Bargaining in Game Theory

A Thesis

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by

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Certificate

This is to certify that this dissertation entitled Bargaining in Game Theory towards the partial fulfilment of the BS-MS dual degree programme at the Indian Institute of Science Education and Research, Pune represents study/work carried out by Venkataram Chebiyyam at Indian Institute of Science Education and Research under the supervision of Dr. Hanzhe Zhang, Associate Professor, Department of Economics, Michigan State University , during the academic year 2022-2023.

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Declaration

I hereby declare that the matter embodied in the report entitled Bargaining in Game Theory are the results of the work carried out by me at the Department of Economics, Michigan State University, under the supervision of Dr. Hanzhe Zhang and the same has not been submitted elsewhere for any other degree.

Venkataram Chebiyyam

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Abstract

We explore the bargaining game beginning with cooperative games, following the Nash Bargaining Solution and understanding the axiomatic approach, and its impact, followed by the Kalai-Smorodinsky solution which also follows the static axiomatic approach. After this, we move over to the realm of non-cooperative games to study the Rubinstein bargaining model which follows a dynamic strategic approach. We see how the Nash and Rubinstein solutions, though different approaches, are connected, and can be combined for applications to problems in the process. Finally, we examine the results of Reputational Bargaining put out by Dilip Abreu and Faruk Gul in the seminal paper on the field, studying the various results and insights they provide to the continuous time incomplete information bargaining model.

Contents

Introduction

Bargaining refers to the phenomenon of argument over the division of a surplus amongst two or more people or parties, where the goal is to come to a cooperative agreement over the division of the same. The problem of bargaining has been studied for a long time by economists, even before the use of game theory. The use of a game theoretical framework to understand the problem of bargaining was an idea first introduced with the birth of game theory itself in von Neumann and Morgenstern's [8].

Before von Neumann and Morgenstern explored using game theory to understand bargaining problems, the problem was deemed indeterminate. The first exploration into bargaining using game theory that is considered to have given birth to the field of study is [7]. In this paper, John F. Nash uses an approach that has come to be known as the Axiomatic Theory of Bargaining. From there, the study evolved to non-cooperative games and then further dealt with games of incomplete information and combined elements with different kinds of games.

Original Contribution

No claim is made to the originality of any result presented in this thesis and it is meant to be solely a literature review. We present the summaries and proofs for the Nash and K-S models mainly from the original papers themselves [7], [4]. For the Rubinstein Bargaining model, most of the details of the extensive form game and the main result are presented from the original paper [10] and relevant chapters from [9], from which the relation between Rubinstein and Nash solutions is explored as well. Finally, Reputational Bargaining is explored mainly from the original paper [1] and [3].

Chapter 1

Axiomatic Bargaining and Nash's Solution

1.1 Introduction

In Nash 1950, a two-player bargaining problem is studied theoretically using game theory. Specifically, Nash builds from the theory of utility and coalitional games introduced in [8] and studies a two-player bargaining game. In this two-player bargaining problem, the players are highly rational; both players have complete information about each other's preferences and have equal bargaining power. Thus, this is a highly ideal look at the problem of bargaining. This theoretical study aims to find a 'solution' to the bargaining problem presented. Here, a solution is defined as the ideal amount the players would get from the distribution after they finish bargaining.

We study the problem and define the structure and elements of the game and the utility functions as follows. The set $N = \{1, 2\}$ represents the set of players, with $S \subseteq \mathbb{R}^2$ being the set of all feasible payoff profiles for the bargaining game. Thus, if $u = (u_1, u_2) \in \mathbb{R}^2$ is a pair of utilities corresponding to a particular outcome of the game, then $S = \{u = (u_1, u_2) | u \in \mathbb{R}^2\}$. We also define a disagreement point, a where $a = (a_1, a_2) \in S$ represents the outcome where the negotiations fall through and there is no cooperation.

1.2 Nash Bargaining

Every bargaining game is represented as the pair (a, S), where the set of feasible payoff profiles satisfies the following conditions:

1. *S is convex.*
2. *S is compact.*
3. *There is at least one point $x \in S$ such that $x^i > a_i$ for $i = 1, 2$*
4. $a \leq x, \forall x \in S$

These restrictions placed on the set S serve to make the problem more realistic, by narrowing the possible set of solutions. Further, this narrowing of the feasible set means the number of potential solutions also decreases, making it so that the solutions have significance.

1.3 The Axioms

In the axiomatic approach, certain axioms are laid regarding the properties of the solutions. The possible solutions to the game (a, S) thus are those pairs $(u_1, u_2) \in S$ such that they satisfy these axioms. For the game (a, S), we represent the solutions by a function $f : U \to S$, where U represents the set of all pairs (a, S) satisfying the restrictions outlined in the previous section. Thus, the solution(s) of the game (a, S) would be given by $f(a, S)$. The following axioms lay the foundations to find the game's solution(s).

Note. For all the axioms, $x = (x_1, x_2) \in S$ where x_1 and x_2 are the utilities of Player 1 and Player 2 respectively

Axiom 1 (Pareto Optimality)**.** There is no $y \in S$ such that $y > f(a, S)$ and $y \neq f(a, S)$

Axiom 2 (Independence of Irrelevant Alternatives)**.** Consider $(a, S), (a, T) \in U$ such that $S \subset T$. If $f(a, T) \in S$, then $f(a, S) = f(a, T)$.

Axiom 3 (Symmetry)**.** If the set S has the property of symmetry, then $f(a, S)$ lies on the line $x_1 = x_2$.

Axiom 4 (Invariance with Respect to Affine Transformations). If $A = (A_1, A_2)$ is an affine transformation, then $f(A(a), A(S)) = A(f(A, S))$.

1.4 The Solution

Definition 1.4.1. *The Nash Bargaining Solution $F : U \to \mathbb{R}^2$ is defined as follows:*

$$F(a, S) = arg\ max_{x_1, x_2 \in S}(x_1 - a_1) \cdot (x_2 - a_2)$$

Theorem 1.4.1. *(Nash [1950]) : A solution $f : U \to \mathbb{R}^2$ satisfies the above four axioms if and only if F=f*

Proof. We begin by transforming the original problem set. We use affine transformations so that the point where the product $(x_1 - a_1).(x_2 - a_2)$ is maximized is transformed to $(1, 1)$. Let us denote this transformed space as $S^{'}$ with $u^{'} = (u_1^{'}, u_2^{'})$ being a general point in this. Axiom 4 allows this since applying the inverse transformation gives back the actual solution.

