Exploring Equivariant Chow Groups for Complexity One T-Varieties via Downgrading Techniques

विद्या वाचस्पति की
उपाधि की अपेक्षाओं की आंशिक पूर्ति में प्रस्तुत शोध प्रबंध

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Certified that the work incorporated in the thesis entitled "*Exploring Equivariant Chow Groups for Complexity One T-Varieties via Downgrading Techniques*", submitted by *Dighe Pavankumar Ramesh* was carried out by the candidate, under my supervision. The work presented here or any part of it has not been included in any other thesis submitted previously for the award of any degree or diploma from any other university or institution.

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Declaration

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Abstract

A T-variety is an algebraic variety X with an effective torus action T. The number $c(X) = \dim(X) - \dim(T)$ is called complexity of T-variety X. Altmann, Hausen and Süss have described these spaces in terms of pp-divisor and divisorial fans. This description of a T-variety involves a variety of dimension $c(X)$ and some combinatorial data encoded in the form of pp-divisors, i.e., divisors where the coefficients come from the Grothendieck group associated with the semigroup of polyhedra having a common tail cone. In case of complete T-variety with $c(X) = 1$, Ilten and Süss have described these spaces in terms of marked fancy divisor. Through this combinatorial description of a T-variety, Ustøl Nødland has provided an description of the Chow group of X.

In the first part of the thesis we study the computation of the equivariant Chow group of T-variety X with $c(X) = 1$. This computation involves the following steps. For a complete complexity 1 T-variety X, we compute combinatorial description of $X \times E_T^N$ as a T-variety, where E_T^N is Nd-dimensional space approximating the contractible space on which T acts freely. Subsequently, through the application of a downgrading technique, we introduce a structure of a T-variety of complexity 1 on the quotient space $\frac{X \times E_T^N}{T}$. By using combinatorial criterion of completeness of a T-variety, we have proved that if X is complete then the quotient space $\frac{X \times E_T^N}{T}$ is complete. Once it has a complete, complexity 1 T-variety structure, one can use Ustøl's result to compute Chow group of $\frac{X \times E_T^N}{T}$.

For an affine T-variety X with the action of a torus T, denoted temporarily, by $T \curvearrowright X$. Assume that T' is a subtorus of T. Then X is a T-variety with respect to the

action of T', $T' \curvearrowright X$. $T' \curvearrowright X$ is called a downgrading of $T \curvearrowright X$. The second part provides a combinatorial description of $T' \curvearrowright X$, in terms of a T/T'-invariant pp-divisor. We also describe the corresponding GIT fan.

Contents

1

Introduction

The thesis has been divided into two parts. The first part includes Chapter 2 to Chapter 4. In the first part, we study the T-equivariant Chow groups of a complete complexity 1 T-variety. The second part is Chapter 5. In the second part, we describe the GIT fan and torus invariant pp-divisor of an affine T-variety with respect to a subtorus of the large torus. Throughout the thesis, by a variety, we mean that integral, separated scheme of a finite type over the field of complex numbers $\mathbb{C}$. A T-variety is an algebraic variety X along with an effective action of an algebraic torus T. The *complexity* of a T-variety is the number $c(X) = \dim X - \dim T$. We are interested in T-varieties, and our approach follows that of Altmann, Hausen, and Süss [AH, AHS]. This description of a T-variety involves a variety of dimension $c(X)$ and some combinatorial data encoded in the form of pp-divisors, i.e., divisors where the coefficients come from the Grothendieck group associated with the semigroup of polyhedra having a common tail cone. These data are derived using Geometric Invariant Theory, as developed by Mumford and others (see [MFK]), applied to the action of the torus on the variety. The general non-affine T-varieties have been described in terms of divisoral fans, which are a generalization of a fan. Quite a lot is known about these spaces, especially if the T-variety is of complexity 1.

The article [AIP$^+$] provides a comprehensive summary of what is known about the geometry of T-varieties including divisors, cohomology of line bundles, intersection theory, etc. Their topology, including a computation of Hodge-Deligne numbers, cohomology ring, and fundamental groups, were studied in [LLM]. Vector bundles over T-varieties with compatible torus actions have also been studied by Ilten and Suß [IS1]. The algebro-combinatorial data describing T-varieties make them a natural source of examples where one can hope to compute various properties.

The natural question is whether one can compute various invariants for these varieties. One such important invariant is the Chow group which also acts as a natural habitat for characteristic classes like Chern classes to live in. This is an old question (see, for example, the work on Chow groups of SL(2) embeddings by Moser-Jauslin [MJ], and Gonzales [Gon]). Under certain conditions, Nødland computed Chow groups of T-varieties [Nød]. It was further studied by Botero [Bot]. Since we are in an equivariant setting, a natural question is to ask if we can compute equivariant versions of these invariants. The study of equivariant invariants of varieties have a rich history; see, for example, Timashev [Tim1, Tim2]. Equivariant Chow groups, modeled after the equivariant cohomology of Borel, were defined by Edidin and Graham [EG], extending previous work done by Briney, Gillet and Vistoli. Note that these Chow groups can be interpreted as the Chow groups of the quotient stack obtained from the action of the torus on the variety [Kre]. We shall describe a way to compute these groups.

The first part of the thesis is organized as follows. Chapter 2 is a summary of the notions we need. The theory of toric variety is reviewed in the section 2.1, followed by a quick review of T-varieties and gluing of T-varieties in section 2.2 and 2.3, respectively. We focus on complete complexity 1 T-varieties. Section 2.4 has been dedicated to the different algebro-combinatorial descriptions of a complete complexity 1 T-variety. In this section, we briefly recall the notions of marked fansy divisor and divisorial polyhedron. Projectivization of a toric vector bundle serves as a valuable source of example of T-variety. In Section 2.5 we briefly recall a combinatorial description of projectivization of a toric vector bundle.

The Chapter 3 based on the papers [FMSS], [FS], [EG], and [Nød]. Let X be a T-variety, and $A_k(X)$ be the k^{th} Chow group. The group $A_k^T(X)$ defined in the section 3.1(using invariant cycles of X).

Theorem 1.0.1. *[FMSS, Theorem 1] If X is a variety with an action of a torus T, then the canonical map $A_k^T(X) \to A_k(X)$ is an isomorphism.*

The initial definition of the equivariant Chow group of X relied solely on invariant cycles. However, it became evident that there is an insufficient number of invariant cycles on X(See example in [EG, Section 3.5]) that possess desirable properties(homotopy invariant,intersection product etc). Consequently, Dan Edidin and William Graham [EG] introduced a novel definition of the equivariant Chow group. The k^{th}-equivariant Chow group of X is usual k^{th} Chow group of the space $(X \times U)/T$, where U is an approximate classifying space for a torus T. For a torus T, we denote approximate space by E_T^N(Totaro's approximation of ET), where N is large enough integer. In section 3.2, we study the space E_T^N as a toric variety with a dense torus T_E. Hence one can think of $(X \times E_T^N)/T$ as variety with $(T \times T_E)/T \cong T_E$ action and use the above theorem;

$$A_k^{T_E}((X \times E_T^N)/T) \to A_k((X \times E_T^N)/T).$$

Let Σ_E be a fan description of space E_T^N. At the end of the Chapter 3 we prove the following proposition.

Proposition 1.0.2. *Suppose Y is a curve, T a torus and E_T^N the Nd-dimensional space approximating the contractible space on which T acts freely.*

1. *(Affine case) Suppose $\mathfrak{D} = \sum_{i=1}^n \Delta_i \otimes \{p_i\}$ is a pp-divisor on Y and $X(\mathfrak{D})$ is the corresponding affine T-variety. For $I \in \mathscr{T}_I$ (see equation (3.3)), define*

$$\mathfrak{D}_I = \sum_{i=1}^n \Big(\Delta_i \times \sigma_I\Big) \otimes \{p_i\}$$

where σ_I is the cone defined in equation (3.4). Let $\mathcal{S}_{\mathfrak{D}}$ be the divsorial fan generated by $\{\mathfrak{D}_I \,|\, I \in \mathscr{T}_I\}$. Then, $X(\mathfrak{D}) \times E_T^N$, considered as a complexity 1 T-variety under the action of $T \times T_E$, is described by $\mathcal{S}_{\mathfrak{D}}$.

2. *(General case) For a T-variety $X = X(\mathcal{S})$, described by a divisorial fan $\mathcal{S}$ over a curve Y, $X \times E_T^N$ is desribed by the divisorial fan generated by $\{\mathcal{S}_{\mathfrak{D}} \,|\, \mathfrak{D} \in \mathcal{S}\}$ where for each $\mathfrak{D} \in \mathcal{S}$, $\mathcal{S}_{\mathfrak{D}}$ is defined as in the affine case .*

In short, the above proposition gives the combinatorial description of $T \times T_E$ action on $X \times E_T^N$ as a complexity 1 T-variety.

