The second moment of a certain pair correlation function for Sato-Tate sequences

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by

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Om ajñāna-timirāndhasya jñānāñjana-śalākayā

chakṣur unmīlitaṁ yena tasmai śrī-gurave namaḥ

(Skanda Puranam, Brahma Samhita, Uttara Khandam)

I offer my respectful obeisances unto the universal teacher,

who with the torchlight of knowledge has opened my eyes,

which were blinded by the darkness of ignorance.

Certificate

Certified that the work incorporated in the thesis entitled "*The second moment of a certain pair correlation function for Sato-Tate sequences*", submitted by *Jewel Mahajan* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

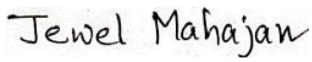
Date: November 5, 2023

Dr. Kaneenika Sinha

Thesis Supervisor

Declaration

I declare that this written submission represents my ideas in my own words and where others' ideas have been included, I have adequately cited and referenced the original sources. I also declare that I have adhered to all principles of academic honesty and integrity and have not misrepresented or fabricated or falsified any idea/data/fact/source in my submission. I understand that violation of the above will be cause for disciplinary action by the institute and can also evoke penal action from the sources which have thus not been properly cited or from whom proper permission has not been taken when needed.



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Abstract

In [BS19], Balasubramanyam and Sinha derived the first moment of the pair correlation function for Hecke angles lying in small subintervals of $[0, 1]$, as one averages over large families of Hecke newforms of weight k with respect to $\Gamma_0(N)$. The goal of this thesis is to study the second moment of this pair correlation function. We also record estimates for lower order error terms in the computation of the second moment and show that the variance goes to 0 under the same growth conditions on weights and levels for the families of Hecke newforms as required for the convergence of the first moment.

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Statement of Originality

The main results of this thesis which constitute original research are Theorems 6.5.15, 6.5.16 and 6.5.17. The content of Chapter 6 is original.

In Chapter 2, we explicitly write down some proofs of the theorems which were known before, but had not been clearly explained in the literature. These include Lemmas 2.1.5, 2.1.7, Corollaries 2.1.6, 2.1.8 and Theorem 2.1.9 in this chapter.

In Chapter 3, we review well-known results about the dimension formula for spaces of modular cusp forms and the Eichler-Selberg trace formula for the Hecke operators acting on these spaces. We recall and present these results systematically in a manner suitable for application to the original results in this thesis.

In Chapter 4, we present Propositions 4.5.2 and 4.5.3 as immediate applications of previously known results in the literature. The results mentioned in Section 4.4 are explicitly presented for the first time in this thesis.

Notations

Symbol	What it represents
$\mathbb{N}$	The set of all positive integers
$\mathbb{Z}$	The ring of all integers
$\mathbb{Q}$	The field of rational numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{C}$	The field of complex numbers
Re z (or, Im z)	The real part (or, imaginary part) of a complex number z
$\lvert z\rvert$ (or, $\bar{z}$)	The absolute value (or, conjugate)of a complex number z
$\mathbb{H}$	The complex upper half-plane
$M_n(\mathbb{Z})$	The group of all $n \times n$ integer matrices
$SL_2(\mathbb{Z})$	The group of all 2×2 integer matrices of determinant 1
$GL_2^+(\mathbb{R})$	The group of all 2×2 real matrices with positive determinant
I_2	The 2×2 identity matrix
(a, b)	The greatest common divisor of two natural numbers a and b
$M_k(N)$	The space of modular forms of weight k and of level N
$S_k(N)$	The space of cusp forms of weight k and of level N
$S_k^{new}(N)$	The space of primitive modular cusp forms of weight k and of level N
$\mathcal{F}_{N,k}$	The space of all Hecke newforms of weight k and level N
$\sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n$	The Fourier expansion of a newform $f \in \mathcal{F}_{N,k}$
$a_f(n)$	The normalised n-th Fourier coefficient of a newform $f \in \mathcal{F}_{N,k}$
T_m	The m-th Hecke operator on the space $M_k(N)$ (or, $S_k(N)$)
$a \mid b$	a divides b
$a \nmid b$	a does not divide b
$\lfloor x \rfloor$	The greatest integer $n \leq x$
$\#M$ or $\lvert M\rvert$	The number of elements in the set M
$\lvert[a, b]\rvert$	The length of the interval $[a, b]$, i.e., $b - a$
$\pi(x)$	The number of primes less than or equal to x
$\pi_N(x)$	The number of primes less than or equal to x, which are coprime to N
$\nu(n)$	The number of distinct prime divisors of n
$\mu(n)$	The Möbius function
$d(n)$ or $\sigma_0(n)$	The number of distinct positive divisors of n
$\sigma_1(n)$	The sum of positive divisors of n
p	A prime number
$\nu_p(n)$	$\max\{e \in \mathbb{Z} : p^e \text{ divides } n\}$
$\psi(n)$	$n \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right)$
$\phi(n)$	$n \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right)$
$C^{\infty}(X)$	The space of functions on X having continuous derivatives of all orders
$C_c(X)$	The space of compactly supported continuous functions on X

Whenever the following notations are used in the thesis, they will bear the meaning explained below.

- Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions with $g \neq 0$. We write $f \sim g$ to mean $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$.
- Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions with $g \neq 0$. We write $f = \mathrm{o}(g)$ to mean $\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$.
- Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions with $g(x) > 0$, for all $x \in \mathbb{R}$. We say $f = \mathrm{O}_K(g)$ (or, $f \ll_K g$), if there exists a positive constant $c = c(K)$, depending only on K such that $|f(x)| \leq c(K)g(x)$, for all x.

  If the implied constant c is absolute, then we simply write $f = \mathrm{O}(g)$ (or, $f \ll g$).
- A function $f \in C^\infty(X)$ is called a smooth function on X.
- For a function $U : \mathcal{F}_{N,k} \to \mathbb{C}$, we define

$$\langle U(f) \rangle := \frac{1}{|\mathcal{F}_{N,k}|} \sum_{f \in \mathcal{F}_{N,k}} U(f).$$

- By $\mathcal{F}_{N,k}$, we denote an orthogonal basis of $S_k^{new}(N)$ consisting of normalised Hecke eigenforms (See Definition 3.1.55).
- Let $\{p_1, p_2, \cdots, p_t\}$ be a finite set of primes. We denote $\sum_{\substack{p_1, \cdots, p_t \leq x \text{ all distinct} \\ (p_1,N)=1, \cdots, (p_t,N)=1}}$ by $\sum'_{p_1, \cdots, p_t \leq x}$.
- Let $\mathbb{N}_0$ be the set of non-negative integers, and for any $n \in \mathbb{N}_0$, let $\mathbb{N}_0^n := \underbrace{\mathbb{N}_0 \times \cdots \times \mathbb{N}_0}_{n \text{ times}}$ be the n-fold Cartesian product. The Schwartz space (or Schwartz class) or space of rapidly decreasing functions on $\mathbb{R}^n$ is the function space

$$\mathcal{S}(\mathbb{R}^n) := \left\{ f \in C^\infty(\mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}^n, \sup_{x \in \mathbb{R}^n} \left| x^\alpha \left(D^\beta f \right)(x) \right| < \infty \right\},$$

  where $C^\infty(\mathbb{R}^n)$ is the function space of smooth functions on $\mathbb{R}^n$, and sup denotes the supremum, and we used multi-index notation, i.e. $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ and $D^\beta := \partial_1^{\beta_1} \partial_2^{\beta_2} \ldots \partial_n^{\beta_n}$.

Organisation of the Thesis

The thesis contains seven chapters. In Chapter 1, we review some basic notions of equidistribution and some equivalent conditions for a sequence to be equidistributed mod 1.

Chapter 2 is devoted to the study of the level spacing distribution function and pair correlation function, or, more generally, k-level correlation of a sequence. We also see some equivalent definitions of these statistics.

In Chapter 3, we give a brief introduction to modular forms, and review some of their properties necessary to understand the thesis problem. We also recall the Eichler-Selberg trace formula for the trace of a Hecke operator $T_n(N,k)$ acting on $S_k(N)$, for all n, obtain precise estimates for the terms appearing in the formula and make their dependence on n explicit. In Section 3.2.3, we recall the formula for the trace of a Hecke operator $T_n^{new}(N,k)$ acting on $S_k^{new}(N)$, for all n with $(n,N) = 1$ and similarly obtain precise estimates for the terms appearing in this formula. Many of the results in Section 3.2 are already available in the literature and some of them, although easily derivable from the available results are not explicitly mentioned in the literature. Hence, they have been written down explicitly for the benefit of the reader. In Section 3.3, we collect the properties of Hecke eigenvalues, which we will use frequently in our estimation.

In Chapter 4, we introduce the thesis problem and also share some motivation for considering the thesis problem. Section 4.1 is dedicated to a question of Katz and Sarnak, which is the primary motivation behind the thesis problem, and modify the question in terms of localized pair correlation function. In Section 4.2, we mention the result available on the first moment of smooth localized pair correlation function ([BS19]). In Section 4.3, we mention the main results of this thesis without proof.

The goal of Section 4.4 is to simplify the pair correlation function to express it in terms of characteristic functions, as mentioned in Theorem 4.4.3. In Section 4.5, we consider the smooth analogue of this pair correlation function by considering a special class of Schwartz class functions, since one can approximate the characteristic functions by functions from such a class (See [Hil22, Lemma 2.1]).

In Chapter 5, we revisit the result obtained for the first moment or average of the smooth localized pair correlation function in [BS19] and record a generalization. We also present the inequalities in a much clearer form so that the optimal choice for the parameters required in our results becomes clear to us.

In Chapter 6, we estimate the second moment of the smooth localized pair correlation function and calculate the variance from this. This is the original contribution of this thesis.

In Chapter 7, we mention some possible directions for future research.

In Appendix A, we give a quick reference to terms mentioned in Chapters 5 and 6 for the convenience of the reader. While reading Chapters 5 and 6, readers can keep track of notations by referring to the appendix whenever needed.

Contents

Chapter 1

Equidistribution and related results

1.1 Introduction

For a sequence of real numbers and any given interval, a common question to ask is whether the given interval contains any elements from the sequence and if it does, the very next question one can ask is how dense the elements of the sequence are in that given interval. Further, are the elements placed "uniformly" across the interval or are they likely to cluster around some specific points? How are the elements of the sequence spaced apart in that interval? To understand this for a given sequence of numbers in the unit interval, we recall the notion of uniform distribution modulo one and more generally, asymptotic distribution of sequences. We also recall the more sophisticated notions of level spacing distribution and pair correlation statistics.

1.2 Introduction to equidistribution

Definition 1.2.1. (Uniform distribution modulo one) *A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be equidistributed $\bmod 1$ or uniformly distributed modulo 1 (abbreviated u.d. $\bmod 1$), if for every pair of real numbers a, b with $0 \leq a < b \leq 1$, we have*

$$\lim_{N\to\infty} \frac{\#\left\{n \leq N : \{x_n\} \in [a,b]\right\}}{N} = b - a,$$

where $\{x_n\} := x_n - \lfloor x_n \rfloor$ denotes the fractional part of x_n.

In particular, a sequence $(x_n)_{n\geq 1}(\subseteq [0,1])$ is said to be uniformly distributed modulo 1 (or, simply uniformly distributed), if for every pair of real numbers a, b with $0 \leq a < b \leq 1$, we have

$$\lim_{N\to\infty} \frac{\#\left\{n \leq N : x_n \in [a,b]\right\}}{N} = b - a.$$

Remark 1.2.2. *For a u.d. $\bmod 1$ sequence of real numbers $(x_n)_{n\geq 1}$, we have*

$$\#\{1 \leq n \leq N : x_n = a\} = o(N), \text{ for each } a \in [0,1].$$

Examples of uniformly distributed modulo 1 sequences :

- (Bohl, Sierpiński and Weyl independently in 1909-1910) $(n\theta)_{n\geq 1}$ is u.d. mod 1 if $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- Let $P(x) = \sum_{k=0}^{m} \alpha_i x^i \in \mathbb{R}[x]$ $(m \geq 1)$ with at least one coefficient $\alpha_j (j > 0)$ in $\mathbb{R} \setminus \mathbb{Q}$. Then $(P(n))_{n\geq 1}$ is u.d. mod 1 (See [KN74, Theorem 3.2]).

- The sequence $0, \theta, -\theta, 2\theta, -2\theta, \ldots,$ is u.d. mod 1, if $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- The sequence $\frac{0}{1}, \frac{0}{2}, \frac{1}{2}, \frac{0}{3}, \frac{1}{3}, \frac{2}{3}, \ldots, \frac{0}{k}, \frac{1}{k}, \ldots, \frac{k-1}{k}, \ldots$ is u.d. mod 1.
- (Csillag, Fejér, 1930) $(\alpha n^\sigma)_{n\geq 1}$, where $\alpha \neq 0$, and $\sigma > 0$ with σ not an integer, is u.d. mod 1.
- $(\alpha n^\sigma (\log n)^\tau)_{n\geq 2}$, where $\alpha \neq 0, \tau \in \mathbb{R}$, and $\sigma > 0$ with σ not an integer, is u.d. mod 1 (Follows from Fejér's theorem).
- (Ivan Vinogradov, 1935) $(p_n\theta)_{n\geq 1}$ is u.d. mod 1, where p_n is the nth prime and $\theta \in \mathbb{R} \setminus \mathbb{Q}$.
- $(\log F_n)_{n\geq 1}$, where F_n is Fibonacci sequence, is u.d. mod 1 $\left(\text{Since} \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}\right)$.
- $(\sqrt{n})_{n\geq 1}$, $(\log n!)_{n\geq 1}$ are u.d. mod 1.
- Let $\log_k(x)$ be recursively defined by $\log_1 x = \log x$ and $\log_k x = \log_{k-1}(\log x)$ for $k \geq 2$. For each $k \geq 1$, $(n \log_k n)_{n\geq n_0(k)}$ is u.d. mod 1, where $n_0(k)$ is the smallest positive integer in the domain of $\log_k x$.

Examples of sequences which are not uniformly distributed mod 1 :

- $(n\theta)_{n\geq 1}$ (or, $(n^2\theta)_{n\geq 1}$) is not u.d. mod 1 if $\theta \in \mathbb{Q}$, (Weyl criterion) but $(\{n\theta\})_{n\geq 1}$ is everywhere dense in $[0,1]$.
- Let $P(x) = \sum_{k=0}^{m} \alpha_i x^i \in \mathbb{R}[x]$ $(m \geq 1)$ with all coefficients $\alpha_j (j > 0)$ in $\mathbb{Q}$. Then $(P(n))_{n\geq 1}$ is not u.d. mod 1.
- $(c \log n)_{n\geq 1}$ is not u.d. mod 1, where c is a constant, but it is dense mod 1 in $[0,1]$.
- The sequence $(n!e)_{n\geq 1}$ has only one limit point and hence it is not dense mod 1. Therefore, it is not u.d. mod 1.
- $(\log p_n)_{n\geq 1}$ is not u.d. mod 1, where p_n is the nth prime.
- $(\log \log n!)_{n\geq 1}$ is not u.d. mod 1.

Not-known : $(e^n)_{n\geq 1}$, $(\pi^n)_{n\geq 1}$, $((\frac{3}{2})^n)_{n\geq 1}$.

We now mention below a special sequence of particular importance, namely a van der Corput sequence which is uniformly distributed mod 1.

Definition 1.2.3. *Let $b \in \mathbb{N}$ and $b \geq 2$.*

- *The b-adic radical inverse function is defined as $\phi_b : \mathbb{N}_0 \to [0,1)$,*

$$\phi_b(n) = \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \cdots,$$

for $n \in \mathbb{N}_0$ with b-adic digit expansion $n = n_0 + n_1 b + n_2 b^2 + \cdots$, where $n_i \in \{0, 1, \cdots, b-1\}$. In other words, is the reflection of the b-adic digit expansion of n at the comma. for example, $n = (100.)_2$ implies $\phi_2(n) = (.001)_2$.

- *The van der Corput sequence in base b is defined as $(x_n)_{n\geq 0}$ with $x_n = \phi_b(n)$.*

Proposition 1.2.4. *The van der Corput sequence in base b is uniformly distributed* mod 1.

Proof. Let us fix $m \in \mathbb{N}$. For every $a \in \{0, 1, \cdots, b^m - 1\}$ with b-adic digit expansion $a = a_0 b^{m-1} + a_1 b^{m-2} + \cdots + a_{m-2} b + a_{m-1}$, we consider the so-called elementary interval in base b of the form $J_a = [\frac{a}{b^m}, \frac{a+1}{b^m})$. For $n \in \mathbb{N}_0$ with b-adic digit expansion $n = n_0 + n_1 b + n_2 b^2 + \cdots$, the element $x_n = \phi_b(n)$ belongs to J_a iff

$$\frac{a}{b^m} \leq \frac{n_0}{b} + \frac{n_1}{b^2} + \frac{n_2}{b^3} + \cdots < \frac{a+1}{b^m},$$

i.e.,

$$a \le Y + t < a + 1,$$

where $Y = n_0 b^{m-1} + n_1 b^{m-2} + \cdots + n_{m-1} \in \mathbb{N}_0$ and $t := \frac{n_m}{b} + \frac{n_{m+1}}{b^2} + \cdots \in [0,1)$. Hence, $a = Y$, which implies, $n_i = a_i$, for $0 \le i < m$. Therefore, $n \equiv a' \pmod{b^m}$, where $a' = a_0 + a_1 b + \cdots + a_{m-1} b^{m-1}$.

Since the congruence $x \equiv a' \pmod{b^m}$ has a unique solution mod b^m, it follows that exactly one of b^m consecutive elements of the van der Corput sequence belongs to J_a. Hence, for $N \in \mathbb{N}$, it holds that

$$\begin{aligned} &\#\{n \le N : \{x_n\} \in J_a\} \\ =&\#\{n \le N : \phi_b(n) \in J_a\} \\ =&\left\lfloor \frac{N}{b^m} \right\rfloor + \theta, \end{aligned}$$

with $\theta \in \{0, 1\}$. Therefore,

$$\lim_{N \to \infty} \frac{1}{N} \#\{n \le N : \{x_n\} \in J_a\} = \frac{1}{b^m} = \lambda(J_a),$$

where λ denotes the one-dimensional Lebesgue measure.

Arbitrary intervals $[a,b] \subseteq [0,1)$ are approximated from the interior and exterior by a finite union of intervals of the form J_a. Therefore, the same result also holds for general intervals of the form $[a,b]$. Therefore, the van der Corput sequence in base b is uniformly distributed mod 1. □

Remark 1.2.5. *In particular, taking $b = 2$, we obtain the sequence $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{5}{8}, \frac{3}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$ is uniformly distributed* mod 1. *It is interesting to note that the same set with lexicographic order, i.e., the sequence $0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \ldots$ is not u.d.* mod 1. *(Ex.* 4.7*, section* 4*, chapter* 2*,[KN74])*

We now mention the following results connecting uniform distribution and density.

Proposition 1.2.6. *The following statements are true.*

(a) *For a uniformly distributed sequence (x_n), the sequence of its fractional parts $(\{x_n\})$ is dense in* $[0,1]$.

(b) *Any dense sequence in* $[0,1]$ *has a rearrangement that is uniformly distributed* mod 1.

(c) *Any sequence has a rearrangement that is not uniformly distributed.*

Proof.

(a) We prove this by contrapositive statement. Suppose the sequence of fractional parts $(\{x_n\})$ of the sequence (x_n) is not dense in $[0,1]$. Then there is an open interval $(a,b) \subseteq [0,1]$, such that $(a,b) \cap \{\{x_n\} : n \in \mathbb{N}\} = \phi$. Let $d := \frac{a+b}{2}$ and $c := \frac{a+d}{2}$. Then

$$\lim_{N \to \infty} \frac{\#\{n \le N : \{x_n\} \in [c,d]\}}{N} = 0 \neq d - c,$$

proves that the sequence (x_n) is not uniformly distributed.

(b) The proof follows from Corollary 1.2.10, as a part of a more general theorem.

(c) Using (a), we note that if the sequence is not dense in $[0,1]$, none of its arrangements can be u.d. We now suppose, the sequence (x_n) is dense. Let $A := \{\{x_n\} : n \in \mathbb{N}\} \cap [0, 1/2]$ and $B := \{\{x_n\} : n \in \mathbb{N}\} \cap (1/2, 1)$. Both A and B are infinite because the sequence is dense. Let $Z := (z_n)$ be the new sequence after rearrangement of the sequence (x_n) such that $A = \{\{z_{10n}\} : n \in \mathbb{N}\}$. Since the numerator in the following limit is non-zero only when n

is a multiple of 10, considering all the 10 subsequences of the form z_{10N+i} ($i = 0, 1, \dots, 10$), we have

$$\lim_{N\to\infty} \frac{\#\{n \leq N : \{z_n\} \in [0, 1/2]\}}{N} = \frac{1}{10} \neq \frac{1}{2}.$$

Hence the new sequence $Z := (z_n)$ is not uniformly distributed.

□

Remark 1.2.7. *The converse of (a) in Proposition 1.2.6 need not be true. For example, one can take $x_n = \log n$ (or, $\sin n$). The sequence $(\{\log n\})$ (or, $(\{\sin n\})$) is dense in* $[0,1]$, *but* $(\log n)$ *(or,* $(\sin n)$ *) is not u.d.* mod 1 *(See Example* 2.4 *and Exercise* 2.7 *of [KN74])*

To prove $(\{\ln n\})_n$ is dense in $[0,1]$, we first claim that at least one of $\ln 2$ and $\ln 3$ is irrational, otherwise $2^p = 3^q$ for some integers p, q, a contradiction. Thus either $(\{n \ln 2\})_n$ or $(\{n \ln 3\})_n$ is dense in $[0,1]$ since $(n\theta)_{n\geq 1}$ is u.d. mod 1 for $\theta \in \mathbb{R}\setminus\mathbb{Q}$. Therefore, we have a subsequence $(\{\ln(2^n)\})_n$ or $(\{\ln(3^n)\})_n$ of $(\{\ln n\})_n$ which is dense in $[0,1]$, and so is $(\{\ln n\})_n$.

To show $\{\sin n\}$ is dense in $[0,1]$, it is enough to show that $\sin(\mathbb{N})$ is dense in $[-1,1]$. For this we first show that, $\sin(\mathbb{Z})$ is dense in $[-1,1]$. We know, any subgroup of $(\mathbb{R}, +)$ is either dense or cyclic. $\mathbb{Z} + \pi\mathbb{Z}$ is a non-cyclic subgroup of $(\mathbb{R}, +)$, and hence dense in $\mathbb{R}$. Since the function $x \to sin x$ is continuous, we deduce that

$$[-1,1] = \sin(\mathbb{R}) = \sin(\overline{\mathbb{Z}+\pi\mathbb{Z}}) \subseteq \overline{\sin(\mathbb{Z}+\pi\mathbb{Z})} = \overline{\sin(\mathbb{Z})}.$$

We now prove a general theorem regarding the rearrangement of a sequence to a uniformly distributed sequence. Let (X, d) be a compact metric space. For a sequence $\omega = (z_n)_{n\geq 1}$ in X, then let $A(\omega)$ denote the set of all accumulation points of ω, i.e., $x \in A(\omega)$ iff every neighbourhood of x contains infinitely many z_n's.

Lemma 1.2.8. *For any sequence $\omega = (z_n)_{n\geq 1}$ containing elements of X, there exists a sequence $(\omega_n)_{n\geq 1}$ containing elements of $A(\omega)$ with $\lim_{n\to\infty} d(z_n, \omega_n) = 0$.*

Proof. Since $A = A(\omega)$ is compact (a closed subset of a compact set is compact), we can define the function $f : X \to \mathbb{R}$, by $f(x) = \min_{a\in A} d(x, a)$, for all $x \in X$. The function f is continuous on X. For any $n \in \mathbb{N}$, there exists $\omega_n \in A$ with $f(z_n) = d(z_n, \omega_n)$. We now prove $\lim_{n\to\infty} f(z_n) = 0$ by contradiction. $\lim_{n\to\infty} f(z_n) \neq 0$ implies that there exists $\epsilon > 0$ such that $f(z_n) \geq \epsilon$ for infinitely many n. Since $S := \{x \in X : f(x) \geq \epsilon\}$ is compact set containing the sequence $\omega = (z_n)_{n\geq 1}$, ω has an accumulation point $z \in S$, i.e., $f(z) \geq \epsilon$. Also, $z \in A$, by definition, which implies $f(z) = d(z, z) = 0$, leading to a contradiction. □

Theorem 1.2.9. *(Harald Niederreiter,* 1984*) For any two sequences $\omega_1 = (x_n)_{n\geq 1}$ and $\omega_2 = (y_n)_{n\geq 1}$ in the compact metric space (X, d), the following are equivalent:*

(a) *There exists a permutation τ of* $\mathbb{N}$ *such that* $\lim_{n\to\infty} d(x_n, y_{\tau(n)}) = 0$.

(b) $A(\omega_1) = A(\omega_2)$.

Proof. The proof of (a) $\Rightarrow$ (b) is straightforward. We now prove (b) $\Rightarrow$ (a). Using the Lemma 1.2.8, we obtain that there exist sequences (x'_n) and (y'_n) in $A(\omega_1) = A(\omega_2)$ such that $\lim_{n\to\infty} d(x_n, x'_n) = 0$ and $\lim_{n\to\infty} d(y_n, y'_n) = 0$. Now, since x'_ns are in $A(\omega_2)$ and y'_ns are in $A(\omega_1)$, we can find strictly increasing sequences $(\alpha(n))$ and $(\beta(n))$ in $\mathbb{N}$ such that $\lim_{n\to\infty} d(x'_n, y_{\alpha(n)}) = 0$ and $\lim_{n\to\infty} d(y'_n, x_{\beta(n)}) = 0$. It follows that

$$\lim_{n\to\infty} d(x_n, y_{\alpha(n)}) = 0 \text{ and } \lim_{n\to\infty} d(y_n, x_{\beta(n)}) = 0. \tag{1.1}$$

Now, both $\alpha : \mathbb{N} \to \mathbb{N}$ and $\beta : \mathbb{N} \to \mathbb{N}$ are injections, we can apply Banach's theorem to get disjoint decompositions $\mathbb{N} = K_1 \cup K_2$ and $\mathbb{N} = L_1 \cup L_2$, for which $\alpha(K_1) = L_1$ and $\beta(L_2) = K_2$.

Then, the map $\tau : \mathbb{N} \to \mathbb{N}$ defined by $\tau(n) = \begin{cases} \alpha(n) & \text{if } n \in K_1 \\ \beta^{-1}(n) & \text{if } n \in K_2, \end{cases}$ is a bijection. Hence, using equation (1.1), we obtain $d(x_n, y_{\tau(n)}) \leq d(x_n, y_{\alpha(n)}) + d(x_n, y_{\beta^{-1}(n)}) \to 0$, as $n \to \infty$. □

The short proof of the above theorem was given by Lech Drewnowski in 1987. As a corollary to the above theorem, we obtain

Corollary 1.2.10. *The sequence $\omega = (z_n)$ of elements of X has a μ-u.d. rearrangement if and only if $A(\omega)$ contains the support of μ.*

Proof. A sequence $(z_n)_{n \geq 1}$ of elements of X is called μ-uniformly distributed (where μ is a Borel probability measure) iff

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(z_n) = \int_X f d\mu,$$

holds for all real-valued continuous functions f on X. Equivalently, $(z_n)_{n \geq 1}$ is μ-uniformly distributed iff

$$\omega_-(B) := \liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \chi_B(z_n) \geq \mu(B)$$

holds for all open sets B in X, where χ_B denoted the characteristic function of B. The support K of the measure μ in X is defined to be the set $K = \{x \in X : \mu(D) > 0 \text{ for all open neighbourhoods } D \text{ of } x\}$. Let $x \in K$ and V be a neighbourhood of x. Then there is an open neighbourhood U of x such that $U \subseteq V$. By definition, $\mu(U) > 0$ and hence $\omega_-(U) > 0$. We claim that $\chi_U(z_n) = 1$, for infinitely many values of n. If not, $\chi_U(z_n) = 0$ eventually, which implies $\omega_-(U) = 0$, a contradiction. Our claim implies, $\chi_U(z_n) = 1$, i.e., $\chi_V(z_n) = 1$, and hence, $z_n \in V$ for infinitely many values of n. Therefore, $K \subseteq A(\omega)$.

We now assume, $K \subseteq A(\omega)$. It follows from Theorems 1.3 and 2.2, Ch. 3 in [KN74], there is a μ-u.d. sequence $(y_n)_{n \geq 1}$ with all $y_n \in K$. We set $x_n = z_p$ for a square $n = p^2$, $p = 1, 2, \ldots$ and $x_n = y_n$ otherwise, we get a μ-u.d. sequence $\omega_1 = (x_n)_{n \geq 1}$ with $A(\omega_1) = A(\omega)$. Hence, using Theorem 1.2.9, we obtain that there exists a permutation τ of $\mathbb{N}$ such that $\lim_{n \to \infty} d(x_n, z_{\tau(n)}) = 0$ and this implies $(z_{\tau(n)})_{n \geq 1}$ is μ-uniformly distributed. □

We now recall the following equivalent criteria for a sequence to be uniformly distributed mod 1, including a well-known criterion of Weyl.

Theorem 1.2.11. *For a sequence $(x_n)_{n=1}^{\infty}$ of real numbers, the following are equivalent:*

(a) *The sequence $(x_n)_{n \geq 1}$ is uniformly distributed* mod 1.

(b) $\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \chi_{[a,b]}(\{x_n\}) = \int_0^1 \chi_{[a,b]}(x) dx$, *for all* $[a, b] \subseteq [0, 1]$.

(c) $\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f(\{x_n\}) = \int_0^1 f(x) dx$, *for all real-valued continuous functions f on* $[0, 1]$.

(d) $\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f(\{x_n\}) = \int_0^1 f(x) dx$, *for all complex-valued continuous functions f on* $[0, 1]$.

(e) $\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f(x_n) = \int_0^1 f(x) dx$, *for all complex-valued continuous functions f on* $\mathbb{R}$ *with period* 1.

(f) $\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} f(x_n) = \int_0^1 f(x) dx$, *for all complex-valued continuous functions* $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$.

(g) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} f(\{x_n\}) = \int_0^1 f(x)dx$, *for all* $f \in H$*, a countable dense subset in* $C([0,1],\mathbb{R})$*, the space of real-valued continuous functions on* $[0,1]$ *with the sup norm.*

(h) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} f(\{x_n\}) = \int_0^1 f(x)dx$, *for all Riemann integrable functions* f *on* $[0,1]$.

(i) *(Weyl criterion,* 1916*)* $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} e^{2\pi i h x_n} = 0$, *for all* $h \in \mathbb{Z}-\{0\}$.

(j) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} e^{2\pi i h x_n} = 0$, *for all* $h \in \mathbb{N}$.

Proof.

- $(a) \iff (b)$ follows from definition.
- $(b) \implies (c)$ follows from the following fact: For $\epsilon > 0$ and $f \in C([0,1],\mathbb{R})$, there exist step functions (finite $\mathbb{R}$-linear combinations of characteristic functions) f_1 and f_2 such that $f_1(x) \le f(x) \le f_2(x)$ and
$$0 \le \int_0^1 (f_2(x) - f_1(x))dx < \epsilon.$$
- $(c) \implies (b)$ follows from the following fact: For $\epsilon > 0$ and closed interval $[a,b] \subseteq [0,1]$, there exist continuous functions $f_1, f_2 \in C([0,1],\mathbb{R})$ such that $f_1(x) \le \chi_{[a,b]}(x) \le f_2(x)$ and
$$0 \le \int_0^1 (f_2(x) - f_1(x))dx < \epsilon.$$
Therefore, $(a) \iff (b) \iff (c)$.
- $(c) \implies (d)$, $(d) \implies (e)$, $(e) \implies (f)$, and $(f) \implies (d)$ and $(d) \implies (c)$ are obvious. Combining all the above, we obtain
$$(a) \iff (b) \iff (c) \iff (d) \iff (e) \iff (f).$$
- $(h) \implies (b)$ is obvious.
- $(b) \implies (h)$ follows from the following fact: For $\epsilon > 0$ and $f \in R([0,1])$, there exist step functions (finite $\mathbb{R}$-linear combinations of characteristic functions) f_1 and f_2 such that $f_1(x) \le f(x) \le f_2(x)$ and
$$0 \le \int_0^1 (f_2(x) - f_1(x))dx < \epsilon.$$
- $(c) \implies (g)$ is obvious.
- $(g) \implies (c)$ follows from the following fact: For $\epsilon > 0$ and $f \in C([0,1])$, there exist $\psi \in H$ such that $\|f-\psi\|_\infty < \epsilon$ (by density), i.e., $\sup_{x\in[0,1]} |f(x) - \psi(x)| < \epsilon$. Now we have
$$\begin{aligned}&\left|\int_0^1 f dx - \frac{1}{N}\sum_{n\le N} f(x_n)\right| \\ &\le \left|\int_0^1 (f-\psi)dx\right| + \left|\int_0^1 \psi dx - \frac{1}{N}\sum_{n\le N}\psi(x_n)\right| + \left|\frac{1}{N}\sum_{n\le N}(f(x_n) - \psi(x_n))\right| < 3\epsilon.\end{aligned}$$
The first term and the third terms on the right are both $< \epsilon$, whatever the value of N using $\|f-\psi\|_\infty < \epsilon$. The second term is also $< \epsilon$ for sufficiently large N using hypothesis.

- $(i) \implies (e)$ Let $\epsilon > 0$. By Weierstrass approximation theorem, there exists a trigonometric polynomial $\psi(x)$, i.e., a finite linear combination of functions of the type $e^{2\pi i h x}, h \in \mathbb{Z}$, with complex coefficients such that $\|f - \psi\|_\infty < \epsilon$. The rest of the proof is the same as calculations mentioned in $(g) \implies (c)$.
- $(e) \implies (i)$ For $h \in \mathbb{Z} \setminus \{0\}$, let us define $f_h(x) := e^{2\pi i h x}$. Then each f_h is a continuous 1 periodic function with $\int_0^1 f_h(x)dx = 0$. This completes the proof.
- $(i) \iff (j)$ follows upon taking complex conjugation.

  This completes the proof.

□

Remark 1.2.12. $(a) \implies (h)$ *does not hold if we replace 'Riemann integrable functions' with 'Lebesgue integrable functions'. In fact, for an arbitrary sequence of real numbers* $(x_n)_{n\geq 1}$, *one can construct a Lebesgue-measurable set* E *of* I, *considering the complement of the set determined by the range of the sequence* $(\{x_n\})_{n\geq 1}$, *such that* $\lambda(E) = 1$ *and*

$$\lim_{N\to\infty} \frac{\#\{n \leq N : \{x_n\} \in E\}}{N} = 0 \neq \lambda(E).$$

We now define the notion of asymptotic distribution functions mod 1, which can be considered as a generalization of the concept of uniform distribution.

Definition 1.2.13. *(Asymptotic distribution functions* mod 1*) A sequence* $(x_n)_{n=1}^\infty$ *of real numbers is said to have the* asymptotic distribution function $\mathbf{mod}\,1$ *(abbreviated a.d.f.* mod 1 *or simply a.d.f.)* $g(x)$ *if for every* $x \in [0,1]$, *we have*

$$\lim_{N\to\infty} \frac{\#\{n \leq N : \{x_n\} \in [0,x]\}}{N} = g(x).$$

Remark 1.2.14. *The function* g *is a non-decreasing function on* $[0,1]$ *with* $g(0) = 0$ *and* $g(1) = 1$.

Remark 1.2.15. *A sequence which is uniformly distributed* mod 1 *has asymptotic distribution function* $g(x) = x$.

Theorem 1.2.16. *For a sequence* $(x_n)_{n=1}^\infty$ *of real numbers, the following are equivalent:*

(a) *The sequence* $(x_n)_{n\geq 1}$ *has the continuous asymptotic distribution function* $g(x)$.

(b) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} \chi_{[a,b]}(\{x_n\}) = \int_0^1 \chi_{[a,b]}(x)dg(x)$, *for all* $[a,b] \subseteq [0,1]$ *and* g *is continuous.*

(c) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} f(\{x_n\}) = \int_0^1 f(x)dg(x)$, *for all real-valued continuous functions* f *on* $[0,1]$.

(d) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} f(\{x_n\}) = \int_0^1 f(x)dg(x)$, *for all complex-valued continuous functions* f *on* $[0,1]$.

(e) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} f(x_n) = \int_0^1 f(x)dg(x)$, *for all complex-valued continuous functions* f *on* $\mathbb{R}$ *with period* 1.

(f) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\leq N} f(x_n) = \int_0^1 f(x)dg(x)$, *for all complex-valued continuous functions* $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$.

(g) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} f(\{x_n\}) = \int_0^1 f(x)dg(x)$, *for all* $f \in H$, *a countable dense subset in* $C([0,1],\mathbb{R})$, *the space of real-valued continuous functions on* $[0,1]$ *with the sup norm.*

(h) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} f(\{x_n\}) = \int_0^1 f(x)dg(x)$, *for all Riemann integrable functions* f *on* $[0,1]$.

(i) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x}dg(x)$, *for all* $h \in \mathbb{Z}\setminus\{0\}$.

(j) $\lim_{N\to\infty} \frac{1}{N}\sum_{n\le N} e^{2\pi i h x_n} = \int_0^1 e^{2\pi i h x}dg(x)$, *for all* $h \in \mathbb{N}$.

Proof. The proofs are similar to the proofs mentioned in Theorem 1.2.11 where we replace the Riemann integration with Riemann-Stieltjes integration w.r.t the non-decreasing function $g(x)$. □

Theorem 1.2.17. *[Wiener – Schoenberg, 1928] The sequence* $(x_n)_{n=1}^{\infty}$ *of real numbers has a continuous a.d.f. if and only if for all* $m \in \mathbb{N}$, *the limit*

$$a_m := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i m x_n}$$

exists and

$$\sum_{m=1}^{N} |a_m|^2 = o(N).$$

Proof. We refer the readers to Chapter 11 of [Mur01] for a proof. □

Chapter 2

Level spacing statistics

2.1 Introduction

Once we know that a sequence is uniformly distributed, we can investigate some finer statistics and compare them with those of suitable random models. In this section, we give definitions of some of those finer level spacing statistics, namely the level spacing distribution function and pair correlation function. Also, there are other level spacing statistics, namely maximal gap and minimal gap statistics. We exclude their discussion from this chapter.

We begin with the definition of the level-spacing distribution function of a sequence. Let $(x_n)_{n\geq 1}$ be a uniformly distributed mod 1 sequence in the unit interval $[0,1]$.

We consider the nearest-neighbour spacings of the sequence as follows:

We order the first N elements of the sequence as $x_{1,N} \leq x_{2,N} \leq \cdots \leq x_{N,N}$. The mean or average spacing between consecutive elements is $\sim 1/N$ as $N \to \infty$. We define the normalised spacings to be

$$\delta_n^{(N)} := N(x_{n+1,N} - x_{n,N}).$$

Definition 2.1.1. *We say that the sequence* $(x_n)_{n\geq 1} \subset [0,1]$ *has the* level spacing distribution function $P(s)$ *on* $[0,\infty)$ *if, for each interval* $[a,b] \subset [0,\infty)$,

$$\lim_{N\to\infty} \frac{1}{N} \# \{1 \leq n \leq N-1 : \delta_n^{(N)} \in [a,b]\} = \int_a^b P(s)ds.$$

Equivalently,

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{1 \leq n \leq N-1 : x_{n+1,N} - x_{n,N} \in \left[\frac{a}{N}, \frac{b}{N}\right]\right\} = \int_a^b P(s)ds.$$

Definition 2.1.2. *Let* $(x_n)_{n\geq 1} \subset [0,1]$ *be a sequence. We say,* $(x_n)_{n\geq 1}$ *has Poissonian level spacing distribution function if* $P(s) = e^{-s}$.

To compute the level-spacing distribution function of a sequence, we need to order the points first and then compute nearest-neighbour gaps, and this makes the analysis of proving some results about the level-spacing distribution function difficult. Hence, we consider the unordered spacings, where we look at the pairwise differences of all elements of the sequences (in the scale of the mean spacing), not just between nearest neighbours and define the pair correlation function of a sequence.

We now give the definition of the pair correlation function of a sequence by considering unordered spacings.

Definition 2.1.3. *Let $X = (x_n)_{n\ge 1}$ be a sequence in* $[0,1]$*. If the limit*

$$\lim_{N\to\infty}\frac{1}{N}\#\left\{(i,j):1\le i\ne j\le N, x_i-x_j\in\left[-\frac{s}{N},\frac{s}{N}\right]+\mathbb{Z}\right\}$$

exists for each $s>0$, the function $R_X:[0,\infty)\to\mathbb{R}$ defined by

$$R_X(s):=\lim_{N\to\infty}\frac{1}{N}\#\left\{(i,j):1\le i\ne j\le N, x_i-x_j\in\left[-\frac{s}{N},\frac{s}{N}\right]+\mathbb{Z}\right\}$$

is called the pair correlation function of the sequence X.

We also denote the pair correlation function by $R(s)$ if the implied sequence is clear to us.

Remark 2.1.4. *Although there are $N(N-1)$ pairs of differences, we only divide by N before taking the limit and not by N^2, in the definition of the pair correlation function. The reason is, if we fix i and try to count the number of pairs (i,j), we only expect one j within distance $\frac{1}{N}$ of x_i, or, in other words, we expect a bounded (i.e., $O(1)$) number of j's such that $x_j\in x_i+[-\frac{s}{N},\frac{s}{N}]$, since the average spacing between nearest neighbours is $\frac{1}{N}$. Summing over all N number of x_i's, we expect our count to be of size roughly $O(N)$.*

We now list some equivalent definitions of the pair correlation function in the theorem below, for which we need the following lemmas.

Lemma 2.1.5. *Let $s>0$ be fixed and $(t_{i,j})_{i\ge 1,j\ge 1}$ be a double sequence. There exists $N_0\in\mathbb{N}$ such that*

$$\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\chi_{[-s,s]}(N\{t_{i,j}\})=\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(t_{i,j}+m))$$

for all $N\ge N_0$.

Proof. There exists $N_0\in\mathbb{N}$ such that $\frac{s}{N_0}<\frac{1}{2}$. Therefore, for all $N\ge N_0$, $\frac{s}{N}\le\frac{s}{N_0}<\frac{1}{2}$.

We claim that if $m\ne 0$, then $\chi_{[-s,s]}(N(\{t_{i,j}\}-m))=0$ for all $N\ge N_0$, for all pairs (i,j) $(i\ne j)$ with $0\le\{t_{i,j}\}<\frac{1}{2}$.

If not, there exist a pair (i,j) $(i\ne j)$ with $0\le\{t_{i,j}\}<\frac{1}{2}$, such that $\chi_{[-s,s]}(N'(\{t_{i,j}\}-m))=1$ for some $m\ne 0$, and for some $N'\ge N_0$.

Therefore, for that specific pair (i,j),

$$\{t_{i,j}\}\in\left[m-\frac{s}{N'},m+\frac{s}{N'}\right]\implies\{t_{i,j}\}\in\left(m-\frac{1}{2},m+\frac{1}{2}\right)\cap\left[0,\frac{1}{2}\right)=\phi,$$

which is a contradiction. This proves our claim.

Therefore, for all $N\ge N_0$,

$$\begin{aligned}
&\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\chi_{[-s,s]}(N(\{t_{i,j}\}))\\
&=\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\sum_{0\ne m\in\mathbb{Z}}\chi_{\left[-\frac{s}{N},\frac{s}{N}\right]}(\{t_{i,j}\}-m)+\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\chi_{\left[-\frac{s}{N},\frac{s}{N}\right]}(\{t_{i,j}\}))\\
&=\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\sum_{m\in\mathbb{Z}}\chi_{\left[-\frac{s}{N},\frac{s}{N}\right]}(\{t_{i,j}\}-m)\\
&=\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(\{t_{i,j}\}+m))\\
&=\sum_{\substack{1\le i\ne j\le N\\ 0\le\{t_{i,j}\}<\frac{1}{2}}}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(t_{i,j}+m)).
\end{aligned}$$

□

Corollary 2.1.6. *Let $s > 0$ be fixed and $(t_{i,j})_{i\geq 1, j\geq 1}$ be a double sequence. There exists $N_0 \in \mathbb{N}$ such that*

$$\sum_{\substack{1\leq i\neq j\leq N\\ 0\leq \{t_{i,j}\}<\frac{1}{2}}} \chi_{[-s,s]}(N\{t_{i,j}\}) = \#\Big\{(i,j) : 1\leq i\neq j\leq N, 0\leq \{t_{i,j}\} < \frac{1}{2}, \{t_{i,j}\} \in \Big[-\frac{s}{N}, \frac{s}{N}\Big] + \mathbb{Z}\Big\}$$

for all $N \geq N_0$.

Lemma 2.1.7. *Let $s > 0$ be fixed and $(t_{i,j})_{i\geq 1, j\geq 1}$ be a double sequence. There exists $N_0 \in \mathbb{N}$ such that*

$$\sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \chi_{[-s,s]}(N(\{t_{i,j}\}-1)) = \sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \sum_{m\in\mathbb{Z}} \chi_{[-s,s]}(N(t_{i,j}+m)),$$

for all $N \geq N_0$.

Proof. Let $N_0 \in \mathbb{N}$ as mentioned in the proof of Lemma 2.1.5. Therefore, for all $N \geq N_0$, $\frac{s}{N} \leq \frac{s}{N_0} < \frac{1}{2}$.

We claim that if $m \neq 1$, then $\chi_{[-s,s]}(N(\{t_{i,j}\} - m)) = 0$ for all $N \geq N_0$, for all pairs (i,j) $(i \neq j)$ with $\frac{1}{2} \leq \{t_{i,j}\} < 1$.

If not, there exist a pair (i,j) $(i \neq j)$ with $\frac{1}{2} \leq \{t_{i,j}\} < 1$, such that $\chi_{[-s,s]}(N''(\{t_{i,j}\} - m)) = 1$ for some $m \neq 1$, and for some $N'' \geq N_0$.

Therefore, for that specific pair (i,j),

$$\{t_{i,j}\} \in \Big[m - \frac{s}{N''}, m + \frac{s}{N''}\Big] \implies \{t_{i,j}\} \in \Big(m - \frac{1}{2}, m + \frac{1}{2}\Big) \cap \Big[\frac{1}{2}, 1\Big) = \phi,$$

which is a contradiction. This proves our claim.

Therefore, for all $N \geq N_0$,

$$\begin{aligned}
&\sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \chi_{[-s,s]}(N(\{t_{i,j}\}-1)) \\
&= \sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \sum_{1\neq m\in\mathbb{Z}} \chi_{\left[-\frac{s}{N},\frac{s}{N}\right]}(\{t_{i,j}\}-m) + \sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \chi_{\left[-\frac{s}{N},\frac{s}{N}\right]}(\{t_{i,j}\}-1)) \\
&= \sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \sum_{m\in\mathbb{Z}} \chi_{\left[-\frac{s}{N},\frac{s}{N}\right]}(\{t_{i,j}\}-m) \\
&= \sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \sum_{m\in\mathbb{Z}} \chi_{[-s,s]}(N(\{t_{i,j}\}+m)) \\
&= \sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \sum_{m\in\mathbb{Z}} \chi_{[-s,s]}(N(t_{i,j}+m)).
\end{aligned}$$

□

Corollary 2.1.8. *Let $s > 0$ be fixed and $(t_{i,j})_{i\geq 1, j\geq 1}$ be a double sequence. There exists $N_0 \in \mathbb{N}$ such that*

$$\begin{aligned}
&\sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq \{t_{i,j}\}<1}} \chi_{[-s,s]}(N(\{t_{i,j}\}-1)) \\
&= \#\Big\{(i,j) : 1\leq i\neq j\leq N, \frac{1}{2} \leq \{t_{i,j}\} < 1, \{t_{i,j}\} \in \Big[-\frac{s}{N}, \frac{s}{N}\Big] + \mathbb{Z}\Big\},
\end{aligned}$$

for all $N \geq N_0$.

Theorem 2.1.9. *Let* $(x_n)_{n\geq 1} \subseteq [0,1]$ *be a sequence of real numbers,* $N \in \mathbb{N}$ *and* $s > 0$ *be fixed. Let*

$$R_2([-s,s],(x_n)_n,N) := \frac{1}{N}\#\left\{1 \leq i \neq j \leq N : x_i - x_j \in \left[-\frac{s}{N},\frac{s}{N}\right] + \mathbb{Z}\right\},$$

$$R_2'([-s,s],(x_n)_n,N) := \frac{1}{N}\sum_{1\leq i\neq j\leq N}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(x_i - x_j + m)),$$

$$R_2''([-s,s],(x_n)_n,N) := \frac{1}{N}\#\left\{1 \leq i \neq j \leq N : \|x_i - x_j\| \leq \frac{s}{N}\right\},$$

$$R_2'''([-s,s],(x_n)_n,N) := \frac{1}{N}\#\left\{1 \leq i \neq j \leq N : ((x_i - x_j)) \in \left[-\frac{s}{N},\frac{s}{N}\right]\right\},$$

where $((\cdot)) : \mathbb{R} \to [-\frac{1}{2},\frac{1}{2})$ *is the signed distance to the nearest integer, i.e.,*

$$((x)) = \begin{cases} \{x\} & \text{if } 0 \leq \{x\} < \frac{1}{2} \\ \{x\} - 1 & \text{if } \frac{1}{2} \leq \{x\} < 1, \end{cases}$$

and $\|\cdot\|$ *denotes the distance to the nearest integer function. If any of the above sequences is convergent as* $N \to \infty$*, all other sequences are also convergent, and they all converge to the same limit, i.e.,*

$$\lim_{N\to\infty} R_2([-s,s],(x_n)_n,N) = \lim_{N\to\infty} R_2'([-s,s],(x_n)_n,N) = \lim_{N\to\infty} R_2''([-s,s],(x_n)_n,N)$$

and

$$\lim_{N\to\infty} R_2([-s,s],(x_n)_n,N) = \lim_{N\to\infty} R_2'''([-s,s],(x_n)_n,N),$$

provided the limit exists in at least one case.

Proof. Let $N_0 \in \mathbb{N}$ as mentioned in Lemma 2.1.5, i.e., $2s < N_0$ and we write $t_{i,j} = x_i - x_j$. Hence, for any $N \geq N_0$, we have

$$\begin{aligned}
&R_2'''([-s,s],(x_n)_n,N) \qquad (2.1)\\
&= \frac{1}{N}\#\left\{1 \leq i \neq j \leq N : ((x_i - x_j)) \in \left[-\frac{s}{N},\frac{s}{N}\right]\right\}\\
&= \frac{1}{N}\#\left\{1 \leq i \neq j \leq N : N((t_{i,j})) \in [-s,s]\right\}\\
&= \frac{1}{N}\sum_{1\leq i\neq j\leq N}\chi_{[-s,s]}(N((t_{i,j})))\\
&= \frac{1}{N}\sum_{\substack{1\leq i\neq j\leq N\\ 0\leq\{t_{i,j}\}<\frac{1}{2}}}\chi_{[-s,s]}(N((t_{i,j}))) + \frac{1}{N}\sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq\{t_{i,j}\}<1}}\chi_{[-s,s]}(N((t_{i,j})))\\
&= \frac{1}{N}\sum_{\substack{1\leq i\neq j\leq N\\ 0\leq\{t_{i,j}\}<\frac{1}{2}}}\chi_{[-s,s]}(N\{t_{i,j}\}) + \frac{1}{N}\sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq\{t_{i,j}\}<1}}\chi_{[-s,s]}(N(\{t_{i,j}\} - 1))\\
&= \frac{1}{N}\sum_{\substack{1\leq i\neq j\leq N\\ 0\leq\{t_{i,j}\}<\frac{1}{2}}}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(t_{i,j} + m)) + \frac{1}{N}\sum_{\substack{1\leq i\neq j\leq N\\ \frac{1}{2}\leq\{t_{i,j}\}<1}}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(t_{i,j} + m))\\
&= \frac{1}{N}\sum_{1\leq i\neq j\leq N}\sum_{m\in\mathbb{Z}}\chi_{[-s,s]}(N(t_{i,j} + m))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N} \sum_{1 \le i \neq j \le N} \sum_{m \in \mathbb{Z}} \chi_{[-s,s]}(N(x_i - x_j + m)) \\
&= R_2'([-s,s],(x_n)_n,N),
\end{aligned}
$$

where we use Lemmas 2.1.5 and 2.1.7 in the last fourth line.

Using Corollaries 2.1.6 and 2.1.8, for any $N \ge N_0$, and for any $s > 0$, we have

$$
\begin{aligned}
&R_2([-s,s],(x_n)_n,N) \\
&= \frac{1}{N} \# \left\{ 1 \le i \neq j \le N : x_i - x_j \in \left[-\frac{s}{N}, \frac{s}{N} \right] + \mathbb{Z} \right\} \\
&= \frac{1}{N} \# \left\{ 1 \le i \neq j \le N : \{t_{i,j}\} \in \left[-\frac{s}{N}, \frac{s}{N} \right] + \mathbb{Z} \right\} \\
&= \frac{1}{N} \# \left\{ (i,j) : 1 \le i \neq j \le N, 0 \le \{t_{i,j}\} < \frac{1}{2}, \{t_{i,j}\} \in \left[-\frac{s}{N}, \frac{s}{N} \right] + \mathbb{Z} \right\} \qquad (2.2) \\
&+ \frac{1}{N} \# \left\{ (i,j) : 1 \le i \neq j \le N, \frac{1}{2} \le \{t_{i,j}\} < 1, \{t_{i,j}\} \in \left[-\frac{s}{N}, \frac{s}{N} \right] + \mathbb{Z} \right\} \\
&= \frac{1}{N} \sum_{\substack{1 \le i \neq j \le N \\ 0 \le \{t_{i,j}\} < \frac{1}{2}}} \chi_{[-s,s]}(N\{t_{i,j}\}) + \frac{1}{N} \sum_{\substack{1 \le i \neq j \le N \\ \frac{1}{2} \le \{t_{i,j}\} < 1}} \chi_{[-s,s]}(N(\{t_{i,j}\} - 1)) \\
&= R_2'([-s,s],(x_n)_n,N),
\end{aligned}
$$

where we use the last five lines of equation (2.1) in the last line.

We can show, for any $a > 0$, $((x)) \in [-a,a] \iff \|x\| \le a$.

This shows, for any $N \in \mathbb{N}$, and for any $s > 0$,

$$
R_2''([-s,s],(x_n)_n,N) = R_2'''([-s,s],(x_n)_n,N). \qquad (2.3)
$$

Combining equations (2.1), (2.2) and (2.3), we obtain that for all $N \ge N_0$,

$$
R_2([-s,s],(x_n)_n,N) = R_2'([-s,s],(x_n)_n,N) = R_2''([-s,s],(x_n)_n,N) = R_2'''([-s,s],(x_n)_n,N),
$$

and this completes the proof.

□

Remark 2.1.10. *The above definition of pair correlation function is in particular* 2*-level correlation function, where the definition of* k*-level correlation is given in the following way.*

Definition 2.1.11. *Let* $k \ge 2$*. Given a bounded set* $B \subset \mathbb{R}^{k-1}$*, we define* k*-level correlation of the sequence* $(x_n)_{n \ge 1}$ *as*

$$
\begin{aligned}
R_k(B,N) &:= \frac{1}{N} \# \{ i_1, \ldots, i_k \le N \text{ all distinct: } N(((x_{i_1} - x_{i_2})), ((x_{i_1} - x_{i_3})), \ldots, ((x_{i_1} - x_{i_k}))) \in B \} \\
&= \frac{1}{N} \sum_{\substack{i_1, \ldots, i_k \le N \\ \text{distinct}}} \chi_B(N((x_{i_1} - x_{i_2})), N((x_{i_1} - x_{i_3})), \ldots, N((x_{i_1} - x_{i_k}))).
\end{aligned}
$$

The following theorem shows that one can also consider the differences $((x_{i_1} - x_{i_2})), ((x_{i_2} - x_{i_3})), \ldots, ((x_{i_{k-1}} - x_{i_k}))$ in the definition instead of the differences $((x_{i_1} - x_{i_2})), ((x_{i_1} - x_{i_3})), \ldots, ((x_{i_1} - x_{i_k}))$.

Theorem 2.1.12. *Let* $(x_n)_{n \ge 1} \subset [0,1]$ *be a sequence and* $k \ge 2$*. The following are equivalent:*

(a) *For all test functions,* $f \in C_c(\mathbb{R}^{k-1})$*, we have*

$$
\lim_{N \to \infty} \frac{1}{N} \sum_{\substack{i_1, \ldots, i_k \le N \\ \text{all distinct}}} f(N((x_{i_1} - x_{i_2})), N((x_{i_1} - x_{i_3})), \ldots, N((x_{i_1} - x_{i_k}))) = \int_{\mathbb{R}^{k-1}} f(x)\, dx
$$

(b) *For all test functions,* $g \in C_c(\mathbb{R}^{k-1})$, *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{i_1,\dots,i_k\le N\\ \text{all distinct}}} g(N((x_{i_1}-x_{i_2})),N((x_{i_2}-x_{i_3})),\dots,N((x_{i_{k-1}}-x_{i_k}))) = \int_{\mathbb{R}^{k-1}} g(x)\,dx$$

(c) *For all rectangles* $B = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_{k-1},b_{k-1}], b_i > a_i, 1\le i\le k-1,$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{i_1,\dots,i_k\le N\\ \text{all distinct}}} \chi_B(N((x_{i_1}-x_{i_2})),N((x_{i_1}-x_{i_3})),\dots,N((x_{i_1}-x_{i_k}))) = \lambda(B),$$

where λ *denotes the* $(k-1)$ *dimensional Lebesgue measure.*

Proof. We refer the readers to [HZ23, Appendix A] for a proof. □

Definition 2.1.13. *Let* $(x_n)_{n\ge1}\subseteq[0,1]$ *be a sequence and* $k\ge2$. *We say, the sequence has Poissonian* k*-th order correlation if the sequence satisfies any of the above criteria mentioned in Theorem 2.1.12.*

Remark 2.1.14. *Let* $k\ge2$ *and* $(x_n)_{n\ge1}\subseteq[0,1]$ *has Poissonian* k*-th order correlation. We know, for any* $a>0$, $((x))\in[-a,a] \iff \|x\|\le a$. *Therefore, using condition* (c) *of Theorem 2.1.12 in the second last line, we obtain*

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\#\{i_1,\dots,i_k\le N \text{ all distinct}: \|x_{i_1}-x_{i_{r+1}}\|\le\frac{s_r}{N}, \text{ for all } 1\le r\le k-1\}\\
=&\lim_{N\to\infty}\frac{1}{N}\#\{i_1,\dots,i_k\le N \text{ all distinct}: N((x_{i_1}-x_{i_{r+1}}))\in[-s_r,s_r], \text{ for all } 1\le r\le k-1\}\\
=&\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{i_1,\dots,i_k\le N\\ \text{all distinct}}}\chi_{[-s_1,s_1]\times\cdots\times[-s_{k-1},s_{k-1}]}(N((x_{i_1}-x_{i_2})),N((x_{i_1}-x_{i_3})),\dots,N((x_{i_1}-x_{i_k})))\\
=&\lambda([-s_1,s_1]\times\cdots\times[-s_{k-1},s_{k-1}])\\
=&(2s_1)(2s_2)\cdots(2s_{k-1}).
\end{aligned}$$

Corollary 2.1.15. *Let* $(x_n)_{n\ge1}\subseteq[0,1]$ *be a sequence. The following are equivalent:*

(a) *For all test functions,* $f\in C_c(\mathbb{R})$, *we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i_1\ne i_2\le N} f(N((x_{i_1}-x_{i_2}))) = \int_{\mathbb{R}} f(x)\,dx$$

(b) *For all intervals* $[a_1,b_1], b_1>a_1$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i_1\ne i_2\le N}\chi_{[a_1,b_1]}(N((x_{i_1}-x_{i_2}))) = b_1-a_1.$$

(c) *For all intervals* $[-s,s], s>0$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i_1\ne i_2\le N}\chi_{[-s,s]}(N((x_{i_1}-x_{i_2}))) = 2s.$$

Proof. In view of Theorem 2.1.12, to prove this corollary, it is enough to show that condition (c) implies condition (b) (the other part is trivial). For this, we show $\#\{(i,j): i\ne j\le N, N((x_i-x_j))\in[0,s]\} = s\,o(N)$, for all $s>0$. Let $s>0$. We start with the following observations:

$$\{-x\} = 1-\{x\}, \text{ for } x\notin\mathbb{Z}, \tag{2.4}$$

and

$$((-x)) = \begin{cases} -((x)), & \text{if } \{x\} \in [0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1) \\ ((x)), & \text{if } \{x\} = \tfrac{1}{2}. \end{cases} \tag{2.5}$$

Using equations (2.4) and (2.5), we get

$$\begin{aligned} &\#\{(i,j) : i \neq j \leq N, \{x_i - x_j\} \neq \frac{1}{2}, N((x_i - x_j)) \in [-s,0)\} \\ =&\#\{(i,j) : i \neq j \leq N, 1 - \{x_j - x_i\} \neq \frac{1}{2}, -N((x_j - x_i)) \in [-s,0)\} \\ =&\#\{(j,i) : i \neq j \leq N, \{x_j - x_i\} \neq \frac{1}{2}, N((x_j - x_i)) \in (0,s]\} \\ =&\#\{(i,j) : i \neq j \leq N, \{x_i - x_j\} \neq \frac{1}{2}, N((x_i - x_j)) \in (0,s]\}, \end{aligned} \tag{2.6}$$

where we use the bijection $\#\{(i,j) : i \neq j \leq N, \cdots\} \leftrightarrow \#\{(j,i) : i \neq j \leq N, \cdots\}$ in the third line. For $N > 2s$, equation 2.1.9 gives

$$\begin{aligned} &\#\{(i,j) : i \neq j \leq N, \{x_i - x_j\} = \frac{1}{2}, N((x_i - x_j)) \in [-s,0)\} \\ =&\#\{(i,j) : i \neq j \leq N, \{x_i - x_j\} = \frac{1}{2}, -N/2 \in [-s,0)\} = 0. \end{aligned} \tag{2.7}$$

For $N > 2s$, adding equations (2.6)and (2.7), we obtain

$$\begin{aligned} &\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [-s,0)\} \\ =&\#\{(i,j) : i \neq j \leq N, \{x_i - x_j\} \neq \frac{1}{2}, N((x_i - x_j)) \in (0,s]\} \\ +&\#\{(i,j) : i \neq j \leq N, \{x_i - x_j\} = \frac{1}{2}, N((x_i - x_j)) \in (0,s]\} \\ =&\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in (0,s]\}, \end{aligned}$$

where the set in the third line is an empty set.

Using the given hypothesis, we obtain

$$\begin{aligned} 0 \leq &\lim_{N\to\infty} \frac{1}{N}\#\{(i,j) : i \neq j \leq N, ((x_i - x_j)) = 0\} \\ \leq &\lim_{N\to\infty} \frac{1}{N}\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [-\epsilon,\epsilon]\} = 2\epsilon, \text{ for any } \epsilon > 0, \text{ i.e.,} \end{aligned}$$

$$\lim_{N\to\infty} \frac{1}{N}\#\{(i,j) : i \neq j \leq N, ((x_i - x_j)) = 0\} = 0. \tag{2.8}$$

Therefore, for $N > 2s$,

$$\begin{aligned} &\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [-s,s]\} \\ =&\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [-s,0)\} + \#\{(i,j) : i \neq j \leq N, ((x_i - x_j)) = 0\} \\ +&\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in (0,s]\} \\ =&\#2\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [0,s]\} - \#\{(i,j) : i \neq j \leq N, ((x_i - x_j)) = 0\}, \end{aligned} \tag{2.9}$$

and hence, using equation (2.8) together with the hypothesis, we obtain

$$\begin{aligned} &\lim_{N\to\infty} \frac{1}{N}\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [0,s]\} \\ =&\frac{1}{2}\lim_{N\to\infty} \frac{1}{N}\#\{(i,j) : i \neq j \leq N, N((x_i - x_j)) \in [-s,s]\} = \frac{1}{2}(2s) = s, \text{ for any } s > 0, \end{aligned}$$

which proves condition (b). □

Condition (c) of Corollary 2.1.15 allows us to say that a sequence $(x_n)_{n\ge 1}\subset[0,1]$ has Poissonian 2-th order correlation (or, Poissonian pair correlation) if for any $s>0$, it satisfies

$$\lim_{N\to\infty}\frac{1}{N}\#\left\{1\le i\ne j\le N : ((x_i-x_j))\in\left[-\frac{s}{N},\frac{s}{N}\right]\right\}=2s.$$

Remark 2.1.16. *It is important to note that we cannot obtain a similar version of the equivalence of Condition* (c) *to any other conditions mentioned in the proof of Corollary 2.1.15 for higher order correlations ($k\ge 3$) since a similar symmetry argument cannot be used to obtain a generalization of* (2.9)*, i.e., for the rectangle $B=[0,s_1]\times\cdots\times[0,s_{k-1}]$, the following is not true in general:*

$$\frac{1}{N}\#\left\{i_1,\dots,i_k\le N \text{ all distinct}, N((x_{i_1}-x_{i_{r+1}}))\in[0,s_r]\ (1\le r<k)\right\}$$
$$\neq\left(\frac{1}{2}\right)^{k-1}\frac{1}{N}\#\left\{i_1,\dots,i_k\le N \text{ all distinct}, N((x_{i_1}-x_{i_{r+1}}))\in[-s_r,s_r]\ (1\le r<k)\right\}.$$

The following theorem shows having Poissonian pair correlation is stronger than uniform distribution mod 1.

Theorem 2.1.17. *Let $(x_n)_{n\ge 1}\subset[0,1]$ be a sequence. If $(x_n)_{n\ge 1}$ has Poissonian pair correlation, then $(x_n)_{n\ge 1}$ is uniformly distributed.*

Remark 2.1.18. *Theorem 2.1.17 was proved independently by Aistleitner, Lachmann and Pausinger [ALP18] and by Grepstad and Larcher [GL17].*

We can recover Poissonian level spacing distribution, if we have Poissonian correlations of all orders $k\ge 2$.

Theorem 2.1.19. *Let $(x_n)_{n\ge 1}\subset[0,1]$ be a sequence. If $(x_n)_{n\ge 1}$ has Poissonian correlations of all orders $k\ge 2$, then the sequence has Poissonian level spacing distribution function.*

Proof. We refer the readers to [KR99, Appendix A] for a proof. □

Chapter 3

Introduction to Modular forms

3.1 Preliminaries

The main goal of this thesis is to study pair correlation statistics in the context of modular forms. For this, we need a brief introduction to modular forms, which we give in this chapter. Modular forms play an essential role not only in Number Theory but also in other parts of Mathematics. Modular forms are used for the construction of Ramanujan graphs, cryptography, and coding theory and are often related to generating functions for partitions of an integer. In this section, we review the properties of modular forms necessary to understand the thesis problem and to tackle it. We refer the readers to [Miy89] and [MDG16] for a more detailed understanding of this topic.

3.1.1 Modular Forms

Let us consider the following matrix group, namely the special linear group over the set of integers $\mathbb{Z}$.

$$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \right\}.$$

For each positive integer N, we define

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

In particular, $\Gamma(1) = SL_2(\mathbb{Z})$. We call $\Gamma(N)$ the principal congruence modular group.

Definition 3.1.1. *A subgroup $\Gamma \subseteq SL_2(\mathbb{Z})$ is called a congruence subgroup if $\Gamma(N) \subseteq \Gamma$ for some $N \in \mathbb{N}$.*

If Γ is a congruence subgroup, the smallest N such that $\Gamma(N) \subseteq \Gamma$ is called the level of Γ.

Remark 3.1.2. *It can be shown that $\Gamma(N)$ has finite index in $SL_2(\mathbb{Z})$. Hence, every congruence subgroup Γ also has finite index in $SL_2(\mathbb{Z})$.*

Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the upper half-plane, considered as an open subset of $\mathbb{C}$ with the usual topology.

The group

$$GL_2^+(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R},\ ad - bc > 0 \right\}$$

acts on H via Möbius transformations (or fractional linear transformations):

$$\begin{aligned} GL_2^+(\mathbb{R}) \times H &\to H \\ \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) &\mapsto \frac{az+b}{cz+d}, \end{aligned} \tag{3.1}$$

i.e.,

$$\gamma \cdot z = \frac{az+b}{cz+d}.$$

Remark 3.1.3. *The action of* $GL_2^+(\mathbb{R})$ *on* H *extends by the same formula* (3.1) *to include its boundary* $\mathbb{R} \cup \{\infty\}$ *as follows:* $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, x\right) \mapsto \frac{ax+b}{cx+d}$, *for* $x \in \mathbb{R}$, *and* $\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \infty\right) \mapsto \frac{a}{c}$.

Definition 3.1.4. *A subgroup* Γ *of a topological group* G *is called a discrete subgroup if* Γ *is discrete with respect to the topology of* G.

Definition 3.1.5. *A non-scalar element* α *of* $GL_2^+(\mathbb{R})$ *is called elliptic, parabolic or hyperbolic, when it satisfies*

$$(tr\ (\alpha))^2 < 4\ det\ (\alpha),\ (tr\ (\alpha))^2 = 4\ det\ (\alpha), or, (tr\ (\alpha))^2 > 4\ det\ (\alpha),$$

respectively.

Definition 3.1.6. *Let* Γ *be a discrete subgroup of* $SL_2(\mathbb{R})$. *An element* $s \in \mathbb{R} \cup \{\infty\}$ *is called a cusp of* Γ, *if* $\gamma \cdot s = s$ *for some parabolic element* $\gamma \in \Gamma$.

Proposition 3.1.7. *Let* $\Gamma \subseteq SL_2(\mathbb{Z})$ *be a subgroup of finite index. Then the set of cusps of* Γ *is given by* $\mathbb{Q} \cup \{\infty\}$.

Proof. We refer the readers to [KL06, Proposition 3.5] for a proof. □

Definition 3.1.8. *Let* k *be an integer. A function* $f : H \to \mathbb{C}$ *is called a* **classical (or "elliptic") modular form** *of weight* k *with respect to the full modular group* $\Gamma = SL_2(\mathbb{Z})$, *if it satisfies the following conditions:*

(1) f *is holomorphic on* H.

(2) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$, *for all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, *and* $z \in H$.

(3) $f(z)$ *is bounded as* $Im\ (z) \to \infty$.

Remark 3.1.9. *Let* f *be a holomorphic function on* H *satisfying Condition* 2 *of the definition 3.1.8, i.e.,* $f(z+1) = f(z)$, *for all* $z \in H$. *This means* f *has a Laurent series expansion (See page* 3 *of [DS05] for a discussion), which we call Fourier series at* ∞ *:*

$$f(z) = \sum_{n=-\infty}^{\infty} a_f(n) q^n,\ q = e^{2\pi i z}.$$

Since $q \to 0$ *as* $Im\ z \to \infty$, *Condition* 3 *of Definition 3.1.8 ensures that* $a_f(n) = 0$ *for* $n < 0$. *Therefore, the Fourier series of* f *is in fact a power series :*

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n,\ q = e^{2\pi i z}.$$

Definition 3.1.10. *Let* f *be a holomorphic function on* H *satisfying condition* 2 *of Definition 3.1.8, and* $a_f(n) = 0$, *for* $n < 0$ *in the Fourier expansion (as mentioned above) of* f *at* ∞. *Then* f *is said to be* **holomorphic at** ∞.

Remark 3.1.11. *Some authors also prefer to give the definition of a modular form with the equivalent Condition* $(3)'$: f *is holomorphic at* ∞, *in place of condition* (3) *in the Definition 3.1.8.*

Remark 3.1.12. *Taking* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ *in Condition 2 of Definition 3.1.8, we get* $f = (-1)^k f$, *which shows that the only modular form with respect to the full modular group* $SL_2(\mathbb{Z})$ *of any odd weight* k, *is the zero function, but non-zero odd weight examples exist in more general contexts.*

Definition 3.1.13. *A classical modular form* $f : \mathbb{H} \to \mathbb{C}$ *of weight* k *with respect to the full modular group* $SL_2(\mathbb{Z})$ *is called a* cusp form *of weight* k *with respect to the full modular group* $SL_2(\mathbb{Z})$, *if* $a_f(0) = 0$ *in the Fourier series expansion of* f *at* ∞.

We now define modular forms for congruence subgroups Γ of $SL_2(\mathbb{Z})$.

For $k \in \mathbb{Z}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$, we introduce the following notation: for any holomorphic function f on $\mathbb{H}$,

$$(f[\gamma]_k)(z) := (\det \gamma)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right). \tag{3.2}$$

Remark 3.1.14. *Sometimes* $f|[\gamma]_k$ *is used in place of* $f[\gamma]_k$, *and it is usually called the "slash" notation because of the slash that is put, but we will use the above notation* $f[\gamma]_k$.

In the following definition, we define the notion of holomorphy at ∞ for a congruence subgroup, which we will need in condition (3) of the definition 3.1.16. We refer the readers to page 16 of [DS05] for a more detailed discussion.

Definition 3.1.15. *Let* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$. *Then* Γ *contains a matrix of the form* $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ *for some minimal* $h \in \mathbb{N}$. *Let* $f : \mathbb{H} \to \mathbb{C}$ *be a holomorphic function which satisfies* $f[\gamma]_k = f$ *for all* $\gamma \in \Gamma$. *Therefore,* $f(z+h) = f(z)$, *and hence* f *has a Laurent expansion.* f *is said to be* holomorphic at ∞ with respect to Γ, *if* f *has a Fourier series expansion:*

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q_h^n, \; q_h = e^{2\pi i z/h}.$$

Definition 3.1.16. *Let* k *be an integer and* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$. *A function* $f : \mathbb{H} \to \mathbb{C}$ *is called a* classical (or "elliptic") modular form *of weight* k *with respect to* Γ, *if it satisfies the following conditions:*

(1) f *is holomorphic on* $\mathbb{H}$.