Now, for no point in this transformed set will the sum $(x_1 - a_1) + (x_2 - a_2)$ or rather $u_1^{'} + u_2^{'}$ be greater than 2. This is because if there was a point such that $u_1^{'} + u_2^{'} > 2$ then there would be a point on the line joining $(1, 1)$ and $(u_1^{'}, u_2^{'})$ such that the product of the two utilities would maximize the product. This however is a contradiction. Thus, every point of this transformed set would be on or below the line $u_1^{'} + u_2^{'} = 2$.

Now we draw a square in the region $u_1^{'} + u_2^{'} \leq 2$, on the plane of payoff profiles, such that it is symmetric about the line $u_1^{'} = u_2^{'}$ and encloses the entire set $S^{'}$. Note that the line $u_1^{'} = u_2^{'}$ will also pass through the transformed disagreement point and the disagreement point has essentially turned into the new origin.

Now consider the region enclosed by this square. This set of profiles is symmetric. Thus, in order to satisfy Axiom 3, the solution for this set must lie on the line $u_1^{'} = u_2^{'}$. Further, we must satisfy the Pareto Optimality condition, i.e., Axiom 1. Thus we need a point on the line $u_1^{'} = u_2^{'}$ such that there is no point on or in this square that has payoffs greater than it. Thus, we would search for the largest point in the intersection of the line and this square. The point in question would be $(1, 1)$. Thus the Nash Solution for this point would be the point $(1, 1)$.

However, the point $(1, 1)$ is also a part of the transformed set $S^{'}$. Thus by using Axiom 2, since $(1, 1)$ is the solution of the square, it is also the solution of $S^{'}$ and hence, the Nash Solution of the game (a, S) is $f(a, S) = argmax_{x_1, x_2 \in S}(x_1 - a_1).(x_2 - a_2)$, after reversing the affine transformation. □

1.5 Impact and Extensions

Given that this was a seminal paper in the field, it is obvious that as time went on and more applications were found, quite a few shortcomings to the model were found, forcing economists to find better ways to tackle the bargaining problem. One such illustration is as follows: Let us consider two different bargaining problems: $(0, S_1)$ and $(0, S_2)$. Where S_1 is the convex shell formed from the points $\{(0.75, 0.75), (1, 0), (0, 1)\}$ and S_2 from the points $\{(1, 0.7), (1, 0), (0, 1)\}$. The NS for these problems is $(0.75, 0, 75)$ and $(1, 0.7)$, respectively. The graph of the feasible utility regions is displayed below. The blue region represents S_1 while the blue and yellow regions together represent S_2 The amount that player 2 receives in the NS for the second game is lower than what they receive in the first game. However, from the yellow region in the figure, it is clear that for every utility level that player 1 can demand, the maximum amount that player 2 could possibly ask for is higher in $(0, S_2)$ than in $(0, S_1)$. Thus we can see that in $(0, S_2)$, a point can be made for player 2 to demand much more than what they receive in the NS.

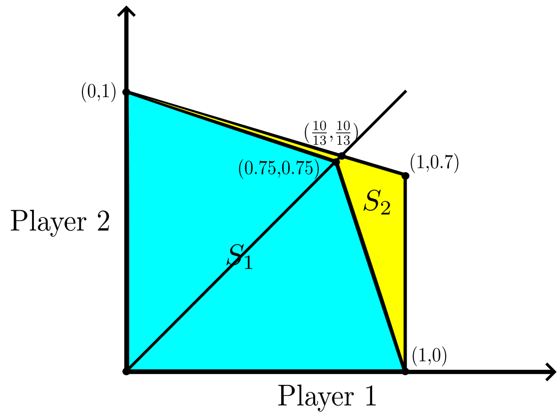

Figure 1.1: A graph highlighting the feasible regions for the games $(0, S_1)$ and $(0, S_2)$. The point $(10/13, 10/13)$ represents the KC solution for $(0, S_2)$

This issue stems from the axiom of independence of irrelevant alternatives which means that we end up ignoring the interpersonal comparisons of utilities. However, there are cases where including this comparison while attempting to find the game's solution would lead to a solution closer to what is realistically observed than the Nash Solution. It is important to note that this does not mean that the Nash Solution is wrong, but rather that other solution concepts may be more suitable under various different circumstances.

Thus in order to avoid such contradictions in similar situations, it makes sense that we use a different axiom in place of IIA to describe the solution of the game. This is the thought process demonstrated in [4].

1.6 Axiom of Monotonicity

For every game (a, S) we define $b(S) = (b_1(S), b_2(S))$ such that:

$$b_1(S) = sup\{x \in \mathbb{R} : \text{for some } y \in \mathbb{R}, (x, y) \in S\}$$
$$b_2(S) = sup\{y \in \mathbb{R} : \text{for some } x \in \mathbb{R}, (x, y) \in S\}$$

We now use $b(S)$ to further define a function $g_S(x)$ on all $x \leq b_1(S)$ such that:

$$g_S(x) = \begin{cases} y & \text{if } (x, y) \text{ is the Pareto of } (a, S) \\ b_2(S) & \text{if there is no such } y \end{cases}$$

To put it in words, the function $g_S(x)$ represents the maximum utility that player 2 can get given that player 1 gets at least x amount of utility. Using this notation to describe the example outlined in the criticism in the previous section, $g_{S_1}(x) \leq g_{S_2}(x)$.

We once again use the function $f(a, S) = (f_1(a, S), f_2(a, S))$ to denote the solution of the problem.

Axiom: *[Axiom of Monotonicity] Consider two bargaining problems (a, S_1) and (a, S_2) such that $b_1(S_1) = b_2(S_2)$ and $g_{S_1} \leq g_{S_2}$. Then, $f_2(a, S_1) \leq f_2(a, S_2)$.

1.7 The Kalai-Smorodinsky Solution

The K-S solution is given by the value of the function $f(a, S)$ that satisfies the following four axioms:

1. Pareto Optimality
2. Symmetry
3. Invariance with respect to affine transformations
4. Monotonicity.

Theorem 1.7.1. *There is one and only one solution, μ, satisfying the axiom of monotonicity. The function μ has to be the following simple representation: for a pair $(a, S) \in U$ consider the line joining a to $b(S)$ - $L(a, b(S))$. The maximal element (using the partial order of $\mathbb{R}^2$) of S on this line is $\mu(a, S)$.*

We divide the proof into three parts. First, we show that the function μ itself is well-defined. Once this is established, we show that μ is indeed a solution for the game and satisfies all the axioms. Finally, we prove the uniqueness of μ in terms of satisfying the axiom of monotonicity.

