\Sigma \times \{t\}$ transversely at one point. Thus if the particles perform a "cosmic dance" in space for a finite time interval $[0, 1]$, their worldlines together will form a **braid** in the spacetime, $\Sigma \times [0, 1]$. Thus, philosophically speaking, each n-string braid can be thought of as an abstract cosmic dance performed by n-particles in some space, represented by a manifold. Joan Birman describes this in a precise mathematical form in [**Bir**]. Thus we could define n-string braid group of any manifold M as a group formed by "motions" of n special points of M. The classical Artin braids appear as braids of the plane $\mathbb{R}^2$. We will be interested more in braids of the 2-sphere S^2, which we call spherical braids.

2.2.1 Spherical braids in $\mathbb{R}P^3$.

Let Σ be a manifold of arbitrary dimension and let n be a positive integer. Consider a set of n distinct points $X := \{p_1, p_2, ..., p_n\} \subset \Sigma$. These are thought of as special n points in space. Consider an isotopy,

$$F : \Sigma \times [0, 1] \to \Sigma,$$

such that, $F_0 = Id_\Sigma$ and F_1 is a diffeomorphism of Σ mapping X to itself. Then we may represent the isotopy in the space $\Sigma \times I$ by considering the map,

$$\begin{aligned} \overline{F} : \Sigma \times I &\to \Sigma \times I, \\ (q, t) &\mapsto (F_t(q), t) \end{aligned}$$

defined by it. Notice that the image of the set $X \times I$ under $\overline{F}$ is a collection of paths each one starting at some $(p_i, 0)$ and ending at some $(p_j, 1)$.

Definition 2.2.1. The topological pair $(\Sigma \times I, \overline{F}(X \times I))$ is refered to, as an n **braid of** Σ.

Consider the case when $\Sigma = S^2$. Each n-braid of S^2 is represented by a set of n paths in $S^2 \times I$ such that each path starts at a point of $S^2 \times \{0\}$ and ends at $S^2 \times \{1\}$ and intersects each of the sections $S^2 \times \{i\}$ at a unique point transversally. We may interpret the strip $S^2 \times I$ as a 2+1 dimensional spacetime where the I direction represents the flow of time. If we do not allow points of S^2 to move back in time their world lines intersect each "spacelike" sphere in the strip, i.e., spheres of the form $S^2 \times \{t\}$ at a unique point, just like the strings of braids. Or in other words, the projection map $f : S^2 \times I \to I$, is monotonic when restricted to each of the strings. Let α and β be two such motions of points in a set $X \subset S^2$. Then clearly we can define a new motion by composing them, that is performing β after α, which we will call $\alpha\beta$. As a braid, it is defined as the braid obtained by gluing the $S^2 \times \{1\}$ of the strip containing α to the $S^2 \times \{0\}$ of the strip containing β matching the indices properly and then rescaling the newly formed strip. This defines a multiplication of braids. The set of isotopy classes of braids forms a group under this operation. We describe this group in detail in Section 2.2.2.

Let B be the 3-ball. We know that, $\partial(S^2 \times I) \approx S^2 \amalg S^2$. Notice that S^3 can be obtained by gluing boundaries of $B \amalg B$ and $S^2 \times I$. Classically the plats in S^3 were constructed [**Bir**] by considering a spherical braid in this strip and certain simple tangles (which we discuss in the next section) in both balls. We wish to discuss a generalization of this construction. Let M denote the mapping cylinder of the canonical two-sheeted covering map $S^2 \to \mathbb{R}P^2$. Notice that, $\partial M \approx S^2$. By gluing the boundaries of $M \amalg B$ and $S^2 \times I$ we can obtain a copy of $\mathbb{R}P^3$. We choose the convention that $S^2 \times \{0\}$ is identified with ∂M and $S^2 \times \{1\}$ is identified with ∂B. When we say "braids in $\mathbb{R}P^3$", we mean the braids in $S^2 \times I$ region in some splitting of $\mathbb{R}P^3$ of this type.

Remark 2.2.1. Every braid in $\mathbb{R}P^3$ may be represented by a diagram drawn on an annulus using the projection defined in Section 2.1.

Proof of the remark: In the ball model of $\mathbb{R}P^3$ presented as the quotient of 3-ball D, let π represent the quotient map. We can consider B as a 3-ball with the same center as D. The 3-ball obtained by gluing B and $S^2 \times I$, say B_1 is a larger ball containing B with the same center as D. The complement of the interior of B_1 maps under the quotient map to the mapping cylinder M. Let C be an equator of $\partial B \approx S^2$. Let D_1 be the unique flat 2-disk containing C in D. ∂D_1 is an equator to ∂D and let N and N' be the corresponding north and south poles. Without loss of generality, we may assume that the special points are lying on an equatorial circle, C of S^2. Then clearly we can project any braid into the annulus, $C \times I$, in a projective plane $\pi(D_1)$ in $\mathbb{R}P^3$. $\square$

Thus we can represent spherical braids by a diagram drawn on some annulus. See Figure 2.6. for an example. The diagram of a composition $\alpha\beta$ will appear as keeping the diagram of β "inside" the diagram of α.

2.2.2 The braid group of S^2

It is easy to see that the composition of motions as defined in Section 2.2.1, defines a product of braids of the 2-sphere. There is always an "identity braid", which corresponds to all points being at rest. For every motion, there is a "inverse" motion which is obtained by reversing

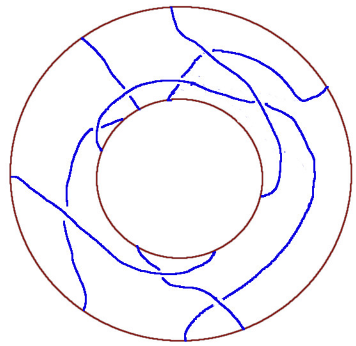

Figure 2.6: Example of a diagram of a spherical braid

the direction of time. Now if we consider the set of all motions of a fixed finite set, say of n points, there is an equivalence relation induced by ambient isotopy relative to the boundary of the strip. The set of isotopy classes clearly forms a group, which is called the braid group of the sphere. The classical Artin braid group may be described similarly, as the braid group of the plane. Refer to [**Bir**].

The braid group of S^2 is very similar [**Bir**], to the Artin braid group, which is the braid group of $\mathbb{R}^2$. Here for the sake of simplicity of notation, we will denote the $n-$string braid group of S^2 as B_n. It should not be confused with the Artin braid group. Since for the work in this thesis, the only braids we use are from the braid group of S^2, there is no chance of confusion.

Let C be an equator for S^2 as before. We may assume that the boundary points of each string lie on the boundary circles of $C \times I$. That is we are thinking of every braid as a motion of finitely many special points on C. If $p_1, p_2, ..., p_n$ are these special points on C, we number both the points $(p_i, 0)$ and $(p_i, 1)$ by i. We choose to index them in clockwise order. Also, it is helpful to think of the indices as elements of $\frac{\mathbb{Z}}{n\mathbb{Z}}$. For keeping the notation simple, we will denote the class of a number say $i + n\mathbb{Z}$, also as just i. Refer to the Figure 2.7.

We can describe the generators of B_n as follows. Consider the braid formed by a crossing between the i^{th} string and the $i + 1^{st}$ string and connecting every other special point to the other point with the same index in a trivial way. Refer to Figure 2.8. Clearly, there are two

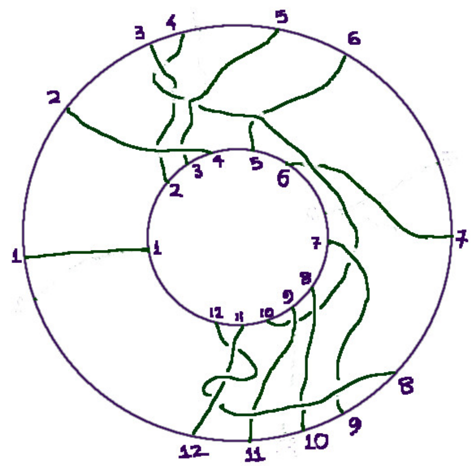

Figure 2.7: Diagram of a braid with indices.

such braids as shown in the diagram and they are inverses of each other in B_n. We denote them as σ_i and σ_i^{-1}. Notice that because of the geometry of the sphere we have a generator σ_n in the n-string braid group connecting the n^{th} point and first point. Thus in B_n, we have a generator σ_n. Clearly, we have the following presentation of B_n.

$$B_n \approx \left\langle \sigma_1, \sigma_2, ..., \sigma_n \;\middle|\; \begin{array}{c} \sigma_i\sigma_j = \sigma_j\sigma_i, i-j>2, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \\ \sigma_1\sigma_2...\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}...\sigma_2\sigma_1 = 1, \\ \sigma_n = \sigma_1^{-1}\sigma_2^{-1}...\sigma_{n-2}^{-1} = \sigma_{n-1}^{-1}\sigma_{n-2}^{-1}...\sigma_2^{-1}. \end{array} \right\rangle$$

In the above presentation, the first two relations are the "far commutativity" and "Yang-Baxter equation" respectively, similar to Artin's braid group. The third relation,

$$\sigma_1\sigma_2...\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}...\sigma_2\sigma_1 = 1$$

appearing in the presentation of B_n is purely arising from the geometry of the sphere. Refer to Figure 2.9.

The fourth relation,

$$\sigma_n = \sigma_1^{-1}\sigma_2^{-1}...\sigma_{n-2}^{-1} = \sigma_{n-1}^{-1}\sigma_{n-2}^{-1}...\sigma_2^{-1}$$

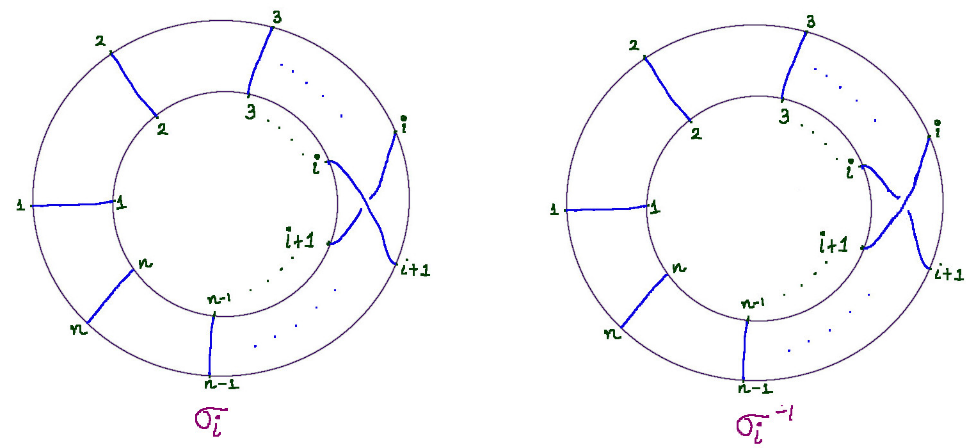

Figure 2.8: Generating braids of B_n

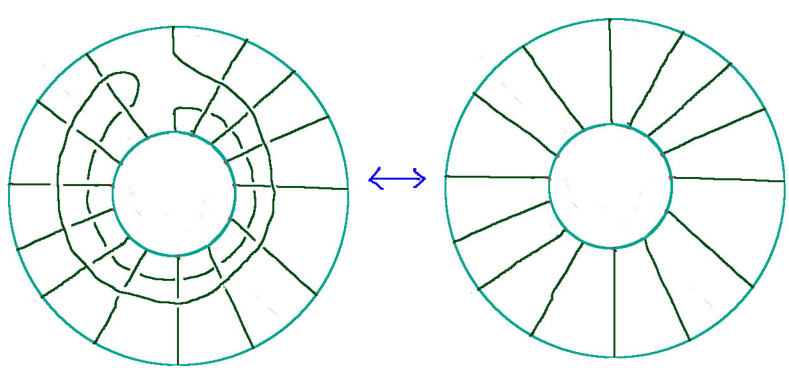

Figure 2.9: $\sigma_1\sigma_2...\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}...\sigma_2\sigma_1 = 1$

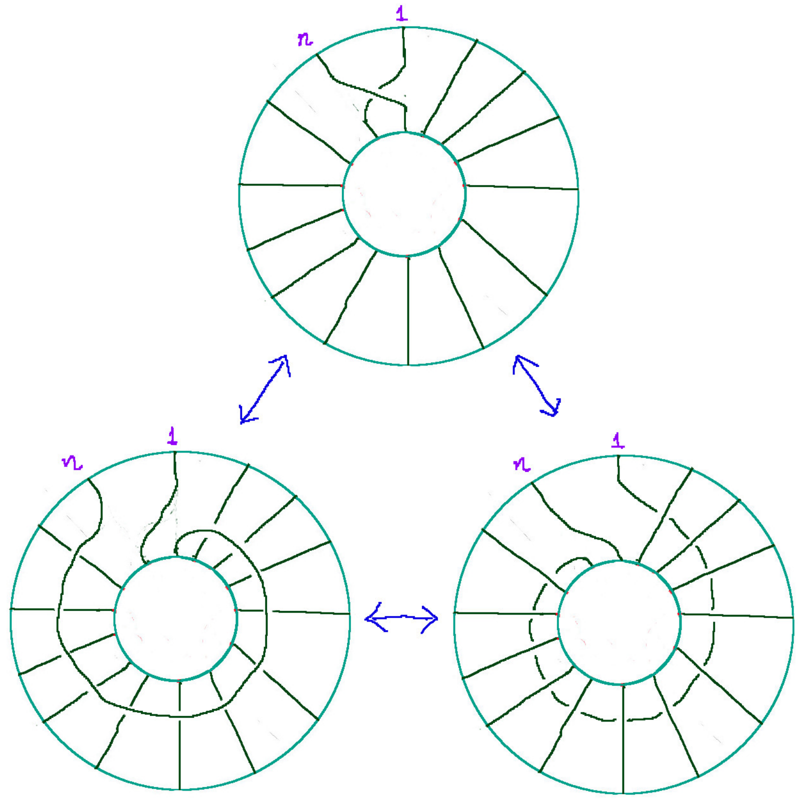

Figure 2.10: $\sigma_n = \sigma_1^{-1}\sigma_2^{-1}...\sigma_{n-2}^{-1} = \sigma_{n-1}^{-1}\sigma_{n-2}^{-1}...\sigma_2^{-1}$

says that we can express this generator as a product of the other generators. Refer to Figure 2.10.

2.3 Reidemeister type moves in $\mathbb{R}P^3$

Several invariants of classical knots were developed using the theory of diagrams. These invariants were easy to compute because of their combinatorial character. The main ingredient in the construction of such invariants are Reidemeister moves [**Rei**]. Refer to Figure 2.11. These are a set of transformations defined on diagrams so that, two links can be isotopic if and only if their diagrams can be transformed to each other using these moves. See [**Kauf**]. It is natural to ask whether such a set of moves can be defined for knot diagrams in $\mathbb{R}P^3$.

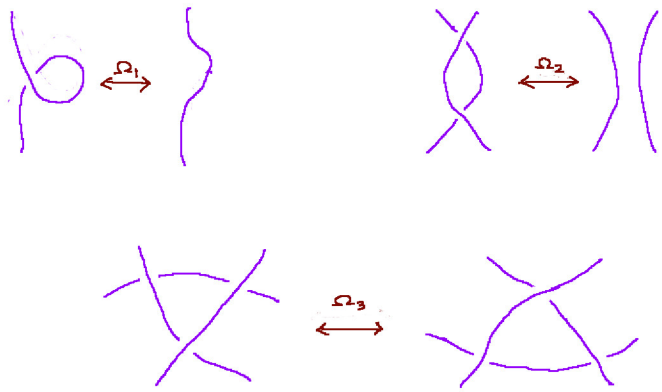

Figure 2.11: Reidemeister Moves

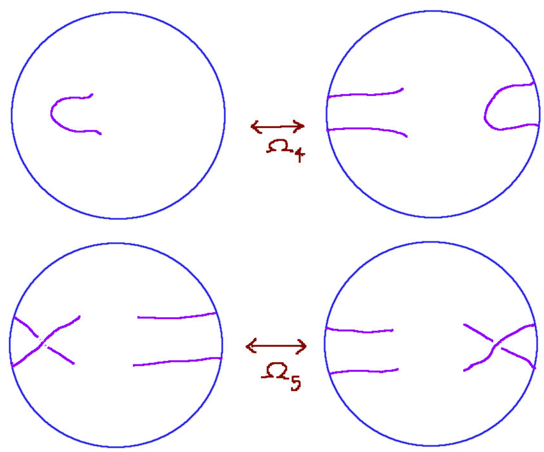

Figure 2.12: Two slide moves

In [**Drob**], Drobotukhina answered this question.

For diagrams of links in $\mathbb{R}P^3$, we need two additional moves and the classical Reidemeister moves in order to capture the whole possibilities of isotopy of links. See Figure 2.12. In [**Drob**], Drobotukhina also constructed a generalized Kauffman bracket polynomial for links in $\mathbb{R}P^3$ using these moves and classical theory of Kauffman states [**K-L**].

Chapter 3

A structure theorem for knots in $\mathbb{R}P^3$

To study knot theory in any space, one crucial aspect is to have several examples of knots. In this chapter, we introduce three important surgery procedures which can be used to produce many links in $\mathbb{R}P^3$. These are named as: Projectivization, Space bending surgery, and Class changing surgery. They are defined in Section 3.1.1, 3.1.3, and 3.1.4 respectively. The first two surgeries are performed on affine links and the last one can be performed on an arbitrary link in $\mathbb{R}P^3$. Projectivization can be seen as a special case of both Space bending surgery and Class changing surgery. In Section 3.1.3, we provide the structure theorem, which gives a complete characterization of links in $\mathbb{R}P^3$ using Space bending surgery. In Section 3.2, we introduce companionship of knots is $\mathbb{R}P^3$ and give a different characterization of affineness of a knot using companionship.

3.1 Three surgeries on links in $\mathbb{R}P^3$

3.1.1 Projectivization of affine knots

Consider a smooth 2-sphere S which bounds a ball in $\mathbb{R}P^3$. Removing a tubular neighbourhood of S produces two connected components, a $3-$ *ball* B and a mapping cylinder, C, of the two sheeted covering map from S^2 to $\mathbb{R}P^2$. Since the boundaries of B and C are naturally identified to S, for the sake of simplicity, most of the time we will refer to both ∂B

and ∂C by S. As mentioned above, if we join two knots in $\mathbb{R}P^3$ by removing an unknotted arc contained in a ball and identifying the boundary points as in the classical case, then the resulting knot is in $M := \mathbb{R}P^3 \# \mathbb{R}P^3$. Similarly, the connected sum of n knots in $\mathbb{R}P^3$ will be a knot in $\mathbb{R}P^3 \# \mathbb{R}P^3 \# ... \# \mathbb{R}P^3$ (n-times). Thus each time we add a knot, we change the ambient manifold.