(2) $f[\gamma]_k = f$ *for all* $\gamma \in \Gamma$.

(3) *For all* $\gamma \in SL_2(\mathbb{Z})$, $f[\gamma]_k$ *is holomorphic at* ∞ *with respect to* $\gamma^{-1}\Gamma\gamma$.

Remark 3.1.17. *Since* $g := f[\gamma]_k$ *is holomorphic on* $\mathbb{H}$ *and* $g[\alpha]_k = g$ *for all* $\alpha \in \gamma^{-1}\Gamma\gamma = \Gamma'$, *where* Γ' *is again a congruence subgroup of* $SL_2(\mathbb{Z})$, *its holomorphy at* ∞ *(as mentioned in Definition 3.1.15) is well defined.*

Definition 3.1.18. *Let* k *be an integer and* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$. *A classical modular form* $f : \mathbb{H} \to \mathbb{C}$ *of weight* k *with respect to* Γ *is called a* cusp form *of weight* k *with respect to* Γ, *if* $a_f(0) = 0$ *in the Fourier series expansion of* $f[\gamma]_k$ *at* ∞, *for all* $\gamma \in SL_2(\mathbb{Z})$.

Definition 3.1.19. *Let* k *be an integer and* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$. *The space of modular forms and cusp forms of weight* k *with respect to* Γ *are denoted by* $M_k(\Gamma)$ *and* $S_k(\Gamma)$.

In this thesis, we will only be dealing with the following special congruence subgroup, namely Hecke congruence subgroup of level N:

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; N \text{ divides } c \right\}.$$

Definition 3.1.20. *Let k and N be integers with $N \geq 1$. A classical modular form (or cusp form) of weight k with respect to $\Gamma_0(N)$ is called a modular form (or cusp form) of weight k and level N.*

Remark 3.1.21. *Let f be a modular form of weight k and level N. Then, $f[\gamma]_k$ is holomorphic at ∞ with respect to $\gamma^{-1}\Gamma_0(N)\gamma$, for all $\gamma \in SL_2(\mathbb{Z})$ (using Definition 3.1.16). In particular, taking $\gamma = I_2$, we have $f[I_2]_k = f$ is holomorphic at ∞ with respect to $\Gamma' = \Gamma_0(N)$.*

Therefore, using Definition 3.1.15 with the observation that $h = 1$ is the minimal natural number such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N) = \Gamma'$, we can say that if f is a modular form of weight k and level N, f has a Fourier series expansion:

$$f(z) = \sum_{n=0}^{\infty} a_f(n) q^n, \;\; q = e^{2\pi i z}.$$

Similarly, if f is a cusp form of weight k and level N, f has a Fourier series expansion with $a_f(0) = 0$:

$$f(z) = \sum_{n=1}^{\infty} a_f(n) q^n, \;\; q = e^{2\pi i z}.$$

Definition 3.1.22. *Let k and N be integers with $N \geq 1$. The space of modular forms and cusp forms of weight k and of level N are denoted by $M_k(N)$ and $S_k(N)$, i.e., $M_k(N) = M_k(\Gamma_0(N))$ and $S_k(N) = S_k(\Gamma_0(N))$.*

Remark 3.1.23. *It can be shown that both $M_k(N)$ and $S_k(N)$ are finite-dimensional complex vector spaces, where $S_k(N)$ is a vector subspace of $M_k(N)$ [CS17, Theorem 7.4.1].*

In the next few sections, we find a basis of eigenforms for the space $S_k(N)$. But for this, we need to make $S_k(N)$ an inner product space, which was first done by the German mathematician Hans Petersson and this inner product is called Petersson Inner Product.

3.1.2 Petersson Inner Product

We recall that the action of the group $GL_2^+(\mathbb{R})$ on $\mathbb{H}$ is given by equation (3.1).

Definition 3.1.24. *In the upper half plane $\mathbb{H}$, we define the hyperbolic measure*

$$d\mu(z) = \frac{dx\,dy}{y^2}, \quad z = x + iy \in \mathbb{H}.$$

We record the following two propositions whose proofs are straightforward.

Proposition 3.1.25. *The hyperbolic measure $d\mu$ is invariant under the action of $GL_2^+(\mathbb{R})$ on $\mathbb{H}$, i.e., for all $\alpha \in GL_2^+(\mathbb{R}), z \in \mathbb{H}$, $d\mu(\alpha \cdot z) = d\mu(z)$. Hence, $d\mu$ is also invariant under $\Gamma = SL_2(\mathbb{Z})$ and so under $\Gamma_0(N)$.*

Proposition 3.1.26. *Let $f, g \in M_k(\Gamma_0(N))$. Then $f(z)\overline{g(z)}(\mathrm{Im}\, z)^k$ is $\Gamma_0(N)$-invariant, i.e., for all $\alpha \in \Gamma_0(N)$,*

$$f(\alpha \cdot z)\overline{g(\alpha \cdot z)}(\mathrm{Im}\,(\alpha \cdot z))^k = f(z)\overline{g(z)}(\mathrm{Im}\, z)^k.$$

Definition 3.1.27. *Let $f \in M_k(N)$, and $g \in S_k(N)$, where $M_k(N)$ and $S_k(N)$ are as mentioned in Definition 3.1.22. We define the Petersson Inner Product as the following:*

$$\langle \, , \, \rangle : M_k(N) \times S_k(N) \to \mathbb{C}$$

given by

$$\langle f, g \rangle := \int_{\Gamma_0(N)\backslash H} f(z)\overline{g(z)}(\text{Im } z)^k d\mu(z), \tag{3.3}$$

where $\Gamma_0(N)\backslash H$ is the orbit space of the action of $\Gamma_0(N)$ on H.

Remark 3.1.28. *The integral in equation* (3.3) *is finite as long as one of f and g belongs to $S_k(N)$ and the other belongs to $M_k(N)$.*

Remark 3.1.29. *Using Propositions 3.1.25 and 3.1.26, we obtain that the integral in equation* (3.3) *is well-defined.*

Proposition 3.1.30. *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$. Let and $f, g \in M_k(\Gamma)$. Then*

(a) *$\langle \, , \, \rangle$ is linear in first argument,*

(b) *$\langle \, , \, \rangle$ is conjugate symmetric, i.e., $\langle f, g \rangle = \overline{\langle g, f \rangle}$,*

(c) *$\langle f, f \rangle > 0$ for $f \neq 0$.*

Therefore, the Petersson inner product defines a Hermitian inner product on $S_k(\Gamma)$.

Proof. We refer the readers to [MDG16, Chapter 7] for a proof. □

Remark 3.1.31. *The Petersson inner product defines a Hermitian inner product on $S_k(N)$, for integers k and N with $N \geq 1$.*

We now state the following standard result from the functional analysis without proof.

Lemma 3.1.32. *A finite-dimensional inner product space V over $\mathbb{C}$ is a Hilbert space.*

Theorem 3.1.33. *The space $S_k(N)$ of cusp forms of weight k and level N, is a Hilbert space with respect to Petersson inner product.*

Proof. The proof follows from Proposition 3.1.30 and Remark 3.1.23. □

3.1.3 Hecke Operators

For each weight k and level N, we define a family of linear operators that preserve the spaces $M_k(N)$ and $S_k(N)$, called the Hecke operators (See [Kob84], Proposition 35 on page 160).

Definition 3.1.34. *Let G be a group acting on a set X. The orbit of an element x in X is the set of elements in X to which x can be moved by the elements of G. The orbit of x is denoted by $G \cdot x$:*

$$G \cdot x = \{g \cdot x : g \in G\}.$$

The set of all orbits of X under the action of G is called the orbit space of the action and is written as $G\backslash X$, i.e.,

$$G\backslash X = \{G \cdot x : x \in X\}.$$

There are different ways of defining Hecke operators on the space of modular forms $M_k(N)$, although all of them are equivalent. We follow the treatment by Cohen and Stromberg [CS17].

For positive integers m and N, we define the set

$$X_m(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}), N | c, (a, N) = 1, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = m \right\}.$$

We observe that the group $\Gamma_0(N)$ acts on the set $X_m(N)$ by the left multiplication of matrices.

Definition 3.1.35. *Let $f \in M_k(N)$. For $m \in \mathbb{N}$, the m-th Hecke operator T_m on the space $M_k(N)$ is defined by*

$$T_m(f) := m^{k/2-1} \sum_{\gamma \in \Gamma_0(N)\backslash X_m(N)} f[\gamma]_k.$$

Remark 3.1.36. *It can be shown that the definition of T_m is independent of the choice of representatives for the cosets $\Gamma_0(N)\backslash X_m(N)$ (See [CS17, Proposition 10.2.3 (a)] for a proof).*

Remark 3.1.37. *In general, it may happen that $f[\gamma]_k \notin M_k(N)$, for arbitrary $\gamma \in GL^+(\mathbb{R})$ and $f \in M_k(N)$. For this reason, we define the Hecke operator where we sum elements of the form $f[\gamma]_k$ for a given modular form f over the orbit representatives to get back a modular form of weight k and level N.*

Theorem 3.1.38. *A system of orbit representatives of $X_m(N)$ for the left action of $\Gamma_0(N)$ is given by the set,*

$$\Delta_m^N := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{Z}) : (a, N) = 1,\, ad = m,\, a > 0,\, 0 \leq b \leq d-1 \right\},$$

that is,

$$X_m(N) = \bigsqcup_{\substack{ad=m \\ (a,N)=1, a>0}} \bigsqcup_{b=0}^{d-1} \Gamma_0(N) \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

Proof. We refer the readers to [CS17, Proposition 6.5.3 (b)] for a proof. □

Using Theorem 3.1.38, we have the following explicit expression for the m-th Hecke operator.

Definition 3.1.39. *Let $f \in M_k(N)$. For $m \in \mathbb{N}$, the m-th Hecke operator T_m on the space $M_k(N)$ is given by*

$$\begin{aligned} T_m(f) &= m^{k/2-1} \sum_{\gamma \in \Delta_m^N} f[\gamma]_k \\ &= m^{k/2-1} \sum_{\substack{ad=m \\ (a,N)=1, a>0}} \sum_{b=0}^{d-1} f\left[\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right]_k \\ &= \frac{1}{m} \sum_{\substack{ad=m \\ (a,N)=1, a>0}} a^k \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right), \end{aligned}$$

where $f[\gamma]_k$ is the slash operator defined in equation (3.2)*.*

Proposition 3.1.40. *Let n and N be positive integers, k be a non-negative even integer and $f = \sum_{m \geq 0} \lambda_f(m) q^m \in M_k(N)$. Then $T_n f(z) = \sum_{m \geq 0} b_f(m) q^m$, where*

$$b_f(m) = \sum_{\substack{d | (m,n), d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mn}{d^2}\right).$$

Proof. We refer the readers to [CS17, Proposition 10.2.5] for a proof. □

Remark 3.1.41. *Let m and N be positive integers and k be a non-negative even integer. The m-th Hecke operator is a well-defined linear map from $M_k(N)$ to $M_k(N)$ and from $S_k(N)$ to $S_k(N)$.*

Corollary 3.1.42. *Let N be a positive integer, p be a prime, k be a non-negative even integer and $f = \sum_{m \geq 0} \lambda_f(m) q^m \in M_k(N)$. Then*

$$T_p f(z) = \begin{cases} \sum_{m \geq 0} \lambda_f(pm) q^m + p^{k-1} \sum_{r \geq 0} \lambda_f(r) q^{pr} & \text{if } p \nmid N \\ \sum_{m \geq 0} \lambda_f(pm) q^m & \text{if } p \mid N. \end{cases}$$

When p divides the level N, the Hecke operator T_p is often denoted by U_p.

Proof. If $p \mid N$, then $(p, N) = p$, and hence,

$$b_f(m) = \sum_{\substack{d \mid (m,p), d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mp}{d^2}\right) = \lambda_f(pm).$$

If $p \nmid N$, then $(p, N) = 1$. Hence, for prime p dividing m,

$$b_f(m) = \sum_{\substack{d \mid (m,p), d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mp}{d^2}\right) = \lambda_f(pm) + p^{k-1} \lambda_f\left(\frac{m}{p}\right), \text{ and}$$

for prime p not dividing m,

$$b_f(m) = \sum_{\substack{d \mid (m,p), d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mp}{d^2}\right) = \lambda_f(pm).$$

□

We now record some important properties of Hecke operators below.

Theorem 3.1.43 (Hecke). *The Hecke operators $\{T_n : n \geq 1\}$ acting on $S_k(N)$ satisfy the following properties :*

(a) *For $m, n \in \mathbb{N}$,*

$$T_m T_n = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} d^{k-1} T_{\frac{mn}{d^2}}.$$

In particular, $T_m T_n = T_n T_m$, for $m, n \in \mathbb{N}$.

(b) *The Hecke operators are multiplicative, i.e., for $m, n \in \mathbb{N}$ with $(m, n) = 1$, $T_m T_n = T_{mn}$.*

(c) *For a positive integer r and a prime p, such that $(p, N) = 1$, $T_{p^r} T_p = T_{p^{r+1}} + p^{k-1} T_{p^{r-1}}$. If $p \mid N$, then $T_{p^r} = (T_p)^r$.*

Proof. We refer the readers to [CS17, Theorem 10.2.9] for a proof. □

Theorem 3.1.44 (Petersson). *The Hecke operators $\{T_n : (n, N) = 1\}_{n \geq 1}$ on $S_k(N)$ are self-adjoint (or, Hermitian) with respect to the Petersson inner product, i.e., for any $f, g \in S_k(N)$, and $(n, N) = 1$, we have $\langle T_n f, g \rangle = \langle f, T_n g \rangle$.*

Proof. For a proof of this theorem, we refer the readers to [Kob84, Proposition 50]. □

Definition 3.1.45. *A cusp form $f \in S_k(N)$ is called a **Hecke eigenform** if it is an eigenfunction for each T_n with $(n, N) = 1$, i.e., there exists a sequence of complex numbers $\{\alpha_n : (n, N) = 1\}$ such that*

$$T_n f = \alpha_n f.$$

*If the first Fourier coefficient $a_f(1)$ of an eigenform is 1, then we say that f is a **normalised Hecke eigenform**.*

Definition 3.1.46. *A basis of $S_k(N)$ consisting of Hecke eigenforms is called a **Hecke eigenbasis**.*

We now begin with the following standard theorem from linear algebra.

Theorem 3.1.47. *Let R be a commutative ring of Hermitian operators on a finite-dimensional Hilbert space V over $\mathbb{C}$. Then V has an orthogonal basis $f_1, f_2, \ldots, f_r$ of eigenvectors of R.*

Proof. For a proof of this theorem, we refer the readers to [MDG16, Theorem 7.4.1]. □

Theorem 3.1.48. *The space of cusp forms $S_k(N)$ has an orthogonal basis consisting of eigenfunctions of T_m for all $(m, N) = 1$. In particular, if $N = 1$, there exists an orthogonal Hecke eigenbasis for all $T_n, n \geq 1$.*

Proof. Using Theorem 3.1.33, we note that the space of cusp forms $S_k(N)$ is a finite-dimensional Hilbert space over $\mathbb{C}$ with respect to Petersson inner product. Also, using Theorems 3.1.43 and 3.1.44, we obtain that $R = \{T_n : (n, N) = 1\}_{n \geq 1}$ is a commutative ring of Hermitian operators on $S_k(N)$. The proof now follows from Theorem 3.1.47. □

3.1.4 Oldforms and newforms

In this section, we discuss the theory of newforms developed by Atkin–Lehner, generalized by W. Li. Newforms are analogues of primitive characters in the context of modular forms. In particular, we will see that there is a natural decomposition of the spaces $S_k(N)$ and that newforms have nicer properties than general modular forms. The advantage of having the space of newforms is that we will have a basis consisting of normalised eigenforms of all Hecke operators.

Theorem 3.1.48 shows that $S_k(N)$ has an orthogonal basis consisting of eigenfunctions of T_m under the restriction, $(m, N) = 1$, and this basis need not be unique. However, on the space of newforms $S_k^{new}(\Gamma_0(N))$ (to be defined below), we will see that there exists a basis of forms (unique, up to normalization and ordering) which are simultaneous eigenfunctions for all the Hecke operators T_p, including those with $p|N$.

In 1970, Atkin and Lehner resolved this difficulty by focusing on forms, which are indeed of level N, i.e., they don't come from lower levels. We observe that if $N'|N$, then $\Gamma_0(N) \subseteq \Gamma_0(N')$, and hence $S_k(N') \subseteq S_k(N)$. More generally, if $d|\frac{N}{N'}$, and $f(z) \in S_k(N')$, it turns out that $f(dz) \in S_k(N)$. We name the subspace containing all the duplicating forms coming from lower levels as the space of oldforms.

Definition 3.1.49. *The $\mathbb{C}$-span of*

$$\bigcup_{\substack{N'|N \\ N \neq N'}} \bigcup_{d|\frac{N}{N'}} \{f(dz) : f \in S_k(N')\}$$

is called the space of oldforms on $\Gamma_0(N)$, and is denoted by $S_k^{old}(N)$. The orthogonal complement of the space of oldforms with respect to the Petersson inner product in $S_k(N)$ is called the space of primitive modular cusp forms on $\Gamma_0(N)$, and is denoted by $S_k^{new}(N)$.

Remark 3.1.50. *We have the following orthogonal decomposition for the space $S_k(N)$,*

$$S_k(N) = S_k^{old}(N) \bigoplus S_k^{new}(N).$$

Remark 3.1.51. *By the Atkin-Lehner decomposition [AL70], we know that*

$$S_k(N) = \bigoplus_{d|N} \bigoplus_{a|\frac{N}{d}} i_{a,d}(S_k^{new}(d)), \tag{3.4}$$

where, for positive integers a and d such that $ad|N$, $i_{a,d}$ denotes the embedding $f(z) \to f(az)$ of $S_k(d)$ into $S_k(N)$.

Remark 3.1.52. *The spaces $S_k^{old}(N)$ and $S_k^{new}(N)$ are stable under the action of all Hecke operators $\{T_n\}_{n \geq 1}$, not only those with $(n, N) = 1$ (See [CS17]).*

The space $S_k^{new}(N)$ has the following nice properties:

Theorem 3.1.53. *The following statements are true:*

(a) *There exists an orthogonal basis of the space $S_k^{new}(N)$ formed by eigenforms of all the Hecke operators T_n.*

(b) *Let $f = \sum_{n=1}^{\infty} \lambda_f(n) q^n \in S_k^{new}(N)$ be a Hecke eigenform. Then, $\lambda_f(1) \neq 0$.*

(c) *If $f = \sum_{n=1}^{\infty} \lambda_f(n) q^n \in S_k^{new}(N)$ is a normalised Hecke eigenform, i.e., $\lambda_f(1) = 1$, then $T_n f = \lambda_f(n) f$ for all integers n.*

(d) *If $f = \sum_{n=1}^{\infty} \lambda_f(n) q^n \in S_k^{new}(N)$ is a normalised Hecke eigenform, then*

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n),d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mn}{d^2}\right).$$

Remark 3.1.54. *We note parts (c) and (d) of Theorem 3.1.53 follow from the following fact: Let $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) q^n \in S_k^{new}(N)$, where $\lambda_f(1) = 1$ and*

$$T_n(f(z)) = \alpha(n) f(z), \; n \geq 1.$$

Using Proposition 3.1.40, we obtain

$$\sum_{\substack{d|(m,n),d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mn}{d^2}\right) = \alpha(m)\lambda_f(n).$$

In particular, $n = 1$ gives $\lambda_f(m) = \alpha(m)\lambda_f(1)$, i.e., $\lambda_f(m) = \alpha(m)$.

Hence, we obtain

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n),d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mn}{d^2}\right).$$

Definition 3.1.55. *Theorem 3.1.53 (a) and (b) guarantee the existence of an orthogonal basis of $S_k^{new}(N)$ consisting of normalised Hecke eigenforms, which we denote by $\mathcal{F}_{N,k}$. Any $f(z) \in \mathcal{F}_{N,k}$ is called a* **Hecke newform** *of weight k and level N.*

Remark 3.1.56. *We note that we don't call every function in $S_k^{new}(N)$ a newform. A normalised cusp form $f \in S_k^{new}(N)$ which belongs to an orthogonal basis of $S_k^{new}(N)$ and is a common eigenfunction for the Hecke operators T_n with $(n, N) = 1$ is called a newform.*

Definition 3.1.57. *A newform $f(z) = \sum_{n \geq 1} \lambda_f(n) q^n$ of level N and weight k is said to be a* **CM form** *(or to have complex multiplication) if there exists an imaginary quadratic field K such that $\lambda_f(p) = 0$ if and only if p is inert in K. For weight $k \geq 2$, the field K is unique and we say that f has CM by K.*

If no such field exists, we say that f is a **non-CM** *newform.*

3.1.5 Ramanujan-Petersson conjecture

Let k and N be positive integers with k even. Let $S_k^{new}(N)$ denote the space of primitive modular cusp forms of weight k with respect to $\Gamma_0(N)$. For $n \geq 1$, let T_n denote the n-th Hecke operator acting on $S_k^{new}(N)$. We denote the set of Hecke newforms of level N in $S_k^{new}(N)$ by $\mathcal{F}_{N,k}$ (as mentioned in Definition 3.1.55). Using Theorem 3.1.53, any $f(z) \in \mathcal{F}_{N,k}$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) q^n, \qquad q = e^{2\pi i z},$$

where $\lambda_f(1) = 1$ and

$$T_n(f(z)) = \lambda_f(n) f(z),\ n \geq 1. \tag{3.5}$$

We now mention below the Ramanujan-Petersson conjecture, which was first conjectured by Ramanujan for the Ramanujan τ function and later generalized by Petersson for more general modular forms. The conjecture was completely proved by Deligne in 1974, using his proof of the general Weil conjectures (See [CS17] for a history of the problem).

Theorem 3.1.58 (Ramanujan–Petersson conjecture). *Let $f(z) = \sum_{n \geq 1} \lambda_f(n) q^n$ be a normalised Hecke eigenform. Then, for any prime p with $(p, N) = 1$,*

$$|\lambda_f(p)| \leq 2p^{\frac{k-1}{2}}.$$

Remark 3.1.59. *It follows from Ramanujan–Petersson conjecture for any $n \in \mathbb{N}$ with $(n, N) = 1$,*

$$|\lambda_f(n)| \leq d(n) n^{\frac{k-1}{2}},$$

where $d(n)$ is the number of positive divisors of n (See Exercise 5.3.2 in [MDG16] for a proof).

If we consider the normalised eigenvalues $a_f(n) = \frac{\lambda_f(n)}{n^{(k-1)/2}}$, then Ramanujan–Petersson bound gives that for any $n \in \mathbb{N}$ with $(n, N) = 1$,

$$|a_f(n)| \leq d(n). \tag{3.6}$$

Hence, using equation (3.5), we obtain that any $f(z) \in F_{N,k}$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n, \qquad q = e^{2\pi i z},$$

where $a_f(1) = 1$ and

$$\frac{T_n(f(z))}{n^{\frac{k-1}{2}}} = a_f(n) f(z),\ n \geq 1. \tag{3.7}$$

Let us fix N and k and consider a newform $f(z)$ in $F_{N,k}$ (as mentioned in Definition 3.1.55). Let p be a prime number with $(p, N) = 1$. Equation (3.6) gives, $|a_f(p)| \leq 2$. Moreover, $a_f(p)$'s being the eigenvalues of a Hermitian operator, $a_f(p) \in \mathbb{R}$. Hence, the eigenvalues $a_f(p)$ lie in the interval $[-2, 2]$ and if we denote $a_f(p) = 2\cos \pi\theta_f(p)$, we have $\theta_f(p) \in [0, 1]$.

Theorem 3.1.60. *If $f = \sum_{n=1}^{\infty} \lambda_f(n) q^n \in S_k^{new}(N)$ is a normalized eigenform, then the Fourier coefficients satisfy*

(a) $\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n), d>0 \\ (d,N)=1}} d^{k-1} \lambda_f\left(\frac{mn}{d^2}\right).$

(b) $\lambda_f(m)\lambda_f(n) = \lambda_f(mn),$ *if* $(m, n) = 1$.

(c) $\lambda_f(p)\lambda_f(p^n) = \lambda_f(p^{n+1}) + p^{k-1}\lambda_f(p^{n-1}),$ *for $n \geq 1$, and prime p such that $(p, N) = 1$.*

Proof. Proof of (b) and (c) directly follows from (a), which is already mentioned in Theorem 3.1.53. □

Corollary 3.1.61. *If $f = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n \in S_k^{new}(N)$ is a normalized eigenform, then the normalised Fourier coefficients satisfy*

(a) $a_f(m)a_f(n) = \sum_{\substack{d|(m,n), d>0 \\ (d,N)=1}} a_f\left(\frac{mn}{d^2}\right).$

(b) $a_f(m)a_f(n) = a_f(mn)$, *if* $(m,n) = 1$.

(c) $a_f(p)a_f(p^n) = a_f(p^{n+1}) + p^{k-1}a_f(p^{n-1})$, *for* $n \geq 1$, *and prime* p *such that* $(p,N) = 1$.

Proof. The proof follows from Theorem 3.1.60 after putting $\lambda_f(n) = n^{\frac{k-1}{2}}a_f(n)$. □

3.2 Eichler-Selberg Trace formula

The Eichler-Selberg trace formula gives us a formula for the trace Tr of the Hecke operator $T_n(N,k)$ acting on $S_k(N)$, for each $n \in \mathbb{N}$, in terms of Kronecker class numbers. Selberg proved this formula for $SL_2(\mathbb{R})$, in the level 1 case, in his famous 1956 paper ([Sel56]). In the same year, Eichler ([Eic56]) obtained a formula for $k = 2$ and square-free level. Hijikata ([Hij74]) gave the formula for traces of Hecke operators for $\Gamma_0(N)$, where $(n,N) = 1$. Joseph Oesterlé gave a more general formula for the space of cusp forms of weight k and level N, which is valid for all $n \in \mathbb{N}$, in his thesis [Oes77]). This explicit formula is known as the Eichler-Selberg trace formula.

3.2.1 Class Numbers

For a negative integer Δ congruent to 0 or 1 (mod 4), we let $B(\Delta)$ denote the set of all positive definite binary quadratic forms (not necessarily primitive) with discriminant Δ (i.e., $\Delta < 0$),

$$B(\Delta) = \{aX^2 + bXY + cY^2, a, b, c \in \mathbb{Z}, a > 0, \text{ and } b^2 - 4ac = \Delta\}.$$

By $b(\Delta)$, we denote primitive such forms,

$$b(\Delta) = \{aX^2 + bXY + cY^2 \in B(\Delta), \text{ gcd } (a,b,c) = 1\}.$$

One can define a right action of the group $SL_2(\mathbb{Z})$ on $B(\Delta)$ by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f(X,Y) := f(\alpha X + \beta Y, \gamma X + \delta Y) \text{ for } f(X,Y) \in B(\Delta).$$

This action respects primitive forms. It is well known that there are only finitely many orbits (See for example [Bak84]). The number of orbits in $b(\Delta)$ is called the class number of Δ and is denoted by $h(\Delta)$. For a negative integer Δ congruent to 0 or 1 (mod 4), the *Hurwitz-Kronecker class number* of Δ is denoted by $H(-\Delta)$, it is defined to be the number of orbits in $B(\Delta)$, but one should count the forms $aX^2 + aY^2$ and $aX^2 + aXY + aY^2$, if at all present in $B(\Delta)$, with multiplicity $\frac{1}{2}$ and $\frac{1}{3}$, respectively. The *Hurwitz-Kronecker class number* H is extended to $\mathbb{Z}$, by defining $H(0) = -\frac{1}{12}$, and $H(n) = 0$ for $-n \equiv 2$ or 3 (mod 4), where $n \in \mathbb{N}$. The relation between the *Hurwitz-Kronecker class number* and the ordinary class numbers is given as follows:

Proposition 3.2.1. *Let* Δ *be a negative integer congruent to* 0 *or* 1 *(mod* 4*) and* h_w *be defined as follows:*

$$h_w(-3) = \frac{1}{3},$$
$$h_w(-4) = \frac{1}{2},$$
$$h_w(\Delta) = h(\Delta), \text{ for } \Delta < -4.$$

Then

$$H(-\Delta) = \sum_f h_w\left(\frac{\Delta}{f^2}\right),$$

where f *runs over all positive divisors of* Δ *for which* $\Delta/f^2 \in \mathbb{Z}$ *is congruent to* 0 *or* 1 *(mod* 4*).*

Proof. This formula follows from the definition of $H(N)$. □

We now mention the following proposition which will be useful while estimating trace formula.

Proposition 3.2.2. *Let n be a positive integer. Then*

$$\sum_{t\in\mathbb{Z},\, t^2<4n} H(4n - t^2) = \begin{cases} 2\sigma_1(n) - \lambda(n) + \frac{1}{6}, & \text{if } n \text{ is a square,} \\ 2\sigma_1(n) - \lambda(n), & \text{otherwise,} \end{cases}$$

where

$$\lambda(n) = \sum_{d|n} \min\left(d, \frac{n}{d}\right) \text{ and } \sigma_1(n) = \sum_{\substack{d|n \\ d>0}} d.$$

Proof. The above recursion formula is due to Kronecker (1857) and Gierster (1879). We refer the readers to [Coh93, Theorem 5.3.8] for a proof. □

Remark 3.2.3. *It can be observed from [Coh93, Corollary 5.3.9] (which follows from Proposition 3.2.2 and we don't mention it here) that to compute individual class numbers, we need to know the preceding ones. Thus Proposition 3.2.2 gives a recursive formula for class numbers.*

3.2.2 Trace formula on the space $S_k(N)$

In this section, we mention the well-known Eichler-Selberg trace formula T_n on the space $S_k(N)$, of modular cusp forms of weight k and level N, for each $n \in \mathbb{N}$.

Theorem 3.2.4. *(Eichler-Selberg trace formula) Let n be a positive integer. The trace of the Hecke operator $T_n(N,k)$ acting on $S_k(N)$ is given by*

$$TrT_n(N,k) = \mathcal{A}_1(n,N,k) + \mathcal{A}_2(n,N,k) + \mathcal{A}_3(n,N,k) + \mathcal{A}_4(n,N,k).$$

where $\mathcal{A}_i(n,N,k)$'s are as follows:

$$\mathcal{A}_1(n,N,k) = \begin{cases} n^{(k/2-1)}\frac{k-1}{12}\psi(N) & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{A}_2(n,N,k) = -\frac{1}{2}\sum_{t\in\mathbb{Z},\, t^2<4n} \frac{\rho^{k-1} - \bar\rho^{k-1}}{\rho - \bar\rho} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right)\mu(t,f,n),$$

$$\mathcal{A}_3(n,N,k) = -\sum_{\substack{d|n \\ 0<d\le\sqrt{n}}}{}' d^{k-1} F(N)_d,$$

$$\mathcal{A}_4(n,N,k) = \begin{cases} \sum_{t|n,\, t>0} t & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where

- $\psi(N) = N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right).$
- *ρ and $\bar\rho$ are the zeros of the polynomial $x^2 - tx + n$, and h_w is as mentioned in the Section 3.2.1.*
- *The inner sum in $\mathcal{A}_2(n,N,k)$ runs over all positive divisors f of $t^2 - 4n$ such that $t^2 - 4n/f^2 \in \mathbb{Z}$ is congruent to 0 or 1 (mod 4).*
- *For a positive integer f, $\mu(t,f,n)$ is given by*

$$\mu(t,f,n) = \frac{\psi(N)}{\psi\left(\frac{N}{N_f}\right)} M(t,n,NN_f),$$

where $N_f = (N,f)$, and $M(t,n,K)$ denotes the number of solutions of the congruence $x^2 - tx + n \equiv 0 \pmod{K}$.

- *The term $F(N)_d$ in $A_3(n,N,k)$ is a multiplicative function of N given by*

$$F(N)_d = \sum_{\substack{c \mid N \\ \left(c,\frac{N}{c}\right) \mid \frac{n}{d} - d}} \phi\left(\left(c, \frac{N}{c}\right)\right).$$

- *The prime on the summation in $A_3(n,N,k)$ indicates that if there is a contribution from the term $d = \sqrt{n}$, it should be multiplied by $\frac{1}{2}$.*

Proof. We refer the readers to [Eic56], [Hij74], and [Oes77] for proofs and to the introduction in Section 3.2 for a history behind proof of this formula. □

Remark 3.2.5. *$A_1(n,N,k)$, $A_2(n,N,k)$ and $A_3(n,N,k)$ are called the identity term, elliptic term and hyperbolic-unipotent term respectively. We refer the readers to Theorem 2.1 and Equation 3.53 of [KL06] for the reasoning behind such nomenclature.*

The rest of this section is dedicated to estimating the terms $A_i(n,N,k)$ arising from the Eichler-Selberg trace formula. We follow [Ser97, Section 4.2] for a proof of Proposition 3.2.7, for which we need the following Lemma 3.2.6 and ultimately prove Theorem 3.2.9.

Lemma 3.2.6. *Let $t^2 - 4n \neq 0$, and $\nu(N)$ denotes the number of distinct prime divisors of N. Given an integer K, let $M(t,n,K)$ denotes the number of solutions of the congruence $x^2 - tx + n \equiv 0$ (mod K). Then,*

$$M(t,n,K) \leq 2^{\nu(K)} |t^2 - 4n|^{1/2}.$$

Proof. We refer the readers to [Hux81] for a proof. One can also check [KL06, Lemma 29.3]. □

Proposition 3.2.7. *In the trace of $T_n(N,k)$ of Theorem 3.2.4, the elliptic term,*

$$|A_2(n,N,k)| = \left| \frac{1}{2} \sum_{t \in \mathbb{Z}, t^2 < 4n} \frac{\rho^{k-1} - \bar\rho^{k-1}}{\rho - \bar\rho} \sum_f h_w\left(\frac{t^2 - 4n}{f^2}\right) \mu(t,f,n) \right|$$
$$\leq 2^{\left(2 + \frac{\ln n}{\ln 4}\right)} n^{(k+1)/2} \left(2\sigma_1(n) - \lambda(n) + \frac{1}{6}\right) 2^{\nu(N)}.$$

Proof. Since $\left|\frac{t^2-4n}{f^2}\right| \leq 4n$, it follows that $h_w\left(\frac{t^2-4n}{f^2}\right)$ is bounded by a constant which depends only on n, but not on k and N.

Also, $\rho^2 - t\rho + n = 0$ implies that $|\rho| = |\bar\rho| = n^{1/2}$, and $|\rho - \bar\rho| = \sqrt{4n - t^2} \geq 1$. Hence,

$$\left|\frac{\rho^{k-1} - \bar\rho^{k-1}}{\rho - \bar\rho}\right| \leq \frac{n^{(k-1)/2}}{\sqrt{4n - t^2}} \leq n^{(k-1)/2}. \tag{3.8}$$

For any divisor d of N,

- $\frac{\psi(N)}{\psi\left(\frac{N}{d}\right)} = \frac{N\prod_{p \mid N}\left(1+\frac{1}{p}\right)}{\frac{N}{d}\prod_{p \mid \frac{N}{d}}\left(1+\frac{1}{p}\right)} \leq d\prod_{p \mid d}\left(1 + \frac{1}{p}\right) = \psi(d)$, and
- $\psi(d) = d\prod_{p \mid d}\left(1 + \frac{1}{p}\right) \leq N\prod_{p \mid N}\left(1 + \frac{1}{p}\right) = \psi(N)$.

Now, $f^2 \mid (4n - t^2)$ implies $f \leq \sqrt{4n - t^2} \leq \sqrt{4n} = 2n^{\frac{1}{2}}$, and hence

$$\frac{\psi(N)}{\psi\left(N/(N,f)\right)} \leq \psi((N,f)) \leq \psi(f) = f\prod_{p \mid f}\left(1 + \frac{1}{p}\right) \leq f2^{\nu(f)} \leq f2^{\frac{\ln f}{\ln 2}} \leq 2^{\left(2 + \frac{\ln n}{\ln 4}\right)}\sqrt{n}. \tag{3.9}$$

The summation over t contains at most $\lfloor 4\sqrt{n} \rfloor + 1$ many terms and the summation over f contains at most $\lfloor 2\sqrt{n} \rfloor + 1$ many terms.

Using Lemma 3.2.6, we obtain

$$|M(t,n,NN_f)| \leq 2^{\nu(NN_f)}|t^2 - 4n|^{1/2} \leq 2^{\nu(N)+1}\sqrt{n}. \tag{3.10}$$

Thus, using equations (3.9) and (3.10), we get

$$|\mu(t,f,n)| = \frac{\psi(N)}{\psi\left(\frac{N}{N_f}\right)}|M(t,n,NN_f)| \leq 2^{\left(3+\frac{\ln n}{\ln 4}\right)} n 2^{\nu(N)}. \tag{3.11}$$

Therefore, using Proposition 3.2.2 and equations (3.8) and (3.11), we obtain

$$\begin{aligned}
&|A_2(n,N,k)| \\
&= \left| -\frac{1}{2} \sum_{t \in \mathbb{Z}, t^2 < 4n} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right)\mu(t,f,n)\right| \\
&\leq \frac{1}{2} n^{(k-1)/2} \sum_{t \in \mathbb{Z}, t^2 < 4n} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) 2^{\left(3+\frac{\ln n}{\ln 4}\right)} n 2^{\nu(N)} \\
&\leq 2^{\left(2+\frac{\ln n}{\ln 4}\right)} n^{(k+1)/2}\left(2\sigma_1(n) - \lambda(n) + \frac{1}{6}\right) 2^{\nu(N)},
\end{aligned}$$

□

Proposition 3.2.8. *In the trace of $T_n(N,k)$ of Theorem 3.2.4, the hyperbolic-unipotent term,*

$$|A_3(n,N,k)| = \left| \sideset{}{'}\sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1}F(N)_d \right| \leq d(N)\sqrt{N} n^{\frac{k-1}{2}} d(n),$$

where $F(N)_d$ is a multiplicative function of N as mentioned in Theorem 3.2.4 and the prime on the summation indicates that if there is a contribution from the term $d = \sqrt{n}$, it should be multiplied by $\frac{1}{2}$.

Proof. Since $\phi\left(\left(c, \frac{N}{c}\right)\right) \leq \left(c, \frac{N}{c}\right) \leq \min\left(c, \frac{N}{c}\right) \leq \sqrt{N}$, we have

$$F(N)_d = \sum_{\substack{c|N \\ \left(c,\frac{N}{c}\right) | \frac{n}{d} - d}} \phi\left(\left(c, \frac{N}{c}\right)\right) \leq \sum_{c|N} \sqrt{N} \leq d(N)\sqrt{N}.$$

Therefore,

$$\sideset{}{'}\sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1}F(N)_d \leq d(N)\sqrt{N} \sideset{}{'}\sum_{\substack{d|n \\ 0<d\leq\sqrt{n}}} d^{k-1} \leq d(N)\sqrt{N} n^{\frac{k-1}{2}} d(n).$$

□

Theorem 3.2.9. *Let n, k and N be positive integers with k even. The trace of the Hecke operator $T_n(N,k)$ acting on $S_k(N)$ is given by*

$$TrT_n(N,k) = \begin{cases} n^{\frac{k}{2}-1}\frac{k-1}{12}\psi(N) + O\left(2^{\left(\frac{\ln n}{\ln 4}\right)} n^{(k+1)/2}\sigma_1(n)d(N)\sqrt{N}d(n)\right) & \text{if } n \text{ is a square,} \\ O\left(2^{\left(\frac{\ln n}{\ln 4}\right)} n^{(k+1)/2}\sigma_1(n)d(N)\sqrt{N}d(n)\right) & \text{otherwise,} \end{cases}$$

where the implied constant is the error term is absolute.

Proof. We note that

$$A_4(n,N,k) = \begin{cases} \sum_{t|n,t>0} t & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases} \quad nd(n). \tag{3.12}$$

The proof now follows from Theorem 3.2.4, where we use Propositions 3.2.7 and 3.2.8 and equation (3.12). Therefore, if n is a square,

$$\begin{aligned} &\mathrm{Tr} T_n(N,k) \quad n^{\frac{k}{2}} \;\; {}^{1}\frac{k \quad 1}{12}\psi(N) \\ &= O\left(2^{\left(2+\frac{\ln n}{\ln 4}\right)} n^{(k+1)/2}\left(2\sigma_1(n) \quad \lambda(n) + \frac{1}{6}\right) 2^{\nu(N)} + d(N)\sqrt{N} n^{\frac{k \; 1}{2}} d(n) + nd(n)\right) \\ &= O\left(2^{\left(\frac{\ln n}{\ln 4}\right)} n^{(k+1)/2}\sigma_1(n) 2^{\nu(N)} + d(N)\sqrt{N} n^{(k \quad 1)/2} d(n) + nd(n)\right) \\ &= O\left(2^{\left(\frac{\ln n}{\ln 4}\right)} n^{(k+1)/2}\sigma_1(n) d(N)\sqrt{N} d(n)\right), \text{ using } 2^{\nu(N)} \quad d(N). \end{aligned}$$

□

Remark 3.2.10. *Since* $d(N) \quad {}_{\epsilon} N^{\epsilon}$, *for any* $\epsilon > 0$, *and* $N < \psi(N)$, *we have*

$$\sqrt{N} d(N) \quad N^{\frac{1}{2}+\frac{1}{3}} \quad N \quad \psi(N),$$

as $k + N \; ! \; 1$. *Also,* $n^{\frac{k}{2}} \quad n^{\frac{k}{2}}(k \quad 1)$, *as* $k + N \; ! \; 1$. *Therefore, the main term in Theorem 3.2.9, is indeed a dominant term as* $k + N \; ! \; 1$.

Corollary 3.2.11. *Let* k *and* N *be positive integers with* k *even. Then the dimension of the space* $S_k(N)$, *of cusp forms of weight* k *and level* N *is given by*

$$\dim S_k(N) = \frac{k \quad 1}{12}\psi(N) + O\left(\sqrt{N} d(N)\right),$$

where the implied constant is the error term is absolute.

Proof. Using Theorem 3.1.40, we obtain that the Hecke operator T_1, is just the identity map on $S_k(N)$. Hence, we have $Tr(T_1) = \dim S_k(N)$. The proof now follows from Theorem 3.2.9. □

3.2.3 Trace formula on the space $S_k^{new}(N)$

We now compute a formula for the trace of T_n acting on $S_k^{new}(N)$, which we denote as $T_n^{new}(N,k)$.

Proposition 3.2.12. *(Murty, Sinha, 2010) Let* n *be a positive integer coprime to* N. *The trace of the Hecke operator* $T_n^{new}(N,k)$ *acting on* $S_k^{new}(N)$ *is given by*

$$TrT_n^{new}(N,k) = A_1(n,N,k) + A_2(n,N,k) + A_3(n,N,k) + A_4(n,N,k).$$

where $A_i(n)$*'s are as follows:*

$$A_1(n,N,k) = \begin{cases} n^{(k/2 \quad 1)} \frac{k \quad 1}{12} N B_1(N) & \text{if } n \text{ is a square,} \\ 0 & \text{otherwise,} \end{cases}$$

$$A_2(n,N,k) = \frac{1}{2}\sum_{t \in \mathbb{Z}, t^2 < 4n} \frac{\rho^{k \quad 1} \quad \rho^{k \quad 1}}{\rho \quad \rho} \sum_f h_w\left(\frac{t^2 \quad 4n}{f^2}\right) B_2(N)_f,$$

$$A_3(n,N,k) = \sum_{\substack{d|n \\ 0<d \quad \sqrt{n}}}{}' d^{k \quad 1} B_3(N)_d,$$

$$A_4(n,N,k) = \begin{cases} \mu(N)\sum_{t|n} t & \text{if } k = 2, \\ 0 & \text{otherwise,} \end{cases}$$

where

- $B_1(N)$ *is a multiplicative function such that for a prime power* p^r,

$$B_1(p^r) = \begin{cases} 1 - \frac{1}{p} & \text{if } r = 1, \\ 1 - \frac{1}{p} - \frac{1}{p^2} & \text{if } r = 2, \\ \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^2}\right) & \text{if } r \geq 3. \end{cases}$$

- ρ *and* $\bar{\rho}$ *are the zeros of the polynomial,* $x^2 - tx + n$.
- *The inner sum in* $A_2(n,N,k)$ *runs over all positive divisors* f *of* $t^2 - 4n$ *such that* $(t^2 - 4n)/f^2 \in \mathbb{Z}$ *is congruent to* 0 *or* 1 *(mod* 4*).*
- *For a positive integer* f, $B_2(N)_f$ *is a multiplicative function of* N *such that*

$$B_2(p)_f = \begin{cases} p - 1 & \text{if } p|f, \\ 1 + \left(\frac{t^2 - 4n}{p}\right) & otherwise, \end{cases}$$

  where $\left(\frac{\cdot}{\cdot}\right)$ *denotes the Legendre symbol. If* $N = p^r$ *for some* $r \geq 2$ *and* $p^b || f$, *then*

$$B_2(p^r)_f = \sum_{i=r-2}^{r} \sigma_0^{-1}(p^{r-i}) \frac{\psi(p^i)}{\psi(p^{i-\min\{i,b\}})} M(t,n,p^{i+\min\{i,b\}}),$$

  where

$$\psi(N) = N \prod_{\substack{p|N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right),$$

  $\sigma_0^{-1}(N)$ *denotes the Dirichlet inverse of* $\sigma_0(N)$ *and* $M(t,n,p^{i+\min\{i,b\}})$ *denotes the number of elements of* $(\mathbb{Z}/p^i\mathbb{Z})$ *which lift to solutions of* $x^2 - tx + n \equiv 0 \bmod (p^{i+\min\{i,b\}})$.
- *The prime on the summation in* $A_3(n,N,k)$ *indicates that if there is a contribution from the term* $d = \sqrt{n}$, *it should be multiplied by* $\frac{1}{2}$.
- $B_3(N)_d$ *is a multiplicative function of* N *such that for a prime power* p^r,

$$B_3(p^r)_d = \begin{cases} \phi(p^{\frac{r-2}{2}}) & \text{if } r \text{ is even and } p^{\frac{r-2}{2}} || (\frac{n}{d} - d), \\ \phi(p^{\frac{r}{2}}) - \phi(p^{\frac{r-2}{2}}) & \text{if } r \text{ is even and } p^{\frac{r}{2}} | (\frac{n}{d} - d), \\ 0 & \text{otherwise.} \end{cases}$$

- $\mu(n)$ *is the Möbius function, defined by*

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^{\nu(n)} & \text{if } n \text{ is squarefree}, \\ 0 & otherwise, \end{cases}$$

  where $\nu(n)$ *denotes the number of distinct prime divisors of* n.

Proof. The proof uses the Eichler-Selberg trace formula for Hecke operators on the space of cusp forms, $S_k(N)$, as mentioned in Theorem 3.2.4. Then, one uses Atkin-Lehner decomposition as mentioned in (3.4) and Möbius inversion formula to obtain the trace formula on the space $S_k^{new}(N)$ of primitive modular cusp forms. We refer the readers to [MS10] for a complete proof. □

3.2.4 Estimation of the terms of trace formula

Lemma 3.2.13. *Let a and b be integers such that $a^2 - 4b \neq 0$. Let p be a prime and $N = p^\alpha$, $\alpha \in \mathbb{N}$. Then, the number of solutions (mod N) of the congruence $x^2 - ax + b \equiv 0$ (mod N) is at most $2^{\lfloor \frac{\nu_p(|a^2-4b|)}{2} \rfloor}$, where $\nu_p(|a^2-4b|)$ is the largest power of the prime dividing $|a^2-4b|$.*

Proof. This lemma is due to M. N. Huxley ([Hux81]). We refer the readers to Corollary 2 of [Ste91, Page 805] or, [Ser97, Page 84] for a proof. □

Corollary 3.2.14. *Let $t, n \in \mathbb{Z}$, $\alpha \in \mathbb{N}$, and p be a prime. Then,*

$$M(t,n,p^\alpha) \le 2^{\lfloor \frac{\nu_p(|t^2-4n|)}{2} \rfloor},$$

where $M(t,n,K)$ is as defined in Theorem 3.2.4.

We now mention the following simple result which we will use frequently.

Lemma 3.2.15. *Let M and N be positive integers. Then, $\prod_{p|N} p^{\nu_p(M)} \le M$.*

Proof.

$$\prod_{p|N} p^{\nu_p(M)} = \prod_{\substack{p|N \\ p|M}} p^{\nu_p(M)} \prod_{\substack{p|N \\ p\nmid M}} p^{\nu_p(M)} = \prod_{\substack{p|N \\ p|M}} p^{\nu_p(M)} \le \prod_{p|M} p^{\nu_p(M)} = M.$$

□

Lemma 3.2.16. *Let $t, n \in \mathbb{Z}$, $f, N \in \mathbb{N}$. Then,*

$$|B_2(N)_f| \le 4^{\nu(N)} f \sqrt{|t^2 - 4n|}.$$

Proof. From the definition of $B_2(N)_f$, in Proposition 3.2.12, we get that if $N = p^r$ for some $r \ge 2$, and $p^b \| f$, then

$$\begin{aligned} B_2(p^r)_f &= \sum_{i=r-2}^{r} \sigma_0^{-1}(p^{r-i}) \frac{\psi(p^i)}{\psi(p^{i-\min\{i,b\}})} M(t,n,p^{i+\min\{i,b\}}) \\ &= \sum_{i=r-2}^{r} \sigma_0^{-1}(p^{r-i}) p^{\min\{i,b\}} M(t,n,p^{i+\min\{i,b\}}), \end{aligned}$$

where $\sigma_0^{-1}(N)$, the inverse of $\sigma_0(N)$ with respect to Dirichlet convolution, is a multiplicative function defined on prime powers as follows:

$$\sigma_0^{-1}(p^r) = \begin{cases} 1 & \text{if } r = 0 \text{ or } 2, \\ -2 & \text{if } r = 1, \\ 0 & \text{if } r > 2. \end{cases}$$

We define $g(i,b) := p^{\min\{i,b\}} M(t,n,p^{i+\min\{i,b\}})$, and note that $g(i,b) \ge 0$.

Hence, if $N = p^r$ for some $r \ge 2$, and $p^b \| f$, then

$$\begin{aligned} B_2(p^r)_f &= \sum_{i=r-2}^{r} \sigma_0^{-1}(p^{r-i}) p^{\min\{i,b\}} M(t,n,p^{i+\min\{i,b\}}) \\ &= \sigma_0^{-1}(p^2) g(r-2,b) + \sigma_0^{-1}(p) g(r-1,b) + \sigma_0^{-1}(1) g(r,b) \\ &= g(r-2,b) - 2g(r-1,b) + g(r,b) \end{aligned}$$

Hence,

$$|B_2(p^r)_f| \le g(r-2,b) + 2g(r-1,b) + g(r,b) \tag{3.13}$$

$$\begin{aligned}
&= p^{\min\{r-2,b\}} M(t,n,p^{r-2+\min\{r-2,b\}}) + 2p^{\min\{r-1,b\}} M(t,n,p^{r-1+\min\{r-1,b\}}) \\
&+ p^{\min\{r,b\}} M(t,n,p^{r+\min\{r,b\}}) \\
&\le p^b 2^{\lfloor \frac{\nu_p(|t^2-4n|)}{2} \rfloor} + 2p^b 2^{\lfloor \frac{\nu_p(|t^2-4n|)}{2} \rfloor} + p^b 2^{\lfloor \frac{\nu_p(|t^2-4n|)}{2} \rfloor} \\
&\le 4p^b 2^{\frac{\nu_p(|t^2-4n|)}{2}} \\
&\le 4p^{\nu_p(f)} \sqrt{p^{\nu_p(|t^2-4n|)}}.
\end{aligned}$$

Also, for a positive integer f,

$$B_2(p)_f \le p \le p^{\nu_p(f)}. \tag{3.14}$$

Combining equations (3.13) and (3.14), we obtain that for any $r \in \mathbb{N}$,

$$|B_2(p^r)_f| \le 4p^{\nu_p(f)} \sqrt{p^{\nu_p(|t^2-4n|)}}.$$

Therefore, using the fact, $B_2(N)_f$ is a multiplicative function of N, we obtain that

$$B_2(N)_f = B_2\left(\prod_{p|N} p^{\nu_p(N)}\right)_f = \prod_{p|N} B_2\left(p^{\nu_p(N)}\right)_f,$$

and hence,

$$|B_2(N)_f| \le \prod_{p|N} |B_2\left(p^{\nu_p(N)}\right)_f| \le \prod_{p|N} 4p^{\nu_p(f)} \sqrt{p^{\nu_p(|t^2-4n|)}} \le 4^{\nu(N)} f \sqrt{|t^2-4n|},$$

where we use Lemma 3.2.15 in the last inequality. □

Proposition 3.2.17. *In the trace of $T_n^{new}(N,k)$ of Theorem 3.2.12, the elliptic term,*

$$A_2(n,N,k) = \frac{1}{2} \sum_{t \in \mathbb{Z}, t^2<4n} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) B_2(N)_f = O(4^{\nu(N)} n^{k/2} \sigma_1(n)),$$

where $\sigma_1(n)$ is the sum of positive divisors of n.

Proof. We observe that the inner sum in $A_2(n,N,k)$ runs over all positive divisors f of t^2-4n such that $t^2-4n/f^2 \in \mathbb{Z}$ is congruent to 0 or 1 (mod 4), i.e., $f \le \sqrt{|4n-t^2|}$. Hence, Lemma 3.2.16 gives,

$$|B_2(N)_f| \le 4^{\nu(N)} f \sqrt{|t^2-4n|} \le 4^{\nu(N)} f \sqrt{|t^2-4n|} \le 4^{\nu(N)} |t^2-4n|.$$

Using equation (3.8) and Proposition 3.2.2, we obtain

$$\begin{aligned}
|A_2(n,N,k)| &= \left| \frac{1}{2} \sum_{t \in \mathbb{Z}, t^2<4n} \frac{\rho^{k-1} - \bar{\rho}^{k-1}}{\rho - \bar{\rho}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) B_2(N)_f \right| \\
&\le \frac{1}{2} \sum_{t \in \mathbb{Z}, t^2<4n} \frac{n^{(k-1)/2}}{\sqrt{4n-t^2}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) |B_2(N)_f| \\
&\le \frac{1}{2} \sum_{t \in \mathbb{Z}, t^2<4n} \frac{n^{(k-1)/2}}{\sqrt{4n-t^2}} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right) 4^{\nu(N)} |t^2-4n| \\
&\le \frac{n^{(k-1)/2}}{2} 4^{\nu(N)} \sum_{t \in \mathbb{Z}, t^2<4n} \sqrt{4n-t^2} \sum_f h_w\left(\frac{t^2-4n}{f^2}\right)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{n^{(k-1)/2}}{2} 4^{\nu(N)} \sqrt{4n} \sum_{t \in \mathbb{Z}, t^2 < 4n} H(4n - t^2)\\
&\le n^{k/2} 4^{\nu(N)} \left(2\sigma_1(n) - \lambda(n) + \frac{1}{6}\right)\\
&\ll n^{k/2} 4^{\nu(N)} \sigma_1(n).
\end{aligned}$$

□

Proposition 3.2.18. *In the trace of $T_n^{new}(N,k)$ of Theorem 3.2.12,*

$$A_3(n,N,k) = \sum_{\substack{d \mid n\\ 0<d \le \sqrt{n}}}{}' d^{k-1} B_3(N)_d = O(n^{k/2} d(n)),$$

where the prime on the summation indicates that if there is a contribution from the term $d = \sqrt{n}$, it should be multiplied by $\frac{1}{2}$.

Proof. $B_3(N)_d$ is a multiplicative function of N such that for a prime power p^r,

$$B_3(p^r)_d = \begin{cases} -\phi(p^{\frac{r-2}{2}}) & \text{if r is even and } p^{\frac{r-2}{2}} \| (\frac{n}{d} - d),\\ \phi(p^{\frac{r}{2}}) - \phi(p^{\frac{r-2}{2}}) & \text{if r is even and } p^{\frac{r}{2}} \mid (\frac{n}{d} - d),\\ 0 & \text{otherwise.}\end{cases}$$

Hence,

$$B_3(p^r)_d = \begin{cases} -p^{\frac{r-2}{2}}\left(1 - \frac{1}{p}\right) & \text{if r is even and } p^{\frac{r-2}{2}} \| (\frac{n}{d} - d),\\ p^{\frac{r}{2}}\left(1 - \frac{1}{p}\right)^2 & \text{if r is even and } p^{\frac{r}{2}} \mid (\frac{n}{d} - d),\\ 0 & \text{otherwise.}\end{cases}$$

Thus,

$$|B_3(p^r)_d| \le \begin{cases} p^{\frac{r-2}{2}} = 2^{\nu_p(\frac{n}{d} - d)} & \text{if r is even and } p^{\frac{r-2}{2}} \| (\frac{n}{d} - d),\\ p^{\frac{r}{2}} \le 2^{\nu_p(\frac{n}{d} - d)} & \text{if r is even and } p^{\frac{r}{2}} \mid (\frac{n}{d} - d),\\ 0 & \text{otherwise.}\end{cases}$$

So, for any $r \in \mathbb{N}$,

$$|B_3(p^r)_d| \le 2^{\nu_p(\frac{n}{d} - d)} \le p^{\nu_p(\frac{n}{d} - d)}.$$

Therefore, using the fact, $B_3(N)_d$ is a multiplicative function of N, we obtain that

$$B_3(N)_d = B_3\left(\prod_{p \mid N} p^{\nu_p(N)}\right)_d = \prod_{p \mid N} B_3\left(p^{\nu_p(N)}\right)_d,$$

and hence,

$$|B_3(N)_d| \le \prod_{p \mid N} \left|B_3\left(p^{\nu_p(N)}\right)_d\right| \le \prod_{p \mid N} p^{\nu_p(\frac{n}{d} - d)} \le \frac{n}{d} - d, \tag{3.15}$$

where we use Lemma 3.2.15 in the last inequality. Therefore, using equation (3.15), we obtain

$$|A_3(n,N,k)| \le \sum_{\substack{d \mid n\\ 0<d \le \sqrt{n}}}{}' d^{k-1} |B_3(N)_d|$$

$$
\begin{aligned}
&\le \sum_{\substack{d|n \\ 0<d\le\sqrt{n}}} d^{k-1}\left(\frac{n}{d}-d\right)\\
&\le \sum_{\substack{d|n \\ 0<d\le\sqrt{n}}} d^{k-2}(n-d^2)\\
&\le n\sum_{\substack{d|n \\ 0<d\le\sqrt{n}}} d^{k-2}\\
&\le nn^{(k-2)/2}d(n)\\
&\le n^{k/2}d(n).
\end{aligned}
$$

□

Proposition 3.2.19. *In the trace of $T_n^{new}(N,k)$ of Theorem 3.2.12,*

$$
A_4(n,N,k) = \begin{cases} \mu(N)\sum_{t|n} t & \text{if } k = 2,\\ 0 & \text{otherwise,}\end{cases} = O(\sigma_1(n)).
$$

Proof.

$$
|A_4(n,N,k)| = \begin{cases} |\mu(N)||\sum_{t|n} t| & \text{if } k = 2,\\ 0 & \text{otherwise,}\end{cases} \le \begin{cases} \sigma_1(n) & \text{if } k = 2,\\ 0 & \text{otherwise.}\end{cases}
$$

We also note that if N is not squarefree, then $\mu(N) = 0$, and hence, $A_4(n,N,k) = 0$. □

Theorem 3.2.20. *Let n be a positive integer coprime to N. The trace of the Hecke operator $T_n^{new}(N,k)$ acting on $S_k^{new}(N)$ is given by*

$$
TrT_n^{new}(N,k) = \begin{cases} n^{(k/2-1)}\frac{k-1}{12}NB_1(N) + O(4^{\nu(N)}n^{k/2}\sigma_1(n)) & \text{if } n \text{ is a square,}\\ O(4^{\nu(N)}n^{k/2}\sigma_1(n)) & \text{otherwise,}\end{cases}
$$

where $B_1(N)$ is as mentioned in Theorem 3.2.12.

Proof. Applying the bounds for $A_i(n,N,k)$ $(i = 2,3,4)$ obtained in Propositions 3.2.17, 3.2.18 and 3.2.19 in Theorem 3.2.12, we obtain

$$
\begin{aligned}
\mathrm{Tr}T_n^{new}(N,k) &= A_1(n,N,k) + A_2(n,N,k) + A_3(n,N,k) + A_4(n,N,k)\\
&= A_1(n,N,k) + O(4^{\nu(N)}n^{k/2}\sigma_1(n) + O(n^{k/2}d(n)) + O(\sigma_1(n))\\
&= A_1(n,N,k) + O(4^{\nu(N)}n^{k/2}\sigma_1(n)) \qquad (3.16)\\
&= \begin{cases} n^{(k/2-1)}\frac{k-1}{12}NB_1(N) + O(4^{\nu(N)}n^{k/2}\sigma_1(n)) & \text{if } n \text{ is a square,}\\ O(4^{\nu(N)}n^{k/2}\sigma_1(n)) & \text{otherwise.}\end{cases}
\end{aligned}
$$

□

Corollary 3.2.21. *Let N and k be positive integers with k even and $(n,N) = 1$. The trace of the normalised Hecke operator $\hat{T}_n^{new}(N,k)$ acting on $S_k^{new}(N)$ is given by*

$$
Tr\hat{T}_n^{new}(N,k) = \begin{cases} \frac{1}{\sqrt{n}}\frac{k-1}{12}NB_1(N) + O(4^{\nu(N)}\sqrt{n}\sigma_1(n)) & \text{if } n \text{ is a square,}\\ O(4^{\nu(N)}\sqrt{n}\sigma_1(n)) & \text{otherwise,}\end{cases}
$$

where $\hat{T}_n^{new}(N,k) := \frac{T_n^{new}(N,k)}{n^{(k-1)/2}}$.

Proof. The proof follows from equation (3.16), when we divide both sides of the equation by $n^{(k-1)/2}$. □

Corollary 3.2.22. *Let N and k be positive integers with k even and $(n,N) = 1$. Let $F_{N,k}$ be the set of Hecke newforms of weight k and level N (as mentioned in Definition 3.1.55) and $n^{\frac{k-1}{2}} a_f(n)$ denote the n-th Fourier coefficient of $f \in F_{N,k}$. Then*

$$\sum_{f \in F_{N,k}} a_f(n) = \begin{cases} \frac{1}{\sqrt{n}} \frac{k-1}{12} N B_1(N) + O(4^{\nu(N)} \sqrt{n}\sigma_1(n)) & \text{if } n \text{ is a square,} \\ O(4^{\nu(N)} \sqrt{n}\sigma_1(n)) & \text{otherwise.} \end{cases}$$

Proof. We note that for each positive integer n such that $(n,N) = 1$, the sum $\sum_{f \in F_{N,k}} a_f(n)$ is precisely the trace of the normalised Hecke operator $\hat{T}_n^{new}(N,k) = \frac{T_n^{new}(N,k)}{n^{(k-1)/2}} = \frac{T_n(N,k)}{n^{(k-1)/2}}$ acting on $S_k^{new}(N)$, by equation (3.7), i.e., $\sum_{f \in F_{N,k}} a_f(n) = \mathrm{Tr}\hat{T}_n^{new}(N,k)$. The proof now follows from Corollary 3.2.21 □

Lemma 3.2.23. *Let N and k be positive integers with k even. Then,*

$$\frac{A_0(k-1)}{12}\phi(N) + O(\sqrt{N}) < |F_{N,k}| < \frac{(k-1)}{12}\phi(N) + O(2^{\nu(N)}), \text{ where}$$

$$A_0 = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^2 - p}\right) \approx 0.373956.$$

Moreover, if N is not a perfect square, the lower bound can be improved to

$$\frac{A_0(k-1)}{12}\phi(N) + O(2^{\nu(N)}) \leq |F_{N,k}|.$$

Proof. We refer the readers to [Mar05, Theorem 6(C)] for a proof. □

Lemma 3.2.24. *Let N and k be positive integers with k even. Then, $|F_{N,k}| \asymp \frac{Nk}{2^{\nu(N)}}$ as $N,k \to \infty$.*

Proof. Using Lemma 3.2.23, we have $|F_{N,k}| > \frac{A_0(k-1)}{12}\phi(N) + O(\sqrt{N})$, for all even integers $k \geq 2$ and all integers $N \geq 1$, where A_0 is a positive constant.

We also know, for all $N \in \mathbb{N}$, $\phi(N) \gg \frac{N}{2^{\nu(N)}}$.

Thus, for $N \in \mathbb{N}$, $|F_{N,k}| > \frac{A_0(k-1)}{12}\phi(N) + O(\sqrt{N}) \gg \frac{A_0(k-1)}{12}\frac{N}{2^{\nu(N)}} + O(\sqrt{N})$.

Hence, $|F_{N,k}|\frac{2^{\nu(N)}}{Nk} > (\frac{A_0(k-1)}{12}\frac{N}{2^{\nu(N)}} + O(\sqrt{N}))\frac{2^{\nu(N)}}{Nk} = \frac{A_0}{12}\left(1 - \frac{1}{k}\right) + O\left(\frac{2^{\nu(N)}}{k\sqrt{N}}\right)$.

Therefore, $\frac{1}{|F_{N,k}|}\frac{Nk}{2^{\nu(N)}} < \frac{1}{\frac{A_0}{12}(1-\frac{1}{k}) + O(\frac{2^{\nu(N)}}{k\sqrt{N}})}$.

We know, $2^{\nu(N)} \leq d(N)$, and $d(N) \ll N^{\epsilon}$, for any $\epsilon > 0$, where $d(N)$ denotes the divisor function and $\nu(N)$ denotes the number of distinct prime divisors of N.

Therefore, $2^{\nu(N)} \ll N^{1/4}$, and this implies $O(\frac{2^{\nu(N)}}{k\sqrt{N}}) = O(\frac{1}{kN^{1/4}}) \to 0$ as $N,k \to \infty$.

Therefore, $\lim_{N,k \to \infty} \frac{1}{|F_{N,k}|}\frac{Nk}{2^{\nu(N)}} \leq \frac{12}{A_0}$. This completes the proof. □

Theorem 3.2.25. *Let n, N and k be positive integers with k even and $(n,N) = 1$. Then,*

(a)

$$\sum_{f \in F_{N,k}} a_f(n) = \begin{cases} \frac{|F_{N,k}|}{\sqrt{n}} + O(4^{\nu(N)} \sqrt{n}\sigma_1(n)) & \text{if } n \text{ is a square,} \\ O(4^{\nu(N)} \sqrt{n}\sigma_1(n)) & \text{otherwise.} \end{cases}$$

(b)

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(n) = \begin{cases} \frac{1}{\sqrt{n}} + O\left(\frac{8^{\nu(N)}\sqrt{n}\sigma_1(n)}{kN}\right) & \text{if } n \text{ is a square,} \\ O\left(\frac{8^{\nu(N)}\sqrt{n}\sigma_1(n)}{kN}\right) & \text{otherwise.} \end{cases}$$

Here, $\sigma_1(n)$ refers to the sum of positive divisors of n and the implied constants in the error terms are absolute.

Proof. We note that $\hat{T}_1^{new}(N,k)$ is the identity map on $S_k^{new}(N)$ by equation (3.7). Hence, we have $Tr(\hat{T}_1^{new}(N,k)) = \sum_{f\in F_{N,k}} a_f(1) = \sum_{f\in F_{N,k}} 1 = |F_{N,k}| = \dim S_k^{new}(N)$, where $F_{N,k}$ is as mentioned in Definition 3.1.55. In particular, if we put $n = 1$ in Corollary 3.2.21, we get

$$|F_{N,k}| = \mathrm{Tr}\hat{T}_1^{new}(N,k) = \frac{k-1}{12}NB_1(N) + O(4^{\nu(N)}).$$

Hence,

$$\frac{k-1}{12}NB_1(N) = |F_{N,k}| + O(4^{\nu(N)}).$$

If n is a square, Corollary 3.2.22 gives,

$$\begin{aligned}\sum_{f\in F_{N,k}} a_f(n) &= \frac{1}{\sqrt{n}}\,\frac{k-1}{12}NB_1(N) + O(4^{\nu(N)}\sqrt{n}\sigma_1(n)) \\ &= \frac{1}{\sqrt{n}}(|F_{N,k}| + O(4^{\nu(N)})) + O(4^{\nu(N)}\sqrt{n}\sigma_1(n)) \\ &= \frac{|F_{N,k}|}{\sqrt{n}} + O(4^{\nu(N)}\sqrt{n}\sigma_1(n)).\end{aligned}$$

This completes the proof of (a).

Dividing both sides of the above equation by $|F_{N,k}|$, we obtain

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(n) = \frac{1}{\sqrt{n}} + O\left(\frac{4^{\nu(N)}\sqrt{n}\sigma_1(n)}{|F_{N,k}|}\right).$$

Using Lemma 3.2.24, we obtain $\frac{1}{|F_{N,k}|} \ll \frac{2^{\nu(N)}}{kN}$ as $N,k \to \infty$, and hence,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(n) = \frac{1}{\sqrt{n}} + O\left(\frac{8^{\nu(N)}\sqrt{n}\sigma_1(n)}{kN}\right).$$

□

Corollary 3.2.26. *Let n, N and k be positive integers with k even and $(n,N) = 1$. Then, for any $c' > \frac{3}{2}$,*

(a)

$$\sum_{f\in F_{N,k}} a_f(n) = \begin{cases} \frac{|F_{N,k}|}{\sqrt{n}} + O(4^{\nu(N)}n^{c'}) & \text{if } n \text{ is a square,} \\ O(4^{\nu(N)}n^{c'}) & \text{otherwise.} \end{cases}$$

(b)

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(n) = \begin{cases} \frac{1}{\sqrt{n}} + O\left(\frac{8^{\nu(N)}n^{c'}}{kN}\right) & \text{if } n \text{ is a square,} \\ O\left(\frac{8^{\nu(N)}n^{c'}}{kN}\right) & \text{otherwise.} \end{cases}$$

Here the implied constants in the error terms are absolute.

Proof. Let n be a square. Since $\sigma_1(n) = \sum_{d|n} d \le nd(n) \ll n^{1+\epsilon}$, for any $\epsilon > 0$, we have

$$\begin{aligned}\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} a_f(n) &= \frac{1}{\sqrt{n}} + O\left(\frac{8^{\nu(N)}\sqrt{n}\sigma_1(n)}{kN}\right)\\ &= \frac{1}{\sqrt{n}} + O\left(\frac{8^{\nu(N)}n^{\frac{3}{2}+\epsilon}}{kN}\right), \text{ for any } \epsilon > 0.\end{aligned}$$

Therefore, if n is a square,

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} a_f(n) = \frac{1}{\sqrt{n}} + O\left(\frac{8^{\nu(N)}n^{c'}}{kN}\right), \text{ for any } c' > \frac{3}{2}.$$

□

Corollary 3.2.27. *Let N and k be positive integers with k even. Let $\{p_1, p_2, \ldots, p_t\}$ be a finite set of distinct primes such that $(p_i, N) = 1$, and $m_i \in \mathbb{N} \cup \{0\}$, for each $i = 1, 2, \ldots, t$. Then, for any $c' > \frac{3}{2}$,*

$$\begin{aligned}&\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} a_f(p_1^{2m_1}p_2^{2m_2}\ldots p_t^{2m_t})\\ &= \frac{1}{p_1^{m_1}p_2^{m_2}\ldots p_t^{m_t}} + O\left(\frac{8^{\nu(N)}(p_1^{2m_1}p_2^{2m_2}\ldots p_t^{2m_t})^{c'}}{kN}\right)\end{aligned}$$

Here the implied constant in the error term is absolute.

Proof. The proof follows from Corollary 3.2.26. □

Corollary 3.2.28. *Let k be a positive even integer and N be a positive integer. For a prime p such that $(p, N) = 1$ and $l \geq 1$, we have*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} a_f(p^{2l}) = \frac{1}{p^l} + O\left(\frac{lp^{3l}4^{\nu(N)}}{|F_{N,k}|}\right).$$

Proof. The proof follows from Theorem 3.2.25 and the fact $\sigma_1(n) \le nd(n)$. □

Remark 3.2.29. *Let N, k and n be positive integers with k even. Then, for any $c' > \frac{3}{2}$ and $\beta \in (0, 1)$,*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} a_f(n) = \begin{cases}\frac{1}{\sqrt{n}} + O\left(\frac{n^{c'}}{kN^\beta}\right) & \text{if } n \text{ is a square,}\\ O\left(\frac{n^{c'}}{kN^\beta}\right) & \text{otherwise.}\end{cases}$$

Proof. We note that $8^{\nu(N)} \le d(N)^3 \ll N^{1-\beta}$, for any $\beta \in (0, 1)$.

Therefore, if n is a square, for any $c' > \frac{3}{2}$ and $\beta \in (0, 1)$,

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} a_f(n) = \frac{1}{\sqrt{n}} + O\left(\frac{n^{c'}N^{1-\beta}}{kN}\right) = \frac{1}{\sqrt{n}} + O\left(\frac{n^{c'}}{kN^\beta}\right).$$

□

3.2.5 Key estimates from the trace formula

The following lemmas form the key to a major part of the calculations in Section 6.2.