Proof. Part 1: μ is well defined

Let (a, S) be a bargaining pair. The line $L(a, b(S))$ constructed as described would have a positive slope. This would mean that the partial order of $\mathbb{R}^2$ induces a total order on $L(a, b(S))$. This implies that if $L(a, b(S))$ intersects S, then there is a unique maximal element of S on it, and $\mu(S)$ is well defined.

The existence of a point $(b_1(S), y) \in S$ such that $y \geqslant a_2$; there is a point $(x_1, b_2(S)) \in S$ such that $x \geqslant a_1, a < b(S)$; and S is convex guarantees that $L(a, b(S))$ intersects S.

Part 2: μ satisfies the axioms

Since S, is convex and compact, we know that Pareto Optimality is satisfied. Let A be an affine transformation. We know that A would preserve the partial ordering of the set $\mathbb{R}^2$. Also, since A is an affine transformation, we know that upon applying it on a straight line, the resulting mapping would be to a straight line as well. Finally, the result of transforming $b(S)$ would be $b(A(S))$. Now since, μ is defined as the maximal point according to the partial order on $\mathbb{R}^2$ on the line as defined above, we see that the solution is invariant under affine transformations.

The monotonicity follows from the following geometric observations. If L_α is a line of slope $\alpha(0 \leqslant \alpha \leqslant \pi/2)$ passing through a and if $(\sigma_1(\alpha), \sigma_2(\alpha))$ is the intersection point of the L_α with the boundary of $\{x \in R^2 : x \geqslant 0 \text{ and } x \leqslant y \text{ for some } y \in S\}$, then if $\beta > \alpha, \sigma_2(\beta) \geqslant \sigma_2(\alpha)$ and if $\left(\sigma_1^{(2)}(\alpha), \sigma_2^{(2)}(\alpha)\right)$ is the corresponding point for (a, S_2), then $\sigma_1^{(2)}(\alpha) \geqslant \sigma_1^{(1)}(\alpha)$.

Part 3: μ is unique

It is enough to prove this fact for normalized bargaining pairs. So let $(0, S)$ be such a pair and f any monotonic solution. Let $S_1 = \{x \in R^2 : x \geqslant 0 \text{ and } x \leqslant y \text{ for some } y \in S\}$. Clearly $(0, S_1)$ is a normalized bargaining pair, $S_1 \supset S$, and there is no point $y \in S_1$ such that $y \geqslant f(0, S_1)$ and $y \neq f(0, S_1)$. Therefore $f(0, S_1) = f(0, S)$. Also the points $(0, 1)$ and $(1, 0)$ are in S_1. Let $S_2 = \text{convex hull } \{(0, 1), (1, 0), \mu(0, S_1)\}$. Then $(0, S_2)$ is a normalized bargaining pair, it is symmetric for the two players, and $S_2 \subset S_1$. Therefore

$f(0, S_2) = \mu(0, S_1)$. Also S_1 contains no point y such that $y \neq f(0, S_2)$ and $y \geqslant f(0, S_2)$. Therefore $f(0, S) = \mu(0, S_1) = \mu(0, S)$, and this completes the proof. $\square$

Chapter 2

Rubinstein Bargaining Solution

2.1 Introduction

The previous chapter used the axiomatic approach to tackle the bargaining problem. While seeing many applications and extensions, the axiomatic approach does have many drawbacks. As seen in the previous chapter, the K-S model addresses issues with how axioms cannot be used for certain scenarios, as the desired result may not be obtained. From this, it is obvious that it would be hard to argue rigorously for the need for some axioms without describing the game's very specific structures. Also, intrinsic to Nash's model is the fact that it does not factor in comparisons of interpersonal utilities, which can be a huge factor in studying various Social Science and real-life scenarios. Modifications like the ones made in the K-C model become increasingly necessary and larger in number, detracting from the validity and significance of the solution set.

Another seminal paper in this field, [10] takes the strategic approach to solving the bargaining game. It details thoroughly the structure of the negotiations of the game, unlike the cooperative game in [7]. This allows for versatile applications of the model to real-life scenarios as well as understanding the negotiation process much better and easier to make extensions. Later on in [2], we also go on to see that in a specific limit of the Rubinstein Solution, we arrive at the Nash solution, connecting these seemingly very opposite approaches to each other and reinforcing the fact that the Nash solution gives us an ideal solution.

The theory and outlines in this chapter are presented from [10], [9] with proofs adapted from [11].

2.2 Structure of the Negotiations

Note. we maintain the same idea from the previous models that two players are bargaining over an excess 'pie' of size 1.

We start our description of the bargaining model by first mathematically defining what an 'agreement' made by the parties in the negotiation process is. An agreement is a tuple (x_1, x_2) where x_i refers to the share of the pie that Player i receives. Building upon this, we also define the set of all possible agreements. This set is defined as:

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 + x_2 = 1 \text{ and } x_1, x_2 \geq 0\}$$

To completely describe the bargaining process, we now need to define two more things: what are the agreements that each player would like to reach and, consequently, like to avoid, and once these preferences are established, how the players would go about with the bargaining process.

This process is vital since, in the strategic approach, unlike the axiomatic approach where we define the properties the solution must have explicitly, thus restricting it, we restrict the game itself in its entirety. Thus, it is crucial that the restrictions placed are carefully chosen since placing too many restrictions would mean the model would not be applicable in more than a few scenarios. However, place too few restrictions, and the equilibrium (solution to the game) becomes too broad and ends up providing little helpful information. So we would ideally like to find a structure of the game where both parties still have a lot of freedom regarding how they negotiate while still making the solution useful.

This is achieved by describing the bargaining process as follows:

1. The game is an infinite horizon game
2. At each timestep $t \in \{0, 1, 2, \dots\}$, a player (Player i) makes an offer of the form x^t

where $x^t \in X$ and the other player (Player j) chooses to either accept the offer (Y) or refuse the offer (N). If the offer is accepted, the game ends at that time step. If the offer is rejected, the game moves to the next time step where everything goes on similarly except that Player j makes the offer this time, and Player i has the option to accept or reject it. The game goes on until either player chooses to accept an offer.

3. If no outcome is reached, i.e., both players keep rejecting each other's offers for infinitely long, we denote the outcome using D standing for Disagreement.

4. When an offer is agreed to, we denote the outcome using the tuple (x, t) where x represents the pie partition agreed upon, and t refers to the timestep in which the agreement was reached/offer was accepted.