The above-mentioned process seems to be the only technique to define a connected sum "operation" for all knots in $\mathbb{R}P^3$. And it comes with the hassle of dealing with knots in an arbitrary 3 manifold M and $\mathbb{R}P^3$ at the same time. But for certain special families of knots in $\mathbb{R}P^3$ we can have some operation very similar to the connected sum. Below is a procedure, that may be regarded as a connected sum of an affine knot with the projective unknot.

Consider an affine knot, K, in $\mathbb{R}P^3$ which is embedded inside a ball, B with a boundary sphere, S. Then we can push a small arc of K out of B in such a way that K now intersects S at exactly two points and the complement C of B intersects K at an unknotted arc disjoint from a projective plane in C. Then clearly this arc is a trivial cycle in $H_1(C, S)$. Now consider a standardly embedded projective unknot, J, in another copy of $\mathbb{R}P^3$. Removing a small ball, B', with a boundary S', containing an unknotted arc of J, produces a copy of the mapping cylinder, C'. And $J \cap C'$ represents a non-trivial cycle in $H_1(C', S')$. Refer to Figure 3.3. Now by gluing B and C' along their boundary spheres in such a way that $K \cap S$ gets identified with $J \cap S'$ produces a knot in $\mathbb{R}P^3$. Refer to Figure 3.1. Notice that the initial knot K is of $class - 0$ while the new one is a $class - 1$ knot. Thus this is a process of producing a $class - 1$ knot from a $class - 0$ knot, which we will call a " **projectivization**". A knot in $\mathbb{R}P^3$ which intersects a projective plane exactly at one point can be thought of as produced from an affine knot via the above surgery. Hence we wish to make the following definition.

Definition 3.1.1. A knot, K in $\mathbb{R}P^3$ is called a **projectivized affine knot**, if there is a separating sphere S intersecting K at exactly two points such that the corresponding $C \cap K$ is isotopic to a standardly embedded non-trivial cycle in $H_1(C, S)$.

Note that an affine knot can be projectivized in many ways. Projectivization is not unique and hence we say a projectivization of an affine knot.

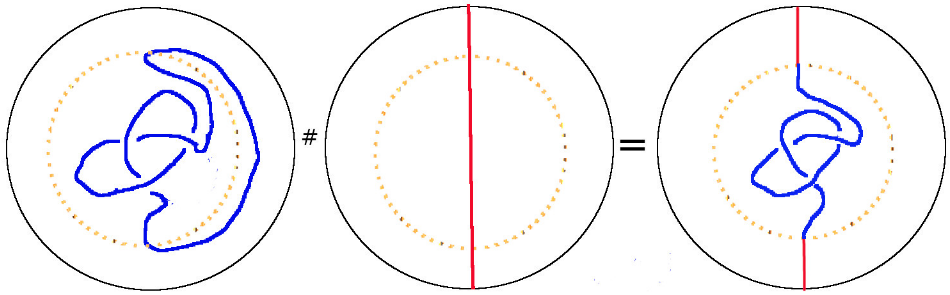

Figure 3.1: Projectivization of an affine trefoil

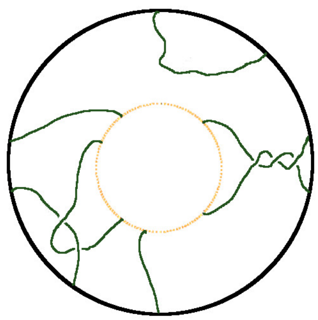

Figure 3.2: A tangle in the mapping cylinder

3.1.2 Tangles in the mapping cylinder

Any knot in $\mathbb{R}P^3$ can intersect the mapping cylinders in a collection of curves. If a knot, K, is not entirely contained in a mapping cylinder C, then $K \cap C$ is a tangle consisting of only intervals meeting the boundary sphere at exactly two points. They also can be linked within C. Refer to Figure 3.2. Thus the problem we are dealing with is the theory of tangles in the mapping cylinder. This tangle is in no way unique to the knot or to the separating sphere. In what follows we wish to study the tangles created by a knot in the mapping cylinders in $\mathbb{R}P^3$. If needed one can indeed push out a small arc from the knot outside the ball and add an extra component for the tangle. Hence it is in no way unique to the knot.

Notice that C is a 3-manifold with a boundary sphere and an embedded projective plane. Tangles, induced by knots, are composed of intervals with their boundary points on the sphere and interiors embedded in the interior of C. The arcs may or may not intersect the projective plane. The patterns of their intersections are interesting to study. Such studies can give topological information on the knot. For example, the $mod-2$ intersection number

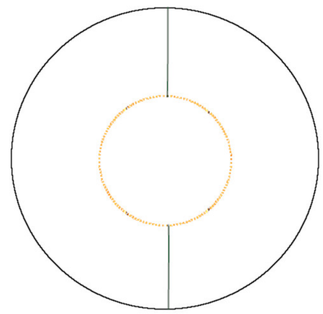

Figure 3.3: Diagram of a class-1 arc in the mapping cylinder

of the tangle and the projective plane is the homology class of the knot in the integral homology of $\mathbb{R}P^3$. Thus the $mod-2$ intersection number of the tangle induced by K in C with the projective plane is independent of the tangle and depends only on the knot.

Thus the projectivized affine knots, defined above, are knots with a simple tangle that consists of just one arc intersecting the projective plane at exactly one point.

Since the affine knots are homologically trivial, one can see that the process of "projectivization" also changes a class-0 knot into a class-1 knot. And indeed this is a way of defining connected sums of a knot with the projective unknot which has a clear variant of connected sum with the affine unknot. But the procedure described above is limited to affine knots. We would, here, like to define a general version of this process. And thereby obtain a classifying theme for all knots in $\mathbb{R}P^3$.

3.1.3 Space bending surgery

Let L^n represent the projective closure of n disjoint non-parallel lines in $\mathbb{R}^3$.

Definition 3.1.2. The tangle defined by L^n in the complement mapping cylinder of some open ball centred at the origin intersecting all lines in L^n will be called the n^{th} **residual tangle** denoted by T^n.

Notice that T^n is composed only of $class-1$ arcs in a mapping cylinder C' intersecting a projective plane at exactly one point. Each arc has two boundary points on $S' := \partial C$. Hence the boundary of T^n is a collection of $2n$ points on S'. It is easy to see that for every

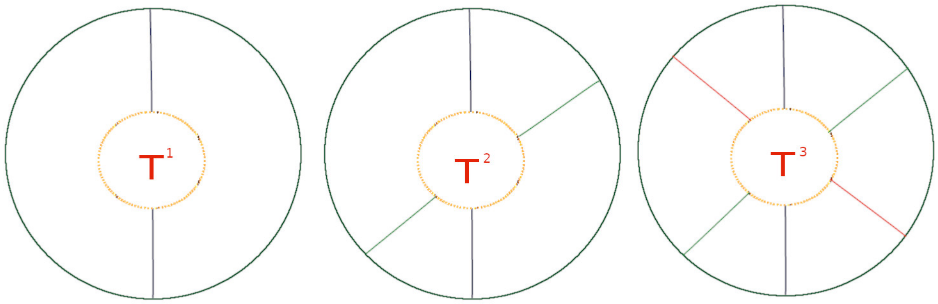

Figure 3.4: Residual tangles

n there is a unique such T^n whose boundary has $2n$ points.
Consider an affine knot K in $\mathbb{R}P^3$ with a non-empty intersection with a 3-ball B. Now the boundary of B is a separating sphere, S, and the complement of B in $\mathbb{R}P^3$ is a mapping cylinder C. Notice that $C \cap K$ is a finite collection of intervals with boundary points on S. Thus $B \cap K$ is a tangle with some even number, say $2m$, boundary points on S.
Let $f : S \to S'$ be a diffeomorphism sending the boundary of $K \cap B$ to ∂T^m. Now gluing B and C' along f produces a copy of $\mathbb{R}P^3$, with a new knot which is formed by gluing $K \cap B$ and T^m along their boundary. We wish to denote the knot obtained by the above surgery as $\Sigma(K, S, f)$. This procedure will be referred to as *space bending surgery.*

Remark 3.1.1. The philosophy behind the name is that an affine knot is basically a classical knot in $\mathbb{R}^3$ with a new identity once we add a projective plane at infinity. Thus, through this surgery, we are connecting this knot with the projective plane at infinity, and we are making it "aware" of the change in the ambient space.

If K is an affine knot and S intersects K at exactly two points then any $\Sigma(K, S, f)$ is a projectivization of K as defined above. Thus we can clearly see that projectivization for a given affine knot is not unique. It depends on the choice of the intersection of K with S and the diffeomorphism f. The following theorem says that every knot in $\mathbb{R}P^3$ is isotopic to a knot of the form $\Sigma(K, S, f)$.

Theorem 3.1.1. (The structure theorem): *Any knot in $\mathbb{R}P^3$ can be obtained from an affine knot by a space bending surgery.*

Geometric Proof: Let J be a knot in $\mathbb{R}P^3$. Choose an embedded projective plane, P,

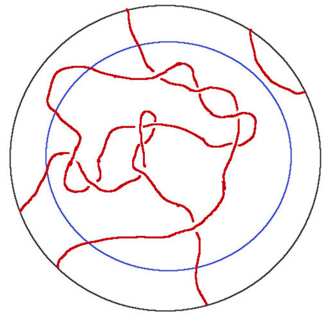

Figure 3.5: An arbitrary diagram with a separating circle.

such that J intersects P at m points transversely. Let C be a closed tubular neighbourhood for P. Then C is a mapping cylinder, which is an I-bundle over P. Taking C sufficiently thin, we may assume that $J \cap C$ is a collection of m fibres of the I-bundle. Replace the arcs in $J \cap C$ with m boundary parallel arcs in C to obtain an affine link, say K, such that J is isotopic to $\Sigma(K, \partial C, f)$ for some f. □

Diagramatic Proof: Let J be a knot in $\mathbb{R}P^3$. Consider a diagram of J on some 2-disk D^2 as in Figure 3.5. Each smoothly embedded circle in the interior of D^2 will separate the diagram into a smaller disk and an annulus. One of the boundary circles of the annulus is the boundary of the initial D^2. Since J is assumed to be in regular position, the set of double points in the diagram will be finite. For each such diagram, we can choose a separating circle σ in D^2 such that all the crossings (double points) are contained only in the disk bounded by σ. And the corresponding annulus contains only a finite set of non-intersecting intervals. There are three kinds of intervals according to the way their boundaries are placed in the annulus.

Case 1: Suppose all the intervals have their one boundary point on σ and the other on ∂D^2. Then the number of intervals is even and corresponding to an interval with one boundary point $p \in \partial D^2$ there is an interval with one boundary point at the antipodal point to p.

Case 2: Suppose both boundary points of an interval in the annulus are on σ. Since there are no double points on the annulus, we can "push" this curve inside the disk (Figure 3.6) bounded by σ and thereby remove it from the annulus.

Case 3: Suppose both the boundary points of an interval τ in the annulus is on ∂D^2. Since there are no double points on the annulus, one can enlarge τ without creating any new double point, in order to meet σ transversally at exactly two points. Refer to Figure 3.7.

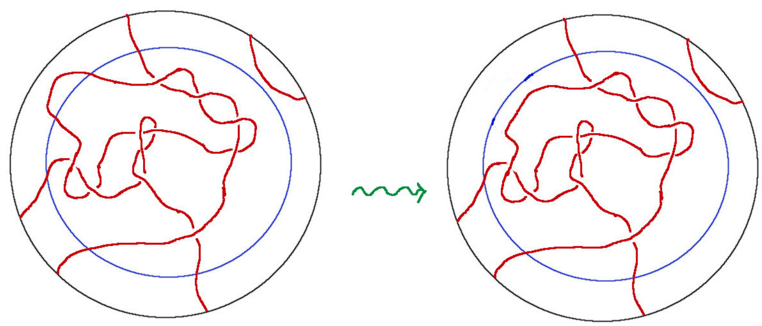

Figure 3.6: Removing a class-0 arc from the annulus.

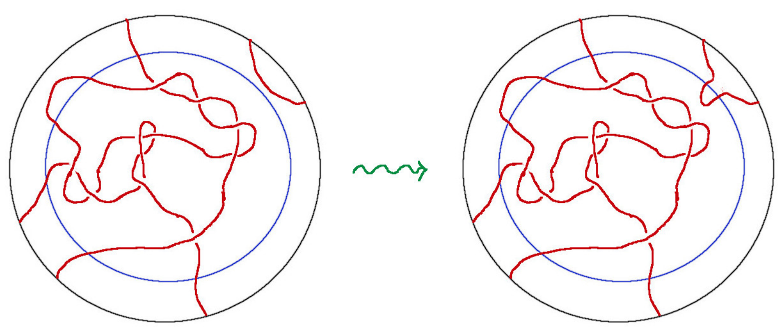

Figure 3.7: Removing a class-0 arc, by creating two class-1 arcs

This removes τ from the annulus and adds two more intervals in the annulus both sharing one of their boundary points with τ and another boundary point on σ and one interval in the disk bounded by σ connecting the boundary points of the new intervals lying on σ. Clearly corresponding to σ there is a unique standardly embedded 2-sphere S in D^3 whose intersection with the chosen D^2 is σ. By the methods suggested in Case 2 and Case 3, one can see that we can always isotopically move J in D^3 so that the diagram we obtain is always in Case 1. In the splitting induced by S, the mapping cylinder C contains a residual tangle isotopic to T^n where n is half of the number of points in the intersection of J with S. In this residual tangle, each $class - 1$ arc is represented in the diagram by two intervals with a pair of antipodally opposite points in their boundary as described in Case 1. Let the corresponding ball in the separation be denoted by B. Then $J \cap B$ is a tangle consisting of only intervals whose endpoints lie on ∂B. By connecting the endpoints by $class - 0$ arcs

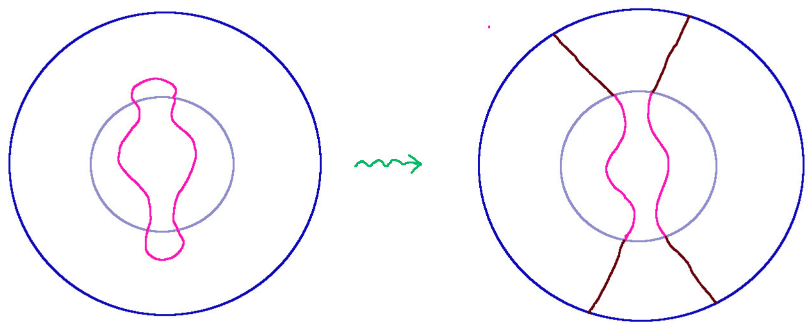

Figure 3.8: Surgery on affine unknot producing affine unknot

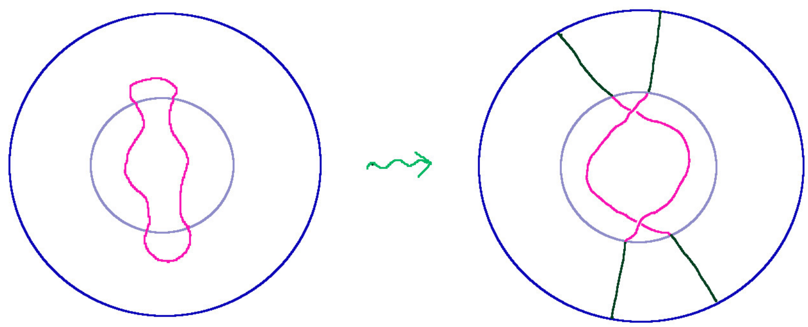

Figure 3.9: Surgery on affine unknot producing a non-affine knot

in C in such a way that the union of the new arcs with the tangle in B is connected, we can obtain an affine knot K. Cutting $\mathbb{R}P^3$ along a small enough tubular neighbourhood of S produces $B' \approx B$ and $C' \approx C$ both whose boundaries are naturally identified by a diffeomorphism $f : \partial B' \to \partial C'$ sending $K \cap \partial B'$ onto $J \cap \partial C'$. Then it is obvious that J is isotopic to $\Sigma(K, \partial B', f)$. $\square$

The following diagram shows an example that demonstrates the dependence on the f in the surgery.

The knots on the left are both obtained from the affine unknot by space bending surgery. The separating spheres are both the same. But by using two different f's we can obtain two distinct knots. The one above is again an affine unknot and the one below is a non-affine class-0 knot.

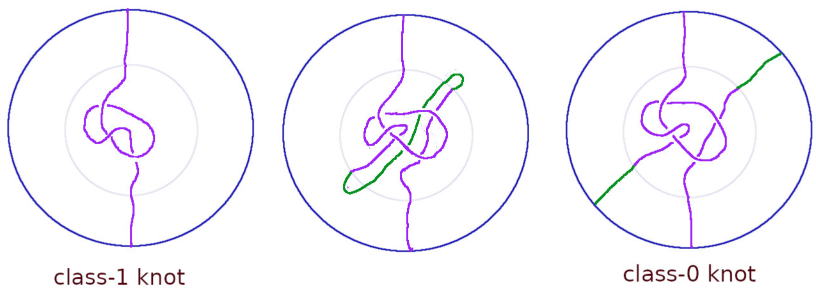

Figure 3.10: Class changing surgery on a projective trefoil producing an affine trefoil

Corollary 3.1.2. *For every knot in $\mathbb{R}P^3$, there exists a separating sphere such that in the corresponding separation, the tangle in the mapping cylinder is a residual tangle.*

Definition 3.1.3. Let K be a knot in $\mathbb{R}P^3$. A separating sphere that induces a splitting such that the part of K in the mapping cylinder is a residual tangle is said to be a "residual sphere" for K.

3.1.4 Class changing surgery

Space bending surgeries are reversible, and can be altered to produce many knots from a given knot. For example, let S be a residual sphere for a knot K. Then in the induced splitting, $\mathbb{R}P^3 := B \cup C$, the tangle in C is a residual tangle. Now by pushing an arc of K from B to C, such that the boundary points are diametrically opposite in S. Now remove this class-0 arc in C and replace it with a class-1 arc in C. Thus we obtain a new knot (Figure 3.10). This can be repeated an arbitrary number of times. This procedure is just a small variation to the space bending surgery. Each time it is performed, it changes the homology class of the knot. Thus we would refer to this procedure as **"class changing surgery"**. The following picture demonstrates such a surgery on a non-affine knot.