Let us fix a few notations for a brief understanding of Chapter 4. Let Y be a smooth curve, and let $\mathcal{S}$ be a divisorial fan. Let $X = X(\mathcal{S})$ be a T-variety of complexity 1 with

d dimensional torus acting on it. Note that $T \times T_E$ acts effectively on $X(\mathcal{S}) \times E_T^N$. The Chapter 4 gives the main results leading to the computation of the equivariant Chow groups using Ilten and Vollmert's technique of downgrading [IV]. Using the downgrading technique one can construct the base space $\widetilde{Y_C}$ of $X \times E_T^N$ as a T-variety, which is birational to the quotient $(X \times E_T^N)/T$.

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi_C"] & X \times E_T^N \\
\widetilde{Y_C} \arrow[d, "\pi"] & \\
Y, &
\end{tikzcd}$$

Lemma 1.0.3. *The divisorial fan $\widetilde{\mathcal{S}_C}$ over Y constructed in 4.2, gives us the T_E-variety $\widetilde{Y_C}$, which corresponds to the good quotient of $\widetilde{X \times E_T^N}$ by the action of T.*

For technical reasons we assumed that $Y = \mathbb{P}^1$, and $X = X(\mathcal{S})$ is a complete T-variety of complexity 1. We have obtained the following result.

Lemma 1.0.4. *The T-variety Y_C constructed in 4.1 is complete and fits into a diagram.*

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi_C"] & X \times E_T^N \arrow[d, "q"] \\
\widetilde{Y_C} \arrow[r] \arrow[d, "\pi"] & Y_C \\
\mathbb{P}^1, &
\end{tikzcd}$$

where r_E is a T-equivariant birational proper morphism, and the maps q and π_C are geometric quotients.

We summarize the part first in the following steps.

1. A combinatorial description of $X \times E_T^N$ as a complexity 1 T-variety with the torus

$T \times T_E$.

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi"] & X \times E_T^N \\
Y. &
\end{tikzcd}$$

2. The T_E-variety $\widetilde{Y_C}$ which corresponds to the good quotient of $\widetilde{X \times E_T^N}$ by the action of T.

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi_C"] & X \times E_T^N \\
\widetilde{Y_C} \arrow[d, "\pi"] & \\
Y. &
\end{tikzcd}$$

3. A construction of geometric quotient of $X \times E_T^N$ by the action of T as a variety with T_E action.

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi_C"] & X \times E_T^N \arrow[d, "q"] \\
\widetilde{Y_C} \arrow[r] \arrow[d, "\pi"] & Y_C \\
\mathbb{P}^1. &
\end{tikzcd}$$

The space Y_C is complete, rational T-variety of complexity 1. The computation of equivariant Chow groups of X is reduces to the computation of usual Chow groups of Y_C. This is done in section 4.2 following Nødland [Nød]. Let X be a complete, rational T-variety of dimension $n+1$ and $\mathcal{S}$ is associated divisorial fan over $\mathbb{P}^1$ with a tail $\Sigma \subset N_{\mathbb{Q}}$. For each point $p \in \mathbb{P}^1$, we have a polyhedral complex associated with it(see Section 2.3 for more details), we denote this polyhedral complex by $\mathcal{S}_p$. There are few important facts about these polyhedral complexes,

- Only finitely many $\mathcal{S}_p$ are different from Σ.

- The tail cone of $\mathcal{S}_p$ is Σ.

Let P be the set of all points in $\mathbb{P}^1$ with $\mathcal{S}_p$ is different from Σ (see Section 3.1 for more details). For a description of a k^{th} Chow group of X, consider the following sets:

- R_k = Cones of dimension $n+1-k$ corresponding to subvarieties not contracted by r .
- V_k = Non-contracted faces of dimension $n-k$ of polyhedral subdivision corresponding to the fiber of points in P.
- T_k = Cones of dimension $n-k$ corresponding to subvarieties contracted by r.

The Chow group $A_k(X)$ is obtained by the following exact sequence:

$$V \longrightarrow \mathbb{Z}^{V_k} \oplus \mathbb{Z}^{R_k} \oplus \mathbb{Z}^{T_k} \longrightarrow A_k(X) \longrightarrow 0\ .$$

Where V is a lattice described in the Section 3.1. A natural source of T-varieties are projectivization of a toric vector bundles. Consider a rank two toric vector bundle $\mathscr{E}$ on a toric variety X_Σ. Then, the projectivization $\mathbb{P}(\mathscr{E})$ is complete complexity one T-variety. Altmann, Hausen and Süss [AHS, Proposition 8.4] described $\mathbb{P}(\mathscr{E})$ in terms of divisorial fan. For $X = \mathbb{P}(\mathscr{E})$, Nødland proved the following result:

$$|R_k| + |V_k| + |T_k| = \begin{cases} \#\Sigma(n-k+1) + 2\#\Sigma(d-i), & \text{if } i < n; \\ \#\Sigma(1) + \#P, & \text{if } i = n; \end{cases}$$

i.e $|R_k| + |V_k| + |T_k|$ is constant (right hand side of the above equation depends on Σ and P). From above we have proved the following result:

Proposition 1.0.5. *For any rank two toric vector bundle $\mathscr{E}$ on a smooth toric variety X_Σ we have $X = P(\mathscr{E})$ and $X_{\mathscr{E}} = X \times E_T^N/T$,The numbers $|r_k|$, $|v_k|$, and $|t_k|$ are associated with $X_{\mathscr{E}}$ then*

$$|r_k| + |v_k| + |t_k| = \sum_{i=0}^{i=k} S_i' S_{k-i},$$

where S_i' and S_{k-i} are define in the Section 4.3.

Let X be a T-variety. As mentioned, Altmann, Hausen and Süss [AH, AHS] described these spaces in terms of pp-divisors and divisorial fans. For an affine X, Altmann and

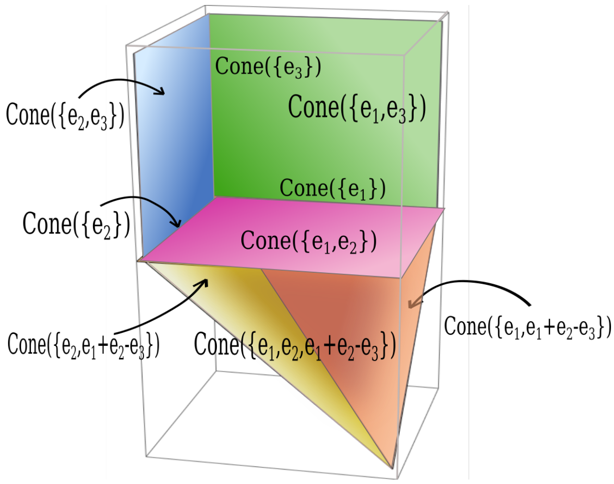

Figure 1.1: T-action

Hausen [AH] was constructed the pp-divisor using Geometric Invariant Theory and called it *GIT constructed pp-divisor* ([AH, Section 9]). Let X be an affine and let T' be a subtorus of the torus T. In [AH, Section 11], Altmann and Hausen was determined a pp-divisor description of X with the action of T', provided that $\dim(X) = \dim(T)$. For $\dim(X) - \dim(T) = 1$, In [IV, Theorem 5.2], Ilten and Vollmert constructed a pp-divisor description of X with the action of T'. In the same paper they predict that such a construction should exist for general case but did not specify details. In Chapter 5 we prove that the GIT constructed pp-divisor is torus invariant. The Chapter has been organized in the following way. In section 5.1.1, we define poset 5.1.1 for an affine toric variety. By using the poset, we have described GIT-data for a toric variety and downgraded affine T-variety. We illustrate this by the example 5.2.3.

Example 1.0.6 (5.2.3)**.** *Consider toric variety* $X = \operatorname{Spec}(\mathbb{C}[u, v, w, uvw^{-1}])$. *Let* T *be the largest torus and let* T' *be a subtorus of* T*, the torus inclusion is given by* $(t_1, t_2) \mapsto (t_1, t_2, t_1)$. *The GIT fan of* X *with the action of* T *and* X *with the action of* T' *are in the Fig 1.1 and 1.2 respectively.*

The above example is a consequence of the following proposition.