Lemma 3.2.30. *Let $k = k(x)$ and $N = N(x)$ be positive integers with k even. If $(m_1, m_2, \ldots, m_t) = (0, 0, \ldots, 0)$, then*

$$\frac{1}{|F_{N,k}|} \sum_{\substack{p_1,p_2,\ldots,p_t \leq x \\ \text{all distinct primes}}} \sum_{f \in F_{N,k}} a_f(p_1^{2m_1} p_2^{2m_2} \ldots p_t^{2m_t}) = \prod_{i=0}^{t-1} (\pi_N(x) - i).$$

On the other hand, for an integer $a \geq 0$, let

$$\sum_{(m_1,m_2,m_3,\ldots,m_t)}^{(a)}$$

denote a sum over a subset of the set of t-tuples

$$\{(m_1, m_2, m_3, \ldots, m_t) \in \mathbb{Z}^t : 1 \leq m_i \leq M_i \text{ if } 1 \leq i \leq t - a \text{ and } m_i = 0 \text{ if } t - a + 1 \leq i \leq t\}.$$

Then,

$$\frac{1}{|F_{N,k}|} \sum_{\substack{p_1,p_2,\ldots,p_t \leq x \\ \text{all distinct primes}}} \sum_{(m_1,m_2,m_3,\ldots,m_t)}^{(a)} \sum_{f \in F_{N,k}} a_f(p_1^{2m_1} p_2^{2m_2} \ldots p_t^{2m_t})$$
$$= O\left(\pi_N(x)^a (\log\log x)^{t-a}\right) + O\left(\frac{\pi_N(x)^t x^{(2M_1+2M_2+\cdots+2M_t)c'} 8^{\nu(N)}}{kN}\right).$$

Here, c' is a fixed number greater than $\frac{3}{2}$ and the implied constant in the error terms is absolute.

Proof. We denote $t' = t - a$.

Using Corollary 3.2.27, we obtain

$$\frac{1}{|F_{N,k}|} \sum_{\substack{p_1,p_2,\ldots,p_t \leq x \\ \text{all distinct primes}}} \sum_{f \in F_{N,k}} a_f(p_1^{2m_1} p_2^{2m_2} \ldots p_t^{2m_t})$$
$$= \frac{1}{|F_{N,k}|} \sum_{\substack{p_1,p_2,\ldots,p_t \leq x \\ \text{all distinct primes}}} \sum_{f \in F_{N,k}} a_f(p_1^{2m_1} p_2^{2m_2} \ldots p_{t'}^{2m_{t'}})$$
$$= \sum_{\substack{p_{t'+1},\ldots,p_{t'+a} \leq x \text{ all distinct} \\ p_{t'+i} \notin \{p_1,\ldots,p_{t'},\ldots,p_{t'+i-1}\}}} 1 \sum_{\substack{p_1,\ldots,p_{t'} \leq x \\ \text{all distinct primes}}} \left(\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} a_f(p_1^{2m_1} p_2^{2m_2} \ldots p_{t'}^{2m_{t'}})\right)$$
$$= (\pi_N(x) - t')(\pi_N(x) - t' - 1)(\pi_N(x) - t' - 2) \ldots (\pi_N(x) - t' - (a-1))$$
$$\sum_{\substack{p_1,p_2,\ldots,p_{t'} \leq x \\ \text{all distinct primes}}} \left(\frac{1}{p_1^{m_1} p_2^{m_2} \ldots p_{t'}^{m_{t'}}} + O\left(\frac{8^{\nu(N)} p_1^{2m_1c'} p_2^{2m_2c'} \ldots p_{t'}^{2m_{t'}c'}}{kN}\right)\right).$$

Therefore,

$$\frac{1}{|F_{N,k}|} \sum_{\substack{p_1,p_2,\ldots,p_t \leq x \\ \text{all distinct primes}}} \sum_{(m_1,m_2,m_3,\ldots,m_t)}^{(a)} \sum_{f \in F_{N,k}} a_f(p_1^{2m_1} p_2^{2m_2} \ldots p_t^{2m_t})$$

$$
\begin{aligned}
&\leq \pi_N(x)^a \left(\sum_{p \leq x} \sum_{m \geq 1} \frac{1}{p^m} \right)^{t'} \\
&+ \pi_N(x)^a \sum_{\substack{p_1,p_2,\ldots,p_{t'} \leq x \\ \text{all distinct primes}}} \sum_{(m_1,m_2,m_3,\ldots,m_t)}^{(a)} \frac{8^{\nu(N)} p_1^{2m_1c'} p_2^{2m_2c'} \ldots p_{t'}^{2m_{t'}c'}}{kN} \\
&\leq \pi_N(x)^a \left(\sum_{p \leq x} \sum_{m \geq 1} \frac{1}{p^m} \right)^{t'} + \pi_N(x)^a \left(\frac{8^{\nu(N)}}{kN} \right) \sum_{\substack{p_1,p_2,\ldots,p_{t'} \leq x \\ \text{all distinct primes}}} \prod_{i=1}^{t'} \left(\sum_{m_i=1}^{M_i} p_i^{2m_ic'} \right) \\
&\leq \pi_N(x)^a \left(\sum_{p \leq x} \sum_{m \geq 1} \frac{1}{p^m} \right)^{t'} + \pi_N(x)^a \left(\frac{8^{\nu(N)}}{kN} \right) \sum_{\substack{p_1,p_2,\ldots,p_{t'} \leq x \\ \text{all distinct primes}}} \left(\prod_{i=1}^{t'} p_i^{2M_ic'} \right) \\
&\leq \pi_N(x)^a \left(\sum_{p \leq x} \frac{1}{p} \right)^{t'} + \pi_N(x)^a \left(\frac{8^{\nu(N)} x^{(2M_1+2M_2+\cdots+2M_{t'})c'}}{kN} \right) \sum_{\substack{p_1,p_2,\ldots,p_{t'} \leq x \\ \text{all distinct primes}}} 1 \\
&\ll \pi_N(x)^a (\log\log x)^{t'} + \pi_N(x)^{a+t'} \left(\frac{8^{\nu(N)} x^{(2M_1+2M_2+\cdots+2M_{t'})c'}}{kN} \right) \\
&\ll \pi_N(x)^a (\log\log x)^{t-a} + \frac{\pi_N(x)^t 8^{\nu(N)} x^{(2M_1+2M_2+\cdots+2M_t)c'}}{kN}.
\end{aligned}
$$

□

3.3 Properties of Hecke eigenvalues and estimates

In this section, we collect properties of Hecke eigenvalues and estimates for their averages. These will be used in the proof of Theorem 4.3.1. The following lemmas will be useful in determining the asymptotic conditions under which our main theorem holds.

Lemma 3.3.1. *Let f, g be functions with $\lim_{x \to \infty} g(x) = \infty$ and there exist numbers $x_0 \in \mathbb{R}$, $M \in (0,1)$ such that $|f(x)| \leq M g(x)$ for all $x > x_0$. Then $e^{f(x)} \ll e^{g(x)}$.*

Proof. $\lim_{x \to \infty} g(x) = \infty$ implies that there exists x_0' such that $g(x) > 0$ for all $x > x_0'$.

Since $t \mapsto e^t$ is an increasing function, the given condition implies, for all $x > x_0'' = \max\{x_0, x_0'\}$,

$$
\begin{aligned}
& |f(x)| \leq M g(x) \\
\Longrightarrow\ & -Mg(x) \leq f(x) \leq Mg(x) \\
\Longrightarrow\ & -(M+1)g(x) \leq f(x) - g(x) \leq (M-1)g(x) \\
\Longrightarrow\ & e^{-(M+1)g(x)} \leq e^{f(x)-g(x)} \leq e^{(M-1)g(x)} \\
\Longrightarrow\ & \lim_{x \to \infty} e^{f(x)-g(x)} = 0,
\end{aligned}
$$

by sandwich theorem.

Therefore, corresponding to some $K > 0$, there exists G such that for all $x > G$, $e^{f(x)-g(x)} \leq K$, i.e., $e^{f(x)} \leq K e^{g(x)}$.

Thus, $e^{f(x)} = O\left(e^{g(x)}\right)$. □

Lemma 3.3.2. *Let f, g be functions such that $g(x) > 0$ in its domain with $\lim_{x\to\infty} f(x) = \infty$ and $f(x) = o(g(x))$. Then $e^{f(x)} = o\left(e^{g(x)}\right)$, as $x \to \infty$.*

Proof. Let $G > 0$ be sufficiently large.

Since $\lim_{x\to\infty}\left(\frac{f(x)}{g(x)} - 1\right) = -1 = \beta(say)$, corresponding to $\epsilon = \frac{1}{2}$, there exists M' such that for all $x > M'$, $\frac{f(x)}{g(x)} - 1 < \beta + \epsilon = -\frac{1}{2}$, i.e., $f(x) - g(x) < -\frac{g(x)}{2}$.

Now, $g(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and $f(x) = o(g(x))$ implies $\lim_{x\to\infty} g(x) = \infty$.

Since $\lim_{x\to\infty} g(x) = \infty$, there exists M'' such that for all $x > M''$, $g(x) > 2G$, i.e., $-\frac{g(x)}{2} < -G$.

Thus, for all $x > M = \max\{M', M''\}$, $f(x) - g(x) < -\frac{g(x)}{2} < -G$.

Therefore, $\lim_{x\to\infty}(f(x) - g(x)) = -\infty$, meaning $\lim_{x\to\infty} e^{f(x)-g(x)} = 0$, i.e., $\lim_{x\to\infty}\frac{e^{f(x)}}{e^{g(x)}} = 0$. □

Corollary 3.3.3. *Let f, g be functions such that $g(x) > 1$ in its domain with $\lim_{x\to\infty} f(x) = \infty$ and $\log f(x) = o(\log g(x))$. Then $f(x) = o(g(x))$, as $x \to \infty$.*

Proof. We note that $g(x) > 1$ implies $\log g(x) > 0$ and $\lim_{x\to\infty} f(x) = \infty$ implies $\lim_{x\to\infty}\log f(x) = \infty$.

The proof now follows from Lemma 3.3.2. □

Lemma 3.3.4. *Let $l \geq 0, N, n \geq 1$, k be a positive even integer and p be a prime coprime to N. Let f be a newform in $F_{N,k}$ and $L_p(l,n) := a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2}))$. Then,*

1) $L_p(l,n; l>n) = a_f(p^{2l-2n}) + a_f(p^{2l+2n})$,

2) $L_p(l,n; l<n) = a_f(p^{2l+2n}) - a_f(p^{2n-2l-2})$,

3) $L_p(l,n; l=n) = 1 + a_f(p^{4l})$, *where the summations $L_p(l,n; l>n)$, $L_p(l,n; l<n)$, and $L_p(l,n : l=n)$ denote different cases $l > n$, $l < n$ and $l = n$ respectively.*

Proof.

1) Since $l > n$ implies that $l - n \geq 1$, we have $\min\{2l, 2n-2\} = 2n-2$ and $\min\{2l, 2n\} = 2n$.

Therefore, using Lemma 3.3.6, we obtain

$$
\begin{aligned}
& L_p(l,n; l>n) \\
=& a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2})) \\
=& a_f(p^{2l})a_f(p^{2n}) - a_f(p^{2l})a_f(p^{2n-2}) \\
=& \sum_{t=0}^{2n} a_f(p^{2l+2n-2t}) - \sum_{t=0}^{2n-2} a_f(p^{2l+2n-2-2t}) \\
=& \sum_{t=l-n}^{l+n} a_f(p^{2t}) - \sum_{t=l-n+1}^{l+n-1} a_f(p^{2t}) \\
=& a_f(p^{2l-2n}) + \sum_{t=l-n+1}^{l+n-1} a_f(p^{2t}) + a_f(p^{2l+2n}) - \sum_{t=l-n+1}^{l+n-1} a_f(p^{2t}) \\
=& a_f(p^{2l-2n}) + a_f(p^{2l+2n}).
\end{aligned}
$$

2) Since $l < n$ implies that $l \leq n - 1$, we have $\min\{2l, 2n-2\} = 2l$ and $\min\{2l, 2n\} = 2l$.

Therefore, using Lemma 3.3.6, we obtain

$$
\begin{aligned}
&L_p(l,n;l<n)\\
=&a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2}))\\
=&a_f(p^{2l})a_f(p^{2n}) - a_f(p^{2l})a_f(p^{2n-2})\\
=&\sum_{t=0}^{2l} a_f(p^{2l+2n-2t}) - \sum_{t=0}^{2l} a_f(p^{2l+2n-2-2t})\\
=&\sum_{t=n-l}^{l+n} a_f(p^{2t}) - \sum_{t=n-l-1}^{l+n-1} a_f(p^{2t})\\
=&\sum_{t=n-l}^{l+n-1} a_f(p^{2t}) + a_f(p^{2l+2n}) - a_f(p^{2n-2l-2}) - \sum_{t=n-l}^{l+n-1} a_f(p^{2t})\\
=&a_f(p^{2l+2n}) - a_f(p^{2n-2l-2}).
\end{aligned}
$$

3) Since $l=n$, using Lemma 3.3.6, we obtain

$$
\begin{aligned}
&L_p(l,n;l=n)\\
=&a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2}))\\
=&a_f(p^{2l})(a_f(p^{2l}) - a_f(p^{2l-2}))\\
=&a_f(p^{2l})a_f(p^{2l}) - a_f(p^{2l})a_f(p^{2l-2})\\
=&\sum_{t=0}^{2l} a_f(p^{2l+2l-2t}) - \sum_{t=0}^{2l-2} a_f(p^{2l+2l-2-2t})\\
=&\sum_{t=0}^{2l} a_f(p^{2t}) - \sum_{t=1}^{2l-1} a_f(p^{2t})\\
=&1+\sum_{t=1}^{2l-1} a_f(p^{2t}) + a_f(p^{4l}) - \sum_{t=1}^{2l-1} a_f(p^{2t})\\
=&1+a_f(p^{4l}).
\end{aligned}
$$

□

Corollary 3.3.5. *Let f be a newform in $F_{N,k}$. For a prime p coprime to the level N and integers $l \geq 0$ and $n \geq 1$,*

$$a_f(p^{2l})\left(a_f(p^{2n}) - a_f(p^{2n-2})\right) = a_f(p^{2l+2n}) + \begin{cases} a_f(p^{2l-2n}) & \text{if } l \geq n,\\ -a_f(p^{2n-2l-2}) & \text{if } l<n. \end{cases}$$

The next result regarding the multiplicative relationship between Hecke eigenvalues is classical.

Lemma 3.3.6. *For primes p_1, p_2 coprime to the level N and non-negative integers i, j,*

$$a_f(p_1^i)a_f(p_2^j) = \begin{cases} a_f(p_1^i p_2^j), & \text{if } p_1 \neq p_2,\\ \displaystyle\sum_{l=0}^{\min(i,j)} a_f(p_1^{i+j-2l}), & \text{if } p_1 = p_2. \end{cases}$$

Moreover, if $p_1 = p_2$, then

$$
\begin{aligned}
&\left(a_f(p_1^{2n_1}) - a_f(p_1^{2n_1-2})\right)\left(a_f(p_2^{2n_2}) - a_f(p_2^{2n_2-2})\right)\\
=&\begin{cases} a_f(p_1^{2n_1+2n_2}) - a_f(p_1^{2n_1+2n_2-2}) + a_f(p_1^{|2n_1-2n_2|}) - a_f(p_1^{|2n_1-2n_2|-2}), & \text{if } n_1 \neq n_2,\\ a_f(p_1^{4n_1}) - a_f(p_1^{4n_1-2}) + 2, & \text{if } n_1 = n_2. \end{cases}
\end{aligned}
$$

Proof. The first part of the proof follows from Corollary 3.1.61 and the second part follows from Lemma 3.3.4. □

Lemma 3.3.7. *Let us consider positive integers* $k = k(x)$ *and* $N = N(x)$ *such that*

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \ ! \ 1 \text{ as } x \ ! \ 1 .$$

Then, for any absolute constant $C > 0$*,*

$$x^{C\pi_N(x)} = o\left(\frac{kN}{8^{\nu(N)}}\right) \text{ as } x \ ! \ 1$$

Proof. Using Corollary 3.3.3, we observe that $x^{C\pi_N(x)c^0} = o\left(kN/8^{\nu(N)}\right)$ holds if $C\pi_N(x)\log x = o\left(\log\left(kN/8^{\nu(N)}\right)\right)$, i.e., if $x = o\left(\log\left(kN/8^{\nu(N)}\right)\right)$, as $x \ ! \ 1$. □

Chapter 4

History of the thesis problem and new results

Let k and N be positive integers with k even. Let $S_k(N)$ denote the space of modular cusp forms of weight k with respect to $\Gamma_0(N)$. For $n \geq 1$, let T_n denote the n-th Hecke operator acting on $S_k(N)$. We denote the set of Hecke newforms in $S_k(N)$ by $\mathcal{F}_{N,k}$. Any $f(z) \in \mathcal{F}_{N,k}$ has a Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} a_f(n) q^n, \qquad q = e^{2\pi i z},$$

where $a_f(1) = 1$ and

$$\frac{T_n(f(z))}{n^{\frac{k-1}{2}}} = a_f(n) f(z), \; n \geq 1.$$

We denote $s(N,k) := |\mathcal{F}_{N,k}|$ and note that $s(N,k)$ is the dimension of the subspace of primitive cusp forms in $S_k(N)$.

Let p be a prime number with $(p,N) = 1$. By a conjecture of Ramanujan, which was later proved by Deligne, the eigenvalues $a_f(p)$ lie in the interval $[-2,2]$. We denote $a_f(p) = 2\cos\pi\,\theta_f(p)$, with $\theta_f(p) \in [0,1]$.

The Sato-Tate conjecture, now a theorem [BLGHT11] due to Barnet-Lamb, David Geraghty, Harris, and Taylor in 2011, is the assertion that if f is a non-CM newform in $\mathcal{F}_{N,k}$, then the Sato-Tate sequence

$$\{\theta_f(p) : p \text{ prime}, (p,N) = 1\} \subset [0,1]$$

is equidistributed in the interval $[0,1]$ with respect to the measure $\mu(t)dt$, where $\mu(t) = 2\sin^2(\pi t)$. That is, for an interval $I = [a,b]$ such that $0 \leq a < b \leq 1$,

$$\lim_{x \to \infty} \frac{1}{\pi_N(x)} \#\{p \leq x : (p,N) = 1, \theta_f(p) \in [a,b]\} = \mu(I), \tag{4.1}$$

where $\pi_N(x)$ denotes the number of primes $p \leq x$ such that $(p,N) = 1$, and $\mu(I)$ denotes the measure $\int_I \mu(t)dt$ of the interval I.

We "straighten out" the Sato-Tate sequence into a uniformly distributed sequence by defining,

$$H(\theta_f(p)) := \int_0^{\theta_f(p)} \mu(t)dt. \tag{4.2}$$

As an immediate consequence of (4.1), we see that the sequence $\{H(\theta_f(p)) : (p,N) = 1\}$ is uniformly distributed in the interval $[0,1]$.

4.1 Katz-Sarnak Conjecture

A study of the moments of the pair correlation function for the sequence $\{H(\theta_f(p)) : (p,N)=1\}$ as one varies f over appropriate families $F_{N,k}$ was initiated in [BS19]. This study is primarily motivated by a question of Katz and Sarnak that compares the spacings between straightened Hecke angles to spacings between points arising from independent and uniformly distributed random variables in the unit interval. As we saw in Chapter 2, one way to address these questions is via the pair correlation function, which looks at the spacings between unordered elements of a uniformly distributed sequence. In this context, the question of Katz and Sarnak can be stated as follows:

Question 4.1.1 (Katz, Sarnak [KS99]). *For any $s>0$, the pair correlation function of the sequence $\{H(\theta_f(p)) : p$ prime, $(p,N)=1\}$ is defined as:*

$$R(x,s)(f) := \frac{1}{\pi_N(x)} \# \left\{ (p,q) : p \neq q \leq x, \begin{array}{c} (p,N)=(q,N)=1, \\ H(\theta_f(p)) - H(\theta_f(q)) \in \left[\frac{-s}{\pi_N(x)}, \frac{s}{\pi_N(x)}\right] + \mathbb{Z} \end{array} \right\}.$$

For any $s>0$, does the limit $\lim_{x\to\infty} R(x,s)(f)$ exist and is it equal to $2s$?

If the answer is yes, we say that the sequence $\{H(\theta_f(p)) : p$ prime, $(p,N)=1\}$ has Poissonnian pair correlation.

A variation of the question above was addressed in [BS19] by restricting $\theta_f(p)$ to short intervals I, such that $|I|\to 0$ as $x\to\infty$.

Question 4.1.2. *Let $0<\psi<1$ and I_δ denote intervals of the form*

$$[\psi-\delta, \psi+\delta],\ \delta=\delta(x)\to 0 \text{ as } x\to\infty.$$

Suppose

$$\#\left\{p \leq x : (p,N)=1,\ \theta_f(p)\in I_\delta\right\} \sim \pi_N(x)\mu(I_\delta) \text{ as } x\to\infty. \tag{4.3}$$

We define

$$\widetilde{R}_\delta(x,s)(f) := \frac{1}{\pi_N(x)\mu(I_\delta)} \# \left\{ (p,q) : p \neq q \leq x, \begin{array}{c} (p,N)=(q,N)=1,\ \theta_f(p),\theta_f(q)) \in I_\delta, \\ H(\theta_f(p)) - H(\theta_f(q)) \in \left[\frac{-s}{\pi_N(x)}, \frac{s}{\pi_N(x)}\right] \end{array} \right\}.$$

Does the limit $\lim_{x\to\infty} \widetilde{R}_\delta(x,s)(f)$ exist and is it equal to $2s$?

To answer the above question meaningfully, we need conditions on $\delta(x)$ for which (4.3) holds. The existence and distribution of Hecke angles in shrinking intervals I with $|I|\to 0$ as $x\to\infty$ is inextricably linked to effective error terms in the Sato-Tate equidistribution theorem (we explain this in detail later). These error terms have been addressed in [Mur85], [RT17], [Tho21] and [HIJS22]. In this context, an unconditional theorem of Thorner leads to the following result:

Theorem 4.1.3 (Thorner, [Tho21]). *Let $F(x)$ be a monotonically increasing function with $\lim_{x\to\infty} F(x)=\infty$. Then, for any interval $I\subseteq[0,1]$ of length,*

$$\mu(I) \gg \frac{\log(kN\log x)F(x)}{\sqrt{\log x}},$$

we have

$$\lim_{x\to\infty} \frac{1}{\pi_N(x)} \#\{p\leq x : (p,N)=1,\ \theta_f(p)\in I\} = \mu(I).$$

In particular, if $\delta(x)\to 0$ is chosen such that

$$\mu(I_\delta) \gg \frac{\log(kN\log x)F(x)}{\sqrt{\log x}},$$

then

$$\#\{p\leq x : (p,N)=1,\ \theta_f(p)\in I_\delta\} \sim \pi_N(x)\mu(I_\delta) \text{ as } x\to\infty.$$

One simplifies Question 4.1.2 as follows: for $0 < \psi < 1$, henceforth, we denote $A := 2\sin^2 \pi\psi$. Let us consider intervals

$$I_L(\psi) := \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right],$$

such that $L = L(x) \to \infty$ as $x \to \infty$, and (4.3) holds for $\delta = 1/AL$. Then, as $x \to \infty$,

$$L_f(\psi) := \#\left\{p \leq x : (p, N) = 1, \theta_f(p) \in I_L(\psi)\right\} \sim \pi_N(x)\mu(I_L(\psi)).$$

The advantage of localizing our intervals around ψ is that the Sato-Tate density $2\sin^2 \pi t \approx A$ is essentially constant in short intervals and the straightening of the Hecke angles is more or less equivalent to rescaling them. That is,

$$L_f(\psi) \sim \pi_N(x) \int_{\psi - \frac{1}{AL}}^{\psi + \frac{1}{AL}} 2\sin^2 \pi t \, dt \approx A \frac{2}{AL} \pi_N(x) = \frac{2\pi_N(x)}{L}$$

and if $\theta_f(p)$, $\theta_f(p) \in I_L(\psi)$, then

$$H(\theta_f(p)) - H(\theta_f(q)) = \int_{\theta_f(q)}^{\theta_f(p)} 2\sin^2 \pi t \, dt \approx A(\theta_f(p) - \theta_f(q)) \text{ as } x \to \infty.$$

Thus,

$$\widetilde{R}_{\frac{1}{AL}}(x, s)(f) = \frac{1}{L_f(\psi)} \# \left\{ (p, q) : p \neq q \leq x, \begin{array}{c} (p, N) = (q, N) = 1, \theta_f(p)), \theta_f(q) \in I_L(\psi), \\ H(\theta_f(p)) - H(\theta_f(q)) \in \left[\frac{-s}{\pi_N(x)}, \frac{s}{\pi_N(x)}\right] \end{array} \right\}$$

$$\approx \frac{1}{L_f(\psi)} \# \left\{ (p, q) : p \neq q \leq x, \begin{array}{c} (p, N) = (q, N) = 1, \theta_f(p), \theta_f(q) \in I_L(\psi), \\ \theta_f(p) - \theta_f(q) \in I_x \end{array} \right\},$$

where

$$I_x = \left[\frac{-s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)}\right].$$

The pair correlation function of a sequence is obtained by evaluating some exponential sums related to the sequence. In the case of Hecke angles, we have to remove the imaginary parts of these sums in order to apply existing techniques. Therefore, we modify the above question and consider the families

$$A_{f,x} := \{\pm\theta_f(p) \bmod 1 : p \leq x, (p, N) = 1\}.$$

As explained in Section 4.4, the pair correlation function of the families $A_{f,x}$ turns out to be asymptotic to

$$R_{1/L}(x, s)(f) := \frac{L}{8\pi_N(x)} \sum_{\substack{(p,q): p \neq q \leq x \\ (p,N)=(q,N)=1}} \Big(\sum_{n \in \mathbb{Z}} \chi_{[-\frac{1}{A}, \frac{1}{A}]}(L(\pm\theta_f(p) - \psi + n))$$
$$\sum_{n \in \mathbb{Z}} \chi_{[-\frac{1}{A}, \frac{1}{A}]}(L(\pm\theta_f(q) - \psi + n)) \sum_{n \in \mathbb{Z}} \chi_{[-\frac{s}{A}, \frac{s}{A}]}(\pi_N(x)(\pm\theta_f(p) \pm \theta_f(q) + n)) \Big)$$

While the function $R_{1/L}(x, s)(f)$ is difficult to study (we explain this in Sections 4.5.1 and 4.5.2), one way to address its convergence can be through the method of moments. That is, one may study the moments

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \left(R_{1/L}(x, s)(f)\right)^r, \; r \in \mathbb{N}.$$

4.2 Motivation for consideration of the thesis problem

The perspective of averaging quantities related to f over all Hecke newforms (or eigenforms) goes back to the work of [Sar87], [CDF97] and [Ser97]. In order to approach difficult arithmetic questions pertaining to a Hecke newform f (such as the distribution or spacing properties of Hecke angles $\theta_f(p)$), one can ask what happens to those questions “on average" over families of eigenforms. Summing over all Hecke newforms (or eigenforms as the case may be) allows us to bring in techniques such as the Eichler-Selberg trace formula for the trace of Hecke operators acting on subspaces of cusp forms of weight k with respect to $\Gamma_0(N)$. For example, Conrey, Duke and Farmer [CDF97] used the trace formula to prove that the Sato-Tate conjecture holds on average over large families. That is, if $k(x) > e^x$, they showed that

$$\lim_{x\to\infty} \frac{1}{|F_{1,k}|}\sum_{f\in F_{1,k}} \left(\frac{1}{\pi_N(x)}\#\{p \le x : \theta_f(p)\in[a,b]\}\right) = \int_I 2\sin^2\pi t\,dt,$$

In [Nag06], it is shown that the above asymptotic holds when $\frac{\log k}{\log x}\to\infty$ as $x\to\infty$.

In [BS19], this approach of averaging is adopted in the investigation of the pair correlation function for the Hecke angles. Since we also let the levels N vary, the growth conditions take into account the contribution coming from them. Moreover, it is feasible to consider a smooth variant of $R_{1/L}(x,s)(f)$. This leads to the following theorem:

Theorem 4.2.1 (Balasubramanyam, Sinha, [BS19]). *Let us consider families $F_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$. Let g, ρ be real-valued, even functions $\in C^\infty(\mathbb{R})$ in the Schwartz class with Fourier transforms supported in the interval $[-1,1]$. Let $0<\psi<1$, $\psi\neq 1/2$. Define $A := 2\sin^2\pi\psi$. Define*

$$\rho_L(\theta) := \sum_{n\in\mathbb{Z}} \rho(L(\theta+n)) \text{ for } L = L(x) \geq 1, \tag{4.4}$$

$$G_x(\theta) := \sum_{n\in\mathbb{Z}} g\left(\pi_N(x)(\theta+n)\right),$$

and the smooth localized pair correlation function,

$$R_2(\rho,g;f)(x) := \frac{L}{8\pi_N(x)} \sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1}} \rho_L(\theta_f(p)-\psi)\rho_L(\theta_f(q)-\psi)G_x(\theta_f(p)-\theta_f(q)). \tag{4.5}$$

[Note: A detailed discussion of the above definitions is presented in Section 4.4.]

(a) *Let $L = L(x)\to\infty$ as $x\to\infty$, and $L \leq \pi_N(x)$. Then*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} R_2(\rho,g;f)(x)$$
$$= \frac{T(g,\rho)}{4L} + O\left(\frac{1}{L}\right) + O\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + O\left(\frac{8^{\nu(N)}x^{(8\pi_N(x)+8)c'}}{kN}\right),$$

as $x\to\infty$, where c' is an absolute positive constant and

$$T(g,\rho) = \sum_{l\geq 1}(U(l)-U(l-1))^2\widehat{g}\left(\frac{l}{\pi_N(x)}\right),$$

with

$$U(l) = \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi) - \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi).$$

(b) *If we choose L such that*

$$L = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right),$$

and consider families $F_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$ such that

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty \text{ as } x \to \infty,$$

then,

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} R_2(\rho, g; f)(x) \sim \frac{T(g,\rho)}{4L} \text{ as } x \to \infty.$$

Furthermore,

$$\frac{T(g,\rho)}{4L} \sim A^2\widehat{g}(0)\rho\ \ \rho(0) \text{ as } x \to \infty.$$

We again revisit this theorem with a proof in Theorem 5.5.3 of Chapter 5. We make a few remarks here.

1. The above theorem tells us that the "expected" value of the smooth localized pair correlation function $R_2(\rho, g; f)(x)$ upon averaging over all newforms $f \in F_{N,k}$ is asymptotic to $A^2\widehat{g}(0)\rho\ \ \rho(0)$. However, we require the size of the families $F_{N,k}$ to grow rapidly for this asymptotic to hold. This limitation comes from the estimation of a term in the Eichler-Selberg trace formula. The elliptic term in the trace formula leads to estimates of the form
$$O\left(\frac{x^{D\pi_N(x)}8^{\nu(N)}}{kN}\right)$$
for a positive constant D in the pair correlation sum. The use of alternative trace formulas such as the Petersson trace formula leads to lower values of D than those obtained by the Eichler-Selberg trace formula, but causes the same problem if we want the above error term to go to zero.

2. In [BS19], the "expected" value of $R_2(\rho, g; f)(x)$ upon averaging over $f \in F_{N,k}$ was obtained for positive integers N and k such that N is prime and k is even. The techniques can be readily generalized to all levels N. Accordingly, a modified version of the result of [BS19] has been stated above.

The above theorem about the expected value

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} R_2(\rho, g; f)(x)$$

naturally leads us to questions about higher moments and the variance of $R_2(\rho, g; f)(x)$. What can we say about

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \left(R_2(\rho, g; f)(x)\right)^2$$

and

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \left(R_2(\rho, g; f)(x) - A^2\widehat{g}(0)\rho\ \ \rho(0)\right)^2?$$

4.3 New Results

The primary objective of this thesis is to address these questions. In this direction, we prove the following theorem.

Theorem 4.3.1. *Let us consider families $\mathcal{F}_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$. Let g, ρ be real-valued, even functions $\in C^\infty(\mathbb{R})$ in the Schwartz class with Fourier transforms supported in $[-1,1]$ and $L = L(x) \to \infty$ as $x \to \infty$. Let $0 < \psi < 1$, $\psi \neq 1/2$, and let $A = 2\sin^2 \pi\psi$.*

(a) *With ρ_L, G_x, $R_2(\rho, g; f)(x)$ and $T(g,\rho)$ as defined in the previous theorem, we have*

$$\frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \big(R_2(\rho,g;f)(x)\big)^2 - \left(\frac{T(g,\rho)}{4L}\right)^2$$
$$\ll \frac{1}{L} + \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x)} + \frac{L^2(\log\log x)}{\pi_N(x)^2} + \frac{L^3(\log\log x)^2}{\pi_N(x)^3} + \frac{(\log\log x)^2}{\pi_N(x)L}$$
$$+ \frac{(\log\log x)^3}{\pi_N(x)L^2} + \frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN},$$

where $c' > \frac{3}{2}$ is an absolute constant.

(b) *Also,*

$$\frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \left(R_2(\rho,g;f)(x) - \frac{T(g,\rho)}{4L}\right)^2$$
$$\ll \frac{1}{L} + \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x)} + \frac{L(\log\log x)^2}{\pi_N(x)} + \frac{L^2(\log\log x)}{\pi_N(x)^2} + \frac{L^3(\log\log x)^2}{\pi_N(x)^3}$$
$$+ \frac{(\log\log x)^2}{\pi_N(x)L} + \frac{(\log\log x)^3}{\pi_N(x)L^2} + \frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN},$$

where $c' > \frac{3}{2}$ is an absolute constant.

(c) *In particular, if we choose $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$, and consider families $\mathcal{F}_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$ such that*

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty \text{ as } x \to \infty,$$

then

$$\lim_{x\to\infty}\frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \big(R_2(\rho,g;f)(x)\big)^2 = (A^2\widehat{g}(0)(0)\rho - \rho(0))^2$$

and

$$\lim_{x\to\infty}\frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \left(R_2(\rho,g;f)(x) - A^2\widehat{g}(0)\rho - \rho(0)\right)^2 = 0.$$

Remark 4.3.2. *The above theorem is proved in Theorems 6.5.15, 6.5.16 and 6.5.17.*

Remark 4.3.3. *The above theorem tells us that $\mathbb{E}[(R_2(\rho,g;f)(x))^2] \sim \mathbb{E}[(R_2(\rho,g;f)(x))]^2$ for very rapidly growing families $\mathcal{F}_{N,k}$. In turn, these are asymptotic to what one would expect from a Poissonnian model. This indicates an affirmative answer to Question 4.1.2 for a random Hecke newform in $S_k(N)$ with appropriate parameters as specified in Theorem 4.3.1(c).*

Remark 4.3.4. *Another version of the above theorem with fewer lower order terms appears in [MS23]. In the preprint, we address $(R_2(\rho,g;f)(x))^r$ for any natural number r. However, since the treatment in this thesis is restricted to the case $r = 2$, the calculations in this thesis give us a better understanding of the lower order terms.*

4.4 The pair correlation function for Hecke angles

Let

$$A_{f,x} := \{\theta_f(p) \bmod 1 : p \le x,\ (p,N) = 1\}, \tag{4.6}$$

and

$$I_L(\psi) := \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right],$$

such that $L = L(x) \to \infty$ as $x \to \infty$. Then, let us consider L such that

$$\begin{aligned} L_f(\psi) &:= \#\left\{p \le x : (p,N) = 1,\ \theta_f(p) \in I_L(\psi)\right\} \\ &\sim \pi_N(x) \int_{\psi - \frac{1}{AL}}^{\psi + \frac{1}{AL}} 2\sin^2 \pi t\, dt \sim \pi_N(x) A \frac{2}{AL} \sim \frac{2\pi_N(x)}{L}, \text{ as } x \to \infty, \end{aligned}$$

and

$$\begin{aligned} L_f(1-\psi) &:= \#\left\{p \le x : (p,N) = 1,\ \theta_f(p) \in I_L(1-\psi)\right\} \\ &\sim \pi_N(x) \int_{1-\psi-\frac{1}{AL}}^{1-\psi+\frac{1}{AL}} 2\sin^2 \pi t\, dt \\ &\sim \pi_N(x) 2\sin^2 \pi(1-\psi) \frac{2}{AL} \sim \frac{2\pi_N(x)}{L}, \text{ as } x \to \infty. \end{aligned}$$

Note that the family $I_L \cap A_{f,x}$ has cardinality

$$\#(I_L \cap A_{f,x}) = L_f(\psi) + L_f(1-\psi) \sim \frac{4\pi_N(x)}{L}, \tag{4.7}$$

as $x \to \infty$. Therefore, the mean spacing of the family $A_{f,x}$ of Hecke angles in the intervals I_L is

$$\frac{|I_L|}{\#(I_L \cap A_{f,x})} \sim \frac{1}{2A\pi_N(x)} \text{ as } x \to \infty.$$

We have the following lemma:

Lemma 4.4.1. *Let $\psi \in (0,1)$ and $A = 2\sin^2 \pi\psi$ and $I_L := \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right]$. Then, for a fixed prime p, and sufficiently large L,*

$$\begin{aligned} &\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1 - \theta_f(p)) \\ &= \sum_{m \in \mathbb{Z}} \chi_{[-\frac{1}{A}, \frac{1}{A}]}(L(\theta_f(p) - \psi + m)) + \sum_{m \in \mathbb{Z}} \chi_{[-\frac{1}{A}, \frac{1}{A}]}(L(-\theta_f(p) - \psi + m)). \end{aligned}$$

Proof. We know, $0 \le \theta_f(p) \le 1$ and $-1 < -\psi < 0$. Hence, $-1 < \theta_f(p) - \psi < 1$.

For $m \ge 2$, and sufficiently large L,

$$\begin{aligned} & 1 \le m - 1 < \theta_f(p) - \psi + m \\ \Rightarrow\ & \theta_f(p) - \psi + m > 1 \\ \Rightarrow\ & \theta_f(p) - \psi + m \notin \left[-\frac{1}{AL}, \frac{1}{AL}\right] \\ \Rightarrow\ & \theta_f(p) + m \notin \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right] \\ \Rightarrow\ & \chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p) + m) = 0. \end{aligned} \tag{4.8}$$

For $m \le -2$, and sufficiently large L,

$$
\begin{aligned}
& \theta_f(p) - \psi + m < m + 1 \le -1 \\
\Longrightarrow\ & \theta_f(p) - \psi + m < -1 \\
\Longrightarrow\ & \theta_f(p) - \psi + m \notin \left[-\frac{1}{AL}, \frac{1}{AL}\right] \\
\Longrightarrow\ & \theta_f(p) + m \notin \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right] \\
\Longrightarrow\ & \chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p) + m) = 0.
\end{aligned}
\tag{4.9}
$$

Now, if $\chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p) + 1) = 1$ for some prime p, we get

$$
\theta_f(p) + 1 \in \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right] \cap [1, 2] \subset (0, 1) \cap [1, 2],
$$

which is a contradiction.

Hence, for all primes p, and sufficiently large L,

$$
\chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p) + 1) = 0. \tag{4.10}
$$

Similarly, for all primes p, and sufficiently large L,

$$
\chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p) - 1) = 0. \tag{4.11}
$$

Combining equations (4.8), (4.9), (4.10) and (4.11), we obtain

$$
\sum_{m \in \mathbb{Z}} \chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p) + m) = \chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(\theta_f(p)). \tag{4.12}
$$

We know, $-1 \le -\theta_f(p) \le 0$ and $-1 < -\psi < 0$. Hence, $-2 < -\theta_f(p) - \psi < 0$.

For $m \ge 3$, and sufficiently large L,

$$
\begin{aligned}
& 1 \le m - 2 < -\theta_f(p) - \psi + m \\
\Longrightarrow\ & -\theta_f(p) - \psi + m > 1 \\
\Longrightarrow\ & -\theta_f(p) - \psi + m \notin \left[-\frac{1}{AL}, \frac{1}{AL}\right] \\
\Longrightarrow\ & -\theta_f(p) + m \notin \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right] \\
\Longrightarrow\ & \chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(-\theta_f(p) + m) = 0.
\end{aligned}
\tag{4.13}
$$

For $m \le -1$, and sufficiently large L,

$$
\begin{aligned}
& -\theta_f(p) - \psi + m < m \le -1 \\
\Longrightarrow\ & -\theta_f(p) - \psi + m < -1 \\
\Longrightarrow\ & -\theta_f(p) - \psi + m \notin \left[-\frac{1}{AL}, \frac{1}{AL}\right] \\
\Longrightarrow\ & -\theta_f(p) + m \notin \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right] \\
\Longrightarrow\ & \chi_{[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}]}(-\theta_f(p) + m) = 0.
\end{aligned}
\tag{4.14}
$$

Now, if $\chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(-\theta_f(p)) = 1$ for some prime p, we have

$$\theta_f(p) \in \left[\psi-\frac{1}{AL},\psi+\frac{1}{AL}\right] \cap [-1,0] \subset (0,1) \cap [-1,0],$$

which is a contradiction.

Hence, for all primes p, and sufficiently large L,

$$\chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(-\theta_f(p)) = 0. \tag{4.15}$$

Similarly, for all primes p, and sufficiently large L,

$$\chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(-\theta_f(p)+2) = 0. \tag{4.16}$$

Combining equations (4.13), (4.14), (4.15) and (4.16), we obtain

$$\sum_{m\in\mathbb{Z}} \chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(-\theta_f(p)+m) = \chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(1-\theta_f(p)). \tag{4.17}$$

Adding equations (4.12) and (4.17), we obtain that for a fixed prime p, and sufficiently large L,

$$\begin{aligned}
&\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1-\theta_f(p)) \\
&= \sum_{m\in\mathbb{Z}} \chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(\theta_f(p)+m) + \sum_{m\in\mathbb{Z}} \chi_{[\psi-\frac{1}{AL},\psi+\frac{1}{AL}]}(-\theta_f(p)+m) \\
&= \sum_{m\in\mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\theta_f(p)-\psi+m)) + \sum_{m\in\mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(-\theta_f(p)-\psi+m)).
\end{aligned}$$

□

Lemma 4.4.2. *Let $\psi \in (0,1)$ and $A = 2\sin^2\pi\psi$ and $I_x = \left[-\frac{s}{A\pi_N(x)},\frac{s}{A\pi_N(x)}\right]$. Then, for a fixed prime p, and sufficiently large x,*

$$\sum_{n\in\mathbb{Z}} \chi_{[-\frac{s}{A},\frac{s}{A}]}(\pi_N(x)(-\theta_f(p)-\theta_f(q)+n)) = 2B(\theta_f(p),\theta_f(q),x,s),$$

where

$$\begin{aligned}
B(\theta_f(p),\theta_f(q),x,s) &= \chi_{I_x}(\theta_f(p)-\theta_f(q)-1) + \chi_{I_x}(\theta_f(p)-\theta_f(q)) + \chi_{I_x}(\theta_f(p)-\theta_f(q)+1) \\
&\quad + \chi_{I_x}(\theta_f(p)+\theta_f(q)) + \chi_{I_x}(\theta_f(p)+\theta_f(q)-1) + \chi_{I_x}(\theta_f(p)+\theta_f(q)-2).
\end{aligned}$$

Proof. Part 1: We know, $0 \le \theta_f(p),\theta_f(q) \le 1$.

Hence, $0 \le \theta_f(p)+\theta_f(q) \le 2$, and this implies $m \le \theta_f(p)+\theta_f(q)+m \le m+2$, for any integer m.

For $m \ge 1$, and sufficiently large x,

$$\begin{aligned}
&1 \le m \le \theta_f(p)+\theta_f(q)+m \\
&\Rightarrow \theta_f(p)+\theta_f(q)+m \notin \left[-\frac{s}{A\pi_N(x)},\frac{s}{A\pi_N(x)}\right] = I_x \\
&\Rightarrow \chi_{I_x}(\theta_f(p)+\theta_f(q)+m) = 0.
\end{aligned} \tag{4.18}$$

For $m \leq -3$, and sufficiently large x,

$$\begin{aligned}
&\theta_f(p) + \theta_f(q) + m \leq m + 2 \leq -1 \\
\implies &\theta_f(p) + \theta_f(q) + m \notin \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)} \right] = I_x \qquad (4.19)\\
\implies &\chi_{I_x}(\theta_f(p) + \theta_f(q) + m) = 0.
\end{aligned}$$

Hence, using equations (4.18) and (4.19), we obtain

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}} \chi_{[-\frac{s}{A}, \frac{s}{A}]}(\pi_N(x)(\theta_f(p) + \theta_f(q) + m)) \\
&= \sum_{m \in \mathbb{Z}} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m) \qquad (4.20)\\
&= \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m).
\end{aligned}$$

<u>Part 2</u>: We know, $0 \leq \theta_f(p), \theta_f(q) \leq 1$.

Hence, $-1 \leq \theta_f(p) - \theta_f(q) \leq 1$, and this implies $m - 1 \leq \theta_f(p) + \theta_f(q) + m \leq m + 1$, for any integer m.

For $m \geq 2$, and sufficiently large x,

$$\begin{aligned}
&1 \leq m - 1 \leq \theta_f(p) - \theta_f(q) + m \\
\implies &\theta_f(p) - \theta_f(q) + m \notin \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)} \right] = I_x \qquad (4.21)\\
\implies &\chi_{I_x}(\theta_f(p) - \theta_f(q) + m) = 0.
\end{aligned}$$

For $m \leq -2$, and sufficiently large x,

$$\begin{aligned}
&\theta_f(p) - \theta_f(q) + m \leq m + 1 \leq -1 \\
\implies &\theta_f(p) - \theta_f(q) + m \notin \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)} \right] = I_x \qquad (4.22)\\
\implies &\chi_{I_x}(\theta_f(p) - \theta_f(q) + m) = 0.
\end{aligned}$$

Hence, using equations (4.21) and (4.22), we obtain

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}} \chi_{[-\frac{s}{A}, \frac{s}{A}]}(\pi_N(x)(\theta_f(p) - \theta_f(q) + m)) \\
&= \sum_{m \in \mathbb{Z}} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m) \qquad (4.23)\\
&= \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m).
\end{aligned}$$

<u>Part 3</u>: Similarly, we obtain

$$\begin{aligned}
&\sum_{m \in \mathbb{Z}} \chi_{[-\frac{s}{A}, \frac{s}{A}]}(\pi_N(x)(-\theta_f(p) + \theta_f(q) + m)) \qquad (4.24)\\
&= \sum_{m=-1}^{1} \chi_{I_x}(-\theta_f(p) + \theta_f(q) + m).
\end{aligned}$$

<u>Part 4 :</u> We know, $0 \le \theta_f(p), \theta_f(q) \le 1$.

Hence, $-2 \le -\theta_f(p) - \theta_f(q) \le 0$, and this implies $m - 2 \le -\theta_f(p) - \theta_f(q) + m \le m$, for any integer m.

For $m \ge 3$, and sufficiently large x,

$$
\begin{aligned}
& 1 \le m - 2 \le -\theta_f(p) - \theta_f(q) + m \\
\Rightarrow\ & -\theta_f(p) - \theta_f(q) + m \notin \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)} \right] = I_x \\
\Rightarrow\ & \chi_{I_x}(-\theta_f(p) - \theta_f(q) + m) = 0.
\end{aligned}
\tag{4.25}
$$

For $m \le -1$, and sufficiently large x,

$$
\begin{aligned}
& -\theta_f(p) - \theta_f(q) + m \le m \le -1 \\
\Rightarrow\ & -\theta_f(p) - \theta_f(q) + m \notin \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)} \right] = I_x \\
\Rightarrow\ & \chi_{I_x}(-\theta_f(p) - \theta_f(q) + m) = 0.
\end{aligned}
\tag{4.26}
$$

Hence, using equations (4.25) and (4.26), we obtain

$$
\begin{aligned}
& \sum_{m \in \mathbb{Z}} \chi_{\left[-\frac{s}{A}, \frac{s}{A}\right]}(\pi_N(x)(-\theta_f(p) - \theta_f(q) + m)) \\
= & \sum_{m \in \mathbb{Z}} \chi_{I_x}(-\theta_f(p) - \theta_f(q) + m) \\
= & \sum_{m=0}^{2} \chi_{I_x}(-\theta_f(p) - \theta_f(q) + m).
\end{aligned}
\tag{4.27}
$$

Adding equations (4.20), (4.23), (4.24) and (4.27), we obtain

$$
\begin{aligned}
& \sum_{n \in \mathbb{Z}} \chi_{\left[-\frac{s}{A}, \frac{s}{A}\right]}(\pi_N(x)(\pm\theta_f(p) \pm \theta_f(q) + n)) \\
= & \sum_{m \in \mathbb{Z}} \chi_{\left[-\frac{s}{A}, \frac{s}{A}\right]}(\pi_N(x)(\theta_f(p) + \theta_f(q) + m)) + \sum_{m \in \mathbb{Z}} \chi_{\left[-\frac{s}{A}, \frac{s}{A}\right]}(\pi_N(x)(\theta_f(p) - \theta_f(q) + m)) \\
+ & \sum_{m \in \mathbb{Z}} \chi_{\left[-\frac{s}{A}, \frac{s}{A}\right]}(\pi_N(x)(-\theta_f(p) + \theta_f(q) + m)) + \sum_{m \in \mathbb{Z}} \chi_{\left[-\frac{s}{A}, \frac{s}{A}\right]}(\pi_N(x)(-\theta_f(p) - \theta_f(q) + m)) \\
= & \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m) + 2 \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m) + \sum_{m=0}^{2} \chi_{I_x}(-\theta_f(p) - \theta_f(q) + m) \\
= & \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m) + 2 \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m) + \sum_{m=0}^{2} \chi_{I_x}(\theta_f(p) + \theta_f(q) - m) \\
= & 2 \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m) + 2 \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m) \\
= & 2B(\theta_f(p), \theta_f(q), x, s).
\end{aligned}
$$

□

Theorem 4.4.3. *Let $0 < \psi < 1$ and $\psi \neq 1/2$. Then,*

$$\#\left\{(i,j) : i \neq j,\ x_i, x_j \in (I_L \cap A_{f,x}),\ x_i - x_j \in \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)}\right]\right\}$$

$$= \frac{1}{2} \sum_{\substack{(p,q): p \neq q \leq x \\ (p,N)=(q,N)=1}} \Big(\sum_{n \in \mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\theta_f(p) - \psi + n)) \sum_{n \in \mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\theta_f(q) - \psi + n))$$

$$\sum_{n \in \mathbb{Z}} \chi_{[-\frac{s}{A},\frac{s}{A}]}(\pi_N(x)(\theta_f(p) - \theta_f(q) + n)) \Big),$$

where

$$A_{f,x} := \{\theta_f(p) \bmod 1 : p \leq x,\ (p,N) = 1\}.$$

Proof. Let $0 < \psi < 1$ and $\psi \neq 1/2$. Then, using Lemmas 4.4.1 and 4.4.2, we obtain

$$\sum_{\substack{(p,q): p \neq q \leq x \\ (p,N)=(q,N)=1}} \Big(\sum_{n \in \mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\theta_f(p) - \psi + n)) \sum_{n \in \mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\theta_f(q) - \psi + n)) \tag{4.28}$$

$$\sum_{n \in \mathbb{Z}} \chi_{[-\frac{s}{A},\frac{s}{A}]}(\pi_N(x)(\theta_f(p) - \theta_f(q) + n)) \Big)$$

$$= \sum_{\substack{(p,q): p \neq q \leq x \\ (p,N)=(q,N)=1}} (\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1 - \theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1 - \theta_f(q)))$$

$$2B(\theta_f(p), \theta_f(q), x, s),$$

where

$$B(\theta_f(p), \theta_f(q), x, s) = \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m) + \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m).$$

Now, for sufficiently large L, and $\psi \neq \frac{1}{2}$, $[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}] \cap [1 - \psi - \frac{1}{AL}, 1 - \psi + \frac{1}{AL}] = \phi$. Hence, for a prime r,

$$\chi_{I_L}(\theta_f(r)) = 1 \tag{4.29}$$

$$\iff \theta_f(r) \in \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right]$$

$$\iff \theta_f(r) \notin \left[1 - \psi - \frac{1}{AL}, 1 - \psi + \frac{1}{AL}\right]$$

$$\iff -\theta_f(r) \notin \left[-1 + \psi - \frac{1}{AL}, -1 + \psi + \frac{1}{AL}\right]$$

$$\iff 1 - \theta_f(r) \notin \left[\psi - \frac{1}{AL}, \psi + \frac{1}{AL}\right]$$

$$\iff \chi_{I_L}(1 - \theta_f(r)) = 0,$$

which implies

$$(\chi_{I_L}(\theta_f(r)), \chi_{I_L}(1 - \theta_f(r))) = (0,0),\ \text{or},\ (0,1),\ \text{or},\ (1,0),\ \text{i.e.},$$
$$\chi_{I_L}(\theta_f(r)) + \chi_{I_L}(1 - \theta_f(r)) = 0,\ \text{or},\ 1.$$

Therefore, for any pair of primes (p,q),

$$(\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1 - \theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1 - \theta_f(q))) \neq 0,\ \text{iff},$$
$$\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1 - \theta_f(p)) = \chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1 - \theta_f(q)) = 1,$$

if and only if exactly one of the following happens:

(i) $\theta_f(p) \in I_L$, and $\theta_f(q) \in I_L$;

(ii) $\theta_f(p) \in I_L$, and $1-\theta_f(q) \in I_L$;

(iii) $1-\theta_f(p) \in I_L$, and $\theta_f(q) \in I_L$;

(iv) $1-\theta_f(p) \in I_L$, and $1-\theta_f(q) \in I_L$.

We now evaluate each of these cases separately.

Part 1: $\theta_f(p) \in I_L$, and $\theta_f(q) \in I_L$. Hence,

$$\begin{aligned}
&\psi - \frac{1}{AL} \le \theta_f(p), \theta_f(q) \le \psi + \frac{1}{AL} \\
\Rightarrow\ & 2\psi - \frac{2}{AL} \le \theta_f(p) + \theta_f(q) \le 2\psi + \frac{2}{AL} \\
\Rightarrow\ & (2\psi - 1) - \frac{2}{AL} \le \theta_f(p) + \theta_f(q) - 1 \le (2\psi - 1) + \frac{2}{AL} \\
\Rightarrow\ & (2\psi - 2) - \frac{2}{AL} \le \theta_f(p) + \theta_f(q) - 2 \le (2\psi - 2) + \frac{2}{AL}.
\end{aligned}$$

Therefore, for sufficiently large L, and for $\psi \in (0,1)$, with $\psi \neq 1/2$,

$$\left[(2\psi + m) - \frac{2}{AL}, (2\psi + m) + \frac{2}{AL}\right] \cap I_x = \phi, \text{ for } m = 0, -1, -2,$$

and hence,

$$\chi_{I_x}(\theta_f(p) + \theta_f(q) + m) = 0, \text{ for } m = 0, -1, -2.$$

Similarly, we obtain that for sufficiently large L, and for $\psi \in (0,1)$, with $\psi \neq 1/2$,

$$\left[m - \frac{2}{AL}, m + \frac{2}{AL}\right] \cap I_x = \phi, \text{ for } m = -1, 1,$$

and hence,

$$\chi_{I_x}(\theta_f(p) - \theta_f(q) + m) = 0, \text{ for } m = -1, 1.$$

Hence,

$$\begin{aligned}
B(\theta_f(p), \theta_f(q), x, s) &= \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p) + \theta_f(q) + m) + \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p) - \theta_f(q) + m) \\
&= \chi_{I_x}(\theta_f(p) - \theta_f(q)).
\end{aligned}$$

Therefore, using equation (4.29), we obtain

$$\begin{aligned}
&\sum_{\substack{(p,q): p \neq q \le x \\ (p,N)=(q,N)=1 \\ \theta_f(p) \in I_L, \theta_f(q) \in I_L}} (\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1 - \theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1 - \theta_f(q))) \\
&\quad B(\theta_f(p), \theta_f(q), x, s) \\
&= \sum_{\substack{(p,q): p \neq q \le x \\ (p,N)=(q,N)=1 \\ \theta_f(p) \in I_L, \theta_f(q) \in I_L}} B(\theta_f(p), \theta_f(q), x, s) \\
&= \sum_{\substack{(p,q): p \neq q \le x \\ (p,N)=(q,N)=1 \\ \theta_f(p) \in I_L, \theta_f(q) \in I_L}} \chi_{I_x}(\theta_f(p) - \theta_f(q)) \\
&= \#\{(p,q) : p \neq q \le x, (p,N) = (q,N) = 1, \theta_f(p) \in I_L, \theta_f(q) \in I_L, \theta_f(p) - \theta_f(q) \in I_x\}.
\end{aligned}$$

<u>Part 2 :</u> $\theta_f(p) \in I_L$, and $1-\theta_f(q) \in I_L$. Hence,

$$\begin{aligned}
&\psi - \frac{1}{AL} \le \theta_f(p), 1-\theta_f(q) \le \psi + \frac{1}{AL}\\
\Longrightarrow\ & 1 - \frac{2}{AL} \le \theta_f(p)+\theta_f(q) \le 1 + \frac{2}{AL}\\
\Longrightarrow\ & -\frac{2}{AL} \le \theta_f(p)+\theta_f(q) - 1 \le \frac{2}{AL}\\
\Longrightarrow\ & -1 - \frac{2}{AL} \le \theta_f(p)+\theta_f(q) - 2 \le -1 + \frac{2}{AL}.
\end{aligned}$$

Therefore, for sufficiently large L, and for $\psi \in (0,1)$, with $\psi \not\equiv 1/2$,

$$\left[m - \frac{2}{AL}, m + \frac{2}{AL}\right] \cap I_x = \phi, \text{ for } m = -1, 1,$$

and hence,

$$\chi_{I_x}(\theta_f(p)+\theta_f(q)+m) = 0, \text{ for } m = 0, -2.$$

Similarly, we obtain that for sufficiently large L, and for $\psi \in (0,1)$, with $\psi \not\equiv 1/2$,

$$\left[(2\psi - m) - \frac{2}{AL}, (2\psi - m) + \frac{2}{AL}\right] \cap I_x = \phi, \text{ for } m = 0,1,2,$$

and hence,

$$\chi_{I_x}(\theta_f(p) - \theta_f(q)+m) = 0, \text{ for } m = 0, -1, -2.$$

Hence,

$$\begin{aligned}
B(\theta_f(p), \theta_f(q), x, s) &= \sum_{m=-2}^{0} \chi_{I_x}(\theta_f(p)+\theta_f(q)+m) + \sum_{m=-1}^{1} \chi_{I_x}(\theta_f(p)-\theta_f(q)+m)\\
&= \chi_{I_x}(\theta_f(p)+\theta_f(q)-1).
\end{aligned}$$

Therefore, using equation (4.29), we obtain

$$\begin{aligned}
&\sum_{\substack{(p,q): p \not\equiv q \pmod x\\ (p,N)=(q,N)=1\\ \theta_f(p)\in I_L, 1-\theta_f(q)\in I_L}} (\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1-\theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1-\theta_f(q)))\\
&\quad B(\theta_f(p), \theta_f(q), x, s)\\
=& \sum_{\substack{(p,q): p \not\equiv q \pmod x\\ (p,N)=(q,N)=1\\ \theta_f(p)\in I_L, 1-\theta_f(q)\in I_L}} B(\theta_f(p), \theta_f(q), x, s)\\
=& \sum_{\substack{(p,q): p \not\equiv q \pmod x\\ (p,N)=(q,N)=1\\ \theta_f(p)\in I_L, 1-\theta_f(q)\in I_L}} \chi_{I_x}(\theta_f(p)+\theta_f(q)-1)\\
=& \#\{(p,q) : p \not\equiv q \pmod x, (p,N)=(q,N)=1, \theta_f(p)\in I_L, 1-\theta_f(q)\in I_L, \theta_f(p)+\theta_f(q)-1 \in I_x\}.
\end{aligned}$$

<u>Part 3 :</u> $1-\theta_f(p) \in I_L$, and $\theta_f(q) \in I_L$. Similarly, as in part 2, we obtain

$$\begin{aligned}
&\sum_{\substack{(p,q): p \not\equiv q \pmod x\\ (p,N)=(q,N)=1\\ 1-\theta_f(p)\in I_L, \theta_f(q)\in I_L}} (\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1-\theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1-\theta_f(q)))\\
&\quad B(\theta_f(p), \theta_f(q), x, s)\\
=& \#\{(p,q) : p \not\equiv q \pmod x, (p,N)=(q,N)=1, 1-\theta_f(p)\in I_L, \theta_f(q)\in I_L, \theta_f(p)+\theta_f(q)-1 \in I_x\}.
\end{aligned}$$

Part 4 : $1-\theta_f(p)\in I_L$, and $1-\theta_f(q)\in I_L$. Hence,

$$\begin{aligned}
&\psi-\frac{1}{AL}\le 1-\theta_f(p),\,1-\theta_f(q)\le\psi+\frac{1}{AL}\\
\Longrightarrow\ &(2-2\psi)-\frac{2}{AL}\le\theta_f(p)+\theta_f(q)\le(2-2\psi)+\frac{2}{AL}\\
\Longrightarrow\ &(1-2\psi)-\frac{2}{AL}\le\theta_f(p)+\theta_f(q)-1\le(1-2\psi)+\frac{2}{AL}\\
\Longrightarrow\ &-2\psi-\frac{2}{AL}\le\theta_f(p)+\theta_f(q)-2\le-2\psi+\frac{2}{AL}.
\end{aligned}$$

Therefore, for sufficiently large L, and for $\psi\in(0,1)$, with $\psi\neq 1/2$,

$$\left[(m-2\psi)-\frac{2}{AL},(m-2\psi)+\frac{2}{AL}\right]\cap I_x=\phi,\ \text{ for } m=0,1,2,$$

and hence,

$$\chi_{I_x}(\theta_f(p)+\theta_f(q)+m)=0,\ \text{ for } m=0,-1,-2.$$

Similarly, we obtain that for sufficiently large L, and for $\psi\in(0,1)$, with $\psi\neq 1/2$,

$$\left[m-\frac{2}{AL},m+\frac{2}{AL}\right]\cap I_x=\phi,\ \text{ for } m=-1,1,$$

and hence,

$$\chi_{I_x}(\theta_f(p)-\theta_f(q)+m)=0,\ \text{ for } m=-1,1.$$

Hence,

$$\begin{aligned}
B(\theta_f(p),\theta_f(q),x,s)&=\sum_{m=-2}^{0}\chi_{I_x}(\theta_f(p)+\theta_f(q)+m)+\sum_{m=-1}^{1}\chi_{I_x}(\theta_f(p)-\theta_f(q)+m)\\
&=\chi_{I_x}(\theta_f(p)-\theta_f(q)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sum_{\substack{(p,q):p\neq q\le x\\(p,N)=(q,N)=1\\1-\theta_f(p)\in I_L,1-\theta_f(q)\in I_L}}(\chi_{I_L}(\theta_f(p))+\chi_{I_L}(1-\theta_f(p)))(\chi_{I_L}(\theta_f(q))+\chi_{I_L}(1-\theta_f(q)))\\
&\qquad B(\theta_f(p),\theta_f(q),x,s)\\
=&\sum_{\substack{(p,q):p\neq q\le x\\(p,N)=(q,N)=1\\1-\theta_f(p)\in I_L,1-\theta_f(q)\in I_L}}B(\theta_f(p),\theta_f(q),x,s)\\
=&\sum_{\substack{(p,q):p\neq q\le x\\(p,N)=(q,N)=1\\1-\theta_f(p)\in I_L,1-\theta_f(q)\in I_L}}\chi_{I_x}(\theta_f(p)-\theta_f(q))\\
=&\#\{(p,q):p\neq q\le x,(p,N)=(q,N)=1,1-\theta_f(p)\in I_L,1-\theta_f(q)\in I_L,\theta_f(p)-\theta_f(q)\in I_x\}.
\end{aligned}$$

Therefore, using equation (4.28) and parts (1), (2), (3), and (4), we obtain

$$\begin{aligned}
\frac{1}{2}\sum_{\substack{(p,q):p\neq q\le x\\(p,N)=(q,N)=1}}\Big(&\sum_{n\in\mathbb{Z}}\chi_{\left[-\frac{1}{A},\frac{1}{A}\right]}(L(\theta_f(p)-\psi+n))\sum_{n\in\mathbb{Z}}\chi_{\left[-\frac{1}{A},\frac{1}{A}\right]}(L(\theta_f(q)-\psi+n))\\
&\sum_{n\in\mathbb{Z}}\chi_{\left[-\frac{s}{A},\frac{s}{A}\right]}(\pi_N(x)(\theta_f(p)-\theta_f(q)+n))\Big)
\end{aligned}$$

$$= \sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1}} (\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1-\theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1-\theta_f(q)))\cdot B(\theta_f(p),\theta_f(q),x,s)$$

$$= \left(\sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1\\ \theta_f(p)\in I_L,\theta_f(q)\in I_L}} + \sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1\\ \theta_f(p)\in I_L,1-\theta_f(q)\in I_L}} + \sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1\\ 1-\theta_f(p)\in I_L,\theta_f(q)\in I_L}} + \sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1\\ 1-\theta_f(p)\in I_L,1-\theta_f(q)\in I_L}} \right)$$
$$(\chi_{I_L}(\theta_f(p)) + \chi_{I_L}(1-\theta_f(p)))(\chi_{I_L}(\theta_f(q)) + \chi_{I_L}(1-\theta_f(q)))\cdot B(\theta_f(p),\theta_f(q),x,s)$$

$$= \#\{(p,q) : p\neq q\le x, (p,N)=(q,N)=1, \theta_f(p)\in I_L, \theta_f(q)\in I_L, \theta_f(p)-\theta_f(q)\in I_x\}$$

$$+ \#\{(p,q) : p\neq q\le x, (p,N)=(q,N)=1, \theta_f(p)\in I_L, 1-\theta_f(q)\in I_L, \theta_f(p)-(1-\theta_f(q))\in I_x\}$$

$$+ \#\{(p,q) : p\neq q\le x, (p,N)=(q,N)=1, \theta_f(p)\in I_L, 1-\theta_f(q)\in I_L, \theta_f(q)-(1-\theta_f(p))\in I_x\}$$

$$+ \#\{(p,q) : p\neq q\le x, (p,N)=(q,N)=1, 1-\theta_f(p)\in I_L, 1-\theta_f(q)\in I_L, (1-\theta_f(p))-(1-\theta_f(q))\in I_x\}$$
$$= \#\left\{(i,j) : i\neq j,\, x_i,x_j\in (I_L\cap A_{f,x}),\, x_i-x_j\in I_x\right\}$$
$$= \#\left\{(i,j) : i\neq j,\, x_i,x_j\in (I_L\cap A_{f,x}),\, x_i-x_j\in \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)}\right]\right\},$$

where

$$\begin{aligned} A_{f,x} &= \{\theta_f(p), 1-\theta_f(p) : p\le x,\ (p,N)=1\} \\ &= \{\pm\theta_f(p) \bmod 1 : p\le x,\ (p,N)=1\}. \end{aligned}$$

□

Remark 4.4.4. *It is important to note that the above theorem does not hold true for $\psi = \frac{1}{2}$. The obstruction comes from the fact that $B(\theta_f(p),\theta_f(q),x,s) = \chi_{I_x}(\theta_f(p)+\theta_f(q)-1) + \chi_{I_x}(\theta_f(p)-\theta_f(q))$, for $\psi = \frac{1}{2}$, and when it is plugged into equation* (4.28), *this doesn't give anything significant in terms of the pair correlation function for Hecke angles.*

Using equation (4.7) and Theorem 4.4.3, the pair correlation function for the families $A_{f,x}\cap I_L$ is given by

$$\frac{1}{|I_L\cap A_{f,x}|}\#\left\{(i,j) : i\neq j,\, x_i,x_j\in (I_L\cap A_{f,x}),\, x_i-x_j\in \left[-\frac{s}{A\pi_N(x)}, \frac{s}{A\pi_N(x)}\right]\right\}$$
$$\approx \frac{L}{8\pi_N(x)} \sum_{\substack{(p,q):p\neq q\le x\\ (p,N)=(q,N)=1}} \sum_{n\in\mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\pm\theta_f(p)-\psi+n)) \sum_{n\in\mathbb{Z}} \chi_{[-\frac{1}{A},\frac{1}{A}]}(L(\pm\theta_f(q)-\psi+n))$$
$$\sum_{n\in\mathbb{Z}} \chi_{[-\frac{s}{A},\frac{s}{A}]}(\pi_N(x)(\pm\theta_f(p)\pm\theta_f(q)+n)). \tag{4.30}$$

4.5 Smooth analogue

We now consider a smooth analogue of the right-hand side of equation (4.30).

Let ρ be an even test function in the Schwartz class (see Notations) such that the Fourier transform $\widehat{\rho}$ of ρ is smooth and compactly supported, and normalised so that

$$\sup\{|t| : \widehat{\rho}(t) \neq 0\} = 1.$$

We define

$$\rho_L(\theta) := \sum_{n \in \mathbb{Z}} \rho(L(\theta + n)).$$

$\rho_L(\theta)$ is a 1-periodic function, localized to a scale of $1/L$, and therefore, effectively counts points θ such that $|\theta| < 1/L$. It has the Fourier expansion

$$\rho_L(\theta) = \sum_{|l| \leq L} \widehat{\rho_L}(l) e(l\theta) = \widehat{\rho_L}(0) + \sum_{1 \leq l \leq L} \widehat{\rho_L}(l) 2\cos(2\pi l\theta),$$

where $\widehat{\rho_L}(l) = \frac{1}{L}\widehat{\rho}\left(\frac{l}{L}\right)$.

Similarly, let g be an even test function satisfying the same properties as ρ, that is, an even test function in the Schwartz class such that the Fourier transform $\widehat{g}$ of g is smooth and compactly supported, and normalised so that

$$\sup\{|t| : \widehat{g}(t) \neq 0\} = 1.$$

We define

$$\begin{aligned} G_x(\theta) &:= \sum_{n \in \mathbb{Z}} g\left(\pi_N(x)(\theta + n)\right) \\ &= \sum_{|n| \leq \pi_N(x)} \widehat{G_x}(n) e(n\theta) \\ &= \widehat{G_x}(0) + \sum_{1 \leq n \leq \pi_N(x)} \widehat{G_x}(n) 2\cos(2\pi n\theta), \end{aligned}$$

where $\widehat{G_x}(n) := \frac{1}{\pi_N(x)}\widehat{g}\left(\frac{n}{\pi_N(x)}\right)$. Similar to the case of ρ_L, the function $G_x(\theta)$ is a 1-periodic function, localized to a scale of $1/\pi_N(x)$, and therefore, effectively counts points θ such that $|\theta| < 1/\pi_N(x)$.

The smooth analogue of the right-hand side of (4.30) is defined as

$$R_2(\rho, g; f)(x) := \frac{L}{8\pi_N(x)} \sum_{\substack{(p,q): p \neq q \leq x \\ (p,N)=(q,N)=1}} \rho_L(\theta_f(p) - \psi)\rho_L(\theta_f(q) - \psi) G_x(\theta_f(p) - \theta_f(q)).$$

We recall the following classical result, which gives a recursive relation between $a_f(p^l)$, $l \geq 0$. For an integer $l \geq 0$,

$$2\cos 2\pi l\,\theta_f(p) = \begin{cases} 2, & \text{if } l = 0, \\ a_f(p^{2l}) - a_f(p^{2l-2}), & \text{if } l \geq 1. \end{cases} \tag{4.31}$$

Denote

$$U(l) = \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi) - \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi), \ (0 \leq l \leq L)$$

and

$$G(n) = \widehat{g}\left(\frac{n}{\pi_N(x)}\right), \ 0 \leq n \leq \pi_N(x).$$

Lemma 4.5.1. *Let ρ, f, L and ψ be as defined earlier. Then, for any prime p coprime to level N,*

$$2\widehat{\rho}(0) + \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)(2\cos 2\pi l\,\theta_f(p)) = \sum_{l \geq 0} U(l) a_f(p^{2l}).$$

Proof. Using equation (4.31), we obtain

$$
\begin{aligned}
&2\widehat{\rho}(0) + \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)(2\cos 2\pi l\,\theta_f(p)) \\
=&2\widehat{\rho}(0) + \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)(a_f(p^{2l}) - a_f(p^{2l-2})) \\
=&2\widehat{\rho}(0) + \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)a_f(p^{2l}) - \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)a_f(p^{2l-2}) \\
=&2\widehat{\rho}(0) + \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)a_f(p^{2l}) - \sum_{l \geq 0} \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi)a_f(p^{2l}) \\
=&2\widehat{\rho}(0) - \widehat{\rho}\left(\frac{1}{L}\right)(2\cos 2\pi\psi) + \sum_{l \geq 1} \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi)a_f(p^{2l}) \\
&- \sum_{l \geq 1} \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi)a_f(p^{2l}) \\
=&2\widehat{\rho}(0) - \widehat{\rho}\left(\frac{1}{L}\right)(2\cos 2\pi\psi) + \sum_{l \geq 1} \left(\widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi) - \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi)\right) a_f(p^{2l}) \\
=&\sum_{l \geq 0} \left(\widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi) - \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi)\right) a_f(p^{2l}) \\
=&\sum_{l \geq 0} U(l)a_f(p^{2l}).
\end{aligned}
$$

□

Using Lemma 4.5.1, we have

$$
\begin{aligned}
&\rho_L(-\theta_f(p) - \psi) = \rho_L(\theta_f(p) - \psi) + \rho_L(-\theta_f(p) - \psi) \\
=&\sum_{|l| \leq L} \widehat{\rho_L}(l)\left\{e(l(\theta_f(p) - \psi)) + e(l(-\theta_f(p) - \psi))\right\} \\
=&\sum_{|l| \leq L} \widehat{\rho_L}(l)e(-l\psi)2\cos(2\pi l\,\theta_f(p)) \\
=&\left(2\widehat{\rho_L}(0) + \sum_{1 \leq l \leq L} \widehat{\rho_L}(l)(2\cos 2\pi l\psi)2\cos(2\pi l\,\theta_f(p))\right). \\
=&\frac{1}{L}\sum_{0 \leq l \leq L} U(l)a_f(p^{2l}).
\end{aligned}
\tag{4.32}
$$

Similarly,

$$
\begin{aligned}
&G_x\left(-\theta_f(p) - \theta_f(q)\right) \\
=&\frac{1}{\pi_N(x)} \sum_{|n| \leq \pi_N(x)} \widehat{g}\left(\frac{n}{\pi_N(x)}\right) e\left(-n\,\theta_f(p) - n\,\theta_f(q)\right)) \\
=&\frac{1}{\pi_N(x)}\left(4G(0) + \sum_{n \geq 1} 2G(n)(2\cos 2\pi n\,\theta_f(p))(2\cos 2\pi n\,\theta_f(q))\right),
\end{aligned}
$$

Using the above Fourier expansions,

$$R_2(\rho, g; f)(x) = \frac{L}{8\pi_N(x)} \frac{1}{L^2\pi_N(x)} \sum_{\substack{(p,q):p\neq q \leq x \\ (p,N)=(q,N)=1}} \left[\sum_{l\geq 0} U(l)a_f(p^{2l})\right] \left[\sum_{l'\geq 0} U(l')a_f(q^{2l'})\right]$$
$$\left[4G(0) + \sum_{n\geq 1} 2G(n)(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2}))\right] \tag{4.33}$$

We now denote

$$T_1(p) := \sum_{l\geq 0} U(l)a_f(p^{2l}),$$

$$T_2(q) := \sum_{l'\geq 0} U(l')a_f(q^{2l'})$$

and

$$T_3(p,q) := \sum_{n\geq 0} G(n)A(p,q,n),$$

where

$$A(p,q,n) = \begin{cases} 4 & \text{if } n = 0 \\ 2(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})) & \text{if } n \geq 1. \end{cases}$$

Thus, we get

$$\begin{aligned} R_2(\rho, g; f)(x) &= \frac{1}{8\pi_N(x)^2 L} \sum_{\substack{(p,q):p\neq q \leq x \\ (p,N)=(q,N)=1}} T_1(p)T_2(q)T_3(p,q) \\ &= \frac{1}{8\pi_N(x)^2 L} \sideset{}{'}\sum_{p,q\leq x} T_1(p)T_2(q)T_3(p,q). \end{aligned} \tag{4.34}$$

Since $\widehat{\rho}$ and $\widehat{g}$ are continuous and compactly supported, we have the bounds $|U(l_i)|, |U(k_i)|, |G(n_i)| \ll 1$, which will be used in the calculations in this thesis.