Notice that the class-1 knot in the example above is a projectivized affine trefoil. We may refer to it as a projective trefoil. It is easy to see that the structure theorem facilitates the surgery to be performed on arbitrary knots, rather than on just affine knots. This is because, when we are pushing out arcs of K out of a ball B, if the boundary of B is a residual sphere, then it can be done in such a way that the arc of K or the class-1 arc which is replacing it, would not be interlinked with the tangle outside.

3.2 Projective inessentiality

The following is a definition of companionship for knots in $\mathbb{R}P^3$ directly generalized from that of S^3. A subset of a solid torus is said to be geometrically essential if every meridinal disk intersects it.

Definition 3.2.1. A knot K_1 in $\mathbb{R}P^3$ is called a companion for a knot K_2 if there exists a tubular neighbourhood, V of K_1 containing K_2 such that K_2 is geometrically essential in V.

Theorem 3.2.1. *The projective unknot is a companion of every knot K, in $\mathbb{R}P^3$.*

Proof: Choose a ball model of $\mathbb{R}P^3$ presented as quotient of a 3-ball D^3. Consider a diagram of K on a closed disk D^2, as in Section 3.1.3. Choose a circle σ which separates the diagram into the diagram of a residual tangle in a closed annulus A and the diagram of the complementary tangle in the closed disk $D \setminus (A \setminus \sigma)$.

Case 1: Suppose K is a non-affine knot. Then it has a non-empty residual tangle in A. Now, as shown in Figure 3.11, choose a pair of closed half disks intersecting the boundary circle of D^2. The two closed half disks represent a closed 2-disk, say B in the quotient projective plane, $P := \pi(D^2)$. Let $\overline{N} \in \mathbb{R}P^3$ be the point on which δ is not defined, as in Section 2.1. Then $V := \pi^{-1}(B) \cup \overline{N}$, is diffeomorphic to a solid torus pinched at a point. Refer to Figure 3.12. We may choose a small enough open ball neighbourhood, say U of $\overline{N}$ such that $U \cup V$ is homeomorphic to a solid torus. It can be smoothed out to be a solid torus, V_2, in $\mathbb{R}P^3$ disjoint from K. Notice that, if x is the center of the disk B, then $L_1 := \delta^{-1}(x) \cup \overline{N}$ is a projective unknot and V_2 is a tubular neighbourhood for L_1. Thus V_2 induces a Heegaard splitting of $\mathbb{R}P^3$. The complement, $V_1 := \mathbb{R}P^3 \setminus V_2$, is an open solid torus containing K. Then, $\mathbb{R}P^3 = V_1 \bigcup V_2$ is a Dehn surgery description of $\mathbb{R}P^3$ as obtained from S^3 by a $(2, 1)$ surgery on the unknot. Notice that the core circle of V_1, say L is a projective unknot in $\mathbb{R}P^3$ and V_1 is a tubular neighbourhood of L. Now since two 1-dimensional submanifolds in V_1 cannot intersect transversely, we may assume that L is disjoint from K. Now any meridinal disk in V_1 shares its boundary with a Möbius strip in V_2 and together

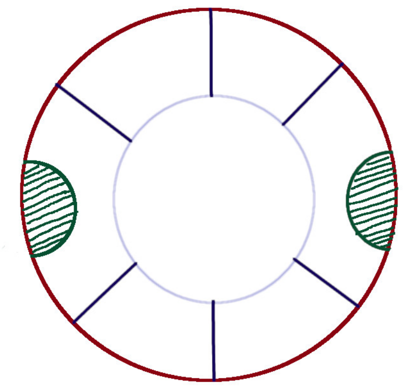

Figure 3.11: The shaded region is a 2-disk in the quotient $\mathbb{R}P^2$

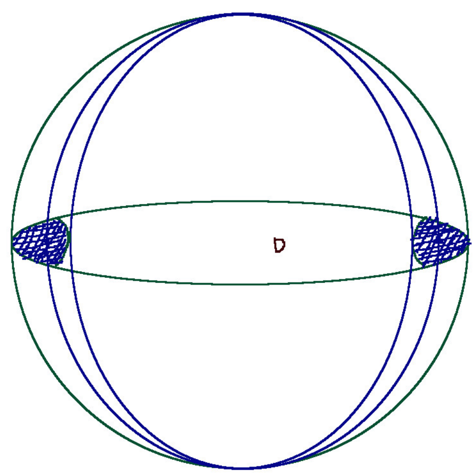

Figure 3.12: The pullback of a disk under the projection representing a solid torus pinched at a point

they form an embedded projective plane. Thus if K was disjoint from a meridinal disk of V_1 then it affine, which is a contradiction to the initial assumption. Thus L is a companion of K.

Case 2: Suppose K is an affine knot. Consider a diagram of K on a closed 2-disk $D^2 \subset D^3$ with a separating circle σ as in the previous case. Let A and B be the corresponding half open annulus and open 2-disk in splitting induced by σ, that is, $D^2 \setminus \sigma = A \sqcup B$. Since K is affine, we may assume the diagram is contained entirely inside B. There is a unique open 3-ball $B^3 \subset D^3$ such that $B^3 \cap D^2 = B$. The quotient map π is injective on B^3 and $\pi(B^3) \approx B^3$. We will refer to $\pi(B^3)$ also as B^3. Then $U := D^3 \setminus \overline{B^3} \approx S^2 \times (0,1)$ is the unique $(0,1)$-bundle over S^2 such that, $U \cap D^2 = A$. Notice that $\delta(B^3) = B$ and hence we may assume $K \subset B^3$. Let $C := \pi(\overline{U})$ which is a mapping cylinder containing the embedded projective plane $P := \pi(\partial D^3)$. Now by pulling out a boundary parallel arc of K out of B^3 into C and moving it through P back into B^3 we may create a non-empty residual tangle, T for K with two arcs in C. Then the new diagram on D^2, $\delta(T)$ is a collection of four arcs in A and $A \setminus \delta(T)$ is four disjoint regions. By isotopically moving K if required, we may assume that there is a diameter γ of D^2 disjoint from the diagram of K. Then γ intersects exactly two of these regions which are diametrically opposite. Choose an open 2-disk E in the Möbius strip $M := \pi(A)$ such that the two half disks in $\pi^{-1}(E)$ are contained in the two regions in $A \setminus \delta(T)$ which are disjoint from γ. Now choose an open solid torus V containing $\delta^{-1}(E)$ as in the previous case. Then $\overline{V}$ is clearly disjoint from K. Let x be the central point of E. The core of V, $\delta^{-1}(x) \cup \{\overline{N}\}$ is a projective unknot. Then $V' := \mathbb{R}P^3 \setminus \overline{V}$ is an open solid torus and $\mathbb{R}P^3 = \overline{V} \cup \overline{V'}$ is a Heegaard splitting. V' contains K and the projective unknot $\pi(\gamma)$. Without loss of generality, we can assume that V' is a tubular neighbourhood for $\pi(\gamma)$.

Now we will show that K is geometrically essential in V'. Let F be a meridinal disk of V' and let $\alpha := \delta(F)$. Note that α is a proper arc with its boundary points in $\partial \overline{E}$ and it intersects γ transversally at one point. Suppose K is disjoint from F. Notice that F is contained in an embedded projective plane Q disjoint from K. Let $P' := \delta^{-1}(\gamma) \cup \{\overline{N}\}$, be another embedded projective plane disjoint from K. We may assume that Q and P' intersect transversally at a projective line. Then, $X := \mathbb{R}P^3 \setminus (P' \cup Q)$ is a disconnected space with exactly two components. Also notice that, $Y := V' \cap X$ and $\delta(Y) = \pi(D^2) \setminus (E \cup \pi(\gamma) \cup \alpha)$ both have two components. Since the diagram of K intersects both components of $\delta(Y)$ non-trivially, K should intersect both components of Y non-trivially. But since K is connected

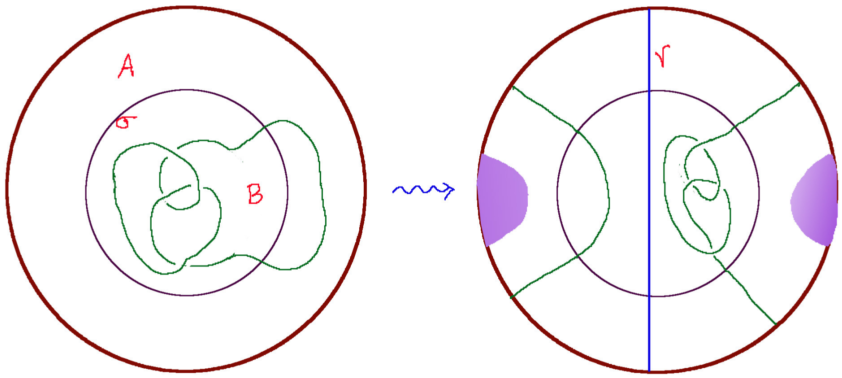

Figure 3.13: The shaded region is $\pi^{-1}(E)$

and is disjoint from both P' and Q, it should lie in exactly one connected component of X. Which is a contradiction. Thus, K intersects every meridinal disk of V'. Thus K is geometrically essential in V' and $\pi(\gamma)$ is a companion for K. □

Definition 3.2.2. A Möbius strip, M, neatly embedded in a solid torus V, such that ∂M represents the $(2,1)$ class in $H_1(\partial V, \mathbb{Z})$ will be called an **untwisted Möbius strip**.

Notice that $\mathbb{R}P^3$ is diffeomorphic with $L(2,1)$, a lens space. A tubular neighbourhood of a projective unknot is a solid torus whose complement is also a solid torus. Thus the boundary of a tubular neighbourhood of a projective unknot induces a Heegaard splitting for $\mathbb{R}P^3$. The meridian of a Heegaard torus is glued to a $(2,1)$ curve on the boundary of the other one in order to construct $L(2,1)$. The Heegaard surface for this splitting is a torus. The characteristic curve on the Heegaard surface bounds a 2-disk in one solid torus and a Möbius strip on the other. They together glue to form an $\mathbb{R}P^2$ in $L(2,1)$.

Definition 3.2.3. Let K be a knot in $\mathbb{R}P^3$. K will be called "**projectively inessential**" if it is contained in a solid torus V from a Heegaard splitting such that there exists an untwisted Möbius strip in V disjoint to K.

Theorem 3.2.2. *A knot in $\mathbb{R}P^3$ is affine if and only if it is projectively inessential.*

Proof: (If) Let K be a projectively inessential knot. By definition, there exists a solid torus V containing K whose complement $U := \mathbb{R}P^3 \setminus V \setminus \partial V$ is also a solid torus. $\mathbb{R}P^3 = V \cup U$

is a Heegaard splitting. And there is an untwisted Möbius strip M in V disjoint from K. The circle ∂M is a characteristic curve for the splitting and hence is a meridian for U which bounds a disk D in U. Then $D \cup M$ is a projective plane and it is disjoint from K. Thus K is an affine knot.

(Only if) Suppose K is an affine knot in $\mathbb{R}P^3$. Choose a projective plane P disjoint from K. There exists a ball model for $\mathbb{R}P^3$ where P is the closure of an equatorial disk, D. Choose a disk D' perpendicular to D and consider a diagram of K on the closure projective plane, P' of D'. The image of P under the projection is a projective line L, disjoint from the image of K. Choose a residual sphere for K which will intersect P' on a circle. Choose a 2-disk on P' disjoint from the diagram of K and L. Refer to Figure 3.13. Then the inverse image of this disk can be modified into a solid torus V, whose complement solid torus U contains both K and L. Clearly, V can be chosen in a way that it is disjoint from K and ∂V is a transversal to P. Clearly, $V \cap P$ is a 2-disk, hence $U \cap P$ is an untwisted Möbius strip. Thus K is projectively inessential. □

This theorem provides a purely geometric criterion for a knot to be affine. We also wish here to mention a theorem announced by J.Viro and O. Viro in [**Vir**]

Theorem 3.2.3 (J.Viro-O.Viro)**.** *A link $L \subset \mathbb{R}P^3$ is contractible if and only if $\pi_1(\mathbb{R}P^3 \setminus L)$ contains a non-trivial element of order 2.*

The notion of a "contractible link" they use exactly the same as the notion of affine links discussed above. This is a purely algebraic criterion for a knot to be affine. Thus we have many techniques to detect whether a given knot is affine.

Chapter 4

Genus for knots in $\mathbb{R}P^3$

How would one talk about the "complexity" of a knot in a given space? For classical knots, Seifert introduced [**Seif**] the notion of a genus defined using the concept of surfaces bounded by knots. These surfaces are now called "Seifert surface". It is easy to see that a knot that is the boundary of some orientable surface would be null-homologous in $\mathbb{R}P^3$. Since $H_1(\mathbb{R}P^3, \mathbb{Z}) \approx \frac{\mathbb{Z}}{2\mathbb{Z}}$, we cannot Seifert surfaces for all knots in $\mathbb{R}P^3$. In this chapter, we introduce the notion of a "**good surface**" for a knot. The philosophy we adopt here is that *the complexity of a knot should also reflect in the complexity of the surfaces on which it can be embedded*. By drawing an analogy from classical knot theory, we define a "genus" for knots in $\mathbb{R}P^3$ using good surfaces. It will be a numerical invariant for knots and links in $\mathbb{R}P^3$. We analyze many of the properties of this genus. It will turn out that, this genus behaves very similarly to the classical genus.

In what follows, a knot that is not an unknot will be called a "non-trivial" knot. Similarly, a link that is not an unlink will be called a non-trivial link. Let S be a closed connected surface in $\mathbb{R}P^3$ containing a knot K. Notice that by choosing a tubular neighbourhood of the knot and pushing it to the boundary isotopically, we can always construct a closed connected surface containing the knot. Recall that we have a canonical two-sheeted cover $h : S^3 \to \mathbb{R}P^3$. Then $\widetilde{S} := h^{-1}(S)$ is a closed surface in S^3 containing the link, $h^{-1}(K) = \widetilde{K}$. By removing any point in the complement of $\widetilde{S}$, we can obtain an embedding of $\widetilde{S}$ in $\mathbb{R}^3$. As every closed surface embedded in $\mathbb{R}^3$ is orientable, $\widetilde{S}$ is orientable. If S is contained in some

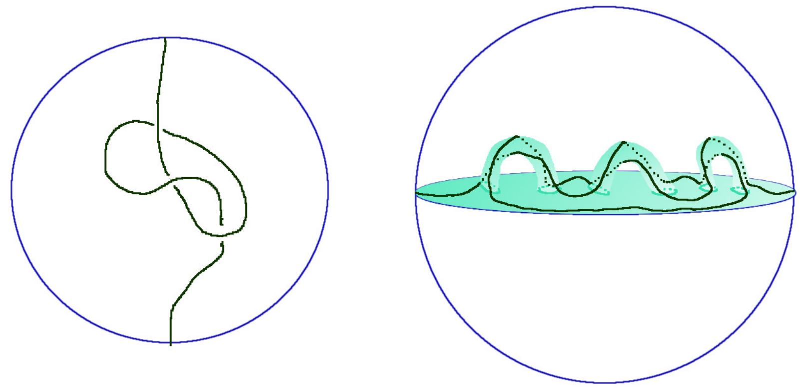

Figure 4.1: A good surface containing projective trefoil

open 3-ball in $\mathbb{R}P^3$, that is, it is contained in the affine part of $\mathbb{R}P^3$, then $\widetilde{S}$ is disconnected.

Definition 4.0.1. A closed surface S containing K in $\mathbb{R}P^3$ is said to be a "good surface for K" if the inverse image $\widetilde{S} := h^{-1}(S)$ is connected.

Notice that, if S is a good surface for K, it may or may not be orientable. As an example, consider the affine unknot, K_0. Clearly, an embedded projective plane P and a torus T intersecting P at only K_0, are both good surfaces for K_0. But T is orientable, while P is non-orientable.

Theorem 4.0.1. *Every link in* $\mathbb{R}P^3$ *has a good surface.*

Proof: Let L be a link in $\mathbb{R}P^3$. Consider a diagram of L on some embedded projective plane P. If L is an unlink this diagram can be assumed to have no crossings. The inverse image of P is a great 2-sphere in S^3 and thus P itself is a good surface for L. If L is not an unlink, then for each crossing we may attach a handle on P containing the crossing as shown in Figure 4.1. Thus, we will obtain a surface, S containing L. Since $h^{-1}(P)$ is connected, $h^{-1}(S)$ is also connected. Thus S is a good surface for L. □

Definition 4.0.2. The minimum genus of an inverse image surface varied over all the good surfaces for K will be called the "**genus** of K".

Remark 4.0.1. It is easy to see that, if K and K' are isotopic links, then the isotopy will throw any good surface for K to a good surface for K'. **Thus genus is an invariant of links under ambient isotopy**.

Theorem 4.0.2. *Every knot in* $\mathbb{R}P^3$ *has an orientable good surface* F *such that* $h^{-1}(F)$ *has a genus less than or equal to 3.*

Proof: Let K be a knot in $\mathbb{R}P^3$. Let N be a closed tubular neighbourhood of K and let $T = \partial N$. Note N is a solid torus and T is a torus. Let K' be a longitude of N, which is a knot on T homologous to K in N. Since K and K' are isotopic in $\mathbb{R}P^3$, it is sufficient to show that there is an orientable good surface F for K' such that the inverse image $h^{-1}(F)$ has genus less than or equal to 3.
If K is of class-1, then let F be the torus T. Note that K' is on T. Since $h^{-1}(T)$ is a torus in S^3 it has genus 1. Thus F is an orientable good surface for K'.
If K is of class-0, then let J be a projective unknot which is disjoint from N. Let $N(J)$ be a closed tubular neighbourhood of J such that $N(J)$ is disjoint from N. Let F be a closed orientable surface of genus 2 in $\mathbb{R}P^3$ obtained from $T \cup \partial N(J)$ by attaching a pipe. Then K' is on F and the inverse image $h^{-1}(F)$ has genus 3. □

Corollary 4.0.3. *The genus of any knot is less than or equal to 3.*

Proof: It follows from proof of Theorem 4.0.2.