Proposition 1.0.7. *Let* $\lambda_T(u)$ *be GIT cone associated with* $u \in M$ *(under the* T*-action)*

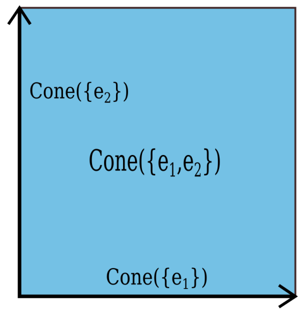

Figure 1.2: T'-action

and $\lambda_{T'}(v')$ *be GIT cone associated to* $v' \in M'$ *(under the* T'*-action) then*

$$\lambda_{T'}(v') = \bigcap_{i(u)=v'} i(\lambda_T(u)).$$

In section 5.3 and 5.3.1, we have proved that the base space Y' is T-variety and for right choice of the section (s', 5.3.1) there is a torus invariant pp-divisor $\mathfrak{D}'$ on Y' such that $X(\mathfrak{D}') \cong X$. i.e,

Theorem 1.0.8. *Consider an affine* T*-variety* X *with the action of the torus* T*, with weight cone* $\omega \subset M_{\mathbb{Q}}$*. Let* T' *be a subtorus of* T*, and the associated lattice map be* $i : M \to M'$*. Then, there exists* $\frac{T}{T'}$*-variety* Y' *and a* $\frac{T}{T'}$*-invariant pp-divisor* $\mathfrak{D}'$ *on* (Y', N') *with tail*$(\mathfrak{D}') = i(\omega)^\vee$ *such that* $X(\mathfrak{D}') \cong X$*.*

The findings presented in the thesis correspond to the material covered in preprints [DMM2] and [DMM1].

Notations and Conventions

Throughout the thesis, we follow the following notations and conventions, if not specified.

- $\mathbb{C}$ denotes field of complex number.
- $\mathbb{Q}$ denotes field of rational number.
- $\mathbb{R}$ denotes field of real number.

- $\mathbb{Z}$ denotes set of integers.
- $\mathbb{N}$ denotes set of positive integers.
- for a lattice N, $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ is associated $\mathbb{Q}$-vector space and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ is associated $\mathbb{R}$-vector space.
- a variety is integral, separated scheme of finite type over a field $\mathbb{C}$.
- a cone is strongly convex rational polyhedral.
- Σ denotes fan in $N_{\mathbb{Q}}$ or $N_{\mathbb{R}}$.
- $\Sigma(d)$ denotes d dimensional cones in the fan Σ.
- X_{Σ} denotes toric variety associated to Σ.
- we will use same notation for a lattice homomorphism and the corresponding vector space homomorphism.
- $\mathrm{CaDiv}(Y)$ denotes group of Cartier divisors on Y.
- $\mathrm{CaDiv}_{\mathbb{Q}}(Y) = \mathrm{CaDiv}(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$ denotes group of rational Cartier divisors on Y.

2

Varieties with torus action

In this chapter, we briefly discuss normal varieties with a torus action and algebra-combinatorial data associated with that. To begin, we will provide a concise review of the fundamental concepts in algebraic geometry essential for comprehending the algebraic-geometric aspects of algebra-combinatorial data..

Let X be a normal variety. A *prime divisor* $D \subset X$ is an irreducible subvariety of dimension $\dim(X) - 1$. A *Weil divisor* is an element of $Z_{\dim(X)-1}(X)$, where $Z_k(X)$ is a free abelian group generated by k-dimensional subvarieties of X.

Notation 2.1. *The field of rational functions on X denoted by $\mathbb{C}(X)$.*

Let D be a prime divisor on X. The local ring corresponding to D is a discrete valuation ring $\mathscr{O}_{X,D}$, with a quotient field $\mathbb{C}(X)$. We denote corresponding discrete valuation by v_D. If $f \in \mathbb{C}(X)^*$ is a rational function, then the following divisor:

$$\operatorname{div}(f) = \sum v_D(f) \cdot D$$

is called a *principal divisor*. Let $D = \sum a_i D_i$ be a Weil divisor and $U \subset X$, then the following Weil divisor on U:

$$D|_U = \sum_{U \cap D_i \neq \phi} U \cap D_i$$

is called a *restriction of D to U*.

Definition 2.0.1. A *Cartier divisor* is a locally principal Weil divisor.

Two Weil divisors D and E are linearly equivalent if $\mathrm{div}(f) = D - E$, for some $f \in \mathbb{C}(X)^*$. We denote it by $D \sim E$.

2.1 Toric Variety

Definition 2.1.1. A normal variety with a torus as a dense open subset and action extends to the variety is called toric variety.

Example 2.1.2. *Let $T = \mathbb{C}[t_1, t_2, t_3]_{t_1 t_2 t_3}$ be a torus acting on $\frac{\mathbb{C}[x,y,z,w]}{<xy-zw>}$. The torus action is given by,*

$$(t_1, t_2, t_3)(x, y, z, w) = (t_1 x, t_2 y, t_3 z, t_1 t_2 t_3^{-1} w).$$

For a torus $T = (\mathbb{C}^*)^d$, $M = \mathrm{Hom}(T, \mathbb{C}^*)$ is a character lattice of rank d. For $m = (m_1, \ldots, m_d) \in M$, an associated character is denoted by χ^m, it is defined as:

$$\chi^m : T \to \mathbb{C}^*, \ \ (t_1, \ldots, t_d) \mapsto t_1^{m_1} \ldots t_d^{m_d}.$$

We briefly recall a combinatorial description of an affine toric variety. Consider the following setup. Let M and N be a pair of dual lattices

$$\langle -, - \rangle : M \times N \to \mathbb{Z}$$

and $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ are the associated $\mathbb{R}$-vector spaces. Let $N_{\mathbb{Q}}$ and $M_{\mathbb{Q}}$ be the associated $\mathbb{Q}$-vector space.

Definition 2.1.3. A rational cone in $N_{\mathbb{R}}$ is set

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u \cdot u \;\middle|\; \lambda_u \in \mathbb{R}_{\geq 0} \right\}$$

where, S is a subset N. We say σ is *polyhedral*, if S is finite. We say it is *strongly convex or pointed* if $\{0\}$ is only linear space contained in σ.

Remark 2.1.4. *Also note that,*

$$\sigma \cap N_{\mathbb{Q}} = \left\{ \sum_{u \in S} \lambda_u \cdot u \,\middle|\, \lambda_u \in \mathbb{Q}_{\geq 0} \right\}$$

is strongly convex polyhedral cone in $N_{\mathbb{Q}}$ i.e. only linear space contained in σ is $\{0\}$ and S is finite.

For this section, by a cone, we mean that strongly convex rational polyhedral cone. Consider $\sigma^{\vee} \subset M_{\mathbb{R}}$ is dual cone of a cone $\sigma \subset N_{\mathbb{R}}$ defined as

$$\sigma^{\vee} = \{ m \in M_{\mathbb{R}} \,|\, \langle m, u \rangle \geq 0, \ \forall \ \ u \in \sigma \} \,.$$

Note that, dual cone of a rational cone is rational.

Definition 2.1.5. An affine semi-group $S \subset M$ is *saturated*, if some positive integer multiple of $m \in M$ is in S then m is in S.

Definition 2.1.6. A *face of a cone* σ in $N_{\mathbb{R}}$ is a set $\{u \in \sigma \,|\, \langle m, u \rangle = 0\}$ provided that $\sigma \subset \{u \in N_{\mathbb{R}} \,|\, \langle m, u \rangle \geq 0\}$ for some $m \neq 0$ in $\sigma^{\vee}$. A face of a rational polyhedral cone is rational polyhedral.

Notation 2.2. *If σ' is a face of σ, we denote it by $\sigma' \prec \sigma$ or $\sigma \succ \sigma'$.*

Remark 2.1.7. *For a rational cone σ, a face of σ is completely determined by $m \in \sigma^{\vee} \cap M_{\mathbb{Q}}$(cf [CLS, Proposition 1.2.8]).*

Let $\mathbb{C}[S_\sigma = \sigma^{\vee} \cap M] = \bigoplus_{u \in \sigma^{\vee} \cap M} \mathbb{C} \cdot \chi^u \subset \mathbb{C}[M] = \bigoplus_{u \in M} \mathbb{C} \cdot \chi^u$ be a semi-group algebra associated to σ. By Gordan's Lemma ([CLS, Proposition 1.2.17]), $\mathbb{C}[S_\sigma = \sigma^{\vee} \cap M]$ is a finitely generated M-graded, $\mathbb{C}$-algebra. The cone σ is a strongly convex, hence S_σ is a saturated affine semigroup. By [CLS, Theorem 1.3.5], $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is a normal variety and hence $U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$ is an affine toric variety. Conversely, every affine toric variety has an associated strongly convex rational polyhedral cone. If $\sigma_1 = \{u \in \sigma_2 \,|\, \langle m, u \rangle = 0\}$ is a face of a cone σ_2 , then $S_{\sigma_1} = S_{\sigma_2} + \mathbb{Z}(-m)$. Thus we have ring homomorphism

$$\mathbb{C}[S_{\sigma_2} = \sigma_2^{\vee} \cap M] \to \mathbb{C}[S_{\sigma_1} = \sigma_1^{\vee} \cap M] = \mathbb{C}[S_{\sigma_2}]_{\chi^m}$$

such that

$$\chi^u \mapsto \chi^u$$

then associated morphism $U_{\sigma_1} \to U_{\sigma_2}$ is an open embedding.