5. The game begins at time step $t = 0$ with Player 1 making an offer and Player 2 choosing to accept or reject it.

2.3 Preferences

Since we do not impose too many restrictions on the structure of the game itself, the key ideas that lead to exciting results arise from our assumptions about the players' preferences. As we described above, the game's preferences and structure were the two things that had to be rigorously defined. Since we have completed talking about the structure of the game, we move on to talking about the players' preferences over the game's outcomes. We define these preferences mathematically as a complete ordering $\succeq_i$ over the set $\{X \times T\} \cup D$ that is also transitive and reflexive.

Thus, we formally define the game as follows:

Definition: A bargaining game of alternating offers is an extensive game with the structure defined in Section, in which each player's preference ordering $\succeq_i$ over $\{X \times T\} \cup D$ is complete, transitive, and reflexive.

Now that we have defined the game, we move on to placing assumptions on the preferences to add restrictions. The assumptions are as follows:

Note. The statements in all of the following assumptions hold $\forall x, y \in X,\ t, t_1, t_2 \in \mathbb{N}$ and $i \in \{1, 2\}$

1. Pie is desirable. if $x_i > y_i$ then $(x,t) \succ_i (y,t)$
2. Time is valuable. if $x_i > 0$ and $t_2 > t_1$ then $(x, t_1) \succ_i (x, t_2)$
3. Continuity.
4. Stationarity. $(x,t) \succ_i (y, t+1) \iff (x, 0) \succ_i (y, 1)$

2.4 Nash Equilibrium of the Game

More often than not, when we want the solution of a game, we turn to the notion of the Nash Equilibrium. However, we do not use the Nash Equilibrium in this particular game. This is because this game has no *unique* Nash Equilibrium. Every agreement $(x, 0) \in X$ would be a valid outcome that the Nash Equilibrium of a game generates. We will now proceed to show this in the current section:

Consider the action profile where Player 1 always proposes the same amount and accepts offers only if they receive more than what they receive in their offer. Let the same be the case for Player 2. We can define this mathematically as follows:

$$\overline{\sigma}^t(x^0, \ldots, x^{t-1}) = \begin{cases} \overline{x} & \text{if } t \text{ is even, for all } (x^0, \ldots, x^{t-1}) \in X^t \\ Y & \text{if } t \text{ is odd and } x^t \geq \bar{x_1} \\ N & \text{if } t \text{ is odd and } x^t < \bar{x_1} \end{cases}$$

Player 2's strategies, which we represent with $\bar{\tau}$ will be similar, with even and odd switched around. From these definitions, once Player 1 makes the offer $\bar{x}$ at $t = 0$, Player 2 immediately accepts this offer and the game ends at $t = 0$.

Now suppose Player 1 uses a different strategy. Perpetual disagreement is the worst outcome, and Player 2 never makes an offer different from x or accepts an agreement x with $x_j < \bar{x_j}$. Thus the best outcome that Player 1 can obtain, given Player 2's strategy, is $(x, 0)$.

Thus, we turn to the Subgame Perfect Nash Equilibrium to find the solution to the game.

2.5 Main Result

Theorem 2.5.1. *Every bargaining game of alternating offers in which the players' preferences satisfy A1 through A6 has a unique subgame perfect equilibrium* (σ^*, τ^*)*. In this equilibrium Player 1 proposes the agreement* x^* *whenever it is their turn to make an offer, and accepts an offer* y *of Player 2 iff* $y_1 \geq y_1^*$ *; Player 2 always proposes* y^* *, and accepts only those offers* x *with* $x_2 \geq x_2^*$*. The outcome is that Player 1 proposes* x^* *in period 0, and Player 2 immediately accepts this offer. (Here* $y_1^* = v_1(x_1^*, 1)$ *and* $x_2^* = v_2(y_2^*, 1)$*) where*

$v_i : [0,1] \times T \rightarrow [0,1]$ *for* $i = 1, 2$ *as follows:*

$$v_i(x_i, t) = \begin{cases} y_i & \text{if } (y,0) \sim_i (x,t) \\ 0 & \text{if } (y,0) \succ_i (x,t) \text{ for all } y \in X \end{cases}$$

From the definition, it is clear that if $v_i(x_i, t) > 0$, then Player i is indifferent between receiving a share of $v_i(x_i, t)$ at time 0 and x_i at time t. Thus, it makes sense to refer to $v_i(x_i, t)$ as the present value of (x, t) for Player i.

Say both players have a discount factor, i.e., the utility received from the same amount of pie decreases as time goes on. Let the discount of the players be δ_1 and δ_2 respectively. Given that these are not constant discount factors, then we get $x^* = \left(\frac{1-\delta_2}{1-\delta_1\delta_2}, \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}\right)$ and $y^* = \left(\frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2}, \frac{1-\delta_1}{1-\delta_1\delta_2}\right)$.

This literature will explore a specific realm rather than a general one as shown in the above theorem. Though specific, the version of the problem discussed is the most used and well-known. In the following section, we show that given δ_1 and δ_2, the above values of x^* and y^* are in fact the values for the unique SPNE of the game.

2.6 Proof for unique SPNE with Discount Factors δ_1 and δ_2

Proof. Part 1: The Equilibrium is Subgame Perfect

We first show that the above-outlined strategies represent those chosen in a subgame perfect Nash equilibrium.

Consider a time period where Player 1 makes the offer of x. This means that if Player 2 accepts, Player 1 receives a share of x_1 and thus a utility of x_1. There is no deviation that Player 1 can make and achieve a utility greater than x_1 since if they make an offer $x^{'}$ with $x_1^{'} > x_1$, then Player 2 will not accept this offer and the offer that Player 2 then makes in the following time period will at-most give a utility of $\delta_1 x_1$. The only other option is to then reject this offer and make one more in the following time period, but then again, the most Player 1 can offer while making sure Player 2 accepts will net them a utility of $\delta_1^2 x_1$, which is even lesser.

Now, we move on to Player 2. In this scenario, if Player 2 accepts the initial offer, they get a utility of $x_2 = 1 - x_1$. If not they reject the offer and in the next period they make an offer where the most they can get will be $\delta_2 x_2$. Hence, Player 2 has no profitable deviation either.

Thus, by the one-step deviation principle, the strategies x^* and y^* give the Subgame Perfect Nash Equilibrium for the Rubinstein Bargaining game and the offer made by Player 1 in the first time period is immediately accepted.

Part 2: Uniqueness

Now, we show that this SPNE is unique for this game.