4.5.1 Equidistribution properties of Hecke angles in small scales

In this section, we explain the connection between the error terms in the Sato-Tate distribution theorem (see equation (4.1)) and the distribution of the families $A_{f,x}$ (defined in (4.6)) in shrinking intervals I_L where $L = L(x) \to \infty$ as $x \to \infty$. As we will see below, this provides an insight into the difficulties in obtaining the pair correlation function for a deterministic $f \in F_{N,k}$, and why it helps to consider a *random* $f \in F_{N,k}$ instead.

The question is, what growth conditions on $L = L(x)$ are sufficient to ensure that

$$\lim_{x\to\infty} \frac{1}{|A_{f,x}|} \sum_{\theta\in A_{f,x}} \rho_L(\theta - \psi) = \int_0^1 \rho_L(t - \psi)\mu(t)dt?$$

Define

$$N_{\rho,L,f}(x) := \sum_{\theta\in A_{f,x}} \rho_L(\theta - \psi).$$

By (4.32),

$$\begin{aligned}\frac{N_{\rho,L,f}(x)}{|A_{f,x}|} &= \frac{1}{2\pi_N(x)}\left(2\widehat{\rho_L}(0)\pi_N(x) + \sum_{1\le l\le L}\widehat{\rho_L}(l)(2\cos 2\pi l\psi)\sum_{\substack{p\le x\\(p,N)=1}}(a_f(p^{2l}) - a_f(p^{2l-2}))\right)\\ &= \frac{1}{2\pi_N(x)}\sum_{0\le l\le L}\frac{U(l)}{L}\sum_{\substack{p\le x\\(p,N)=1}}a_f(p^{2l}).\end{aligned} \tag{4.35}$$

It is easy to see that

$$\frac{U(0)}{2L} = \int_0^1 \rho_L(t-\psi)\mu(t)dt.$$

The following proposition is a consequence of Thorner's discrepancy estimates.

Proposition 4.5.2. *Let $N \ge 1$ and $k \ge 2$ be integers with k even. Let $f \in \mathcal{F}_{N,k}$ be a non-CM newform. Let ρ be as defined in (4.4). If $0<\psi<1$ and L is chosen such that*

$$L \asymp \frac{c_{11}\sqrt{\log x}}{2\sqrt{\log(kN\log x)}}$$

for a suitably small constant c_{11}, we have

$$\frac{N_{\rho,L,f}(x)}{2\pi_N(x)} \to \int_0^1 \rho_L(t-\psi)\mu(t)dt \text{ as } x \to \infty. \tag{4.36}$$

Proof. The key ingredient in the proof is the following estimate, which follows from [Tho21, Proposition 2.1]. Let $f \in \mathcal{F}_{N,k}$ be a non-CM newform. Then there exist constants c_9 (suitably large) and c_{10} and c_{11} (suitably small) such that if

$$2l \le c_{11}\sqrt{\log x}/\sqrt{\log(kN\log x)},$$

then

$$\sum_{\substack{p\le x\\(p,N)=1}} a_f\left(p^{2l}\right) \ll l^2\pi_N(x)\left(x^{-\frac{1}{2c_9 l}} + e^{-\frac{c_{10}\log x}{4l^2\log(2kNl)}} + e^{-\frac{c_{10}\sqrt{\log x}}{2l}}\right). \tag{4.37}$$

Note that $\widehat{\rho}$ is a compactly supported, continuous function and therefore, absolutely bounded. Thus,

$$\frac{U(l)}{L} = \frac{1}{L}\left[\widehat{\rho}\left(\frac{l}{L}\right)\cos 2\pi l\psi - \widehat{\rho}\left(\frac{l+1}{L}\right)\cos 2\pi(l+1)\psi\right] \ll \frac{1}{L}.$$

Choosing

$$L \asymp \frac{c_{11}\sqrt{\log x}}{2\sqrt{\log(kN\log x)}},$$

we have

$$\begin{aligned}&\frac{N_{\rho,L,f}(x)}{2\pi_N(x)} - \frac{U(0)}{2L}\\ &= \frac{1}{2\pi_N(x)}\sum_{1\le l\le L}\frac{U(l)}{L}\sum_{\substack{p\le x\\(p,N)=1}}a_f(p^{2l})\\ &\ll \frac{1}{L\pi_N(x)}\sum_{1\le l\le L}l^2\left(x^{-\frac{1}{2c_9 l}} + e^{-\frac{c_{10}\log x}{4l^2\log(2kNl)}} + e^{-\frac{c_{10}\sqrt{\log x}}{2l}}\right)\\ &\ll \frac{L^2}{\pi_N(x)}\left(x^{-\frac{1}{2c_9 L}} + e^{-\frac{c_{10}\log x}{4L^2\log(2kNL)}} + e^{-\frac{c_{10}\sqrt{\log x}}{2L}}\right).\end{aligned}$$

The above term $\to 0$ as $x \to \infty$, if $L \ll \sqrt{\log x}/\sqrt{\log(kN \log x)}$. This proves the proposition. □

The limitation of the above proposition is in the range of L for which it holds. Is it possible to obtain the asymptotic (4.36) for a larger range of L, for example, $L(x) \sim x^{\alpha}$ for some $\alpha > 0$? It turns out that this can be done if one assumes strong analytic hypotheses on symmetric power L-functions corresponding to a non-CM newform f with squarefree level N. In this respect, using conditional discrepancy estimates of Rouse and Thorner [RT17], we have the following proposition:

Proposition 4.5.3. *Let $N \geq 1$ and $k \geq 2$ be integers with N squarefree and k even. Let $f \in \mathcal{F}_{N,k}$ be a non-CM newform such that for each $l \geq 0$, the following hypotheses hold:*

1. *The symmetric power L-function $L(s, Sym^l f)$ is the L-function of a cuspidal automorphic representation on $GL_{l+1}(\mathbb{A}_{\mathbb{Q}})$.*
2. *The Generalized Riemann hypothesis holds for $L(s, Sym^l f)$.*

Let ρ be as defined in (4.4). If $0 < \psi < 1$ and L is chosen such that

$$L = \mathrm{o}\left(\frac{x^{1/2c}}{(\log x)^{2/c}}\right)$$

for a constant $c > 1$, we have

$$\frac{N_{\rho,L,f}(x)}{2\pi_N(x)} \sim \int_0^1 \rho_L(t-\psi)\mu(t)dt \text{ as } x \to \infty.$$

Proof. By [RT17, Proposition 3.3], if $l \geq 1$ and $x \geq 5 \times 10^5$,

$$\sum_{\substack{p \leq x \\ (p,N)=1}} a_f\left(p^{2l}\right) \ll (l \log l)\sqrt{x} \log x \log(N(k-1)). \tag{4.38}$$

As in the proof of Proposition 4.5.2, we obtain that for $x \geq 5 \times 10^5$,

$$\begin{aligned} \frac{N_{\rho,L,f}(x)}{2\pi_N(x)} - \frac{U(0)}{2L} &= \frac{1}{2\pi_N(x)} \sum_{1 \leq l \leq L} \frac{U(l)}{L} \sum_{\substack{p \leq x \\ (p,N)=1}} a_f(p^{2l}) \\ &\ll L \log L \frac{\sqrt{x} \log x}{\pi_N(x)} \log(N(k-1)) \\ &\ll L \log L \frac{(\log x)^2}{\sqrt{x}} \log(N(k-1)). \end{aligned}$$

Let us choose $L(x)$ such that

$$L(x) = \mathrm{o}\left(\frac{x^{1/2c}}{(\log x)^{2/c}}\right) \text{ for } c > 1.$$

Then,

$$L \log L \frac{(\log x)^2}{\sqrt{x}} \log(N(k-1)) \to 0 \text{ as } x \to \infty.$$

□

By the results of Newton and Thorne [NT21a], [NT21b], the hypothesis (1) in the above proposition is now known to be true for all $l \geq 1$. Therefore, in comparison to Proposition 4.5.2, we have a larger range of L for which (4.36) holds, under GRH.

In what follows, we make some remarks here about $N_{\rho,L,f}(x)$ for a random $f \in \mathcal{F}_{N,k}$. We derive the expected value of $N_{\rho,L,f}(x)$ as we average over all $f \in \mathcal{F}_{N,k}$ (not just non-CM newforms). As we will see, averaging enables us to obtain the asymptotic in (4.36) over a more flexible range of L.

We first introduce the following notation: for any $\phi : \mathcal{F}_{N,k} \to \mathbb{C}$, we denote

$$\langle \phi \rangle := \frac{1}{|\mathcal{F}_{N,k}|} \sum_{f \in \mathcal{F}_{N,k}} \phi(f).$$

The following proposition tells us that an average version of (4.36) holds over a range of L that grows with the size of the families $\mathcal{F}_{N,k}$ under consideration.

Proposition 4.5.4. *Consider families $\mathcal{F}_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$ such that*

$$\frac{\log\left(|\mathcal{F}_{N,k}|/4^{\nu(N)}\right)}{\log x} \to \infty \text{ as } x \to \infty.$$

Let ρ be as chosen above. If $0 < \psi < 1$ and L is chosen such that

$$L = o\left(\frac{\log\left(|\mathcal{F}_{N,k}|/4^{\nu(N)}\right)}{\log x}\right),$$

we have

$$\left\langle \frac{N_{\rho,L,f}(x)}{2\pi_N(x)} \right\rangle \to \int_0^1 \rho_L(t-\psi)\mu(t)dt \text{ as } x \to \infty.$$

Proof. Applying (4.35), Corollary 3.2.28 and the estimate $U(l) \ll 1$,

$$\begin{aligned}
&\left\langle \frac{N_{\rho,L,f}(x)}{2\pi_N(x)} - \frac{U(0)}{2L} \right\rangle \\
&= \frac{1}{\pi_N(x)} \sum_{1 \le l \le L} \frac{U(l)}{L} \sum_{\substack{p \le x \\ (p,N)=1}} \left\langle a_f(p^{2l}) \right\rangle \\
&= \frac{1}{2\pi_N(x)} \sum_{1 \le l \le L} \frac{U(l)}{L} \sum_{\substack{p \le x \\ (p,N)=1}} \left(\frac{1}{p^l} + O\left(\frac{lp^{3l}4^{\nu(N)}}{|\mathcal{F}_{N,k}|}\right)\right) \\
&= \frac{1}{2\pi_N(x)L} \sum_{1 \le l \le L} \left(\sum_{\substack{p \le x \\ (p,N)=1}} \frac{U(l)}{p^l} + O\left(\frac{4^{\nu(N)}x^{3L}\pi_N(x)}{|\mathcal{F}_{N,k}|}\right)\right) \\
&= O\left(\frac{\log\log x}{\pi_N(x)}\right) + O\left(\frac{4^{\nu(N)}x^{3L}}{|\mathcal{F}_{N,k}|}\right)
\end{aligned}$$

$$\text{If } L = o\left(\frac{\log\left(|\mathcal{F}_{N,k}|/4^{\nu(N)}\right)}{\log x}\right), \text{ then, } x^{3L} = o\left(\frac{|\mathcal{F}_{N,k}|}{4^{\nu(N)}}\right).$$

Thus, we have

$$\left\langle \frac{N_{\rho,L,f}(x)}{2\pi_N(x)} - \frac{U(0)}{2L} \right\rangle \to 0 \text{ as } x \to \infty.$$

□

Corollary 4.5.5. *We consider families $F_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$ such that*

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{\log x} \to \infty \text{ as } x \to \infty.$$

Let ρ be as chosen above. If $0 < \psi < 1$ and L is chosen such that

$$L = \mathrm{o}\left(\frac{\log\left(kN/8^{\nu(N)}\right)}{\log x}\right),$$

we have

$$\left\langle \frac{N_{\rho,L,f}(x)}{2\pi_N(x)} \right\rangle \sim \int_0^1 \rho_L(t-\psi)\mu(t)dt \text{ as } x \to \infty.$$

Proof. The proof follows from Lemma 3.2.24. □

Remark 4.5.6. *If N varies over prime levels, then the above asymptotic will hold for families $F_{N,k}$ such that*

$$\frac{\log kN}{\log x} \to \infty \text{ as } x \to \infty.$$

4.5.2 Remarks on the pair correlation function

In Section 4.5.1, we saw that Thorner's unconditional estimate (4.37) and the conditional estimate of Rouse and Thorner (4.38) for sums $\sum_{p \leq x} a_f(p^{2u})$ play a pivotal role in deriving equidistribution properties of Hecke angles. By (4.33), these sums also appear in the pair correlation function $R_2(\rho, g; f)(x)$. But, we need estimates for these sums for u as large as $\pi_N(x)$, whereas (4.37) holds for

$$u \leq \sqrt{\log x}/\sqrt{\log(kN \log x)}.$$

The conditional estimate (4.38) holds for all $u \geq 1$, if x is sufficiently large. However, when we apply this estimate to (4.33), we get

$$R_2(\rho, g; f)(x) \ll x(\log x)^2 \log^2(N(k-1)),$$

which is not enough to determine the convergence of $R_2(\rho, g; f)(x)$ as $x \to \infty$.

In [BS19], the trace formula estimates in Corollary 3.2.26 were applied to obtain the limit

$$\lim_{x \to \infty} \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} R_2(\rho, g; f)(x),$$

albeit for rapidly increasing families $F_{N,k}$, parametrized by

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty.$$

The main theorem of this article, Theorem 4.3.1 makes the following fundamental observations over and above the results of [BS19].

1. The first observation is that in these rapidly increasing families, the trace formula estimates are versatile enough to accommodate the convergence of the second moment of $R_2(\rho, g; f)(x)$. To simplify the second moment

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} (R_2(\rho, g; f)(x))^2$$

addressed in this article, we require a delicate balancing act between estimates for several sums of type

$$\sum_{p,q,r,s} a_f(p^{2u}q^{2v}r^{2w}s^{2t})$$

and the ranges of u, v, w and t in each of these sums. These calculations are carried out in Section 6.2.

2. The second observation is that we obtain the convergence of the second moment of $R_2(\rho, g; f)(x)$ with the same choice of L and the same growth conditions for $\mathcal{F}_{N,k}$ as those required for the convergence of the first moment of $R_2(\rho, g; f)(x)$.

3. The Katz-Sarnak conjecture predicts that

$$\lim_{x\to\infty} \frac{1}{|\mathcal{F}_{N,k}|} \sum_{f\in\mathcal{F}_{N,k}} \big(R_2(\rho, g; f)(x)\big)^2 = \left(\lim_{x\to\infty} \frac{1}{|\mathcal{F}_{N,k}|} \sum_{f\in\mathcal{F}_{N,k}} (R_2(\rho, g; f)(x))\right)^2.$$

That is,

$$\mathrm{E}[(R_2(\rho, g; f)(x))^2] \sim \mathrm{E}[(R_2(\rho, g; f)(x))]^2 \text{ as } x \to \infty.$$

In Theorem 4.3.1, we are able to obtain this asymptotic, for (N, k) such that

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty.$$

In the next two chapters, we will focus on the first and second moments of the smooth localized pair correlation function $R_2(g, \rho; f)(x)$. In Chapter 5, we revisit the result on first moment [BS19] and mention the main result for any level N. In Chapter 6, we find the second moment and variance of $R_2(g, \rho; f)(x)$, and record estimates for lower order error terms in their computations, which is the main contribution of this thesis. We also show that the variance goes to 0 under the same growth conditions on weights and levels for the families of Hecke newforms as required for the convergence of the first moment.

Chapter 5

First moment of the pair correlation function $R_2(g,\rho;f)(x)$

We revisit Theorem 4.2.1 obtained for the first moment or average of the smooth localized pair correlation function $R_2(g,\rho;f)(x)$ in [BS19]. The goal of this chapter is to present inequalities and estimates for any level N. In [BS19], only prime levels were addressed. However, some modifications need to be made taking into account the levels N which are not prime. With our current estimates, it becomes clear to us what the optimal choice for L should be to obtain the convergence of the first moment of $R_2(\rho,g;f)(x)$ (defined in equation (4.5)). The idea used in the proof is the same as in [BS19].

We recall the definition of $R_2(\rho,g;f)(x)$ from equation (4.33) and simplify it to obtain

$$\begin{aligned}
&R_2(\rho,g;f)(x)\\
&=\frac{1}{8\pi_N(x)^2L}\sideset{}{'}\sum_{p,q\le x}\left(\sum_{l\ge 0}U(l)a_f(p^{2l})\right)\left(\sum_{l'\ge 0}U(l')a_f(q^{2l'})\right)\\
&\qquad\left(4G(0)+\sum_{n\ge 1}2G(n)(a_f(p^{2n})-a_f(p^{2n-2}))(a_f(q^{2n})-a_f(q^{2n-2}))\right)\\
&=\frac{1}{8\pi_N(x)^2L}\sideset{}{'}\sum_{p,q\le x}\left(4G(0)\sum_{l,l'\ge 0}U(l)U(l')a_f(p^{2l})a_f(q^{2l'})\right.\\
&+U(0)^2\sum_{n\ge 1}2G(n)(a_f(p^{2n})-a_f(p^{2n-2}))(a_f(q^{2n})-a_f(q^{2n-2}))\\
&\left.+\sum_{\substack{l,l'\ge 0,n\ge 1\\(l,l')\ne(0,0)}}2U(l)U(l')G(n)a_f(p^{2l})(a_f(p^{2n})-a_f(p^{2n-2}))a_f(q^{2l'})(a_f(q^{2n})-a_f(q^{2n-2}))\right).
\end{aligned}\tag{5.1}$$

To find the average of $R_2(\rho,g;f)(x)$ over newforms $f\in F_{N,k}$, we take the average over each of these three subparts separately. The averages over newforms of these three sums which we denote by P, Q, and T, are estimated in Propositions 5.1.1, 5.1.2, and 5.4.1 respectively.

Therefore,

$$P(\rho,g;f)(x)=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{8\pi_N(x)^2L}\sideset{}{'}\sum_{p,q\le x}4G(0)\sum_{l,l'\ge 0}U(l)U(l')a_f(p^{2l})a_f(q^{2l'}),$$

$$
\begin{aligned}
Q(\rho, g; f)(x) = {} & \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} {\sum_{p,q \leq x}}' U(0)^2 \sum_{n \geq 1} 2G(n) \\
& (a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})), \\
T(\rho, g; f)(x) = {} & \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} {\sum_{p,q \leq x}}' \sum_{\substack{l,l' \geq 0, n \geq 1 \\ (l,l') \neq (0,0)}} 2U(l)U(l')G(n) \\
& a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2}))a_f(q^{2l'})(a_f(q^{2n}) - a_f(q^{2n-2})).
\end{aligned} \tag{5.2}
$$

Hence,

$$
\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} R_2(\rho, g; f)(x) = P(\rho, g; f)(x) + Q(\rho, g; f)(x) + T(\rho, g; f)(x).
$$

T is estimated by further breaking it into two sums, namely R and S in equation (5.3), where R and S are defined in equations (5.4) and (5.5) respectively.

The estimation for R is done in Proposition 5.2.1. The estimation for S is done in Section 5.3 and the final estimate is mentioned in Proposition 5.3.7.

Adding $2R$ and S, we find estimate for T in Proposition 5.4.1. Finally, we find an estimation for the average of $R_2(\rho, g; f)(x)$ over newforms for all levels in Theorem 5.5.3. We note that the term S contributes the main term (see Proposition 5.3.7). In Proposition 5.5.2, we find an asymptotic limit for the main term.

Propositions 5.1.1, 5.1.2, 5.2.1 and 5.3.7 are generalisations of Proposition 15 of [BS19] to all levels N. We omit their proofs, as the proofs are similar to that of prime levels mentioned in [BS19] and can be obtained as a direct application of the Eichler-Selberg trace formula.

Section A.1 in Appendix A gives a quick reference to the terms mentioned in this chapter.

5.1 Revisiting the Pair correlation sum

We now evaluate $\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} R_2(\rho, g; f)(x)$ by finding estimates for each of the sums mentioned in equation (5.1). Before we start estimating the sums separately, we record the following lemmas without proof.

The following two propositions estimate $P(\rho, g; f)$ and $Q(\rho, g; f)$ respectively.

Proposition 5.1.1. *Let N, k be positive integers with k even and $L = L(x) \to \infty$ as $x \to \infty$. Then,*

$$
\begin{aligned}
P(\rho, g; f) &= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} {\sum_{p,q \leq x}}' 4G(0) \sum_{l,l' \geq 0} U(l)U(l')a_f(p^{2l})a_f(q^{2l'}) \\
&= O\left(\frac{1}{L}\right) + O\left(\frac{8^{\nu(N)} L}{kN} x^{4Lc'}\right),
\end{aligned}
$$

where both the summations over l and l' run up to $\lfloor Lc \rfloor$.

Proposition 5.1.2. *Let N, k be positive integers with k even and $L = L(x) \to \infty$ as $x \to \infty$. Then,*

$$
\begin{aligned}
Q(\rho, g; f) = {} & \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} {\sum_{p,q \leq x}}' U(0)^2 \sum_{n \geq 1} 2G(n)(a_f(p^{2n}) - a_f(p^{2n-2})) \\
& (a_f(q^{2n}) - a_f(q^{2n-2})) = O\left(\frac{1}{L}\right) + O\left(\frac{\pi_N(x)}{L} \frac{8^{\nu(N)} x^{4\pi_N(x)c'}}{kN}\right),
\end{aligned}
$$

where the summation over n runs up to $\pi_N(x)$.

We now find an estimate for $T(\rho, g; f)(x)$ (defined in equation (5.2)), where the summations over l and l' run up to $\lfloor L \rfloor$ and the summation over n runs up to $\pi_N(x)$. Since the summation is over $(l, l') \neq (0, 0)$, we can break the summation into the following three parts:

1) $l \geq 1, l' = 0, n \geq 1$,

2) $l = 0, l' \geq 1, n \geq 1$,

3) $l \geq 1, l' \geq 1, n \geq 1$.

By interchanging the variables l and l' first and then interchanging the variables p and q in the summation over $l = 0, l' \geq 1, n \geq 1$, we note that the summation over $l = 0, l' \geq 1, n \geq 1$ is exactly the same as the summation over $l \geq 1, l' = 0, n \geq 1$.

We denote the summations over $l \geq 1, l' = 0, n \geq 1$ and over $l \geq 1, l' \geq 1, n \geq 1$ by $R(\rho, g; f)(x)$ and $S(\rho, g; f)(x)$ respectively, i.e.,

Therefore,

$$T(\rho, g; f)(x) = 2R(\rho, g; f)(x) + S(\rho, g; f)(x). \tag{5.3}$$

5.2 Estimation for $R(\rho, g; f)(x)$

From the definition of $R(\rho, g; f)(x)$, we have

$$R(\rho, g; f)(x) = \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} \sum_{p,q \leq x}' \sum_{l,n \geq 1} 2U(l)U(0)G(n)a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2})) \cdot (a_f(q^{2n}) - a_f(q^{2n-2})). \tag{5.4}$$

The following proposition gives estimates for $R(\rho, g; f)(x)$.

Proposition 5.2.1. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$R(\rho, g; f)(x) = O\left(\frac{1}{L}\right) + O\left(\frac{8^{\nu(N)} x^{8\pi_N(x)c'} \pi_N(x) L}{kN}\right).$$

5.3 Estimation for $S(\rho, g; f)(x)$

From the definition of $S(\rho, g; f)(x)$, we have

$$S(\rho, g; f)(x) = \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} \sum_{p,q \leq x}' \sum_{l,l',n \geq 1} 2U(l)U(l')G(n)a_f(p^{2l}) (a_f(p^{2n}) - a_f(p^{2n-2}))a_f(q^{2l'})(a_f(q^{2n}) - a_f(q^{2n-2})). \tag{5.5}$$

Since the summation is over $l, l', n \geq 1$, where the indexes l and l' run up to $\lfloor L \rfloor$ and the index n runs up to $\pi_N(x)$, we can break the summation into the following four parts:

1) $l \neq n, l' \neq n$,

2) $l = n, l' \neq n$,

3) $l \neq n, l' = n$,

4) $l = n, l' = n$, i.e., $l = l' = n$.

We also denote the summation in the i-th part by $S_i(\rho, g; f)(x)$, $i = 1, 2, 3, 4$ respectively.

By interchanging the variables l and l' first and then interchanging the variables p and q in the summation over $l \neq n, l' = n$, we note that the summation over $l \neq n, l' = n$ is exactly the same as the summation over $l = n, l' \neq n$.

Therefore,

$$S(\rho, g; f)(x) = \sum_{i=1}^{4} S_i(\rho, g; f)(x) = S_1(\rho, g; f)(x) + 2S_2(\rho, g; f)(x) + S_4(\rho, g; f)(x). \quad (5.6)$$

We also set the notation $S_i(.)$, where the condition(s) within the first bracket indicates the summation has been taken over all l, l', n satisfying the condition(s) within the bracket, although this notation has no specific meaning in general (because there will be no l, l', n after the sum is estimated). For example, $S_1(l \neq n, l' > n)$ will mean the summation has been taken over all l, l', n satisfying the condition $l \neq n, l' > n$.

5.3.1 Estimation for $S_1(\rho, g; f)(x)$

We now estimate $S_1(\rho, g; f)(x)$. With the notations mentioned in Lemma 3.3.4, we have

$$S_1(\rho, g; f)(x) = \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L} \sideset{}{'}\sum_{p,q \leq x} \sum_{\substack{l,l',n \geq 1 \\ l \neq n, l' \neq n}} 2U(l)U(l')G(n)L_p(l,n) \quad L_q(l',n).$$

Using the same idea similar to that of equation (5.6), we get

$$\begin{aligned} &S_1(\rho, g; f)(x) \\ =&S_1(l > n, l' > n) + S_1(l > n, l' < n) + S_1(l < n, l' > n) + S_1(l < n, l' < n) \\ =&S_1(l > n, l' > n) + 2S_1(l > n, l' < n) + S_1(l < n, l' < n). \end{aligned}$$

We estimate all three sums separately in the following lemmas.

Lemma 5.3.1. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$S_1(l > n, l' > n) = O\left(\frac{(\log\log x)^2}{\pi_N(x)^2}\right) + O\left(\frac{8^{\nu(N)} x^{8Lc'} L^2}{kN}\right).$$

Lemma 5.3.2. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$S_1(l > n, l' < n) = O\left(\frac{L \log\log x}{\pi_N(x)}\right) + O\left(\frac{8^{\nu(N)} x^{8Lc'} L^2}{kN}\right).$$

Lemma 5.3.3. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$\begin{aligned} &S_1(l < n, l' < n) \\ =&\frac{\pi_N(x)(\pi_N(x) - 1)}{8\pi_N(x)^2 L} \sum_{l \geq 1} 2U(l)^2 G(l+1) + O\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + O\left(\frac{1}{\pi_N(x) L}\right) \\ &+ O\left(\frac{8^{\nu(N)} x^{(8L+4)c'}}{kN}\right) + O\left(\frac{8^{\nu(N)} x^{8\pi_N(x)c'} L\pi_N(x)}{kN}\right), \end{aligned}$$

as $x \to \infty$, where $L \leq \pi_N(x)$.

Combining Lemmas 5.3.1, 5.3.2 and 5.3.3, we obtain the following proposition.

Proposition 5.3.4. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$\begin{aligned}
&S_1(\rho, g; f)(x) \\
&= \frac{\pi_N(x)(\pi_N(x)-1)}{4\pi_N(x)^2 L}\sum_{l=1}^{\lfloor L \rfloor} U(l)^2 G(l+1) + O\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + O\left(\frac{1}{\pi_N(x)L}\right) \\
&+ O\left(\frac{8^{\nu(N)} x^{(8\pi_N(x)+4)c'}}{kN}\right),
\end{aligned}$$

as $x \to \infty$, where $L \le \pi_N(x)$.

5.3.2 Estimation for $S_2(\rho, g; f)(x) = S_3(\rho, g; f)(x)$

We now estimate $S_2(\rho, g; f)(x)$. With the notations mentioned in Lemma 3.3.4, we have

$$S_2(\rho, g; f)(x) = \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L}\sideset{}{'}\sum_{p,q \le x}\sum_{\substack{l,l',n \ge 1 \\ l=n,\, l' \neq n}} 2U(l)U(l')G(n)L_p(l,n) \cdot L_q(l',n).$$

Therefore,

$$S_2(\rho, g; f)(x) = S_2(l = n, l' > n) + S_2(l = n, l' < n).$$

We estimate both sums $S_2(l = n, l' > n)$, and $S_2(l = n, l' < n)$ separately and then add them together to obtain the following proposition.

Proposition 5.3.5. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$\begin{aligned}
&S_2(\rho, g; f)(x) \\
&= \frac{\pi_N(x)(\pi_N(x)-1)}{4\pi_N(x)^2 L}\sum_{l=1}^{\lfloor L \rfloor} U(l+1)U(l)G(l+1) + O\left(\frac{8^{\nu(N)} x^{(8L+8)c'}}{kN}\right) \\
&+ O\left(\frac{1}{\pi_N(x)L}\right) + O\left(\frac{\log\log x}{\pi_N(x)}\right).
\end{aligned}$$

5.3.3 Estimation for $S_4(\rho, g; f)(x)$

We first calculate $S_4(\rho, g; f)(x)$. With the notations mentioned in Lemma 3.3.4, we have

$$S_4(\rho, g; f)(x) = \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \frac{1}{8\pi_N(x)^2 L}\sideset{}{'}\sum_{p,q \le x}\sum_{\substack{l,l',n \ge 1 \\ l=n=l'}} 2U(l)U(l')G(n)L_p(l,n) \cdot L_q(l',n).$$

Proposition 5.3.6. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$S_4(\rho, g; f)(x) = \frac{\pi_N(x)(\pi_N(x)-1)}{4\pi_N(x)^2 L}\sum_{l=1}^{\lfloor L \rfloor} U(l)^2 G(l) + O\left(\frac{1}{\pi_N(x)L}\right) + O\left(\frac{8^{\nu(N)} x^{8Lc'}}{kN}\right).$$

Using equation (5.6), and combining Propositions 5.3.4, 5.3.5, and 5.3.6, we obtain the following proposition.

Proposition 5.3.7. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$\begin{aligned}
&S(\rho, g; f)(x)\\
&= \frac{\pi_N(x)(\pi_N(x) - 1)}{4\pi_N(x)^2 L}\left(\sum_{l=1}^{\lfloor L \rfloor} U(l)^2 G(l+1) - \sum_{l=1}^{\lfloor L \rfloor} 2U(l+1)U(l)G(l+1) + \sum_{l=1}^{\lfloor L \rfloor} U(l)^2 G(l)\right)\\
&+ O\left(\frac{1}{\pi_N(x)L}\right) + O\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + O\left(\frac{8^{\nu(N)} x^{(8\pi_N(x)+8)c'}}{kN}\right).
\end{aligned}$$

5.4 Estimation for $T(\rho, g; f)(x) = S(\rho, g; f)(x) + 2R(\rho, g; f)(x)$

Combining Propositions 5.2.1 and 5.3.7, with equation (5.3), we obtain the following proposition.

Proposition 5.4.1. *Let N and k be positive integers with k even. With ρ and g as defined earlier, we have*

$$\begin{aligned}
&T(\rho, g; f)(x)\\
&= \frac{\pi_N(x)(\pi_N(x) - 1)}{4\pi_N(x)^2 L}\left(\sum_{l=1}^{\lfloor L \rfloor} U(l)^2 G(l+1) - \sum_{l=1}^{\lfloor L \rfloor} 2U(l+1)U(l)G(l+1) + \sum_{l=1}^{\lfloor L \rfloor} U(l)^2 G(l)\right)\\
&+ O\left(\frac{1}{L}\right) + O\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + O\left(\frac{8^{\nu(N)} x^{(8\pi_N(x)+8)c'}}{kN}\right),
\end{aligned}$$

as $x \to \infty$, where $L \sim \pi_N(x)$.

5.5 Average of $R_2(\rho, g; f)(x)$ over newforms for all levels

We now revisit Theorem 4.2.1 and restate it in the Theorem 5.5.3 for the convenience of the reader.

5.5.1 Estimating the main term

Lemma 5.5.1. *Let $0 < \psi < 1$. With ρ as defined earlier,*

$$(1)\ \lim_{x \to \infty} \frac{1}{L}\sum_{n=1}^{\lfloor L \rfloor} \widehat{\rho}\left(\frac{n}{L}\right)^2 \cos 4n\pi\psi = \begin{cases} \frac{(\rho * \rho)(0)}{2}, & \text{if } \psi = \frac{1}{2},\\ 0, & \text{if } \psi \neq \frac{1}{2}\end{cases},$$

$$(2)\ \lim_{x \to \infty} \frac{1}{L}\sum_{n=1}^{\lfloor L \rfloor} \widehat{\rho}\left(\frac{n}{L}\right)^2 = \frac{(\rho * \rho)(0)}{2},$$

$$(3)\ \lim_{x \to \infty} \frac{1}{L}\sum_{n=1}^{\lfloor L \rfloor} \widehat{\rho}\left(\frac{n}{L}\right)^2 \sin 4n\pi\psi = 0, \text{ where } (\rho * \rho)(0) = \int_{-\infty}^{\infty} \rho(t)^2\, dt.$$

Proof. The proof follows from Theorem 2 of [BS19]. □

Using Lemma 5.5.1, we prove the following proposition.

Proposition 5.5.2. *Let N and k be positive integers with k even and L is such that $L = L(x) \to \infty$ as $x \to \infty$. Then, with ρ, g and f as defined earlier and for $\psi \in (0,1)$,*

$$\frac{T(g,\rho)}{4L} \sim C_\psi A^2 \widehat{g}(0)(\rho * \rho)(0),$$

where $A = 2\sin^2 \pi\psi$, *and*

$$C_\psi = \begin{cases} 2 & \text{if } \psi = \frac{1}{2}, \\ 1 & \text{if } \psi \neq \frac{1}{2}. \end{cases}$$

Proof. The proof follows from Theorem 2 of [BS19]. □

Theorem 5.5.3. *Let us consider families $F_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$. Let g, ρ be real-valued, even functions $\in C^\infty(\mathbb{R})$ in the Schwartz class with Fourier transforms supported in the interval $[-1, 1]$. Let $0 < \psi < 1$, $\psi \neq 1/2$. Define $A := 2\sin^2 \pi\psi$.*

(a) *Let $L = L(x) \to \infty$ as $x \to \infty$, and $L \ll \pi_N(x)$. Then*

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} R_2(\rho, g; f)(x)$$
$$= \frac{T(g,\rho)}{4L} + O\left(\frac{1}{L}\right) + O\left(\frac{L(\log\log x)^2}{\pi_N(x)}\right) + O\left(\frac{8^{\nu(N)} x^{(8\pi_N(x)+8)c'}}{kN}\right),$$

where c' is an absolute positive constant and

$$T(g,\rho) = \sum_{l \geq 1} (U(l) - U(l-1))^2 \widehat{g}\left(\frac{l}{\pi_N(x)}\right)$$

with

$$U(l) = \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi) - \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi (l+1)\psi).$$

(b) *In particular, if we choose $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$ with $L = L(x) \to \infty$ as $x \to \infty$ and families $F_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$ such that*

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty, \text{ as } x \to \infty,$$

then

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} R_2(\rho, g; f)(x) \sim \frac{T(g,\rho)}{4L}, \text{ as } x \to \infty.$$

Furthermore,

$$\frac{T(g,\rho)}{4L} \sim A^2 G(0)(\rho * \rho)(0), \text{ as } x \to \infty.$$

Proof. Using equation (5.3), and Propositions 5.1.1, 5.1.2 and 5.3.7, we obtain the proof of (a).

We note that $\frac{8^{\nu(N)} x^{(8\pi_N(x)+8)c'}}{kN} \ll \frac{8^{\nu(N)} x^{E\pi_N(x)c'}}{kN}$, for some constant E. Also, $\lim_{x\to\infty} \frac{8^{\nu(N)} x^{E\pi_N(x)c'}}{kN} = 0$ holds, if $x^{E\pi_N(x)c'} = o\left(kN/8^{\nu(N)}\right)$, as $x \to \infty$, which is true using Lemma 3.3.7.

Thus, $\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty$, as $x \to \infty$, implies $\lim_{x\to\infty} \frac{8^{\nu(N)} x^{(8\pi_N(x)+8)c'}}{kN} = 0$.

Since $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$ implies $L(x) \ll \pi_N(x)$, and $\frac{L(\log\log x)^2}{\pi_N(x)}$, the proof follows from Proposition 5.5.2 and part (a). □

Chapter 6

Second moment of the pair correlation function $R_2(g, \rho; f)(x)$

6.1 A brief overview

The goal of this chapter is to prove Theorem 4.3.1, which is the original contribution of this thesis. As we saw before, Theorem 4.3.1 addresses the second moment of the pair correlation function $R_2(\rho, g; f)(x)$, where

$$R_2(\rho, g; f)(x) := \frac{L}{8\pi_N(x)} \sum_{\substack{(p,q):p\neq q \quad x \\ (p,N)=(q,N)=1}} \rho_L(\quad \theta_f(p) \quad \psi)\rho_L(\quad \theta_f(q) \quad \psi)G_x(\quad \theta_f(p) \quad \theta_f(q)),$$

as defined in equation (4.5).

In this chapter, we will frequently use $\tilde{}$ to denote the function $\quad(\rho, g; f)(x)$. To find the second moment of $R_2(\rho, g; f)(x)$ over newforms $f\ 2\ F_{N,k}$, we first break $R_2(\rho, g; f)(x)^2$ in three parts, namely $K(\rho, g; f)(x)$, $L(\rho, g; f)(x)$, and $M(\rho, g; f)(x)$ in equation (6.1) and then take the average of each part over newforms $f\ 2\ F_{N,k}$.

Section A.2 in Appendix A gives a quick reference to the terms mentioned in this chapter.

6.2 Second moment of $R_2(\rho, g; f)(x)$ and variance

The goal of this section is to prove Theorem 4.3.1.

Henceforth, in all the sums below, p, q, r and s will denote distinct primes coprime to N. Also, for a finite set of primes $f p_1, p_2, \quad , p_t g$, we denote $\sum\limits_{\substack{p_1, \quad ,p_t \quad x \text{ all distinct} \\ (p_1,N)=1, \quad ,(p_t,N)=1}}$ by $\sum\limits_{p_1, \quad ,p_t \quad x}{}^{0}$.

Using equation (4.34), we obtain

$$\begin{aligned}
\big(R_2(\rho, g; f)(x)\big)^2 &= \left(\frac{1}{8\pi_N(x)^2 L} \sum_{p,q \quad x}{}^{0} T_1(p)T_2(q)T_3(p,q)\right)^2 \\
&= \frac{1}{64\pi_N(x)^4 L^2} \sum_{p,q \quad x}{}^{0} \sum_{r,s \quad x}{}^{0} T_1(p)T_2(q)T_3(p,q)T_1(r)T_2(s)T_3(r,s) \\
&= K(\rho, g; f)(x) + L(\rho, g; f)(x) + M(\rho, g; f)(x), \qquad (6.1)
\end{aligned}$$

where

$$K(\rho, g; f)(x) := \frac{2}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q \le x} T_1^2(p) T_2^2(q) T_3^2(p,q),$$

$$L(\rho, g; f)(x) := \frac{4}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r \le x} T_1^2(p) T_2(q) T_2(r) T_3(p,q) T_3(p,r),$$

and

$$M(\rho, g; f)(x) := \frac{1}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} T_1(p) T_2(q) T_3(p,q) T_1(r) T_2(s) T_3(r,s).$$

The averages of each of these sums over newforms (denoted by $\langle\ \rangle$) are evaluated in separate sections below.

The estimation for the average of $K(\rho, g; f)(x)$, which we denoted by $\langle K(\rho, g; f)(x) \rangle$, is addressed in Section 6.3 and the final estimate for $\langle K(\rho, g; f)(x) \rangle$ is presented in in Proposition 6.3.26.

The estimation for the average of $L(\rho, g; f)(x)$ is addressed in Section 6.4 and the final estimate for $\langle L(\rho, g; f)(x) \rangle$ is presented in Proposition 6.4.31.

The estimation for the average of $M(\rho, g; f)(x)$ is addressed in Section 6.5 and the final estimate for $\langle M(\rho, g; f)(x) \rangle$ is mentioned in Proposition 6.5.14. We also note that the contribution for the main term comes from the sum with four distinct primes, i.e., $\langle M(\rho, g; f)(x) \rangle$ (see equation (6.141)).

Finally, we combine all the estimates obtained in Propositions 6.3.26, 6.4.31 and 6.5.14 together to prove Theorem 6.5.15, which proves the first part of our main theorem on the second moment of the local pair correlation function $R_2(\rho, g; f)(x)$, previously stated as Theorem 4.3.1 (a).

In Theorem 6.5.16, we find the variance of the local pair correlation function $R_2(\rho, g; f)(x)$, which proves the second part of our main theorem, previously stated as Theorem 4.3.1 (b).

In Theorem 6.5.17, we show that the variance goes to 0 under the same growth conditions on weights and levels for the families of Hecke newforms, as required for the convergence of the first moment, and this proves the third part of our main theorem (Theorem 4.3.1 (c)). Theorem 6.5.17 gives us the optimal choice of $L = L(x)$ and the growth condition on weights $k = k(x)$ and levels $N = N(x)$, for which the second moment of $R_2(\rho, g; f)(x)$ is asymptotic to the square of the expected value of $R_2(\rho, g; f)(x)$ and the variance goes to zero, as $x \to \infty$.

6.3 Estimation for $\langle K(\rho, g; f)(x) \rangle = \langle (K_1 + 2K_2 + K_4)(\rho, g; f)(x) \rangle$

We first address

$$K(\rho, g; f)(x) = \frac{1}{32\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q \le x} T_1^2(p) T_2^2(q) T_3^2(p,q).$$

Here,

$$T_1^2(p) = \sum_{l_1, l_2 \ge 0} U(l_1) U(l_2) a_f(p^{2l_1}) a_f(p^{2l_2}),$$

$$T_2^2(q) = \sum_{k_1, k_2 \ge 0} U(k_1) U(k_2) a_f(q^{2k_1}) a_f(q^{2k_2}),$$

and

$$T_3^2(p,q) = \sum_{n_1, n_2 \ge 0} G(n_1) G(n_2) A(p,q,n_1) A(p,q,n_2).$$

Thus,

$$K(\rho, g; f)(x) \tag{6.2}$$

$$=\frac{1}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}' T_1^2(p)T_2^2(q)T_3^2(p,q)$$
$$=\frac{1}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}'\sum_{l_1,l_2\ge 0}\sum_{k_1,k_2\ge 0}\sum_{n_1,n_2\ge 0}U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)$$
$$a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p,q,n_1)A(p,q,n_2).$$

The indices n_1,n_2 run up to $\pi_N(x)$ and we can break the summation into the following four parts:

1) $n_1=0, n_2=0$,

2) $n_1\neq 0, n_2=0$,

3) $n_1=0, n_2\neq 0$,

4) $n_1\neq 0, n_2\neq 0$.

We also denote the sum in the i-th part by $K_i(\rho,g;f)(x)$, $i=1,2,3,4$ respectively. We note that the sum over n_1,n_2 where $n_1=0, n_2\neq 0$, is exactly the same as the sum over n_1,n_2 where $n_1\neq 0, n_2=0$, that is, $K_2(\rho,g;f)(x)=K_3(\rho,g;f)(x)$.

Therefore,

$$K(\rho,g;f)(x)=\sum_{i=1}^{4}K_i(\rho,g;f)(x)=K_1(\rho,g;f)(x)+2K_3(\rho,g;f)(x)+K_4(\rho,g;f)(x), \quad (6.3)$$

where $K_1(\rho,g;f)(x)$, $K_3(\rho,g;f)(x)$, and $K_4(\rho,g;f)(x)$ are defined in Sections 6.3.1, 6.3.2, and 6.3.3 respectively.

We now prove a few propositions to evaluate each of the above sums. The following estimates follow from Lemma 3.2.30, and will be used in Propositions 6.3.1 - 6.16.

Let u and v denote non-negative integers.

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q\le x}{}'\sum^{(1)}_{\substack{(u,v)\neq(0,0)\\ u=0 \text{ or } v=0\\ 0\le u\le U\\ 0\le v\le V}}a_f(p^{2u}q^{2v}) \ll \pi_N(x)\log\log x+\frac{\pi_N(x)^2x^{(2U+2V)c'}8^{\nu(N)}}{kN}, \quad (6.4)$$

and

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q\le x}{}'\sum^{(0)}_{\substack{(u,v)\\ 1\le u\le U\\ 1\le v\le V}}a_f(p^{2u}q^{2v}) \ll (\log\log x)^2+\frac{\pi_N(x)^2x^{(2U+2V)c'}8^{\nu(N)}}{kN}. \quad (6.5)$$

6.3.1 Estimation for $\langle K_1(\rho,g;f)(x)\rangle$

Proposition 6.3.1. *Let ρ,f,g be as defined earlier. For positive integers k and N with k even,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}K_1(\rho,g;f)(x) \quad (6.6)$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{32\pi_N(x)^4L^2}\sum_{l_1,l_2\ge 0}\sum_{k_1,k_2\ge 0}16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2$$
$$\sum_{p,q\le x}{}' a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})$$
$$\ll \frac{1}{\pi_N(x)^2}+\frac{L\log\log x}{\pi_N(x)^3}+\frac{L^2(\log\log x)^2}{\pi_N(x)^4}+\frac{L^2x^{8Lc'}8^{\nu(N)}}{\pi_N(x)^3kN},$$

where $c' > \frac{3}{2}$ is an absolute constant.

Proof. We note that each of the indices in the above sum l_1, l_2, k_1 and k_2 run up to L.

By Lemma 3.3.6,

$$a_f(p^{2l_1})a_f(p^{2l_2}) = \sum_{i=0}^{\min\{2l_1,2l_2\}} a_f(p^{2l_1+2l_2-2i}).$$

Thus,

$$\sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0} 16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2 \sideset{}{'}\sum_{p,q\leq x} a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2}) \quad (6.7)$$

$$= \sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0} 16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2$$

$$\sideset{}{'}\sum_{p,q\leq x}\sum_{i=0}^{\min\{2l_1,2l_2\}}\sum_{j=0}^{\min\{2k_1,2k_2\}} a_f(p^{2l_1+2l_2-2i}q^{2k_1+2k_2-2j}).$$

The innermost part in each of the above terms is of the form $a_f(p^{2u}q^{2v})$ where both $2u$ and $2v$ are at most $4L$. We first collect those terms with $u = v = 0$ in the sum

$$\sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0}\sum_{i=0}^{\min\{2l_1,2l_2\}} a_f(p^{2l_1+2l_2-2i})\sum_{j=0}^{\min\{2k_1,2k_2\}} a_f(q^{2k_1+2k_2-2j}).$$

Note that the exponent $u = 0$ only appears when $l_1 = l_2$ and when $i = 2l_1 = 2l_2$. So, the part of the sum

$$\sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0} 16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2 \sideset{}{'}\sum_{p,q\leq x} a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})$$

with $u = v = 0$ is

$$\sum_{l_1\geq 0}\sum_{k_1\geq 0} 16U(l_1)^2U(k_1)^2G(0)^2\pi_N(x)(\pi_N(x)-1).$$

We see that

$$\frac{1}{32\pi_N(x)^4L^2}\sum_{l_1\geq 0}\sum_{k_1\geq 0} 16U(l_1)^2U(k_1)^2G(0)^2\pi_N(x)(\pi_N(x)-1)$$
$$\ll \frac{L^2\pi_N(x)^2}{\pi_N(x)^4L^2} \ll \frac{1}{\pi_N(x)^2}.$$

Next, we collect those terms with $(u,v) \neq (0,0)$, but either $u = 0$ or $v = 0$. If $u = 0$, then $l_1 = l_2$ and $i = 2l_1 = 2l_2$. Since $v \neq 0$, the contribution of these terms to (6.7) is

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{32\pi_N(x)^4L^2}\sum_{l_1\geq 0}\sum_{\substack{k_1,k_2\geq 0\\ k_1\neq k_2}} 16U(l_1)^2U(k_1)U(k_2)G(0)^2$$

$$\sideset{}{'}\sum_{p,q\leq x}\sum_{j=0}^{\min\{2k_1,2k_2\}} a_f(p^0)a_f(q^{2k_1+2k_2-2j})$$

$$=\frac{1}{2\pi_N(x)^4L^2}\sum_{l_1\geq 0}\sum_{\substack{k_1,k_2\geq 0\\ k_1\neq k_2}} U(l_1)^2U(k_1)U(k_2)G(0)^2$$

$$\sideset{}{'}\sum_{p,q\leq x}\sum_{j=0}^{\min\{2k_1,2k_2\}}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(q^{2k_1+2k_2-2j})$$

$$
\begin{aligned}
&= \frac{1}{2\pi_N(x)^4L^2}\sum_{l_1\geq 0}\sum_{\substack{k_1,k_2\geq 0\\ k_1\neq k_2}} U(l_1)^2U(k_1)U(k_2)G(0)^2\left(\pi_N(x)\log\log x + \pi_N(x)^2x^{4Lc'}\frac{8^{\nu(N)}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sum_{l_1\geq 0}\sum_{\substack{k_1,k_2\geq 0\\ k_1\neq k_2}}\left(\pi_N(x)\log\log x + \pi_N(x)^2x^{4Lc'}\frac{8^{\nu(N)}}{kN}\right)\\
&\ll \frac{L^3}{\pi_N(x)^4L^2}\left(\pi_N(x)\log\log x + \pi_N(x)^2x^{4Lc'}\frac{8^{\nu(N)}}{kN}\right)\\
&\ll \frac{L\log\log x}{\pi_N(x)^3} + \frac{Lx^{4Lc'}}{\pi_N(x)^2}\frac{8^{\nu(N)}}{kN}.
\end{aligned}
$$

by (6.4).

The estimate for the contribution of terms with $u \neq 0$ and $v = 0$ is similar.

We now collect those terms in (6.7) such that u and v are both non-zero. By (6.5), the contribution of these terms to (6.7) is

$$
\begin{aligned}
&\ll \frac{L^4}{\pi_N(x)^4L^2}\left((\log\log x)^2 + \pi_N(x)^2x^{8Lc'}\frac{8^{\nu(N)}}{kN}\right)\\
&\ll \frac{L^2(\log\log x)^2}{\pi_N(x)^4} + \frac{L^2x^{8Lc'}}{\pi_N(x)^2}\frac{8^{\nu(N)}}{kN}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K_1(\rho, g; f)(x)\\
&\ll \frac{1}{\pi_N(x)^2} + \frac{L\log\log x}{\pi_N(x)^3} + \frac{Lx^{4Lc'}}{\pi_N(x)^2}\frac{8^{\nu(N)}}{kN} + \frac{L^2(\log\log x)^2}{\pi_N(x)^4} + \frac{L^2x^{8Lc'}}{\pi_N(x)^2}\frac{8^{\nu(N)}}{kN}\\
&\ll \frac{1}{\pi_N(x)^2} + \frac{L\log\log x}{\pi_N(x)^3} + \frac{L^2(\log\log x)^2}{\pi_N(x)^4} + \frac{L^2x^{8Lc'}8^{\nu(N)}}{\pi_N(x)^2kN}.
\end{aligned}
$$

□

6.3.2 Estimation for $\langle (K_2 + K_3)(\rho, g; f)(x)\rangle$

We now look at the part of the sum $K(\rho, g; f)(x)$ with $n_1 = 0$ and $n_2 \neq 0$, i.e., we now estimate $K_3(\rho, g; f)(x)$. In this case, the innermost term

$$
\begin{aligned}
&a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p,q,n_1)A(p,q,n_2)\\
&= 8a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))(a_f(q^{2n_2}) - a_f(q^{2n_2-2}))\\
&= 8\{a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\}\{a_f(q^{2k_1})a_f(q^{2k_2})(a_f(q^{2n_2}) - a_f(q^{2n_2-2}))\}.
\end{aligned}
$$

We want to find an estimate for

$$
\begin{aligned}
&K_3(\rho, g; f)(x) \qquad (6.8)\\
&= \frac{1}{32\pi_N(x)^4L^2}\sum_{p,q\leq x}{}'\sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0}\sum_{n_2\geq 1} U(l_1)U(l_2)U(k_1)U(k_2)G(0)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p,q,0)A(p,q,n_2)\\
&= \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\leq x}{}'\sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0}\sum_{n_2\geq 1} U(l_1)U(l_2)U(k_1)U(k_2)G(0)G(n_2)
\end{aligned}
$$

$$a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))(a_f(q^{2n_2}) - a_f(q^{2n_2-2}))$$

$$
\begin{aligned}
=&\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}^{'}\sum_{n_2\ge 1}G(0)G(n_2)\sum_{l_1,l_2\ge 0}U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
&\sum_{k_1,k_2\ge 0}U(k_1)U(k_2)a_f(q^{2k_1})a_f(q^{2k_2})(a_f(q^{2n_2}) - a_f(q^{2n_2-2}))\\
=&\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}^{'}\sum_{n_2\ge 1}G(0)G(n_2)A(\rho,g;f;n_2,p)A(\rho,g;f;n_2,q),
\end{aligned}
$$

where for $n\ge 1$, and for any prime r,

$$A(\rho,g;f;n,r) := \sum_{l_1,l_2\ge 0}U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n}) - a_f(r^{2n-2})). \tag{6.9}$$

Using Lemma 3.3.5, for $l_2\ge n_2$, we have

$$
\begin{aligned}
&a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
=&a_f(p^{2l_1})(a_f(p^{2l_2-2n_2}) + a_f(p^{2l_2+2n_2}))\\
=&a_f(p^{2l_1})a_f(p^{2l_2-2n_2}) + a_f(p^{2l_1})a_f(p^{2l_2+2n_2}).
\end{aligned}
$$

We note that

1) The product $a_f(p^{2l_1})a_f(p^{2l_2-2n_2})$ gives 1, only if, $l_1 = l_2 - n_2$, i.e., $l_1 - l_2 = -n_2$, and

2) The product $a_f(p^{2l_1})a_f(p^{2l_2+2n_2})$ gives 1, only if, $l_1 = l_2 + n_2$, i.e., $l_1 - l_2 = n_2$.

Using Lemma 3.3.5, for $l_2 < n_2$, we have

$$
\begin{aligned}
&a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
=&a_f(p^{2l_1})(a_f(p^{2l_2+2n_2}) - a_f(p^{2n_2-2l_2-2}))\\
=&a_f(p^{2l_1})a_f(p^{2l_2+2n_2}) - a_f(p^{2l_1})a_f(p^{2n_2-2l_2-2})).
\end{aligned}
$$

We note that

1) The product $a_f(p^{2l_1})a_f(p^{2l_2+2n_2})$ gives 1, only if, $l_1 = l_2 + n_2$, i.e., $l_1 - l_2 = n_2$, and

2) The product $a_f(p^{2l_1})a_f(p^{2n_2-2l_2-2})$ gives 1, only if, $l_1 = n_2 - l_2 - 1$, i.e., $l_1 + l_2 = n_2 - 1$.

For any prime r and positive integer n_2, we define

$$A_1(\rho,g;f;n_2,r) := \sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2}}U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2+2n_2}), \tag{6.10}$$

$$A_2(\rho,g;f;n_2,r) := \sum_{\substack{l_1,l_2\ge 0,\, l_2\ge n_2\\ l_1-l_2\neq -n_2}}U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2-2n_2}),$$

$$A_3(\rho,g;f;n_2,r) := \sum_{\substack{l_1,l_2\ge 0,\, l_2<n_2\\ l_1+l_2\neq n_2-1}}U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2n_2-2l_2-2}).$$

For a positive integer n_2, we define

$$B_1(\rho,g;f;n_2) := 2\sum_{l\ge 0}U(l)U(l+n_2), \tag{6.11}$$

$$B_2(\rho,g;f;n_2) := \sum_{l=0}^{n_2-1} U(l)U(n_2-1-l). \tag{6.12}$$

We will write $A_i(n_2,r)$ and $B_j(n_2)$ for $A_i(\rho,g;f;n_2,r)$ and $B_j(\rho,g;f;n_2)$ respectively, $i=1,2,3$ and $j=1,2$.

For any prime r and fixed integer $n_2 \ge 1$,

$$\begin{aligned}
&A(\rho,g;f;n_2,r) \qquad\qquad (6.13)\\
=&\sum_{l_1,l_2\ge 0} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\
=&\sum_{l_1,l_2\ge 0,l_2\ge n_2} U(l_1)U(l_2)(a_f(r^{2l_1})a_f(r^{2l_2-2n_2})+a_f(r^{2l_1})a_f(r^{2l_2+2n_2}))\\
+&\sum_{l_1,l_2\ge 0,l_2<n_2} U(l_1)U(l_2)(-a_f(r^{2l_1})a_f(r^{2n_2-2l_2-1})+a_f(r^{2l_1})a_f(r^{2l_2+2n_2}))\\
=&\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2=-n_2}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2\ne -n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2-2n_2})\\
+&\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2=n_2}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2\ne n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2+2n_2})\\
+&\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1-l_2=n_2}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1-l_2\ne n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2+2n_2})\\
-&\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1+l_2=n_2-1}} U(l_1)U(l_2)-\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1+l_2\ne n_2-1}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2n_2-2l_2-2})\\
=&\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2=-n_2}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2\ne -n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2-2n_2})\\
+&\sum_{\substack{l_1,l_2\ge 0,\\ l_1-l_2=n_2}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\ne n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2+2n_2})\\
-&\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1+l_2=n_2-1}} U(l_1)U(l_2)-\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1+l_2\ne n_2-1}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2n_2-2l_2-2})\\
=&\sum_{l\ge 0} U(l)U(l+n_2)+\sum_{\substack{l_1,l_2\ge 0,l_2\ge n_2\\ l_1-l_2\ne -n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2-2n_2})\\
+&\sum_{l\ge 0} U(l)U(l+n_2)+\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\ne n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2+2n_2})\\
-&\sum_{l=0}^{n_2-1} U(l)U(n_2-1-l)-\sum_{\substack{l_1,l_2\ge 0,l_2<n_2\\ l_1+l_2\ne n_2-1}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2n_2-2l_2-2})\\
=&2\sum_{l\ge 0} U(l)U(l+n_2)-\sum_{l=0}^{n_2-1} U(l)U(n_2-1-l)+\sum_{i=1}^{3} A_i(n_2,r)\\
=&B_1(n_2)-B_2(n_2)+\sum_{i=1}^{3} A_i(n_2,r).
\end{aligned}$$

We note that

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} G(0)G(n_2)A_1(n_2,p)A_1(n_2,q)$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} G(0)G(n_2)$$
$$\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2+2n_2}) \sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2}} U(k_1)U(k_2)a_f(q^{2k_1})a_f(q^{2k_2+2n_2})$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} G(0)G(n_2)\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2}} U(l_1)U(l_2)U(k_1)U(k_2)$$
$$\sum_{t=|l_1-(l_2+n_2)|}^{l_1+l_2+n_2} a_f(p^{2t}) \sum_{t'=|k_1-(k_2+n_2)|}^{k_1+k_2+n_2} a_f(q^{2t'})$$
$$=\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} G(0)G(n_2)\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2}} U(l_1)U(l_2)U(k_1)U(k_2)$$
$$\sum_{t=|l_1-(l_2+n_2)|}^{l_1+l_2+n_2}\sum_{t'=|k_1-(k_2+n_2)|}^{k_1+k_2+n_2}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(p^{2t}q^{2t'})\right)$$
$$\ll\frac{1}{\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1}\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2}}$$
$$\sum_{t=|l_1-(l_2+n_2)|}^{l_1+l_2+n_2}\sum_{t'=|k_1-(k_2+n_2)|}^{k_1+k_2+n_2}\left(\frac{1}{p^tq^{t'}}+\frac{8^{\nu(N)}p^{2tc'}q^{2t'c'}}{kN}\right)$$
$$\ll\frac{1}{\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1}\left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right)L\left(\frac{1}{q}+\frac{1}{q^2}+\cdots\right)L+\frac{1}{\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1}$$
$$\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2}}(l_1+l_2+n_2)(k_1+k_2+n_2)\left(\frac{8^{\nu(N)}p^{2(l_1+l_2+n_2)c'}q^{2(k_1+k_2+n_2)c'}}{kN}\right)$$
$$\ll\frac{1}{\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1}\left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right)L\left(\frac{1}{q}+\frac{1}{q^2}+\cdots\right)L$$
$$+\frac{1}{\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} L^4\pi_N(x)^2\left(\frac{8^{\nu(N)}(pq)^{2(2L+n_2)c'}}{kN}\right)$$
$$\ll\frac{1}{\pi_N(x)^4}\sum_{n_2\ge 1}\left(\sum_{p,q\le x}{}'\frac{1}{pq}\right)+\frac{1}{\pi_N(x)^2}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} L^2\left(\frac{8^{\nu(N)}(x^2)^{2(2L+\pi_N(x))c'}}{kN}\right)$$
$$\ll\frac{(\log\log x)^2}{\pi_N(x)^3}+\pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{(8L+4\pi_N(x))c'}}{kN}\right).$$

A similar calculation holds when $A_1(n_2,p)A_1(n_2,q)$ is replaced by $A_i(n_2,p)A_j(n_2,q)$, where $i=1,2,3$ and $j=1,2,3$. Hence,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{p,q\le x}{}' \sum_{n_2\ge 1} G(0)G(n_2)A_i(n_2,p)A_j(n_2,q) \qquad (6.14)$$

$$\frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{(8L+4\pi_N(x))c^0}}{kN}\right).$$

Therefore, using equations (6.8) and (6.14), we obtain

$$\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}} K_3(\rho, g; f)(x)$$
$$=\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1} G(0)G(n_2)A(\rho, g; f; n_2, p)A(\rho, g; f; n_2, q)$$
$$=\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1} G(0)G(n_2)\left(B_1(n_2)\quad B_2(n_2)+\sum_{i=1}^{3}A_i(n_2,p)\right)\left(B_1(n_2)\quad B_2(n_2)+\sum_{i=1}^{3}A_i(n_2,q)\right)$$
$$=\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1} G(0)G(n_2)\left(B_1(n_2)\quad B_2(n_2)\right)^2$$
$$+\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{16}{32\pi_N(x)^4L^2}\sum_{i=1}^{3}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1} G(0)G(n_2)A_i(n_2,p)B_1(n_2)$$
$$\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{16}{32\pi_N(x)^4L^2}\sum_{i=1}^{3}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1} G(0)G(n_2)A_i(n_2,p)B_2(n_2)$$
$$+\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1} G(0)G(n_2)A_i(n_2,p)A_j(n_2,q)$$
$$\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1}(2L+L)^2$$
$$+\sum_{i=1}^{3}\sum_{j=1}^{2}\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}\frac{16}{32\pi_N(x)^4L^2}\sum_{p,q}{}^{0}\sum_{x\; n_2\; 1}LL^2$$
$$+\frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{((8L+4)+4\pi_N(x))c^0}}{kN}\right)$$
$$\frac{1}{\pi_N(x)^2L^2}\sum_{n_2\; 1}(2L+L)^2 + \frac{1}{\pi_N(x)^2L^2}\sum_{n_2\; 1}LL^2$$
$$+\frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{(16L+4\pi_N(x))c^0}}{kN}\right)$$
$$\frac{1}{\pi_N(x)^2L^2}L^3\pi_N(x) + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{((8L+4)+4\pi_N(x))c^0}}{kN}\right)$$
$$\frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{(16L+4\pi_N(x))c^0}}{kN}\right).$$

Thus,

$$\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}}(K_2+K_3)(\rho, g; f)(x) \tag{6.15}$$
$$=\frac{2}{jF_{N,k}j}\sum_{f2F_{N,k}}K_2(\rho, g; f)(x)$$

$$\frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{(16L+4\pi_N(x))c'}}{kN}\right).$$

6.3.3 Estimation for $\langle K_4(\rho, g; f)(x)\rangle$

We now look at the part of the sum $K(\rho, g; f)(x)$ with $n_1 \neq 0$ and $n_2 \neq 0$, i.e., we now estimate $K_4(\rho, g; f)(x)$.

For $n_1, n_2 \geq 1$ and for any prime p, we define

$$T(p, n_1, n_2) := (a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2})).$$

Thus,

$$A(p, q, n_1)A(p, q, n_2) = 4T(p, n_1, n_2)T(q, n_1, n_2).$$

In this case, the innermost term then becomes

$$\begin{aligned}
&a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p, q, n_1)A(p, q, n_2)\\
=&4a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
&\quad a_f(q^{2k_1})a_f(q^{2k_2})(a_f(q^{2n_1}) - a_f(q^{2n_1-2}))(a_f(q^{2n_2}) - a_f(q^{2n_2-2}))\\
=&4a_f(p^{2l_1})a_f(p^{2l_2})T(p, n_1, n_2)a_f(q^{2k_1})a_f(q^{2k_2})T(q, n_1, n_2)\\
=&4k(p, n_1, n_2, l_1, l_2)k(q, n_1, n_2, l_1, l_2),
\end{aligned}$$

where for $n_1, n_2 \geq 1, l_1, l_2 \geq 0$ and for any prime r,

$$k(r, n_1, n_2, l_1, l_2) := a_f(r^{2l_1})a_f(r^{2l_2})T(r, n_1, n_2).$$

For $n_1, n_2 \geq 1$ and for any prime r, we define

$$k(r, n_1, n_2) := \sum_{l_1, l_2 \geq 0} U(l_1)U(l_2)k(r, n_1, n_2, l_1, l_2).$$

Therefore, $k(r, n_1, n_2) = k(r, n_2, n_1)$.

We want to find an estimate for

$$\begin{aligned}
&K_4(\rho, g; f)(x) &(6.16)\\
=&\frac{1}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q \leq x}\sum_{l_1,l_2 \geq 0}\sum_{k_1,k_2 \geq 0}\sum_{n_1,n_2 \geq 1} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p, q, n_1)A(p, q, n_2)\\
=&\frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q \leq x}\sum_{l_1,l_2 \geq 0}\sum_{k_1,k_2 \geq 0}\sum_{n_1,n_2 \geq 1} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)\\
&\qquad k(p, n_1, n_2, l_1, l_2)k(q, n_1, n_2, l_1, l_2)\\
=&\frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q \leq x}\sum_{n_1,n_2 \geq 1} G(n_1)G(n_2)\sum_{l_1,l_2 \geq 0} U(l_1)U(l_2)k(p, n_1, n_2, l_1, l_2)\\
&\qquad \sum_{k_1,k_2 \geq 0} U(k_1)U(k_2)k(q, n_1, n_2, l_1, l_2)\\
=&\frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q \leq x}\sum_{n_1,n_2 \geq 1} G(n_1)G(n_2)k(p, n_1, n_2)k(q, n_1, n_2)\\
=&\frac{8}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q \leq x}\sum_{\substack{n_1,n_2 \geq 1\\ n_2 > n_1}} G(n_1)G(n_2)k(p, n_1, n_2)k(q, n_1, n_2)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_1=n_2}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2)\\
=&C(\rho,g;f)(x) + D(\rho,g;f)(x),
\end{aligned}
$$

where

$$
C(\rho,g;f)(x) := \frac{8}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2). \tag{6.17}
$$

and

$$
D(\rho,g;f)(x) := \frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_1=n_2}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2). \tag{6.18}
$$

We now find an estimate for

$$
C(\rho,g;f)(x) := \frac{8}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2).
$$

To estimate $C(\rho, g; f)(x)$, we first find estimate for $k(p, n_1, n_2)$ in Proposition 6.3.8, where

$$
\begin{aligned}
&k(p,n_1,n_2)\\
=&\sum_{l_1,l_2\ge 0} U(l_1)U(l_2)k(p,n_1,n_2,l_1,l_2)\\
=&\sum_{l_1,l_2\ge 0} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})T(p,n_1,n_2)\\
=&\sum_{l_1,l_2\ge 0} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_1})-a_f(p^{2n_1-2}))(a_f(p^{2n_2})-a_f(p^{2n_2-2})).
\end{aligned}
$$

In what follows below in this section, we always have $n_2 > n_1$.

We also note that since the summation is over l_1, l_2, where the indexes l_1 and l_2 run up to $\lfloor L\rfloor$, we can break the summation into the following four parts:

1) $l_1 \ge n_1, l_2 \ge n_2$,

2) $l_1 \ge n_1, l_2 < n_2$,

3) $l_1 < n_1, l_2 \ge n_2$,

4) $l_1 < n_1, l_2 < n_2$.

We denote the summation in the i-th part by $\alpha_i(\rho, g; f, n_1, n_2, p)$, $i = 1, 2, 3, 4$ respectively.

We will write $\alpha_i(n_1, n_2, p)$ for $\alpha_i(\rho, g; f, n_1, n_2, p)$, $i = 1, 2, 3, 4$, in short.

Therefore,

$$
k(p,n_1,n_2) = \sum_{i=1}^{4}\alpha_i(n_1,n_2,p).
$$

We now estimate $\alpha_i(n_1, n_2, r)$ for each $i = 1, 2, 3, 4$ in the following lemmas. We begin with $\alpha_1(n_1, n_2, r)$.

Lemma 6.3.2. *Let $\alpha_1(n_1, n_2, r)$ be the part of $k(r, n_1, n_2)$, where the summation is taken over $l_1, l_2 \ge 0$, with $l_1 \ge n_1, l_2 \ge n_2$, and the indexes l_1 and l_2 run up to $\lfloor L\rfloor$. Then, for any prime r and positive integers n_1, n_2 with $n_2 \ge n_1$,*

$$
\alpha_1(n_1,n_2,r)
$$

$$= \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 = n_2 - n_1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 = -n_1 - n_2}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 = n_1 + n_2}} U(l_1)U(l_2)$$
$$+ \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 = n_1 - n_2}} U(l_1)U(l_2) + (X_1 + X_2 + X_3 + X_4)(\rho, g; f; n_1, n_2, r),$$

where $X_i(\rho, g; f; n_1, n_2, r)$ $(i = 1, 2, 3, 4)$ *are defined in equations* (6.19), (6.20), (6.21), *and* (6.22) *respective]y.*

Proof. Using Corollary 3.3.5, for any prime r and integers l_1, l_2 with $l_1 \leq n_1, l_2 \leq n_2$, we have

$$\begin{aligned}
& a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_1}) - a_f(r^{2n_1-2}))(a_f(r^{2n_2}) - a_f(r^{2n_2-2})) \\
=& a_f(r^{2l_1})(a_f(r^{2n_1}) - a_f(r^{2n_1-2}))a_f(r^{2l_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2})) \\
=& (a_f(r^{2l_1-2n_1}) + a_f(r^{2l_1+2n_1}))(a_f(r^{2l_2-2n_2}) + a_f(r^{2l_2+2n_2})) \\
=& a_f(r^{2l_1-2n_1})a_f(r^{2l_2-2n_2}) + a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2}) \\
+& a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2}) + a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}).
\end{aligned}$$

We note that

1) The product $a_f(r^{2l_1-2n_1})a_f(r^{2l_2-2n_2})$ gives 1, only if $l_1 - n_1 = l_2 - n_2$, i.e., $l_1 - l_2 = n_1 - n_2$,

2) The product $a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2})$ gives 1, only if $l_1 - n_1 = l_2 + n_2$, i.e., $l_1 - l_2 = n_1 + n_2$,

3) The product $a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2})$ gives 1, only if $l_1 + n_1 = l_2 - n_2$, i.e., $l_1 - l_2 = -n_1 - n_2$, and

4) The product $a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})$ gives 1, only if $l_1 + n_1 = l_2 + n_2$, i.e., $l_1 - l_2 = n_2 - n_1$.

For any prime r and positive integers n_1, n_2 with $n_2 \leq n_1$, we define

$$X_1(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 \neq n_2 - n_1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}), \tag{6.19}$$

$$X_2(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 \neq -n_1 - n_2}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2}), \tag{6.20}$$

$$X_3(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 \neq n_1 + n_2}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2}), \tag{6.21}$$

and

$$X_4(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 \leq n_2 \\ l_1 - l_2 \neq n_1 - n_2}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2l_2-2n_2}). \tag{6.22}$$

We will write $X_i(n_1, n_2, r)$ for $X_i(\rho, g; f; n_1, n_2, r)$, $i = 1, 2, 3, 4$.