Definition 2.1.8 (Fan)**.** A finite collection of cones Σ with following properties:

- If $\sigma_2 \in \Sigma$, and $\sigma_1 \prec \sigma_2$ then $\sigma_1 \in \Sigma$
- If σ_1 and σ_2 are elements of Σ then $\sigma_1 \cap \sigma_2 \prec \sigma_1$ and $\sigma_1 \cap \sigma_2 \prec \sigma_2$

is called a fan.

By [CLS, Proposition 3.1.3], if σ_1, $\sigma_2 \in \Sigma$ then

$$S_{\sigma_1 \cap \sigma_2} = S_{\sigma_1} + S_{\sigma_2}.$$

Thus,

$$\mathbb{C}[S_{\sigma_1}] \longrightarrow \mathbb{C}[S_{\sigma_1}]_{\chi^m} = \mathbb{C}[S_{\sigma_1 \cap \sigma_2}] = \mathbb{C}[S_{\sigma_2}]_{\chi^{-m}} \longleftarrow \mathbb{C}[S_{\sigma_2}]$$

$$U_{\sigma_1} \longleftarrow (U_{\sigma_1})_{\chi^m} = U_{\sigma_1 \cap \sigma_2} = (U_{\sigma_2})_{\chi^{-m}} \longrightarrow U_{\sigma_2}$$

where $m \in \operatorname{relint}(\sigma_1^\vee \cap (-\sigma_2)^\vee)$

Let Σ be a fan. If σ_1, σ_2, and σ_3 are elements of Σ then

$$\sigma_1 \cap \sigma_2 \cap \sigma_3 \prec \sigma_1 \cap \sigma_2 \prec \sigma_1 \succ \sigma_1 \cap \sigma_3 \succ \sigma_1 \cap \sigma_2 \cap \sigma_3.$$

We have the following open embeddings,

$$\begin{tikzcd}
 & & U_{\sigma_1 \cap \sigma_2 \cap \sigma_3} \arrow[dll] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr] & & \\
U_{\sigma_2} & U_{\sigma_1 \cap \sigma_2} \arrow[l] \arrow[r] & U_{\sigma_1} & U_{\sigma_1 \cap \sigma_3} \arrow[l] \arrow[r] & U_{\sigma_3}.
\end{tikzcd}$$

Note that, all cones are strongly convex, hence $0 \in \Sigma$. An affine toric variety corresponding to $\{0\}$ is a torus. Hence $U_0 \to U_\sigma$ is an open embedding for all $\sigma \in \Sigma$. A variety associated with Σ is obtained by gluing affine toric varieties U_σ, where $\sigma \in \Sigma$. We denote it by $X(\Sigma)$. From [CLS, Theorem 3.1.5], $X(\Sigma)$ is a toric variety. Conversely, every toric variety is of this form.

Example 2.1.9. *Consider the Example 2.1.2, take* $\sigma = Cone(\{e_1, e_2, e_1 + e_3, e_2 + e_3\})$.

$$\mathbb{C}[S_\sigma] = \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C} \cdot \chi^u = \mathbb{C}[u, v, w, uvw^{-1}] \equiv \frac{\mathbb{C}[x, y, z, w]}{< xy - zw >}$$

where $\sigma^\vee = Cone(\{e_1, e_2, e_3, e_1 + e_2 - e_3\})$.

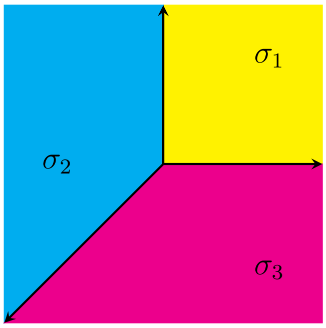

Figure 2.1: Fan Description of Projective Plane.

Example 2.1.10. *Consider an example of a non-affine toric variety* $\mathbb{P}^2$ *given by fan description shown in the following figure , where* $\sigma_1 = Cone(\{e_1, e_2)\}$, $\sigma_2 = Cone(\{e_2, -e_1 - e_2)\}$, $\sigma_3 = Cone(\{e_1, -e_1 - e_2\})$.

2.2 Affine T-varieties with higher complexity

From here on, we will work with the dual vector spaces $M_\mathbb{Q}$ and $N_\mathbb{Q}$ over field $\mathbb{Q}$.

Definition 2.2.1. A normal variety X with an effective action of a torus T is called T-variety of complexity $\dim(X) - \dim(T)$.

Example 2.2.2 ([IV]). *Let* X *be the hypersurface given by the equation* $x + x^2y + z^2 + w^3$ *in* $\mathbb{C}^4$ *with* $\mathbb{C}^*$*-action,*

$$t \cdot (x, y, z, w) = (t^6x, t^{-6}y, t^3z, t^2w).$$

This is complexity 2 *affine* T*-variety.*

Definition 2.2.3 (Morphism of T-varieties)**.** Let X and X' be T-varieties with tori action T and T'. A morphism $\psi : X \to X'$ is called T-equivariant morphism if ψ along with a morphism $\dot{\psi} : T \to T'$ such that $\psi(t \cdot x) = \dot{\psi}(t) \cdot \psi(x)$.

Example 2.2.4. *Let $\psi_1 \colon N \to N'$ be a lattice homomorphism. Consider cones $\sigma \subset N_{\mathbb{Q}}$ and $\sigma' \subset N'_{\mathbb{Q}}$ such that $\psi_1(\sigma \cap N) \subset \sigma' \cap N'$. Then we have following ring homomorphism*

$$\mathbb{C}[M'] \to \mathbb{C}[M] \ \textit{such that} \ \chi^{u'} \mapsto \chi^{u' \circ \psi_1}$$

where M and M' are dual lattices of N and N' respectively. This map restricts to

$$\mathbb{C}[S_{\sigma'}] \to \mathbb{C}[S_\sigma]$$

hence we have the following morphism of toric variety:

$$\dot{\psi} \colon \operatorname{Spec}(\mathbb{C}[M]) \to \operatorname{Spec}(\mathbb{C}[M']) \ \textit{and} \ \psi \colon U_\sigma \to U_{\sigma'}.$$

In the following step, we will revisit the concept of a proper polyhedral divisor. But first, let's examine the following example:

Example 2.2.5. *Let $A = \bigoplus_{u \in \mathbb{Z}_{\geq 0}} A_u$ be an integral, finitely generated, $\mathbb{Z}$-graded, $\mathbb{C}$-algebra. Then integral closure of A is completely determined by the Proj(A) and $\mathscr{O}(1)$, i.e*

$$\bar{A} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \Gamma(\mathit{Proj}(A), \mathscr{O}(1)^{\otimes n}).$$

Note that, $Y := \operatorname{Proj}(A)$ is a semi-projective variety, and the normal variety $\operatorname{Spec}(\bar{A})$ is completely determined by Y and collection of line bundles on Y.

Definition 2.2.6 (Semi-projective Variety)**.** A variety which is projective over its global section is called a *semi-projective variety.*

The line bundle of a Cartier divisor

Let D be a Cartier divisors on a normal variety Y. If U is an open subset of Y, then

$$\mathscr{O}_Y(U) = \{f \in \mathbb{C}(Y)^* \mid \operatorname{div}(f)|_U \geq 0\} \cup \{0\},$$

and

$$\mathscr{O}_Y(D)(U) = \{f \in \mathbb{C}(Y)^* \,|\, (\mathrm{div}(f) + D)|_U \geq 0\} \cup \{0\}\,.$$

By [CLS, Proposition 8.0.7], a Weil divisor D on Y is Cartier if and only $\mathscr{O}_Y(D)$ is a line bundle on Y. Conversely, every line bundle is isomorphic to $\mathscr{O}_Y(D)$, for some Cartier divisor D on Y(See [CLS, Theorem 6.0.20] for more details).

Definition 2.2.7. A half space in $N_{\mathbb{Q}}$ is set $\{u \in N_{\mathbb{Q}} \,|\, \langle m, u\rangle \geq 0\}$ for some $m \in M_{\mathbb{Q}}$

Definition 2.2.8. A polyhedron in $N_{\mathbb{Q}}$ is a intersection of finitely many half spaces in $N_{\mathbb{Q}}$.