Let $\bar{u_1}$ be the maximum utility that Player 1 can get in any SPNE starting from a time period where they make the offer. Similarly, let $\underline{u_1}$ be the minimum utility.

Consider a game state where Player 2 makes the offer.

Then, we know that Player 1 will:

1. Accept any offer that is $\geq \delta_1 \bar{u_1}$
2. Reject any offer that is $< \delta_1 \underline{u_1}$

Thus, player 2 can secure:

1. Atleast $(1 - \delta_1 \bar{u_1})$

2. Atmost $(1 - \delta_1 \underline{u_1})$

Similarly, consider a gamestate where Player 1 makes the offer.

Then, Player 2 will accept any offer which gives them utility that is atleast $\delta 2(1 - \delta_1 \bar{u_1})$.

Thus, we get:

$$\bar{u_1} \leq 1 - \delta 2(1 - \delta_1 \bar{u_1})$$

Since Player 2 will also accept any offer that nets them more than $\delta_2(1 - \delta_1 \underline{u_1})$, we have:

$$\underline{u_1} \geq 1 - \delta_2(1 - \delta_1 \underline{u_1})$$

From eqn. 1 we have $\bar{u_1} \leq \frac{1-\delta_2}{1-\delta_1\delta_2}$

and from eqn. 2 we have $\underline{u_1} \geq \frac{1-\delta_2}{1-\delta_1\delta_2} \implies \bar{u_1} \leq \frac{1-\delta_2}{1-\delta_1\delta_2} \leq \underline{u_1} \implies \bar{u_1} = \underline{u_1}$ □

2.7 Nash Program for SPNE of Rubinstein Problem

In [6], the idea that there is a connection between strategic and axiomatic models and they could be related was first introduced. Here, we see how the equilibrium discussed in the previous section and the Nash solution from Chapter 1 are related. We begin this exploration by first describing common underlying models and observing the equilibrium at certain limits.

2.7.1 Bargaining with the risk of Breakdown

Consider a model similar to the one considered in Section 2.2, with the change being that there is now an exogenous added probability of the negotiations breaking down. Let this probability be α, and the payoffs at this breakdown point represented by $B = (b_1, b_2)$. We also consider that in this model, the players are indifferent about the time at which the agreement is reached. We have now defined a strategic extensive form game. Given α, it follows that given disagreement at a given timestep, the game moves on to the next with probability $(1-\alpha)$. Thus, if a strategy σ^t leads to an outcome (x,t) in the previous game, here this strategy leads to (x,t) with probability $(1-\alpha)^t$. Otherwise, the outcome reached is B.

We see that we can also model a game with the axiomatic approach to fit the above scenario using the following axioms:

1. Axiom 1: Pie is Desirable
2. Axiom 2: B is the worst outcome
3. Axiom 3: Players are risk-averse

We further define

$$S=\left\{(s_1,s_2)\in\mathbb{R}^2:(s_1,s_2)=(u_1(x_1),u_2(x_2))\text{ for some }x\in X\right\}$$

and $b=(b_1,b_2)=(0,0)$ as the disagreement point. Then, (b,S) is a bargaining game similar to those described in Section 1.2, satisfying the axioms described above.

Following an analysis similar to that seen in Sections 2.5 and 2.6, it is obvious that the equilibrium strategies for this game would be given by

$$x_1^*=\frac{(1-b_2)+(1-\alpha)b_1}{2-\alpha}$$

$$y_2^*=\frac{(1-b_1)+(1-\alpha)b_2}{2-\alpha}$$

where Player 1 always makes the offer $x^*=(x_1^*,1-x_1^*)$ and never accepts an offer y with $y_1<x_1^*$ and similarly for Player 2. Thus, as described in Theorem 2.5.1, expanding on the definition of $v_i(x_i,t)$, we have

$$u_1(y^*) = (1 - \alpha) \cdot u_1(x^*) \text{ and } u_2(x^*) = (1 - \alpha) \cdot u_2(y^*)$$

Now consider the limit where the exogenous probability diminishes to zero. In this case, we have

$$\lim_{\alpha \to 0} \left[u_i\left(x^*\right) - u_i\left(y^*\right)\right] = 0$$

Therefore, we see that at this limit, x^* converges to the maximizer of $u_1(x_1) \cdot u_2(x_2)$, which in turn is the Nash Solution to (b, S).

Chapter 3

Reputational Bargaining

3.1 Introduction

As discussed earlier, the Rubinstein model explores the phenomenon of bargaining by exploring a non-cooperative game with an extensive form. Now, we move on to explore a game where the information players have about each other are incomplete. In the reputational bargaining model put out by Dilip Abreu and Faruk Gul in their paper [1], the bargaining model takes place in continuous time as a slightly modified war of attrition. In this game, while bargaining, the players gain more information about each other as time goes on. The war-of-attrition model arises from a model where the players could be considered either to be rational or irrational. An irrational player is one who ends up being committed to one offer and never accepting another or is willing to deviate to a different one. Thus, it is easy to see how the bargaining process could be interpreted as a standard war-of-attrition game with incomplete information. The reputation here refers to the reputation of the opponent being committed to a single offer. Building on the results of the war of attrition and economic theories like the Coase conjecture, Abreu and Gul analyze the equilibrium of this game, and its properties and similar to previous models, find a connection with the Rubinstein solution at the limit of multiple different offers being available for a player to make during the negotiation process.

The theory for this chapter is presented from [1], [3] and [5].

3.2 The Structure of the Game

We begin by looking at a simplified version of the game. In this simplified version, the two players have only one possible type they can take, and the bargaining is in continuous time. Let the players be i and j. Let C_i and C_j be the set of all possible 'irrational types' a player can take. $C_i \subset (0,1)$ is a finite set of possible irrational types for Player i.

The irrationality of a player is probabilistic as well. With probabilities z_i and z_j, the players are irrational. Thus, a player can take on a 'type' from α_i or α_j if and only if they are irrational, which happens with probability z_i or z_j. If they are indeed irrational, we represent the type of the player using $\alpha_i \in C_i$ and $\alpha_j \in C_j$. We categorize a player as irrational in this game when they make the same offer every time it is their turn, regardless of the other player's offer. We use $\pi_i(\alpha_i)$ to represent the conditional probability that Player i is irrational of type α_i, given that Player i ends up being irrational.