Theorem 4.0.4. *A link in* $\mathbb{R}P^3$ *is an unlink if and only if it has genus 0.*

Proof: Let K be an unlink. Without loss of generality, we may assume that there exists a diagram of K with no crossings. This may be regarded as its embedding in the projective plane on which it is being projected. Since any embedded projective plane is covered by a 2-sphere in S^3, the projective plane is a good surface for K and hence K has genus 0.
Conversely, suppose a link K has genus 0. Then it has a good surface which is covered by a 2-sphere in S^3. This implies the good surface is an embedded projective plane, say G. We may consider a ball model where $\mathbb{R}P^3$ is presented as the quotient of a 3-ball D, such that G is represented by a flat 2-disk in D. Now by choosing any point $\overline{N}$ in ∂D disjoint from G, we can define a projection, $\delta : \mathbb{R}P^3 \setminus \{\overline{N}\} \to G$ as in Section 2.1. Notice that δ maps every point of G to itself and therefore K is its own image. Thus K has an image with no crossings and hence is an unlink. □

As a simple consequence of the theorem above, we derive the following "non-cancellation property" for both the surgery procedures we discussed in the previous chapter.

Corollary 4.0.5. *A space bending surgery or a class changing surgery performed on a non-trivial link will always yield a non-trivial link.*

Proof: Let K' be a link obtained by performing a space bending surgery on a non-trivial link K. Let B and C be the 3-ball and the mapping cylinder used in the surgery. If K' is an unlink, then by theorem 4.0.4, it follows that it has a good surface Σ which is an embedded projective plane. This implies, $B \cap K = B \cap K'$ is a tangle embedded in the region, $X := B \cap \Sigma$. We may assume that this B intersects Σ transversally. Since B is a 3-ball and hence X should be a finite collection of closed 2-disks contained in Σ. Thus all arcs in X are unknotted. This implies that all arcs in the tangle $B \cap K$ also were unknotted. We know that $K \cap C$ is a collection of boundary parallel arcs which are all unknotted. Hence K should be an unlink which contradicts the initial hypothesis. Thus, K' is a non-trivial link. The proof in the case of class changing surgery readily follows from similar considerations. □

Theorem 4.0.6. *All non-trivial class-*1 *knots in* $\mathbb{R}P^3$ *have genus 1.*

Proof: Let K be a class-1 knot in $\mathbb{R}P^3$. Let U be a tubular neighbourhood of K. Notice that $T := \partial \overline{U}$ is a torus. By choosing an arbitrary non-vanishing normal vector field, X on K we can push K into T along X. Thus we can obtain a parallel of K inside T. We will refer to this knot which is clearly isotopic to K also as K. Let $\widetilde{K}$ be the knot in S^3 covering K. Now $h^{-1}(U)$ is a solid torus in S^3 and its boundary is a torus covering T. Thus T is a good surface for K. And since K is non-trivial, the genus of K is 1 by Theorem 4.0.4. □

At this point, it is natural to ask the question, of whether the converse of this theorem is true. That is, do all genus-1 knots belong to class-1? Theorem 4.0.11 below, proves that it is not the case.

Definition 4.0.3. Suppose a knot K_1 is a companion to a knot K_2. If K_1 has a tubular neighbourhood N, which is a solid torus such that K_2 is contained in $\partial \overline{N}$ and is not null homologous in N. Then, we say K_2 is a "**cable knot** with companion K_1".

Theorem 4.0.7. *Suppose K is a non-trivial knot in $\mathbb{R}P^3$. Then K has genus 1 if and only if it is isotopic to a cable knot with companion J such that J is a class-1 knot.*

Proof: We first show the if part. We may assume K is lying on the boundary, T, of a closed tubular neighbourhood N of a class-1 knot J. Notice that both T and $h^{-1}(T)$ are tori since J. Thus T is a good surface for K and thus the genus of K can be at most 1. Since K is not an unknot K has genus 1 by Theorem 4.0.4.
Now we prove the only if part. Suppose K had genus 1. This means a torus in S^3 under h is a two-sheeted cover of a minimal good surface of K. Thus the good surface should be a torus or a Klein bottle. Since the Klein bottle does not embed into $\mathbb{R}P^3$, see [**Bre**], it should be a torus. Then we may choose a torus T containing K in $\mathbb{R}P^3$ such that $T' := h^{-1}(T)$ in S^3 is also a torus. Clearly, T' bounds a solid torus N' in S^3. Then $N := h(N')$ is a solid torus whose boundary is T. Let J be the core of N which is covered by the core of N' under h. Then clearly J is of class-1. Suppose K was null homologous in N. Then, it bounds a 2-disk in N which contradicts that K is not an unknot. Thus K is not null homologous in N. Clearly K is a cable knot with companion J. □

Now we will study the properties of genus for affine knots. Notice that any affine knot has two lifts in S^3 which are unlinked. Consider a closed orientable surface, S, contained in an affine region of $\mathbb{R}P^3$. Let P be an embedded projective plane disjoint from it. By removing an open 2-disk from both S and P, and connecting them by a tube, we can always obtain a surface in $\mathbb{R}P^3$ which is homeomorphic to the connected sum, $P\#S$. The inverse image of this surface would be a great sphere (the inverse image of P) attached with two copies of S with tubes, one "inside" and one "outside". This is a connected surface. Thus for any knot K contained in a surface S in an affine region, the connected sum of S with a projective plane is always a good surface for K.

Lemma 4.0.8. *Every non-trivial affine knot has a genus strictly greater than* 1.

Proof: Let K be a non-trivial affine knot. Then K cannot have genus 0 by Theorem 4.0.4. Now suppose K had genus 1. Then by Theorem 4.0.7, there exists a class-1 knot J such that K is a cable knot with companion J. Let U be a tubular neighbourhood of J such that K is embedded in $\partial\overline{U}$ and K is not null homologous in U. Now since K is affine, there exists an embedded projective plane, P disjoint from K. But, since J is class-1, J has to intersect P transversally at least at one point, say x. We may assume that $U \cap P$ has at least one meridinal disk of the solid torus U containing x, say D. Notice that since K is disjoint from P, K does not intersect D. This implies K will be null homologous in U since it is

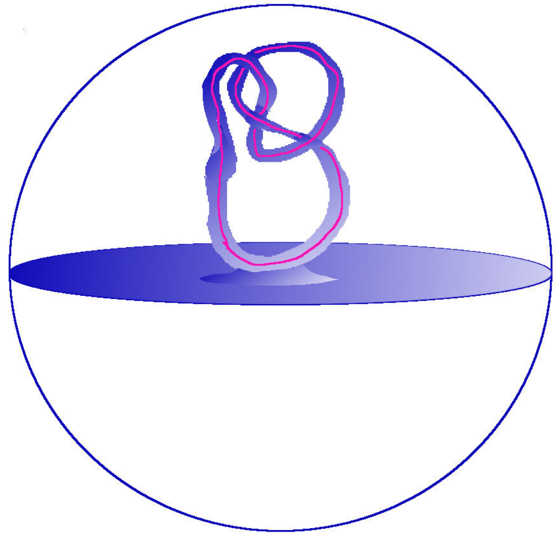

Figure 4.2: A minimal good surface for affine trefoil

contractible in U. This contradicts the assumption that K is a cable knot with companion J. Hence, K cannot have genus 1. Thus the genus of K has to be strictly greater than 1. $\square$

Theorem 4.0.9. *The genus of any non-trivial affine knot in* $\mathbb{R}P^3$ *is* 2.

Proof: Let K be an affine knot, contained in the affine part of $\mathbb{R}P^3$. Consider a torus (possibly knotted) containing it in the affine part. This may be obtained by choosing a tubular neighbourhood for K and pushing K to its boundary torus. We can connect this torus with a projective plane by a pipe. See Figure 4.2. Thus we can get a good surface, say G for K in $\mathbb{R}P^3$. G is covered in S^3 by a genus 2 surface. This implies that the genus of any affine knot is less than or equal to 2. Then by Lemma 4.0.8, the genus of K is exactly 2. $\square$

Corollary 4.0.10. *All knots of genus* 1 *are non-affine.*

Remark 4.0.2. Thus, genus detects affineness of knots.

Definition 4.0.4. We will call a torus T^2 in $\mathbb{R}P^3$ to be class-1 if it is the boundary of a tubular neighbourhood of a class-1 knot. In particular, the boundary of the tubular neighbourhood of a projective unknot will be called a standard class-1 torus.

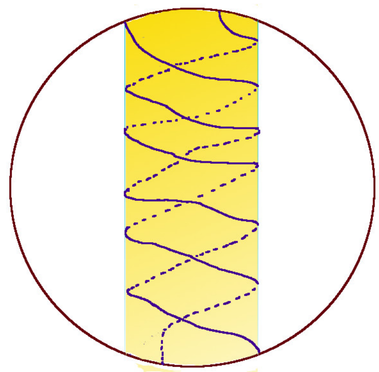

Figure 4.3: A class-0 non-affine knot of genus 1 on a class-1 torus

A class-1 torus will be covered by a single torus in S^3. Thus for a knot embedded in a class-1 torus, it is a good surface. Since every knot of genus 0 is an unknot, a non-trivial knot embedded on a class-1 torus has genus 1.

Theorem 4.0.11. *There are infinitely many class-*0 *non-affine knots with genus* 1.

Proof: The first homology of the torus, $H_1(T^2, \mathbb{Z}) \approx \mathbb{Z} \oplus \mathbb{Z}$. Every knot on a torus T^2 will represent a homology class of the form (p, q) where p and q are coprime. Choose a ball model of $\mathbb{R}P^3$ as the quotient of a 3-ball D^3. Consider the knot shown in Figure 4.3 as a diagram drawn on a closed 2-disk D^2. Let d be a diameter of D^2, then $\pi(d)$ is a projective unknot. Choose a tubular neighbourhood U of $\pi(d)$. Without loss of generality, we may assume, the image of U under δ is a region in D^2 which is bounded by two parallel chords of the circle ∂D^2. See the shaded region in Figure 4.3. Clearly $T := \partial\overline{U}$ is a standard class-1 torus intersecting the embedded projective plane $P := \pi(\partial D^3)$ at exactly one meridinal circle, say m, of the solid torus $\overline{U}$. Let K be a (p, q) knot in T which is the quotient of a torus braid embedded in the cylinder $\pi^{-1}(T)$ (Figure 4.3 gives an example of a braid whose quotient is the $(2, 5)$ knot in T). All points in $K \cap P$ are contained in m. We can choose K in such a way that there are exactly p points in $K \cap P$. Thus K is class-0 whenever p is even. Notice that K represent the class p in $H_1(\overline{U}, \mathbb{Z}) \approx \mathbb{Z}$. Hence if $p \neq 0$, then it is a cable knot with the projective unknot as the companion. Hence by Theorem 4.0.7, it has genus 1 and Corollary 4.0.10 implies that it should be non-affine. By changing different values of p and q we can construct infinitely many such knots. Hence there exist infinitely many class-0 non-affine knots. □

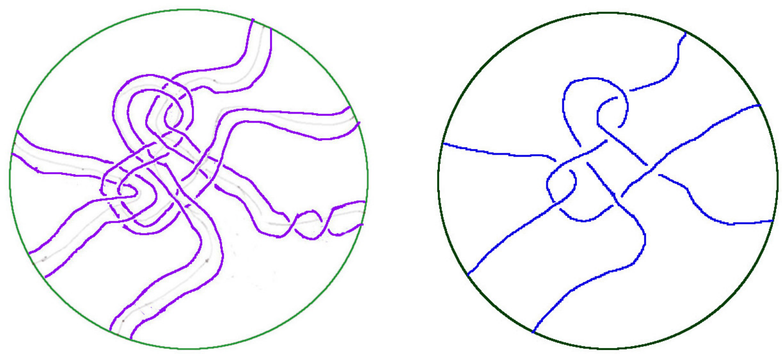

Figure 4.4: A class-1 knot (right) and one of its class-0 cable knots (left)

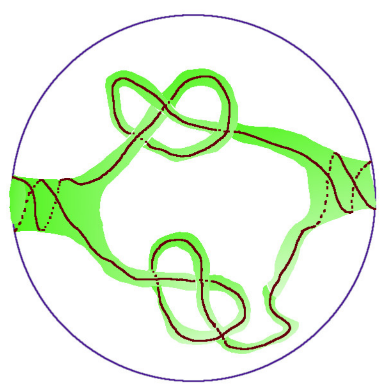

Figure 4.5: A non-affine knot on a good surface

Thus the family of knots with genus 1 contains all class-1 knots and several class-0 non-affine knots. The family of knots with genus 2 has all the affine knots. It is not clear whether there can be a class-0 non-affine knot in this family. Any knot with genus 3 is a class-0 non-affine knot. But it is not known to the authors whether there exist knots with genus 3.

The knot shown on the right of Figure 4.4 is of class-1. It is obvious that the one on the left of Figure 4.4 is of class-0 and is a cable knot with the one on the right as its companion. Thus by Theorem 4.0.7, the one on the left has genus 1. Then by Corollary 4.0.10, it is a class-0 non-affine knot.

Conjecture 4.0.12. *There exist class-*0 *non-affine knots, with genus not equal to* 1.

As a support for the conjecture, in Figure 4.5, we present a diagram of a knot, which we suspect to be of genus 3. The surface on which the knot is lying is good and its inverse image is a genus-3 surface in S^3.

Chapter 5

A braid theory for links in $\mathbb{R}P^3$

In this chapter, we use the concept of **spherical braids** to define a braid theory for links in $\mathbb{R}P^3$. In Section 5.1 we introduce the concept of the projective plat closure of an even stranded spherical braid in $\mathbb{R}P^3$ and prove the main theorem (Theorem 5.1.2), which states that every link in $\mathbb{R}P^3$ can be constructed by closing a spherical braid in this way. In Theorem 5.1.3, we state a criterion for affineness of a knot using this braid representation. In Section 5.2 we introduce the notion of "residual permutation" and prove the equivalence of the number of cycles in the permutation of the braid and the number of components in the plat closure. We also provide a few examples. In Section 5.3, we propose a set of moves on spherical braids, which we hope will work to provide an analogue of the Markov theorem for projective plat closures.

5.1 Projective plat closure of spherical braids in $\mathbb{R}P^3$.

Joan Birman introduce the concept of plat closures in $\mathbb{R}^3$, (see [**Bir**]). This is a different way to close a braid into a link, than the classical closure defined by J.W. Alexander, in [**Alex**]. Birman proved that every classical link is isotopic to the plat closure of an Artin braid. We introduce the idea of plat closures in $\mathbb{R}P^3$ and prove that every link in $\mathbb{R}P^3$ is isotopic to the plat closure of a spherical braid (Theorem 5.1.2). A key ingredient in this is the structure theorem from Chapter 3 (Theorem 3.1.1). For this purpose, we need to discuss some more geometric concepts related to $\mathbb{R}P^3$ as follows.

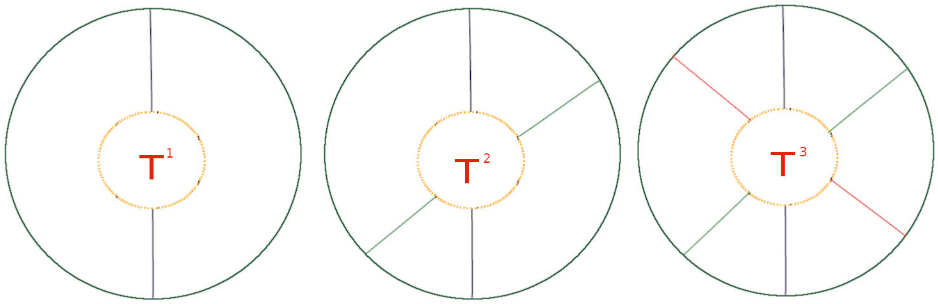

Figure 5.1: Residual tangles

Let B be the 3-ball. We know that, $\partial(S^2 \times I) \approx S^2 \amalg S^2$. Notice that S^3 can be obtained by gluing boundaries of $B \amalg B$ and $S^2 \times I$. Classically the plats in S^3 were constructed [**Bir**] by considering a spherical braid in this strip and certain simple tangles in both the balls. The tangles used there coincide with "internal tangles" that we define below. We wish to discuss a generalization of this construction. Let M denote the mapping cylinder of the canonical two-sheeted covering map $S^2 \to \mathbb{R}P^2$. Notice that, $\partial M \approx S^2$. By gluing the boundaries of $M \amalg B$ and $S^2 \times I$ we can obtain a copy of $\mathbb{R}P^3$. We choose the convention that $S^2 \times \{0\}$ is identified with ∂M and $S^2 \times \{1\}$ is identified with ∂B. When we say "braids in $\mathbb{R}P^3$", we mean the braids in $S^2 \times I$ region in some splitting of $\mathbb{R}P^3$ of this type. Now by considering a braid in $S^2 \times I$ and gluing its boundary with some tangles in B and M we can form a collection of linked knotted curves in $\mathbb{R}P^3$.

Now for this purpose, we need some natural choices for tangles in B and M. Already we studied a family of natural tangles in M, which are called **residual tangles**. Refer to Section 3.1.3. for the definition. Recall that the residual tangle with exactly n components is denoted by T^n. See Figure 5.1 for some examples. Choose an equator C for ∂B and let D be the flat disk in B with boundary C. Let A^n represent the tangle in B formed by n unknotted unlinked arcs neatly embedded in D. We will call them "internal tangles". See Figure 5.2. for some examples.