A *tail cone*, $\mathrm{tail}(\Delta)$, of any polyhedron $\Delta \subset N_{\mathbb{Q}}$ is defined as

$$\mathrm{tail}(\Delta) := \{v \in N_{\mathbb{Q}} \,|\, v + \Delta \subseteq \Delta\}\,.$$

Given a cone $\sigma \subset N_{\mathbb{Q}}$, a σ-polyhedron is a polyhedron whose tail cone is σ. Let σ be a pointed cone in $N_{\mathbb{Q}}$ and $\mathrm{Pol}^+_{\sigma}(N)$ is a collection of σ-polyhedra; with respect to the *Minkowski sum*, it is semi-group. The Grothendieck group of $\mathrm{Pol}^+_{\sigma}(N)$ is denoted by $\mathrm{Pol}_{\sigma}(N)$.

Definition 2.2.9 (Polyhedral Divisor)**.** A polyhedral divisor with a tail cone $\sigma \subset N_{\mathbb{Q}}$ on a normal variety Y is a formal sum $\mathfrak{D} = \sum \Delta_Z \otimes Z$ with only finitely many Δ_Z different from σ, where $\sum$ runs over all prime divisors on Y and $\Delta_Z \in \mathrm{Pol}_{\sigma}(N)$.

For $u \in \sigma^{\vee}$, there is a map $\langle u, -\rangle^+ : \mathrm{Pol}^+_{\sigma}(N) \to \mathbb{Q}$ such that

$$\Delta \mapsto \mathrm{min}_{v\in\Delta}\langle u, v\rangle.$$

Then $\langle u, -\rangle^+$ induces a map $\langle u, -\rangle : \mathrm{Pol}_{\sigma}(N) \to \mathbb{Q}$. Let $\mathfrak{D} = \sum \Delta_Z \otimes Z$ be a polyhedral divisor and $\sigma^{\vee} \subset M_{\mathbb{Q}}$ be a dual of cone $\sigma \subset N_{\mathbb{Q}}$. For all $u \in \sigma^{\vee}$ we have,

$$\mathfrak{D}(u) = \sum \langle u, \Delta_Z\rangle Z$$

is rational Weil divisor on Y. To a polyhedral divisor $\mathfrak{D}$, one can associate an M-graded $\mathscr{O}_Y$-algebra and an affine scheme. Consider sheaf of M-graded $\mathscr{O}_Y$-algebras,

$$\mathscr{A}(-) = \bigoplus_{u\in\sigma^{\vee}\cap M} \Gamma(-, \mathscr{O}_Y(\mathfrak{D}(u))).$$

Define, the affine scheme associated with $\mathfrak{D}$ to be:

$$X(\mathfrak{D}) = \mathrm{Spec}(\mathscr{A}(Y)) = \mathrm{Spec}\Big(\bigoplus_{u\in\sigma^\vee\cap M}\Gamma(Y,\mathscr{O}_Y(\mathfrak{D}(u))\Big).$$

Let D be a Weil divisor on Y and $f \in \Gamma(Y, \mathscr{O}_Y(D))$, the non-vanishing locus of f is $Y_f := Y \setminus \mathrm{Supp}(\mathrm{div}(f) + D)$.

Definition 2.2.10 (Semi-ample Divisor)**.** A Weil divisor D on Y is semi-ample divisor, if Y_f cover Y, where $f \in \Gamma(Y, \mathscr{O}_Y(nD))$ for some non-negative integer n.

Definition 2.2.11 (Big Divisor)**.** A Weil divisor D is big, if some multiple of D has non-vanishing locus.

Definition 2.2.12 (Proper Polyhedral Divisor or pp-divisor)**.** Let Y be a normal algebraic variety and $\sigma \subset N_{\mathbb{Q}}$ be a pointed cone. A proper polyhedral divisor with a tail cone σ is a formal sum $\mathfrak{D} = \sum \Delta_i \otimes D_i$, where D_i are prime divisors of Y and Δ_i are σ-polyhedra with only finitely many Δ_i different from σ, such that

1. for each $u \in \sigma^\vee$, the evaluation $\mathfrak{D}(u) = \sum\langle u, \Delta_i\rangle \cdot D_i$ are semi-ample, rational Cartier divisors on Y.

2. for each $u \in \mathrm{relint}(\sigma^\vee)$, the evaluation $\mathfrak{D}(u)$ is a big divisor on Y.

We call $\mathfrak{D}$ is pp-divisor on (Y, N) with a tail cone σ and denote it by $\mathrm{tail}(\mathfrak{D}) = \sigma$.

The pp-divisor $\mathfrak{D}$ on (Y, N) with tail cone σ corresponds to a convex, piecewise linear map $h_{\mathfrak{D}} : \sigma^\vee \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$ such that $h_{\mathfrak{D}}(u)$ is strictly semi-ample for $u \in \sigma^\vee$ and big for $u \in \mathrm{relint}(\sigma^\vee)$,

$$h_{\mathfrak{D}}(u) = \mathfrak{D}(u).$$

By [AH, proposition 2.11], every convex piecewise linear map $h : \omega \to \mathrm{CaDiv}_{\mathbb{Q}}(Y)$[1], such that $h(u)$ is strictly semi-ample for $u \in \omega$ and big for $u \in \mathrm{relint}(\omega)$ corresponds to certain pp-divisor on (Y, N) with tail cone $\omega^\vee$.

Definition 2.2.13. Suppose a T-variety is described by $\mathfrak{D}$ on (Y, N). We call Y the base space of the pp-divisor.

[1]A cone ω is a full dimensional cone in the lattice M, and let N be its dual lattice

Let us recall two important theorems, [AH, Theorem 3.1] and [AH, Theorem 3.4] by Altmann and Hausen.

Theorem 2.2.14 ([AH]). *Let Y be a normal semi-projective variety and $\mathfrak{D}$ be a pp-divisor on $(Y, N_{\mathbb{Q}})$. Consider $\mathscr{O}_Y$-algebra $\mathscr{A}$ defined above 2.2. The algebraic torus $T = Spec(\mathbb{C}[M])$, and the relative spectrum $\tilde{X} = Spec_Y(\mathscr{A})$. Then the following statements hold :*

1. *The scheme $\tilde{X}$ is a normal algebraic variety of dimension $dim(Y) + dim(T)$, and the grading of $\mathscr{A}$ defines an effective torus action $T \times \tilde{X} \to \tilde{X}$ having the canonical map $\pi : \tilde{X} \to Y$ as a good quotient.*
2. *The ring of global sections $A = \Gamma(\tilde{X}, \mathscr{O}_{\tilde{X}}) = \Gamma(Y, \mathscr{A})$ is a finitely generated M-graded normal $\mathbb{C}$-algebra, and we have a proper, birational T-equivariant contraction morphism $\tilde{X} \to X$ with $X = Spec(A)$.*

$$\begin{tikzcd} \tilde{X} \arrow[r, "r"] \arrow[d, "\pi"] & X \\ Y. & \end{tikzcd} \tag{2.3}$$

If r is identity map then we call X contraction-free.

Theorem 2.2.15 ([AH]). *Let X be a normal affine variety and suppose that $T = Spec(\mathbb{C}[M])$ acts effectively on X with weight cone $\omega \subset M_{\mathbb{Q}}$. Then there exist normal semiprojective variety Y and pp-divisor on $(Y, \omega^{\vee})$ such that we have an isomorphism of graded algebra:*

$$\Gamma(X, \mathscr{O}_X) \cong \bigoplus_{u \in \omega \cap M} \Gamma(Y, \mathscr{O}_Y(\mathfrak{D}(u)))$$

We will briefly recall the construction of the base space Y, and a pp-divisor of the form of a map $\mathfrak{D} : \omega \to \mathrm{CaDiv}_Q(Y)$ in the chapter 5

Example 2.2.16 ([AH]). *Consider the following example of T-variety of complexity 1,*

$$A = \mathbb{C}[x, y, z, \frac{x^3 + y^4}{z}] \cong \frac{\mathbb{C}[x, y, z, w]}{\langle x^3 + y^4 + zw \rangle}.$$

Consider the following two dimension torus action on A,

$$(t_1, t_2)(x, y, z, w) = (t_1^4 x, t_1^3 y, t_2 z, t_1^{12} t_2^{-1} w).$$

with respect to this action we have the following graded components,

$$A_{(4,0)} = \mathbb{C} \cdot x, A_{(3,0)} = \mathbb{C} \cdot y, A_{(0,1)} = \mathbb{C} \cdot z, A_{(12,-1)} = \mathbb{C} \cdot \frac{x^3 + y^4}{z}.$$

The pp-divisor associated with this T-variety is $\mathfrak{D}$ on $(Y = \mathbb{P}^1, \mathbb{Z}^2)$ with $\sigma = \langle (1,0), (1,12) \rangle$ is,

$$\mathfrak{D} = \Delta_0 \otimes \{0\} + \Delta_1 \times \{1\} + \Delta_\infty \otimes \{\infty\}.$$

where $\Delta_0 = (\frac{1}{3}, 0) + \sigma$, $\Delta_1 = (-\frac{1}{4}, 0) + \sigma$, $\Delta_\infty = (\{0\} \times [0,1]) + \sigma$. See [AHS, Section 11] for a construction of this pp-divisor.