3.2.1 The Game at $t = 0$

Once again, like all of our previous models, we consider that the bargaining problem involves the players splitting a pie of size 1. At time 0, Player 1 will choose to make an offer of α_1. If the player is irrational, they are making this offer since they are of type α_1. If they are rational, however, this offer is a calculated one. Now, Player 2 would accept this offer in two scenarios. The first case is that in which they are rational (check once again). The second is if they are irrational of type α_2 such that $\alpha_2 < 1 - \alpha_1$, since this means they are getting more than what they would have asked for anyways. If not, Player 2 rejects the offer and makes their own offer of α_2 such that $\alpha_2 > 1 - \alpha_1$.

After this, if Player 1 accepts the offer, the game ends. If not, a war of attrition ensues over the split.

3.2.2 The War of Attrition

The game now proceeds on to a war of attrition in continuous time. This means that at any time, one or both of the players can choose to accept the offer of the other player.

Thus, choosing not to accept would mean they choose to continue the war of attrition. As mentioned previously, irrational players always make the same offer. A further property is that in the war of attrition, they never choose to concede. Thus, we call α_i the 'commitment type' of Player i. Further, we use r_i to represent the players' exponential discount rates in this war of attrition.

We define μ_i as a probability distribution on the set of possible types C_i and $F^1_{\alpha_1,\alpha_2}$ as a collection of cumulative probability distributions on $\mathbb{R}^+ \cup \{\infty\} \forall (\alpha_1, \alpha_2) \in C_1 \times C_2$, such that $\alpha_1 + \alpha_2 > 1$. Here $F^1_{\alpha_1,\alpha_2}(t)$ represents the probability of Player 1 accepting Player 2's offer by time t.

3.3 Equilibrium with Single Type for Both Players

We know that an irrational player would never accept the opponent's offer. Thus, the more time that goes on without Player i conceding, the higher is Player i's 'reputation' of being irrational. Thus we can express this reputation as a function of time as follows:

$$z_i(t) = \frac{z_i}{1 - F^i_{\alpha_1,\alpha_2}(t)}$$

From [Hendricks, Weiss, Wilson 1988] we know that for a war of attrition as that described in the previous subsection, an equilibrium is characterized by the following three properties:

1. At most one player concedes with positive probability at time 0.
2. At any time $t \neq 0$, the concession of Player i is at a constant hazard rate $\lambda_i = \frac{r_j \cdot (1-\alpha_i)}{(\alpha_j + \alpha_i - 1)}$. This is the rate at which the opponent would become indifferent between the strategies of conceding and staying.
3. Unless the game ends with probability 1 at time 0, concession by both players continues forever.

However, the bargaining game differs slightly from a general war of attrition in that we have non-zero prior probabilities of being of a particular commitment type (irrational). Due to this, Property 3 is replaced by the following:

1. Both players' reputations reach the value 1 at the same time (say) T_0.

We now show that the equilibrium does indeed possess the above properties. We begin with Property 3. Consider that this property did not hold and without loss of generality, Player i reached reputation 1 at some time T before Player j. Then, this would mean that Player j at time T realizes that Player i is irrational. However, knowing Player j is irrational, it would be much more beneficial for a rational Player i to concede at a time before T. And this would be a contradiction since we are dealing with the equilibrium.

We move on to property 1. If this property did not hold, then it would mean that a player could wait to see if their opponent concedes at time $t = 0$, and if they do not, concede at the instant right after, which would lead to a higher payoff.

Using the same reasoning, the concession must be continuous for every instant after time $t = 0$. In order for this to be possible, the players must be indifferent between staying in the war of attrition or conceding. In order for this to be the case, the payoff received from conceding at the current instant must be equal to that received in the next instant from staying on. Thus, we have:

$$r_i \cdot (1 - \alpha_j) = \lambda_j(\alpha_i - (1 - \alpha_j)) \tag{3.1}$$

From the above equation, for $t \leq T_0$ we have:

$$1 - F^j_{\alpha_1,\alpha_2}(t) = (1 - F^j_{\alpha_1,\alpha_2}(0)) \cdot e^{-\lambda_j t}$$

Now, from the definition of T_0 we have:

$$z_j(T_0) = \frac{z_j}{1 - F^j_{\alpha_1,\alpha_2}(t)} = \frac{z_j e^{\lambda_j t}}{1 - F^j_{\alpha_1,\alpha_2}(0)} = 1$$

As described above, since we are analyzing the equilibrium and we have assumed that Player j is conceding at time 0 with non-zero probability, Player i will not concede at time 0.

Thus, we will have $F^i_{\alpha_1,\alpha_2}(0) = 0$. Let T_i be the time at which Player i's reputation reaches 1. Then, we have:

$$z_i e^{\lambda_i T_i} = 1$$
$$\implies T_i = -\frac{\log z_i}{\lambda_i}$$

And carrying along the same line of reasoning, since at most one player can concede with positive probability at time $t = 0$, and we need both Players reputation to reach the value of 1 at the same time, we have $T_0 = min\{T_1, T_2\}$.

This implies then that $1 - F^i_{\alpha_1,\alpha_2}(0) = max\left\{0, 1 - z_i z_j^{-\lambda_i/\lambda_j}\right\}$.

Thus, the equilibrium strategy profile for the single-type bargaining game is characterized by the value of T_0 and $F^i_{\alpha_1,\alpha_2}(0)$, i $= 1, 2$.

From this, we obtain the payoffs of the Players in the equilibrium of the game as follows:

$$u_i = F^j_{\alpha_1,\alpha_2}(0) \cdot \alpha_i + (1 - F^j_{\alpha_1,\alpha_2}(0)) \cdot (1 - \alpha_j)$$

3.4 Other Properties of the equilibrium

3.4.1 Inefficiency

An essential property of this equilibrium is that it has an inefficiency associated with it. By this, we mean that the equilibrium payoff is not necessarily the maximum amount a player could obtain at any step of the game. This means that part of the surplus being split between the players is simply lost. With the help of an illustration, we show this inefficiency and also show that this inefficiency is a consequence of a 'delay' in the equilibrium.

Consider a symmetric model such that $r_1 = r_2$, $\alpha_1 = \alpha_2$ and $z_1 = z_2$. This would mean that $\lambda_1 = \lambda_2$, thus implying $F^1_{\alpha_1,\alpha_2}(0) = F^2_{\alpha_1,\alpha_2}(0) = 0$ at equilibrium. From the previous result, this implies that the expected payoff for Player 1 would be $(1-\alpha_1)$ if they are rational since the best strategy for a rational player would be to concede at time $t = 0$. However, the expected payoff of an irrational Player 1 would be lesser than that of $(1 - \alpha_1)$, as they would not immediately concede and allow the game to move on. Thus the expected payoff for Player 1, in general, would be lesser than $(1 - \alpha_1)$. This occurs due to the fact that the probability of reaching an agreement where both players immediately concede has a probability of z^2, which is extremely small as $z < 1$. Thus this inefficiency arises due to the fact that there is a delay in reaching an agreement.