Notice that boundaries of both T^n and A^n are composed of $2n$ points. Now consider gluing the boundaries of M with a residual n-tangle and B with an internal $n-$tangle, using a diffeomorphism, $f : \partial M \to \partial B$, which sends $2n$ points on the boundary of T^n to the $2n$ points on the boundary of A^n. By identifying ∂M and ∂B with S^2, we can find an isotopy $H : S^2 \times I \to S^2$, of f to the identity map of S^2. By representing the image of ∂T^n under

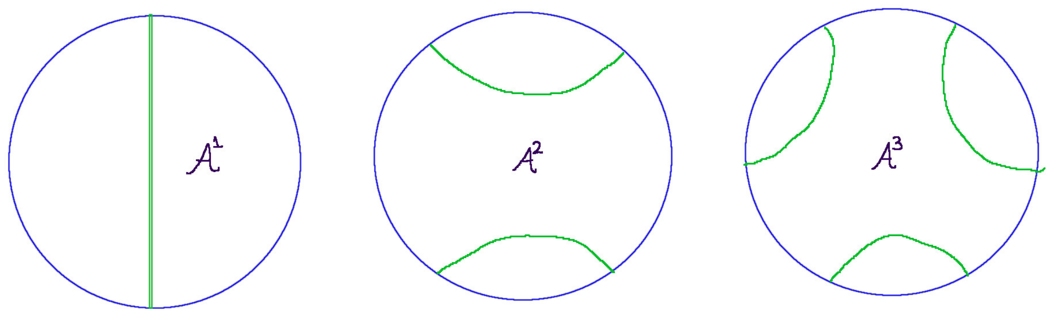

Figure 5.2: Internal tangles

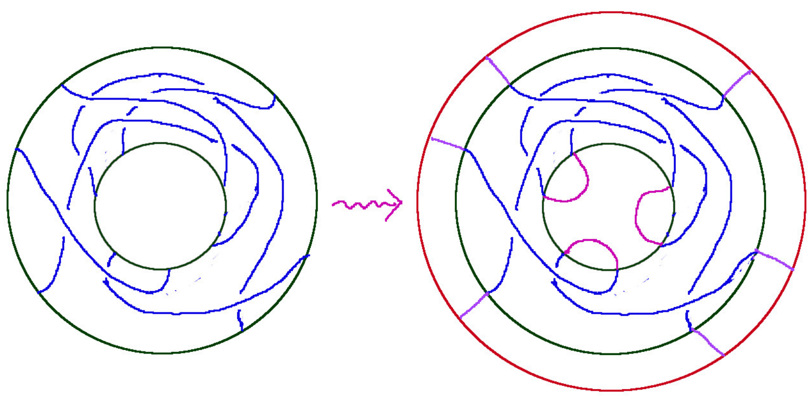

Figure 5.3: Example of closing a typical braid

each of the maps, $h_t(x) := H(x,t)$ on the sphere $S^2 \times \{t\}$ in $S^2 \times I$, we can obtain a braid in $S^2 \times I$. Arcs in the internal tangle and residual tangle will join the boundary points of the braid. Thus we get a link in $\mathbb{R}P^3$. We will refer to this "closure" of braids as **projective plat closure**. For simplicity we will refer it by just plat closure. See Figure 5.3 for a demonstration.

Many examples of projective plat closures of standard braids such as identity braids, generating braids, torus braids, and weaving braids are provided in the Appendix.

Remark 5.1.1. Notice that plat closures can be defined only for braids with an even number of strings.

Remark 5.1.2. Let $1, 2, ..., 2n$ represent the indices of points on the boundary of a spherical braid as defined above. We choose the convention that the internal tangles connect the points 1 to 2, 3 to 4,...and $2n - 1$ to $2n$.

The following lemma will be used to prove the main theorem of this chapter.

Lemma 5.1.1. *Given any link K in $\mathbb{R}P^3$, there exists a separating sphere, which will split $\mathbb{R}P^3$ into two pieces a ball B and a mapping cylinder M such that $K \cap M$ is a residual tangle.*

Proof: From the structure theorem (Theorem 3.1.1) we know that K is isotopic to some $\Sigma(J, S', f)$. Where J is an affine knot and S' is a 2-sphere which separates $\mathbb{R}P^3$ into a ball B' and a mapping cylinder M'. Then, $\Sigma(J, S, f) \cap M'$ is a residual tangle. Now the ambient isotopy which turns $\Sigma(J, S', f)$ into K will also turn S' into a 2-sphere S. Then it is obvious that in the splitting induced by S, the intersection of the mapping cylinder M and K is a residual tangle. $\square$

Theorem 5.1.2. *Every link in $\mathbb{R}P^3$ is isotopic to the projective plat closure of some spherical braid.*

Proof of the theorem: Let K be a link in $\mathbb{R}P^3$. Let $S \subset \mathbb{R}P^3$ be the separating sphere provided by Lemma 5.1.1. Let B and M denote the ball and the mapping cylinder in the corresponding splitting respectively. The part of K inside M is already a residual tangle, say T^n. Let $B' \subset B$ be a smaller closed ball with the same center. Refer to Figure 5.4. Notice that the region outside B' in B homeomorphic to $S^2 \times I$ with $S^2 \times 0$ mapped to $\partial B'$ and $S^2 \times 1$ mapped to ∂B. We will refer to this region as "**strip**" in what follows.
We can move the link isotopically so that all the crossings appear in the projection of the strip in the diagram.
Now the tangle inside B' is just a collection of untwisted unlinked arcs all of whose boundaries are on $\partial B'$. The tangle inside the strip now has many arcs all of whose boundaries lie on ∂B and $\partial B'$. The arcs may also be knotted. Refer to Figure 5.5. There are three types of arcs in the strip based on where their boundary points are placed. Both the boundary points of an arc may be on $\partial B'$. We will call them "type 1" arcs. The arcs with both boundary points on ∂B will be called "type 2" arcs. The arcs with one boundary point on ∂B and another on $\partial B'$ will be called "type 3" arcs.
The projection,

$$f : S^2 \times I \to I$$

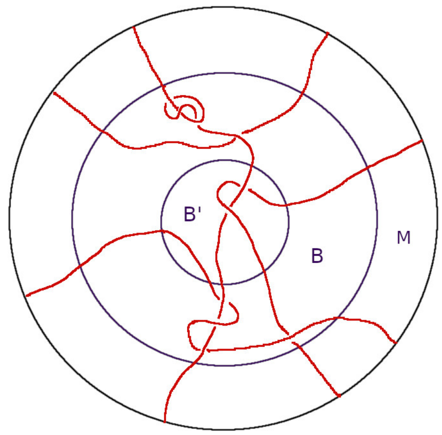

Figure 5.4: A residual tangle in M and a generic tangle inside B

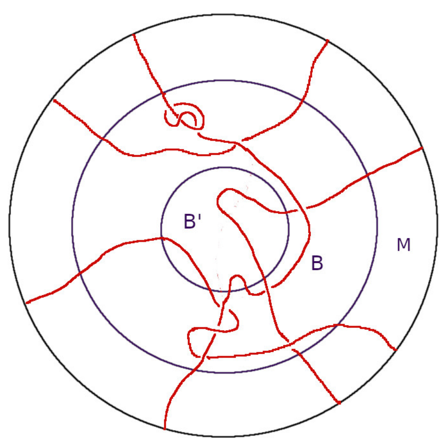

Figure 5.5: After pushing all the crossings to the strip.

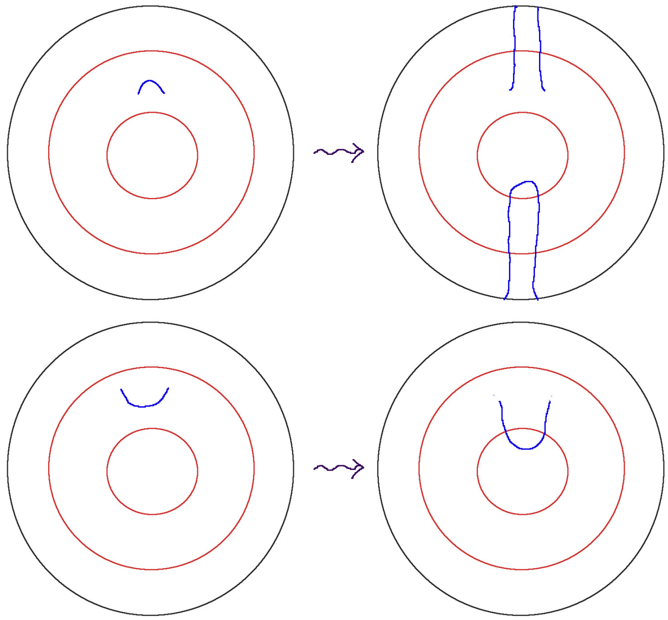

Figure 5.6: Transfering the extremum points to B'.

on the strip can be restricted to the arcs. We will call this restriction, the "height function" on the tangle. We shall denote this function also by f. Note that if f has any point of inflection on an arc, the arc can be isotopically moved in order to remove the inflection point and make f monotonic locally. Hence, in what follows we will always assume that f has no points of inflection on the tangle and extrema will mean either maxima or minima. Clearly, on a type 1 arc, there exists at least one maximum point for f. Similarly, type 2 arcs have to have at least one minimum point. If any of these arcs are knotted, they will contain more extrema points of f. Type 3 arcs that are not knotted may be isotopically moved so that f is monotonic on the modified arc. Now all the extrema points may be removed from the strip by moving them inside B'. Refer to Figure 5.6. It is easy to see that these operations can be done by ambient isotopies of $\mathbb{R}P^3$. Once all the extremum points are removed, f will be monotonic on all the arcs in the strip.

Let γ be an equator for $\partial B'$. It is easy to see that, we can isotopically move the link so that the tangle in B' is an internal tangle with all the boundary points on γ. That is, all the boundary points on γ are connected to their immediate neighbour. Refer to Figure 5.7.

Now it is easy to see that the tangle inside the strip is a braid. The residual tangle in M and the internal tangle in B' are "closing" this braid into a projective plat closure. $\square$

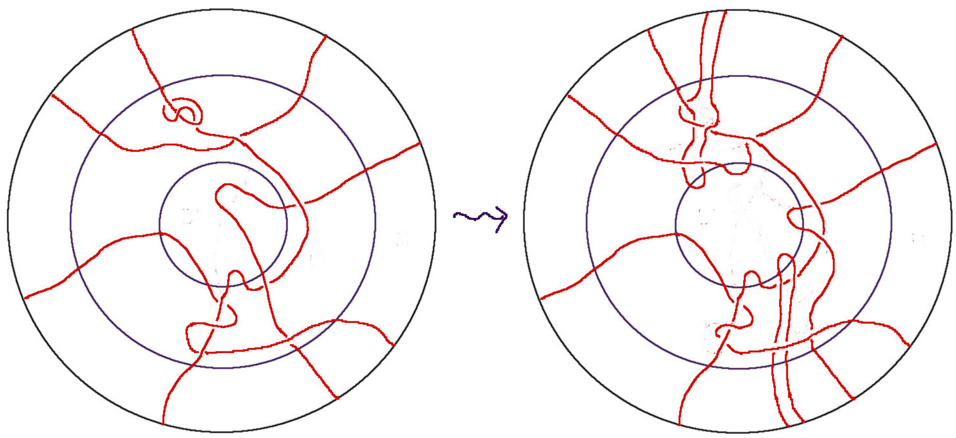

Figure 5.7: A plat closure diagram from an arbitrary diagram.

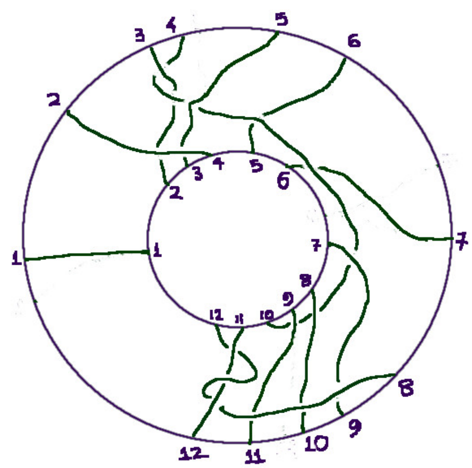

Figure 5.8: The braid appearing in the proof of Theorem 5.1.2.

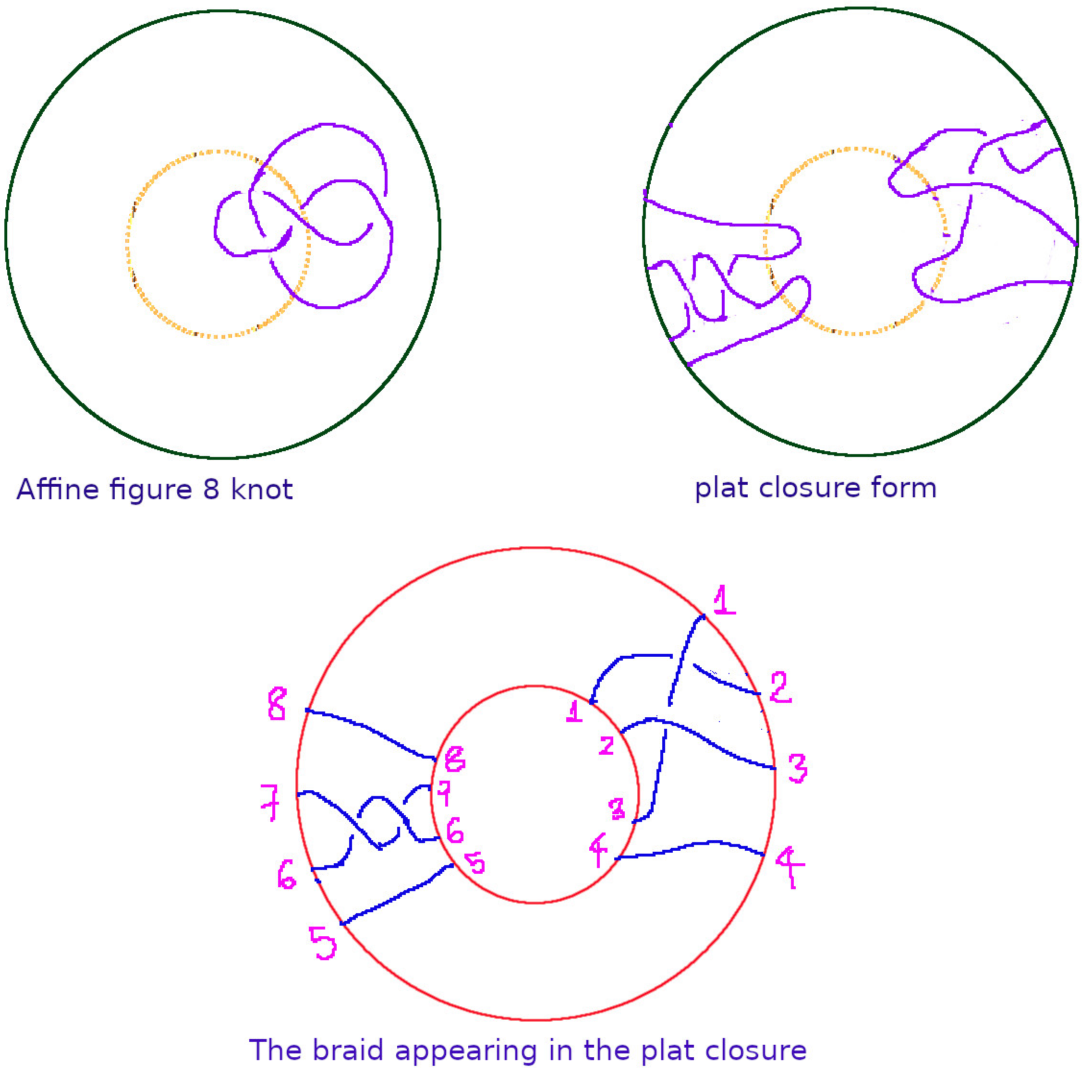

Figure 5.9: The braid in B_8 is $\sigma_1\sigma_2^{-1}\sigma_6^{-2}$
The word does not contain σ_4 and σ_8.

Given a braid, a natural question one would like to ask is about the nature of the link formed by closing the braid. Like its homological properties, affineness, and so on. It is obvious that if the projective plat closure of a $2n$ string braid is a knot, then it will be in homology class $n\,(mod\,2)$ in $H_1(\mathbb{R}P^3)$. The following theorem studies the conditions for the link to be affine.

Theorem 5.1.3. *Let L be link in $\mathbb{R}P^3$. Then L is affine if and only if L is isotopic to the plat closure of a $k\,(k = 2n)$ braid $\beta = \sigma_{i_1}\sigma_{i_2}\cdots\sigma_{i_l}$ such that for some even integer j, $j, j+n \notin \{i_1, i_2, \ldots, i_l\}$.*

Proof: Suppose j is such an even class. Then in a diagram of the closure link, drawn on a disk, we may draw a diametrical line on the diagram which passes through the region of the strip between the points j and $j+1$. The line does not intersect the link. And hence, by pulling it back under the projection, we can construct a projective plane disjoint from the link. Thus the link is affine, by Remark 2.1.3.

Now we will prove the converse. If L is affine, choose a projection, with a disjoint diameter d for the disk. There are four points in $d \cap C \times \{0,1\}$, two on each of the circles. Notice that the techniques used in the proof of Theorem 5.1.2, can be performed by moving the diagrams without intersecting d and we may obtain a diagram of a braid β in the strip. In the strip, we can choose indexing for the boundary points of β so that, when we move clockwise from an intersection point on $C \times \{0\}$, the first boundary point of β we reach is indexed as 1. Notice that the choice of this intersection point is arbitrary, but any of the two points in $d \cap C \times \{0\}$ will work. Then, the boundary point we see first in the anti-clockwise direction on $C \times \{0\}$ is indexed as $2n$, an even class. This indexing induces an indexing on $C \times \{1\}$. Then choose, $j = 2n$, then it will satisfy the condition above. □

5.2 Residual permutations

There are certain natural questions one would like to ask about the plat representations of links in projective space. For example, by looking at the braid, can we predict the number of components of the closure link? We try to answer this question here.

The indexing on the boundary points of a braid, $\beta \in B_k$, gives a bijection,

$$f_\beta : \frac{\mathbb{Z}}{k\mathbb{Z}} \to \frac{\mathbb{Z}}{k\mathbb{Z}},$$

which we choose to be the one sending the indices of points on $C \times \{0\}$ to the indices of points on $C \times \{1\}$. This is also the projective analogue of the permutation assosiated to a classical braid. We also consider another permutaion,

$$g : \frac{\mathbb{Z}}{k\mathbb{Z}} \to \frac{\mathbb{Z}}{k\mathbb{Z}}$$

defined as follows,

$$\begin{aligned} g(i) &= i+1, \text{ if i is an odd class,} \\ &= i-1, \text{ if i is an even class.} \end{aligned}$$

Since $k = 2n$ is even, this is a well defined permutation of $\frac{\mathbb{Z}}{k\mathbb{Z}}$. Let G denote the group $\frac{\mathbb{Z}}{k\mathbb{Z}}$. Notice that n is an order 2 element in G and let H denote the subgroup generated by n. Then we have,

$$\frac{G}{H} \approx \frac{\mathbb{Z}}{n\mathbb{Z}}.$$

For brevity of notation, we will denote the point p_i on both $C \times \{0\}$ and $C \times \{1\}$ by simply i. We may assume that, the points on both $C \times \{0\}$ and $C \times \{1\}$ are arranged symmetrically like numbers on a clock. Then the points i and $i+n$ are diametrically opposite. Thus they belong to the same coset in $\frac{G}{H}$. We will denote by $[i]$ the coset where the element i belongs. Consider the permutation $f_\beta^{-1} g f_\beta$. Notice that this induces a permutation,

$$h_\beta : \frac{G}{H} \to \frac{G}{H}.$$

We call this the **residual permutation** of β.