Example 2.2.17 ($\mathbb{C}^*$-surfaces [AH])**.** *Consider X = Spec(A) normal affine surface with an effective $\mathbb{C}^*$-action. Consider $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is grading induced by effective $\mathbb{C}^*$-action. We denote $A_- = \bigoplus_{i<0} A_i$.*

1. *(Elliptic $\mathbb{C}^*$-surface) A $\mathbb{C}^*$-surface X is called elliptic if the $\mathbb{C}^*$-action has an attractive fixed point. Equivalently $A_0 = \mathbb{C}$ and $A_- = 0$. A pp-divisor associated with this is given by projective curve Y and $\sigma = \mathbb{Q}_{\geq 0}$ and*

$$\mathfrak{D} = \sum_i (\sigma + u_i) \otimes \{y_i\},$$

where $u_i \in \mathbb{Q}$ with finitely many different from 0 such that $\sum u_i > 0$, $y_i \in Y$.

2. *(Parabolic $\mathbb{C}^*$-Surface) If $\mathbb{C}^*$-action has infinitely many fixed points. Equivalently $A_0 \neq \mathbb{C}$ and $A_- = 0$. Similarly In this case Y is affine curve, and $\sigma = \mathbb{Q}_{\geq 0}$,*

$$\mathfrak{D} = \sum_i (\sigma + u_i) \otimes \{y_i\},$$

where $u_i \in \mathbb{Q}$ with finitely many different from 0 and $y_i \in Y$.

3. *(Hyperbolic $\mathbb{C}^*$-Surface) If $\mathbb{C}^*$-action has finitely many fixed points. Equivalently*

$A_+ = \bigoplus_{i \geq 0} A_i \neq 0$ and $A_- \neq 0$. In this case Y is affine and $\sigma = \{0\}$,

$$\mathfrak{D} = \sum_i [u_i, v_i] \otimes \{y_i\} ,$$

with $y_i \in Y$ and $u_i \leq v_i$ and only finitely many u_i and v_i are different from 0.

2.3 Gluing of affine T-varities

First we are going to recall the morphism of pp-divisors, and then we will talk about a divisorial fan.

Definition 2.3.1 (Plurifunction)**.** Let Y be a normal semi-projective variety and N be a lattice. A plurifunction with respect to Y and N is an element of $\mathbb{C}(Y, N)^* = N \otimes \mathbb{C}(Y)^*$.

Definition 2.3.2 (Polyhedral Principal Divisor)**.** A polyhedral principle divisor with respect to $\sigma \subset N_{\mathbb{Q}}$ of a plurifunction $f = \sum v_i \otimes f_i$ is

$$\mathrm{div}(f) = \sum (v_i + \sigma) \otimes \mathrm{div}(f_i)$$

Let Y and Y' be normal semiprojective varieties, N and N' be lattices. Let $\sigma \subset N_{\mathbb{Q}}$ and $\sigma' \subset N'_{\mathbb{Q}}$ be pointed cones. Consider the pp-divisor

$$\mathfrak{D} = \sum \Delta_i \otimes D_i \ , \ \mathfrak{D}' = \sum \Delta'_i \otimes D'_i$$

On (Y, N) and (Y', N') respectively. Tail cones of $\mathfrak{D}$ and $\mathfrak{D}'$ are σ and σ' respectively.

Definition 2.3.3 (Polyhedral Pull Back)**.** Let $\phi : Y \to Y'$ be a morphism with none of the supports $\mathrm{Supp}(D'_i)$ contains $\phi(Y)$, the polyhedral pull back is polyhedral divisor

$$\phi^*(\mathfrak{D}') = \sum \Delta'_i \otimes \phi^*(D'_i).$$

Definition 2.3.4 (Polyhedral Push Forward)**.** Let $L : N \to N'$ a linear map with image of σ is subset of σ', the polyhedral push forward is

$$L_*(\mathfrak{D}) = \sum (L(\Delta_i) + \sigma') \otimes D_i$$

Definition 2.3.5 (Morphism of pp-divisors)**.** A morphism $\mathfrak{D} \to \mathfrak{D}'$ is a triple (ϕ, L, f) where ϕ is dominant morphism from $Y \to Y'$, L is a linear map from $N \to N'$ and f is plurifunction $f \in \mathbb{C}(Y, N')^*$ such that,

$$\phi^*(\mathfrak{D}') \leq L_*(\mathfrak{D}) + \mathrm{div}(f).$$

From [AH, Proposition 8.6], for a morphism of pp-divisors $\mathfrak{D} \to \mathfrak{D}'$, we have an induced homomorphism $X(\mathfrak{D}) \to X(\mathfrak{D}')$ of T-varieties is given by

$$\Gamma(Y', \mathscr{O}_{Y'}(\mathfrak{D}'(v))) \to \Gamma(Y, \mathscr{O}_Y \mathfrak{D}(L^*(v)))$$

where L^* is dual map associated to L,

$$s \to f(v)\phi^*(s).$$

Following [AHS, Section 5], for the gluing of affine T-varieties, we will allow the empty set ϕ as a coefficient to a pp-divisor with

$$\phi + \Delta = \phi \ , \ 0 \cdot \phi = \sigma.$$

Consider formal sum $\mathfrak{D} = \sum \Delta_D \otimes D$ with D are prime divisor and we are allowing empty coefficient to D. then defined locus of $\mathfrak{D}$ as

$$\mathrm{Loc}(\mathfrak{D}) = Y \setminus \bigcup_{\Delta_D = \phi} D.$$

A formal sum $\mathfrak{D} = \sum \Delta_D \otimes D$ is called pp-divisor if $\mathfrak{D} \restriction_{\mathrm{Loc}(\mathfrak{D})}$ is a pp-divisor on $(\mathrm{Loc}(\mathfrak{D}), N)$ with tail cone $\mathrm{tail}(\mathfrak{D})$.

Let $\mathfrak{D} = \sum \Delta_D \otimes D$ be a pp-divisor on (Y, N), with $\mathrm{tail}(\mathfrak{D}) = \sigma \in N_{\mathbb{Q}}$. Consider a T-variety $X(\mathfrak{D}) = \mathrm{Spec}(A)$, where $A = \bigoplus_{u \in \sigma^\vee \cap M} A_u$ induced by effective torus action on X.

Definition 2.3.6 (Zero set and Principal set)**.** For $v \in \sigma^\vee \cap M$ and $h \in A_v$,

$$Z(h) = \mathrm{Supp}(\mathrm{div}(h) + \mathfrak{D}(v)) \ , \ Y_h = Y \setminus Z(h).$$

For v we have face of Δ_D given by set,

$$\Delta_{(D,v)} = \{u \in \Delta_D \,|\, \langle v, u' - u\rangle \quad \text{for all } u' \in \Delta_D\}\,,\ \text{tail}(\Delta_{(D,v)}) = \sigma \cap v^{\perp}.$$

Consider T-invariant open subset $X_h = \text{Spec}(A_h)$, pp-divisor for this T-variety is

$$\mathfrak{D}_h = \sum \Delta_{(D,v)} \otimes D \restriction_{Y_h} := \phi \otimes (\text{div}(h) + \mathfrak{D}(v)) + \sum \Delta_{(D,v)} \otimes D$$

Example 2.3.7 (Morphism of a pp divisors)**.** *The torus invariant inclusion $X_h \to X$ is given by the morphism $(i_h, I, 1) : \mathfrak{D}_h \to \mathfrak{D}$, where i_h is an inclusion map $Y_h \to Y$, I is an identity of lattice homomorphism and* 1 *is trivial plurifunction.*

Example 2.3.8. *Consider a triple $(id_Y, id_N, \sum e_i \otimes t_i)$, where $t_i \in \mathbb{C}^*$. An automorphism associated with the above triple is a multiplication automorphism $t = (t_1 \dots t_n) : X(\mathfrak{D}) \to X(\mathfrak{D})$.*

Consider two pp-divisors $\mathfrak{D} = \sum \Delta_D \otimes D$ and $\mathfrak{D}' = \sum \Delta'_D \otimes D$ on (Y, N) with $\text{tail}(\mathfrak{D}) = \sigma$ and $\text{tail}(\mathfrak{D}') = \sigma'$ respectively.

Definition 2.3.9 (Intersection of pp-divisors)**.** Intersection is defined as:

$$\mathfrak{D} \cap \mathfrak{D}' = \sum(\Delta_D \cap \Delta'_D) \otimes D.$$

Definition 2.3.10. We say that $\mathfrak{D}'$ is subset of $\mathfrak{D}$, i.e. $\mathfrak{D}' \subset \mathfrak{D}$ if $\Delta'_D \subset \Delta_D$ holds for every prime divisor D (consequently $\sigma' \subset \sigma$).