For any prime r and positive integers n_1,n_2 with $n_2 \ge n_1$,

$$\begin{aligned}
&\alpha_1(n_1,n_2,r)\\
=&\alpha_1(\rho,g;f;n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_1})-a_f(r^{2n_1-2}))(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\
=&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2}} U(l_1)U(l_2)\{a_f(r^{2l_1-2n_1})a_f(r^{2l_2-2n_2})+a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2})+\\
&\qquad a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2})+a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})\}\\
=&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=n_2-n_1}} U(l_1)U(l_2)+X_1(\rho,g;f;n_1,n_2,r)+\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=-n_1-n_2}} U(l_1)U(l_2)+X_2(\rho,g;f;n_1,n_2,r)\\
+&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=n_1+n_2}} U(l_1)U(l_2)+X_3(\rho,g;f;n_1,n_2,r)+\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=n_1-n_2}} U(l_1)U(l_2)+X_4(\rho,g;f;n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=n_2-n_1}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=-n_1-n_2}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=n_1+n_2}} U(l_1)U(l_2)\\
+&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2\le n_2\\ l_1-l_2=n_1-n_2}} U(l_1)U(l_2)+(X_1+X_2+X_3+X_4)(\rho,g;f;n_1,n_2,r).
\end{aligned} \tag{6.23}$$

□

Lemma 6.3.3. *Let $\alpha_2(n_1,n_2,r)$ be the part of $k(r,n_1,n_2)$, where the summation is taken over $l_1,l_2 \ge 0$, with $l_1 \le n_1, l_2 < n_2$, and the indexes l_1 and l_2 run up to $\lfloor L \rfloor$. Then, for any prime r and positive integers n_1,n_2 with $n_2 \ge n_1$,*

$$\begin{aligned}
&\alpha_2(n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2<n_2\\ l_1-l_2=n_2-n_1}} U(l_1)U(l_2)-\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2<n_2\\ l_1+l_2=n_2-n_1-1}} U(l_1)U(l_2)+\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2<n_2\\ l_1-l_2=n_1+n_2}} U(l_1)U(l_2)\\
&-\sum_{\substack{l_1,l_2\ge 0\\ l_1\le n_1,l_2<n_2\\ l_1+l_2=n_1+n_2-1}} U(l_1)U(l_2)+(Y_1+Y_2+Y_3+Y_4)(\rho,g;f;n_1,n_2,r),
\end{aligned}$$

where $Y_i(\rho,g;f;n_1,n_2,r)$ $(i=1,2,3,4)$ are defined in equations (6.24), (6.25), (6.26), *and* (6.27) *respective]y.*

Proof. Using Corollary 3.3.5, for any prime r and integers l_1,l_2 with $l_1 \le n_1, l_2 < n_2$, we have

$$\begin{aligned}
&a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_1})-a_f(r^{2n_1-2}))(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\
=&a_f(r^{2l_1})(a_f(r^{2n_1})-a_f(r^{2n_1-2}))a_f(r^{2l_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\
=&(a_f(r^{2l_1-2n_1})+a_f(r^{2l_1+2n_1}))(a_f(r^{2l_2+2n_2})-a_f(r^{2n_2-2l_2-2}))\\
=&a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})-a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2})\\
+&a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2})-a_f(r^{2l_1-2n_1})a_f(r^{2n_2-2l_2-2}).
\end{aligned}$$

We note that

1) The product $a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})$ gives 1, only if $l_1+n_1 = l_2+n_2$, i.e., $l_1 - l_2 = n_2 - n_1$,

2) The product $a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2})$ gives 1, only if $l_1 + n_1 = n_2 - l_2 - 1$, i.e., $l_1 + l_2 = n_2 - n_1 - 1$,

3) The product $a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2})$ gives 1, only if $l_1 - n_1 = l_2+n_2$, i.e., $l_1 - l_2 = n_2+n_1$, and

4) The product $a_f(r^{2l_1-2n_1})a_f(r^{2n_2-2l_2-2})$ gives 1, only if $l_1 - n_1 = n_2 - l_2 - 1$, i.e., $l_1 + l_2 = n_2 + n_1 - 1$.

For any prime r and positive integers n_1, n_2 with $n_2 \geq n_1$, we define

$$Y_1(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 - l_2 \neq n_2 - n_1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}), \tag{6.24}$$

$$Y_2(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 + l_2 \neq n_2 - n_1 - 1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2}), \tag{6.25}$$

$$Y_3(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 - l_2 \neq n_1 + n_2}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2}), \tag{6.26}$$

and

$$Y_4(\rho, g; f; n_1, n_2, r) := \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 + l_2 \neq n_1 + n_2 - 1}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2n_2-2l_2-2}). \tag{6.27}$$

We will write $Y_i(n_1, n_2, r)$ for $Y_i(\rho, g; f; n_1, n_2, r)$, $i = 1, 2, 3, 4$.

For any prime r and positive integers n_1, n_2 with $n_2 \geq n_1$,

$$\begin{aligned}
&\alpha_2(n_1, n_2, r) \qquad (6.28)\\
=&\alpha_2(\rho, g; f; n_1, n_2, r)\\
=& \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
=& \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2}} U(l_1)U(l_2)\{a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}) - a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2}) +\\
&\qquad a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2}) - a_f(r^{2l_1-2n_1})a_f(r^{2n_2-2l_2-2})\}\\
=& \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 - l_2 = n_2 - n_1}} U(l_1)U(l_2) + Y_1(\rho, g; f; n_1, n_2, r) - \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 + l_2 = n_2 - n_1 - 1}} U(l_1)U(l_2) + Y_2(\rho, g; f; n_1, n_2, r)\\
&+ \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 - l_2 = n_1 + n_2}} U(l_1)U(l_2) + Y_3(\rho, g; f; n_1, n_2, r) - \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 + l_2 = n_1 + n_2 - 1}} U(l_1)U(l_2) + Y_4(\rho, g; f; n_1, n_2, r)\\
=& \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 - l_2 = n_2 - n_1}} U(l_1)U(l_2) - \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 + l_2 = n_2 - n_1 - 1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \geq 0 \\ l_1 \leq n_1, l_2 < n_2 \\ l_1 - l_2 = n_1 + n_2}} U(l_1)U(l_2)
\end{aligned}$$

$$-\sum_{\substack{l_1,l_2\geq 0\\ l_1\geq n_1,l_2<n_2\\ l_1+l_2=n_1+n_2-1}} U(l_1)U(l_2) + (Y_1+Y_2+Y_3+Y_4)(\rho,g;f;n_1,n_2,r).$$

□

Lemma 6.3.4. *Let $\alpha_3(n_1,n_2,r)$ be the part of $k(r,n_1,n_2)$, where the summation is taken over $l_1,l_2\geq 0$, with $l_1<n_1, l_2\geq n_2$, and the indexes l_1 and l_2 run up to $\lfloor L\rfloor$. Then, for any prime r and positive integers n_1,n_2 with $n_2\geq n_1$,*

$$\begin{aligned}&\alpha_3(n_1,n_2,r)\\ =&\sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1-l_2=n_2-n_1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1-l_2=-n_1-n_2}} U(l_1)U(l_2) - \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1+l_2=n_1-n_2-1}} U(l_1)U(l_2)\\ &-\sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1+l_2=n_1+n_2-1}} U(l_1)U(l_2) + (Z_1+Z_2+Z_3+Z_4)(\rho,g;f;n_1,n_2,r),\end{aligned}$$

where $Z_i(\rho,g;f;n_1,n_2,r)$ $(i=1,2,3,4)$ are defined in equations (6.29), (6.30), (6.31), *and* (6.32) *respective]y.*

Proof. Using Corollary 3.3.5, for any prime r and integers l_1,l_2 with $l_1<n_1,l_2\geq n_2$, we have

$$\begin{aligned}&a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_1})-a_f(r^{2n_1-2}))(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\ =&a_f(r^{2l_1})(a_f(r^{2n_1})-a_f(r^{2n_1-2}))a_f(r^{2l_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\ =&(a_f(r^{2l_1+2n_1})-a_f(r^{2n_1-2l_1-2}))(a_f(r^{2l_2-2n_2})+a_f(r^{2l_2+2n_2}))\\ =&a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})+a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2})\\ &-a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2})-a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2-2n_2}).\end{aligned}$$

We note that

1) The product $a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})$ gives 1, only if $l_1+n_1=l_2+n_2$, i.e., $l_1-l_2=n_2-n_1$,

2) The product $a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2})$ gives 1, only if $l_1+n_1=l_2-n_2$, i.e., $l_1-l_2=-n_1-n_2$,

3) The product $a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2})$ gives 1, only if $n_1-l_1-1=l_2+n_2$, i.e., $l_1+l_2=n_1-n_2-1$, and

4) The product $a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2-2n_2})$ gives 1, only if $n_1-l_1-1=l_2-n_2$, i.e., $l_1+l_2=n_1+n_2-1$.

For any prime r and positive integers n_1,n_2 with $n_2\geq n_1$, we define

$$Z_1(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1-l_2\not\equiv n_2-n_1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}), \tag{6.29}$$

$$Z_2(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1-l_2\not\equiv -n_1-n_2}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2}), \tag{6.30}$$

$$Z_3(\rho,g;f;n_1,n_2,r) := -\sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2\geq n_2\\ l_1+l_2\not\equiv n_1-n_2-1}} U(l_1)U(l_2)a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2}), \tag{6.31}$$

and

$$Z_4(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\ \ 0 \\ l_1<n_1,l_2\ \ n_2 \\ l_1+l_2 \neq n_1+n_2\ \ 1}} U(l_1)U(l_2)a_f(r^{2n_1\ \ 2l_1\ \ 2})a_f(r^{2l_2\ \ 2n_2}). \tag{6.32}$$

We will write $Z_i(n_1,n_2,r)$ for $Z_i(\rho,g;f;n_1,n_2,r)$, $i = 1,2,3,4$.

For any prime r and positive integers n_1,n_2 with $n_2\ \ n_1$,

$$\begin{aligned}
&\alpha_3(n_1,n_2,r) \qquad (6.33)\\
=&\alpha_3(\rho,g;f;n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_1})\ \ a_f(p^{2n_1\ \ 2}))(a_f(p^{2n_2})\ \ a_f(p^{2n_2\ \ 2}))\\
=&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2}} U(l_1)U(l_2)\{a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}) + a_f(r^{2l_1+2n_1})a_f(r^{2l_2\ \ 2n_2})\\
&\qquad a_f(r^{2n_1\ \ 2l_1\ \ 2})a_f(r^{2l_2+2n_2})\ \ a_f(r^{2n_1\ \ 2l_1\ \ 2})a_f(r^{2l_2\ \ 2n_2})\}\\
=&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1\ \ l_2=n_2\ \ n_1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1\ \ l_2=\ \ n_1\ \ n_2}} U(l_1)U(l_2)\\
&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1+l_2=n_1\ \ n_2\ \ 1}} U(l_1)U(l_2)\ \ \sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1+l_2=n_1+n_2\ \ 1}} U(l_1)U(l_2)\\
&+Z_1(\rho,g;f;n_1,n_2,r) + Z_2(\rho,g;f;n_1,n_2,r) + Z_3(\rho,g;f;n_1,n_2,r) + Z_4(\rho,g;f;n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1\ \ l_2=n_2\ \ n_1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1\ \ l_2=\ \ n_1\ \ n_2}} U(l_1)U(l_2)\ \ \sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1+l_2=n_1\ \ n_2\ \ 1}} U(l_1)U(l_2)\\
&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2\ \ n_2\\ l_1+l_2=n_1+n_2\ \ 1}} U(l_1)U(l_2) + (Z_1+Z_2+Z_3+Z_4)(\rho,g;f;n_1,n_2,r).
\end{aligned}$$

□

Lemma 6.3.5. *Let $\alpha_3(n_1,n_2,r)$ be the part of $k(r,n_1,n_2)$, where the summation is taken over $l_1,l_2\ \ 0$, with $l_1<n_1, l_2<n_2$, and the indexes l_1 and l_2 run up to bLc. Then, for any prime r and positive integers n_1,n_2 with $n_2\ \ n_1$,*

$$\begin{aligned}
&\alpha_4(n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2<n_2\\ l_1\ \ l_2=n_2\ \ n_1}} U(l_1)U(l_2)\ \ \sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_2\ \ n_1\ \ 1}} U(l_1)U(l_2)\ \ \sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_1\ \ n_2\ \ 1}} U(l_1)U(l_2)\\
+&\sum_{\substack{l_1,l_2\ \ 0\\ l_1<n_1,l_2<n_2\\ l_1\ \ l_2=n_1\ \ n_2}} U(l_1)U(l_2) + (W_1+W_2+W_3+W_4)(\rho,g;f;n_1,n_2,r),
\end{aligned}$$

where $W_i(\rho,g;f;n_1,n_2,r)$ ($i = 1,2,3,4$) are defined in equations (6.34), (6.35), (6.36), and (6.37) respective]y.

Proof. Using Corollary 3.3.5, for any prime r and integers l_1,l_2 with $l_1<n_1,l_2<n_2$, we have

$$\begin{aligned}
&a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_1})\ \ a_f(r^{2n_1\ \ 2}))(a_f(r^{2n_2})\ \ a_f(r^{2n_2\ \ 2}))\\
=&a_f(r^{2l_1})(a_f(r^{2n_1})\ \ a_f(r^{2n_1\ \ 2}))a_f(r^{2l_2})(a_f(r^{2n_2})\ \ a_f(r^{2n_2\ \ 2}))
\end{aligned}$$

$$
\begin{aligned}
&=(a_f(r^{2l_1+2n_1}) - a_f(r^{2n_1-2l_1-2}))(a_f(r^{2l_2+2n_2}) - a_f(r^{2n_2-2l_2-2}))\\
&=a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}) - a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2})\\
&\quad - a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2}) + a_f(r^{2n_1-2l_1-2})a_f(r^{2n_2-2l_2-2}).
\end{aligned}
$$

We note that

1) The product $a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2})$ gives 1, only if $l_1+n_1 = l_2+n_2$, i.e., $l_1 - l_2 = n_2 - n_1$,

2) The product $a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2})$ gives 1, only if $l_1 + n_1 = n_2 - l_2 - 1$, i.e., $l_1 + l_2 = n_2 - n_1 - 1$,

3) The product $a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2})$ gives 1, only if $n_1 - l_1 - 1 = l_2 + n_2$, i.e., $l_1 + l_2 = n_1 - n_2 - 1$, and

4) The product $a_f(r^{2n_1-2l_1-2})a_f(r^{2n_2-2l_2-2})$ gives 1, only if $n_1 - l_1 - 1 = n_2 - l_2 - 1$, i.e., $l_1 + l_2 = n_1 - n_2$.

For any prime r and positive integers n_1, n_2 with $n_2 \geq n_1$, we define

$$W_1(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1-l_2\neq n_2-n_1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}), \tag{6.34}$$

$$W_2(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2\neq n_2-n_1-1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2}), \tag{6.35}$$

$$W_3(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2\neq n_1-n_2-1}} U(l_1)U(l_2)a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2}), \tag{6.36}$$

$$W_4(\rho,g;f;n_1,n_2,r) := \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1-l_2\neq n_1-n_2}} U(l_1)U(l_2)a_f(r^{2n_1-2l_1-2})a_f(r^{2n_2-2l_2-2}). \tag{6.37}$$

We will write $W_i(n_1,n_2,r)$ for $W_i(\rho,g;f;n_1,n_2,r)$, $i = 1,2,3,4$.

For any prime r and positive integers n_1, n_2 with $n_2 \geq n_1$,

$$
\begin{aligned}
&\alpha_4(n_1,n_2,r) \qquad (6.38)\\
=&\alpha_4(\rho,g;f;n_1,n_2,r)\\
=&\sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
=&\sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2}} U(l_1)U(l_2)\{a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}) - a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2})\\
&\qquad - a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2}) + a_f(r^{2n_1-2l_1-2})a_f(r^{2n_2-2l_2-2})\}\\
=&\sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1-l_2=n_2-n_1}} U(l_1)U(l_2) - \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_2-n_1-1}} U(l_1)U(l_2)\\
&- \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_1-n_2-1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,l_2<n_2\\ l_1-l_2=n_1-n_2}} U(l_1)U(l_2)\\
&+W_1(\rho,g;f;n_1,n_2,r) + W_2(\rho,g;f;n_1,n_2,r) + W_3(\rho,g;f;n_1,n_2,r) + W_4(\rho,g;f;n_1,n_2,r)
\end{aligned}
$$

$$= \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1 \quad l_2=n_2 \quad n_1}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_2 \quad n_1 \quad 1}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_1 \quad n_2 \quad 1}} U(l_1)U(l_2)$$

$$+ \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1 \quad l_2=n_1 \quad n_2}} U(l_1)U(l_2) + (W_1+W_2+W_3+W_4)(\rho,g;f;n_1,n_2,r).$$

□

Lemma 6.3.6. *Let ρ, f, g be as defined earlier. Then for any prime r and integers $n_1, n_2 \quad 1$ with $n_2 \quad n_1$, we have*

$$\sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2 \quad n_2\\ l_1 \quad l_2=n_2 \quad n_1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2 \quad n_2\\ l_1 \quad l_2= \quad n_1 \quad n_2}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2 \quad n_2\\ l_1 \quad l_2=n_1+n_2}} U(l_1)U(l_2)$$

$$+ \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2 \quad n_2\\ l_1 \quad l_2=n_1 \quad n_2}} U(l_1)U(l_2)$$

$$+ \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2<n_2\\ l_1 \quad l_2=n_2 \quad n_1}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2<n_2\\ l_1+l_2=n_2 \quad n_1 \quad 1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2<n_2\\ l_1 \quad l_2=n_1+n_2}} U(l_1)U(l_2)$$

$$\sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad n_1,l_2<n_2\\ l_1+l_2=n_1+n_2 \quad 1}} U(l_1)U(l_2))$$

$$+ \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2 \quad n_2\\ l_1 \quad l_2=n_2 \quad n_1}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2 \quad n_2\\ l_1 \quad l_2= \quad n_1 \quad n_2}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2 \quad n_2\\ l_1+l_2=n_1 \quad n_2 \quad 1}} U(l_1)U(l_2)$$

$$\sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2 \quad n_2\\ l_1+l_2=n_1+n_2 \quad 1}} U(l_1)U(l_2)$$

$$+ \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1 \quad l_2=n_2 \quad n_1}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_2 \quad n_1 \quad 1}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1+l_2=n_1 \quad n_2 \quad 1}} U(l_1)U(l_2)$$

$$+ \sum_{\substack{l_1,l_2 \quad 0\\ l_1<n_1,l_2<n_2\\ l_1 \quad l_2=n_1 \quad n_2}} U(l_1)U(l_2)$$

$$=2\sum_{l \quad 0} U(l)U(l+n_2 \quad n_1) + 2\sum_{l \quad 0} U(l)U(l+n_2+n_1)$$

$$\sum_{l=0}^{n_1+n_2 \quad 1} U(l)U(n_1+n_2 \quad 1 \quad l) \qquad \sum_{l=0}^{n_2 \quad n_1 \quad 1} U(l)U(n_2 \quad n_1 \quad 1 \quad l).$$

Proof. We denote the terms (with their corresponding signs) in the lemma by $J_i(\rho, n_1, n_2)$, $i = 1, 2, \quad , 16$, so that the required sum $= \sum_{i=1}^{16} J_i(\rho, n_1, n_2)$.

We note that

$$J_1(\rho,n_1,n_2) + J_5(\rho,n_1,n_2) + J_9(\rho,n_1,n_2) + J_{13}(\rho,n_1,n_2) \tag{6.39}$$

$$= \sum_{\substack{l_1,l_2 \quad 0\\ l_1 \quad l_2=n_2 \quad n_1}} U(l_1)U(l_2)$$

$$= \sum_{l \quad 0} U(l)U(l + n_2 \quad n_1).$$

$$\begin{aligned} & J_2(\rho,n_1,n_2) + J_{10}(\rho,n_1,n_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1, l_2 \quad n_2 \\ l_1 \quad l_2 = \quad n_1 \quad n_2}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 < n_1, l_2 \quad n_2 \\ l_1 \quad l_2 = \quad n_1 \quad n_2}} U(l_1)U(l_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_2 \quad n_2 \\ l_1 \quad l_2 = \quad n_1 \quad n_2}} U(l_1)U(l_2) \\ = & \sum_{l \quad 0} U(l)U(l + n_2 + n_1). \end{aligned} \tag{6.40}$$

$$\begin{aligned} & J_3(\rho,n_1,n_2) + J_7(\rho,n_1,n_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1, l_2 \quad n_2 \\ l_1 \quad l_2 = n_1 + n_2}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1, l_2 < n_2 \\ l_1 \quad l_2 = n_1 + n_2}} U(l_1)U(l_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1 \\ l_1 \quad l_2 = n_1 + n_2}} U(l_1)U(l_2) \\ = & \sum_{l \quad 0} U(l)U(l + n_2 + n_1). \end{aligned} \tag{6.41}$$

$$\begin{aligned} & J_4(\rho,n_1,n_2) + J_{16}(\rho,n_1,n_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1, l_2 \quad n_2 \\ l_1 \quad l_2 = n_1 \quad n_2}} U(l_1)U(l_2) + \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 < n_1, l_2 < n_2 \\ l_1 \quad l_2 = n_1 \quad n_2}} U(l_1)U(l_2) \\ = & \sum_{\substack{l \quad 0 \\ l \quad n_1}} U(l)U(l + n_2 \quad n_1) + \sum_{\substack{l \quad 0 \\ l < n_1}} U(l)U(l + n_2 \quad n_1) \\ = & \sum_{l \quad 0} U(l_1)U(l + n_2 \quad n_1). \end{aligned} \tag{6.42}$$

$$\begin{aligned} & J_6(\rho,n_1,n_2) + J_{14}(\rho,n_1,n_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1, l_2 < n_2 \\ l_1 + l_2 = n_2 \quad n_1 \quad 1}} U(l_1)U(l_2) \qquad \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 < n_1, l_2 < n_2 \\ l_1 + l_2 = n_2 \quad n_1 \quad 1}} U(l_1)U(l_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_2 < n_2 \\ l_1 + l_2 = n_2 \quad n_1 \quad 1}} U(l_1)U(l_2) \\ = & \sum_{l=0}^{n_2 \quad n_1 \quad 1} U(l)U(n_2 \quad n_1 \quad 1 \quad l). \end{aligned} \tag{6.43}$$

$$\begin{aligned} & J_8(\rho,n_1,n_2) + J_{12}(\rho,n_1,n_2) \\ = & \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad n_1, l_2 < n_2 \\ l_1 + l_2 = n_1 + n_2 \quad 1}} U(l_1)U(l_2)) \qquad \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 < n_1, l_2 \quad n_2 \\ l_1 + l_2 = n_1 + n_2 \quad 1}} U(l_1)U(l_2) \end{aligned} \tag{6.44}$$

$$
\begin{aligned}
&= -\sum_{l=0}^{n_2-1} U(l)U(n_1+n_2-1-l) - \sum_{l=0}^{n_1-1} U(l)U(n_1+n_2-1-l) \\
&= -\sum_{l=n_1}^{n_1+n_2-1} U(l)U(n_1+n_2-1-l) - \sum_{l=0}^{n_1-1} U(l)U(n_1+n_2-1-l) \\
&= -\sum_{l=0}^{n_1+n_2-1} U(l)U(n_1+n_2-1-l).
\end{aligned}
$$

We also note that the indexing sets in $J_{11}(\rho, n_1, n_2)$ and $J_{15}(\rho, n_1, n_2)$ are empty sets, because for $n_2 > n_1$, we have $0 \leq l_2 + l_2 = n_1 - n_2 - 1 < -1$, which is a contradiction. Hence, $J_{11}(\rho, n_1, n_2) = 0 = J_{15}(\rho, n_1, n_2)$.

Therefore, adding equations (6.39), (6.40), (6.41), (6.42), (6.43), and (6.44), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{16} J_i(\rho, n_1, n_2) \\
=&2\sum_{l\geq 0} U(l)U(l+n_2-n_1) + 2\sum_{l\geq 0} U(l)U(l+n_2+n_1) \\
&-\sum_{l=0}^{n_1+n_2-1} U(l)U(n_1+n_2-1-l) - \sum_{l=0}^{n_2-n_1-1} U(l)U(n_2-n_1-1-l).
\end{aligned}
$$

□

Lemma 6.3.7. *Let ρ, f, g be as defined earlier. X_i ($i = 1,2,3,4$) be as defined in equations* (6.19), (6.20), (6.21), *and* (6.22) *respectively. Let Y_i ($i = 1,2,3,4$) be as defined in equations* (6.24), (6.25), (6.26), *and* (6.27) *respectively. Let Z_i ($i = 1,2,3,4$) be as defined in equations* (6.29), (6.30), (6.31), *and* (6.32) *respectively. Let W_i ($i = 1,2,3,4$) be as defined in equations* (6.34), (6.35), (6.36), *and* (6.37) *respectively. Then for any prime r and integers $n_1, n_2 \geq 1$ with $n_2 \geq n_1$, we have*

$$
\sum_{i=1}^{4} (X_i + Y_i + Z_i + W_i)(\rho, g; f; n_1, n_2, r) = \sum_{i=1}^{9} V_i(\rho, g; f; n_1, n_2, r),
$$

where,

$$
V_1(\rho, g; f; n_1, n_2, r) = \sum_{\substack{l_1, l_2 \geq 0 \\ l_1 - l_2 \neq n_2 - n_1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2+2n_2}),
$$

$$
V_2(\rho, g; f; n_1, n_2, r) = \sum_{\substack{l_1, l_2 \geq 0 \\ l_2 \geq n_2 \\ l_1 - l_2 \neq -n_1 - n_2}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2l_2-2n_2}),
$$

$$
V_3(\rho, g; f; n_1, n_2, r) = \sum_{\substack{l_1, l_2 \geq 0 \\ l_1 \geq n_1 \\ l_1 - l_2 \neq n_1+n_2}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2l_2+2n_2}),
$$

$$
V_4(\rho, g; f; n_1, n_2, r) = \sum_{\substack{l_1, l_2 \geq 0 \\ l_1 \geq n_1, l_2 \geq n_2 \\ l_1 - l_2 \neq n_1 - n_2}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2l_2-2n_2}),
$$

$$
V_5(\rho, g; f; n_1, n_2, r) = \sum_{\substack{l_1, l_2 \geq 0 \\ l_2 < n_2 \\ l_1 + l_2 \neq n_2 - n_1 - 1}} U(l_1)U(l_2)a_f(r^{2l_1+2n_1})a_f(r^{2n_2-2l_2-2}),
$$

$$V_6(\rho,g;f;n_1,n_2,r) = \sum_{\substack{l_1,l_2\geq 0\\ l_1\geq n_1,\, l_2<n_2\\ l_1+l_2\not\equiv n_1+n_2-1}} U(l_1)U(l_2)a_f(r^{2l_1-2n_1})a_f(r^{2n_2-2l_2-2}),$$

$$V_7(\rho,g;f;n_1,n_2,r) = \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1\\ l_1+l_2\not\equiv n_1-n_2-1}} U(l_1)U(l_2)a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2+2n_2}),$$

$$V_8(\rho,g;f;n_1,n_2,r) = \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,\, l_2\geq n_2\\ l_1+l_2\not\equiv n_1+n_2-1}} U(l_1)U(l_2)a_f(r^{2n_1-2l_1-2})a_f(r^{2l_2-2n_2}),$$

$$V_9(\rho,g;f;n_1,n_2,r) = \sum_{\substack{l_1,l_2\geq 0\\ l_1<n_1,\, l_2<n_2\\ l_1-l_2\not\equiv n_1-n_2}} U(l_1)U(l_2)a_f(r^{2n_1-2l_1-2})a_f(r^{2n_2-2l_2-2}).$$

Proof. The proof follows from the following facts:

$$V_1(\rho,g;f;n_1,n_2,r) = (X_1+Y_1+Z_1+W_1)(\rho,g;f;n_1,n_2,r),$$
$$V_2(\rho,g;f;n_1,n_2,r) = (X_2+Z_2)(\rho,g;f;n_1,n_2,r),$$
$$V_3(\rho,g;f;n_1,n_2,r) = (X_3+Y_3)(\rho,g;f;n_1,n_2,r),$$
$$V_4(\rho,g;f;n_1,n_2,r) = X_4(\rho,g;f;n_1,n_2,r),$$
$$V_5(\rho,g;f;n_1,n_2,r) = (Y_2+W_2)(\rho,g;f;n_1,n_2,r),$$
$$V_6(\rho,g;f;n_1,n_2,r) = Y_4(\rho,g;f;n_1,n_2,r),$$
$$V_7(\rho,g;f;n_1,n_2,r) = (Z_3+W_3)(\rho,g;f;n_1,n_2,r),$$
$$V_8(\rho,g;f;n_1,n_2,r) = Z_4(\rho,g;f;n_1,n_2,r),$$
$$V_9(\rho,g;f;n_1,n_2,r) = W_4(\rho,g;f;n_1,n_2,r).$$

□

Proposition 6.3.8. *Let* ρ, f, g *be as defined earlier and* $V_i(\rho,g;f;n_1,n_2,r)$ $(i=1,\dots,9)$ *be as defined in Lemma 6.3.7. Then for any prime* r *and integers* $n_1,n_2\geq 1$ *with* $n_2>n_1$*, we have*

$$\begin{aligned}
&k(r,n_1,n_2)\\
=&2\sum_{l\geq 0}U(l)U(l+n_2-n_1)+2\sum_{l\geq 0}U(l)U(l+n_2+n_1)\\
&-\sum_{l=0}^{n_1+n_2-1}U(l)U(n_1+n_2-1-l)-\sum_{l=0}^{n_2-n_1-1}U(l)U(n_2-n_1-1-l)+\sum_{i=1}^{9}V_i(\rho,g;f;n_1,n_2,r)\\
=&\sum_{i=1}^{4}E_i(\rho,g;f;n_1,n_2)+\sum_{i=1}^{9}V_i(\rho,g;f;n_1,n_2,r)\\
=&\sum_{i=1}^{4}E_i(n_1,n_2)+\sum_{i=1}^{9}V_i(n_1,n_2,r),
\end{aligned}$$

where

$$E_1(\rho,g;f;n_1,n_2) := 2\sum_{l\geq 0}U(l)U(l+n_2-n_1),$$

$$E_2(\rho,g;f;n_1,n_2) := 2\sum_{l\geq 0}U(l)U(l+n_2+n_1),$$

$$E_3(\rho,g;f;n_1,n_2) := -\sum_{l=0}^{n_1+n_2-1}U(l)U(n_1+n_2-1-l),$$

$$E_4(\rho, g; f; n_1, n_2) := \sum_{l=0}^{n_2 \quad n_1 \quad 1} U(l)U(n_2 \quad n_1 \quad 1 \quad l).$$

We write $V_i(n_1, n_2, r)$ for $V_i(\rho, g; f; n_1, n_2, r)$ $(i = 1, \quad , 9)$. We also write $E_j(n_1, n_2)$ for $E_j(\rho, g; f; n_1, n_2)$ $(j = 1, \quad , 4)$.

Proof. Adding equations (6.23),(6.28),(6.33) and (6.38) and using Lemma 6.3.6 and Lemma 6.3.7, we obtain that for any prime r and integers $n_1, n_2 \quad 1$ with $n_2 > n_1$,

$$\begin{aligned}
&k(r, n_1, n_2) \\
&= \sum_{i=1}^{4} \alpha_i(n_1, n_2, r) \\
&= \sum_{i=1}^{16} J_i(\rho, n_1, n_2) + \sum_{i=1}^{4} (X_i + Y_i + Z_i + W_i)(\rho, g; f; n_1, n_2, r) \\
&= 2 \sum_{l \quad 0} U(l)U(l + n_2 \quad n_1) + 2 \sum_{l \quad 0} U(l)U(l + n_2 + n_1) \\
&\quad \sum_{l=0}^{n_1+n_2 \quad 1} U(l)U(n_1 + n_2 \quad 1 \quad l) \quad \sum_{l=0}^{n_2 \quad n_1 \quad 1} U(l)U(n_2 \quad n_1 \quad 1 \quad l) + \sum_{i=1}^{9} V_i(\rho, g; f; n_1, n_2, r).
\end{aligned}$$

□

Corollary 6.3.9. *Let $\rho, f, g, V_i(\rho, g; f; n_1, n_2, r)$ $(i = 1, \quad , 9)$ be as defined in Lemma 6.3.7. Then for any prime r and integers $n_1, n_2 \quad 1$ with $n_2 = n_1 = n$, we have*

$$\begin{aligned}
&k(r, n_1, n_2) \\
&= 2 \sum_{l \quad 0} U(l)U(l + n_2 \quad n_1) + 2 \sum_{l \quad 0} U(l)U(l + n_2 + n_1) \\
&\quad \sum_{l=0}^{n_1+n_2 \quad 1} U(l)U(n_1 + n_2 \quad 1 \quad l) + \sum_{i=1}^{9} V_i(\rho, g; f; n_1, n_2, r) \\
&= \sum_{i=1}^{3} E_i(\rho, g; f; n_1, n_2) + \sum_{i=1}^{9} V_i(\rho, g; f; n_1, n_2, r) \\
&= \sum_{i=1}^{3} E_i(n_1, n_2) + \sum_{i=1}^{9} V_i(n_1, n_2, r),
\end{aligned}$$

where $E_i(\rho, g; f; n_1, n_2)$ is as defined in Proposition 6.3.8.

Proof. The proof is the same as the proof in Proposition 6.3.8, the only exception being the term $\sum_{l=0}^{n_2 \quad n_1 \quad 1} U(l)U(n_2 \quad n_1 \quad 1 \quad l)$ does not appear here, since $n_1 = n_2$. □

With the estimates of $k(r, n_1, n_2)$ in our hand, we are now ready to find $\langle \mathsf{C}(\rho, g; f)(x) \rangle$, where $\mathsf{C}(\rho, g; f)(x)$ is defined in equation (6.17).

Lemma 6.3.10. *Let ρ, f, g be as defined earlier. Let $E_i(\rho, g; f; n_1, n_2)$ $(i = 1, \quad , 4)$ be as defined in Proposition 6.3.8. Let $V_j(\rho, g; f; n_1, n_2, r)$, $(i = 1, \quad , 9)$, and $\mathsf{C}(\rho, g; f)(x)$ be as defined in Lemma 6.3.7 and equation* (6.17) *respectively. Then,*

$$\mathsf{C}(\rho, g; f)(x) = \sum_{i=1}^{3} \mathsf{C}_i(\rho, g; f)(x),$$

where

$$C_1(\rho,g;f)(x) := \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}E_i(n_1,n_2)\right)^2, \tag{6.45}$$

$$C_2(\rho,g;f)(x) := \frac{16}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,p)\right), \tag{6.46}$$

$$C_3(\rho,g;f)(x) := \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{9}\sum_{j=1}^{9}V_i(n_1,n_2,p)V_j(n_1,n_2,q)\right). \tag{6.47}$$

Proof.

$$\begin{aligned}
&C(\rho,g;f)(x)\\
&=\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2)\\
&=\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}E_i(n_1,n_2)+\sum_{i=1}^{9}V_i(n_1,n_2,p)\right)\\
&\qquad\qquad\left(\sum_{i=1}^{4}E_i(n_1,n_2)+\sum_{i=1}^{9}V_i(n_1,n_2,q)\right)\\
&=\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}E_i(n_1,n_2)\right)^2\\
&+\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,p)\right)\\
&+\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,q)\right)\\
&+\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{9}\sum_{j=1}^{9}V_i(n_1,n_2,p)V_j(n_1,n_2,q)\right)\\
&=\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}E_i(n_1,n_2)\right)^2\\
&+\frac{16}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,p)\right)\\
&+\frac{8}{32\pi_N(x)^4L^2}\sum_{p,q}^{0}\sum_{\substack{x\; n_1,n_2\; 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{9}\sum_{j=1}^{9}V_i(n_1,n_2,p)V_j(n_1,n_2,q)\right)
\end{aligned}$$

$$=C_1(\rho, g; f)(x) + C_2(\rho, g; f)(x) + C_3(\rho, g; f)(x).$$

□

We now find $\langle C_1(\rho, g; f)(x)\rangle$ in Lemma 6.3.12 using the next lemma.

Lemma 6.3.11. *Let $E_i(n_1, n_2)$ $(i = 1, 2, 3, 4)$ be as mentioned in Proposition 6.3.8. Then, $E_i(n_1, n_2) = 0$, $(i = 1, 2, 3, 4)$ for $n_2 \; n_1 > 2bLc + 1$.*

Proof. We note that if $n_2 \; n_1 > 2bLc + 1$, then $n_2 + n_1 > 2bLc + 1$ and hence,

1) $l + n_2 \; n_1 > 2bLc + 1$,

2) $l + n_2 + n_1 > 2bLc + 1$,

3) $n_2 \; n_1 \; 1 \; l \; n_2 \; n_1 \; 1 \; bLc > 2bLc \; bLc = bLc$, i.e., $n_2 \; n_1 \; 1 \; l > bLc$, and

4) $n_2 + n_1 \; 1 \; l \; n_2 + n_1 \; 1 \; bLc > 2bLc \; bLc = bLc$, i.e., $n_2 + n_1 \; 1 \; l > bLc$.

Since $l^0 > bLc$, i.e., $l^0 \; bLc + 1$, implies $U(l^0) = 0$, we have $E_i(n_1, n_2) = 0$, $(i = 1, 2, 3, 4)$ for $n_2 \; n_1 > 2bLc + 1$. □

Lemma 6.3.12. *Let ρ, f, g be as defined earlier and $C_1(\rho, g; f)(x)$ be as defined in equation (6.45). Then, $C_1(\rho, g; f)(x) = O\left(\frac{L}{\pi_N(x)}\right)$ and hence,*

$$\frac{1}{jF_{N,k}j} \sum_{f \, 2 \, F_{N,k}} C_1(\rho, g; f)(x) \qquad \frac{L}{\pi_N(x)}.$$

Proof. We want to find an estimate for

$$C_1(\rho, g; f)(x) = \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}^{0} \sum_{\substack{x \; n_1, n_2 \; 1 \\ n_2 > n_1}} G(n_1)G(n_2) \left(\sum_{i=1}^{4} E_i(n_1, n_2)\right)^2,$$

where $E_i(n_1, n_2)$ $(i = 1, 2, 3, 4)$ are mentioned in Proposition 6.3.8.

We know, for $l \; 0$, $jU(l)j = j\widehat{\rho}\left(\frac{l}{L}\right) 2\cos 2\pi l\psi \; \widehat{\rho}\left(\frac{l+1}{L}\right) 2\cos 2\pi(l+1)\psi j \; 2k_1 + 2k_1 = 4k_1$, for some $k_1 > 0$ and $U(l) = 0$ for $l > bLc$. Therefore, $jE_i(n_1, n_2)j \; 32k_1^2(L + 1) \; 64k_1^2 L$.

Also, for $n \; 0$, $jG(n)j \; k_2$.

Hence, using Lemma 6.3.11,

$$C_1(\rho, g; f)(x)$$

$$= \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}^{0} \sum_{\substack{x \; n_1, n_2 \; 1 \\ n_2 > n_1}} G(n_1)G(n_2) \left(\sum_{i=1}^{4} E_i(n_1, n_2)\right)^2$$

$$= \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}^{0} \sum_{\substack{x \; n_1, n_2 \; 1, n_2 > n_1 \\ n_2 \; n_1 \; 2bLc + 1}} G(n_1)G(n_2) \left(\sum_{i=1}^{4} E_i(n_1, n_2)\right)^2$$

$$\frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}^{0} \sum_{\substack{x \; n_1, n_2 \; 1 \\ 0 < n_2 \; n_1 \; 2bLc + 1}} \left(\sum_{i=1}^{4} 64k_1^2 L\right)^2$$

$$\frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}^{0} \sum_{x \; n_1 = 1}^{\pi_N(x)} \sum_{n_2 = n_1 + 1}^{2bLc + n_1 + 1} \left(\sum_{i=1}^{4} 64k_1^2 L\right)^2$$

$$\frac{L^3\pi_N(x)^3}{\pi_N(x)^4L^2} = \frac{L}{\pi_N(x)}.$$

Therefore,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} C_1(\rho,g;f)(x) \ll \frac{L}{\pi_N(x)}. \tag{6.48}$$

□

We now find $\langle C_2(\rho,g;f)(x)\rangle$ in Lemma 6.3.14 using the next lemma.

Lemma 6.3.13. *Let $n_1, n_2 \geq 1$ be integers with $n_2 \geq n_1$ and p be prime with $p \leq x$. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} V_1(n_1,n_2,p) \ll \frac{1}{p}L + (L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4n_2c'}}{kN},$$

where $V_1(n_1,n_2,p)$ is defined in Lemma 6.3.7.

Proof. For prime p with $p \leq x$, we have

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} V_1(n_1,n_2,p)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2-n_1}} U(l_1)U(l_2)a_f(p^{2l_1+2n_1})a_f(p^{2l_2+2n_2})\\
&= \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2-n_1}} U(l_1)U(l_2)\sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(p^{2t})\right)\\
&= \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2-n_1}} U(l_1)U(l_2)\sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\left\{\frac{1}{p^t}+O\left(\frac{8^{\nu(N)}p^{2tc'}}{kN}\right)\right\}\\
&\ll \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2-n_1}}\left(\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|}}+\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|+1}}+\cdots\right)\\
&+\sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2-n_1}}\sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\frac{8^{\nu(N)}p^{2(l_1+l_2+n_1+n_2)c'}}{kN}\\
&\ll \left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right)L+\sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2-n_1}}(l_1+l_2+n_1+n_2)\frac{8^{\nu(N)}p^{4Lc'+4n_2c'}}{kN}\\
&\ll \frac{1}{p}L+(L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4n_2c'}}{kN}.
\end{aligned}$$

□

Lemma 6.3.14. *Let ρ, f, g be as defined earlier and $C_2(\rho,g;f)(x)$ be as defined in equation* (6.46)*. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} C_2(\rho,g;f)(x) \ll \frac{L(\log\log x)}{\pi_N(x)^2}+\frac{L^2(L+\pi_N(x))}{\pi_N(x)}\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}.$$

Proof. We want to find an estimate for the average of

$$C_2(\rho,g;f)(x) = \frac{16}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,p)\right),$$

where $E_i(n_1,n_2)$ $(i=1,2,3,4)$ are mentioned in Proposition 6.3.8.

We know, for $l,n\ge 0$, $|U(l)| = |\widehat{\rho}\left(\frac{l}{L}\right)2\cos 2\pi l\psi - \widehat{\rho}\left(\frac{l+1}{L}\right)2\cos 2\pi(l+1)\psi| \le 2k_1+2k_1 = 4k_1$, $|G(n)|\le k_2$ for some $k_1,k_2>0$ and $U(l)=0$ for $l>\lfloor L\rfloor$.

We note that for $i=1,2,3,4$,

$$|E_i(n_1,n_2)| \le 32k_1^2(L+1) \le 64k_1^2L.$$

Hence, using Lemma 6.3.11 and Lemma 6.3.13, we obtain for $i=1,2,3,4$,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{16}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)E_i(n_1,n_2)V_1(n_1,n_2,p)$$

$$= \frac{1}{2\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1, n_2>n_1\\ n_2-n_1\le 2\lfloor L\rfloor+1}} G(n_1)G(n_2)E_i(n_1,n_2)\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}V_1(n_1,n_2,p)\right)$$

$$\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ 0<n_2-n_1\le 2\lfloor L\rfloor+1}} L\left(\frac{1}{p}L + (L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4n_2c'}}{kN}\right)$$

$$\ll \frac{L}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{2\lfloor L\rfloor+n_1+1}\left(\frac{1}{p}L + (L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}\right)$$

$$\ll \frac{L}{\pi_N(x)^4L^2}L^2\pi_N(x)\sideset{}{'}\sum_{p,q\le x}\frac{1}{p}$$

$$+\frac{L}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{2\lfloor L\rfloor+n_1+1}(L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}$$

$$\ll \frac{L}{\pi_N(x)^3}(\log\log x)\pi_N(x) + \frac{L}{\pi_N(x)^4L^2}\pi_N(x)^3(L+\pi_N(x))L^3\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}$$

$$\ll \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{L^2(L+\pi_N(x))}{\pi_N(x)}\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}.$$

We note that other terms also give the same estimate.

Therefore,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}C_2(\rho,g;f)(x)$$

$$= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{16}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{4}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,p)\right) \tag{6.49}$$

$$\ll \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{L^2(L+\pi_N(x))}{\pi_N(x)}\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}.$$

□

We now find $\langle C_3(\rho,g;f)(x)\rangle$ in Lemma 6.3.17 using the next two lemmas, where

$$C_3(\rho,g;f)(x) = \frac{8}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\le x}\sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{9}\sum_{j=1}^{9}V_i(n_1,n_2,p)V_j(n_1,n_2,q)\right).$$

Lemma 6.3.15. *Let $n_1,n_2 \ge 1$ be integers and p and q be distinct primes with $p \neq q \le x$. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} V_1(n_1,n_2,p)V_1(n_1,n_2,q) \ll \frac{1}{pq}L^2 + (L+\pi_N(x))^2L^4\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN},$$

where $V_1(n_1,n_2,p)$ is defined in Lemma 6.3.7.

Proof.

For distinct primes p and q with $p \neq q \le x$, we have

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} V_1(n_1,n_2,p)V_1(n_1,n_2,q)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2-n_1}} U(l_1)U(l_2)a_f(p^{2l_1+2n_1})a_f(p^{2l_2+2n_2})\\
&\qquad\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2-n_1}} U(k_1)U(k_2)a_f(q^{2k_1+2n_1})a_f(q^{2k_2+2n_2})\\
&=\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2-n_1}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2-n_1}} U(l_1)U(l_2)U(k_1)U(k_2)\\
&\qquad\sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\ \sum_{t'=|(k_1-k_2)-(n_2-n_1)|}^{k_1+k_2+n_1+n_2}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(p^{2t}q^{2t'})\right)\\
&=\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2-n_1}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2-n_1}} U(l_1)U(l_2)U(k_1)U(k_2)\\
&\qquad\sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\ \sum_{t'=|(k_1-k_2)-(n_2-n_1)|}^{k_1+k_2+n_1+n_2}\left\{\frac{1}{p^tq^{t'}}+O\left(\frac{8^{\nu(N)}p^{2tc'}q^{2t'c'}}{kN}\right)\right\}\\
&\ll \sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2-n_1}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2-n_1}}\left(\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|}}+\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|+1}}+\cdots\right)\\
&\qquad\left(\frac{1}{q^{|(k_1-k_2)-(n_2-n_1)|}}+\frac{1}{q^{|(k_1-k_2)-(n_2-n_1)|+1}}+\cdots\right)\\
&+\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2-n_1}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2-n_1}}\ \sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\ \sum_{t'=|(k_1-k_2)-(n_2-n_1)|}^{k_1+k_2+n_1+n_2}\frac{8^{\nu(N)}(pq)^{2(2L+n_1+n_2)c'}}{kN}\\
&\ll \left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right)L\left(\frac{1}{q}+\frac{1}{q^2}+\cdots\right)L\\
&+\sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\neq n_2-n_1}}\sum_{\substack{k_1,k_2\ge 0\\ k_1-k_2\neq n_2-n_1}}(l_1+l_2+n_1+n_2)(k_1+k_2+n_1+n_2)\frac{8^{\nu(N)}(pq)^{4Lc'+4\pi_N(x)c'}}{kN}\\
&\ll \frac{1}{pq}L^2+(L+\pi_N(x))^2L^4\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}$$

□

Lemma 6.3.16. *Let $N \geq 1$ be a positive integer and k be even integer. Then*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q \quad x}{}' \sum_{\substack{n_1,n_2 \quad 1\\ n_2>n_1}} G(n_1)G(n_2)V_1(n_1,n_2,p)V_1(n_1,n_2,q)$$
$$\ll \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^4\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.$$

Proof. Using Lemma 6.3.15, we obtain

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q \quad x}{}' \sum_{\substack{n_1,n_2 \quad 1\\ n_2>n_1}} G(n_1)G(n_2)V_1(n_1,n_2,p)V_1(n_1,n_2,q)$$
$$= \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q \quad x}{}' \sum_{\substack{n_1,n_2 \quad 1\\ n_2>n_1}} G(n_1)G(n_2)\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} V_1(n_1,n_2,p)V_1(n_1,n_2,q)$$
$$\ll \frac{1}{\pi_N(x)^4L^2}\sum_{p,q \quad x}{}' \sum_{\substack{n_1,n_2 \quad 1\\ n_2>n_1}} \left(\frac{1}{pq}L^2 + (L+\pi_N(x))^2L^4\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\right)$$
$$\ll \frac{1}{\pi_N(x)^2L^2}\sum_{p,q \quad x}{}' \frac{1}{pq}L^2 + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}$$
$$\ll \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.$$

□

We note that the other terms also give the same estimate. Therefore, we have the following lemma.

Lemma 6.3.17. *Let ρ, f, g be as defined earlier and $C_3(\rho,g;f)(x)$ be as defined in equation* (6.47)*. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} C_3(\rho,g;f)(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}. \tag{6.50}$$

Proposition 6.3.18. *Let ρ, f, g be as defined earlier. Let $C(\rho,g;f)(x)$ be as defined in equation* (6.17)*. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} C(\rho,g;f)(x)$$
$$\ll \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.$$

Proof. Combining inequations (6.48), (6.49) and (6.50), and using Lemma 6.3.10, we obtain

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} C(\rho,g;f)(x) \tag{6.51}$$
$$= \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \left(\sum_{i=1}^{3} C_i(\rho,g;f)(x)\right)$$

$$
\begin{aligned}
&= \sum_{i=1}^{3}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} C_i(\rho,g;f)(x)\right)\\
&\quad \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{L^2(L+\pi_N(x))}{\pi_N(x)}\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}\\
&\quad + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\\
&\quad \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

□

We now find an estimate for $\langle D(\rho,g;f)(x)\rangle$, where $D(\rho,g;f)(x)$ is defined in equation (6.18), i.e.,

$$
\begin{aligned}
&D(\rho,g;f)(x)\\
&=\frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\ \ x}\ \sum_{\substack{n_1,n_2\ \ 1\\ n_1=n_2}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2)\\
&=\frac{4}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\ \ x}\ \sum_{n\ \ 1} G(n)G(n)k(p,n,n)k(q,n,n).
\end{aligned}
$$

Lemma 6.3.19. *Let* $\rho, f, g, E_i(\rho,g;f;n_1,n_2)(i=1,2,3), V_j(\rho,g;f;n_1,n_2,r), (i=1, \quad ,9)$, *and* $D(\rho,g;f)(x)$ *be as defined earlier. Then,*

$$D(\rho,g;f)(x) = \sum_{i=1}^{3} D_i(\rho,g;f)(x),$$

where

$$D_1(\rho,g;f)(x) := \frac{8}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\ \ x}\ \sum_{\substack{n_1,n_2\ \ 1\\ n_2=n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{3}E_i(n_1,n_2)\right)^2, \tag{6.52}$$

$$D_2(\rho,g;f)(x) := \frac{16}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\ \ x}\ \sum_{\substack{n_1,n_2\ \ 1\\ n_2=n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{3}\sum_{j=1}^{9}E_i(n_1,n_2)V_j(n_1,n_2,p)\right), \tag{6.53}$$

$$D_3(\rho,g;f)(x) := \frac{8}{32\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q\ \ x}\ \sum_{\substack{n_1,n_2\ \ 1\\ n_2=n_1}} G(n_1)G(n_2)\left(\sum_{i=1}^{9}\sum_{j=1}^{9}V_i(n_1,n_2,p)V_j(n_1,n_2,q)\right). \tag{6.54}$$

Proof. The proof is similar to the proof of Lemma 6.3.10. □

We find an estimate for $\langle D_1(\rho,g;f)(x)\rangle$ in the following lemma.

Lemma 6.3.20. *Let* ρ, f, g *be as defined earlier. Let* $D_1(\rho,g;f)(x)$ *be as defined in equation* (6.52)*. Then,* $D_1(\rho,g;f)(x) = O\left(\frac{1}{\pi_N(x)}\right)$ *and hence,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D_1(\rho,g;f)(x) \qquad \frac{1}{\pi_N(x)}.$$

Proof. We want to find an estimate for

$$D_1(\rho, g; f)(x) = \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q \quad x}^{0} \sum_{\substack{n_1, n_2 \quad 1 \\ n_2 = n_1}} G(n_1)G(n_2) \left(\sum_{i=1}^{3} E_i(n_1, n_2) \right)^2 ,$$

where $E_i(n_1, n_2)$ $(i = 1, 2, 3)$ are mentioned in Proposition 6.3.8.

We know, for $l \quad 0$, $jU(l)j = j\widehat{\rho}\left(\frac{l}{L}\right) 2\cos 2\pi l\psi \quad \widehat{\rho}\left(\frac{l+1}{L}\right) 2\cos 2\pi(l+1)\psi j \quad 2k_1 + 2k_1 = 4k_1,$ for some $k_1 > 0$ and $U(l) = 0$ for $l > bLc$. Therefore, $jE_i(n_1, n_2)j \quad 32k_1^2(L+1) \quad 64k_1^2 L.$

Also, for $n \quad 0$, $jG(n)j \quad k_2$.

Hence,

$$\begin{aligned} & D_1(\rho, g; f)(x) \\ = & \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q \quad x}^{0} \sum_{\substack{n_1, n_2 \quad 1 \\ n_2 = n_1}} G(n_1)G(n_2) \left(\sum_{i=1}^{3} E_i(n_1, n_2) \right)^2 \\ & \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q \quad x}^{0} \sum_{n_1 = 1}^{\pi_N(x)} \left(\sum_{i=1}^{3} 64k_1^2 L \right)^2 \\ & \frac{L^2 \pi_N(x)^3}{\pi_N(x)^4 L^2} = \frac{1}{\pi_N(x)}. \end{aligned}$$

Therefore,

$$\frac{1}{jF_{N,k}j} \sum_{f \, 2 \, F_{N,k}} D_1(\rho, g; f)(x) \qquad \frac{1}{\pi_N(x)}. \tag{6.55}$$

□

We find an estimate for $\langle D_2(\rho, g; f)(x) \rangle$ in the following lemma.

Lemma 6.3.21. *Let ρ, f, g be as defined earlier. Let $D_2(\rho, g; f)(x)$ be as defined in equation* (6.53). *Then, $D_2(\rho, g; f)(x) = O\left(\frac{L}{\pi_N(x)}\right)$ and hence,*

$$\frac{1}{jF_{N,k}j} \sum_{f \, 2 \, F_{N,k}} D_2(\rho, g; f)(x) \qquad \frac{L}{\pi_N(x)}.$$

Proof. We want to find an estimate for

$$D_2(\rho, g; f)(x) = \frac{16}{32\pi_N(x)^4 L^2} \sum_{p,q \quad x}^{0} \sum_{\substack{n_1, n_2 \quad 1 \\ n_2 = n_1}} G(n_1)G(n_2) \left(\sum_{i=1}^{3} \sum_{j=1}^{9} E_i(n_1, n_2) V_j(n_1, n_2, p) \right) ,$$

where $E_i(n_1, n_2)$ $(i = 1, 2, 3)$ are mentioned in Proposition 6.3.8.

We know, for $l, n \quad 0$, $jU(l)j = j\widehat{\rho}\left(\frac{l}{L}\right) 2\cos 2\pi l\psi \quad \widehat{\rho}\left(\frac{l+1}{L}\right) 2\cos 2\pi(l+1)\psi j \quad 2k_1 + 2k_1 = 4k_1$, $jG(n)j \quad k_2$ for some $k_1, k_2 > 0$ and $U(l) = 0$ for $l > bLc$.

We note that for $i = 1, 2, 3, 4$,

$$jE_i(n_1, n_2)j \quad 32k_1^2(L+1) \quad 64k_1^2 L.$$

Also, for $j = 1, \quad , 9$,

$$jV_j(n_1, n_2, p)j \quad 64k_1^2 \sum_{l_1, l_2 \quad 0} 1 \quad 64k_1^2 L^2.$$

Therefore, for $i = 1, 2, 3$, and $j = 1, \quad , 9$,

$$|E_i(n_1, n_2)V_j(n_1, n_2, p)| \quad 64^2 k_1^4 L^3.$$

Hence,

$$\begin{aligned}
&D_2(\rho, g; f)(x) \\
&= \frac{16}{32\pi_N(x)^4 L^2} \sum_{p,q}{}' \sum_{x \ n_1, n_2 \ 1 \atop n_2 = n_1} G(n_1)G(n_2) \left(\sum_{i=1}^{3} \sum_{j=1}^{9} E_i(n_1, n_2)V_j(n_1, n_2, p) \right) \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q}{}' \sum_{x \ n_1=1}^{\pi_N(x)} \left(\sum_{i=1}^{3} \sum_{j=1}^{9} 64^2 k_1^4 L^3 \right) \\
&\quad \frac{L^3 \pi_N(x)^3}{\pi_N(x)^4 L^2} = \frac{L}{\pi_N(x)}.
\end{aligned}$$

Therefore,

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} D_2(\rho, g; f)(x) \qquad \frac{L}{\pi_N(x)}. \tag{6.56}$$

□

To find an estimate for $\langle D_3(\rho, g; f)(x)\rangle$, we use the following lemma.

Lemma 6.3.22. *Let N and k be positive integers with k even. Then*

$$\begin{aligned}
&\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}' \sum_{x \ n_1, n_2 \ 1 \atop n_2 = n_1} G(n_1)G(n_2)V_1(n_1, n_2, p)V_1(n_1, n_2, q) \\
&\quad \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN},
\end{aligned}$$

whenever $L \quad \pi_N(x)$, where $V_1(n_1, n_2, p)$ is defined in Lemma 6.3.7.

Proof. Using Lemma 6.3.15, we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}' \sum_{x \ n_1, n_2 \ 1 \atop n_2 = n_1} G(n_1)G(n_2)V_1(n_1, n_2, p)V_1(n_1, n_2, q) \\
&= \frac{8}{32\pi_N(x)^4 L^2} \sum_{p,q}{}' \sum_{x \ n_1, n_2 \ 1 \atop n_2 = n_1} G(n_1)G(n_2) \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} V_1(n_1, n_2, p)V_1(n_1, n_2, q) \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q}{}' \sum_{x \ n_1, n_2 \ 1 \atop n_2 = n_1} \left(\frac{1}{pq} L^2 + (L + \pi_N(x))^2 L^4 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN} \right) \\
&\quad \frac{1}{\pi_N(x)^3 L^2} \sum_{p,q \ \ x}{}' \frac{1}{pq} L^2 + \frac{(L + \pi_N(x))^2}{\pi_N(x)} L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN} \\
&\quad \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN}.
\end{aligned}$$

□

We note that the other terms also give the same estimate. Therefore, we have the following lemma.

Lemma 6.3.23. *Let ρ, f, g be as defined earlier. Let $D_3(\rho, g; f)(x)$ be as defined in equation* (6.54)*. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D_3(\rho,g;f)(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}, \tag{6.57}$$

whenever $L \ll \pi_N(x)$.

We find $\langle D(\rho, g; f)(x)\rangle$ in the following proposition.

Proposition 6.3.24. *Let ρ, f, g be as defined earlier. Let $D(\rho, g; f)(x)$ be as defined in equation* (6.18)*. Then,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D(\rho,g;f)(x) \ll \frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN},$$

whenever $L \ll \pi_N(x)$.

Proof. Combining inequations (6.55), (6.56) and (6.57), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D(\rho,g;f)(x) &(6.58)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(\sum_{i=1}^{3} D_i(\rho,g;f)(x)\right)\\
&=\sum_{i=1}^{3}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D_i(\rho,g;f)(x)\right)\\
&\ll \frac{1}{\pi_N(x)} + \frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\\
&\ll \frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}$$

□

Proposition 6.3.25. *Let ρ, f, g be as defined earlier. Let $K_4(\rho, g; f)(x)$ be as defined in equation* (6.16)*. For positive integers k and N with k even,*

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K_4(\rho,g;f)(x)\\
&\ll \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN},
\end{aligned}$$

where $c' > \frac{3}{2}$ is an absolute constant and $L \ll \pi_N(x)$.

Proof. Combining equations (6.51) and (6.58), and using equation (6.16), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K_4(\rho,g;f)(x) &(6.59)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} C(\rho,g;f)(x) + \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D(\rho,g;f)(x)\\
&\ll \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\\
&+\frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}
\end{aligned}$$

$$\frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.$$

□

Proposition 6.3.26. *Let ρ, f, g be as defined earlier. Let $K(\rho,g;f)(x)$ be as defined in equation* (6.2)*. For positive integers k and N with k even,*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K(\rho,g;f)(x)$$
$$\ll \frac{L}{\pi_N(x)} + \frac{L\log\log x}{\pi_N(x)^2} + \frac{L^2(\log\log x)^2}{\pi_N(x)^4} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN},$$

where $c' > \frac{3}{2}$ is an absolute constant and $L \ll \pi_N(x)$.

Proof. Combining estimates in equations (6.6), (6.15), (6.59) and using equation (6.3), we have

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K(\rho,g;f)(x) \tag{6.60}$$
$$= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(\sum_{i=1}^{4}K_i(\rho,g;f)(x)\right)$$
$$= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K_1(\rho,g;f)(x) + \frac{2}{|F_{N,k}|}\sum_{f\in F_{N,k}} K_2(\rho,g;f)(x) + \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} K_4(\rho,g;f)(x)$$
$$\ll \frac{1}{\pi_N(x)^2} + \frac{L\log\log x}{\pi_N(x)^3} + \frac{L^2(\log\log x)^2}{\pi_N(x)^4} + \frac{L^2x^{8Lc'}8^{\nu(N)}}{\pi_N(x)^3kN}$$
$$+ \frac{L}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)^3} + \pi_N(x)L^2\left(\frac{8^{\nu(N)}x^{(8L+4\pi_N(x))c'}}{kN}\right)$$
$$+ \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}$$
$$\ll \frac{L}{\pi_N(x)} + \frac{L\log\log x}{\pi_N(x)^2} + \frac{L^2(\log\log x)^2}{\pi_N(x)^4} + \frac{(\log\log x)^2}{\pi_N(x)^2} + (L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.$$

□

6.4 Estimation for $\left\langle L(\rho,g;f)(x)\right\rangle = \left\langle (L_1+2L_2+L_4)(\rho,g;f)(x)\right\rangle$

We now find estimate for

$$L(\rho,g;f)(x) = \frac{1}{16\pi_N(x)^4L^2}\sum_{p,q,r\leq x}{}' T_1^2(p)T_2(q)T_2(r)T_3(p,q)T_3(p,r).$$

Here,

$$T_1^2(p) = \sum_{l_1,l_2\geq 0} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2}),$$

$$T_2(q) = \sum_{k_1\geq 0} U(k_1)a_f(q^{2k_1}),$$

$$T_2(r) = \sum_{k_2\geq 0} U(k_2)a_f(r^{2k_2}),$$

$$\begin{aligned} &T_3(p,q) \\ =&4G(0) + \sum_{n \geq 1} 2\widehat{g}\left(\frac{n}{\pi_N(x)}\right)(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})) \\ =& \sum_{n_1 \geq 0} \widehat{g}\left(\frac{n_1}{\pi_N(x)}\right) A(p,q,n_1), \end{aligned}$$

and

$$T_3(p,r) = \sum_{n_2 \geq 0} \widehat{g}\left(\frac{n_2}{\pi_N(x)}\right) A(p,r,n_2),$$

where

$$A(p,q,n) = \begin{cases} 4 & \text{if } n = 0 \\ 2(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})) & \text{if } n \geq 1, \end{cases}$$

and $G(n) = \widehat{g}\left(\frac{n}{\pi_N(x)}\right)$, as defined earlier.

Thus,

$$\begin{aligned} &L(\rho,g;f)(x) \\ =&\frac{1}{16\pi_N(x)^4 L^2} \sum_{p,q,r \leq x}{}' T_1^2(p) T_2(q) T_2(r) T_3(p,q) T_3(p,r) \\ =&\frac{1}{16\pi_N(x)^4 L^2} \sum_{p,q,r \leq x}{}' \sum_{l_1,l_2 \geq 0} \sum_{k_1,k_2 \geq 0} \sum_{n_1,n_2 \geq 0} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2) \\ & \qquad a_f(p^{2l_1}) a_f(p^{2l_2}) a_f(q^{2k_1}) a_f(r^{2k_2}) A(p,q,n_1) A(p,r,n_2). \end{aligned}$$

Since the summation is over n_1, n_2, where the indexes n_1, n_2 run up to $\pi_N(x)$, we can break the summation into the following four parts:

1) $n_1 = 0, n_2 = 0$,

2) $n_1 \neq 0, n_2 = 0$,

3) $n_1 = 0, n_2 \neq 0$,

4) $n_1 \neq 0, n_2 \neq 0$.

We also denote the summation in the i-th part by $L_i(\rho,g;f)(x)$, $i = 1,2,3,4$ respectively.

We will also write L_i for $L_i(\rho,g;f)(x)$, $i = 1,2,3,4$, in short.

By interchanging the variables n_1, n_2 first and then interchanging the variables q, r and at last interchanging the variables k_1, k_2, in $L_3(\rho,g;f)(x)$, the summation over n_1, n_2 where $n_1 = 0, n_2 \neq 0$, we note that the summation over n_1, n_2 where $n_1 = 0, n_2 \neq 0$, is exactly the same as the summation over n_1, n_2 where $n_1 \neq 0, n_2 = 0$, i.e., $L_2(\rho,g;f)(x) = L_3(\rho,g;f)(x)$.

Therefore,

$$L(\rho,g;f)(x) = \sum_{i=1}^{4} L_i(\rho,g;f)(x) = L_1(\rho,g;f)(x) + 2L_3(\rho,g;f)(x) + L_4(\rho,g;f)(x), \quad (6.61)$$

where $L_1(\rho,g;f)(x)$, $L_3(\rho,g;f)(x)$, and $L_4(\rho,g;f)(x)$ are defined in Sections 6.4.1, 6.4.2, and 6.4.3 respectively.

6.4.1 Estimation for $\left\langle L_1(\rho,g;f)(x)\right\rangle$

If $(n_1,n_2)=(0,0)$, then the innermost term is

$$16a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2}).$$

Thus,

$$L_1(\rho,g;f)(x)=\frac{1}{16\pi_N(x)^4L^2}\sum_{l_1,l_2\ge 0}\sum_{k_1,k_2\ge 0}16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2 \sum_{p,q,r\le x}{}' a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2}). \tag{6.62}$$

We also note that since the summation is over l_1,l_2,k_1,k_2, where the indexes l_1, l_2, k_1 and k_2 run up to $\lfloor L\rfloor$, we can break the summation into the following four parts:

1) $l_1=l_2, k_1=k_2$,

2) $l_1\ne l_2, k_1=k_2$,

3) $l_1=l_2, k_1\ne k_2$,

4) $l_1\ne l_2, k_1\ne k_2$.

We also denote the summation in the i-th part by $L_{1i}(\rho,g;f)(x)$, $i=1,2,3,4$ respectively, i.e.,

$$\begin{aligned}&16\pi_N(x)^4L^2L_{11}(\rho,g;f)(x)\\&=\sum_{\substack{l_1,l_2\ge 0\\ l_1=l_2}}\sum_{\substack{k_1,k_2\ge 0\\ k_1=k_2}}16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\sum_{p,q,r\le x}{}' a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2}),\end{aligned}\tag{6.63}$$

$$\begin{aligned}&16\pi_N(x)^4L^2L_{12}(\rho,g;f)(x)\\&=\sum_{\substack{l_1,l_2\ge 0\\ l_1\ne l_2}}\sum_{k_1\ge 0}16U(l_1)U(l_2)U(k_1)^2G(0)^2\sum_{p,q,r\le x}{}'\sum_{i=0}^{\min\{2l_1,2l_2\}} a_f(p^{2l_1+2l_2-2i})a_f(q^{2k_1}r^{2k_1}),\end{aligned}\tag{6.64}$$

$$\begin{aligned}&16\pi_N(x)^4L^2L_{13}(\rho,g;f)(x)\\&=\sum_{l_1\ge 0}\sum_{\substack{k_1,k_2\ge 0\\ k_1\ne k_2}}16U(l_1)^2U(k_1)U(k_2)G(0)^2\sum_{p,q,r\le x}{}'\sum_{i=0}^{2l_1}a_f(p^{4l_1-2i})a_f(q^{2k_1}r^{2k_2}),\end{aligned}\tag{6.65}$$

$$\begin{aligned}&16\pi_N(x)^4L^2L_{14}(\rho,g;f)(x)\\&=\sum_{\substack{l_1,l_2\ge 0\\ l_1\ne l_2}}\sum_{\substack{k_1,k_2\ge 0\\ k_1\ne k_2}}16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\\&\qquad\sum_{p,q,r\le x}{}'\sum_{i=0}^{\min\{2l_1,2l_2\}} a_f(p^{2l_1+2l_2-2i})a_f(q^{2k_1}r^{2k_2}).\end{aligned}\tag{6.66}$$

Therefore,

$$\begin{aligned}&L_1(\rho,g;f)(x)\\&=\sum_{i=1}^{4}L_{1i}(\rho,g;f)(x)\end{aligned}$$

$$= L_{11}(\rho, g; f)(x) + L_{12}(\rho, g; f)(x) + L_{13}(\rho, g; f)(x) + L_{14}(\rho, g; f)(x).$$

We note that

$$a_f(p^{2l_1})a_f(p^{2l_2}) = \sum_{i=0}^{\min\{2l_1, 2l_2\}} a_f(p^{2l_1+2l_2-2i}).$$

We see that $2l_1 + 2l_2 - 2i = 0$ only if $i = 2l_1 = 2l_2$.