Theorem 5.2.1. *The number of components in the plat closure link of a braid β is equal to the number of disjoint cycles in its residual permutation.*

Proof: Notice that, the point $f_\beta^{-1} g f_\beta(i)$ is connected to the point i by the arc formed by the string of β connecting i to $f_\beta(i)$ followed by the string in the internal tangle connecting $f_\beta(i)$ to $g(f_\beta(i))$ and then by the string of β connecting $g(f_\beta(i))$ to $f_\beta^{-1}(g(f_\beta(i)))$. Also notice that each coset in $\frac{G}{H}$ has two points on $C \times \{0\}$ which are connected by one string in the residual

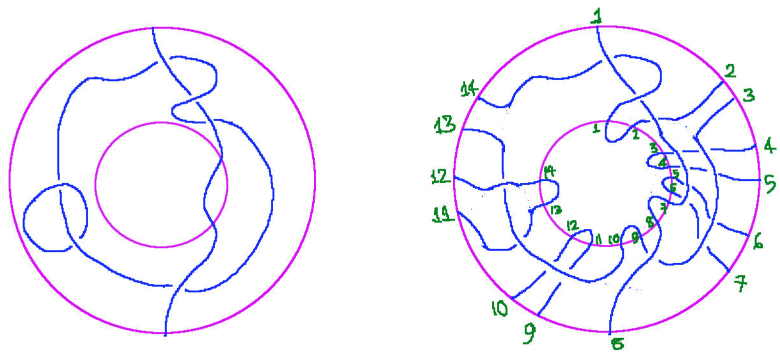

Figure 5.10: A three-component link and its residual permutation: (1)(2 7 6 3)(4 5)

tangle.

Now suppose $([i_1]\ [i_2]\ ...\ [i_l])$ is a disjoint cycle in the cycle decomposition of h_β. Choose any element j_1 in the coset $[i_1]$. Then j_1 and $j_2 := f_\beta^{-1}(g(f_\beta(j_1)))$ are connected by an arc as described above. Note that j_2 is an element of $[i_2]$ by definition of h_β. We can follow this arc from j_2 through the string of residual tangle to the point $j_2' := j_2 + n \in [i_2]$. If $[i_2] = [i_1]$, in which case h_β fixes the $[i_1]$, then this $j_2' = j_1$ and we have a closed loop. Thus the cycle above was $([i_1])$. Otherwise, we may start again from the point by the above method to the point $j_3 := f_\beta^{-1}(g(f_\beta(j_2')))$. By following the string of the residual tangle, we can reach the point $j_3' := j_3 + n$. Then, if $j_3' = j_1$, we have a closed loop and the cycle was a transposition $([i_1]\ [i_2])$. Similarly by following this procedure we can see that when we start from the point j_1 and reach the end of the cycle at the point j_l', we obtain a knot in the closure link. It is easy to see that if we had chosen to begin at the other element $j_1 + n \in [i_1]$ then we would be moving through the same knot, but in the opposite direction. Also if we had represented the cycle with a different ordering of points, then we will again be on the same knot, but starting and ending at a different point.

Thus every cycle appearing in the cycle decomposition of h_β corresponds to a knot in the plat closure of β. That is, disjoint cycles in the residual permutation of β and the knots in the plat closure of β are in one to one correspondence. $\square$

Some more examples can be found in Appendix 2.

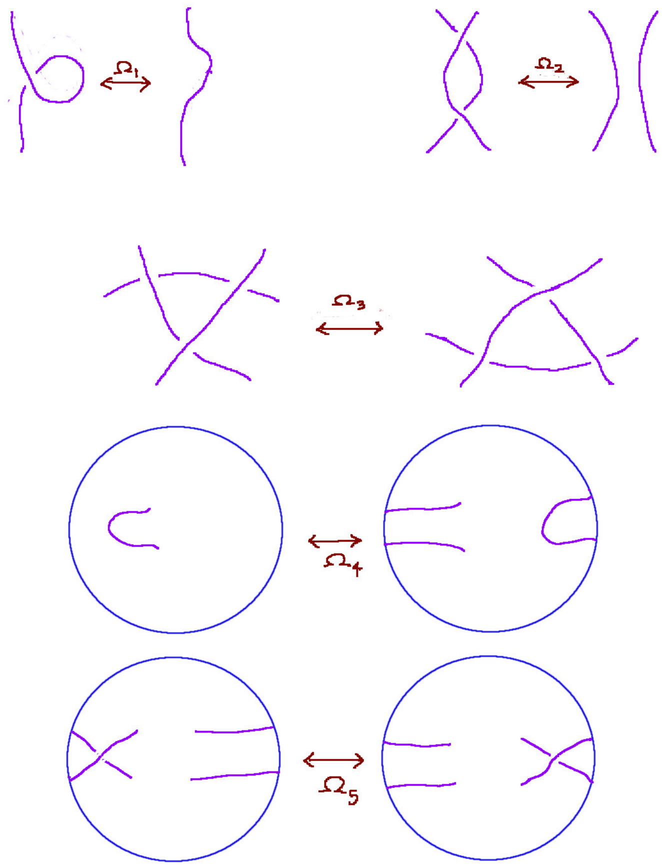

Figure 5.11: Reidemeister type moves on projective diagrams

5.3 Moves on braids

As usual in the classical case, the same link may be represented by plat closures of multiple braids. We will now explore the equivalence relation between the braids which have isotopic closures.

The diagrammatic moves in $\mathbb{R}P^3$, as described in [**Drob**] are shown in Figure 5.11.

Remark 5.3.1 (Drobotukhina, Yulia V., [**Drob**])**.** Two links in $\mathbb{R}P^3$ are isotopic if and only if their diagrams can be transformed from one to the other by a finite sequence of the moves $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and Ω_5.

That is, as sets, the set of equivalence classes of diagrams induced by the above-described moves is in one-to-one correspondence with the set of isotopy classes of links.
Suppose $\beta := \sigma_{i_1}\sigma_{i_2}...\sigma_{i_k}$ is a braid. Notice that the move Ω_2 applied on β is equivalent to deleting a pair of the form $\sigma_{i_l}\sigma_{i_l}^{-1}$ from the word representing β. And performing a Ω_3 move on β is equivalent replacing a sub-word of the form $\sigma_i\sigma_{i+1}\sigma_i$ with a word $\sigma_{i+1}\sigma_i\sigma_{i+1}$ for an arbitrary i. Now we know this is already a relation in the braid group B_k. Clearly the inverse of both these processes can also be described in a similar manner. That is, the moves Ω_2 and Ω_3 are already incorporated in the group structure of the braid group.

Consider the following moves on braids. Suppose $k = 2n$ is even. Let $\beta = \sigma_{i_1}\sigma_{i_2}...\sigma_{i_m}$ be a braid in B_k. Then there will be exactly n odd classes, i.e, $(2l+1) + k\mathbb{Z}$ in $\frac{\mathbb{Z}}{k\mathbb{Z}}$.

Let M_0^i be a move as follows,

$$M_0^i : \beta\sigma_{i-1}\sigma_i \longleftrightarrow \beta\sigma_{i-1}^{-1}\sigma_i^{-1}, \text{if } i \text{ is odd.}$$
$$\overline{M_0^i} : \beta\sigma_{i-1}^{-1}\sigma_i^{-1} \longleftrightarrow \beta\sigma_{i-1}\sigma_i, \text{if } i \text{ is even.}$$

Refer to Figure 5.12. It is easy to see that the plat closures of two braids which are related by a move of this type are equivalent by two Ω_2-moves.

For an odd i, let M_1^i be the move,

$$M_1^i : \beta \longleftrightarrow \beta\sigma_i,$$
$$\overline{M_1^i} : \beta \longleftrightarrow \beta\sigma_i^{-1}.$$

Notice that the internal tangle for a plat has strings connecting every odd class i with $i+1$. Then the diagrams of plat closures of two braids which are related by an M_1^i move are related by an Ω_1 move. Thus the braids have isotopic plat closures. Refer to Figure 5.13.

Now suppose $\beta = \sigma_l\beta'$ is a braid in B_k where $k = 2n$ as above. Then we have the

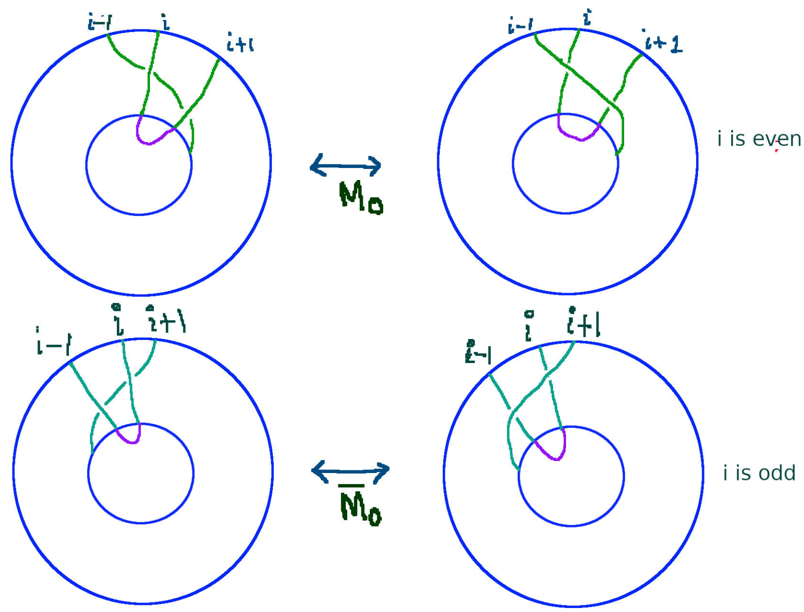

Figure 5.12: M_0^i moves

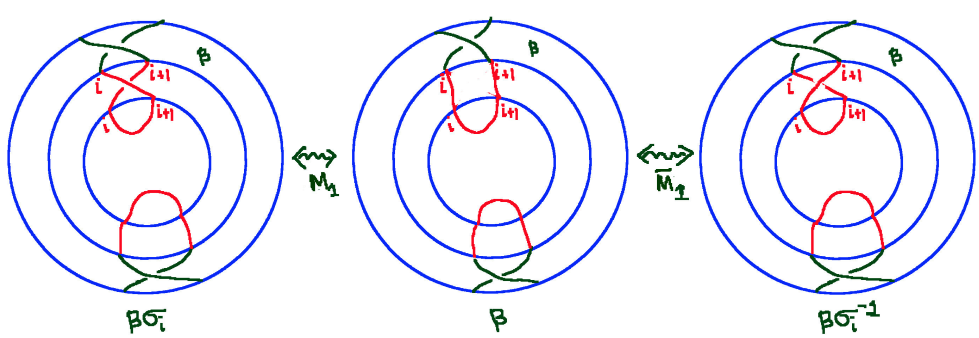

Figure 5.13: A typical M_1-move

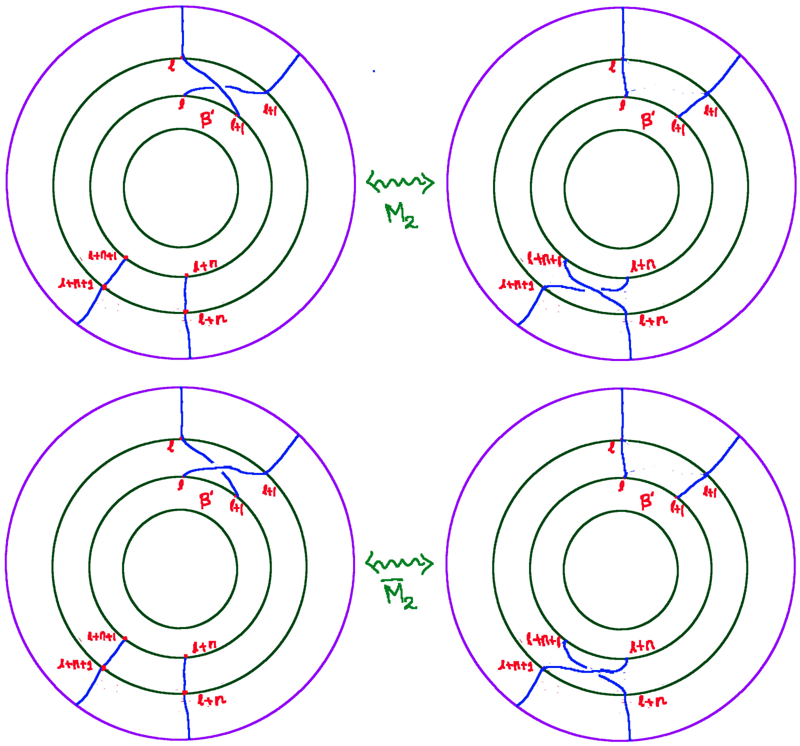

Figure 5.14: M_2-moves

following move on the B_k.

$$M_2 : \sigma_l \beta' \longleftrightarrow \beta' \sigma_{l+n}$$
$$\overline{M_2} : \sigma_l^{-1} \beta' \longleftrightarrow \beta' \sigma_{l+n}^{-1}$$

Clearly, the plat closures of two braids related by an of M_2 or an $\overline{M_2}$ moves, are isotopic since their diagrams are equivalent through an Ω_5 move. See Figure 5.14 for a demonstration of M_2 move.

Suppose we think of the braid group B_{k+4} as the group of motion of the points $p_1, p_2, ..., p_n$, p_{n+1}, p_{n+2}, $-p_1$, $-p_2$, ..., $-p_n$, $-p_{n+1}$, $-p_{n+2}$ on C where each p_i is next to p_{i+1} for $i < n+2$. Let γ be a braid formed by a motion where the points $p_1, p_2, -p_1, -p_2$ were still. Then each of the strings formed by these four points in γ is not braided with any other string in γ. If we remove these four strings from γ it would not disturb the motion of the other k points which will form a k braid. Thus we may think of each of the $k+4$ string braids formed

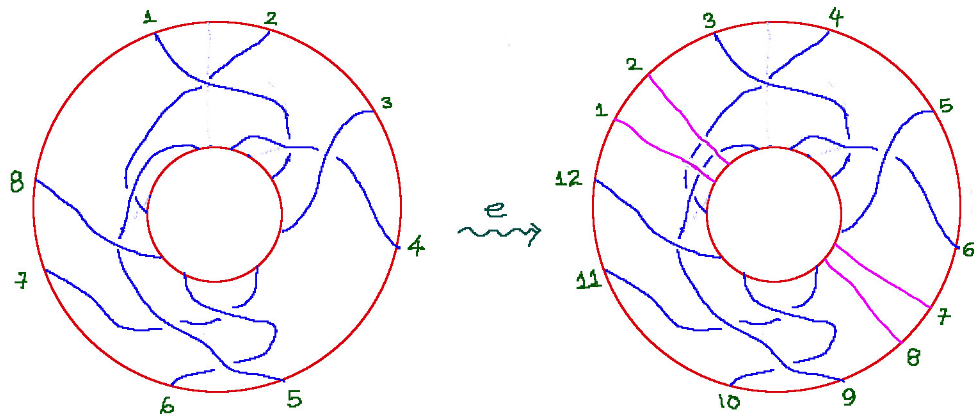

Figure 5.15: A typical case of applying the map $e : B_8 \hookrightarrow B_{12}$

by a motion where the points $p_1, p_2, -p_1, -p_2$ are still as obtained from a k string braid by introducing four still points on C. Thus we have a natural map,

$$e : B_k \hookrightarrow B_{k+4}$$

induced by introducing four points on the equator C of the type x, y, $-x$, $-y$ such that no points of B_k are lying between x and y. Refer to Figure 5.15. Suppose the braids in B_k were described as the motion of $q_1, q_2, ..., q_n, -q_1, -q_2, ..., -q_n$. The image of each k-braid under e is a $k+4$ braid with the four more strings $\{p_1\} \times I$, $\{p_2\} \times I$, $\{-p_1\} \times I$ and $\{p_1\} \times I$ in the strip. We choose to rename the points in the strip in such a way that, x, y, $-x$, and $-y$ will be relabeled as p_1, p_2, $-p_1$ and $-p_2$. Then the rest of the points will be relabeled according to the ordering. That is $q_1, q_2, ..., q_n$ will be labeled as $p_3, p_4, ...p_{n+2}$ and $-q_1, -q_2, ..., -q_n$ as $-p_3, -p_4, ..., -p_{n+2}$ respectively.

For any $3 \leq l \leq k+2$, define

$$\alpha_l := \sigma_2\sigma_3\sigma_4...\sigma_{l-2}\sigma_{l-1}\sigma_{l-2}^{-1}\sigma_{l-3}^{-1}...\sigma_2^{-1},$$
$$\overline{\alpha_l} := \sigma_2^{-1}\sigma_3^{-1}\sigma_4^{-1}...\sigma_{l-2}^{-1}\sigma_{l-1}^{-1}\sigma_{l-2}\sigma_{l-3}...\sigma_2.$$

certain family of braids in B_{k+4}.

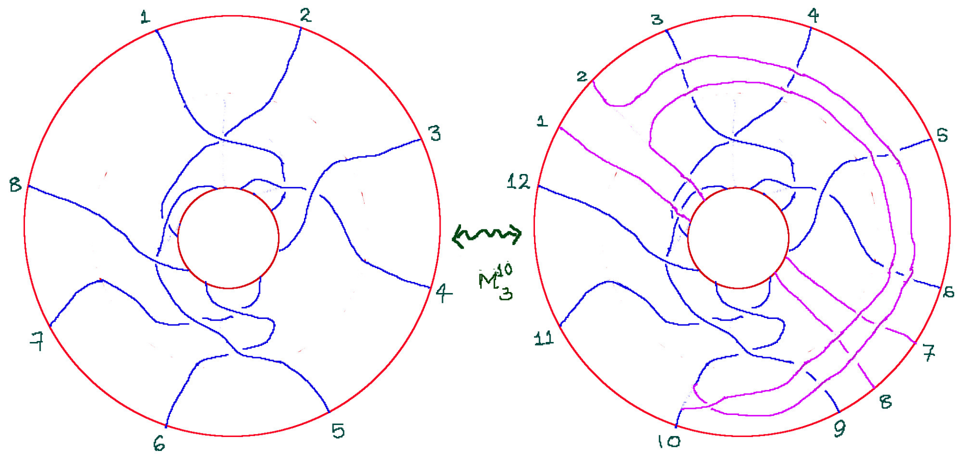

Figure 5.16: M_3^{10}-move performed on a braid in B_8 resulting in B_{12}

Then we define,

$$M_3^l : \beta \longleftrightarrow \alpha_l e(\beta),$$
$$\overline{M_3}^l : \beta \longleftrightarrow \overline{\alpha_l} e(\beta).$$

Which is a move relating braids of B_k and B_{k+4}. Refer to Figure 5.16. Each instance of performing the above move and obtaining a braid β' from a $\beta \in B_k$, the diagram of their plat closure of β change by an Ω_4 move after an Ω_1 move. Thus clearly, the plat closures of braids that are related by these moves are isotopic.