Definition 2.3.11 (Face of a pp-divisor $\mathfrak{D}$)**.** A pp-divisor $\mathfrak{D}'$ is subset of $\mathfrak{D}$, we say $\mathfrak{D}'$ is face of $\mathfrak{D}$, if the morphism $X(\mathfrak{D}') \to X(\mathfrak{D})$ corresponding to an inclusion

$$\bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathscr{O}_Y(\mathfrak{D}(u))) \subset \bigoplus_{u \in \sigma'^\vee \cap M} \Gamma(Y, \mathscr{O}_Y(\mathfrak{D}'(u)))$$

is an open embedding.

Remark 2.3.12 ([AH])**.** *If $\mathfrak{D}' \prec \mathfrak{D}$ then associated open embedding $X(\mathfrak{D}') \to X(\mathfrak{D})$ is torus equivariant open embedding*

Definition 2.3.13 (Divisorial Fan)**.** A divisorial fan $\mathcal{S}$ is finite collection of pp-divisors, such that intersection of two pp-divisors is face for both, i.e., if $\mathfrak{D}, \mathfrak{D}' \in \mathcal{S}$ then $\mathfrak{D} \succ \mathfrak{D} \cap \mathfrak{D}' \prec \mathfrak{D}'$ and if $\mathfrak{D} \in \mathcal{S}$ and $\mathfrak{D}'$ is a face of $\mathfrak{D}$ then $\mathfrak{D}' \in \mathcal{S}$.

Note that, $\{\text{tail}(\mathfrak{D}) \,|\, \mathfrak{D} \in \mathcal{S}\}$ is fan and we say it is tail fan of $\mathcal{S}$. Now we are going recall [AHS, Theorem 5.3] briefly. Given a divisorial fan $\mathcal{S}$, consider $\mathfrak{D}_i \in \mathcal{S}$ and associated affine T-variety $X_i = X(\mathfrak{D}_i)$. By definition of a face we have the following commutative diagram,

$$\begin{tikzcd}
 & X(\mathfrak{D}_1 \cap \mathfrak{D}_2 \cap \mathfrak{D}_3) \arrow[dl] \arrow[d] \arrow[dr] & \\
X(\mathfrak{D}_1 \cap \mathfrak{D}_2) \arrow[r] & X(\mathfrak{D}_1) & X(\mathfrak{D}_1 \cap \mathfrak{D}_3). \arrow[l]
\end{tikzcd}$$

hence we have associated variety

$$X(\mathcal{S}) = \bigsqcup_{\mathfrak{D} \in S} X(\mathfrak{D}) / \sim$$

with an effective torus action. Where $\sim$ is given by, $x \in J_{12}(X(\mathfrak{D}_1 \cap \mathfrak{D}_2)) \sim J_{12} \circ J_{21}^{-1}(x) \in J_{21}(X(\mathfrak{D}_1 \cap \mathfrak{D}_2))$.

$$X(\mathfrak{D}_1) \xleftarrow[J_{12}]{} X(\mathfrak{D}_1 \cap \mathfrak{D}_2) \xrightarrow{J_{21}} X(\mathfrak{D}_2).$$

Consider the following subdivision of a plane.

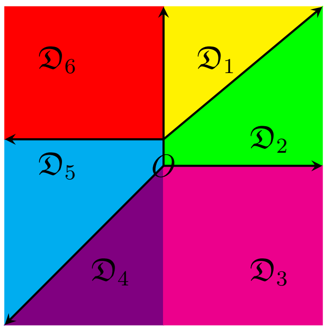

Figure 2.2: 0

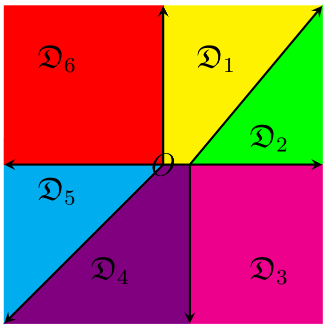

Figure 2.3: 1

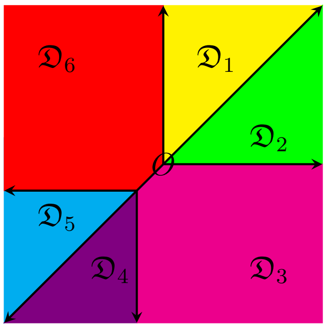

Figure 2.4: ∞

Example 2.3.14 ([AHS]). *Consider the divisorial fan generated by 6 maximal pp-divisor*

$$\left\{\mathfrak{D}_i = \delta_0^i \otimes \{0\} + \delta_1^i \otimes \{1\} + \delta_\infty^i \otimes \{\infty\} \,\middle|\, 0 \leq i \leq 6,\ \delta_p^i \ \textit{ is } \mathfrak{D}_i \textit{ component in } p \textit{ subdivision}\right\}.$$

Example 2.3.15. *Consider $\mathbb{A}^2 - \{(0,0)\}$, which is union of two hyperbolic $\mathbb{C}^*$-surface. Let $Y = \mathbb{P}^1$ be the base space. Consider the following divisorial fan*

$$\{\mathfrak{D}_{a,b} = a\,\{0\} + b\,\{\infty\} \,|\, a, b \in \{1, \phi\} \ \textit{except}\ a = b = 1\}\,.$$

$$X(\mathfrak{D}_{1,\phi}) = \operatorname{Spec}(\mathbb{C}[x,y]_{(x)}),\ X(\mathfrak{D}_{\phi,1}) = \operatorname{Spec}(\mathbb{C}[x,y]_{(y)}),\ X(\mathfrak{D}_{\phi,\phi}) = \operatorname{Spec}(\mathbb{C}[x,y]_{(xy)}).$$

In this paragraph, we will discuss the criterion for completeness for a T-variety $X(\mathcal{S})$, where $\mathcal{S}$ is a divisorial fan. This criterion contains notion of a slice. Lets recall that.

Definition 2.3.16. 1. A polyhedral complex is a finite collection of polyhedra which is closed under faces, and intersection of two polyhedra is a face of both.

2. Support of polyhedral complex is union of all polyhedra.

3. A polyhedral complex with support $N_{\mathbb{Q}}$ is called complete subdivision of $N_{\mathbb{Q}}$ or complete.

Definition 2.3.17 (Slice)**.** For each $y \in Y$, we define polyhedral complex

$$\mathcal{S}_y = \{\mathfrak{D}_y \mid \mathfrak{D} \in \mathcal{S}\},$$

where, for $\mathfrak{D} = \sum \Delta_D \otimes D \in \mathcal{S}$, the polyhedron $\mathfrak{D}_y = \sum\limits_{y \in D} \Delta_D$ is a slice of $\mathfrak{D}$ and $\mathcal{S}_y$ is a slice of $\mathcal{S}$.

Definition 2.3.18. A divisorial fan $\mathcal{S}$ is complete, if it's base space Y is complete and each slice $\mathcal{S}_y$ is complete.

From [AHS], we have criterion for completeness.

Theorem 2.3.19. *A T-variety $X(\mathcal{S})$ is complete if and only if $\mathcal{S}$ is complete.*

From the above theorem, for a complete divisorial fan $\mathcal{S}$(or equivalently for a complete T-variety), we have an associated base space Y and a complete slice for each point of Y (not necessarily closed). Conversely, if we have Y and for each point in Y we have an associate complete polyhedral complex; then, can we construct a complete divisorial fan?(or equivalently a complete T-variety?). Our next section is dedicated to answer this question. A complete T-variety of complexity one $X(\mathcal{S})$(or equivalently a complete divisorial fan) is completely determined by a combinatorial data given by base space and complete slices.

2.4 Complexity one T-variety.

This section briefly discusses combinatorial data associated with a complexity one T-variety. For a complete complexity on T-variety, we have a marked fansy divisor description. Before that let us recall some equivalent definitions. A divisor on a curve is big if and only if it has a positive degree, and semiample if and only if it is big or some multiple is a principal divisor. From these equivalent definitions, we have an identical description of a pp-divisor on a curve.

Definition 2.4.1 ([AH]). Let $\mathfrak{D} = \sum_{p \in Y} \Delta_p \otimes p$ be a polyhedral divisor on (Y, N) with tail cone σ, $\mathfrak{D}$ is pp-divisor if the following holds:

1. A polyhedral divisor $\mathfrak{D} = \sum \Delta_p \otimes \{p\}$, with p are pairwaise disjoint closed points and Δ_p are polyhedron with tailcone σ and only finitely many different from σ.