Proposition 6.4.1. *With* $L_{11}(\rho, g; f)(x)$ *defined in equation* (6.63), *we have*

$$16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} L_{11}(\rho, g; f)(x)$$
$$\ll L^2\pi_N(x)^3 + L^3\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)} L^3 \pi_N(x)^3 x^{8Lc'}}{kN}.$$

Proof.

$$16\pi_N(x)^4 L^2 L_{11}(\rho, g; f)(x)$$
$$= \sum_{\substack{l_1, l_2 \ge 0 \\ l_1 = l_2}} \sum_{\substack{k_1, k_2 \ge 0 \\ k_1 = k_2}} 16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2 \sum_{p,q,r \le x}' a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})$$
$$= \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} 16U(l_1)^2U(k_1)^2G(0)^2 \sum_{p,q,r \le x}' \sum_{i=0}^{2l_1} a_f(p^{4l_1-2i})a_f(q^{2k_1}r^{2k_1})$$
$$= \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} 16U(l_1)^2U(k_1)^2G(0)^2 \pi_N(x)(\pi_N(x) - 1)(\pi_N(x) - 2)$$
$$+ \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} 16U(l_1)^2U(k_1)^2G(0)^2 \sum_{p,q,r \le x}' \sum_{\substack{i=0 \\ (i,k_1) \ne (2l_1, 0)}}^{2l_1} a_f(p^{4l_1-2i}q^{2k_1}r^{2k_1}).$$

Now, using proof of Lemma 3.2.30, if $(i, k_1) \ne (2l_1, 0)$,

$$\sum_{p,q,r \le x}' \left\langle a_f(p^{4l_1-2i}q^{2k_1}r^{2k_1}) \right\rangle = O(\pi_N(x)^2 \log\log x) + O\left(\frac{8^{\nu(N)}\pi_N(x)^3 x^{(4l_1+4k_1-2i)c'}}{kN}\right).$$

Thus,

$$16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} L_{11}(\rho, g; f)(x) \tag{6.67}$$
$$= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} 16U(l_1)^2U(k_1)^2G(0)^2\pi_N(x)(\pi_N(x) - 1)(\pi_N(x) - 2)$$
$$+ \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} 16U(l_1)^2U(k_1)^2G(0)^2 \sum_{p,q,r \le x}' \sum_{\substack{i=0 \\ (i,k_1) \ne (2l_1, 0)}}^{2l_1} \left\langle a_f(p^{4l_1-2i}q^{2k_1}r^{2k_1}) \right\rangle$$
$$\ll \pi_N(x)(\pi_N(x) - 1)(\pi_N(x) - 2)L^2$$
$$+ \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} \sum_{\substack{i=0 \\ (i,k_1) \ne (2l_1, 0)}}^{2l_1} \left(\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)}\pi_N(x)^3 x^{(4l_1+4k_1-2i)c'}}{kN}\right)$$
$$\ll L^2\pi_N(x)^3 + \pi_N(x)^2 \log\log x \sum_{l_1 \ge 0} \sum_{k_1 \ge 0} l_1$$

$$+\sum_{l_1 \geq 0}\sum_{k_1 \geq 0} l_1 \frac{8^{\nu(N)}\pi_N(x)^3 x^{(4l_1+4k_1)c'}}{kN}$$

$$\ll L^2\pi_N(x)^3 + L^3\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)}L^3\pi_N(x)^3 x^{8Lc'}}{kN}.$$

□

Proposition 6.4.2. *With $L_{12}(\rho, g; f)(x)$ defined in equation* (6.64), *we have*

$$16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_{12}(\rho, g; f)(x)$$
$$\ll L^4\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)}L^4\pi_N(x)^3 x^{8Lc'}}{kN}.$$

Proof. We find an estimate for $\langle L_{12}(\rho, g; f)(x)\rangle$, where

$$16\pi_N(x)^4 L^2 L_{12}(\rho, g; f)(x)$$
$$= \sum_{\substack{l_1,l_2 \geq 0\\ l_1\neq l_2}}\sum_{k_1 \geq 0} 16U(l_1)U(l_2)U(k_1)^2 G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x}\sum_{i=0}^{\min\{2l_1,2l_2\}} a_f(p^{2l_1+2l_2-2i})a_f(q^{2k_1}r^{2k_1})$$
$$= \sum_{\substack{l_1,l_2 \geq 0\\ l_1<l_2}}\sum_{k_1 \geq 0} 32U(l_1)U(l_2)U(k_1)^2 G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x}\sum_{i=0}^{2l_1} a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_1}).$$

Now, using proof of the Lemma 3.2.30, for $l_1, l_2, k_1 \geq 0$, $l_1 < l_2$ and $0 \leq i \leq 2l_1$,

$$\sideset{}{'}\sum_{p,q,r \leq x}\left\langle a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_1})\right\rangle$$
$$= O(\pi_N(x)^2 \log\log x) + O\left(\frac{8^{\nu(N)}\pi_N(x)^3 x^{(2l_1+2l_2+4k_1-2i)c'}}{kN}\right).$$

Thus,

$$16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_{12}(\rho, g; f)(x) \tag{6.68}$$
$$= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{l_1,l_2 \geq 0\\ l_1<l_2}}\sum_{k_1 \geq 0} 32U(l_1)U(l_2)U(k_1)^2 G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x}\sum_{i=0}^{2l_1} a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_1})$$
$$= \sum_{\substack{l_1,l_2 \geq 0\\ l_1<l_2}}\sum_{k_1 \geq 0} 32U(l_1)U(l_2)U(k_1)^2 G(0)^2 \sum_{i=0}^{2l_1}\sideset{}{'}\sum_{p,q,r \leq x}\left\langle a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_1})\right\rangle$$
$$\ll \sum_{\substack{l_1,l_2 \geq 0\\ l_1<l_2}}\sum_{k_1 \geq 0}\sum_{i=0}^{2l_1}\left(\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)}\pi_N(x)^3 x^{(2l_1+2l_2+4k_1-2i)c'}}{kN}\right)$$
$$\ll \pi_N(x)^2 \log\log x \sum_{\substack{l_1,l_2 \geq 0\\ l_1<l_2}}\sum_{k_1 \geq 0} l_1 + \sum_{\substack{l_1,l_2 \geq 0\\ l_1<l_2}}\sum_{k_1 \geq 0} l_1 \frac{8^{\nu(N)}\pi_N(x)^3 x^{(2l_1+2l_2+4k_1)c'}}{kN}$$
$$\ll L^4\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)}L^4\pi_N(x)^3 x^{8Lc'}}{kN}.$$

□

Proposition 6.4.3. *With* $L_{13}(\rho, g; f)(x)$ *defined in equation* (6.65)*, we have*

$$16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} L_{13}(\rho, g; f)(x)$$
$$\ll L^4 \pi_N(x)^2 \log\log x + \frac{8^{\nu(N)} L^4 \pi_N(x)^3 x^{8Lc'}}{kN}.$$

Proof. Here, we find estimate for $\langle L_{13}(\rho, g; f)(x)\rangle$, where

$$\begin{aligned}
&16\pi_N(x)^4 L^2 L_{13}(\rho, g; f)(x) \\
&= \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 \neq k_2}} 16U(l_1)^2 U(k_1)U(k_2)G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x} \sum_{i=0}^{2l_1} a_f(p^{4l_1-2i}) a_f(q^{2k_1} r^{2k_2}) \\
&= \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} 32U(l_1)^2 U(k_1)U(k_2)G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x} \sum_{i=0}^{2l_1} a_f(p^{4l_1-2i}) a_f(q^{2k_1} r^{2k_2}) \\
&= \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} 32U(l_1)^2 U(k_1)U(k_2)G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x} \sum_{i=0}^{2l_1} a_f(p^{4l_1-2i} q^{2k_1} r^{2k_2}).
\end{aligned}$$

Now, by Lemma 3.2.30, for $k_1, k_2, l_1 \geq 0$ with $k_2 > k_1$, i.e., $k_2 \neq 0$, and $0 \leq i \leq 2l_1$,

$$\begin{aligned}
&\sideset{}{'}\sum_{p,q,r \leq x} \left\langle a_f(p^{4l_1-2i} q^{2k_1} r^{2k_2}) \right\rangle \\
&= \mathrm{O}(\pi_N(x)^2 \log\log x) + \mathrm{O}\left(\frac{8^{\nu(N)} \pi_N(x)^3 x^{(2k_1+2k_2+4l_1-2i)c'}}{kN}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} L_{13}(\rho, g; f)(x) \qquad (6.69) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} 32U(l_1)^2 U(k_1)U(k_2)G(0)^2 \sideset{}{'}\sum_{p,q,r \leq x} \sum_{i=0}^{2l_1} a_f(p^{4l_1-2i} q^{2k_1} r^{2k_2}) \\
&= \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} 32U(l_1)^2 U(k_1)U(k_2)G(0)^2 \sum_{i=0}^{2l_1} \sideset{}{'}\sum_{p,q,r \leq x} \left\langle a_f(p^{4l_1-2i} q^{2k_1} r^{2k_2}) \right\rangle \\
&\ll \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} \sum_{i=0}^{2l_1} \left(\pi_N(x)^2 \log\log x + \frac{8^{\nu(N)} \pi_N(x)^3 x^{(2k_1+2k_2+4l_1-2i)c'}}{kN} \right) \\
&\ll \pi_N(x)^2 \log\log x \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} l_1 + \sum_{l_1 \geq 0} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 < k_2}} l_1 \frac{8^{\nu(N)} \pi_N(x)^3 x^{(2k_1+2k_2+4l_1)c'}}{kN} \\
&\ll L^4 \pi_N(x)^2 \log\log x + \frac{8^{\nu(N)} L^4 \pi_N(x)^3 x^{8Lc'}}{kN}.
\end{aligned}$$

□

Proposition 6.4.4. *With* $L_{14}(\rho, g; f)(x)$ *defined in equation* (6.66)*, we have*

$$16\pi_N(x)^4 L^2 \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} L_{14}(\rho, g; f)(x)$$

$$L^5\pi_N(x)(\log\log x)^2 + \frac{8^{\nu(N)}L^5\pi_N(x)^3 x^{8Lc'}}{kN}.$$

Proof. We find estimate for $\langle L_{14}(\rho, g; f)(x)\rangle$, where

$$\begin{aligned}
&16\pi_N(x)^4L^2L_{14}(\rho, g; f)(x)\\
&= \sum_{\substack{l_1,l_2\geq 0\\ l_1\neq l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1\neq k_2}} 16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\\
&\qquad \sideset{}{'}\sum_{p,q,r\leq x}\sum_{i=0}^{\min\{2l_1,2l_2\}} a_f(p^{2l_1+2l_2-2i})a_f(q^{2k_1}r^{2k_2})\\
&= \sum_{\substack{l_1,l_2\geq 0\\ l_1<l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1<k_2}} 64U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\\
&\qquad \sideset{}{'}\sum_{p,q,r\leq x}\sum_{i=0}^{2l_1} a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_2}).
\end{aligned}$$

For $l_1, l_2 \geq 0$ with $l_2 > l_1$ and $0 \leq i \leq 2l_1$, we have $2l_1 + 2l_2 - 2i \neq 0$.

Similarly, for $k_1, k_2 \geq 0$ with $k_2 > k_1$, we obtain $k_2 \neq 0$.

Now, using proof of the Lemma 3.2.30, for $0 \leq i \leq 2l_1$, $l_1 < l_2$, $k_1 < k_2$, we obtain

$$\begin{aligned}
&\sideset{}{'}\sum_{p,q,r\leq x}\left\langle a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_2})\right\rangle\\
&= O(\pi_N(x)(\log\log x)^2) + O\left(\frac{8^{\nu(N)}\pi_N(x)^3x^{(2k_1+2k_2+2l_1+2l_2-2i)c'}}{kN}\right)\\
&= O(\pi_N(x)(\log\log x)^2) + O\left(\frac{8^{\nu(N)}\pi_N(x)^3x^{(4k_2+4l_2-2i)c'}}{kN}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&16\pi_N(x)^4L^2\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_{14}(\rho, g; f)(x) \qquad (6.70)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{l_1,l_2\geq 0\\ l_1<l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1<k_2}} 64U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\\
&\qquad \sideset{}{'}\sum_{p,q,r\leq x}\sum_{i=0}^{2l_1} a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_2})\\
&= \sum_{\substack{l_1,l_2\geq 0\\ l_1<l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1<k_2}} 64U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\sum_{i=0}^{2l_1}\sideset{}{'}\sum_{p,q,r\leq x}\left\langle a_f(p^{2l_1+2l_2-2i}q^{2k_1}r^{2k_2})\right\rangle\\
&\ll \sum_{\substack{l_1,l_2\geq 0\\ l_1<l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1<k_2}}\sum_{i=0}^{2l_1}\left(\pi_N(x)(\log\log x)^2 + \frac{8^{\nu(N)}\pi_N(x)^3x^{(4k_2+4l_2-2i)c'}}{kN}\right)\\
&\ll \pi_N(x)(\log\log x)^2\sum_{\substack{l_1,l_2\geq 0\\ l_1<l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1<k_2}} l_1 + \sum_{\substack{l_1,l_2\geq 0\\ l_1<l_2}}\sum_{\substack{k_1,k_2\geq 0\\ k_1<k_2}} l_1\frac{8^{\nu(N)}\pi_N(x)^3x^{(4l_2+4k_2)c'}}{kN}
\end{aligned}$$

$$L^5\pi_N(x)(\log\log x)^2 + \frac{8^{\nu(N)}L^5\pi_N(x)^3x^{8Lc^0}}{kN}.$$

□

Proposition 6.4.5. *With $L_1(\rho,g;f)(x)$ defined in equation* (6.62)*, we have*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_1(\rho,g;f)(x)$$
$$\frac{1}{\pi_N(x)} + \frac{L^2\log\log x}{\pi_N(x)^2} + \frac{L^3(\log\log x)^2}{\pi_N(x)^3} + \frac{1}{\pi_N(x)}\frac{8^{\nu(N)}L^3x^{8Lc^0}}{kN}.$$

Proof. Adding equations (6.67),(6.68),(6.69) and (6.70), we obtain

$$16\pi_N(x)^4L^2\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_1(\rho,g;f)(x)$$
$$=\sum_{i=1}^{4}\left(16\pi_N(x)^4L^2\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_{1i}(\rho,g;f)(x)\right)$$
$$L^2\pi_N(x)^3 + L^3\pi_N(x)^2\log\log x + \frac{8^{\nu(N)}L^3\pi_N(x)^3x^{8Lc^0}}{kN}$$
$$+L^4\pi_N(x)^2\log\log x + \frac{8^{\nu(N)}L^4\pi_N(x)^3x^{8Lc^0}}{kN}$$
$$+L^4\pi_N(x)^2\log\log x + \frac{8^{\nu(N)}L^4\pi_N(x)^3x^{8Lc^0}}{kN}$$
$$+L^5\pi_N(x)(\log\log x)^2 + \frac{8^{\nu(N)}L^5\pi_N(x)^3x^{8Lc^0}}{kN}$$
$$L^2\pi_N(x)^3 + L^4\pi_N(x)^2\log\log x + L^5\pi_N(x)(\log\log x)^2 + \frac{8^{\nu(N)}L^5\pi_N(x)^3x^{8Lc^0}}{kN}.$$

Therefore,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_1(\rho,g;f)(x) \tag{6.71}$$
$$\frac{1}{\pi_N(x)^4L^2}\left(L^2\pi_N(x)^3 + L^4\pi_N(x)^2\log\log x + L^5\pi_N(x)(\log\log x)^2 + \frac{8^{\nu(N)}L^5\pi_N(x)^3x^{8Lc^0}}{kN}\right)$$
$$\frac{1}{\pi_N(x)} + \frac{L^2\log\log x}{\pi_N(x)^2} + \frac{L^3(\log\log x)^2}{\pi_N(x)^3} + \frac{1}{\pi_N(x)}\frac{8^{\nu(N)}L^3x^{8Lc^0}}{kN}.$$

□

6.4.2 Estimation for $\left\langle (L_2 + L_3)(\rho,g;f)(x)\right\rangle$

We now look at the part of the sum $L(\rho,g;f)(x)$ with $n_1 = 0$ and $n_2 \neq 0$, i.e., we now estimate $L_3(\rho,g;f)(x)$. In this case, the innermost term

$$a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})A(p,q,0)A(p,r,n_2)$$
$$=8a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))$$
$$=8\{a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\}a_f(q^{2k_1})\{(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))\}a_f(r^{2k_2}).$$

We want to find an estimate for

$$
\begin{aligned}
&(L_2 + L_3)(\rho,g;f)(x)\\
=&2L_3(\rho,g;f)(x)\\
=&\frac{2}{16\pi_N(x)^4L^2}\sum_{p,q,r\leq x}' \sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0}\sum_{n_2\geq 1} U(l_1)U(l_2)U(k_1)U(k_2)G(0)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})A(p,q,0)A(p,r,n_2)\\
=&\frac{16}{16\pi_N(x)^4L^2}\sum_{p,q,r\leq x}' \sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0}\sum_{n_2\geq 1} U(l_1)U(l_2)U(k_1)U(k_2)G(0)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))\\
=&\frac{G(0)}{\pi_N(x)^4L^2}\sum_{p,q,r\leq x}' \sum_{n_2\geq 1} G(n_2)\sum_{l_1,l_2\geq 0} U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
&\qquad \sum_{k_1,k_2\geq 0} U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})a_f((a_f(r^{2n_2}) - a_f(r^{2n_2-2}))\\
=&\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\leq x}' \sum_{n_2\geq 1} G(0)G(n_2)A(\rho,g;f;n_2,p)A_1(\rho,g;f;n_2,q,r),
\end{aligned}
\tag{6.72}
$$

where, for $n \geq 1$, and primes q and r,

$$A_1(\rho,g;f;n,q,r) := \sum_{k_1,k_2\geq 0} U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n}) - a_f(r^{2n-2})), \tag{6.73}$$

and

$$A(\rho,g;f;n,r) = \sum_{l_1,l_2\geq 0} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n}) - a_f(r^{2n-2})),$$

as defined in equation (6.9). We now estimate $A_1(\rho,g;f;n,q,r)$.

Using Corollary 3.3.5, for $k_2 \geq n_2$, we have

$$
\begin{aligned}
&a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))\\
=&a_f(q^{2k_1})(a_f(r^{2k_2-2n_2}) + a_f(r^{2k_2+2n_2})).
\end{aligned}
$$

We note that the product $a_f(q^{2k_1})(a_f(r^{2k_2-2n_2}) + a_f(r^{2k_2+2n_2}))$ gives $1 = a_f(q^0r^0)$, only if $k_1 = 0$ and $k_2 - n_2 = 0$, i.e., $k_1 = 0$ and $k_2 = n_2$, i.e., $(k_1,k_2) = (0,n_2)$.

Using Corollary 3.3.5, for $k_2 < n_2$, we have

$$
\begin{aligned}
&a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))\\
=&a_f(q^{2k_1})(a_f(r^{2k_2+2n_2}) - a_f(r^{2n_2-2k_2-2})).
\end{aligned}
$$

We note that the product $a_f(q^{2k_1})(a_f(r^{2k_2+2n_2}) - a_f(r^{2n_2-2k_2-2}))$ gives $-1 = -a_f(q^0r^0)$, only if $k_1 = 0$ and $n_2 = k_2 + 1$, i.e., $(k_1,k_2) = (0,n_2-1)$.

For primes r, q and positive integer n_2, we define

$$F_1(\rho,g;f;n_2,q,r) := \sum_{\substack{k_1,k_2\geq 0, k_2\geq n_2\\ (k_1,k_2)\neq(0,n_2)}} U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2})), \tag{6.74}$$

$$F_2(\rho,g;f;n_2,q,r) := \sum_{\substack{k_1,k_2\ge 0,k_2<n_2\\ (k_1,k_2)\neq(0,n_2-1)}} U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2})). \tag{6.75}$$

For any primes q,r and fixed integer $n_2\ge 1$,

$$\begin{aligned}
&A_1(\rho,g;f;n_2,q,r) \\
=&\sum_{k_1,k_2\ge 0} U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n})-a_f(r^{2n-2}))\\
=&\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\ (k_1,k_2)=(0,n_2)}} U(k_1)U(k_2)+F_1(\rho,g;f;n_2,q,r)\\
&-\sum_{\substack{k_1,k_2\ge 0,k_2<n_2\\ (k_1,k_2)=(0,n_2-1)}} U(k_1)U(k_2)+F_2(\rho,g;f;n_2,q,r)\\
=&U(0)U(n_2)+F_1(\rho,g;f;n_2,q,r)-U(0)U(n_2-1)+F_2(\rho,g;f;n_2,q,r)\\
=&U(0)(U(n_2)-U(n_2-1))+F_1(n_2,q,r)+F_2(n_2,q,r),
\end{aligned} \tag{6.76}$$

where we write $F_i(n_2,q,r)$ for $F_i(\rho,g;f;n_2,q,r)$, $i=1,2$.

We note that for $i=1,2$, and $j=1,2,3$, $|B_i(n_2)|\le 32k_1^2L$, $|F_i(n_2,q,r)|\le 256k_1^2L^2$ and $|A_j(n_2,p)|\le 64k_1^2L^2$.

We define, $F(n):=U(n)-U(n-1)$.

For primes p,q,r and fixed integer $n_2\ge 1$, using equation (6.13), we obtain

$$\begin{aligned}
&A(\rho,g;f;n_2,p)\,A_1(\rho,g;f;n_2,q,r)\\
=&\left(B_1(n_2)+B_2(n_2)+\sum_{i=1}^{3}A_i(n_2,p)\right)\big(U(0)(U(n_2)-U(n_2-1))+F_1(n_2,q,r)+F_2(n_2,q,r)\big)\\
=&\left(B_1(n_2)+B_2(n_2)+\sum_{i=1}^{3}A_i(n_2,p)\right)\big(U(0)F(n_2)+F_1(n_2,q,r)+F_2(n_2,q,r)\big)\\
=&U(0)F(n_2)(B_1(n_2)+B_2(n_2))+\sum_{i=1}^{3}U(0)F(n_2)A_i(n_2,p)\\
&+F_1(n_2,q,r)(B_1(n_2)+B_2(n_2))+\sum_{i=1}^{3}F_1(n_2,q,r)A_i(n_2,p)\\
&+F_2(n_2,q,r)(B_1(n_2)+B_2(n_2))+\sum_{i=1}^{3}F_2(n_2,q,r)A_i(n_2,p).
\end{aligned} \tag{6.77}$$

Using equation (6.72), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}(L_2(\rho,g;f)(x)+L_3(\rho,g;f)(x))\\
=&2\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}L_3(\rho,g;f)(x)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2}{\pi_N(x)^4L^2}\sum_{p,q,r}{}'\sum_{x\ n_2\ 1}G(0)G(n_2)A(\rho,g;f;n_2,p)A_1(\rho,g;f;n_2,q,r)
\end{aligned} \tag{6.78}$$

$$= \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2)U(0)F(n_2)(B_1(n_2) + B_2(n_2))$$

$$+ \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2) \sum_{i=1}^{3} U(0)F(n_2)A_i(n_2, p)$$

$$+ \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2)F_1(n_2, q, r)(B_1(n_2) + B_2(n_2))$$

$$+ \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2) \sum_{i=1}^{3} F_1(n_2, q, r)A_i(n_2, p)$$

$$+ \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2)F_2(n_2, q, r)(B_1(n_2) + B_2(n_2))$$

$$+ \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2) \sum_{i=1}^{3} F_2(n_2, q, r)A_i(n_2, p)$$

$$= \sum_{t=1}^{6} \lambda_t(x),$$

where $\sum_{t=1}^{6} \lambda_t(x)$ are defined in accordance with the order of terms in the previous line.

Proposition 6.4.6. *With λ_1 defined in equation* (6.78)*, we have*

$$\lambda_1(x) \quad \frac{1}{\pi_N(x)}.$$

Proof. We know, there exists $k_1 > 0$ such that $\left|F(n)\right| \quad 8k_1$ for all positive integers n.

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) \quad U(n \quad 1) = 0$.

Using equation (6.77), we obtain

$$\lambda_1(x) \tag{6.79}$$

$$= \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\,n_2\quad 1} G(0)G(n_2)U(0)F(n_2)(B_1(n_2) + B_2(n_2))$$

$$\frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\quad n_2=1}^{1+bLc} jB_1(n_2) + B_2(n_2)j$$

$$\frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{0} \sum_{x\quad n_2=1}^{1+bLc} L$$

$$\frac{1}{\pi_N(x)^4 L^2} \pi_N(x)^3 L^2$$

$$= \frac{1}{\pi_N(x)}.$$

□

Proposition 6.4.7. *With λ_2 defined in equation* (6.78)*, we have*

$$\lambda_2(x) \quad \frac{L}{\pi_N(x)}.$$

Proof. We know, there exists $k_1 > 0$ such that $|F(n)| \le 8k_1$ for all positive integers n.

Now, $n > 1 + \lfloor Lc \rfloor$, implies $U(n) = 0$ and hence, $F(n) = U(n) - U(n-1) = 0$.

Using equation (6.77), we obtain

$$\lambda_2(x) \tag{6.80}$$

$$= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r \le x}{}' \sum_{n_2 \ge 1} G(0)G(n_2) \sum_{i=1}^{3} U(0)F(n_2)A_i(n_2,p)$$

$$= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r \le x}{}' \sum_{n_2=1}^{1+\lfloor Lc \rfloor} G(0)G(n_2)U(0)F(n_2) \sum_{i=1}^{3} A_i(n_2,p)$$

$$\ll \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r \le x}{}' \sum_{n_2=1}^{1+\lfloor Lc \rfloor} \left| \sum_{i=1}^{3} A_i(n_2,p) \right|$$

$$\ll \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r \le x}{}' \sum_{n_2=1}^{1+\lfloor Lc \rfloor} L^2$$

$$\ll \frac{1}{\pi_N(x)^4 L^2} \pi_N(x)^3 L^3$$

$$= \frac{L}{\pi_N(x)}.$$

□

To find an estimate for $\lambda_3(x)$ (defined in equation (6.78), we first prove the following lemma.

Lemma 6.4.8.

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2,p,r) \ll \left(\frac{1}{r} + \frac{1}{p} + \frac{1}{pr} \right) + L^3 \frac{8^{\nu(N)} (pr)^{4Lc'}}{|F_{N,k}|}.$$

Proof.

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2,p,r)$$

$$= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{\substack{k_1,k_2 \ge 0, k_2 \ge n_2 \\ (k_1,k_2) \notin (0,n_2)}} U(k_1)U(k_2)a_f(p^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))$$

$$= \sum_{\substack{k_1,k_2 \ge 0, k_2 \ge n_2 \\ (k_1,k_2) \notin (0,n_2)}} U(k_1)U(k_2) \left(\left\langle a_f(p^{2k_1} r^{2k_2-2n_2}) \right\rangle + \left\langle a_f(p^{2k_1} r^{2k_2+2n_2}) \right\rangle \right)$$

$$= \sum_{\substack{k_1,k_2 \ge 0, k_2 \ge n_2 \\ (k_1,k_2) \notin (0,n_2)}} U(k_1)U(k_2) \left(\frac{1}{p^{k_1} r^{k_2-n_2}} + \frac{1}{p^{k_1} r^{k_2+n_2}} + O\left(\frac{8^{\nu(N)} p^{2k_1 c'} r^{2k_2 c' + 2n_2 c'}}{kN} \right) \right)$$

$$\ll \sum_{\substack{k_1,k_2 \ge 0, k_2 \ge n_2 \\ (k_1,k_2) \notin (0,n_2)}} \left(\frac{2}{p^{k_1} r^{k_2-n_2}} + \frac{8^{\nu(N)} p^{2k_1 c'} r^{2k_2 c' + 2n_2 c'}}{kN} \right)$$

$$\ll \left(\frac{1}{r} + \quad \right) + \left(\frac{1}{p} + \quad \right) + \left(\frac{1}{p} + \quad \right)\left(\frac{1}{r} + \quad \right) + L^3 \frac{8^{\nu(N)} (pr)^{4Lc'}}{kN}$$

$$\ll \left(\frac{1}{r} + \frac{1}{p} + \frac{1}{pr} \right) + L^3 \frac{8^{\nu(N)} (pr)^{4Lc'}}{kN}.$$

□

Proposition 6.4.9. *With* λ_3 *defined in equation* (6.78), *we have*

$$\lambda_3(x) \ll \frac{\log\log x}{\pi_N(x)L} + L^2\frac{8^{\nu(N)}x^{8Lc'}}{kN}.$$

Proof. Using Lemma 6.4.8, we obtain

$$\begin{aligned}
&\lambda_3(x) \qquad (6.81)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\leq x}\sum_{n_2\geq 1}G(0)G(n_2)F_1(n_2,q,r)(B_1(n_2)+B_2(n_2))\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\leq x}\sum_{n_2=1}^{\pi_N(x)}G(0)G(n_2)F_1(n_2,q,r)(B_1(n_2)+B_2(n_2))\\
&= \frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\leq x}\sum_{n_2=1}^{\pi_N(x)}G(0)G(n_2)(B_1(n_2)+B_2(n_2))\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_1(n_2,q,r)\\
&\leq \frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\leq x}\sum_{n_2=1}^{\pi_N(x)}|B_1(n_2)+B_2(n_2)|\left|\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_1(n_2,q,r)\right|\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\leq x}\sum_{n_2=1}^{\pi_N(x)}L\left(\left(\frac{1}{r}+\frac{1}{q}+\frac{1}{qr}\right)+L^3\frac{8^{\nu(N)}(qr)^{4Lc'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}L\pi_N(x)\sideset{}{'}\sum_{p,q,r\leq x}\left(\left(\frac{1}{r}+\frac{1}{q}+\frac{1}{qr}\right)+L^3\frac{8^{\nu(N)}(qr)^{4Lc'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^3L}\sideset{}{'}\sum_{p,q,r\leq x}\frac{1}{r}+L^2\frac{8^{\nu(N)}x^{8Lc'}}{kN}\\
&\ll \frac{1}{\pi_N(x)^3L}\pi_N(x)(\pi_N(x)-1)\sum_{r\leq x}\frac{1}{r}+L^2\frac{8^{\nu(N)}x^{8Lc'}}{kN}\\
&\ll \frac{\log\log x}{\pi_N(x)L}+L^2\frac{8^{\nu(N)}x^{8Lc'}}{kN}.
\end{aligned}$$

□

To find an estimate for $\lambda_4(x)$ (defined in equation (6.78), we first prove the following lemma.

Lemma 6.4.10.

$$\sum_{i=1}^{3}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_1(n_2,q,r)A_i(n_2,p) \ll \left(\frac{1}{pr}+\frac{1}{pq}+\frac{1}{pqr}\right)L + L^5\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN}.$$

Proof. Using equation (6.74) and definition (6.10), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_1(n_2,q,r)A_1(n_2,p)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{k_1,k_2\geq 0,k_2\leq n_2\\(k_1,k_2)\neq(0,n_2)}}U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\
&\qquad\sum_{\substack{l_1,l_2\geq 0\\l_1-l_2\neq n_2}}U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2+2n_2})
\end{aligned}$$

$$= \sum_{\substack{k_1,k_2 \quad 0, k_2 \quad n_2 \\ (k_1,k_2) \neq (0,n_2)}} \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad l_2 \neq n_2}} U(l_1)U(l_2)U(k_1)U(k_2)$$

$$\sum_{t=|l_1 \quad l_2 \quad n_2|}^{l_1+l_2+n_2} \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \left(a_f(p^{2t}q^{2k_1}) \quad (a_f(r^{2k_2 \quad 2n_2}) + a_f(r^{2k_2+2n_2})) \right)$$

$$= \sum_{\substack{k_1,k_2 \quad 0, k_2 \quad n_2 \\ (k_1,k_2) \neq (0,n_2)}} \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad l_2 \neq n_2}} U(l_1)U(l_2)U(k_1)U(k_2)$$

$$\sum_{t=|l_1 \quad l_2 \quad n_2|}^{l_1+l_2+n_2} \left(\left\langle a_f(p^{2t}q^{2k_1}r^{2k_2 \quad 2n_2}) \right\rangle + \left\langle a_f(p^{2t}q^{2k_1}r^{2k_2+2n_2}) \right\rangle \right)$$

$$\sum_{\substack{k_1,k_2 \quad 0, k_2 \quad n_2 \\ (k_1,k_2) \neq (0,n_2)}} \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad l_2 \neq n_2}}$$

$$\sum_{t=|l_1 \quad l_2 \quad n_2|}^{l_1+l_2+n_2} \left(\frac{1}{p^t}\frac{1}{q^{k_1}} \left(\frac{1}{r^{k_2 \quad n_2}} + \frac{1}{r^{k_2+n_2}} \right) + \left(\frac{8^{\nu(N)} p^{2tc'} q^{2k_1c'} r^{2k_2c'+2n_2c'}}{kN} \right) \right)$$

$$\sum_{\substack{k_1,k_2 \quad 0, k_2 \quad n_2 \\ (k_1,k_2) \neq (0,n_2)}} \frac{1}{q^{k_1}} \frac{1}{r^{k_2 \quad n_2}} \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad l_2 \neq n_2}} \sum_{t=|l_1 \quad l_2 \quad n_2|}^{l_1+l_2+n_2} \frac{1}{p^t}$$

$$+ \sum_{\substack{k_1,k_2 \quad 0, k_2 \quad n_2 \\ (k_1,k_2) \neq (0,n_2)}} \sum_{\substack{l_1,l_2 \quad 0 \\ l_1 \quad l_2 \neq n_2}} \left((l_1 + l_2 + n_2) \frac{8^{\nu(N)} p^{2(l_1+l_2+n_2)c'} q^{2k_1c'} r^{2k_2c'+2n_2c'}}{kN} \right)$$

$$\left(\left(\frac{1}{r} + \quad \right) + \left(\frac{1}{q} + \quad \right) + \left(\frac{1}{q} + \quad \right) \left(\frac{1}{r} + \quad \right) \right) \left(\frac{1}{p} + \quad \right) L$$

$$+ L^4 L \frac{8^{\nu(N)} p^{4Lc'+2n_2c'} q^{2Lc'} r^{2Lc'+2n_2c'}}{kN}$$

$$\left(\frac{1}{pr} + \frac{1}{pq} + \frac{1}{pqr} \right) L + L^5 \frac{8^{\nu(N)} p^{4Lc'+2n_2c'} q^{2Lc'} r^{2Lc'+2n_2c'}}{kN}.$$

We also note that a similar calculation leads to the same estimate for $i = 2, 3$ and we have

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2, q, r) A_i(n_2, p)$$

$$\left(\frac{1}{pr} + \frac{1}{pq} + \frac{1}{pqr} \right) L + L^5 \frac{8^{\nu(N)} p^{4Lc'+2n_2c'} q^{2Lc'} r^{2Lc'+2n_2c'}}{kN}.$$

and hence, adding all three inequations, we obtain

$$\sum_{i=1}^{3} \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2, q, r) A_i(n_2, p) \tag{6.82}$$

$$\left(\frac{1}{pr} + \frac{1}{pq} + \frac{1}{pqr} \right) L + L^5 \frac{8^{\nu(N)} p^{4Lc'+2n_2c'} q^{2Lc'} r^{2Lc'+2n_2c'}}{kN}.$$

□

Proposition 6.4.11. *With λ_4 defined in equation* (6.78)*, we have*

$$\lambda_4(x) \quad \frac{(\log \log x)^2}{\pi_N(x)^2 L} + L^3 \frac{8^{\nu(N)} x^{8Lc' + 4\pi_N(x)c'}}{kN}.$$

Proof. Using equation (6.82), we have

$$
\begin{aligned}
&\lambda_4(x) &(6.83)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{2}{\pi_N(x)^4L^2}\sum_{p,q,r \quad x}{}' \sum_{n_2 \quad 1} G(0)G(n_2)\sum_{i=1}^{3} F_1(n_2,q,r)A_i(n_2,p)\\
&= \frac{2}{\pi_N(x)^4L^2}\sum_{p,q,r \quad x}{}' \sum_{n_2 \quad 1} G(0)G(n_2)\sum_{i=1}^{3}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} A_i(n_2,p)F_1(n_2,q,r)\\
&\quad \frac{2}{\pi_N(x)^4L^2}\sum_{p,q,r \quad x}{}' \sum_{n_2 \quad 1}\left|\sum_{i=1}^{3}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} A_i(n_2,p)F_1(n_2,q,r)\right|\\
&\quad \frac{2}{\pi_N(x)^4L^2}\sum_{p,q,r \quad x}{}' \sum_{n_2 \quad 1}\left(\left(\frac{1}{pr}+\frac{1}{pq}+\frac{1}{pqr}\right)L + L^5\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN}\right)\\
&\quad \frac{1}{\pi_N(x)^4L^2}\pi_N(x)L\sum_{p,q,r \quad x}{}'\frac{1}{pr} + \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r \quad x}{}' L^5\pi_N(x)\frac{8^{\nu(N)}(x^2)^{4Lc'+2\pi_N(x)c'}}{kN}\\
&\quad \frac{1}{\pi_N(x)^4L^2}\pi_N(x)L(\log\log x)^2\pi_N(x) + \frac{1}{\pi_N(x)^4L^2}L^5\pi_N(x)^4\frac{8^{\nu(N)}(x^2)^{4Lc'+2\pi_N(x)c'}}{kN}\\
&\quad \frac{(\log\log x)^2}{\pi_N(x)^2L} + L^3\frac{8^{\nu(N)}x^{8Lc'+4\pi_N(x)c'}}{kN}.
\end{aligned}
$$

□

To find an estimate for $\lambda_5(x)$ (defined in equation (6.78), we first prove the following lemma.

Lemma 6.4.12.

$$
\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_2(n_2,p,r) \qquad \left(\frac{1}{r}+\frac{1}{p}+\frac{1}{pr}\right) + L^2\frac{8^{\nu(N)}p^{2Lc'}r^{2Lc'+2n_2c'}}{kN}.
$$

Proof. Using equations (6.75), (6.11) and (6.12), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_2(n_2,p,r)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \sum_{\substack{k_1,k_2 \quad 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2 \quad 1)}} U(k_1)U(k_2)a_f(p^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2}) \quad a_f(r^{2n_2 \quad 2}))\\
&= \sum_{\substack{k_1,k_2 \quad 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2 \quad 1)}} U(k_1)U(k_2)\left(\left\langle a_f(p^{2k_1}r^{2k_2+2n_2})\right\rangle \quad \left\langle a_f(p^{2k_1}r^{2n_2 \quad 2k_2 \quad 2})\right\rangle\right)\\
&= \sum_{\substack{k_1,k_2 \quad 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2 \quad 1)}} U(k_1)U(k_2)\left(\frac{1}{p^{k_1}r^{k_2+n_2}} + \frac{1}{p^{k_1}r^{n_2 \quad k_2 \quad 1}} + \mathrm{O}\left(\frac{8^{\nu(N)}p^{2k_1c'}r^{2k_2c'+2n_2c'}}{kN}\right)\right)\\
&\quad \sum_{\substack{k_1,k_2 \quad 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2 \quad 1)}} \left(\frac{2}{p^{k_1}r^{n_2 \quad k_2 \quad 1}} + \frac{8^{\nu(N)}p^{2k_1c'}r^{2k_2c'+2n_2c'}}{kN}\right)\\
&\quad \left(\frac{1}{r}+ \quad\right) + \left(\frac{1}{p}+ \quad\right) + \left(\frac{1}{p}+ \quad\right)\left(\frac{1}{r}+ \quad\right) + L^2\frac{8^{\nu(N)}p^{2Lc'}r^{2Lc'+2n_2c'}}{kN}\\
&\quad \left(\frac{1}{r}+\frac{1}{p}+\frac{1}{pr}\right) + L^2\frac{8^{\nu(N)}p^{2Lc'}r^{2Lc'+2n_2c'}}{kN}.
\end{aligned}
$$

□

Proposition 6.4.13. *With* λ_5 *defined in equation* (6.78)*, we have*

$$\lambda_5(x) \ll \frac{\log\log x}{\pi_N(x)L} + L\frac{8^{\nu(N)}x^{4Lc'+2\pi_N(x)c'}}{kN}.$$

Proof. $l > 1 + bLc$, implies $U(l) = 0$ and hence, $B_1(n_2) + B_2(n_2) = 0$, or $n_2 > 1 + bLc$.

Using Lemma 6.4.12, we obtain

$$\begin{aligned}
&\lambda_5(x) \qquad (6.84)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\le x}\sum_{n_2=1}^{\pi_N(x)}G(0)G(n_2)F_2(n_2,q,r)(B_1(n_2)+B_2(n_2))\\
&= \frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\le x}\sum_{n_2=1}^{\pi_N(x)}G(0)G(n_2)(B_1(n_2)+B_2(n_2))\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_2(n_2,q,r)\\
&\le \frac{2}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\le x}\sum_{n_2=1}^{\pi_N(x)}|B_1(n_2)+B_2(n_2)|\left|\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_2(n_2,q,r)\right|\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\le x}\sum_{n_2=1}^{\pi_N(x)}L\left(\left(\frac{1}{r}+\frac{1}{q}+\frac{1}{qr}\right)+L^2\frac{8^{\nu(N)}p^{2Lc'}r^{2Lc'+2n_2c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}L\pi_N(x)\sideset{}{'}\sum_{p,q,r\le x}\left(\left(\frac{1}{r}+\frac{1}{q}+\frac{1}{qr}\right)+L^2\frac{8^{\nu(N)}p^{2Lc'}r^{2Lc'+2\pi_N(x)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^3L}\sideset{}{'}\sum_{p,q,r\le x}\frac{1}{r}+L\frac{8^{\nu(N)}x^{4Lc'+2\pi_N(x)c'}}{kN} \ll \frac{\log\log x}{\pi_N(x)L}+L\frac{8^{\nu(N)}x^{4Lc'+2\pi_N(x)c'}}{kN}.
\end{aligned}$$

□

To find an estimate for $\lambda_6(x)$ (defined in equation (6.78), we first prove the following lemma.

Lemma 6.4.14.

$$\begin{aligned}
&\sum_{i=1}^{3}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_2(n_2,q,r)A_i(n_2,p)\\
&\ll \left(\frac{1}{pr}+\frac{1}{pq}+\frac{1}{pqr}\right)L+L^4\pi_N(x)\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN}.
\end{aligned}$$

Proof. Using equation (6.75) and definition (6.10), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}F_2(n_2,q,r)A_1(n_2,p)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{k_1,k_2\ge 0,k_2<n_2\\(k_1,k_2)\ne(0,n_2-1)}}U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2}))\\
&\qquad\sum_{\substack{l_1,l_2\ge 0\\l_1-l_2\ne n_2}}U(l_1)U(l_2)a_f(p^{2l_1})a_f(p^{2l_2+2n_2})\\
&= \sum_{\substack{k_1,k_2\ge 0,k_2<n_2\\(k_1,k_2)\ne(0,n_2-1)}}\sum_{\substack{l_1,l_2\ge 0\\l_1-l_2\ne n_2}}U(l_1)U(l_2)U(k_1)U(k_2)
\end{aligned}$$

$$\sum_{t=|l_1-l_2-n_2|}^{l_1+l_2+n_2} \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \Big(a_f(p^{2t}q^{2k_1})\quad (a_f(r^{2k_2+2n_2})\quad a_f(r^{2n_2-2k_2-2}))\Big)$$

$$= \sum_{\substack{k_1,k_2\geq 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2-1)}} \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2}} U(l_1)U(l_2)U(k_1)U(k_2)$$

$$\sum_{t=|l_1-l_2-n_2|}^{l_1+l_2+n_2}\Big(\Big\langle a_f(p^{2t}q^{2k_1}r^{2k_2+2n_2})\Big\rangle \quad \Big\langle a_f(p^{2t}q^{2k_1}r^{2n_2-2k_2-2})\Big\rangle\Big)$$

$$\ll \sum_{\substack{k_1,k_2\geq 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2-1)}} \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2}}$$

$$\sum_{t=|l_1-l_2-n_2|}^{l_1+l_2+n_2}\left(\frac{1}{p^t}\frac{1}{q^{k_1}}\left(\frac{1}{r^{k_2+n_2}}+\frac{1}{r^{n_2-k_2-1}}\right)+\left(\frac{8^{\nu(N)}p^{2tc'}q^{2k_1c'}r^{2k_2c'+2n_2c'}}{kN}\right)\right)$$

$$\ll \sum_{\substack{k_1,k_2\geq 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2-1)}} \frac{1}{q^{k_1}}\frac{1}{r^{n_2-k_2-1}} \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2}} \sum_{t=|l_1-l_2-n_2|}^{l_1+l_2+n_2}\frac{1}{p^t}$$

$$+ \sum_{\substack{k_1,k_2\geq 0, k_2<n_2\\ (k_1,k_2)\neq(0,n_2-1)}} \sum_{\substack{l_1,l_2\geq 0\\ l_1-l_2\neq n_2}} \left((l_1+l_2+n_2)\frac{8^{\nu(N)}p^{2(l_1+l_2+n_2)c'}q^{2k_1c'}r^{2k_2c'+2n_2c'}}{kN}\right)$$

$$\ll \left(\left(\frac{1}{r}+\quad\right)+\left(\frac{1}{q}+\quad\right)+\left(\frac{1}{q}+\quad\right)\left(\frac{1}{r}+\quad\right)\right)\left(\frac{1}{p}+\quad\right)L$$

$$+ L^4\pi_N(x)\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN}$$

$$\ll \left(\frac{1}{pr}+\frac{1}{pq}+\frac{1}{pqr}\right)L + L^4\pi_N(x)\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN}.$$

We also note that a similar calculation leads to the same estimate for $i = 2, 3$ and we have

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_2(n_2,q,r)A_i(n_2,p)$$

$$\ll \left(\frac{1}{pr}+\frac{1}{pq}+\frac{1}{pqr}\right)L + L^4\pi_N(x)\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN},$$

and hence, adding all three inequations, we obtain

$$\sum_{i=1}^{3}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_2(n_2,q,r)A_i(n_2,p) \tag{6.85}$$

$$\ll \left(\frac{1}{pr}+\frac{1}{pq}+\frac{1}{pqr}\right)L + L^4\pi_N(x)\frac{8^{\nu(N)}p^{4Lc'+2n_2c'}q^{2Lc'}r^{2Lc'+2n_2c'}}{kN}.$$

□

Proposition 6.4.15. *With λ_6 defined in equation* (6.78)*, we have*

$$\lambda_6(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2 L} + L^2\pi_N(x)\frac{8^{\nu(N)}x^{8Lc'+4\pi_N(x)c'}}{kN}.$$

Proof. Using equation (6.85), we have

$$\lambda_6(x) \tag{6.86}$$

$$
\begin{aligned}
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{'} \sum_{x\; n_2\; 1} G(0)G(n_2) \sum_{i=1}^{3} F_2(n_2,q,r)A_i(n_2,p) \\
&= \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{'} \sum_{x\; n_2\; 1} G(0)G(n_2) \sum_{i=1}^{3} \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} A_i(n_2,p)F_2(n_2,q,r) \\
&\quad \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{'} \sum_{x\; n_2\; 1} \left| \sum_{i=1}^{3} \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} A_i(n_2,p)F_2(n_2,q,r) \right| \\
&\quad \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{'} \sum_{x\; n_2\; 1} \left(\frac{1}{pr} + \frac{1}{pq} + \frac{1}{pqr} \right) L \\
&\quad + \frac{2}{\pi_N(x)^4 L^2} \sum_{p,q,r}{}^{'} \sum_{x\; n_2\; 1} L^4 \pi_N(x) \frac{8^{\nu(N)} p^{4Lc'+2n_2c'} q^{2Lc'} r^{2Lc'+2n_2c'}}{kN} \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \pi_N(x) L \sum_{p,q,r\; x}{}^{'} \frac{1}{pr} + \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r\; x}{}^{'} L^4 \pi_N(x)^2 \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN} \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \pi_N(x) L (\log\log x)^2 \pi_N(x) + \frac{1}{\pi_N(x)^4 L^2} L^4 \pi_N(x)^5 \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN} \\
&\quad \frac{(\log\log x)^2}{\pi_N(x)^2 L} + L^2 \pi_N(x) \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN}.
\end{aligned}
$$

□

Proposition 6.4.16. *With* $(L_2 + L_3)(\rho, g; f)(x)$ *defined in equation* (6.72), *we have*

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} (L_2 + L_3)(\rho, g; f)(x) \\
&\quad \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x) L} + L^2 (L + \pi_N(x)) \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Proof. Adding equations (6.79), (6.80), (6.81), (6.83), (6.84) and (6.86), and using equation (6.78), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} (L_2(\rho, g; f)(x) + L_3(\rho, g; f)(x)) \\
&= \sum_{t=1}^{6} \lambda_t(x) \\
&\quad \frac{1}{\pi_N(x)} + \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x) L} + L^2 \frac{8^{\nu(N)} x^{8Lc'}}{kN} + \frac{(\log\log x)^2}{\pi_N(x)^2 L} + L^3 \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN} \\
&+ \frac{\log\log x}{\pi_N(x) L} + L \frac{8^{\nu(N)} x^{4Lc'+2\pi_N(x)c'}}{kN} + \frac{(\log\log x)^2}{\pi_N(x)^2 L} + L^2 \pi_N(x) \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN} \\
&\quad \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x) L} + L^2 (L + \pi_N(x)) \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN},
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} (L_2 + L_3)(\rho, g; f)(x) \qquad (6.87) \\
&\quad \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x) L} + L^2 (L + \pi_N(x)) \frac{8^{\nu(N)} x^{8Lc'+4\pi_N(x)c'}}{kN}.
\end{aligned}
$$

□

6.4.3 Estimation for $\langle L_4(\rho, g; f)(x)\rangle$

We now look at the part of the sum $L(\rho, g; f)(x)$ with $n_1 \neq 0$ and $n_2 \neq 0$, i.e., we now estimate $L_4(\rho, g; f)(x)$.

For $l \geq 0$, $n, n_1, n_2 \geq 1$ and for any prime p, let

$$L_p(l, n) = a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2})),$$

and

$$T(p, n_1, n_2) = (a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2})),$$

be the same as defined earlier.

Thus,

$$A(p, q, n_1)A(p, r, n_2) = 4T(p, n_1, n_2)(a_f(q^{2n_1}) - a_f(q^{2n_1-2}))(a_f(r^{2n_2}) - a_f(r^{2n_2-2})).$$

In this case, the innermost term is

$$\begin{aligned}
&a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})A(p, q, n_1)A(p, r, n_2)\\
=&4a_f(p^{2l_1})a_f(p^{2l_2})(a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2}))\\
&\quad a_f(q^{2k_1})(a_f(q^{2n_1}) - a_f(q^{2n_1-2}))\\
&\quad a_f(r^{2k_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2}))\\
=&4a_f(p^{2l_1})a_f(p^{2l_2})T(p, n_1, n_2)L_q(k_1, n_1)L_r(k_2, n_2)\\
=&4k(p, n_1, n_2, l_1, l_2)L_q(k_1, n_1)L_r(k_2, n_2),
\end{aligned}$$

where for $n_1, n_2 \geq 1, l_1, l_2 \geq 0$ and for any prime r,

$$k(r, n_1, n_2, l_1, l_2) = a_f(r^{2l_1})a_f(r^{2l_2})T(r, n_1, n_2),$$

as defined earlier.

Now, $T(r, n_1, n_2) = T(r, n_2, n_1)$ implies $k(r, n_1, n_2, l_1, l_2) = k(r, n_2, n_1, l_1, l_2)$.

Hence, we have

$$k(r, n_1, n_2) = k(r, n_2, n_1). \tag{6.88}$$

For $n_1, n_2 \geq 1$ and for primes q, r, we define

$$L(q, r, n_1, n_2) := \sum_{k_1, k_2 \geq 0} U(k_1)U(k_2)L_q(k_1, n_1)L_r(k_2, n_2),$$

and

$$k(r, n_1, n_2) = \sum_{l_1, l_2 \geq 0} U(l_1)U(l_2)k(r, n_1, n_2, l_1, l_2),$$

as defined earlier.

We also note that

$$\begin{aligned}
&L(q, r, n_2, n_1) \\
=&\sum_{k_1, k_2 \geq 0} U(k_1)U(k_2)L_q(k_1, n_2)L_r(k_2, n_1)\\
=&\sum_{k_1, k_2 \geq 0} U(k_2)U(k_1)L_r(k_1, n_1)L_q(k_2, n_2)\\
=&L(r, q, n_1, n_2).
\end{aligned} \tag{6.89}$$

Therefore, using equations (6.88) and (6.89) in the third line, we obtain

$$\begin{aligned}
&\sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1,n_2 \ge 1 \\ n_2<n_1}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2) \qquad (6.90)\\
&= \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1',n_2' \ge 1 \\ n_1'<n_2'}} G(n_2')G(n_1')k(p,n_2',n_1')L(q,r,n_2',n_1')\\
&= \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1',n_2' \ge 1 \\ n_1'<n_2'}} G(n_2')G(n_1')k(p,n_1',n_2')L(r,q,n_1',n_2')\\
&= \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1',n_2' \ge 1 \\ n_1'<n_2'}} G(n_2')G(n_1')k(p,n_1',n_2')L(q,r,n_1',n_2')\\
&= \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1,n_2 \ge 1 \\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2),
\end{aligned}$$

where the fourth line is obtained by interchanging the variables q and r.

Using equation (6.90), we obtain

$$\begin{aligned}
&L_4(\rho,g;f)(x) \qquad (6.91)\\
&= \frac{1}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{l_1,l_2 \ge 0} \sum_{k_1,k_2 \ge 0} \sum_{n_1,n_2 \ge 1} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})A(p,q,n_1)A(p,r,n_2)\\
&= \frac{4}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{l_1,l_2 \ge 0} \sum_{k_1,k_2 \ge 0} \sum_{n_1,n_2 \ge 1} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)\\
&\qquad k(p,n_1,n_2,l_1,l_2)L_q(k_1,n_1)L_r(k_2,n_2)\\
&= \frac{4}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{n_1,n_2 \ge 1} G(n_1)G(n_2) \sum_{l_1,l_2 \ge 0} U(l_1)U(l_2)k(p,n_1,n_2,l_1,l_2)\\
&\qquad \sum_{k_1,k_2 \ge 0} U(k_1)U(k_2)L_q(k_1,n_1)L_r(k_2,n_2)\\
&= \frac{4}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{n_1,n_2 \ge 1} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2)\\
&= \frac{8}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1,n_2 \ge 1 \\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2)\\
&+ \frac{4}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1,n_2 \ge 1 \\ n_1=n_2}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2)\\
&= E(\rho,g;f)(x) + F(\rho,g;f)(x),
\end{aligned}$$

where

$$E(\rho,g;f)(x) := \frac{8}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1,n_2 \ge 1 \\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2), \qquad (6.92)$$

and

$$F(\rho,g;f)(x) := \frac{4}{16\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r \le x} \sum_{\substack{n_1,n_2 \ge 1 \\ n_1=n_2}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2). \qquad (6.93)$$

We now find an estimate for $E(\rho, g; f)(x)$. To estimate $\langle E(\rho, g; f)(x)\rangle$, we first find estimate for $L(q, r, n_1, n_2)$ in the following lemmas.

Lemma 6.4.17. *Let ρ, f, g be as defined earlier. Then for primes q, r, and integers $n_1, n_2 \geq 1$ with $n_2 \geq n_1$, we have*

$$\begin{aligned}
&L(q, r, n_1, n_2)\\
=&F(n_1)F(n_2) + F(n_1)\sum_{i=1}^{3} M_i(n_1, q) + F(n_2)\sum_{i=1}^{3} M_i(n_2, r) + \sum_{i=1}^{3}\sum_{j=1}^{3} M_i(n_1, q)M_j(n_2, r),
\end{aligned}$$

where $M_i(n, s) = M_i(\rho, g; f; n, s)$ $(i = 1, 2, 3)$, are given by

$$M_1(\rho, g; f; n, s) := \sum_{k \geq 0, k > n} U(k)a_f(s^{2k-2n}),$$

$$M_2(\rho, g; f; n, s) := \sum_{k \geq 0} U(k)a_f(s^{2k+2n}),$$

$$M_3(\rho, g; f; n, s) := \sum_{k \geq 0, k < n-1} U(k)a_f(s^{2n-2k-2}),$$

and

$$F(n) := U(n) - U(n-1),$$

for any positive integer n and any prime s.

Proof.

$$\begin{aligned}
&L(q, r, n_1, n_2) &(6.94)\\
=&\sum_{k_1, k_2 \geq 0} U(k_1)U(k_2)L_q(k_1, n_1)L_r(k_2, n_2)\\
=&\sum_{k_1 \geq 0} U(k_1)L_q(k_1, n_1)\sum_{k_2 \geq 0} U(k_2)L_r(k_2, n_2)\\
=&\left(\sum_{k_1 \geq 0, k_1 \geq n_1} U(k_1)L_q(k_1, n_1) + \sum_{k_1 \geq 0, k_1 < n_1} U(k_1)L_q(k_1, n_1)\right)\\
&\times\left(\sum_{k_2 \geq 0, k_2 \geq n_2} U(k_2)L_r(k_2, n_2) + \sum_{k_2 \geq 0, k_2 < n_2} U(k_2)L_r(k_2, n_2)\right)\\
=&(D_1 + D_2)(q, n_1) \times (D_1 + D_2)(r, n_2),
\end{aligned}$$

where $D_i(q, n_1)$, and $D_i(r, n_2)$, $(i = 1, 2)$ are defined in the following way.

For any prime s and any positive integer n,

$$D_1(s, n) := \sum_{k \geq 0, k \geq n} U(k)L_s(k, n, k \geq n),$$

and

$$D_2(s, n) := \sum_{k \geq 0, k < n} U(k)L_s(k, n, k < n),$$

We now estimate $D_1(s, n)$, for any prime s and positive integer n.

Using Corollary 3.3.5, we have

$$D_1(s, n) \qquad (6.95)$$

$$= \sum_{k \quad 0, k \quad n} U(k) L_s(k, n, k \quad n)$$
$$= \sum_{k \quad 0, k \quad n} U(k)(a_f(s^{2k \quad 2n}) + a_f(s^{2k+2n}))$$
$$= U(n) + U(n) a_f(s^{4n}) + \sum_{k \quad 0, k > n} U(k)(a_f(s^{2k \quad 2n}) + a_f(s^{2k+2n}))$$
$$= U(n) + \sum_{k \quad 0, k > n} U(k) a_f(s^{2k \quad 2n}) + \sum_{k \quad 0, k \quad n} U(k) a_f(s^{2k+2n}).$$

We now estimate $D_2(s, n)$, for any prime s and positive integer n.

Using Corollary 3.3.5, we have

$$D_2(s, n) \tag{6.96}$$
$$= \sum_{k \quad 0, k < n} U(k) L_s(k, n, k < n)$$
$$= \sum_{k \quad 0, k < n} U(k)(a_f(s^{2k+2n}) \quad a_f(s^{2n \quad 2k \quad 2}))$$
$$= \quad U(n \quad 1) + U(n \quad 1) a_f(s^{4n \quad 2}) + \sum_{k \quad 0, k < n \quad 1} U(k)(a_f(s^{2k+2n}) \quad a_f(s^{2n \quad 2k \quad 2}))$$
$$= \quad U(n \quad 1) + \sum_{k \quad 0, k < n} U(k) a_f(s^{2k+2n}) \quad \sum_{k \quad 0, k < n \quad 1} U(k) a_f(s^{2n \quad 2k \quad 2}).$$

Adding equations (6.95) and (6.96), we obtain

$$(D_1 + D_2)(s, n) \tag{6.97}$$
$$= \sum_{k \quad 0} U(k) L_s(k, n)$$
$$= U(n) + \sum_{k \quad 0, k > n} U(k) a_f(s^{2k \quad 2n}) + \sum_{k \quad 0, k \quad n} U(k) a_f(s^{2k+2n})$$
$$U(n \quad 1) + \sum_{k \quad 0, k < n} U(k) a_f(s^{2k+2n}) \quad \sum_{k \quad 0, k < n \quad 1} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= U(n) \quad U(n \quad 1) + \sum_{k \quad 0, k > n} U(k) a_f(s^{2k \quad 2n}) + \sum_{k \quad 0} U(k) a_f(s^{2k+2n})$$
$$\sum_{k \quad 0, k < n \quad 1} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= U(n) \quad U(n \quad 1) + \sum_{i=1}^{3} M_i(\rho, g; f; n, s),$$
$$= U(n) \quad U(n \quad 1) + \sum_{i=1}^{3} M_i(n, s)$$
$$= F(n) + \sum_{i=1}^{3} M_i(n, s),$$

where $F(n) := U(n) \quad U(n \quad 1)$.

Putting the estimate for $(D_1 + D_2)(s, n)$ in equation (6.94), we obtain

$$L(q, r, n_1, n_2)$$
$$= (D_1 + D_2)(q, n_1) \quad (D_1 + D_2)(r, n_2)$$

$$= \left(U(n_1) \quad U(n_1 \quad 1) + \sum_{i=1}^{3} M_i(n_1, q) \right) \left(U(n_2) \quad U(n_2 \quad 1) + \sum_{i=1}^{3} M_i(n_2, r) \right)$$
$$= (D_1 + D_2)(q, n_1) \quad (D_1 + D_2)(r, n_2)$$
$$= \big(U(n_1) \quad U(n_1 \quad 1)\big)\big(U(n_2) \quad U(n_2 \quad 1)\big)$$
$$+ \big(U(n_1) \quad U(n_1 \quad 1)\big) \sum_{i=1}^{3} M_i(n_1, q)$$
$$+ \big(U(n_2) \quad U(n_2 \quad 1)\big) \sum_{i=1}^{3} M_i(n_2, r)$$
$$+ \sum_{i=1}^{3} \sum_{j=1}^{3} M_i(n_1, q) M_j(n_2, r)$$
$$= F(n_1)F(n_2) + F(n_1) \sum_{i=1}^{3} M_i(n_1, q) + F(n_2) \sum_{i=1}^{3} M_i(n_2, r) + \sum_{i=1}^{3} \sum_{j=1}^{3} M_i(n_1, q) M_j(n_2, r).$$

□

Lemma 6.4.18. *For any prime s and positive integer n,*

$$\sum_{i=1}^{3} M_i(n, s) = \sum_{t=1}^{bLc+1+n} U(t, n) a_f(s^{2t}),$$

where

$$U(t, n) = \begin{cases} U_1(t, n), & \text{if } n = 1 \\ U_2(t, n), & \text{if } 2 \quad n \quad \frac{bLc+1}{2} \\ U_3(t, n), & \text{if } \frac{bLc+3}{2} \quad n \quad bLc \\ U_4(t, n), & \text{if } bLc + 1 \quad n \quad \pi_N(x). \end{cases}$$

Proof.

Let $n = 1$.

Hence, for any prime s,

$$\sum_{i=1}^{3} M_i(n, s) \tag{6.98}$$
$$= \sum_{k \quad 0, k>n} U(k) a_f(s^{2k \quad 2n}) + \sum_{k \quad 0} U(k) a_f(s^{2k+2n}) \quad \sum_{k \quad 0, k<n \quad 1} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= \sum_{k=n+1}^{bLc+1} U(k) a_f(s^{2k \quad 2n}) + \sum_{k=0}^{bLc+1} U(k) a_f(s^{2k+2n})$$
$$= \sum_{t=1}^{bLc+1 \quad n} U(t+n) a_f(s^{2t}) + \sum_{t=n}^{bLc+1+n} U(t \quad n) a_f(s^{2t})$$
$$= \sum_{t=1}^{bLc+1+n} U_1(t, n) a_f(s^{2t}),$$

where

$$U_1(t, n) = \begin{cases} U(t+n) + U(t \quad n), & \text{if } (1 =)n \quad t \quad bLc + 1 \quad n \\ U(t \quad n), & \text{if } bLc + 2 \quad n \quad t \quad bLc + 1 + n. \end{cases}$$

Let $2 \quad n \quad \frac{bLc+1}{2}$.

Hence, for any prime s,

$$\sum_{i=1}^{3} M_i(n,s) \tag{6.99}$$
$$= \sum_{k \quad 0, k>n} U(k) a_f(s^{2k \quad 2n}) + \sum_{k \quad 0} U(k) a_f(s^{2k+2n}) \quad \sum_{k \quad 0, k<n \quad 1} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= \sum_{k=n+1}^{bLc+1} U(k) a_f(s^{2k \quad 2n}) + \sum_{k=0}^{bLc+1} U(k) a_f(s^{2k+2n}) \quad \sum_{k=0}^{n \quad 2} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= \sum_{t=1}^{bLc+1 \quad n} U(t+n) a_f(s^{2t}) + \sum_{t=n}^{bLc+1+n} U(t \quad n) a_f(s^{2t}) \quad \sum_{t=1}^{n \quad 1} U(n \quad t \quad 1) a_f(s^{2t})$$
$$= \sum_{t=1}^{bLc+1+n} U_2(t,n) a_f(s^{2t}),$$

where

$$U_2(t,n) = \begin{cases} U(t+n) \quad U(n \quad t \quad 1), & \text{if } 1 \quad t \quad n \quad 1 \\ U(t+n) + U(t \quad n), & \text{if } n \quad t \quad bLc+1 \quad n \\ U(t \quad n), & \text{if } bLc+2 \quad n \quad t \quad bLc+1+n. \end{cases}$$

Let $\frac{bLc+3}{2} \quad n \quad bLc$.

Hence, for any prime s,

$$\sum_{i=1}^{3} M_i(n,s) \tag{6.100}$$
$$= \sum_{k \quad 0, k>n} U(k) a_f(s^{2k \quad 2n}) + \sum_{k \quad 0} U(k) a_f(s^{2k+2n}) \quad \sum_{k \quad 0, k<n \quad 1} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= \sum_{k=n+1}^{bLc+1} U(k) a_f(s^{2k \quad 2n}) + \sum_{k=0}^{bLc+1} U(k) a_f(s^{2k+2n}) \quad \sum_{k=0}^{n \quad 2} U(k) a_f(s^{2n \quad 2k \quad 2})$$
$$= \sum_{t=1}^{bLc+1 \quad n} U(t+n) a_f(s^{2t}) + \sum_{t=n}^{bLc+1+n} U(t \quad n) a_f(s^{2t}) \quad \sum_{t=1}^{n \quad 1} U(n \quad t \quad 1) a_f(s^{2t})$$
$$= \sum_{t=1}^{bLc+1+n} U_3(t,n) a_f(s^{2t}),$$

where

$$U_3(t,n) := \begin{cases} \quad U(n \quad t \quad 1) + U(t+n), & \text{if } 1 \quad t \quad bLc+1 \quad n \\ \quad U(n \quad t \quad 1), & \text{if } bLc+2 \quad n \quad t \quad n \quad 1 \\ U(t \quad n), & \text{if } n \quad t \quad bLc+1+n, \end{cases}$$

Let $bLc + 1 \quad n \quad \pi_N(x)$. Then, $n + 1 \quad bLc + 2$, and hence, $U(k) = 0$, for all $k \quad n + 1$.

Hence, for any prime s,

$$\sum_{i=1}^{3} M_i(n,s) \tag{6.101}$$

$$
\begin{aligned}
&= \sum_{k\ge 0, k>n} U(k)a_f(s^{2k-2n}) + \sum_{k\ge 0} U(k)a_f(s^{2k+2n}) - \sum_{k\ge 0, k<n-1} U(k)a_f(s^{2n-2k-2})\\
&= \sum_{k=0}^{\lfloor L\rfloor+1} U(k)a_f(s^{2k+2n}) - \sum_{k=0}^{n-2} U(k)a_f(s^{2n-2k-2})\\
&= \sum_{t=n}^{\lfloor L\rfloor+1+n} U(t-n)a_f(s^{2t}) - \sum_{t=1}^{n-1} U(n-t-1)a_f(s^{2t})\\
&= \sum_{t=1}^{\lfloor L\rfloor+1+n} U_4(t,n)a_f(s^{2t}),
\end{aligned}
$$

where

$$
U_4(t,n) = \begin{cases} -U(n-t-1), & \text{if } 1\le t\le n-1\\ U(t-n), & \text{if } n\le t\le \lfloor L\rfloor+1+n. \end{cases}
$$

Hence, for $1\le n\le \pi_N(x)$, combining equations (6.98), (6.99), (6.100) and (6.101), we obtain

$$
\sum_{i=1}^{3} M_i(n,s) = \sum_{t=1}^{\lfloor L\rfloor+1+n} U(t,n)a_f(s^{2t}),
$$

where

$$
U(t,n) = \begin{cases} U_1(t,n), & \text{if } n=1\\ U_2(t,n), & \text{if } 2\le n\le \frac{\lfloor L\rfloor+1}{2}\\ U_3(t,n), & \text{if } \frac{\lfloor L\rfloor+3}{2}\le n\le \lfloor L\rfloor\\ U_4(t,n), & \text{if } \lfloor L\rfloor+1\le n\le \pi_N(x). \end{cases}
$$

□

Corollary 6.4.19. *Let s be any prime and n be a positive integer such that $1\le n\le \pi_N(x)$. Then,*

$$
\sum_{k\ge 0} U(k)L_s(k,n) = F(n) + \sum_{t=1}^{\lfloor L\rfloor+1+n} U(t,n)a_f(s^{2t}),
$$

where $F(n) = U(n) - U(n-1)$ and for $1\le n\le \pi_N(x)$ and $1\le t\le \lfloor L\rfloor+1+n$,

$$
|U(t,n)| \le 12k_1.
$$

Proof. Using equation (6.97) and Lemma 6.4.18, we obtain

$$
\begin{aligned}
&\sum_{k\ge 0} U(k)L_s(k,n)\\
=&F(n) + \sum_{i=1}^{3} M_i(n,s)\\
=&F(n) + \sum_{t=1}^{\lfloor L\rfloor+1+n} U(t,n)a_f(s^{2t}),
\end{aligned}
$$

where $F(n) = U(n) - U(n-1)$ and

$$
U(t,n) = \begin{cases} U_1(t,n), & \text{if } n=1\\ U_2(t,n), & \text{if } 2\le n\le \frac{\lfloor L\rfloor+1}{2}\\ U_3(t,n), & \text{if } \frac{\lfloor L\rfloor+3}{2}\le n\le \lfloor L\rfloor\\ U_4(t,n), & \text{if } \lfloor L\rfloor+1\le n\le \pi_N(x). \end{cases}
$$

We note that for $l \quad 0$, $jU(l)j = j\widehat{\rho}\left(\frac{l}{L}\right) 2\cos 2\pi l\psi \quad \widehat{\rho}\left(\frac{l+1}{L}\right) 2\cos 2\pi(l+1)\psi j \quad 2k_1 + 2k_1 = 4k_1$, for some $k_1 > 0$ (In particular, $k_1 = 1$ with our assumption). Hence, for $1 \quad n \quad \pi_N(x)$ and $1 \quad t \quad bLc + 1 + n$, $jU(t,n)j \quad 12k_1$. □

Also, using Proposition 6.3.8, we have

$$k(p,n_1,n_2) = \sum_{i=1}^{4} E_i(n_1,n_2) + \sum_{i=1}^{9} V_i(n_1,n_2,p),$$

where $E_i(n_1,n_2)$ ($i = 1,2,3,4$) and $V_j(n_1,n_2,p)$ ($j = 1, \quad ,9$), are mentioned in Proposition 6.3.8 and Lemma 6.3.7 respectively.

Hence,

$$\begin{aligned}
&k(p,n_1,n_2) \quad L(q,r,n_1,n_2)\\
&= \left(F(n_1)F(n_2) + F(n_1)\sum_{i=1}^{3} M_i(n_1,q) + F(n_2)\sum_{i=1}^{3} M_i(n_2,r) + \sum_{i=1}^{3}\sum_{j=1}^{3} M_i(n_1,q)M_j(n_2,r)\right)\\
&\quad\left(\sum_{l=1}^{4} E_l(n_1,n_2) + \sum_{l=1}^{9} V_l(n_1,n_2,p)\right)\\
&= F(n_2)F(n_1)\sum_{l=1}^{4} E_l(n_1,n_2) + F(n_2)F(n_1)\sum_{l=1}^{9} V_l(n_1,n_2,p)\\
&+ F(n_2)\sum_{i=1}^{3}\sum_{l=1}^{4} M_i(n_2,r)E_l(n_1,n_2) + F(n_2)\sum_{i=1}^{3}\sum_{l=1}^{9} M_i(n_2,r)V_l(n_1,n_2,p)\\
&+ F(n_1)\sum_{i=1}^{3}\sum_{l=1}^{4} M_i(n_1,q)E_l(n_1,n_2) + F(n_1)\sum_{i=1}^{3}\sum_{l=1}^{9} M_i(n_1,q)V_l(n_1,n_2,p)\\
&+ \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{4} M_i(n_1,q)M_j(n_2,r)E_l(n_1,n_2) + \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{9} M_i(n_1,q)M_j(n_2,r)V_l(n_1,n_2,p)\\
&= \sum_{t=1}^{8} H_t(p,q,r,n_1,n_2),
\end{aligned} \tag{6.102}$$

where $H_t(p,q,r,n_1,n_2)$ ($t = 1, \quad ,8$) are defined in respective order by the terms in the previous line.

Hence, using equation (6.92), we obtain

$$\begin{aligned}
&\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}} E(\rho,g;f)(x)\\
&= \frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}} \frac{8}{16\pi_N(x)^4L^2}\sum_{p,q,r}{}^{0} \sum_{\substack{x \; n_1,n_2 \; 1\\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2)\\
&= \sum_{t=1}^{8}\frac{1}{jF_{N,k}j}\sum_{f2F_{N,k}} \frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r}{}^{0} \sum_{\substack{x \; n_1,n_2 \; 1\\ n_2>n_1}} G(n_1)G(n_2)H_t(p,q,r,n_1,n_2)\\
&= \sum_{t=1}^{8}\beta_t(x), \text{ where}
\end{aligned} \tag{6.103}$$

$$\beta_t(x) = \frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r}{}^{0} \sum_{\substack{x \; n_1,n_2 \; 1\\ n_2>n_1}} G(n_1)G(n_2)hH_t(p,q,r,n_1,n_2)i, \tag{6.104}$$

for all $i = 1, \quad , 8$.

Proposition 6.4.20. *With* $\beta_1(x)$ *defined in equation* (6.104), *we have* $\beta_1(x) \quad \frac{L}{\pi_N(x)}$.

Proof. With the observations made in Lemma 6.3.14, we note that for $i = 1, 2, 3, 4$,

$$|E_i(n_1, n_2)| \quad 64k_1^2 L,$$

and also, $|F(n)| \quad 8k_1$, for a positive integer n.

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) \quad U(n \quad 1) = 0$.

Using equation (6.102) and equation (6.103), we obtain

$$\begin{aligned}
&\beta_1(x)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2}\sum_{p,q,r}{}^{0} \sum_{x} \sum_{\substack{n_1,n_2 \quad 1\\ n_2>n_1}} G(n_1)G(n_2)H_1(p,q,r,n_1,n_2)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2}\sum_{p,q,r}{}^{0} \sum_{x} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)F(n_2)F(n_1)\sum_{l=1}^{4} E_l(n_1,n_2)\\
&\quad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{\pi_N(x)^4 L^2}\sum_{p,q,r}{}^{0} \sum_{x} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{1+bLc} \sum_{l=1}^{4} L\\
&\quad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{\pi_N(x)^4 L^2}\pi_N(x)^3 L^3\\
&\quad \frac{L}{\pi_N(x)}.
\end{aligned}$$

Hence,

$$\beta_1(x) \quad \frac{L}{\pi_N(x)}. \tag{6.105}$$

□

Proposition 6.4.21. *With* $\beta_2(x)$ *defined in equation* (6.104), *we have*

$$\beta_2(x) \quad \frac{L(\log\log x)}{\pi_N(x)^2} + L^2\frac{8^{\nu(N)}x^{12Lc^0}}{kN}.$$

Proof. With the observations made in Lemma 6.3.14, we note that for $j = 1, \quad , 9$,

$$|V_j(n_1, n_2, p)| \quad 64k_1^2(L)^2,$$

and also, $|F(n)| \quad 8k_1$, for a positive integer n.

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) \quad U(n \quad 1) = 0$.

Using Lemma 6.3.13, we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2}\sum_{p,q,r}{}^{0} \sum_{x} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)\\
&\qquad F(n_2)F(n_1)V_1(n_1,n_2,p)\\
&= \frac{1}{2\pi_N(x)^4 L^2}\sum_{p,q,r}{}^{0} \sum_{x} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)\\
&\qquad F(n_2)F(n_1)\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} V_1(n_1,n_2,p)\right)
\end{aligned}$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{n_1=1}^{1+bLc}\sum_{n_2=n_1+1}^{1+bLc}\sideset{}{'}\sum_{p,q,r\quad x}\left(\frac{1}{p}L+(L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4n_2c'}}{kN}\right)$$
$$\frac{1}{\pi_N(x)^4L^2}\sum_{n_1=1}^{1+bLc}\sum_{n_2=n_1+1}^{1+bLc}\sideset{}{'}\sum_{p,q,r\quad x}\frac{1}{p}L$$
$$+\frac{1}{\pi_N(x)^4L^2}\sum_{n_1=1}^{1+bLc}\sum_{n_2=n_1+1}^{1+bLc}\sideset{}{'}\sum_{p,q,r\quad x}(L+\pi_N(x))L^2\frac{8^{\nu(N)}x^{4Lc'+4(L+1)c'}}{kN}$$
$$\frac{1}{\pi_N(x)^4L^2}L^3\sideset{}{'}\sum_{p,q,r\quad x}\frac{1}{p}+\frac{1}{\pi_N(x)^4L^2}L^2\pi_N(x)^3\pi_N(x)L^2\frac{8^{\nu(N)}x^{12Lc'}}{kN}$$
$$\frac{1}{\pi_N(x)^4}L(\log\log x)\pi_N(x)^2+L^2\frac{8^{\nu(N)}x^{12Lc'}}{kN}$$
$$\frac{L(\log\log x)}{\pi_N(x)^2}+L^2\frac{8^{\nu(N)}x^{12Lc'}}{kN},$$

using $L \quad \pi_N(x)$ in the last third line.