3.4.2 Strength of Players and other Interpretations

The parameters T_i and T_j can also be interpreted as measurements for comparison of strength between the players. This is because as seen, the one who concedes with positive probability at time $t = 0$ is decided by r_i and z_i. Thus, an interpretation of the game in equilibrium could be that of a race between the two players - whoever reaches reputation 1 first being the winner. Thus, if $T_i > T_j$, then Player j wins the race. However, in the equilibrium, we know that both players achieve reputation 1 at the same time. Thus, we interpret that this is achieved through a boost at the very beginning by Player i, by having concession as a possible strategy at the start. This is showcased in Figure 3.1. Here at the very start, we see that there is a huge jump in the reputation of Player 2. With the parameters as shown in the figure, we find that $T_2 > T_1$. We can say that the player with lesser T_i has a greater bargaining strength.

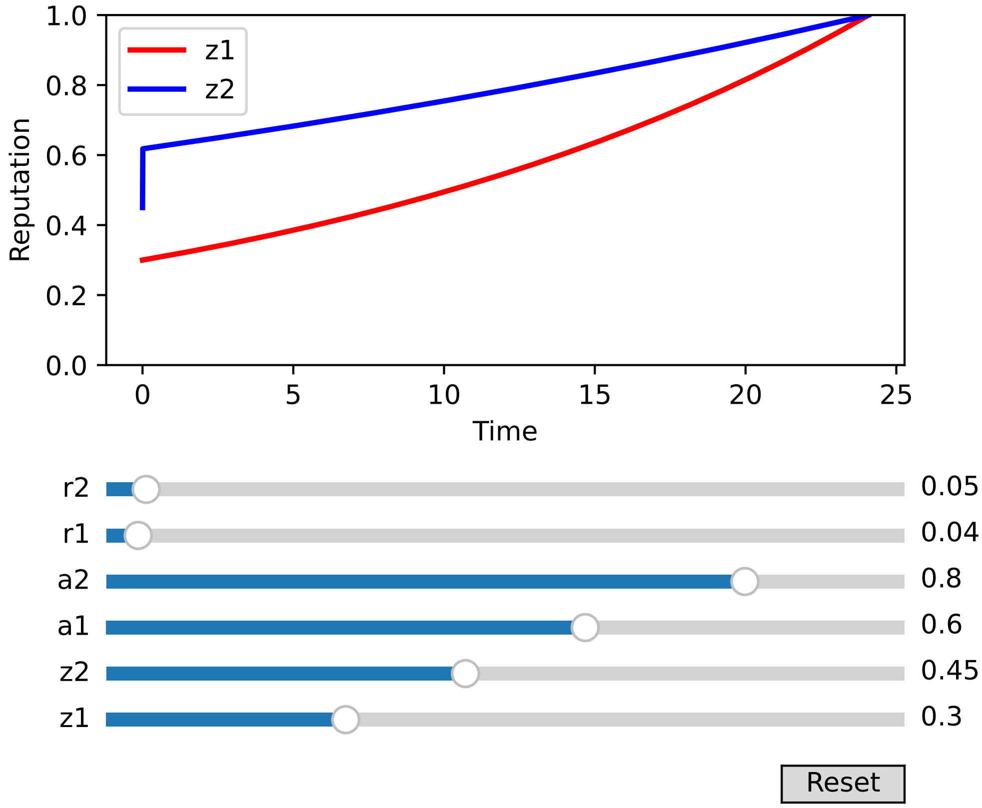

Figure 3.1: A graph representing the change in the reputation of the players with time. A point to note is the jump in reputation for Player 2 (represented by blue) at time 0.

However, as mentioned earlier, z_i is not the only term that makes up T_i. When $T_i > T_j$, we have

$$\frac{\ln z_i}{\ln z_j}\frac{r_i}{r_j}\frac{1-\alpha_j}{1-\alpha_i} > 1$$

Thus not only does the initial reputation decide who can win the race, but also r_i and α_i. From the inequation above, we can see that the players can compensate by having a smaller r_i, which translates to having more patience. They can also make their offer smaller to compensate and catch up.

Another observation that can be made is that while the concession rates affect the time of reaching reputation one linearly, the initial reputation affects it logarithmically.

3.5 Equilibrium with Multiple Types

We now build upon the model and results obtained in the previous section in this current section. We extend the existing model by allowing players multiple commitment(irrational) types. In order to do this, we first examine the case where one player has only one possible type and the other player has any number and then examine the case where both players are allowed to have multiple types. This allows us to explore a more general version of the problem from the previous section, thus allowing us to make some of the properties from the previous section more robust. The significant results from the analysis of this further generalized game are equilibrium properties that are independent of the probability distributions π_i. The significance of these results is that this distribution is an exogenous one. Thus, the results turn out to be independent of the exogenous factors of the game.

For the multiple-type game, we have the following result:

Proposition 3.5.1. *For any bargaining game* $B = \left\{(C^i, z^i, \pi^i, r^i)_{i=1}^2\right\}$ *a sequential equilibrium* $\left(\left(\mu^1, F^1_{\alpha^1,\alpha^2}\right), \left(\mu^2_{\alpha^1}, F^2_{\alpha^1,\alpha^2}\right)\right)$ *exists. Furthermore, all equilibria yield the same distribution over outcomes.*

We give the intuition of the proof for the above proposition. As outlined earlier, we begin by examining the case where Player 1 has only one commitment type and Player 2 has multiple types. We represent the probability that Player 1 is irrational by $p \in (0,1)$. We have $C_1 = \{\alpha_1\}$. Thus, $\mu_1(\alpha_1) = 1$. Thus, at time $t = 0$, Player 1 will make the offer α_1. Player 2 has to possible strategies to play: concede at time 0 with a non-zero probability (play D) or play some $\alpha_2 \in C_2$, where $\alpha_1 + \alpha_2 > 1$. Similar to the previous model, there is a constant rate of concession for both players at all times given by $\lambda_i = \frac{r_j \cdot (1-\alpha_i)}{(\alpha_j + \alpha_i - 1)}$. Let us assume that Player 1 does not concede at $t = 0$. Then we know that Player 1's reputation reaches 1 at time $T_1 = \frac{(-\log x)}{\lambda_1}$. Similarly for Player 2 we have $T_2 = -\frac{1}{\lambda_2} \log \frac{z_2 \pi_2(\alpha_2)}{z_2 \pi_2(\alpha_2) + (1-z_2)\mu^2_{\alpha_1}(\alpha_2)}$.