There is another move that also changes the index of the braid. Suppose one string in a braid is isotopically moved to form a pair of maxima and minima of the projection, $f : S^2 \times I \to I$. Notice that then it is no longer monotonic on this string. We may bring the extrema points into the ball region by isotopically moving them. Refer to Figure 5.17 and 5.18 where it is demonstrated for the cases when half of the braid index is odd or even. We refer to these M_4 and $\overline{M_4}$ moves. When n is odd, the move is just an application of the map, $e : B_k \to B_{k+4}$ defined above. For an even n, $\overline{M_4}$ clealy defines a map from B_k to B_{k+4}. But if β is a k-braid, the form of the braid $M_4(\beta)$ depends on the form of the braid β and writing a closed expression may not be possible, just like the map e mentioned above.

Notice that all the moves described above with a name of the type M_i always come with

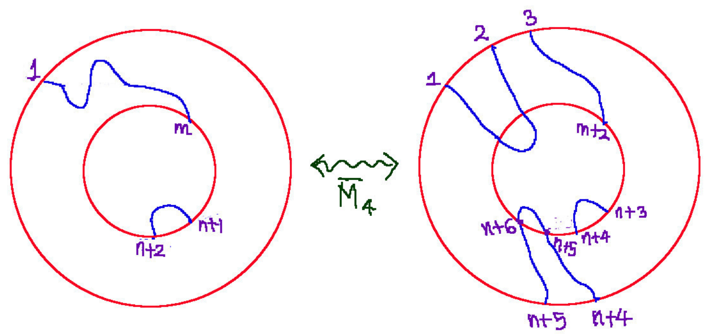

Figure 5.17: When n is even

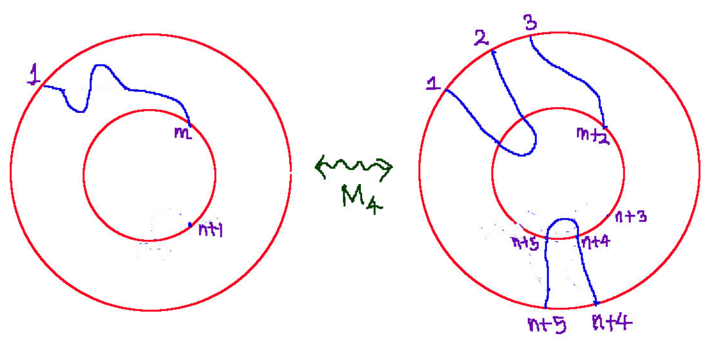

Figure 5.18: When n is odd

a pair $\overline{M_i}$. In each case, it is obvious to see that each one is just another version of the other. In what follows we will drop the overlines and refer to the moves as just M_i since all that we are saying applies to both with some obvious modifications. We would refer to the set of all the operations as just, M-moves.

Definition 5.3.1. If a braid β can be turned into another braid β' by a finite sequence of M-moves, we will say β and β' are M**-equivalent**.

Proposition 5.3.1. *If β and β' are M-equivalent spherical braids then they have isotopic projective plat closures.*

Proof: It is obvious from the definition of each M-move that the diagrams of the plat closures before and after performing them are equivalent under Ω-moves. Hence if β and β' are M-equivalent then clearly their projective plat closures are isotopic. $\square$

Conjecture 5.3.2. *The projective plat closures of two braids are isotopic only if they are M-equivalent.*

Refer to Figure 5.19 for an example in support of this conjecture.

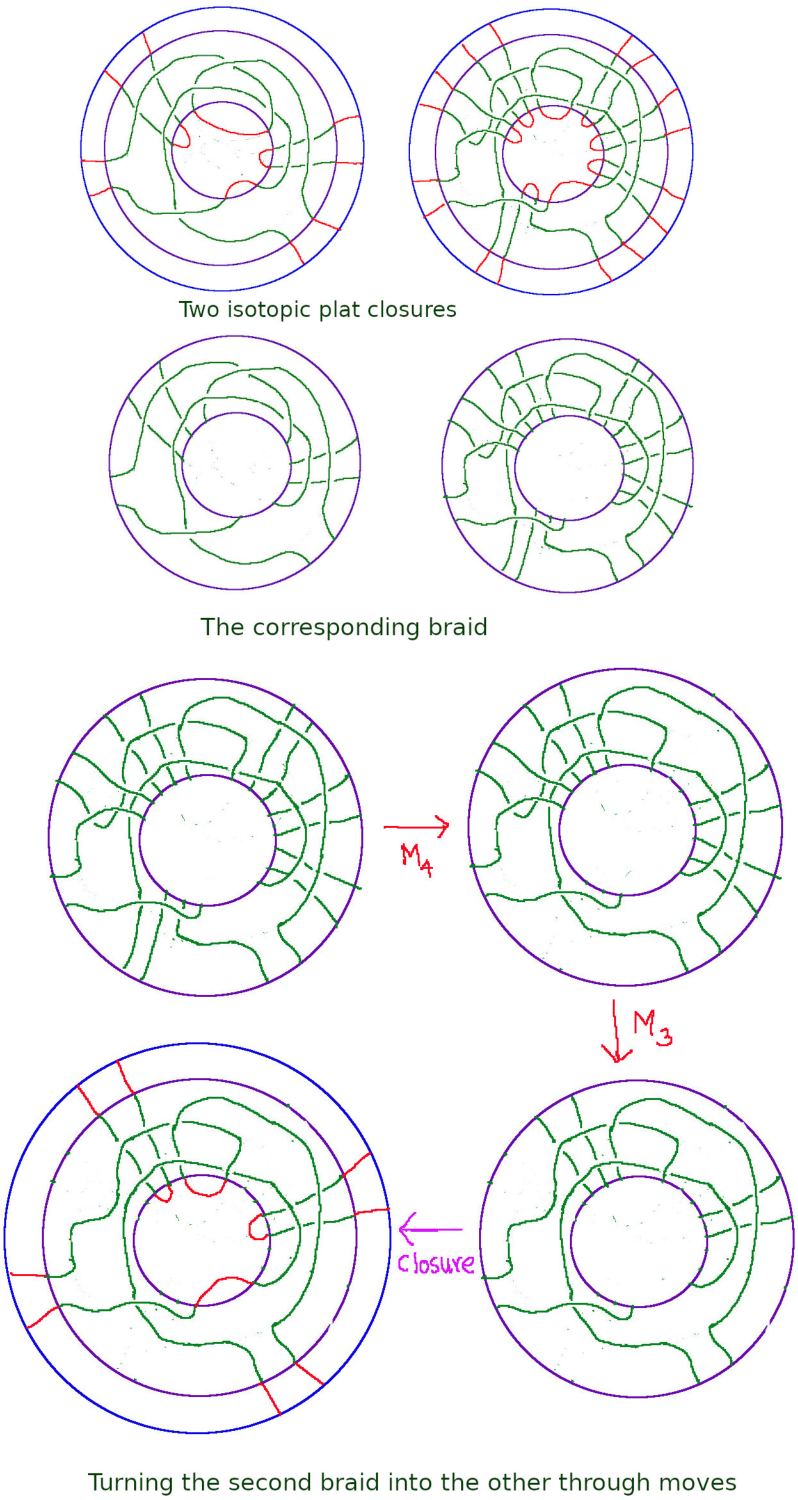

Figure 5.19: An example of applying M-moves

Chapter 6

Concluding remarks

This thesis is a study of several properties of knots and links in the projective 3-space. These properties are mostly geometric in nature. Some of these are analogues of the properties of classical links. But through this study, we understand that some of the properties of these knots are unique and are arising from the geometry of $\mathbb{R}P^3$.

Primarily, classical knot theory was developed by representing knots in two distinct ways. Which are, knot diagrams and closure of braids. We have developed both these approaches for knots is $\mathbb{R}P^3$. We first discuss the procedure for constructing diagrams for knots in $\mathbb{R}P^3$. This was developed in [**Drob**]. Our exposition is about having a special presentation of a knot diagram connecting a classical knot to a projective plane.

We initially introduce a surgery procedure and prove a structure theorem (Theorem 3.1.1) for knots in $\mathbb{R}P^3$ using it. This theorem can be proved with both, a diagrammatic and a geometric approach See Section 3.1.3. In short, this theorem shows that all the "knotting" of any knot in $\mathbb{R}P^3$ can exist only in an affine region and each knot has a simple structure near a projective plane at infinity.

Then in Chapter 4 we develop an invariant called genus for knots and links in $\mathbb{R}P^3$ and study several properties of this invariant. As observed in the analysis of genus, non-affine class-0 knots are a truly different type than affine and class-1 knots. It would be interesting to work on Conjecture 4.0.12 about the existence of non-affine class-0 knots of genus other than 1. Another interesting question to ask is about the *existence of links with genus higher than three.*

Then we go on to describe a braid theory for links in $\mathbb{R}P^3$ in Chapter 5. The braids appearing here are spherical braids. An exciting application of this work is in trying to *construct invariants of knots and links in* $\mathbb{R}P^3$ *arising from representations of braid groups.* Inspired by the classical case, the idea will be to first prove the Conjecture 5.3.2 and look for a normalized "trace" of some appropriate representation [**Jon**] of the spherical braid group, which is invariant under all M-moves.

The braid theory we developed here is arising naturally from the splitting of $\mathbb{R}P^3$ as a 3-ball and a mapping cylinder. The mapping cylinders under consideration carry residual tangles and the 3-balls carry internal tangles. See Section 5.1. Consider M_1 and M_2 to be three-manifolds both with one connected boundary surface, Σ. Suppose we also had a set of arcs embedded in both M_1 and M_2 such that the boundary points of each arc lies on the respective boundaries. Now if we glue M_1 and M_2 along $\partial M_2 \approx \Sigma \approx \partial M_1$ using a self diffeomorphism of f of Σ, we can obtain a link in the manifold $M_1 \sqcup_f M_2$. If f is isotopic to 1_Σ, then we can plot this isotopy in a regular neighbourhood of Σ inside $M_1 \sqcup_f M_2$ which is diffeomorphic to $\Sigma \times I$. This is a braid. Any such diffeomorphism, thus produces braids in $M_1 \sqcup_f M_2$.

Thus *one can construct a braid theory for any three-manifold* by considering an appropriate splitting and some standard family of tangles in the corresponding components. The above construction would use the braid group of the surface Σ. Since every closed three-manifold admits a Heegaard splitting, we have a corresponding braid theory with braid groups of any oriented surface. It will be interesting to take this approach to study knots and links in a general closed three-manifold. One may construct invariants from such theories and ask questions about which properties of three-dimensional manifolds give rise to them. For example, some of them may be coming from algebraic topology, like the Alexander polynomial while some others may have their origins in *topological quantum field theories*, like the Jones polynomial.

An interesting variation of the definition of genus defined above could be considered. We can restrict the class of good surfaces of a knot to a smaller family of some special ones. For any knot in $\mathbb{R}P^3$, there is a covering link in S^3. Consider connected surfaces in S^3 bounded by the covering link of a knot in $\mathbb{R}P^3$. Now if we consider the image of such a surface Σ in $\mathbb{R}P^3$, it gives a singular surface in $\mathbb{R}P^3$. The singular points in this set are precisely the images of points of $\Sigma \cap a(\Sigma)$, where a is the antipodal map.

Notice that the covering link itself is a subset of $\Sigma \cap a(\Sigma)$ and its image is the knot. This part of the singular surface can be smoothened since locally they can be characterized as two half-planes meeting at their boundary line. Hence these singularities are "removable" by deforming Σ isotopically. But if there are other points in $\Sigma \cap a(\Sigma)$ than the points of the covering links, they locally form double points, characterized by two planes meeting at a line. These singularities may not be removable by deforming Σ. But for surfaces bounded by the covering link such that, they have no intersection with their antipodes in their interior, the images in $\mathbb{R}P^3$ are always properly embedded closed surfaces.

It is easy to see that, these surfaces also contain the knot inside, and hence they are good surfaces for the knot. See Chapter 4 for the definition of a good surface. Let's call these, "push down good surfaces". Clearly, all good surfaces for a knot are not push-down. Now we may look at the minimal genus in this restricted family as a "push-down genus" of a knot. If such surfaces exist for all knots in $\mathbb{R}P^3$, we can define this genus for all knots and it would be a finer invariant than the genus defined by all good surfaces. Some of the properties can be seen readily. For example, the unknots can be distinguished by this genus. The projective unknot is covered above by a great circle of S^3. This great circle is sitting on a "great 2-sphere" as an equator, and the corresponding hemispheres are antipodes of each other. The image of any of these hemispheres is a projective plane, containing the projective unknot we started with. Thus, The projective unknot has push-down genus 0. Now suppose the affine unknot has push-down genus 0. Then, its covering link in S^3 is the boundary of some connected surface Σ, such that $S := \Sigma \cup a(\Sigma)$ is a 2-sphere.

Notice that the covering link consists of two unknots which are unlinked. Any circle on S bounds a disk. Hence, both of the unknots in the covering link bounds disks. These disks are antipodes of each other. Thus they are disjoint disks in S^3. By definition, Σ will contain at least one of these disks. But since the corresponding unknot is the boundary of Σ, the disk has to be the whole of Σ. This is a contradiction since $\Sigma \cup a(\Sigma) = S$ is connected and cannot be the union of two disjoint closed sets. Thus the affine unknot cannot have push-down genus 0. It is easy to see that there is a torus in S^3 which will cover a torus in $\mathbb{R}P^3$. Clearly, we can choose this torus in such a way that it contains an affine unknot such that the above torus is cut into two cylinders by the covering unlink. Thus the push-down genus of the affine unknot is 1. Hence the push-down genus distinguishes the two unknots, already revealing that it is a powerful invariant. It would be exciting to work on the existence problem of push-down good surfaces for all knots in $\mathbb{R}P^3$. Also, I would like to point out that this genus can be defined for knots but does not have an obvious generalization for links in $\mathbb{R}P^3$.

Here we would also like to make a remark about a new relation between links arising from the braid theory. Given a plat closure in $\mathbb{R}P^3$, notice that the indexing involves an arbitrary choice of which point is index 1. Refer to Section 2.2.2. The way we add internal tangles is dependent on the indexing chosen. Now it must be clear that if we shift all indices by adding 1 to them (which is possible because the indices are from a cyclic group), the internal tangles will be changed. Let's call this operation "shift". Then in most cases, we get a different link as the braid closure. Notice that if we again add 1 to the indices, we will get back the initial link as the plat closure. Thus given on braid, by this procedure, we can get at most two plat closure links. It is easy to see that if we perform an M-move on the braid, and shift indices, we may obtain a different plat closure link. Clearly, there is a whole family of links that one can generate from one braid by performing shifts and M-moves in some arbitrary order. One question naturally arising from this is to characterize this family of links given by one braid. Can one produce every link in this fashion? If not, what is the equivalence that it generates?

There are many more invariants of classical knots which can be generalized to the knots in $\mathbb{R}P^3$. One of the obvious ones is the complement three manifold obtained by removing a link from $\mathbb{R}P^3$. Invariants of the complement are invariants of the knots in $\mathbb{R}P^3$. Refer to

[**Vir**] for a study of knot group in $\mathbb{R}P^3$. Clearly, one could also analyze other invariants of the complement such as hyperbolic volume, covering spaces of the complement, Reidemeister torsion, and so on in this setting.

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Appendix 1

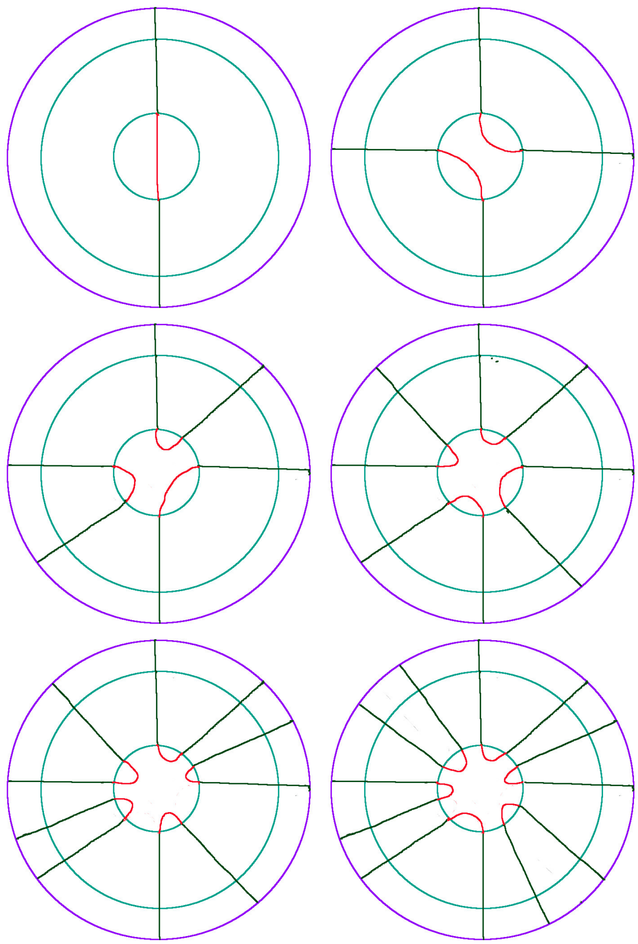

Figure 6.1: Closure of identity braids
Each odd braid close to form a projective unknot,
Every even braid close to form an affine unknot.