Since other sums also give the same estimate, using equation (6.102) and equation (6.103), we obtain

$$\beta_2(x)$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\quad x}\sum_{\substack{n_1,n_2\quad 1\\ n_2>n_1}}G(n_1)G(n_2)H_2(p,q,r,n_1,n_2)$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r\quad x}\sum_{n_1=1}^{1+bLc}\sum_{n_2=n_1+1}^{1+bLc}G(n_1)G(n_2)$$
$$F(n_2)F(n_1)\sum_{l=1}^{9}V_l(n_1,n_2,p)$$
$$\frac{L(\log\log x)}{\pi_N(x)^2}+L^2\frac{8^{\nu(N)}x^{12Lc'}}{kN}.$$

Hence,

$$\beta_2(x)\qquad\frac{L(\log\log x)}{\pi_N(x)^2}+L^2\frac{8^{\nu(N)}x^{12Lc'}}{kN}. \tag{6.106}$$

□

Proposition 6.4.22. *With $\beta_3(x)$ defined in equation* (6.104)*, we have*

$$\beta_3(x)\qquad\frac{\log\log x}{\pi_N(x)}+L\frac{8^{\nu(N)}x^{6Lc'}}{kN}.$$

Proof. We know, for $l \quad 0$, $|U(l)|=|\widehat{\rho}\left(\frac{l}{L}\right)2\cos 2\pi l\psi \quad \widehat{\rho}\left(\frac{l+1}{L}\right)2\cos 2\pi(l+1)\psi| \quad 2k_1+2k_1=4k_1$, for some $k_1>0$ and $U(l)=0$ for $l>bLc$. Therefore, $|E_i(n_1,n_2)| \quad 32k_1^2(L+1) \quad 64k_1^2L$, and also, $\left|F(n)\right| \quad 8k_1$, for a positive integer n.

Now, $n>1+bLc$, implies $U(n)=0$ and hence, $F(n)=U(n) \quad U(n \quad 1)=0$.

Using equation (6.102) and equation (6.103), we obtain

$$\beta_3(x)$$

$$= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r\quad x}{}' \sum_{\substack{n_1,n_2\ \ 1\\ n_2>n_1}} G(n_1)G(n_2)H_3(p,q,r,n_1,n_2)$$

$$= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r\quad x}{}' \sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} G(n_1)G(n_2)$$

$$F(n_2)\sum_{i=1}^{3}\sum_{l=1}^{4}M_i(n_2,r)E_l(n_1,n_2)$$

$$= \sum_{i=1}^{3}\sum_{l=1}^{4}\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} G(n_1)G(n_2)F(n_2)E_l(n_1,n_2)$$

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\quad x}{}' M_i(n_2,r).$$

It is enough to find an estimate for

$$\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} G(n_1)G(n_2)F(n_2)E_l(n_1,n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\quad x}{}' M_2(n_2,r),$$

for each $l\in\{1,2,3,4\}$, since the other two sums corresponding to $M_1(n_2,r)$ and $M_3(n_2,r)$ can be estimated similarly and give the same estimate.

We have,

$$\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} G(n_1)G(n_2)F(n_2)E_l(n_1,n_2)$$

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\quad x}{}' M_2(n_2,r)$$

$$= \frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} G(n_1)G(n_2)F(n_2)E_l(n_1,n_2)$$

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\quad x}{}'\sum_{k_2\ \ 0} U(k_2)a_f(r^{2k_2+2n_2})$$

$$= \frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} G(n_1)G(n_2)F(n_2)E_l(n_1,n_2)\sum_{p,q,r\quad x}{}'$$

$$\sum_{k_2\ \ 0} U(k_2)\left\{\frac{1}{r^{k_2+n_2}} + \mathrm{O}\left(\frac{8^{\nu(N)}r^{2k_2c'+2n_2c'}}{kN}\right)\right\}$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor} L\sum_{p,q,r\quad x}{}'$$

$$\sum_{k_2\ \ 0}\left\{\frac{1}{r^{k_2+n_2}} + \left(\frac{8^{\nu(N)}r^{2k_2c'+2n_2c'}}{kN}\right)\right\}$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)} L\sum_{p,q,r\quad x}{}'$$

$$\sum_{n_2=n_1+1}^{1+\lfloor Lc\rfloor}\left\{\left(\frac{1}{r^{n_2}} + \frac{1}{r^{n_2+1}} + \quad\right) + \left(L\frac{8^{\nu(N)}r^{2Lc'+2n_2c'}}{kN}\right)\right\}$$

$$
\begin{aligned}
&\ll \frac{L}{\pi_N(x)^4 L^2} \sum_{n_1=1}^{\pi_N(x)} \sum_{p,q,r}{}' \sum_{x} \left\{ L\left(\frac{1}{r} + \frac{1}{r^2} + \quad \right) + \left(L^2 \frac{8^{\nu(N)} r^{2Lc' + 2Lc' + 2c'}}{kN} \right) \right\} \\
&\ll \frac{1}{\pi_N(x)^3 L} \left\{ L \sum_{p,q,r}{}' \sum_{x} \frac{1}{r} + \sum_{p,q,r}{}' \sum_{x} \left(L^2 \frac{8^{\nu(N)} r^{2Lc' + 2Lc' + 2c'}}{kN} \right) \right\} \\
&\ll \frac{1}{\pi_N(x)^3 L} \left\{ L\pi_N(x)^2 \log\log x + \pi_N(x)^3 \left(L^2 \frac{8^{\nu(N)} x^{6Lc'}}{kN} \right) \right\} \\
&\ll \frac{\log\log x}{\pi_N(x)} + L \frac{8^{\nu(N)} x^{6Lc'}}{kN}.
\end{aligned}
$$

Hence,

$$
\beta_3(x) \ll \frac{\log\log x}{\pi_N(x)} + L \frac{8^{\nu(N)} x^{6Lc'}}{kN}. \tag{6.107}
$$

□

Proposition 6.4.23. *With* $\beta_4(x)$ *defined in equation* (6.104)*, we have*

$$
\beta_4(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2} + \pi_N(x) L^2 \frac{8^{\nu(N)} x^{6Lc' + 6\pi_N(x)c'}}{kN}.
$$

Proof. We know, for $l \geq 0$, $|U(l)| \leq 4k_1$, for some $k_1 > 0$ and $U(l) = 0$ for $l > bLc$.

Also, $|F(n)| \leq 8k_1$, for a positive integer n.

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) - U(n-1) = 0$.

Using equation (6.102) and equation (6.103), we obtain

$$
\begin{aligned}
&\beta_4(x) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2} \sum_{p,q,r}{}' \sum_{x} \sum_{\substack{n_1, n_2 \geq 1 \\ n_2 > n_1}} G(n_1) G(n_2) H_4(p,q,r,n_1,n_2) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2} \sum_{p,q,r}{}' \sum_{x} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{1+bLc} G(n_1) G(n_2) \\
&\qquad F(n_2) \sum_{i=1}^{3} \sum_{l=1}^{9} M_i(n_2, r) V_l(n_1, n_2, p) \\
&= \sum_{i=1}^{3} \sum_{l=1}^{9} \frac{1}{2\pi_N(x)^4 L^2} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{1+bLc} G(n_1) G(n_2) F(n_2) \\
&\qquad \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{p,q,r}{}' \sum_{x} M_i(n_2, r) V_l(n_1, n_2, p).
\end{aligned}
$$

It is enough to find an estimate for

$$
\begin{aligned}
&\frac{1}{2\pi_N(x)^4 L^2} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{1+bLc} G(n_1) G(n_2) F(n_2) \\
&\qquad \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{p,q,r}{}' \sum_{x} M_2(n_2, r) V_1(n_1, n_2, p),
\end{aligned}
$$

since the other sums can be estimated similarly and give the same upper bound.

We have,

$$\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)F(n_2)$$

$$\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\sum_{p,q,r\quad x}^{0} M_2(n_2,r)V_1(n_1,n_2,p)$$

$$=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)F(n_2)\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\sum_{p,q,r\quad x}^{0}$$

$$\sum_{k_2\quad 0} U(k_2)a_f(r^{2k_2+2n_2})\sum_{\substack{l_1,l_2\quad 0\\ l_1\quad l_2 6= n_2\quad n_1}} U(l_1)U(l_2)a_f(p^{2l_1+2n_1})a_f(p^{2l_2+2n_2})$$

$$=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)F(n_2)\sum_{p,q,r\quad x}^{0}$$

$$\sum_{\substack{l_1,l_2,k_2\quad 0\\ l_1\quad l_2 6= n_2\quad n_1}} U(l_1)U(l_2)U(k_2)\sum_{t=j(l_1\quad l_2)\quad (n_2\quad n_1)j}^{l_1+l_2+n_1+n_2}\left(\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}} a_f(p^{2t}r^{2k_2+2n_2})\right)$$

$$=\frac{1}{\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+bLc} G(n_1)G(n_2)F(n_2)\sum_{p,q,r\quad x}^{0}$$

$$\sum_{\substack{l_1,l_2,k_2\quad 0\\ l_1\quad l_2 6= n_2\quad n_1}} U(l_1)U(l_2)U(k_2)\sum_{t=j(l_1\quad l_2)\quad (n_2\quad n_1)j}^{l_1+l_2+n_1+n_2}\left\{\frac{1}{p^tr^{k_2+n_2}} + \mathrm{O}\left(\frac{8^{\nu(N)}p^{2tc^0}r^{2k_2c^0+2n_2c^0}}{kN}\right)\right\}$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\quad x}^{0}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+bLc}\sum_{\substack{l_1,l_2\quad 0\\ l_1\quad l_2 6= n_2\quad n_1}}$$

$$\left\{\left(\frac{1}{p^{j(l_1\quad l_2)\quad (n_2\quad n_1)j}} + \frac{1}{p^{j(l_1\quad l_2)\quad (n_2\quad n_1)j+1}} + \quad\right)\left(\frac{1}{r^{n_2}} + \frac{1}{r^{n_2+1}} + \quad\right)\right.$$

$$\left. + \left(L(l_1 + l_2 + n_1 + n_2)\frac{8^{\nu(N)}p^{2(l_1+l_2+n_1+n_2)c^0}r^{2Lc^0+2n_2c^0}}{kN}\right)\right\}$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\quad x}^{0}\left\{\left(\frac{1}{p} + \frac{1}{p^2} + \quad\right)L\left(\frac{1}{r} + \frac{1}{r^2} + \quad\right)L\pi_N(x)\right.$$

$$\left. + \left(L^3\pi_N(x)\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{1+bLc}\frac{8^{\nu(N)}p^{4Lc^0+4n_2c^0}r^{2Lc^0+2n_2c^0}}{kN}\right)\right\}$$

$$\frac{1}{\pi_N(x)^3}\sum_{p,q,r\quad x}^{0}\frac{1}{pr} + \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\quad x}^{0}\left(\pi_N(x)^2L^4\frac{8^{\nu(N)}x^{6Lc^0+6\pi_N(x)c^0}}{kN}\right)$$

$$\frac{1}{\pi_N(x)^3}\pi_N(x)(\log\log x)^2 + \left(\pi_N(x)L^2\frac{8^{\nu(N)}x^{6Lc^0+6\pi_N(x)c^0}}{kN}\right)$$

$$\frac{(\log\log x)^2}{\pi_N(x)^2} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{6Lc^0+6\pi_N(x)c^0}}{kN},$$

using $L \quad \pi_N(x)$ in the last fourth line.

Hence,

$$\beta_4(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{6Lc^0+6\pi_N(x)c^0}}{kN}. \tag{6.108}$$

□

Proposition 6.4.24. *With* $\beta_5(x)$ *defined in equation* (6.104), *we have*

$$\beta_5(x) \ll \frac{\log\log x}{\pi_N(x)} + L\frac{8^{\nu(N)}x^{6Lc^0}}{kN}.$$

Proof. We know, for $l \geq 0$, $|U(l)| = |\hat{\rho}\left(\frac{l}{L}\right)2\cos 2\pi l\psi - \hat{\rho}\left(\frac{l+1}{L}\right)2\cos 2\pi(l+1)\psi| \leq 2k_1 + 2k_1 = 4k_1$, for some $k_1 > 0$ and $U(l) = 0$ for $l > \lfloor Lc \rfloor$. Therefore, $|E_i(n_1,n_2)| \leq 32k_1^2(L+1) \leq 64k_1^2 L$, and also, $|F(n)| \leq 8k_1$, for a positive integer n.

Now, $n > 1 + \lfloor Lc \rfloor$, implies $U(n) = 0$ and hence, $F(n) = U(n) - U(n-1) = 0$.

Using equation (6.102) and equation (6.103), we obtain

$$\begin{aligned}
&\beta_5(x)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r\leq x}{}' \sum_{\substack{n_1,n_2\geq 1\\ n_2>n_1}} G(n_1)G(n_2)H_5(p,q,r,n_1,n_2)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r\leq x}{}' \sum_{n_1=1}^{1+\lfloor Lc\rfloor}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\\
&\qquad F(n_1)\sum_{i=1}^{3}\sum_{l=1}^{4}M_i(n_1,q)E_l(n_1,n_2)\\
&= \sum_{i=1}^{3}\sum_{l=1}^{4}\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{1+\lfloor Lc\rfloor}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1)E_l(n_1,n_2)\\
&\qquad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\leq x}{}' M_i(n_1,q).
\end{aligned}$$

It is enough to find an estimate for

$$\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{1+\lfloor Lc\rfloor}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1)E_l(n_1,n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\leq x}{}' M_2(n_1,q),$$

for each $l \in \{1,2,3,4\}$, since the other two sums corresponding to $M_1(n_1,q)$ and $M_3(n_1,q)$ can be estimated similarly and give the same estimate.

We have,

$$\begin{aligned}
&\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{1+\lfloor Lc\rfloor}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1)E_l(n_1,n_2)\\
&\qquad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\leq x}{}' M_2(n_1,q)\\
&= \frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{1+\lfloor Lc\rfloor}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1)E_l(n_1,n_2)\\
&\qquad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\leq x}{}' \sum_{k_2\geq 0} U(k_2)a_f(q^{2k_2+2n_1})
\end{aligned}$$

$$= \frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1)E_l(n_1,n_2) \sum_{p,q,r \quad x}{}' $$

$$\sum_{k_2 \quad 0} U(k_2) \left\{ \frac{1}{q^{k_2+n_1}} + O\left(\frac{8^{\nu(N)} q^{2k_2c' + 2n_1c'}}{kN} \right) \right\}$$

$$\frac{1}{\pi_N(x)^4L^2} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{\pi_N(x)} L \sum_{p,q,r \quad x}{}' \sum_{k_2 \quad 0} \left\{ \frac{1}{q^{k_2+n_1}} + \left(\frac{8^{\nu(N)} q^{2k_2c' + 2n_1c'}}{kN} \right) \right\}$$

$$\frac{1}{\pi_N(x)^4L^2} \sum_{n_1=1}^{1+bLc} L \sum_{p,q,r \quad x}{}'$$

$$\sum_{n_2=n_1+1}^{\pi_N(x)} \left\{ \left(\frac{1}{q^{n_1}} + \frac{1}{q^{n_1+1}} + \quad \right) + \left(L \frac{8^{\nu(N)} q^{2Lc' + 2n_1c'}}{kN} \right) \right\}$$

$$\frac{L}{\pi_N(x)^4L^2} \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{\pi_N(x)} \sum_{p,q,r \quad x}{}' \left\{ \left(\frac{1}{q} + \frac{1}{q^2} + \quad \right) + \left(L \frac{8^{\nu(N)} q^{2Lc' + 2Lc' + 2c'}}{kN} \right) \right\}$$

$$\frac{L}{\pi_N(x)^4L^2} L\pi_N(x) \left\{ \sum_{p,q,r \quad x}{}' \frac{1}{q} + \sum_{p,q,r \quad x}{}' \left(L \frac{8^{\nu(N)} q^{2Lc' + 2Lc' + 2c'}}{kN} \right) \right\}$$

$$\frac{1}{\pi_N(x)^3} \left\{ \pi_N(x)^2 \log\log x + \pi_N(x)^3 \left(L \frac{8^{\nu(N)} x^{6Lc'}}{kN} \right) \right\}$$

$$\frac{\log\log x}{\pi_N(x)} + L\frac{8^{\nu(N)} x^{6Lc'}}{kN}.$$

Hence,

$$\beta_5(x) \qquad \frac{\log\log x}{\pi_N(x)} + L\frac{8^{\nu(N)} x^{6Lc'}}{kN}. \tag{6.109}$$

□

Proposition 6.4.25. *With* $\beta_6(x)$ *defined in equation* (6.104)*, we have*

$$\beta_6(x) \qquad \frac{(\log\log x)^2}{\pi_N(x)^2} + \pi_N(x)L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN}.$$

Proof. We know, for $l \quad 0$, $|U(l)| \quad 4k_1$, for some $k_1 > 0$ and $U(l) = 0$ for $l > bLc$.

Also, $\left|F(n)\right| \quad 8k_1$, for a positive integer n.

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) \quad U(n \quad 1) = 0$.

Using equation (6.102) and equation (6.103), we obtain

$$\beta_6(x)$$

$$= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{2\pi_N(x)^4L^2} \sum_{p,q,r \quad x}{}' \sum_{\substack{n_1,n_2 \quad 1 \\ n_2 > n_1}} G(n_1)G(n_2)H_6(p,q,r,n_1,n_2)$$

$$= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{2\pi_N(x)^4L^2} \sum_{p,q,r \quad x}{}' \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)$$

$$F(n_1) \sum_{i=1}^{3} \sum_{l=1}^{9} M_i(n_1,q)V_l(n_1,n_2,p)$$

$$= \sum_{i=1}^{3}\sum_{l=1}^{9} \frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1) \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} \sideset{}{'}\sum_{p,q,r\le x} M_i(n_1,q)V_l(n_1,n_2,p).$$

It is enough to find an estimate for

$$\frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1) \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} \sideset{}{'}\sum_{p,q,r\le x} M_2(n_1,q)V_1(n_1,n_2,p),$$

since the other sums can be estimated similarly and give the same upper bound.

We have,

$$\frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1) \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} \sideset{}{'}\sum_{p,q,r\le x} M_2(n_1,q)V_1(n_1,n_2,p)$$

$$= \frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1) \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} \sideset{}{'}\sum_{p,q,r\le x} \sum_{k_2\ge 0} U(k_2)a_f(q^{2k_2+2n_1}) \sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\ne n_2-n_1}} U(l_1)U(l_2)a_f(p^{2l_1+2n_1})a_f(p^{2l_2+2n_2})$$

$$= \frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1) \sideset{}{'}\sum_{p,q,r\le x} \sum_{\substack{l_1,l_2,k_2\ge 0\\ l_1-l_2\ne n_2-n_1}} U(l_1)U(l_2)U(k_2) \sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2} \left(\frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} a_f(p^{2t}q^{2k_2+2n_1}) \right)$$

$$= \frac{1}{2\pi_N(x)^4L^2} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)F(n_1) \sideset{}{'}\sum_{p,q,r\le x} \sum_{\substack{l_1,l_2,k_2\ge 0\\ l_1-l_2\ne n_2-n_1}} U(l_1)U(l_2)U(k_2) \sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2} \left\{ \frac{1}{p^tq^{k_2+n_1}} + O\left(\frac{8^{\nu(N)}p^{2tc'}q^{2k_2c'+2n_1c'}}{kN} \right) \right\}$$

$$\ll \frac{1}{\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r\le x} \sum_{n_1=1}^{1+\lfloor L\rfloor} \sum_{n_2=n_1+1}^{\pi_N(x)} \sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\ne n_2-n_1}} \left\{ \left(\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|}} + \frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|+1}} + \cdots \right)\left(\frac{1}{q^{n_1}} + \frac{1}{q^{n_1+1}} + \cdots \right) + \left(L(l_1+l_2+n_1+n_2)\frac{8^{\nu(N)}p^{2(l_1+l_2+n_1+n_2)c'}q^{2Lc'+2n_1c'}}{kN} \right) \right\}$$

$$\ll \frac{1}{\pi_N(x)^4L^2} \sideset{}{'}\sum_{p,q,r\le x} \left\{ \left(\frac{1}{p} + \frac{1}{p^2} + \cdots \right)\left(\frac{1}{q} + \frac{1}{q^2} + \cdots \right) L^2\pi_N(x)\right.$$

$$+ \left(L^3 \pi_N(x) \sum_{n_1=1}^{1+bLc} \sum_{n_2=n_1+1}^{\pi_N(x)} \frac{8^{\nu(N)}(pq)^{4Lc^0+4\pi_N(x)c^0}}{kN} \right) \Bigg\}$$

$$\frac{1}{\pi_N(x)^3} \sum_{p,q,r \quad x}^{0} \frac{1}{pq} + \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r \quad x}^{0} \left(\pi_N(x)^2 L^4 \frac{8^{\nu(N)} x^{8Lc^0+8\pi_N(x)c^0}}{kN} \right)$$

$$\frac{1}{\pi_N(x)^3} \pi_N(x)(\log\log x)^2 + \left(\pi_N(x) L^2 \frac{8^{\nu(N)} x^{8Lc^0+8\pi_N(x)c^0}}{kN} \right)$$

$$\frac{(\log\log x)^2}{\pi_N(x)^2} + \pi_N(x) L^2 \frac{8^{\nu(N)} x^{8Lc^0+8\pi_N(x)c^0}}{kN}.$$

Hence,

$$\beta_6(x) \qquad \frac{(\log\log x)^2}{\pi_N(x)^2} + \pi_N(x) L^2 \frac{8^{\nu(N)} x^{8Lc^0+8\pi_N(x)c^0}}{kN}. \tag{6.110}$$

□

Proposition 6.4.26. *With* $\beta_7(x)$ *defined in equation* (6.104)*, we have*

$$\beta_7(x) \qquad \frac{(\log\log x)^2}{\pi_N(x) L} + L\pi_N(x) \frac{8^{\nu(N)} x^{4Lc^0+4\pi_N(x)c^0}}{kN}.$$

Proof. We know, for $l \quad 0$, $jU(l)j \quad 4k_1$, for some $k_1 > 0$ and $U(l) = 0$ for $l > bLc$.

Using equation (6.102) and equation (6.103), we obtain

$$\begin{aligned}
&\beta_7(x) \\
&= \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2} \sum_{p,q,r \quad x}^{0} \sum_{\substack{n_1,n_2 \quad 1 \\ n_2 > n_1}} G(n_1)G(n_2)H_7(p,q,r,n_1,n_2) \\
&= \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \frac{1}{2\pi_N(x)^4 L^2} \sum_{p,q,r \quad x}^{0} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2) \\
&\qquad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{4} M_i(n_1,q)M_j(n_2,r)E_l(n_1,n_2) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{4} \frac{1}{2\pi_N(x)^4 L^2} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)E_l(n_1,n_2) \\
&\qquad \frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \sum_{p,q,r \quad x}^{0} M_i(n_1,q)M_j(n_2,r).
\end{aligned}$$

It is enough to find an estimate for

$$\frac{1}{2\pi_N(x)^4 L^2} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)E_l(n_1,n_2)$$
$$\frac{1}{jF_{N,k}j} \sum_{f\,2\,F_{N,k}} \sum_{p,q,r \quad x}^{0} M_2(n_1,q)M_2(n_2,r),$$

since the other terms can be estimated similarly and give the same upper bound.

We have,

$$\frac{1}{2\pi_N(x)^4 L^2} \sum_{n_1=1}^{\pi_N(x)} \sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)E_l(n_1,n_2)$$

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\quad x}{}^{'} M_2(n_1,q)M_2(n_2,r)\\
&=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)E_l(n_1,n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\quad x}{}^{'}\\
&\qquad\sum_{k_1,k_2\quad 0} U(k_1)U(k_2)a_f(q^{2k_1+2n_1})a_f(r^{2k_2+2n_2})\\
&=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)E_l(n_1,n_2)\sum_{k_1,k_2\quad 0}U(k_1)U(k_2)\\
&\qquad\sum_{p,q,r\quad x}{}^{'}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(q^{2k_1+2n_1}r^{2k_2+2n_2})\right)\\
&=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)E_l(n_1,n_2)\sum_{k_1,k_2\quad 0}U(k_1)U(k_2)\\
&\qquad\sum_{p,q,r\quad x}{}^{'}\left\{\frac{1}{q^{k_1+n_1}r^{k_2+n_2}}+O\left(\frac{8^{\nu(N)}q^{2k_1c^{'}+2n_1c^{'}}r^{2k_2c^{'}+2n_2c^{'}}}{kN}\right)\right\}\\
&\quad\frac{L}{\pi_N(x)^4L^2}\sum_{p,q,r\quad x}{}^{'}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)}\left\{\left(\frac{1}{q^{n_1}}+\frac{1}{q^{n_1+1}}+\quad\right)\left(\frac{1}{r^{n_2}}+\frac{1}{r^{n_2+1}}+\quad\right)\right.\\
&\qquad\left.+\left(L^2\frac{8^{\nu(N)}q^{2Lc^{'}+2n_1c^{'}}r^{2Lc^{'}+2n_2c^{'}}}{kN}\right)\right\}\\
&\quad\frac{L}{\pi_N(x)^4L^2}\sum_{p,q,r\quad x}{}^{'}\left\{\left(\frac{1}{q}+\frac{1}{q^2}+\quad\right)\pi_N(x)\left(\frac{1}{r}+\frac{1}{r^2}+\quad\right)\pi_N(x)\right.\\
&\qquad\left.+\left(L^2\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)}\frac{8^{\nu(N)}(qr)^{2Lc^{'}+2\pi_N(x)c^{'}}}{kN}\right)\right\}\\
&\quad\frac{L}{\pi_N(x)^2L^2}\sum_{p,q,r\quad x}{}^{'}\frac{1}{qr}+\frac{L}{\pi_N(x)^4L^2}\sum_{p,q,r\quad x}{}^{'}\left(\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{4Lc^{'}+4\pi_N(x)c^{'}}}{kN}\right)\\
&\quad\frac{L}{\pi_N(x)^2L^2}\pi_N(x)(\log\log x)^2+\left(L\pi_N(x)\frac{8^{\nu(N)}x^{4Lc^{'}+4\pi_N(x)c^{'}}}{kN}\right)\\
&\quad\frac{(\log\log x)^2}{\pi_N(x)L}+L\pi_N(x)\frac{8^{\nu(N)}x^{4Lc^{'}+4\pi_N(x)c^{'}}}{kN}.
\end{aligned}
$$

Hence,

$$\beta_7(x)\quad \frac{(\log\log x)^2}{\pi_N(x)L}+L\pi_N(x)\frac{8^{\nu(N)}x^{4Lc^{'}+4\pi_N(x)c^{'}}}{kN}. \tag{6.111}$$

□

Proposition 6.4.27. *With $\beta_8(x)$ defined in equation* (6.104), *we have*

$$\beta_8(x)\quad \frac{(\log\log x)^3}{\pi_N(x)^2L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc^{'}+8\pi_N(x)c^{'}}}{kN}.$$

Proof. We know, for $l\quad 0$, $|U(l)|\quad 4k_1$, for some $k_1>0$ and $U(l)=0$ for $l>bLc$.

Using equation (6.102) and equation (6.103), we obtain

$$
\begin{aligned}
&\beta_8(x)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)H_8(p,q,r,n_1,n_2)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{2\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\\
&\qquad\qquad \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{9} M_i(n_1,q)M_j(n_2,r)V_l(n_1,n_2,p)\\
&=\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{l=1}^{9}\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\\
&\qquad\qquad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\le x}{}' M_i(n_1,q)M_j(n_2,r)V_l(n_1,n_2,p).
\end{aligned}
$$

It is enough to find an estimate for

$$
\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)
\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\le x}{}' M_2(n_1,q)M_2(n_2,r)V_1(n_1,n_2,p),
$$

since the other terms can be estimated similarly and give the same upper bound.

We have,

$$
\begin{aligned}
&\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\\
&\qquad \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\le x}{}' M_2(n_1,q)M_2(n_2,r)V_1(n_1,n_2,p)\\
&=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{p,q,r\le x}{}'\\
&\quad \sum_{k_1,k_2\ge 0} U(k_1)U(k_2)a_f(q^{2k_1+2n_1})a_f(r^{2k_2+2n_2}) \sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\ne n_2-n_1}} U(l_1)U(l_2)a_f(p^{2l_1+2n_1})a_f(p^{2l_2+2n_2})\\
&=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\sum_{p,q,r\le x}{}'\\
&\quad \sum_{\substack{l_1,l_2,k_1,k_2\ge 0\\ l_1-l_2\ne n_2-n_1}} U(l_1)U(l_2)U(k_1)U(k_2)\sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} a_f(p^{2t}q^{2k_1+2n_1}r^{2k_2+2n_2})\\
&=\frac{1}{2\pi_N(x)^4L^2}\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} G(n_1)G(n_2)\sum_{p,q,r\le x}{}'\sum_{\substack{l_1,l_2,k_1,k_2\ge 0\\ l_1-l_2\ne n_2-n_1}} U(l_1)U(l_2)U(k_1)U(k_2)\\
&\quad \sum_{t=|(l_1-l_2)-(n_2-n_1)|}^{l_1+l_2+n_1+n_2}\left\{\frac{1}{p^tq^{k_1+n_1}r^{k_2+n_2}}+O\left(\frac{8^{\nu(N)}p^{2tc'}q^{2k_1c'+2n_1c'}r^{2k_2c'+2n_2c'}}{kN}\right)\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\ll \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)} \sum_{\substack{l_1,l_2\ge 0\\ l_1-l_2\ne n_2-n_1}} \left\{\left(\frac{1}{q^{n_1}}+\frac{1}{q^{n_1+1}}+\cdots\right)\left(\frac{1}{r^{n_2}}+\frac{1}{r^{n_2+1}}+\cdots\right)\right.\\
&\quad \left(\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|}}+\frac{1}{p^{|(l_1-l_2)-(n_2-n_1)|+1}}+\cdots\right)\\
&\quad \left.+\left(L^2(l_1+l_2+n_1+n_2)\frac{8^{\nu(N)}p^{2(l_1+l_2+n_1+n_2)c'}q^{2Lc'+2n_1c'}r^{2Lc'+2n_2c'}}{kN}\right)\right\}\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \left\{\left(\frac{1}{p}+\frac{1}{p^2}+\cdots\right)\left(\frac{1}{q}+\frac{1}{q^2}+\cdots\right)\left(\frac{1}{r}+\frac{1}{r^2}+\cdots\right)L\pi_N(x)^2\right.\\
&\qquad\qquad \left.+\left(L^4\pi_N(x)\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=n_1+1}^{\pi_N(x)}\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\right)\right\}\\
&\ll \frac{1}{\pi_N(x)^2L}\sum_{p,q,r\le x}{}'\frac{1}{pqr}+\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}'\left(\pi_N(x)^3L^4\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^2L}(\log\log x)^3+\left(\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\right)\\
&\ll \frac{(\log\log x)^3}{\pi_N(x)^2L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Hence,

$$\beta_8(x)\ll\frac{(\log\log x)^3}{\pi_N(x)^2L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}. \tag{6.112}$$

□

Proposition 6.4.28. *Let $L=L(x)\to\infty$, as $x\to\infty$ such that $L\ll\pi_N(x)$. With $E(\rho,g;f)(x)$ defined in equation* (6.92)*, we have*

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}E(\rho,g;f)(x) \qquad (6.113)\\
&\ll \frac{L}{\pi_N(x)}+\frac{L(\log\log x)}{\pi_N(x)^2}+\frac{\log\log x}{\pi_N(x)}+\frac{(\log\log x)^2}{\pi_N(x)L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Proof. Adding equations (6.105), (6.106), (6.107), (6.108), (6.109), (6.110), (6.111) and (6.112), and using equation (6.103), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}E(\rho,g;f)(x)\\
&=\sum_{t=1}^{8}\beta_t(x)\\
&=\frac{L}{\pi_N(x)}+\frac{L(\log\log x)}{\pi_N(x)^2}+L^2\frac{8^{\nu(N)}x^{12Lc'}}{kN}\\
&+\frac{\log\log x}{\pi_N(x)}+L\frac{8^{\nu(N)}x^{6Lc'}}{kN}+\frac{(\log\log x)^2}{\pi_N(x)^2}+\pi_N(x)L^2\frac{8^{\nu(N)}x^{6Lc'+6\pi_N(x)c'}}{kN}\\
&+\frac{\log\log x}{\pi_N(x)}+L\frac{8^{\nu(N)}x^{6Lc'}}{kN}+\frac{(\log\log x)^2}{\pi_N(x)^2}+\pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\\
&+\frac{(\log\log x)^2}{\pi_N(x)L}+L\pi_N(x)\frac{8^{\nu(N)}x^{4Lc'+4\pi_N(x)c'}}{kN}+\frac{(\log\log x)^3}{\pi_N(x)^2L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}
\end{aligned}
$$

$$\frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{\log\log x}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)L} + \pi_N(x)^2 L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN}.$$

□

Proposition 6.4.29. *Let $L = L(x) \to \infty$, as $x \to \infty$ such that $L \ll \pi_N(x)$. With $F(\rho, g; f)(x)$ defined in equation* (6.93)*, we have*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} F(\rho, g; f)(x) \tag{6.114}$$
$$\ll \frac{1}{\pi_N(x)}\left(\frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{\log\log x}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)L} + \pi_N(x)^2 L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN}\right).$$

Proof. Using equations (6.93) and (6.102), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} F(\rho, g; f)(x)\\
&= \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \frac{4}{16\pi_N(x)^4 L^2} \sum_{p,q,r}{}' \sum_{\substack{x \le n_1, n_2 < \infty \\ n_1 = n_2}} G(n_1)G(n_2)k(p, n_1, n_2)L(q, r, n_1, n_2)\\
&= \sum_{t=1}^{8} \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} \frac{1}{4\pi_N(x)^4 L^2} \sum_{p,q,r}{}' \sum_{\substack{x \le n_1, n_2 < \infty \\ n_2 = n_1}} G(n_1)G(n_2)H_t(p, q, r, n_1, n_2)\\
&= \sum_{t=1}^{8} \omega_t(x),
\end{aligned}$$

where $\sum_{t=1}^{8} \omega_t(x)$ are defined in respective order by the terms in the equation (6.102).

We note that after replacing "$\sum_{\substack{n_1, n_2 \ge 1 \\ n_2 > n_1}}$", i.e., "$\sum_{n_1=1}^{\pi_N(x)} \sum_{n_2 = n_1 + 1}^{\pi_N(x)}$" with "$\sum_{n_1=1}^{\pi_N(x)}$" in $\beta_i(x)$, we obtain $\omega_i(x)$,for each $i = 1, 2, \ldots, 8$.

Hence, using inequation (6.113) in the last line, we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} F(\rho, g; f)(x)\\
&= \sum_{t=1}^{8} \omega_t(x)\\
&\ll \frac{1}{\pi_N(x)}\left(\frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{\log\log x}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)L} + \pi_N(x)^2 L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN}\right).
\end{aligned}$$

□

Proposition 6.4.30. *With $L_4(\rho, g; f)(x)$ defined in equation* (6.91)*, we have*

$$\frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} L_4(\rho, g; f)(x) \tag{6.115}$$
$$= \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} E(\rho, g; f)(x) + \frac{1}{|F_{N,k}|}\sum_{f \in F_{N,k}} D(\rho, g; f)(x)$$
$$\ll \frac{L}{\pi_N(x)} + \frac{L(\log\log x)}{\pi_N(x)^2} + \frac{\log\log x}{\pi_N(x)} + \frac{(\log\log x)^2}{\pi_N(x)L} + \pi_N(x)^2 L^2 \frac{8^{\nu(N)} x^{8Lc' + 8\pi_N(x)c'}}{kN}.$$

Proof. Adding equations (6.113) and (6.114), and using equation (6.91), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_4(\rho,g;f)(x)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} E(\rho,g;f)(x)+\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} D(\rho,g;f)(x)\\
&\frac{L}{\pi_N(x)}+\frac{L(\log\log x)}{\pi_N(x)^2}+\frac{\log\log x}{\pi_N(x)}+\frac{(\log\log x)^2}{\pi_N(x)L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

□

Proposition 6.4.31. *For positive integers k and N with k even, and $L \quad \pi_N(x)$,*

$$
\begin{aligned}
\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L(\rho,g;f)(x)\quad &\frac{L}{\pi_N(x)}+\frac{L^2(\log\log x)}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{(\log\log x)^2}{\pi_N(x)L}\\
&+\frac{\log\log x}{\pi_N(x)}+L^2\pi_N(x)^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Proof. Adding equations (6.71), (6.87) and (6.115) and using equation (6.61), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L(\rho,g;f)(x) \qquad (6.116)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(\sum_{i=1}^{4} L_i(\rho,g;f)(x)\right)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_1(\rho,g;f)(x)+\frac{2}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_2(\rho,g;f)(x)+\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} L_4(\rho,g;f)(x)\\
&\frac{1}{\pi_N(x)}+\frac{L^2\log\log x}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{1}{\pi_N(x)}\frac{8^{\nu(N)}L^3x^{8Lc'}}{kN}\\
+&\frac{L}{\pi_N(x)}+\frac{\log\log x}{\pi_N(x)L}+L^2(L+\pi_N(x))\frac{8^{\nu(N)}x^{8Lc'+4\pi_N(x)c'}}{kN}\\
+&\frac{L}{\pi_N(x)}+\frac{L(\log\log x)}{\pi_N(x)^2}+\frac{\log\log x}{\pi_N(x)}+\frac{(\log\log x)^2}{\pi_N(x)L}+\pi_N(x)^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}\\
&\frac{L}{\pi_N(x)}+\frac{L^2(\log\log x)}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{(\log\log x)^2}{\pi_N(x)L}\\
+&\frac{\log\log x}{\pi_N(x)}+L^2\pi_N(x)^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

□

6.5 Estimation for $\left\langle M(\rho,g;f)(x)\right\rangle = \left\langle (M_1+2M_2+M_4)(\rho,g;f)(x)\right\rangle$

We now address

$$
M(\rho,g;f)(x)=\frac{1}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \ \ _x\, T_1(p)T_2(q)T_3(p,q)T_1(r)T_2(s)T_3(r,s).
$$

Here,

$$
T_1(p)=\sum_{l_1\quad 0} U(l_1)a_f(p^{2l_1}),
$$

$$T_2(q) = \sum_{l_2 \geq 0} U(l_2) a_f(q^{2l_2}),$$

$$T_1(r) = \sum_{k_1 \geq 0} U(k_1) a_f(r^{2k_1}),$$

$$T_2(s) = \sum_{k_2 \geq 0} U(k_2) a_f(s^{2k_2}),$$

$$\begin{aligned} &T_3(p,q) \\ =&4G(0) + \sum_{n \geq 1} 2\widehat{g}\left(\frac{n}{\pi_N(x)}\right)(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})) \\ =& \sum_{n_1 \geq 0} \widehat{g}\left(\frac{n_1}{\pi_N(x)}\right) A(p,q,n_1), \end{aligned}$$

and

$$T_3(r,s) = \sum_{n_2 \geq 0} \widehat{g}\left(\frac{n_2}{\pi_N(x)}\right) A(r,s,n_2),$$

where

$$A(p,q,n) = \begin{cases} 4 & \text{if } n = 0 \\ 2(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})) & \text{if } n \geq 1, \end{cases}$$

and $G(n) = \widehat{g}\left(\frac{n}{\pi_N(x)}\right)$, as defined earlier.

Thus,

$$\begin{aligned} &M(\rho, g; f)(x) \qquad (6.117) \\ =&\frac{1}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \leq x} T_1(p) T_2(q) T_3(p,q) T_1(r) T_2(s) T_3(r,s) \\ =&\frac{1}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \leq x} \sum_{l_1,l_2 \geq 0} \sum_{k_1,k_2 \geq 0} \sum_{n_1,n_2 \geq 0} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2) \\ & \qquad a_f(p^{2l_1}) a_f(q^{2l_2}) a_f(r^{2k_1}) a_f(s^{2k_2}) A(p,q,n_1) A(r,s,n_2). \end{aligned}$$

Since the summation is over n_1, n_2, where the indexes n_1, n_2 run up to $\pi_N(x)$, we can break the summation into the following four parts:

1) $n_1 = 0, n_2 = 0$,

2) $n_1 \neq 0, n_2 = 0$,

3) $n_1 = 0, n_2 \neq 0$,

4) $n_1 \neq 0, n_2 \neq 0$.

We also denote the summation in the i-th part by $M_i(\rho, g; f)(x)$, $i = 1, 2, 3, 4$ respectively.

We will also write M_i for $M_i(\rho, g; f)(x)$, $i = 1, 2, 3, 4$, in short.

By interchanging the variables n_1, n_2 first and then replacing the variables p, q, r and s with r, s, p and q respectively and at last interchanging the variables k_1, k_2, with l_1, l_2 in $M_3(\rho, g; f)(x)$, the summation over n_1, n_2 where $n_1 = 0, n_2 \neq 0$, we note that the summation over n_1, n_2 where $n_1 = 0, n_2 \neq 0$, is exactly the same as the summation over n_1, n_2 where $n_1 \neq 0, n_2 = 0$, i.e., $M_2(\rho, g; f)(x) = M_3(\rho, g; f)(x)$.

Therefore,

$$
\begin{aligned}
& M(\rho, g; f)(x) \qquad (6.118)\\
&= \sum_{i=1}^{4} M_i(\rho, g; f)(x)\\
&= M_1(\rho, g; f)(x) + 2M_3(\rho, g; f)(x) + M_4(\rho, g; f)(x),
\end{aligned}
$$

where $M_1(\rho, g; f)(x)$, $M_3(\rho, g; f)(x)$, and $M_4(\rho, g; f)(x)$ are defined in Sections 6.5.1, 6.5.2, and 6.5.3 respectively.

6.5.1 Estimation for $\left\langle M_1(\rho, g; f)(x)\right\rangle$

If $(n_1, n_2) = (0, 0)$, then the innermost term is

$$16 a_f(p^{2l_1}) a_f(q^{2l_2}) a_f(r^{2k_1}) a_f(s^{2k_2}).$$

Thus, using equation (6.117), we obtain

$$
\begin{aligned}
& \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} M_1(\rho, g; f)(x)\\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{1}{64\pi_N(x)^4 L^2} \sum'_{p,q,r,s \le x} \sum_{l_1,l_2 \ge 0} \sum_{k_1,k_2 \ge 0} U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\\
&\qquad a_f(p^{2l_1}) a_f(q^{2l_2}) a_f(r^{2k_1}) a_f(s^{2k_2}) A(p,q,0)A(r,s,0)\\
&= \frac{16}{64\pi_N(x)^4 L^2} \sum'_{p,q,r,s \le x} \sum_{l_1,l_2 \ge 0} \sum_{k_1,k_2 \ge 0} U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2\\
&\qquad \left(\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} a_f(p^{2l_1} q^{2l_2} r^{2k_1} s^{2k_2}) \right)\\
&= \frac{G(0)^2}{4\pi_N(x)^4 L^2} \sum'_{p,q,r,s \le x} \Bigg\{ U(0)^4 + \sum_{\substack{l_1,l_2,k_1,k_2 \ge 0\\ (l_1,l_2,k_1,k_2) \ne (0,0,0,0)}} U(l_1)U(l_2)U(k_1)U(k_2)\\
&\qquad \left(\frac{1}{p^{l_1} q^{l_2} r^{k_1} s^{k_2}} + \frac{8^{\nu(N)} p^{2l_1c'} q^{2l_2c'} r^{2k_1c'} s^{2k_2c'}}{kN} \right) \Bigg\}\\
& \frac{1}{\pi_N(x)^4 L^2} \sum'_{p,q,r,s \le x} \Bigg\{ 1 +\\
&\qquad \sum_{\substack{l_1,l_2,k_1,k_2 \ge 0\\ (l_1,l_2,k_1,k_2) \ne (0,0,0,0)}} \left(\frac{1}{p^{l_1} q^{l_2} r^{k_1} s^{k_2}} + \frac{8^{\nu(N)} p^{2l_1c'} q^{2l_2c'} r^{2k_1c'} s^{2k_2c'}}{kN} \right) \Bigg\}\\
& \frac{1}{\pi_N(x)^4 L^2} \sum'_{p,q,r,s \le x} \Bigg\{ \sum_{l_1,l_2,k_1,k_2 \ge 0} \frac{1}{p^{l_1} q^{l_2} r^{k_1} s^{k_2}}\\
&\qquad + \sum_{\substack{l_1,l_2,k_1,k_2 \ge 0\\ (l_1,l_2,k_1,k_2) \ne (0,0,0,0)}} \frac{8^{\nu(N)} p^{2l_1c'} q^{2l_2c'} r^{2k_1c'} s^{2k_2c'}}{kN} \Bigg\}\\
& \frac{1}{\pi_N(x)^4 L^2} \sum'_{p,q,r,s \le x} \left\{ \sum_{l_1 \ge 0} \frac{1}{p^{l_1}} \sum_{l_2 \ge 0} \frac{1}{q^{l_2}} \sum_{k_1 \ge 0} \frac{1}{r^{k_1}} \sum_{k_2 \ge 0} \frac{1}{s^{k_2}} + L^4 \frac{8^{\nu(N)} (pqrs)^{2Lc'}}{kN} \right\}\\
& \frac{1}{\pi_N(x)^4 L^2} \left\{ \sum'_{p,q,r,s \le x} 2^4 + \sum'_{p,q,r,s \le x} L^4 \frac{8^{\nu(N)} (pqrs)^{2Lc'}}{kN} \right\}
\end{aligned}
$$

$$\frac{1}{\pi_N(x)^4L^2}\left\{\pi_N(x)^4 + \pi_N(x)^4L^4\frac{8^{\nu(N)}x^{8Lc^0}}{kN}\right\}$$
$$\frac{1}{L^2} + L^2\frac{8^{\nu(N)}x^{8Lc^0}}{kN}.$$

Hence,

$$\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}} M_1(\rho,g;f)(x) \quad \frac{1}{L^2} + L^2\frac{8^{\nu(N)}x^{8Lc^0}}{kN}. \tag{6.119}$$

6.5.2 Estimation for $\langle (M_2 + M_3)(\rho,g;f)(x)\rangle$

We now look at the part of the sum $M(\rho,g;f)(x)$ with $n_1 = 0$ and $n_2 \neq 0$, i.e., we now estimate $M_3(\rho,g;f)(x)$. In this case, the innermost term

$$\begin{aligned}
&a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})A(p,q,0)A(r,s,n_2)\\
=&8a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})(a_f(r^{2n_2}) \quad a_f(r^{2n_2\ \ 2}))(a_f(s^{2n_2}) \quad a_f(s^{2n_2\ \ 2}))\\
=&8a_f(p^{2l_1})a_f(q^{2l_2})fa_f(r^{2k_1})(a_f(r^{2n_2}) \quad a_f(r^{2n_2\ \ 2}))gfa_f(s^{2k_2})(a_f(s^{2n_2}) \quad a_f(s^{2n_2\ \ 2}))g.
\end{aligned}$$

We want to find an estimate for

$$\begin{aligned}
&M_2(\rho,g;f)(x) + M_3(\rho,g;f)(x) \qquad\qquad (6.120)\\
=&2M_3(\rho,g;f)(x)\\
=&\frac{2}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}{}^{0}\ \sum_{l_1,l_2\quad 0}\ \sum_{k_1,k_2\quad 0}\ \sum_{n_2\quad 1} U(l_1)U(l_2)U(k_1)U(k_2)G(0)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})A(p,q,0)A(p,r,n_2)\\
=&\frac{16}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}{}^{0}\ \sum_{l_1,l_2\quad 0}\ \sum_{k_1,k_2\quad 0}\ \sum_{n_2\quad 1} U(l_1)U(l_2)U(k_1)U(k_2)G(0)G(n_2)\\
&\qquad a_f(p^{2l_1})a_f(q^{2l_2})fa_f(r^{2k_1})(a_f(r^{2n_2}) \quad a_f(r^{2n_2\ \ 2}))gfa_f(s^{2k_2})(a_f(s^{2n_2}) \quad a_f(s^{2n_2\ \ 2}))g\\
=&\frac{G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}{}^{0}\ \sum_{n_2\quad 1} G(n_2)\sum_{l_1,k_1\quad 0} U(l_1)U(k_1)a_f(p^{2l_1})a_f(r^{2k_1})(a_f(r^{2n_2}) \quad a_f(r^{2n_2\ \ 2}))\\
&\qquad \sum_{l_2,k_2\quad 0} U(l_2)U(k_2)a_f(q^{2l_2})a_f(s^{2k_2})(a_f(s^{2n_2}) \quad a_f(s^{2n_2\ \ 2}))\\
=&\frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}{}^{0}\ \sum_{n_2\quad 1} G(0)G(n_2)A_1(\rho,g;f;n_2,p,r)A_1(\rho,g;f;n_2,q,s),
\end{aligned}$$

where $A_1(\rho,g;f;n_2,p,r)$ is defined in equation (6.73) and thus,

$$A_1(\rho,g;f;n_2,u,v) = \sum_{l_1,k_1\quad 0} U(l_1)U(k_1)a_f(u^{2l_1})a_f(v^{2k_1})(a_f(v^{2n_2}) \quad a_f(v^{2n_2\ \ 2})).$$

Using equation (6.76), we obtain

$$A_1(\rho,g;f;n_2,p,r) = U(0)(U(n_2) \quad U(n_2 \quad 1)) + F_1(n_2,p,r) + F_2(n_2,p,r),$$

and

$$A_1(\rho,g;f;n_2,q,s) = U(0)(U(n_2) \quad U(n_2 \quad 1)) + F_1(n_2,q,s) + F_2(n_2,q,s).$$

For primes p, q, r, s and fixed integer $n_2 \geq 1$, using the above equations, we obtain

$$\begin{aligned}
&A_1(\rho, g; f; n_2, p, r) \cdot A_1(\rho, g; f; n_2, q, s)\\
&= \big(U(0)(U(n_2) - U(n_2 - 1)) + F_1(n_2, p, r) + F_2(n_2, p, r)\big)\\
&\qquad \big(U(0)(U(n_2) - U(n_2 - 1)) + F_1(n_2, q, s) + F_2(n_2, q, s)\big)\\
&= \big(U(0)F(n_2) + F_1(n_2, p, r) + F_2(n_2, p, r)\big)\\
&\qquad \big(U(0)F(n_2) + F_1(n_2, q, s) + F_2(n_2, q, s)\big)\\
&= U(0)^2F(n_2)^2 + U(0)F(n_2)(F_1 + F_2)(n_2, p, r) + U(0)F(n_2)(F_1 + F_2)(n_2, q, s)\\
&+ F_1(n_2, p, r)F_1(n_2, q, s) + F_1(n_2, p, r)F_2(n_2, q, s)\\
&+ F_1(n_2, q, s)F_2(n_2, p, r) + F_2(n_2, p, r)F_2(n_2, q, s),
\end{aligned}$$

where

$$F(n) = U(n) - U(n - 1).$$

Therefore, using equation (6.120), we obtain

$$\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \left(M_2(\rho, g; f)(x) + M_3(\rho, g; f)(x)\right) \qquad (6.121)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)A_1(\rho, g; f; n_2, p, r)A_1(\rho, g; f; n_2, q, s)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)U(0)^2F(n_2)^2\\
&+ \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{2}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)U(0)F(n_2)(F_1 + F_2)(n_2, p, r)\\
&+ \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)F_1(n_2, p, r)F_1(n_2, q, s)\\
&+ \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{2}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)F_1(n_2, p, r)F_2(n_2, q, s)\\
&+ \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)F_2(n_2, p, r)F_2(n_2, q, s)\\
&= \sum_{t=1}^{5} \gamma_t(x),
\end{aligned}$$

where $\sum_{t=1}^{5} \gamma_t(x)$ are defined in respective order with the order in the previous line.

Proposition 6.5.1. *With $\gamma_1(x)$ defined in equation* (6.121)*, we have* $\gamma_1(x) \ll \frac{1}{L}$.

Proof. We know, $|F(n)| \leq 8k_1$, for a positive integer n.

Now, $n > 1 + \lfloor Lc \rfloor$, implies $U(n) = 0$ and hence, $F(n) = U(n) - U(n - 1) = 0$.

Using equation (6.121), we obtain

$$\begin{aligned}
&\gamma_1(x) \qquad (6.122)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{x \gg n_2 \geq 1} G(0)G(n_2)U(0)^2F(n_2)^2\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{'} \sum_{n_2=1}^{1+\lfloor Lc \rfloor} G(0)G(n_2)U(0)^2F(n_2)^2
\end{aligned}$$

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2=1}^{1+bLc}1$$
$$\frac{1}{\pi_N(x)^4L^2}\pi_N(x)^4L$$
$$=\frac{1}{L}.$$

□

Proposition 6.5.2. *With* $\gamma_2(x)$ *defined in equation* (6.121), *we have*

$$\gamma_2(x) \qquad \frac{(\log\log x)}{\pi_N(x)L}+L\frac{8^{\nu(N)}x^{6Lc^{\prime}}}{kN}.$$

Proof. We know, $|F(n)| \quad 8k_1$, for a positive integer n.

Now, $n>1+bLc$, implies $U(n)=0$ and hence, $F(n)=U(n) \quad U(n \quad 1)=0$.

Hence,

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2\quad 1}G(0)G(n_2)U(0)F(n_2)F_1(n_2,p,r) \tag{6.123}$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2=1}^{1+bLc}G(n_2)F(n_2)F_1(n_2,p,r)$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2=1}^{1+bLc}G(n_2)F(n_2)$$
$$\sum_{\substack{k_1,k_2\quad 0,k_2\quad n_2\\(k_1,k_2)\neq(0,n_2)}}U(k_1)U(k_2)a_f(p^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2})\quad a_f(r^{2n_2\quad 2}))$$
$$=\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2=1}^{1+bLc}G(n_2)F(n_2)$$
$$\sum_{\substack{k_1,k_2\quad 0,k_2\quad n_2\\(k_1,k_2)\neq(0,n_2)}}U(k_1)U(k_2)\left(\left\langle a_f(p^{2k_1}r^{2k_2\quad 2n_2})\right\rangle+\left\langle a_f(p^{2k_1}r^{2k_2+2n_2})\right\rangle\right)$$
$$=\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2=1}^{1+bLc}G(n_2)F(n_2)$$
$$\sum_{\substack{k_1,k_2\quad 0,k_2\quad n_2\\(k_1,k_2)\neq(0,n_2)}}U(k_1)U(k_2)\left(\frac{1}{p^{k_1}r^{k_2\quad n_2}}+\frac{1}{p^{k_1}r^{k_2+n_2}}+\mathrm{O}\left(\frac{8^{\nu(N)}p^{2k_1c^{\prime}}r^{2k_2c^{\prime}+2n_2c^{\prime}}}{kN}\right)\right)$$
$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\sum_{n_2=1}^{1+bLc}\sum_{\substack{k_1,k_2\quad 0,k_2\quad n_2\\(k_1,k_2)\neq(0,n_2)}}\left(\frac{2}{p^{k_1}r^{k_2\quad n_2}}+\frac{8^{\nu(N)}p^{2k_1c^{\prime}}r^{2k_2c^{\prime}+2n_2c^{\prime}}}{kN}\right)$$
$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{\prime}\left(\left(\frac{1}{r}+\quad\right)L+\left(\frac{1}{p}+\quad\right)L+\left(\frac{1}{p}+\quad\right)\left(\frac{1}{r}+\quad\right)L\right.$$
$$\left.+L^3\frac{8^{\nu(N)}p^{2Lc^{\prime}}r^{4Lc^{\prime}}}{kN}\right)$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{0}\left(\frac{1}{r}L+\frac{1}{p}L+\frac{1}{pr}L+L^3\frac{8^{\nu(N)}x^{6Lc^0}}{kN}\right)$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{0}\left(\frac{1}{r}L\right)+L\frac{8^{\nu(N)}x^{6Lc^0}}{kN}\qquad\frac{(\log\log x)}{\pi_N(x)L}+L\frac{8^{\nu(N)}x^{6Lc^0}}{kN}.$$

We also have,

$$\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\frac{2}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;n_2\quad 1}G(0)G(n_2)U(0)F(n_2)F_2(n_2,p,r)\tag{6.124}$$

$$=\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;\;n_2=1}^{1+bLc}G(n_2)F(n_2)F_2(n_2,p,r)$$

$$=\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;\;n_2=1}^{1+bLc}G(n_2)F(n_2)$$

$$\sum_{\substack{k_1,k_2\quad 0,k_2<n_2\\(k_1,k_2)\not\in(0,n_2\quad 1)}}U(k_1)U(k_2)a_f(p^{2k_1})a_f(r^{2k_2})(a_f(r^{2n_2})\quad a_f(r^{2n_2\quad 2}))$$

$$=\frac{2U(0)G(0)}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;\;n_2=1}^{1+bLc}G(n_2)F(n_2)$$

$$\sum_{\substack{k_1,k_2\quad 0,k_2<n_2\\(k_1,k_2)\not\in(0,n_2\quad 1)}}U(k_1)U(k_2)\left(\left\langle a_f(p^{2k_1}r^{2k_2+2n_2})\right\rangle\quad\left\langle a_f(p^{2k_1}r^{2n_2\quad 2k_2\quad 2})\right\rangle\right)$$

$$=\frac{2U(0)G(0)}{A^2\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;\;n_2=1}^{1+bLc}G(n_2)F(n_2)$$

$$\sum_{\substack{k_1,k_2\quad 0,k_2<n_2\\(k_1,k_2)\not\in(0,n_2\quad 1)}}U(k_1)U(k_2)\left(\frac{1}{p^{k_1}r^{k_2+n_2}}\quad\frac{1}{p^{k_1}r^{n_2\quad k_2\quad 1}}+\mathrm{O}\left(\frac{8^{\nu(N)}p^{2k_1c^0}r^{2k_2c^0+2n_2c^0}}{kN}\right)\right)$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;\;n_2=1}^{1+bLc}\sum_{\substack{k_1,k_2\quad 0,k_2<n_2\\(k_1,k_2)\not\in(0,n_2\quad 1)}}\left(\frac{2}{p^{k_1}r^{n_2\quad k_2\quad 1}}+\frac{8^{\nu(N)}p^{2k_1c^0}r^{2k_2c^0+2n_2c^0}}{kN}\right)$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{0}\left(\left(\frac{1}{r}+\quad\right)L+\left(\frac{1}{p}+\quad\right)L+\left(\frac{1}{p}+\quad\right)\left(\frac{1}{r}+\quad\right)L\right.$$
$$\left.+L^3\frac{8^{\nu(N)}p^{2Lc^0}p^{4Lc^0}}{kN}\right)$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{0}\left(\frac{1}{r}L+\frac{1}{p}L+\frac{1}{pr}L+L^3\frac{8^{\nu(N)}x^{6Lc^0}}{kN}\right)$$

$$\frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s\quad x}^{0}\left(\frac{1}{r}L\right)+L\frac{8^{\nu(N)}x^{6Lc^0}}{kN}\qquad\frac{(\log\log x)}{\pi_N(x)L}+L\frac{8^{\nu(N)}x^{6Lc^0}}{kN}.$$

Adding equations (6.123) and (6.124) and using equation (6.121), we obtain

$$\gamma_2(x)\tag{6.125}$$
$$=\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\frac{2}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}^{0}\sum_{x\;n_2\quad 1}G(0)G(n_2)U(0)F(n_2)(F_1+F_2)(n_2,p,r)$$

$$\frac{(\log\log x)}{\pi_N(x)L} + L\frac{8^{\nu(N)}x^{6Lc'}}{kN}.$$

□

Proposition 6.5.3. *With $\gamma_3(x)$ defined in equation* (6.121), *we have*

$$\gamma_3(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2L} + \pi_N(x)^4L^5\frac{8^{\nu(N)}x^{12Lc'}}{kN}.$$

Proof. We first find an estimate for $\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_1(n_2,p,r)F_1(n_2,q,s)$.

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_1(n_2,p,r)F_1(n_2,q,s) \tag{6.126}$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{l_1,l_2\ge 0,l_2\le n_2\\(l_1,l_2)\neq(0,n_2)}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_2})-a_f(r^{2n_2-2}))$$
$$\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\(k_1,k_2)\neq(0,n_2)}} U(k_1)U(k_2)a_f(q^{2k_1})a_f(s^{2k_2})(a_f(s^{2n_2})-a_f(s^{2n_2-2}))$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\sum_{\substack{l_1,l_2\ge 0,l_2\le n_2\\(l_1,l_2)\neq(0,n_2)}} U(l_1)U(l_2)a_f(p^{2l_1})(a_f(r^{2l_2-2n_2})+a_f(r^{2l_2+2n_2}))$$
$$\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\(k_1,k_2)\neq(0,n_2)}} U(k_1)U(k_2)a_f(q^{2k_1})(a_f(s^{2k_2-2n_2})+a_f(s^{2k_2+2n_2}))$$
$$=\sum_{\substack{l_1,l_2\ge 0,l_2\le n_2\\(l_1,l_2)\neq(0,n_2)}}\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\(k_1,k_2)\neq(0,n_2)}} U(l_1)U(l_2)U(k_1)U(k_2)\Big\langle a_f(p^{2l_1}q^{2k_1})$$
$$(a_f(r^{2l_2-2n_2})+a_f(r^{2l_2+2n_2}))\cdot(a_f(s^{2k_2-2n_2})+a_f(s^{2k_2+2n_2}))\Big\rangle$$
$$\ll\sum_{\substack{l_1,l_2\ge 0,l_2\le n_2\\(l_1,l_2)\neq(0,n_2)}}\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\(k_1,k_2)\neq(0,n_2)}}\left(\frac{1}{p^{l_1}q^{k_1}}\left(\frac{1}{r^{l_2-n_2}}+\frac{1}{r^{l_2+n_2}}\right)\left(\frac{1}{s^{k_2-n_2}}+\frac{1}{s^{k_2+n_2}}\right)\right.$$
$$\left.+\frac{8^{\nu(N)}p^{2l_1c'}q^{2k_1c'}r^{2l_2c'+2n_2c'}s^{2k_2c'+2n_2c'}}{kN}\right)$$
$$\ll\sum_{\substack{l_1,l_2\ge 0,l_2\le n_2\\(l_1,l_2)\neq(0,n_2)}}\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\(k_1,k_2)\neq(0,n_2)}}\left(\frac{1}{p^{l_1}q^{k_1}}\frac{1}{r^{l_2-n_2}}\frac{1}{s^{k_2-k_2}}\right)+L^4\frac{8^{\nu(N)}(pq)^{2Lc'}(rs)^{4Lc'}}{kN}.$$

We now use equation (6.126) and the fact $n_2\ge l_2$, (i.e., $n_2\le\lfloor L\rfloor$ in the following sum) to obtain

$$\sum_{p,q,r,s}{}' \sum_{x\ n_2\ 1} G(0)G(n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_1(n_2,p,r)F_1(n_2,q,s) \tag{6.127}$$
$$\ll\sum_{p,q,r,s}{}' \sum_{x\ n_2\ 1}\sum_{\substack{l_1,l_2\ge 0,l_2\le n_2\\(l_1,l_2)\neq(0,n_2)}}\frac{1}{p^{l_1}}\frac{1}{r^{l_2-n_2}}\sum_{\substack{k_1,k_2\ge 0,k_2\le n_2\\(k_1,k_2)\neq(0,n_2)}}\frac{1}{q^{k_1}}\frac{1}{s^{k_2-n_2}}$$
$$+\sum_{p,q,r,s}{}' \sum_{x\ n_2\ 1} L^4\frac{8^{\nu(N)}(pq)^{2Lc'}(rs)^{4Lc'}}{kN}$$

$$
\begin{aligned}
&\ll \sum_{p,q,r,s}{}' \sum_{x \geq n_2 \geq 1} \left(\frac{1}{p} + \frac{1}{r} + \frac{1}{pr}\right)\left(\frac{1}{q} + \frac{1}{s} + \frac{1}{qs}\right) \\
&\qquad + \sum_{p,q,r,s \leq x}{}' L^5 \frac{8^{\nu(N)} x^{4Lc' + 8Lc'}}{kN} \\
&\ll \sum_{p,q,r,s \leq x}{}' L \frac{1}{pq} + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{12Lc'}}{kN} \\
&\ll (\log\log x)^2 L \pi_N(x)^2 + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{12Lc'}}{kN}.
\end{aligned}
$$

Using equation (6.127), we obtain

$$
\begin{aligned}
&\gamma_3(x) \qquad (6.128)\\
&= \frac{1}{4\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \geq n_2 \geq 1} G(0)G(n_2) \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2,p,r)F_1(n_2,q,s) \\
&\ll \frac{1}{\pi_N(x)^4 L^2} \left((\log\log x)^2 L \pi_N(x)^2 + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{12Lc'}}{kN} \right) \\
&= \frac{(\log\log x)^2}{\pi_N(x)^2 L} + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{12Lc'}}{kN}.
\end{aligned}
$$

□

Proposition 6.5.4. *With $\gamma_4(x)$ defined in equation (6.121), we have*

$$
\gamma_4(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2 L} + L^3 \frac{8^{\nu(N)} x^{6Lc'}}{kN}.
$$

Proof. We first find an estimate for $\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2,p,r)F_2(n_2,q,s)$.

Using the fact $n_2 \leq l_2$, (i.e., $n_2 \leq bLc$ in the following sum), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_1(n_2,p,r)F_2(n_2,q,s) \qquad (6.129)\\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{\substack{l_1,l_2 \geq 0, l_2 \leq n_2 \\ (l_1,l_2) \neq (0,n_2)}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_2}) - a_f(r^{2n_2-2})) \\
&\qquad \sum_{\substack{k_1,k_2 \geq 0, k_2 < n_2 \\ (k_1,k_2) \neq (0,n_2-1)}} U(k_1)U(k_2)a_f(q^{2k_1})a_f(s^{2k_2})(a_f(s^{2n_2}) - a_f(s^{2n_2-2})) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \sum_{\substack{l_1,l_2 \geq 0, l_2 \leq n_2 \\ (l_1,l_2) \neq (0,n_2)}} U(l_1)U(l_2)a_f(p^{2l_1})(a_f(r^{2l_2-2n_2}) + a_f(r^{2l_2+2n_2})) \\
&\qquad \sum_{\substack{k_1,k_2 \geq 0, k_2 < n_2 \\ (k_1,k_2) \neq (0,n_2-1)}} U(k_1)U(k_2)a_f(q^{2k_1})(a_f(s^{2k_2+2n_2}) - a_f(s^{2n_2-2k_2-2})) \\
&= \sum_{\substack{l_1,l_2 \geq 0, l_2 \leq n_2 \\ (l_1,l_2) \neq (0,n_2)}} \sum_{\substack{k_1,k_2 \geq 0, k_2 < n_2 \\ (k_1,k_2) \neq (0,n_2-1)}} U(l_1)U(l_2)U(k_1)U(k_2)\Big\langle a_f(p^{2l_1}q^{2k_1}) \\
&\qquad (a_f(r^{2l_2-2n_2}) + a_f(r^{2l_2+2n_2})) \quad (a_f(s^{2k_2+2n_2}) - a_f(s^{2n_2-2k_2-2}))\Big\rangle
\end{aligned}
$$

$$\ll \sum_{\substack{l_1,l_2\ge 0, l_2\le n_2\\(l_1,l_2)\ne(0,n_2)}} \sum_{\substack{k_1,k_2\ge 0, k_2<n_2\\(k_1,k_2)\ne(0,n_2-1)}} \left(\frac{1}{p^{l_1}q^{k_1}}\left(\frac{1}{r^{l_2-n_2}}+\frac{1}{r^{l_2+n_2}}\right)\left(\frac{1}{s^{k_2+n_2}}-\frac{1}{s^{n_2-k_2-1}}\right)\right.$$
$$\left. + \frac{8^{\nu(N)}p^{2l_1c'}q^{2k_1c'}r^{2l_2c'+2n_2c'}s^{2k_2c'+2n_2c'}}{kN}\right)$$
$$\ll \sum_{\substack{l_1,l_2\ge 0, l_2\le n_2\\(l_1,l_2)\ne(0,n_2)}} \sum_{\substack{k_1,k_2\ge 0, k_2<n_2\\(k_1,k_2)\ne(0,n_2-1)}} \left(\frac{1}{p^{l_1}q^{k_1}}\frac{1}{r^{l_2-n_2}}\frac{1}{s^{n_2-k_2-1}}\right) + L^4\frac{8^{\nu(N)}(pq)^{2Lc'}(rs)^{4Lc'}}{kN}.$$

We now use equation (6.129) to obtain

$$\begin{aligned}
&\sum_{p,q,r,s}{}' \sum_{x^{n_2}\ge 1} G(0)G(n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_1(n_2,p,r)F_2(n_2,q,s) \qquad (6.130)\\
&\ll \sum_{p,q,r,s}{}' \sum_{x^{n_2}\ge 1}\left(\sum_{\substack{l_1,l_2\ge 0, l_2\le n_2\\(l_1,l_2)\ne(0,n_2)}}\frac{1}{p^{l_1}}\frac{1}{r^{l_2-n_2}}\right)\left(\sum_{\substack{k_1,k_2\ge 0, k_2<n_2\\(k_1,k_2)\ne(0,n_2-1)}}\frac{1}{q^{k_1}}\frac{1}{s^{n_2-k_2-1}}\right)\\
&\quad + \sum_{p,q,r,s}{}' \sum_{x^{n_2}\ge 1} L^4\frac{8^{\nu(N)}(pq)^{2Lc'}(rs)^{4Lc'}}{kN}\\
&\ll \sum_{p,q,r,s}{}' \sum_{x^{n_2}\ge 1}\left(\frac{1}{p}+\frac{1}{r}+\frac{1}{pr}\right)\left(\frac{1}{q}+\frac{1}{s}+\frac{1}{qs}\right) + \sum_{p,q,r,s\le x}{}' L^5\frac{8^{\nu(N)}x^{6Lc'}}{kN}\\
&\ll \sum_{p,q,r,s\le x}{}' L\frac{1}{pq} + \pi_N(x)^4L^5\frac{8^{\nu(N)}x^{6Lc'}}{kN}\\
&\ll (\log\log x)^2\pi_N(x)^2L + \pi_N(x)^4L^5\frac{8^{\nu(N)}x^{6Lc'}}{kN}.
\end{aligned}$$

Using equation (6.130), we obtain

$$\begin{aligned}
&\gamma_4(x) \qquad (6.131)\\
&= \frac{2}{4\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x^{n_2}\ge 1} G(0)G(n_2)\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_1(n_2,p,r)F_2(n_2,q,s)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\left((\log\log x)^2\pi_N(x)^2L + \pi_N(x)^4L^5\frac{8^{\nu(N)}x^{6Lc'}}{kN}\right)\\
&= \frac{(\log\log x)^2}{\pi_N(x)^2L} + L^3\frac{8^{\nu(N)}x^{6Lc'}}{kN}.
\end{aligned}$$

□

Proposition 6.5.5. *With* γ_5 *defined in equation* (6.121)*, we have*

$$\gamma_5(x) \ll \frac{(\log\log x)^2}{\pi_N(x)L^2} + \pi_N(x)L^2\frac{8^{\nu(N)}x^{8Lc'+4\pi_N(x)c'}}{kN}.$$

Proof. We first find estimate for $\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_2(n_2,p,r)F_2(n_2,q,s)$.