In order for Player 2 to be able to choose to make an offer α_2 at $t = 0$, it is obvious that $T_1 \geq T_2$. If this inequality is strict, then player 1 must concede with positive probability at time 0 so that conditional on not conceding the probabilities of irrationality reach 1 for both agents simultaneously. That is,

$$\frac{1}{\lambda_1}\log\frac{z_1}{z_1+(1-z_1)(1-d)} = \frac{1}{\lambda_2}\log\frac{z_2\pi_2(\alpha_2)}{z_2\pi_2(\alpha_2)+(1-z_2)\mu^2_{\alpha_1}(\alpha_2)}$$

Here, d is the probability that Player 1 concedes at time 0. Given that the utility expression follows from the previous section as well, it is clear that the payoff of Player 2 if they are of type α_2 is increasing in d. This means that Player 2 obtains a payoff greater than $1-\alpha_1$ whenever $d>0$. If the strategy of Player 2 is to be an equilibrium strategy, it is clear that this must be a mixed strategy equilibrium. This means that the probabilities would have to be distributed so that for every type present in the strategy profile, the resulting payoff must be the same. Due to this, the distribution and hence the strategy become unique.

3.6 Relation between the Rubinstein model and the Current Model

Proposition 3.6.1. *For any bargaining game* $B_n = \left\{(C_i, z^i_n, \pi_i, r_i)^2_{i=1}\right\}$ *be a sequence of continuous-time bargaining games.*

If (a) $\lim z^1_n = \lim z^2_n = 0$, $\lim \frac{z^1_n}{z^1_n+z^2_n} \in (0,1)$ *and (b)* v^i_n *is the sequential equilibrium payoff for Player* i *in the game* B_n $\left(\text{where } v_i = \frac{r_1}{r_i+r_j}\right)$*, then* $\lim inf\, v^i_n \geq \underline{v}_i$*.* $(\underline{v}_i max\{\alpha \in C_i \cup \{0\} | \alpha < v_i\})$

Proof. Without loss of generality, assume that $\left(\mu^1_n, \mu^2_{\alpha_2,n}\right)$ converges to some $\left(\mu^1, \mu^2_{\alpha_2}\right)$.

Assume the following:

1. $\mu_1(\alpha_1) > 0$

2. $\alpha_1 > \frac{r_2}{r_1+r_2}$

3. $\alpha_2 < \frac{r_1}{r_1+r_2}$

Consider the situation where α_1, α_2 are the demands by the players at the time $t = 0$.

Let p_1 be the conditional probability that Player 1 does not concede to Player 2. Then, we know that p_1 solves:

$$\frac{\log \bar{\pi}_1(\alpha_1)}{r_2(1-\alpha_1)} = \frac{\log \bar{\pi}^2_{\alpha_1}(\alpha_2)}{r_1(1-\alpha_2)}$$

where

$$\bar{\pi}_1(\alpha_1) = \frac{z_1 \pi_1(\alpha_1)}{z_1 \pi_1(\alpha_1) + (1-z_1) p_1} \quad \text{and}$$
$$\bar{\pi}^2_{\alpha_1}(\alpha_2) = \frac{z_2 \pi_2(\alpha_2)}{z_2 \pi_2(\alpha_2) + (1-z_2) \mu^2_{\alpha_2}(\alpha_2)}$$

that is

$$\frac{\gamma_1}{\gamma_2} = \frac{\log\left(1 + \frac{(1-z^1)p_1}{z^1 \pi_1(\alpha_1)}\right)}{\log\left(1 + \frac{(1-z_2)\mu_{\alpha_2}(\alpha_2)}{z_2 \pi_2(\alpha_2)}\right)} \quad \text{where} \quad \gamma_i = \frac{r_i}{1-\alpha_i}$$

From assumption 2 and 3, $\gamma_1/\gamma_2 > 1$.

But since $\mu_1(\alpha_1) > 0$ and z_1 and z_2 are converging to 0 at the same rate, p_1 must converge to 0 as well. If the conditional probability of 1 not conceding after (α_1, α_2) is realized, is going to 0, the unconditional probability of conceding must go to $\mu_1(\alpha_1)$.

Now, Player 1 must accept the offer of Player 2 with probability $\mu_1(\alpha_1)$.

Thus, by choosing any $\alpha_2 < r_1/(r_1 + r_2)$ player 2 can guarantee that his opponent concedes immediately if he is rational and has initially demanded $\alpha_1 > r_2/(r_1 + r_2)$. If 1 has demanded $\alpha_1 < r_2/(r_1 + r_2)$, then 2 can guarantee at least $r_1/(r_1 + r_2)$ by accepting this demand. Hence 2 can guarantee a payoff of

$$\underline{v}_2 = \max\left\{\alpha_2 \in C_2 \mid \alpha_2 < \frac{r_1}{r_1 + r_2}\right\}$$

A similar argument establishes that player 1 can guarantee $\underline{v}_1$.

□

From the above result, it is clear that if the space of commitment types is sufficiently rich for both players, then Player i's equilibrium approaches the limit of $\underline{v}_1$, which would be $\frac{r_2}{r_1+r_2}$, the payoff obtained at the equilibrium by the same player in the Rubinstein bargaining model. Thus, we see that filling up the strategy space and making offers as frequently as possible eliminates the delay inherent to this model.

Chapter 4

Conclusion

This thesis has covered the study of the phenomenon of bargaining under the framework of game theory, in its many realms and approaches. The equilibrium payoffs discussed in all of these models are essentially the ideal payoffs in those particular types of games. However, as seen throughout this thesis, it is also important to note that the Nash Solution provides the most ideal solution to a bargaining problem, with the equilibrium only depending on the players' preferences and not going into any details of the negotiation processes while maintaining complete information throughout. We see how changing the situation would call for changes in the models with the K-S solution and the Rubinstein model. Finally, we explored the model of reputational bargaining and the results of the initial paper from Abreu and Gul. Thus, throughout this thesis, we have built up on the fundamentals of the field, placing us in a position to understand recent developments.

This thesis, however, does not include many applications and extensions of reputational bargaining and the apparent future background would be to gain relevant fundamental knowledge in the field of Economics and begin exploring extensions of reputational bargaining, which is where the field is currently located.

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