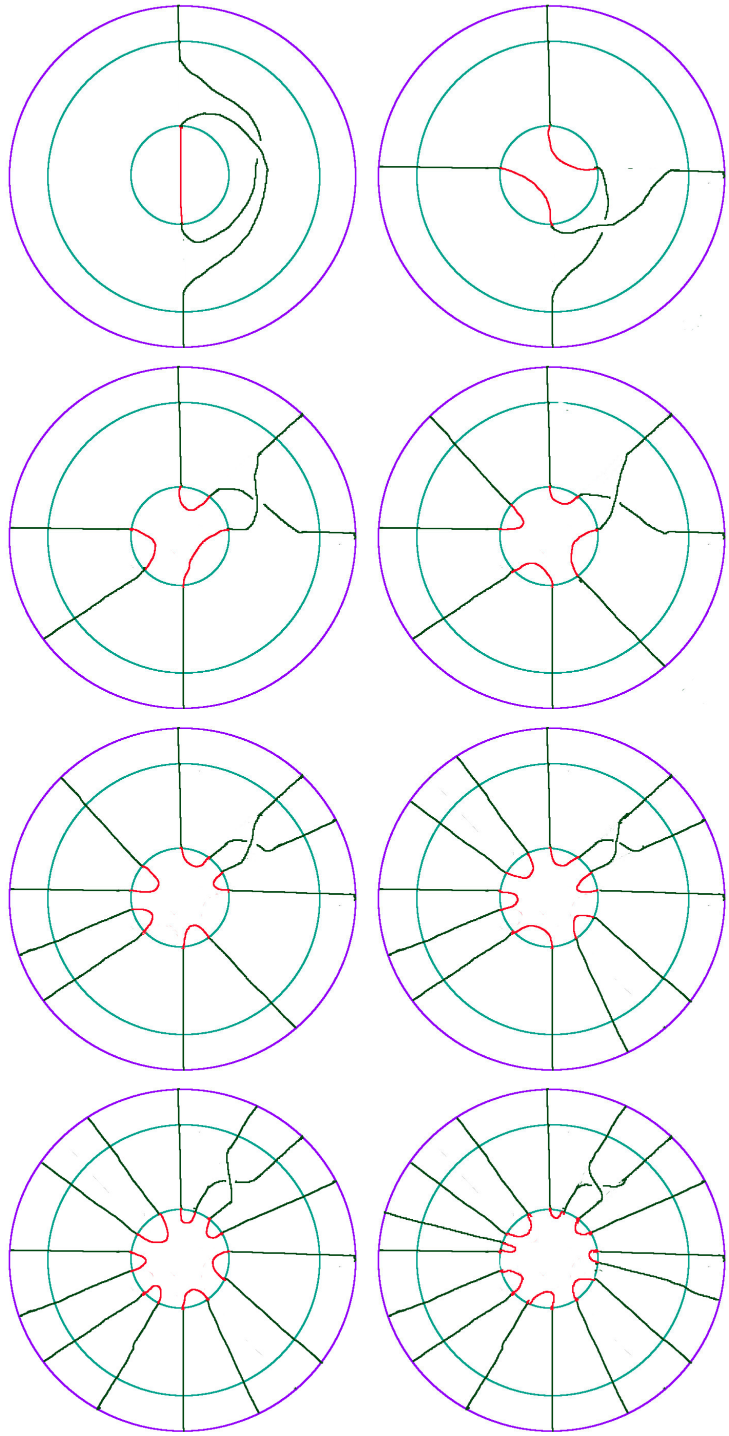

Figure 6.2: Closure of generating braids in each B_n
Odd braids close to projective lines
Even braids close to for unlinks except in B_4, where it closes to a link of two projective unknots.

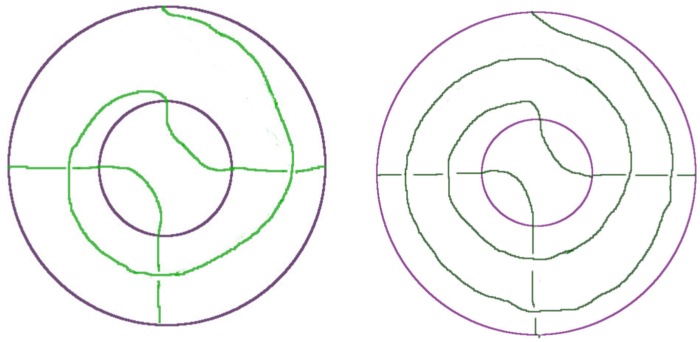

Figure 6.3: Closures of $T(4,1) := \sigma_1\sigma_2\sigma_3$ and $T(4,2) := (\sigma_1\sigma_2\sigma_3)^2$
Clearly the closure of any $T(4,n) := (\sigma_1\sigma_2\sigma_3)^n$ is isotopic to an affine unknot.

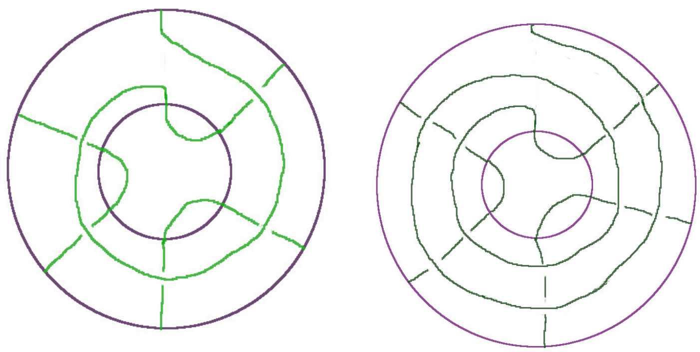

Figure 6.4: Closures of $T(6,1) := \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ and $T(6,2) = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^2$
Clearly the closure of any $T(6,n) = (\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^n$ will be isotopic to a projective unknot.

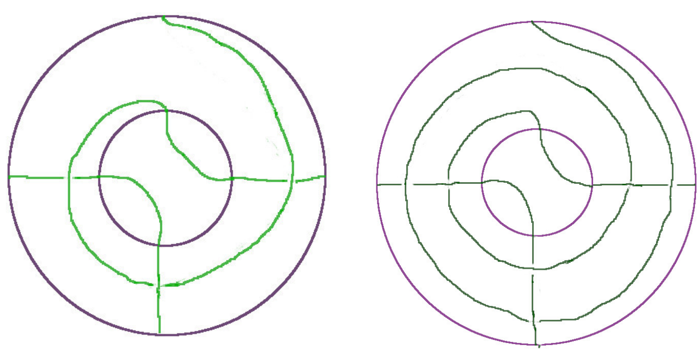

Figure 6.5: Closures of $W(4,1) := \sigma_1\sigma_2^{-1}\sigma_3$ and $W(4,2) := (\sigma_1\sigma_2^{-1}\sigma_3)^2$
Clearly the closure of any $W(4,n) := (\sigma_1\sigma_2^{-1}\sigma_3)^n$ is a class-0 knot.

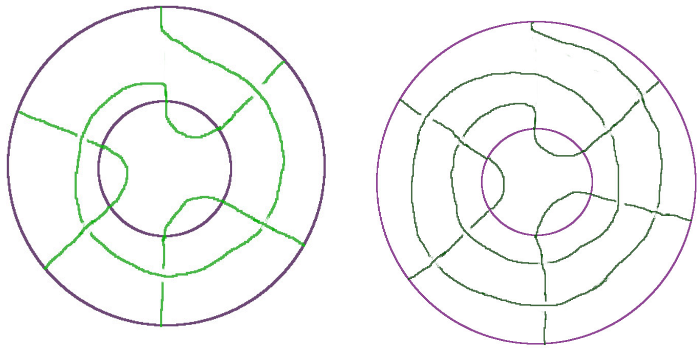

Figure 6.6: Closures of $W(6,1) := \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5$ and
$W(6,2) := (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5)^2$.
Clearly the closure of any $W(6,n) = (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5)^n$ is a class-1 knot.

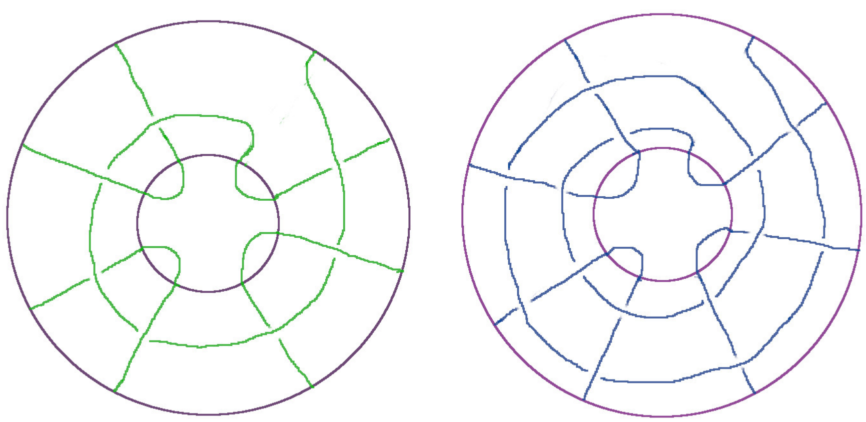

Figure 6.7: Closures of $W(8,1) := \sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5\sigma_6^{-1}\sigma_7$ and
$W(8,2) := (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5\sigma_6^{-1}\sigma_7)^2$.
Clearly the closure of any $W(8,n) = (\sigma_1\sigma_2^{-1}\sigma_3\sigma_4^{-1}\sigma_5\sigma_6^{-1}\sigma_7)^n$ is a link with two components.
One component is the closure of $W(4,n)$ and the other is an affine unknot.

Appendix 2

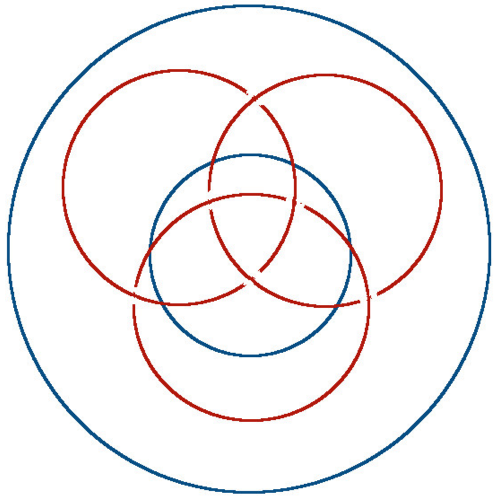

Figure 6.8: Borromean rings

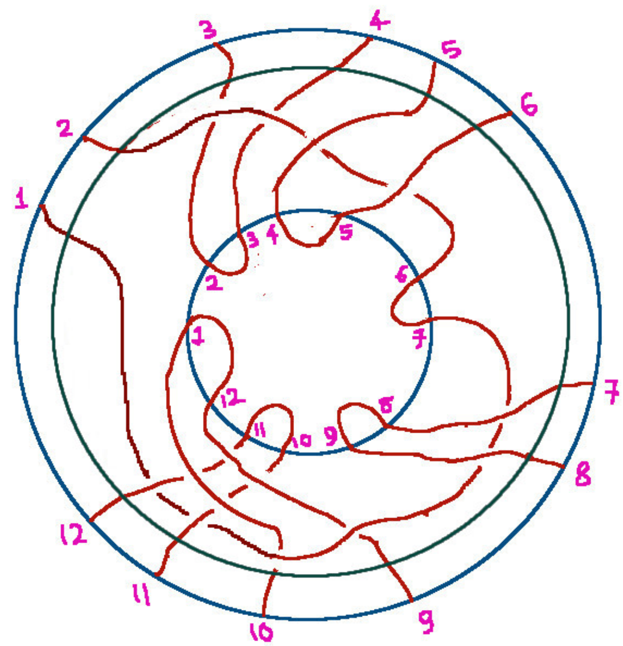

Figure 6.9: A plat representation for Borromean rings.
Residual permutation: (1 2)(3 4)(5 6)

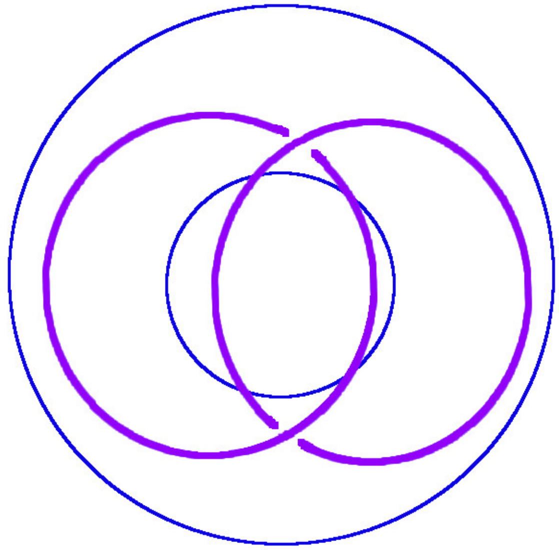

Figure 6.10: Hopf link

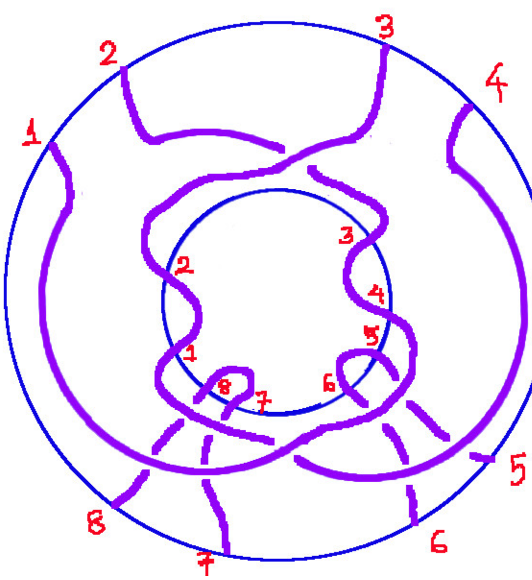

Figure 6.11: A plat representation of Hopf link.The residual permutation is: (1 2)(3 4)

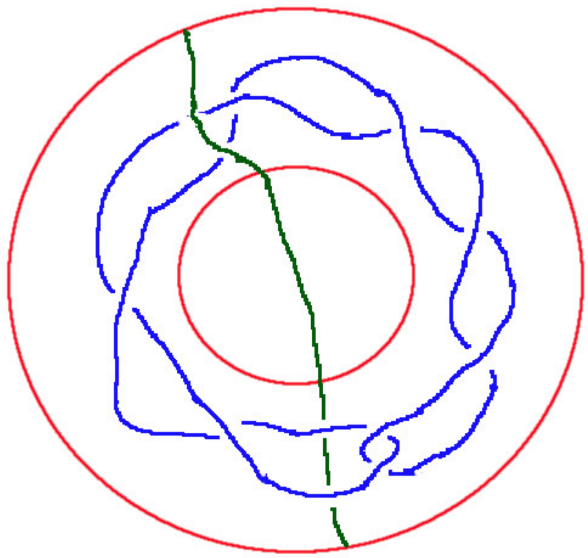

Figure 6.12: A link of a twist knot and the projective unknot.

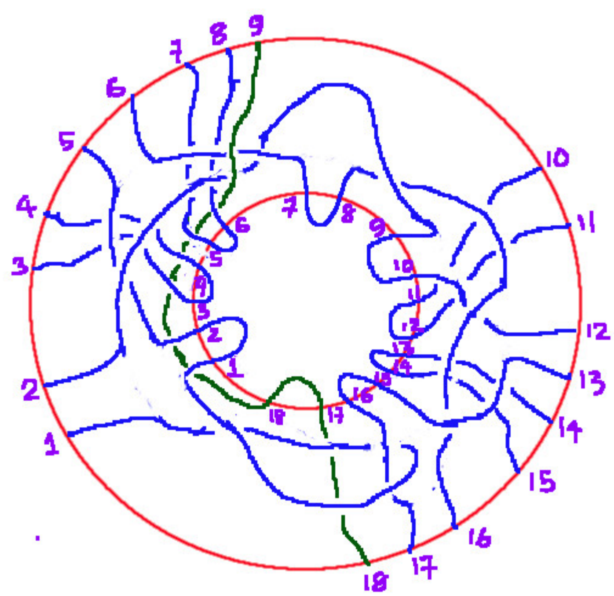

Figure 6.13: A plat representation of the above link.
The residual permutation is: (1 5 6 7 8 4 3 2)(9)

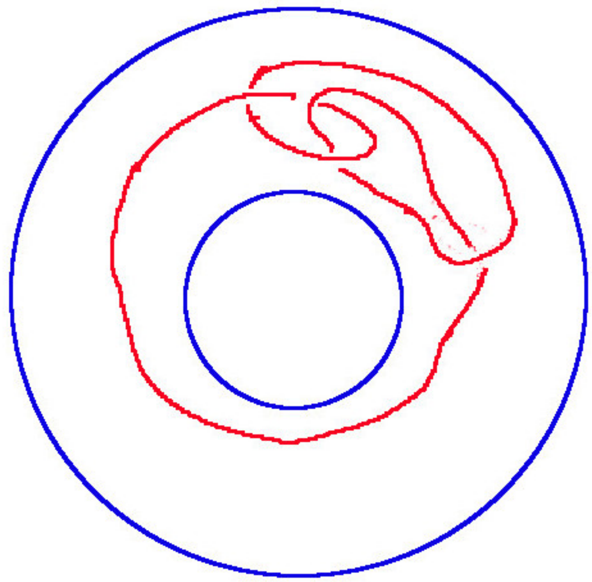

Figure 6.14: Affine figure-8 knot.

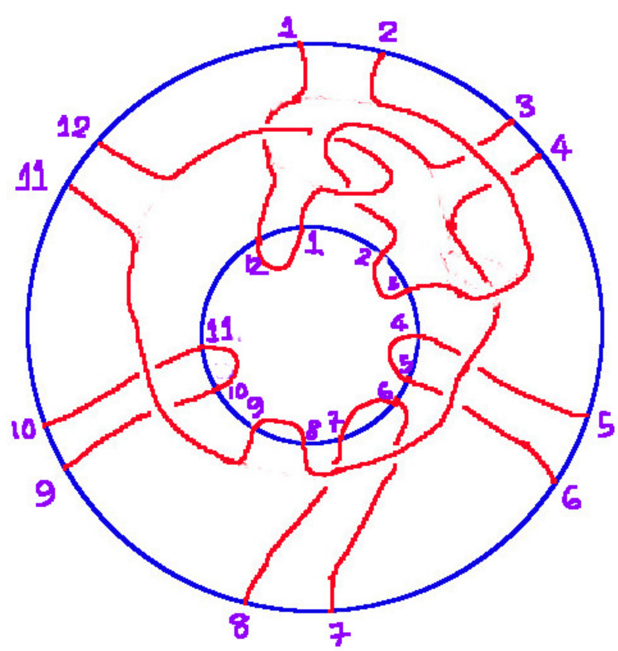

Figure 6.15: A plat representation of the above.The residual permutation is: (1 6 5 4 3 2)