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} F_2(n_2,p,r)F_2(n_2,q,s) \qquad (6.132)$$

$$= \frac{1}{jF_{N,k}j} \sum_{f 2 F_{N,k}} \sum_{\substack{l_1,l_2 \quad 0, l_2 < n_2 \\ (l_1,l_2) \not= (0, n_2 \quad 1)}} U(l_1)U(l_2)a_f(p^{2l_1})a_f(r^{2l_2})(a_f(r^{2n_2}) \quad a_f(r^{2n_2 \quad 2}))$$

$$\sum_{\substack{k_1,k_2 \quad 0, k_2 < n_2 \\ (k_1,k_2) \not= (0, n_2 \quad 1)}} U(k_1)U(k_2)a_f(q^{2k_1})a_f(s^{2k_2})(a_f(s^{2n_2}) \quad a_f(s^{2n_2 \quad 2}))$$

$$= \frac{1}{jF_{N,k}j} \sum_{f 2 F_{N,k}} \sum_{\substack{l_1,l_2 \quad 0, l_2 < n_2 \\ (l_1,l_2) \not= (0, n_2 \quad 1)}} U(l_1)U(l_2)a_f(p^{2l_1})(a_f(r^{2l_2+2n_2}) \quad a_f(r^{2n_2 \quad 2l_2 \quad 2}))$$

$$\sum_{\substack{k_1,k_2 \quad 0, k_2 < n_2 \\ (k_1,k_2) \not= (0, n_2 \quad 1)}} U(k_1)U(k_2)a_f(q^{2k_1})(a_f(s^{2k_2+2n_2}) \quad a_f(s^{2n_2 \quad 2k_2 \quad 2}))$$

$$= \sum_{\substack{l_1,l_2 \quad 0, l_2 < n_2 \\ (l_1,l_2) \not= (0, n_2 \quad 1)}} \sum_{\substack{k_1,k_2 \quad 0, k_2 < n_2 \\ (k_1,k_2) \not= (0, n_2 \quad 1)}} U(l_1)U(l_2)U(k_1)U(k_2)\Big\langle a_f(p^{2l_1}q^{2k_1})$$

$$(a_f(r^{2l_2+2n_2}) \quad a_f(r^{2n_2 \quad 2l_2 \quad 2})) \quad (a_f(s^{2k_2+2n_2}) \quad a_f(s^{2n_2 \quad 2k_2 \quad 2}))\Big\rangle$$

$$\sum_{\substack{l_1,l_2 \quad 0, l_2 < n_2 \\ (l_1,l_2) \not= (0, n_2 \quad 1)}} \sum_{\substack{k_1,k_2 \quad 0, k_2 < n_2 \\ (k_1,k_2) \not= (0, n_2 \quad 1)}} \left(\frac{1}{p^{l_1}q^{k_1}} \left(\frac{1}{r^{l_2+n_2}} + \frac{1}{r^{n_2 \quad l_2 \quad 1}} \right) \left(\frac{1}{s^{k_2+n_2}} + \frac{1}{s^{n_2 \quad k_2 \quad 1}} \right) \right.$$

$$\left. + \frac{8^{\nu(N)} p^{2l_1c^0} q^{2k_1c^0} r^{2l_2c^0+2n_2c^0} s^{2k_2c^0+2n_2c^0}}{kN} \right)$$

$$\sum_{\substack{l_1,l_2 \quad 0, l_2 < n_2 \\ (l_1,l_2) \not= (0, n_2 \quad 1)}} \sum_{\substack{k_1,k_2 \quad 0, k_2 < n_2 \\ (k_1,k_2) \not= (0, n_2 \quad 1)}} \left(\frac{1}{p^{l_1}q^{k_1}} \frac{1}{r^{n_2 \quad l_2 \quad 1}} \frac{1}{s^{n_2 \quad k_2 \quad 1}} \right) + L^4 \frac{8^{\nu(N)}(pq)^{2Lc^0}(rs)^{2Lc^0+2n_2c^0}}{kN}.$$

We now use equation (6.132) to obtain

$$\sum_{p,q,r,s}{}^{0} \sum_{x \; n_2 \quad 1} G(0)G(n_2)\frac{1}{jF_{N,k}j} \sum_{f 2 F_{N,k}} F_2(n_2,p,r)F_2(n_2,q,s) \tag{6.133}$$

$$\sum_{p,q,r,s}{}^{0} \sum_{x \; n_2=1}^{\pi_N(x)} \left(\sum_{\substack{l_1,l_2 \quad 0, l_2 < n_2 \\ (l_1,l_2) \not= (0, n_2 \quad 1)}} \frac{1}{p^{l_1}} \frac{1}{r^{n_2 \quad l_2 \quad 1}} \right) \left(\sum_{\substack{k_1,k_2 \quad 0, k_2 < n_2 \\ (k_1,k_2) \not= (0, n_2 \quad 1)}} \frac{1}{q^{k_1}} \frac{1}{s^{n_2 \quad k_2 \quad 1}} \right)$$

$$+ \sum_{p,q,r,s}{}^{0} \sum_{x \; n_2=1}^{\pi_N(x)} L^4 \frac{8^{\nu(N)}(pq)^{2Lc^0}(rs)^{2Lc^0+2n_2c^0}}{kN}$$

$$\sum_{p,q,r,s}{}^{0} \sum_{x \; n_2=1}^{\pi_N(x)} \left(\frac{1}{p} + \frac{1}{r} + \frac{1}{pr} \right) \left(\frac{1}{q} + \frac{1}{s} + \frac{1}{qs} \right)$$

$$+ \sum_{p,q,r,s}{}^{0}{}_{x} \; \pi_N(x)L^4 \frac{8^{\nu(N)} x^{4Lc^0} x^{4Lc^0+4\pi_N(x)c^0}}{kN}$$

$$\sum_{p,q,r,s}{}^{0}{}_{x} \; \pi_N(x)\frac{1}{pq} + \pi_N(x)^5 L^4 \frac{8^{\nu(N)} x^{8Lc^0+4\pi_N(x)c^0}}{kN}$$

$$(\log\log x)^2 \pi_N(x)^3 + \pi_N(x)^5 L^4 \frac{8^{\nu(N)} x^{8Lc^0+4\pi_N(x)c^0}}{kN}.$$

Using equation (6.133), we obtain

$$\gamma_5(x) \tag{6.134}$$

$$= \frac{1}{4\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s} \sum_{x \geq n_2 \geq 1} G(0)G(n_2) \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} F_2(n_2,p,r)F_2(n_2,q,s)$$
$$\ll \frac{1}{\pi_N(x)^4 L^2}\left((\log\log x)^2 \pi_N(x)^3 + \pi_N(x)^5 L^4 \frac{8^{\nu(N)} x^{8Lc' + 4\pi_N(x)c'}}{kN} \right)$$
$$= \frac{(\log\log x)^2}{\pi_N(x) L^2} + \pi_N(x) L^2 \frac{8^{\nu(N)} x^{8Lc' + 4\pi_N(x)c'}}{kN}.$$

□

Proposition 6.5.6. *Let* $L = L(x) \to \infty$, *as* $x \to \infty$ *such that* $L \leq \pi_N(x)$. *With* $M_2(\rho,g;f)(x)$, *and* $M_3(\rho,g;f)(x)$ *defined in equation* (6.120), *we have*

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} (M_2(\rho,g;f)(x) + M_3(\rho,g;f)(x))$$
$$\ll \frac{1}{L} + \frac{(\log\log x)}{\pi_N(x) L} + \frac{(\log\log x)^2}{\pi_N(x) L^2} + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{8Lc' + 4\pi_N(x)c'}}{kN}.$$

Proof. Adding equations (6.122), (6.125), (6.128), (6.131) and (6.134), and using equation (6.121), we obtain

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} (M_2(\rho,g;f)(x) + M_3(\rho,g;f)(x)) \tag{6.135}$$
$$= \frac{2}{|F_{N,k}|} \sum_{f \in F_{N,k}} M_3(\rho,g;f)(x)$$
$$= \sum_{t=1}^{5} \gamma_t(x)$$
$$\ll \frac{1}{L} + \frac{(\log\log x)}{\pi_N(x) L} + L\frac{8^{\nu(N)} x^{6Lc'}}{kN} + \frac{(\log\log x)^2}{\pi_N(x)^2 L} + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{12Lc'}}{kN}$$
$$+ \frac{(\log\log x)^2}{\pi_N(x)^2 L} + L^3 \frac{8^{\nu(N)} x^{6Lc'}}{kN} + \frac{(\log\log x)^2}{\pi_N(x) L^2} + \pi_N(x) L^2 \frac{8^{\nu(N)} x^{8Lc' + 4\pi_N(x)c'}}{kN}$$
$$\ll \frac{1}{L} + \frac{(\log\log x)}{\pi_N(x) L} + \frac{(\log\log x)^2}{\pi_N(x) L^2} + \pi_N(x)^4 L^5 \frac{8^{\nu(N)} x^{8Lc' + 4\pi_N(x)c'}}{kN},$$

using $L \leq \pi_N(x)$ in the last line. □

6.5.3 Estimation for $\langle M_4(\rho,g;f)(x)\rangle$

We now look at the part of the sum $M(\rho,g;f)(x)$ with $n_1 \neq 0$ and $n_2 \neq 0$, i.e., we now estimate $\langle M_4(\rho,g;f)(x)\rangle$.

For $l \geq 0, n, n_1, n_2 \geq 1$ and for any prime p, let

$$L_p(l,n) = a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2})),$$

be the same as defined earlier.

Also,

$$A(p,q,n_1)A(r,s,n_2)$$
$$= 4(a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(q^{2n_1}) - a_f(q^{2n_1-2}))$$
$$\times (a_f(r^{2n_2}) - a_f(r^{2n_2-2}))(a_f(s^{2n_2}) - a_f(s^{2n_2-2})).$$

In this case, the innermost term is

$$\begin{aligned} & a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})A(p,q,n_1)A(r,s,n_2) \qquad (6.136)\\ =& 4a_f(p^{2l_1})(a_f(p^{2n_1}) \quad a_f(p^{2n_1 \quad 2}))\\ & a_f(q^{2l_2})(a_f(q^{2n_1}) \quad a_f(q^{2n_1 \quad 2}))\\ & a_f(r^{2k_1})(a_f(r^{2n_2}) \quad a_f(q^{2n_2 \quad 2}))\\ & a_f(s^{2k_2})(a_f(s^{2n_2}) \quad a_f(s^{2n_2 \quad 2}))\\ =& 4L_p(l_1,n_1)L_q(l_2,n_1)L_r(k_1,n_2)L_s(k_2,n_2). \end{aligned}$$

For $n \quad 1$ and for any primes $p,$ let

$$K(p,n) := \sum_{l_1 \quad 0} U(l_1)L_p(l_1,n).$$

Hence,

$$K(p,n_1) = \sum_{l_1 \quad 0} U(l_1)L_p(l_1,n_1).$$

$$K(q,n_1) = \sum_{l_2 \quad 0} U(l_2)L_q(l_2,n_1),$$

$$K(r,n_2) = \sum_{k_1 \quad 0} U(k_1)L_r(k_1,n_2),$$

and

$$K(s,n_2) = \sum_{k_2 \quad 0} U(k_2)L_s(k_2,n_2).$$

Using equations (6.136), we obtain

$$\begin{aligned} & M_4(\rho,g;f)(x) \qquad (6.137)\\ =& \frac{1}{64\pi_N(x)^4L^2} \sum_{p,q,r,s \quad x}^{0} \sum_{l_1,l_2 \quad 0} \sum_{k_1,k_2 \quad 0} \sum_{n_1,n_2 \quad 1} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)\\ & \qquad a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})A(p,q,n_1)A(r,s,n_2)\\ =& \frac{4}{64\pi_N(x)^4L^2} \sum_{p,q,r,s \quad x}^{0} \sum_{l_1,l_2 \quad 0} \sum_{k_1,k_2 \quad 0} \sum_{n_1,n_2 \quad 1} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)\\ & \qquad L_p(l_1,n_1)L_q(l_2,n_1)L_r(k_1,n_2)L_s(k_2,n_2)\\ =& \frac{4}{64\pi_N(x)^4L^2} \sum_{p,q,r,s \quad x}^{0} \sum_{n_1,n_2 \quad 1} G(n_1)G(n_2) \left(\sum_{l_1 \quad 0} U(l_1)L_p(l_1,n_1)\right)\\ & \left(\sum_{k_1 \quad 0} U(k_1)L_r(k_1,n_2)\right) \left(\sum_{l_2 \quad 0} U(l_2)L_q(l_2,n_1)\right) \left(\sum_{k_2 \quad 0} U(k_2)L_s(k_2,n_2)\right)\\ =& \frac{4}{64\pi_N(x)^4L^2} \sum_{p,q,r,s \quad x}^{0} \sum_{n_1,n_2 \quad 1} G(n_1)G(n_2)K(p,n_1)K(q,n_1)K(r,n_2)K(s,n_2). \end{aligned}$$

We now find estimate for $K(p,n_1)K(q,n_1)$.

Using Corollary 6.4.19, we have for any two distinct primes p,q and positive integer n,

$$
\begin{aligned}
&K(p,n_1)K(q,n_1)\\
=&\left(F(n_1)+\sum_{t_1=1}^{bLc+1+n_1}U(t_1,n_1)a_f(p^{2t_1})\right)\left(F(n_1)+\sum_{t_2=1}^{bLc+1+n_1}U(t_2,n_1)a_f(q^{2t_2})\right)\\
=&F(n_1)^2+F(n_1)\sum_{t_2=1}^{bLc+1+n_1}U(t_2,n_1)a_f(q^{2t_2})\\
&+F(n_1)\sum_{t_1=1}^{bLc+1+n_1}U(t_1,n_1)a_f(p^{2t_1})+\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}U(t_1,n_1)U(t_2,n_1)a_f(p^{2t_1}q^{2t_2})\\
=&F(n_1)^2+F(n_1)\sum_{t=1}^{bLc+1+n_1}U(t,n_1)(a_f(p^{2t})+a_f(q^{2t}))\\
&+\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}U(t_1,n_1)U(t_2,n_1)a_f(p^{2t_1}q^{2t_2})\\
=&F(n_1)^2+F(n_1)\sum_{t=1}^{bLc+1+n_1}G(t,n_1,p,q)+\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}H(t_1,t_2,n_1,p,q),
\end{aligned}
$$

where

$$G(t^0,n,r,s):=U(t^0,n)(a_f(r^{2t^0})+a_f(s^{2t^0})),$$

and

$$H(t,t^0,n,r,s):=U(t,n)U(t^0,n)a_f(r^{2t}s^{2t^0}),$$

for distinct primes r,s, and positive integer t,t^0,n.

Hence,

$$
\begin{aligned}
&K(r,n_2)K(s,n_2)\\
=&\left(F(n_2)+\sum_{t_3=1}^{bLc+1+n_2}U(t_3,n_2)a_f(r^{2t_3})\right)\left(F(n_2)+\sum_{t_4=1}^{bLc+1+n_2}U(t_4,n_2)a_f(s^{2t_4})\right)\\
=&F(n_2)^2+F(n_2)\sum_{t^0=1}^{bLc+1+n_2}G(t^0,n_2,r,s)+\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2}H(t_3,t_4,n_2,r,s).
\end{aligned}
$$

Therefore, distinct primes p,q,r,s and positive integers n_1,n_2,

$$
\begin{aligned}
&K(p,n_1)K(q,n_1)K(r,n_2)K(s,n_2) \qquad (6.138)\\
=&\left(F(n_1)^2+F(n_1)\sum_{t=1}^{bLc+1+n_1}G(t,n_1,p,q)+\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}H(t_1,t_2,n_1,p,q)\right)\\
&\left(F(n_2)^2+F(n_2)\sum_{t^0=1}^{bLc+1+n_2}G(t^0,n_2,r,s)+\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2}H(t_3,t_4,n_2,r,s)\right)\\
=&F(n_1)^2F(n_2)^2+F(n_1)^2F(n_2)\sum_{t^0=1}^{bLc+1+n_2}G(t^0,n_2,r,s)\\
&+F(n_1)^2\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2}H(t_3,t_4,n_2,r,s)+F(n_2)^2F(n_1)\sum_{t=1}^{bLc+1+n_1}G(t,n_1,p,q)
\end{aligned}
$$

$$+F(n_1)F(n_2)\sum_{t=1}^{bLc+1+n_1}\sum_{t^0=1}^{bLc+1+n_2} G(t,n_1,p,q)G(t^0,n_2,r,s)$$

$$+F(n_1)\sum_{t=1}^{bLc+1+n_1}\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2} G(t,n_1,p,q)H(t_3,t_4,n_2,r,s)$$

$$+F(n_2)^2\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1} H(t_1,t_2,n_1,p,q)$$

$$+F(n_2)\sum_{t^0=1}^{bLc+1+n_2}\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1} G(t^0,n_2,r,s)H(t_1,t_2,n_1,p,q)$$

$$+\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2} H(t_1,t_2,n_1,p,q)H(t_3,t_4,n_2,r,s)$$

$$=\sum_{i=1}^{9} w_i(n_1,n_2,p,q,r,s),$$

where $w_i(n_1,n_2,p,q,r,s)$ are defined in respective order by the terms in the previous equation.

Using equation (6.137) and equation (6.138), we obtain

$$\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}} M_4(\rho,g;f)(x) \tag{6.139}$$

$$=\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{0}\sum_{x\ n_1,n_2\ \ 1} G(n_1)G(n_2)K(p,n_1)K(q,n_1)K(r,n_2)K(s,n_2)$$

$$=\sum_{t=1}^{9}\frac{1}{jF_{N,k}j}\sum_{f\,2\,F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{0}\sum_{x\ n_1,n_2\ \ 1} G(n_1)G(n_2)w_t(n_1,n_2,p,q,r,s)$$

$$=\sum_{t=1}^{9}\delta_t(x),\ \text{where}$$

$$\delta_t(x) = \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}^{0}\sum_{x\ n_1,n_2\ \ 1} G(n_1)G(n_2)hw_t(n_1,n_2,p,q,r,s)i, \tag{6.140}$$

for all $i = 1, \quad , 9$.

Proposition 6.5.7. *With $\delta_1(x)$ defined in equation (6.140), we have*

$$\delta_1(x) = \left(\frac{T(g,\rho)}{4L}\right)^2 + O\left(\frac{1}{\pi_N(x)}\right),$$

where

$$T(g,\rho) = \sum_{l=1}^{1+bLc}(U(l) \quad U(l \quad 1))^2\widehat{g}\left(\frac{l}{\pi_N(x)}\right).$$

Proof. We note that $\left|F(n)\right| \quad 8k_1$, for a positive integer n.

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) \quad U(n \quad 1) = 0$.

Hence, using equation (6.138) and equation (6.139), we obtain

$$\delta_1(x) \tag{6.141}$$

$$
\begin{aligned}
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)w_1(n_1,n_2,p,q,r,s)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)w_1(n_1,n_2,p,q,r,s)\\
&= \frac{1}{16\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)F(n_1)^2F(n_2)^2\\
&= \frac{\pi_N(x)(\pi_N(x)-1)(\pi_N(x)-2)(\pi_N(x)-3)}{16\pi_N(x)^4L^2}\sum_{n_1,n_2\geq 1} G(n_1)G(n_2)F(n_1)^2F(n_2)^2\\
&= \left(1+O\left(\frac{1}{\pi_N(x)}\right)\right)\frac{1}{16L^2}\sum_{n_1,n_2\geq 1} G(n_1)G(n_2)F(n_1)^2F(n_2)^2\\
&= \left(1+O\left(\frac{1}{\pi_N(x)}\right)\right)\frac{1}{4L}\sum_{n_1\geq 1} G(n_1)F(n_1)^2\ \frac{1}{4L}\sum_{n_2\geq 1} G(n_2)F(n_2)^2\\
&= \left(1+O\left(\frac{1}{\pi_N(x)}\right)\right)\left(\frac{1}{4L}\sum_{n\geq 1} G(n)F(n)^2\right)^2\\
&= \left(1+O\left(\frac{1}{\pi_N(x)}\right)\right)\left(\frac{1}{4L}\sum_{n=1}^{1+\lfloor L\rfloor} \widehat{g}\left(\frac{n}{\pi_N(x)}\right)(U(n)-U(n-1))^2\right)^2\\
&= \left(1+O\left(\frac{1}{\pi_N(x)}\right)\right)\left(\frac{T(g,\rho)}{4L}\right)^2\\
&= \left(\frac{T(g,\rho)}{4L}\right)^2 + O\left(\frac{1}{\pi_N(x)}\right).
\end{aligned}
$$

□

Proposition 6.5.8. *With $\delta_2(x)$, and $\delta_4(x)$, defined in equation* (6.140), *we have*

$$\delta_2(x)+\delta_4(x) \ll \frac{(\log\log x)}{\pi_N(x)} + \frac{8^{\nu(N)}x^{4(L+1)c'}}{kN}.$$

Proof.

Using equation (6.138) and equation (6.139), we obtain

$$
\begin{aligned}
&\delta_4(x)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)w_4(n_1,n_2,p,q,r,s)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)\\
&\qquad F(n_2)^2F(n_1)\sum_{t=1}^{\lfloor L\rfloor+1+n_1} G(t,n_1,p,q)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}' \sum_{x\geq n_1,n_2\geq 1} G(n_2)G(n_1)\\
&\qquad F(n_1)^2F(n_2)\sum_{t=1}^{\lfloor L\rfloor+1+n_2} G(t,n_2,p,q)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{4}{64\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1,n_2 \quad 1} G(n_2)G(n_1) \\
&\qquad\qquad F(n_1)^2 F(n_2) \sum_{t=1}^{bLc+1+n_2} G(t, n_2, r, s) \\
&= \delta_2(x),
\end{aligned}
$$

where we first interchange variables n_1, n_2 in the third line, and then we replace the dummy variable by p, q, r, s by r, s, p, q respectively in the fourth line.

We note that $|F(n)| \quad 8k_1,$ for a positive integer $n.$

Now, $n > 1 + bLc$, implies $U(n) = 0$ and hence, $F(n) = U(n) \quad U(n \quad 1) = 0.$

Hence,

$$
\begin{aligned}
&\delta_2(x) + \delta_4(x) \\
&= 2\delta_2(x) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{8}{64\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1,n_2 \quad 1} G(n_1)G(n_2) w_2(n_1, n_2, p, q, r, s) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{8}{64\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1,n_2 \quad 1} G(n_1)G(n_2) \\
&\qquad\qquad F(n_1)^2 F(n_2) \sum_{t'=1}^{bLc+1+n_2} G(t', n_2, r, s) \\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{8}{64\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1,n_2 \quad 1} G(n_1)G(n_2) \\
&\qquad\qquad F(n_1)^2 F(n_2) \sum_{t'=1}^{bLc+1+n_2} U(t', n)(a_f(r^{2t'}) + a_f(s^{2t'})) \\
&= \frac{16}{64\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1=1}^{1+bLc} \sum_{n_2=1}^{1+bLc} G(n_1)G(n_2) \\
&\qquad\qquad F(n_1)^2 F(n_2) \sum_{t'=1}^{bLc+1+n_2} U(t', n) \left(\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} a_f(r^{2t'}) \right) \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1=1}^{1+bLc} \sum_{n_2=1}^{1+bLc} \sum_{t=1}^{bLc+1+n_2} \left(\frac{1}{r^t} + \frac{8^{\nu(N)} r^{2t'c'}}{kN} \right) \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r,s}{}' \sum_{x \quad n_1=1}^{1+bLc} \sum_{n_2=1}^{1+bLc} \left(\frac{1}{r} + \frac{1}{r^2} + \quad + \frac{8^{\nu(N)} r^{2(bLc+1+n_2)c'}}{kN} \right) \\
&\quad \frac{1}{\pi_N(x)^4 L^2} \sum_{p,q,r,s \quad x}{}' \left(\frac{1}{r} L^2 + L^2 \frac{8^{\nu(N)} r^{2(L+1+L+1)c'}}{kN} \right) \\
&\quad \frac{1}{\pi_N(x)^4 L^2} (\log\log x) \pi_N(x)^3 L^2 + \frac{8^{\nu(N)} x^{4(L+1)c'}}{kN} \\
&\quad \frac{(\log\log x)}{\pi_N(x)} + \frac{8^{\nu(N)} x^{4(L+1)c'}}{kN}.
\end{aligned}
$$

Therefore,

$$
\delta_2(x) + \delta_4(x) \quad \frac{(\log\log x)}{\pi_N(x)} + \frac{8^{\nu(N)} x^{4(L+1)c'}}{kN}. \tag{6.142}
$$

□

Proposition 6.5.9. *With* $\delta_3(x)$*, and* $\delta_7(x)$*, defined in equation* (6.140)*, we have*

$$\delta_3(x)+\delta_7(x) \ll \frac{(\log\log x)^2}{L\pi_N(x)} + \frac{\pi_N(x)}{L}\frac{8^{\nu(N)}x^{2(L+1)c'+2\pi_N(x)c'}}{kN}.$$

Proof. Using equation (6.138) and equation (6.139), we obtain

$$\begin{aligned}
&\delta_7(x)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1,n_2\ge 1} G(n_1)G(n_2)w_7(n_1,n_2,p,q,r,s)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1,n_2\ge 1} G(n_1)G(n_2)\\
&\qquad F(n_2)^2\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}H(t_1,t_2,n_1,p,q)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1,n_2\ge 1} G(n_2)G(n_1)\\
&\qquad F(n_1)^2\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2}H(t_3,t_4,n_2,p,q)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1,n_2\ge 1} G(n_1)G(n_2)\\
&\qquad F(n_1)^2\sum_{t_3=1}^{bLc+1+n_2}\sum_{t_4=1}^{bLc+1+n_2}H(t_3,t_4,n_2,r,s)\\
=&\delta_3(x),
\end{aligned}$$

where we first interchange variables n_1,n_2 in the third line, and then we replace the dummy variable by p,q,r,s by r,s,p,q respectively in the fourth line.

We note that $|F(n)| \le 8k_1$, for a positive integer n.

Now, $n > 1+bLc$, implies $U(n)=0$ and hence, $F(n)=U(n)-U(n-1)=0$.

Hence,

$$\begin{aligned}
&\delta_3(x)+\delta_7(x)\\
=&2\delta_7(x)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{8}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1,n_2\ge 1} G(n_1)G(n_2)w_7(n_1,n_2,p,q,r,s)\\
=&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{8}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1,n_2\ge 1} G(n_1)G(n_2)\\
&\qquad F(n_2)^2\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}H(t_1,t_2,n_1,p,q)\\
\ll&\frac{8}{64\pi_N(x)^4L^2}\sum_{p,q,r,s\le x}{}' \sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=1}^{1+bLc} G(n_1)G(n_2)\\
&\qquad F(n_2)^2\sum_{t_1=1}^{bLc+1+n_1}\sum_{t_2=1}^{bLc+1+n_1}U(t_1,n_1)U(t_2,n_1)\left\langle a_f(p^{2t_1}q^{2t_2})\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&\ll \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=1}^{1+\lfloor L\rfloor}\sum_{t_1=1}^{\lfloor L\rfloor+1+n_1}\sum_{t_2=1}^{\lfloor L\rfloor+1+n_1}\left(\frac{1}{p^{t_1}q^{t_2}}+\frac{8^{\nu(N)}p^{2t_1c'}q^{2t_2c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\sum_{n_1=1}^{\pi_N(x)}\sum_{n_2=1}^{1+\lfloor L\rfloor}\left(\frac{1}{(p-1)(q-1)}+\frac{8^{\nu(N)}x^{2(\lfloor L\rfloor+1+n_1)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\left(\frac{1}{pq}L\pi_N(x)+L\pi_N(x)\frac{8^{\nu(N)}x^{2(L+1+\pi_N(x))c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}(\log\log x)^2\pi_N(x)^2L\pi_N(x)+\frac{\pi_N(x)}{L}\frac{8^{\nu(N)}x^{2(L+1+\pi_N(x))c'}}{kN}\\
&\ll \frac{(\log\log x)^2}{L\pi_N(x)}+\frac{\pi_N(x)}{L}\frac{8^{\nu(N)}x^{2(L+1)c'+2\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Therefore,

$$
\delta_3(x)+\delta_7(x)\ll\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{\pi_N(x)}{L}\frac{8^{\nu(N)}x^{2(L+1)c'+2\pi_N(x)c'}}{kN}. \tag{6.143}
$$

□

Proposition 6.5.10. *With* $\delta_6(x)$, *and* $\delta_8(x)$, *defined in equation* (6.140), *we have*

$$
\delta_6(x)+\delta_8(x)\ll\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{6(L+1)c'+6\pi_N(x)c'}}{kN}.
$$

Proof. Using equation (6.138) and equation (6.139), we obtain

$$
\begin{aligned}
&\delta_8(x)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\sum_{n_1,n_2\ge 1}G(n_1)G(n_2)w_8(n_1,n_2,p,q,r,s)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\sum_{n_1,n_2\ge 1}G(n_1)G(n_2)\\
&\qquad F(n_2)\sum_{t'=1}^{\lfloor L\rfloor+1+n_2}\sum_{t_1=1}^{\lfloor L\rfloor+1+n_1}\sum_{t_2=1}^{\lfloor L\rfloor+1+n_1}G(t',n_2,r,s)H(t_1,t_2,n_1,p,q)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\sum_{n_1,n_2\ge 1}G(n_2)G(n_1)\\
&\qquad F(n_1)\sum_{t=1}^{\lfloor L\rfloor+1+n_1}\sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\sum_{t_4=1}^{\lfloor L\rfloor+1+n_2}G(t,n_1,r,s)H(t_3,t_4,n_2,p,q)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s}{}'\sum_x\sum_{n_1,n_2\ge 1}G(n_2)G(n_1)\\
&\qquad F(n_1)\sum_{t=1}^{\lfloor L\rfloor+1+n_1}\sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\sum_{t_4=1}^{\lfloor L\rfloor+1+n_2}G(t,n_1,p,q)H(t_3,t_4,n_2,r,s)\\
&=\delta_6(x),
\end{aligned}
$$

where we first interchange variables n_1,n_2 in the third line, and then we replace the dummy variable by p,q,r,s by r,s,p,q respectively in the fourth line.

Hence,

$$
\delta_6(x)+\delta_8(x)
$$

$$
\begin{aligned}
&= 2\delta_6(x)\\
&= \frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \frac{8}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\ge n_1,n_2\ge 1} G(n_1)G(n_2)w_6(n_1,n_2,p,q,r,s)\\
&= \frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \frac{8}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\ge n_1,n_2\ge 1} G(n_1)G(n_2)\\
&\qquad\qquad F(n_1)\sum_{t=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} G(t,n_1,p,q)H(t_3,t_4,n_2,r,s)\\
&= \frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \frac{8}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\ge n_1,n_2\ge 1} G(n_1)G(n_2)F(n_1)\\
&\quad \sum_{t=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} U(t,n_1)U(t_3,n_2)U(t_4,n_2)(a_f(p^{2t})+a_f(q^{2t}))(a_f(r^{2t_3}s^{2t_4}))\\
&= \frac{1}{|\mathcal{F}_{N,k}|}\sum_{f\in\mathcal{F}_{N,k}} \frac{8}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\ge n_1,n_2\ge 1} G(n_1)G(n_2)F(n_1)\\
&\quad \sum_{t=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} U(t,n_1)U(t_3,n_2)U(t_4,n_2)(a_f(p^{2t}r^{2t_3}s^{2t_4})+a_f(q^{2t}r^{2t_3}s^{2t_4}))\\
&= \frac{16}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\ge n_1,n_2\ge 1} G(n_1)G(n_2)F(n_1)\\
&\quad \sum_{t=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} U(t,n_1)U(t_3,n_2)U(t_4,n_2)\left\langle a_f(p^{2t}r^{2t_3}s^{2t_4})\right\rangle\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\ge n_1,n_2\ge 1}\ \sum_{t=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_3,t_4=1}^{\lfloor L\rfloor+1+n_1}\left(\frac{1}{p^tr^{t_3}s^{t_4}}+\frac{8^{\nu(N)}p^{2tc'}r^{2t_3c'}s^{2t_4c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s\le x}\ \sum_{n_1=1}^{\pi_N(x)}\ \sum_{n_2=1}^{\pi_N(x)}\left(\frac{1}{(p-1)(r-1)(s-1)}+\frac{8^{\nu(N)}x^{6(\lfloor L\rfloor+1+n_1)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s\le x}\left(\frac{1}{prs}\pi_N(x)^2+\pi_N(x)^2\frac{8^{\nu(N)}x^{6(L+1+\pi_N(x))c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}(\log\log x)^3\pi_N(x)\pi_N(x)^2+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{6(L+1)c'+6\pi_N(x)c'}}{kN}\\
&\ll \frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{6(L+1)c'+6\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Hence,

$$
\delta_6(x)+\delta_8(x)\ll \frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{6(L+1)c'+6\pi_N(x)c'}}{kN}. \tag{6.144}
$$

□

Proposition 6.5.11. *With* $\delta_5(x)$ *defined in equation* (6.140), *we have*

$$
\delta_5(x)\ll \frac{(\log\log x)^2}{\pi_N(x)^2}+\frac{8^{\nu(N)}x^{8(L+1)c'}}{kN}.
$$

Proof. We note that $|F(n)|\le 8k_1$, for a positive integer n.

Now, $n>1+\lfloor L\rfloor$, implies $U(n)=0$ and hence, $F(n)=U(n)-U(n-1)=0$.

Hence, using equation (6.138) and equation (6.139), we obtain

$$
\begin{aligned}
&\delta_5(x)\\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{4}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1,n_2 \ge 1} G(n_1)G(n_2)w_5(n_1,n_2,p,q,r,s)\\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{4}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1,n_2 \ge 1} G(n_1)G(n_2)\\
&\qquad F(n_1)F(n_2) \sum_{t=1}^{\lfloor bLc+1+n_1\rfloor} \sum_{t'=1}^{\lfloor bLc+1+n_2\rfloor} G(t,n_1,p,q)G(t',n_2,r,s)\\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{4}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1,n_2 \ge 1} G(n_1)G(n_2)F(n_1)F(n_2)\\
&\qquad \sum_{t=1}^{\lfloor bLc+1+n_1\rfloor} \sum_{t'=1}^{\lfloor bLc+1+n_2\rfloor} U(t,n_1)(a_f(p^{2t}) + a_f(q^{2t}))U(t',n_2)(a_f(r^{2t'}) + a_f(s^{2t'}))\\
&= \frac{16}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1=1}^{1+\lfloor bLc\rfloor} \sum_{n_2=1}^{1+\lfloor bLc\rfloor} G(n_1)G(n_2)F(n_1)F(n_2)\\
&\qquad \sum_{t=1}^{\lfloor bLc+1+n_1\rfloor} \sum_{t'=1}^{\lfloor bLc+1+n_2\rfloor} U(t,n_1)U(t',n_2) \left\langle a_f(p^{2t}r^{2t'})\right\rangle\\
&\ll \frac{1}{\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1=1}^{1+\lfloor bLc\rfloor} \sum_{n_2=1}^{1+\lfloor bLc\rfloor} \sum_{t=1}^{\lfloor bLc+1+n_1\rfloor} \sum_{t'=1}^{\lfloor bLc+1+n_2\rfloor} \left(\frac{1}{p^t r^{t'}} + \frac{8^{\nu(N)} p^{2tc'} r^{2t'c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1=1}^{1+\lfloor bLc\rfloor} \sum_{n_2=1}^{1+\lfloor bLc\rfloor} \left(\frac{1}{(p-1)(r-1)} + \frac{8^{\nu(N)} x^{2(\lfloor bLc\rfloor+1+n_1)c'} x^{2(\lfloor bLc\rfloor+1+n_2)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \left(\frac{1}{pr}L^2 + L^2 \frac{8^{\nu(N)} x^{8(\lfloor bLc\rfloor+1)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4 L^2} (\log\log x)^2 \pi_N(x)^2 L^2 + \frac{8^{\nu(N)} x^{8(L+1)c'}}{kN}\\
&\ll \frac{(\log\log x)^2}{\pi_N(x)^2} + \frac{8^{\nu(N)} x^{8(L+1)c'}}{kN}.
\end{aligned}
$$

Hence,

$$
\delta_5(x) \ll \frac{(\log\log x)^2}{\pi_N(x)^2} + \frac{8^{\nu(N)} x^{8(L+1)c'}}{kN}. \tag{6.145}
$$

□

Proposition 6.5.12. *With* $\delta_9(x)$ *defined in equation* (6.140)*, we have*

$$
\delta_9(x) \ll \frac{(\log\log x)^4}{\pi_N(x)^2 L^2} + \frac{\pi_N(x)^2}{L^2} \frac{8^{\nu(N)} x^{8(L+1)c' + 8\pi_N(x)c'}}{kN}.
$$

Proof. Using equation (6.138) and equation (6.139), we obtain

$$
\begin{aligned}
&\delta_9(x)\\
&= \frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \frac{4}{64\pi_N(x)^4 L^2} \sideset{}{'}\sum_{p,q,r,s \le x} \sum_{n_1,n_2 \ge 1} G(n_1)G(n_2)w_9(n_1,n_2,p,q,r,s)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)\\
&\qquad \sum_{t_1=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_2=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} H(t_1,t_2,n_1,p,q)H(t_3,t_4,n_2,r,s)\\
&= \frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} \frac{4}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)\sum_{t_1=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_2=1}^{\lfloor L\rfloor+1+n_1}\\
&\qquad \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} U(t_1,n_1)U(t_2,n_1)a_f(p^{2t_1}q^{2t_2})U(t_3,n_2)U(t_4,n_2)a_f(r^{2t_3}s^{2t_4})\\
&= \frac{4}{64\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s}\ \sum_{x\geq n_1,n_2\geq 1} G(n_1)G(n_2)\sum_{t_1=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_2=1}^{\lfloor L\rfloor+1+n_1}\\
&\qquad \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2} U(t_1,n_1)U(t_2,n_1)U(t_3,n_2)U(t_4,n_2)\left\langle a_f(p^{2t_1}q^{2t_2}r^{2t_3}s^{2t_4})\right\rangle\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s\leq x}\ \sum_{n_1=1}^{\pi_N(x)}\ \sum_{n_2=1}^{\pi_N(x)}\ \sum_{t_1=1}^{\lfloor L\rfloor+1+n_1}\ \sum_{t_2=1}^{\lfloor L\rfloor+1+n_1}\\
&\qquad \sum_{t_3=1}^{\lfloor L\rfloor+1+n_2}\ \sum_{t_4=1}^{\lfloor L\rfloor+1+n_2}\left(\frac{1}{p^{t_1}q^{t_2}r^{t_3}s^{t_4}} + \frac{8^{\nu(N)}p^{2t_1c'}q^{2t_2c'}r^{2t_3c'}s^{2t_4c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s\leq x}\ \sum_{n_1=1}^{\pi_N(x)}\ \sum_{n_2=1}^{\pi_N(x)}\left(\frac{1}{(p-1)(q-1)(r-1)(s-1)}\right.\\
&\qquad\qquad \left. + \frac{8^{\nu(N)}x^{4(\lfloor L\rfloor+1+n_1)c'}x^{4(\lfloor L\rfloor+1+n_2)c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}\sideset{}{'}\sum_{p,q,r,s\leq x}\left(\frac{1}{pqrs}\pi_N(x)^2 + \pi_N(x)^2\frac{8^{\nu(N)}x^{8(\lfloor L\rfloor+1+\pi_N(x))c'}}{kN}\right)\\
&\ll \frac{1}{\pi_N(x)^4L^2}(\log\log x)^4\pi_N(x)^2 + \frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1+\pi_N(x))c'}}{kN}\\
&\ll \frac{(\log\log x)^4}{\pi_N(x)^2L^2} + \frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Hence,

$$
\delta_9(x) \ll \frac{(\log\log x)^4}{\pi_N(x)^2L^2} + \frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}. \tag{6.146}
$$

□

Proposition 6.5.13. *With* $\delta_t(x)$ $(t=1,..,9)$ *defined in equation* (6.140)*, we have*

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} M_4(\rho,g;f)(x) - \left(\frac{T(g,\rho)}{4L}\right)^2\\
&\ll \frac{(\log\log x)}{\pi_N(x)} + \frac{(\log\log x)^2}{L\pi_N(x)} + \frac{(\log\log x)^3}{\pi_N(x)L^2} + \frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Proof. Adding equations (6.142), (6.143), (6.144), (6.145), and (6.146), and using equation

(6.139), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} M_4(\rho,g;f)(x)\quad\left(\frac{T(g,\rho)}{4L}\right)^2 \qquad (6.147)\\
&=\sum_{t=1}^{9}\delta_t(x)\quad\left(\frac{T(g,\rho)}{4L}\right)^2\\
&\quad \frac{1}{\pi_N(x)}+\frac{(\log\log x)}{\pi_N(x)}+\frac{8^{\nu(N)}x^{4(L+1)c'}}{kN}+\frac{(\log\log x)^2}{L\pi_N(x)}\\
&+\frac{\pi_N(x)}{L}\frac{8^{\nu(N)}x^{2(L+1)c'+2\pi_N(x)c'}}{kN}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{6(L+1)c'+6\pi_N(x)c'}}{kN}\\
&+\frac{(\log\log x)^2}{\pi_N(x)^2}+\frac{8^{\nu(N)}x^{8(L+1)c'}}{kN}+\frac{(\log\log x)^4}{\pi_N(x)^2L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}\\
&\quad \frac{(\log\log x)}{\pi_N(x)}+\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN},
\end{aligned}
$$

using the fact, $\log\log x \quad \pi_N(x)$ in the last line. □

Proposition 6.5.14. *For positive integers k and N with k even, and $L \quad \pi_N(x)$,*

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} M(\rho,g;f)(x)\quad\left(\frac{T(g,\rho)}{4L}\right)^2\\
&\quad \frac{1}{L}+\frac{(\log\log x)}{\pi_N(x)}+\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}.
\end{aligned}
$$

Proof. Adding inequations (6.119), (6.135) and (6.147) and using equation (6.118), we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} M(\rho,g;f)(x)\quad\left(\frac{T(g,\rho)}{4L}\right)^2 \qquad (6.148)\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(\sum_{i=1}^{4} M_i(\rho,g;f)(x)\right)\quad\left(\frac{T(g,\rho)}{4L}\right)^2\\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} M_1(\rho,g;f)(x)+\frac{2}{|F_{N,k}|}\sum_{f\in F_{N,k}} M_2(\rho,g;f)(x)\\
&+\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}} M_4(\rho,g;f)(x)\quad\left(\frac{T(g,\rho)}{4L}\right)^2\\
&\quad \frac{1}{L^2}+L^2\frac{8^{\nu(N)}x^{8Lc'}}{kN}\\
&+\frac{1}{L}+\frac{(\log\log x)}{\pi_N(x)L}+\frac{(\log\log x)^2}{\pi_N(x)L^2}+\pi_N(x)^4L^5\frac{8^{\nu(N)}x^{8Lc'+4\pi_N(x)c'}}{kN}\\
&+\frac{(\log\log x)}{\pi_N(x)}+\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}\\
&\quad \frac{1}{L}+\frac{(\log\log x)}{\pi_N(x)}+\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN},
\end{aligned}
$$

using the fact $\pi_N(x)^2L^7 \quad x^{8c'+4\pi_N(x)c'}$, which holds since $\pi_N(x)^2L^7 \quad x^2x^{\frac{7L}{2}} \quad x^{8c'}x^{4Lc'}$ $x^{8c'+4\pi_N(x)c'}$, where $c' > \frac{3}{2}$. □

Theorem 4.3.1 has been restated again in Theorems 6.5.15 and 6.5.16 for the convenience of the reader.

Theorem 6.5.15. *Let us consider families $F_{N,k}$ with levels $N=N(x)$ and even weights $k=k(x)$. Let g, ρ be real-valued, even functions $\in C^{\infty}(\mathbb{R})$ in the Schwartz class with compactly supported Fourier transforms and $L \ll \pi_N(x)$, be such that $L=L(x)\to\infty$ as $x\to\infty$, Then, for $0<\psi<1$, $\psi\neq 1/2$ we have*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x)\right)^2-\left(\frac{T(g,\rho)}{4L}\right)^2$$
$$\ll \frac{1}{L}+\frac{L}{\pi_N(x)}+\frac{\log\log x}{\pi_N(x)}+\frac{L^2(\log\log x)}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{(\log\log x)^2}{\pi_N(x)L}$$
$$+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN},$$

where $c'>\frac{3}{2}$ is an absolute constant.

Proof. Adding inequations (6.60), (6.116) and (6.148), and using equation (6.1), we obtain

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x)\right)^2-\left(\frac{T(g,\rho)}{4L}\right)^2$$
$$=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}K(\rho,g;f)(x)+\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}L(\rho,g;f)(x)$$
$$+\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}M(\rho,g;f)(x)-\left(\frac{T(g,\rho)}{4L}\right)^2$$
$$\ll\frac{L}{\pi_N(x)}+\frac{L\log\log x}{\pi_N(x)^2}+\frac{L^2(\log\log x)^2}{\pi_N(x)^4}+\frac{(\log\log x)^2}{\pi_N(x)^2}+(L+\pi_N(x))^2L^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}$$
$$+\frac{L}{\pi_N(x)}+\frac{L^2(\log\log x)}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{(\log\log x)^2}{\pi_N(x)L}$$
$$+\frac{\log\log x}{\pi_N(x)}+L^2\pi_N(x)^2\frac{8^{\nu(N)}x^{8Lc'+8\pi_N(x)c'}}{kN}$$
$$+\frac{1}{L}+\frac{(\log\log x)}{\pi_N(x)}+\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{\pi_N(x)^2}{L^2}\frac{8^{\nu(N)}x^{8(L+1)c'+8\pi_N(x)c'}}{kN}$$
$$\ll\frac{1}{L}+\frac{L}{\pi_N(x)}+\frac{\log\log x}{\pi_N(x)}+\frac{L^2(\log\log x)}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}+\frac{(\log\log x)^2}{L\pi_N(x)}+\frac{(\log\log x)^3}{\pi_N(x)L^2}$$
$$+\frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN},$$

with the choice of L satisfying $L\ll\pi_N(x)$. □

Theorem 6.5.16. *Let us consider families $F_{N,k}$ with levels $N=N(x)$ and even weights $k=k(x)$. Let g, ρ be real-valued, even functions $\in C^{\infty}(\mathbb{R})$ in the Schwartz class with compactly supported Fourier transforms and $L \ll \pi_N(x)$, be such that $L=L(x)\to\infty$ as $x\to\infty$, Then, for $0<\psi<1$, $\psi\neq 1/2$ we have*

$$\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x)-\frac{T(g,\rho)}{4L}\right)^2$$
$$\ll\frac{1}{L}+\frac{L}{\pi_N(x)}+\frac{\log\log x}{\pi_N(x)}+\frac{L(\log\log x)^2}{\pi_N(x)}+\frac{L^2(\log\log x)}{\pi_N(x)^2}+\frac{L^3(\log\log x)^2}{\pi_N(x)^3}$$
$$+\frac{(\log\log x)^2}{\pi_N(x)L}+\frac{(\log\log x)^3}{\pi_N(x)L^2}+\frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN},$$

where $c'>\frac{3}{2}$ is an absolute constant.

Proof. Using Theorems 5.5.3 and 6.5.15 with the fact $\frac{T(g,\rho)}{4L} \ll 1$, we obtain

$$
\begin{aligned}
&\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x) - \frac{T(g,\rho)}{4L}\right)^2 \\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x)\right)^2 + \left(\frac{T(g,\rho)}{4L}\right)^2 - 2\frac{T(g,\rho)}{4L}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}R_2(\rho,g;f)(x) \\
&=\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x)\right)^2 - \left(\frac{T(g,\rho)}{4L}\right)^2 \\
&\quad - 2\frac{T(g,\rho)}{4L}\left(\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}R_2(\rho,g;f)(x) - \frac{T(g,\rho)}{4L}\right) \\
&\ll \frac{1}{L} + \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x)} + \frac{L^2(\log\log x)}{\pi_N(x)^2} + \frac{L^3(\log\log x)^2}{\pi_N(x)^3} + \frac{(\log\log x)^2}{\pi_N(x)L} \\
&\quad + \frac{(\log\log x)^3}{\pi_N(x)L^2} + \frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN} + \frac{1}{L} + \frac{L(\log\log x)^2}{\pi_N(x)} + \frac{8^{\nu(N)}x^{(8\pi_N(x)+8)c'}}{kN} \\
&\ll \frac{1}{L} + \frac{L}{\pi_N(x)} + \frac{\log\log x}{\pi_N(x)} + \frac{L^2(\log\log x)}{\pi_N(x)^2} + \frac{L^3(\log\log x)^2}{\pi_N(x)^3} + \frac{L(\log\log x)^2}{\pi_N(x)} \\
&\quad + \frac{(\log\log x)^2}{\pi_N(x)L} + \frac{(\log\log x)^3}{\pi_N(x)L^2} + \frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN},
\end{aligned}
$$

with the choice of L satisfying $L \ll \pi_N(x)$. □

Theorem 6.5.17. *Let us consider families $F_{N,k}$ with levels $N = N(x)$ and even weights $k = k(x)$ such that*

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty \text{ as } x \to \infty.$$

Let g, ρ be real-valued, even functions $\in C^\infty(\mathbb{R})$ in the Schwartz class with compactly supported Fourier transforms supported in $[-1,1]$ and $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$ be such that $L = L(x) \to \infty$ as $x \to \infty$. Then, for $0<\psi<1$, $\psi \neq 1/2$, and $A = 2\sin^2\pi\psi$, we have

$$\lim_{x\to\infty}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x)\right)^2 = (A^2\widehat{g}(0)\rho - \rho(0))^2,$$

and

$$\lim_{x\to\infty}\frac{1}{|F_{N,k}|}\sum_{f\in F_{N,k}}\left(R_2(\rho,g;f)(x) - A^2\widehat{g}(0)\rho - \rho(0)\right)^2 = 0.$$

Proof. We note that $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$ implies that $L \ll \pi_N(x)$ and $\lim_{x\to\infty}\frac{L(\log\log x)^2}{\pi_N(x)} = 0$.

Hence, all the lower order terms except the first and last terms in Theorem 6.5.15 goes to 0, as $x \to \infty$.

Also, using Lemma 3.3.7 to the condition

$$\frac{\log\left(kN/8^{\nu(N)}\right)}{x} \to \infty \text{ as } x \to \infty,$$

we obtain that the last term $\frac{1}{L^2}\frac{8^{\nu(N)}x^{26\pi_N(x)c'}}{kN}$ also goes to 0, as $x \to \infty$.

Using Proposition 5.5.2, we have

$$\lim_{x\to\infty} \frac{T(g,\rho)}{4L} = A^2\widehat{g}(0)\rho \ast \rho(0).$$

Combining all the results obtained in Theorem 6.5.15, we obtain

$$\lim_{x\to\infty} \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} R_2(\rho, g; f)(x)^2 = (A^2\widehat{g}(0)\rho \ast \rho(0))^2. \tag{6.149}$$

We denote $\beta = A^2\widehat{g}(0)\rho \ast \rho(0)$.

Using equation (6.149) with the growth condition on weights and levels, and $L(x) = o\left(\frac{\pi_N(x)}{(\log\log x)^2}\right)$, we obtain

$$\begin{aligned}
&\lim_{x\to\infty} \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} \Big(R_2(\rho, g; f)(x) - A^2\,\widehat{g}(0)(\rho \ast \rho)(0)\Big)^2 \\
=& \lim_{x\to\infty} \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} R_2(\rho, g; f)(x)^2 + \beta^2 - 2\beta \lim_{x\to\infty} \frac{1}{|F_{N,k}|} \sum_{f\in F_{N,k}} R_2(\rho, g; f)(x) \\
=&\beta^2 + \beta^2 - 2\beta^2 \\
=&0.
\end{aligned}$$

□

Remark 6.5.18. *The above theorem tells us that* $\mathrm{E}[(R_2(\rho, g; f)(x))^2] \sim \mathrm{E}[(R_2(\rho, g; f)(x))]^2$ *for very rapidly growing families* $F_{N,k}$. *In turn, these are asymptotic to what one would expect from a Poissonnian model. This indicates an affirmative answer to Question 4.1.2 for a random Hecke newform in* $S_k(N)$ *with appropriate parameters as specified in Theorem 4.3.1(c).*

Chapter 7

Future Research plans

There are several interesting directions for future research. We start by introducing questions of Katz and Sarnak. Let $H(\theta_f(p))$ be as in equation (4.2). We arrange the set $\{H(\theta_f(p)),\ p \leq x,\ (p,N) = 1\}$ in ascending order:

$$0 \leq H(\theta_f(p))_1 \leq H(\theta_f(p))_2 \leq \cdots \leq H(\theta_f(p))_{\pi_N(x)} \leq 1$$

and consider the consecutive spacings $H(\theta_f(p))_{i+1} - H(\theta_f(p))_i$, $1 \leq i \leq \pi_N(x)$. Katz asked the following question.

Question 7.0.1 (Katz). *Is the level spacing distribution of the sequence $\{H(\theta_f(p)), p \leq x, (p,N) = 1\}$ Poissonnian? That is, for any $[a,b] \subset [0, \infty)$, is the limit*

$$\lim_{x \to \infty} \frac{1}{\pi_N(x)} \# \left\{ 1 \leq i \leq \pi_N(x) - 1 : H(\theta_f(p))_{i+1} - H(\theta_f(p))_i \in \left[\frac{a}{\pi_N(x)}, \frac{b}{\pi_N(x)} \right] \right\} = \int_a^b e^{-t} dt?$$

Katz and Sarnak [KS99, Page 9] also considered a vertical variant of the above problem. Let p be a prime and let N be a positive integer such that $(p,N) = 1$. One defines the multiset

$$A_p(N,k) = \{H(\theta_f(p)),\ f \in \mathcal{F}_{N,k}\} \subseteq [0,1].$$

The multiset $A_p(N,k)$ is then arranged in ascending order as follows:

$$0 \leq H(\theta_{f_1}(p)) \leq H(\theta_{f_2}(p)) \leq \cdots \leq H(\theta_{f_r}(p)) \leq 1.$$

Here, $r = |\mathcal{F}_{N,k}|$. Katz and Sarnak consider the level spacings among the multisets $A_p(N,k)$ for $k = 2$, $N \to \infty$ (N coprime to p) and ask if the level spacing distribution matches that of a sequence of independent and uniform random points on $[0,1]$. More precisely, they ask the following question.

Question 7.0.2 ([KS99, Page 9]). *Let p be a fixed prime. Let k be an even positive integer and N be a positive integer coprime to p. Is the level spacing distribution of the multisets $A_p(N,k)$ Poissonnian as N ranges over positive integers coprime to p? That is, is it true that for any $[a,b] \subset [0, \infty)$,*

$$\lim_{\substack{N \to \infty \\ (p,N)=1}} \frac{1}{|\mathcal{F}_{N,k}|} \# \left\{ 1 \leq i \leq |\mathcal{F}_{N,k}| - 1 : H(\theta_{f_{i+1}}(p)) - H(\theta_{f_i}(p)) \in \left[\frac{a}{|\mathcal{F}_{N,k}|}, \frac{b}{|\mathcal{F}_{N,k}|} \right] \right\}$$
$$= \int_a^b e^{-t} dt?$$

We note that Katz and Sarnak state the above question for prime levels N and $k = 2$. A generalization of their question to higher weights and a larger class of levels N is what has been stated above. As a partial answer to their question (for $k = 2$ and N prime), they average over primes and state the following theorem:

Theorem 7.0.3 ([KS99, Page 9]). *For any* $[a,b] \subset [0, \infty)$,

$$\lim_{\substack{x \to \infty \\ N \to \infty \\ N \text{ prime}}} \frac{1}{\pi(x)} \sum_{\substack{p \leq x \\ p \nmid N}} \frac{1}{|F_{N,k}|} \# \left\{ 1 \leq i \leq |F_{N,k}| - 1 : H(\theta_{f_{i+1}}(p)) - H(\theta_{f_i}(p)) \in \left[\frac{a}{|F_{N,k}|}, \frac{b}{|F_{N,k}|} \right] \right\} = \int_a^b e^{-t} dt.$$

Remark 6.5.18 motivates us to think of Question 7.0.4, i.e., what happens to the expected value of the higher moments of smooth localised pair correlation function $R_2(\rho, g; f)(x)$.

If Question 4.1.2 for $\delta = \frac{1}{AL}$ has an affirmative answer, then for a fixed $f \in F_{N,k}$,

$$(R_2(\rho, g; f)(x))^r \approx (C_\psi A^2 \widehat{g}(0)\rho - \rho(0))^r,$$

for each $r \geq 1$. Therefore, in the spirit of Theorem 6.5.15, it would be natural to ask what $(R_2(\rho, g; f)(x))^r$ is, for a random $f \in F_{N,k}$.

Question 7.0.4. *Can we show that*

$$\frac{1}{|F_{N,k}|} \sum_{f \in F_{N,k}} \left(R_2(\rho, g; f)(x) \right)^r \approx (C_\psi A^2 \widehat{g}(0)\rho - \rho(0))^r$$

for $r \geq 3$, under the same growth conditions on L, N, and k as in Theorem 6.5.17? In other words, we ask the following:

Is it true that for any integer $r \geq 3$,

$$\mathrm{E}[(R_2(\rho, g; f)(x))^r] \approx \mathrm{E}[(R_2(\rho, g; f)(x))]^r \text{ as } x \to \infty?$$

We expect that the techniques used in the proof of Theorem 4.3.1 can be generalized to answer Question 7.0.4, but the range of L might change. We have partially answered this question in [MS23].

Question 7.0.5. *Can we answer Question 7.0.1 of Katz? We would like to first study the m-level correlation of the Sato-Tate sequence for integers $m \geq 2$ and use the road map provided by Kurlberg and Rudnick [KR99, Appendix A] to derive the level spacing distribution function of this sequence from these.*

Question 7.0.6. *Can we answer Question 7.0.2 of Katz and Sarnak for the vertical Sato-Tate families $A_p(N,k)$? As above, we would have to first investigate the m-level correlation of the vertical Sato-Tate families.*

Question 7.0.7. *Can we investigate the local analogues of the pair correlation function of vertical Sato-Tate families?*

Question 7.0.8. *Can we investigate the local analogues of the level spacing distribution function of Sato-Tate sequences as well as vertical Sato-Tate families?*

Question 7.0.9. *Can we answer Question 7.0.5 in the context of Hilbert modular forms and modular forms on hyperbolic 3-spaces?*

Appendix A

Quick reference for the meaning of terms in Chapters 5 and 6

This appendix aims to give a quick reference to the terms mentioned in Chapters 5 and 6, so the reader doesn't have to go back and forth between the pages in both chapters. In Section A.2, the notation $\widetilde{K}(\ )$ means that the sum under consideration is $\widetilde{K}$ with additional condition(s) " ". For example, $\widetilde{K}(n_1 \neq 0, n_2 = 0)$ means that the sum under consideration is $\widetilde{K}$ with additional condition $n_1 \neq 0, n_2 = 0$.

A.1 References for terms mentioned in Chapter 5

1. $$U(l) = \widehat{\rho}\left(\frac{l}{L}\right)(2\cos 2\pi l\psi) \quad \widehat{\rho}\left(\frac{l+1}{L}\right)(2\cos 2\pi(l+1)\psi),\ 0 \quad l \quad L.$$

2. $$G(n) = \widehat{g}\left(\frac{n}{\pi_N(x)}\right),\ 0 \quad n \quad \pi_N(x).$$

3. $$A(p,q,n) = \begin{cases} 4 & \text{if } n = 0 \\ 2(a_f(p^{2n}) \quad a_f(p^{2n\ \ 2}))(a_f(q^{2n}) \quad a_f(q^{2n\ \ 2})) & \text{if } n \quad 1. \end{cases}$$

4. $$L_p(l,n) = a_f(p^{2l})(a_f(p^{2n}) \quad a_f(p^{2n\ \ 2})).$$

5. $$R(\rho,g;f)(x) = \frac{1}{8\pi_N(x)^2 L}\sum_{p,q \quad x}{}' \sum_{l,n \quad 1} 2U(l)U(0)G(n)\left\langle L_p(l,n)(a_f(q^{2n}) \quad a_f(q^{2n\ \ 2}))\right\rangle.$$

6. $$S(\rho,g;f)(x) = \frac{1}{8\pi_N(x)^2 L}\sum_{p,q \quad x}{}' \sum_{l,l',n \quad 1} 2U(l)U(l')G(n)\left\langle L_p(l,n)L_q(l',n)\right\rangle.$$

7. $$T(\rho,g;f)(x) = 2R(\rho,g;f)(x) + S(\rho,g;f)(x).$$

8. $$S_1(\rho,g;f)(x) = \frac{1}{8\pi_N(x)^2 L}\sum_{p,q \quad x}{}' \sum_{\substack{l,l',n \quad 1 \\ l\neq n, l'\neq n}} 2U(l)U(l')G(n)\left\langle L_p(l,n)L_q(l',n)\right\rangle.$$

9. $$S_2(\rho,g;f)(x) = \frac{1}{8\pi_N(x)^2 L}\sum_{p,q \quad x}{}' \sum_{\substack{l,l',n \quad 1 \\ l=n, l'\neq n}} 2U(l)U(l')G(n)\left\langle L_p(l,n)L_q(l',n)\right\rangle.$$

10. $$S_4(\rho,g;f)(x) = \frac{1}{8\pi_N(x)^2 L}\sum_{p,q \quad x}{}' \sum_{l \quad 1} 2U(l)^2 G(l)\left\langle L_p(l,n;l=n) \quad L_q(l,n;l=n)\right\rangle.$$

11. $$S(\rho,g;f)(x) = S_1(\rho,g;f)(x) + 2S_2(\rho,g;f)(x) + S_4(\rho,g;f)(x).$$

A.2 References for terms mentioned in Chapter 6

1. $$A(p,q,n) = \begin{cases} 4 & \text{if } n = 0 \\ 2(a_f(p^{2n}) - a_f(p^{2n-2}))(a_f(q^{2n}) - a_f(q^{2n-2})) & \text{if } n \geq 1. \end{cases}$$

2. $$K(\rho,g;f)(x) = \frac{1}{32\pi_N(x)^4L^2}\sum_{p,q\leq x}{}' \sum_{l_1,l_2\geq 0}\sum_{k_1,k_2\geq 0}\sum_{n_1,n_2\geq 0} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)$$
$$a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p,q,n_1)A(p,q,n_2).$$

3. $$\widetilde{K} := K(\rho,g;f)(x);\ K_1(\rho,g;f)(x) = \widetilde{K}(n_1 = 0, n_2 = 0).$$

4. $$K_3(\rho,g;f)(x) = \widetilde{K}(n_1 \neq 0, n_2 = 0)\,;\ K_4(\rho,g;f)(x) = \widetilde{K}(n_1 \neq 0, n_2 \neq 0).$$

5. $$K(\rho,g;f)(x) = K_1(\rho,g;f)(x) + 2K_3(\rho,g;f)(x) + K_4(\rho,g;f)(x).$$

6. $$A(\rho,g;f;n,r) = \sum_{l_1,l_2\geq 0} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2})(a_f(r^{2n}) - a_f(r^{2n-2})).$$

7. $$K_3(\rho,g;f)(x) = \frac{1}{4\pi_N(x)^4L^2}\sum_{p,q\leq x}{}' \sum_{n_2\geq 1} G(0)G(n_2)A(\rho,g;f;n_2,p)A(\rho,g;f;n_2,q).$$

8. $$A_1(n_2,r) = A_1(\rho,g;f;n_2,r) = \sum_{\substack{l_1,l_2\geq 0\\ l_1 - l_2\neq n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2+2n_2}).$$

9. $$A_2(n_2,r) = A_2(\rho,g;f;n_2,r) = \sum_{\substack{l_1,l_2\geq 0,\, l_2\geq n_2\\ l_1 - l_2\neq -n_2}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2l_2-2n_2}).$$

10. $$A_3(n_2,r) = A_3(\rho,g;f;n_2,r) = \sum_{\substack{l_1,l_2\geq 0,\, l_2<n_2\\ l_1+l_2\neq n_2-1}} U(l_1)U(l_2)a_f(r^{2l_1})a_f(r^{2n_2-2l_2-2}).$$

11. $$B_1(n_2) = B_1(\rho,g;f;n_2) = 2\sum_{l\geq 0} U(l)U(l+n_2).$$

12. $$B_2(n_2) = B_2(\rho,g;f;n_2) = \sum_{l\geq 0} U(l)U(n_2 - 1 - l).$$

13. $$A(\rho,g;f;n_2,r) = B_1(n_2) - B_2(n_2) + A_1(n_2,r) + A_2(n_2,r) + A_3(n_2,r).$$

14. $$T(p,n_1,n_2) = (a_f(p^{2n_1}) - a_f(p^{2n_1-2}))(a_f(p^{2n_2}) - a_f(p^{2n_2-2})).$$

15. $$k(r,n_1,n_2,l_1,l_2) = a_f(r^{2l_1})a_f(r^{2l_2})T(r,n_1,n_2).$$

16. $$a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(q^{2k_2})A(p,q,n_1)A(p,q,n_2)$$
$$= 4k(p,n_1,n_2,l_1,l_2)k(q,n_1,n_2,l_1,l_2).$$

17. $$\widetilde{k} := k(p,n_1,n_2) = \sum_{l_1,l_2\geq 0} U(l_1)U(l_2)k(p,n_1,n_2,l_1,l_2).$$

18. $$\alpha_1(n_1,n_2,p) = \widetilde{k}(l_1 \geq n_1, l_2 \geq n_2);\ \alpha_2(n_1,n_2,p) = \widetilde{k}(l_1 \geq n_1, l_2 < n_2).$$

19. $$\alpha_3(n_1,n_2,p) = \widetilde{k}(l_1 < n_1, l_2 \geq n_2);\ \alpha_4(n_1,n_2,p) = \widetilde{k}(l_1 < n_1, l_2 < n_2).$$

20. $$k(p,n_1,n_2) = (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)(n_1,n_2,p).$$

21. $$C(\rho,g;f)(x) = \frac{8}{32\pi_N(x)^4L^2}\sum_{p,q\leq x}{}' \sum_{\substack{n_1,n_2\geq 1\\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2).$$

22. $$D(\rho,g;f)(x) = \frac{4}{32\pi_N(x)^4L^2}\sum_{p,q\le x}{}' \sum_{\substack{n_1,n_2\ge 1\\ n_1 = n_2}} G(n_1)G(n_2)k(p,n_1,n_2)k(q,n_1,n_2).$$

23. $$K_4(\rho,g;f)(x) = C(\rho,g;f)(x) + D(\rho,g;f)(x).$$

24. C_i's ($i = 1,2,3$) are defined in equations 6.45, (6.46), (6.47) respectively.

25. $$C(\rho,g;f)(x) = C_1(\rho,g;f)(x) + C_2(\rho,g;f)(x) + C_3(\rho,g;f)(x).$$

26. D_i's ($i = 1,2,3$) are defined in equations 6.52, (6.53), (6.54) respectively.

27. $$D(\rho,g;f)(x) = (D_1 + D_2 + D_3)(\rho,g;f)(x).$$

28. $$L(\rho,g;f)(x) = \frac{1}{16\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{l_1,l_2\ge 0}\sum_{k_1,k_2\ge 0}\sum_{n_1,n_2\ge 0} U(l_1)U(l_2)U(k_1)U(k_2)G(n_1)G(n_2)$$
$$a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})A(p,q,n_1)A(p,r,n_2).$$

29. $$\widetilde{L} := L(\rho,g;f)(x);\ L_1(\rho,g;f)(x) = \widetilde{L}(n_1 = 0, n_2 = 0).$$

30. $$L_3(\rho,g;f)(x) = \widetilde{L}(n_1 \neq 0, n_2 = 0)\,;\ L_4(\rho,g;f)(x) = \widetilde{L}(n_1 \neq 0, n_2 \neq 0).$$

31. $$\widetilde{L_1} := L_1(\rho,g;f)(x) = \frac{1}{16\pi_N(x)^4L^2}\sum_{l_1,l_2\ge 0}\sum_{k_1,k_2\ge 0} 16U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2$$
$$\sum_{p,q,r\le x}{}' a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2}).$$

32. $$L_{11}(\rho,g;f)(x) = \widetilde{L_1}(l_1 = l_2, k_1 = k_2);\ L_{12}(\rho,g;f)(x) = \widetilde{L_1}(l_1 \neq l_2, k_1 = k_2).$$

33. $$L_{13}(\rho,g;f)(x) = \widetilde{L_1}(l_1 = l_2, k_1 \neq k_2);\ L_{14}(\rho,g;f)(x) = \widetilde{L_1}(l_1 \neq l_2, k_1 \neq k_2).$$

34. $$L_1(\rho,g;f)(x) = (L_{11} + L_{12} + L_{13} + L_{14})(\rho,g;f)(x).$$

35. $$A_1(\rho,g;f;n,q,r) = \sum_{k_1,k_2\ge 0} U(k_1)U(k_2)a_f(q^{2k_1})a_f(r^{2k_2})(a_f(r^{2n}) - a_f(r^{2n-2})).$$

36. $$2L_3(\rho,g;f)(x) = \frac{1}{\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{n_2\ge 1} G(0)G(n_2)A(\rho,g;f;n_2,p)A_1(\rho,g;f;n_2,q,r).$$

37. F_i's are defined in equations (6.74) and (6.75).

38. $$A_1(\rho,g;f;n_2,q,r) = U(0)(U(n_2) - U(n_2-1)) + F_1(\rho,g;f;n_2,q,r) + F_2(\rho,g;f;n_2,q,r).$$

39. $$\langle 2L_3(\rho,g;f)(x)\rangle = (\lambda_1 + \lambda_2 + \cdots + \lambda_6)(x).$$

40. λ_i's ($i = 1,\ldots,6$) are defined in equation (6.78).

41. $$L_4(\rho,g;f)(x) = \frac{1}{16\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{l_1,l_2\ge 0}\sum_{k_1,k_2\ge 0}\sum_{n_1,n_2\ge 1} U(l_1)U(l_2)U(k_1)U(k_2)$$
$$G(n_1)G(n_2)a_f(p^{2l_1})a_f(p^{2l_2})a_f(q^{2k_1})a_f(r^{2k_2})A(p,q,n_1)A(p,r,n_2).$$

42. $$L(\rho,g;f)(x) = (L_1 + 2L_3 + L_4)(\rho,g;f)(x).$$

43. $$L_p(l,n) = a_f(p^{2l})(a_f(p^{2n}) - a_f(p^{2n-2})).$$

44. $$L(q,r,n_1,n_2) = \sum_{k_1,k_2\ge 0} U(k_1)U(k_2)L_q(k_1,n_1)L_r(k_2,n_2).$$

45. $$E(\rho,g;f)(x) = \frac{8}{16\pi_N(x)^4L^2}\sum_{p,q,r\le x}{}' \sum_{\substack{n_1,n_2\ge 1\\ n_2>n_1}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2).$$

46. $$F(\rho,g;f)(x) = \frac{4}{16\pi_N(x)^4L^2}\sum_{p,q,r \;\; x}{}^{'} \sum_{\substack{n_1,n_2 \;\; 1\\ n_1=n_2}} G(n_1)G(n_2)k(p,n_1,n_2)L(q,r,n_1,n_2).$$

47. $$L_4(\rho,g;f)(x) = E(\rho,g;f)(x) + F(\rho,g;f)(x).$$

48. $F(n) = U(n) \quad U(n \quad 1)$ and M_i's are defined in Lemma 6.4.17 .

49. $$L(q,r,n_1,n_2)$$
$$=F(n_1)F(n_2) + F(n_1)\sum_{i=1}^{3}M_i(n_1,q) + F(n_2)\sum_{i=1}^{3}M_i(n_2,r) + \sum_{i=1}^{3}\sum_{j=1}^{3}M_i(n_1,q)M_j(n_2,r).$$

50. $U(t,n)$'s are defined in Lemma 6.4.18.

51. $$\sum_{i=1}^{3}M_i(n,s) = \sum_{t=1}^{\lfloor L\rfloor+1+n} U(t,n)a_f(s^{2t}).$$

52. $$\sum_{k \;\; 0}U(k)L_s(k,n) = F(n) + \sum_{t=1}^{\lfloor L\rfloor+1+n} U(t,n)a_f(s^{2t}).$$

53. $E_i(n_1,n_2)$ $(i = 1,...,4)$ are defined in Proposition 6.3.8.

54. $V_j(n_1,n_2,p)$ $(j = 1,...,9)$ are defined in Lemma 6.3.7.

55. $$k(p,n_1,n_2) = \sum_{i=1}^{4}E_i(n_1,n_2) + \sum_{i=1}^{9}V_i(n_1,n_2,p).$$

56. β_i's $(i = 1,...,8)$ are defined in equation (6.104).

57. $$\langle E(\rho,g;f)(x)\rangle = (\beta_1 + \quad + \beta_8)(x).$$

58. $$M(\rho,g;f)(x) = \frac{1}{64\pi_N(x)^4L^2}\sum_{\substack{p,q,r,s \;\; x\\ \text{all distinct}}}\sum_{l_1,l_2 \;\; 0}\sum_{k_1,k_2 \;\; 0}\sum_{n_1,n_2 \;\; 0} U(l_1)U(l_2)U(k_1)U(k_2)$$
$$G(n_1)G(n_2)a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})A(p,q,n_1)A(r,s,n_2).$$

59. $$\widetilde{M} := M(\rho,g;f)(x);\ M_1(\rho,g;f)(x) = \widetilde{M}(n_1 = 0, n_2 = 0).$$

60. $$M_2(\rho,g;f)(x) = \widetilde{M}(n_1 \neq 0, n_2 = 0);\ M_4(\rho,g;f)(x) = \widetilde{M}(n_1 \neq 0, n_2 \neq 0).$$

61. $$M_1(\rho,g;f)(x) = \frac{1}{64\pi_N(x)^4L^2}\sum_{p,q,r,s \;\; x}{}^{'}\sum_{l_1,l_2 \;\; 0}\sum_{k_1,k_2 \;\; 0} U(l_1)U(l_2)U(k_1)U(k_2)G(0)^2$$
$$a_f(p^{2l_1})a_f(q^{2l_2})a_f(r^{2k_1})a_f(s^{2k_2})A(p,q,0)A(r,s,0).$$

62. $$M_3(\rho,g;f)(x) = \frac{1}{8\pi_N(x)^4L^2}\sum_{p,q,r,s \;\; x}{}^{'}\sum_{n_2 \;\; 1} G(0)G(n_2)A_1(\rho,g;f;n_2,p,r)A_1(\rho,g;f;n_2,q,s).$$

63. γ_i's $(i = 1,...,5)$ are defined in equation (6.121).

64. $$\langle 2M_3(\rho,g;f)(x)\rangle = (\gamma_1 + \gamma_2 + \quad + \gamma_5)(x).$$

65. $$K(p,n) = \sum_{l \;\; 0}U(l)L_p(l,n).$$

66. $$M_4(\rho,g;f)(x) = \frac{4}{64\pi_N(x)^4L^2}\sum_{p,q,r,s \;\; x}{}^{'}\sum_{n_1,n_2 \;\; 1} G(n_1)G(n_2)$$
$$K(p,n_1)K(q,n_1)K(r,n_2)K(s,n_2).$$

67. $$M(\rho,g;f)(x) = M_1(\rho,g;f)(x) + 2M_3(\rho,g;f)(x) + M_4(\rho,g;f)(x).$$

68. $w_i(n_1,n_2,p,q,r,s)$'s $(i = 1,...,9)$ are defined in equation (6.138).

69. $$\delta_t(x) = \frac{1}{16\pi_N(x)^4 L^2} \sum_{p,q,r,s}^{0} \sum_{x\, n_1,n_2 \quad 1} G(n_1)G(n_2) h w_t(n_1, n_2, p, q, r, s) i.$$

70. $$h M_4(\rho, g; f)(x) i = (\delta_1 + \delta_2 + \quad + \delta_9)(x).$$

71. $$\left(R_2(\rho, g; f)(x)\right)^2 = K(\rho, g; f)(x) + L(\rho, g; f)(x) + M(\rho, g; f)(x).$$

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