2. A polyhedron, $\deg(\mathfrak{D}) = \sum \Delta_p$ is a proper subset of the cone σ.

3. If $\mathrm{eval}_u(\deg(\mathfrak{D})) := \min\langle u, \deg(\mathfrak{D})\rangle = 0$, then u is in on the boundary of $\sigma^\vee$, and some multiple of $\mathfrak{D}(u)$ is principle.

Definition 2.4.2 (Marked Fansy Divisor [IS2]). A marked fansy divisor on a curve Y is a formal sum $\Xi = \sum \Xi_p \cdot p$ with a fan Σ and some subset $C \subset \Sigma$, such that

1. Ξ_p is complete polyhedral subdivision of $N_\mathbb{Q}$ and $\mathrm{tail}(\Xi_p) = \Sigma$ for all $p \in Y$. And only finitely many Ξ_p are different from Σ.

2. For full-dimension $\sigma \in C$ the polyhedral divisor $D^\sigma = \sum \Delta_p^\sigma \otimes p$ is proper, where Δ_p^σ is a unique element of Ξ_p with tail cone σ.

3. For $\sigma \in C$ of a full dimension and $\tau \prec \sigma$ we have $\tau \in C$ if and only if $\deg D^\sigma \cap \tau \neq \phi$.

4. If $\tau \prec \sigma$ and $\tau \in C$ then $\sigma \in C$.

Elements of C are call marked or contracted cone. The set $P := \{p \in Y \,|\, \Xi_p \neq \Sigma\}$ is call support of Ξ. For a complete divisorial fan one can associate marked fansy divisor. For a divisorial fan $\mathcal{S}$ and for each $p \in Y$, we have a slice $\mathcal{S}_p$ then the associated marked fansy divisor is $\Xi(\mathcal{S}) = \sum \mathcal{S}_p \cdot p$. Given a marked fansy divisor $(\Xi, C \subset \Sigma)$, there is a complete complete divisorial fan $\mathcal{S}$ generated by the set

$$\{\mathfrak{D}^\sigma \,|\, \sigma \in C\} \bigcup \left\{ \Delta \otimes p + \sum_{q \in P, q \neq p} \phi \otimes q \;\middle|\; \Delta \in \Xi_p \neq \Sigma, \text{ and } \mathrm{tail}(\Delta) \notin C \right\} \tag{2.4}$$

such that $\Xi(\mathcal{S}) = \Xi$.

Remark 2.4.3. *A complete divisorial fan on a curve is contraction free if and only if the set of contracted cones is empty.*

In the following paragraph, we briefly recall the construction of a divisorial fan from a divisorial polyhedron ([IV, Section 4]).

Definition 2.4.4. Let Y be a smooth curve, a *divisorial polyhedron* consists of a pair $(\mathrm{L}, \square)$, where $\square$ is a polyhedron in $M_{\mathbb{Q}}$, and L is piecewise affine concave map from $\square$ to $\mathrm{CaDiv}_{\mathbb{Q}} Y$ taking values in samiample divisors.

Let's recall the construction of a complexity one T-variety from a divisorial polyhedron. For any $u \in \square$ and $P \in Y$ prime, $\mathrm{Lin}_P : \mathrm{tail}(\square) \to \mathbb{Q}$ is defined by

$$\mathrm{Lin}_P(v) = \lim_{\lambda \to \infty} L_P(u + \lambda.v)/\lambda$$

and we set

$$\square_P^* = \{v \in N_{\mathbb{Q}} \mid (v, w) \geq \mathrm{Lin}_P(w) \ \forall w \in \square\}; \qquad \text{and } \mathrm{L}_P^* : \square_P^* \to Q$$

$$\mathrm{L}_P^*(v) = \min_{u \in \square} \Big(\langle u, v \rangle - \mathrm{L}_P(u) \Big)$$

Then, for any P, $\Xi(\mathrm{L}_P^*)$ is subdivision of $\square_P^*$ consisting of pointed polyhedra. Now from the above construction, we can associate a divisorial fan. Define a set $K = \{P \in Y |\ \mathrm{L}_P \not\equiv 0\}$ and $E = \sum_{P \in K} P$. Then the divisorial fan $\mathcal{S}$ is generated by

$$C_{\mathrm{L}} = \{\Delta_P \otimes P + \phi \otimes (E - P) |\ P \in K,\ \Delta_P \in \Xi(\mathrm{L}_P^*)\}.$$

From remark 2.4.3 and equation 2.4:

Remark 2.4.5 ([IV])**.** *The divisorial fan $\mathcal{S}$ is a contraction free.*

Definition 2.4.6 (Inner Normal fan)**.** A inner normal fan of a polyhedron Δ with a tail cone $\sigma \subset N_{\mathbb{Q}}$ is the linearity regions of the function

$$\min \langle -, \Delta \rangle \ : \ \sigma^{\vee} \to \mathbb{Q}.$$

Let us recall the definition of *toric bouquets* before ending this chapter. Let Δ be a

σ-polyhedron in $N_{\mathbb{Q}}$. The map

$$\min \langle -, \Delta \rangle \, : \, \sigma^{\vee} \to \mathbb{Q}$$

gives the *inner normal fan* which is a subdivision of $\sigma^{\vee}$. Define the $\mathbb{C}$-algebra $\mathbb{C}[\sigma^{\vee} \cap M]$, note that the multiplication is given by

$$\chi^u \cdot \chi^v = \begin{cases} \chi^{u+v}, & \text{if } u \text{ and } v \text{ are in the same cone of inner normal fan} \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

A *toric bouquets* is a scheme of the form $\mathrm{TB}(\Delta) := \mathrm{Spec}(\mathbb{C}[\sigma^{\vee} \cap M]$. Similarly toric bouquets associated with a polyhedral complex S is $\mathrm{TB}(S) := \bigsqcup_{\sim} \mathrm{TB}(\Delta)$, where $\sim$ is given by the following diagram

$$\begin{tikzcd}
 & \mathrm{TB}(\Delta_1 \cap \Delta_2 \cap \Delta_3) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{TB}(\Delta_1 \cap \Delta_2) \arrow[r] & \mathrm{TB}(\Delta_1)) & \mathrm{TB}(\Delta_1 \cap \Delta_3). \arrow[l]
\end{tikzcd}.$$

2.5 Projectivization of a toric vector bundle

Definition 2.5.1. A toric vector bundle on X_Σ is a vector bundle $\pi : \mathscr{E} \to X_\Sigma$, such that the torus of X_Σ acts on $\mathscr{E}$ such that the projection map π is torus equivariant and action is linear on fiber.

Torus invariant divisor of a character

Let X_Σ be a toric variety of the fan Σ in $N_{\mathbb{Q}}$. By [CLS, Theorem 3.2.6], there is one-to-one correspondence between k dimensional cones in Σ and $\dim(X) - k$ dimensional orbits in X_Σ. Let us consider 1 dimensional cone ρ in Σ. The orbit closure corresponding to ρ is a torus invariant prime divisor, and we denote it by D_ρ. A minimal generator of ρ is denoted u_ρ. A *torus invariant Weil divisor* is an element of $Z^T_{\dim(X_\Sigma)-1}(X_\Sigma)$, Where $Z^T_k(X_\Sigma)$ is a free abelian group generated by k-dimensional torus invariant subvarieties of X_Σ. Similarly, *torus invariant Cartier divisor* is locally torus invariant Weil divisor. Every Weil divisor on X_Σ is linearly equivalent to torus invariant Weil divisor. The character χ^m is a rational function on X_Σ, then divisor of a character is:

$$\mathrm{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m,\, u_\rho \rangle\, D_\rho.$$

By [CLS, Proposition 4.2.2], every torus invariant Cartier divisor on U_σ is the divisor of a character.

Let $\pi : V \to X$ be a vector bundle of rank k. Then X has an open cover $\{U_i\}$ with isomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^k$. Furthermore, the transition map: $t_{ij} \in \mathrm{GL}_n(\Gamma(U_i \cap U_j, \mathscr{O}_X))$ such that:

$$\phi_i|_{\pi^{-1}(U_i \cap U_j)} = (1 \times t_{ij}) \circ \phi_j|_{\pi^{-1}(U_i \cap U_j)},$$

where $1 \times t_{ij} : (U_i \cap U_j) \times \mathbb{C}^k \to (U_i \cap U_j) \times \mathbb{C}^k$ is gluing data. The map $1 \times t_{ij}$ induces:

$$1 \times \bar{t_{ij}} : (U_i \cap U_j) \times \mathbb{P}^{k-1} \to (U_i \cap U_j) \times \mathbb{P}^{k-1}.$$

This gluing data induces projective vector bundle $\bar{\pi} : \mathbb{P}(V) \to X$.

Definition 2.5.2. A projectivization of a vector bundle $\pi : \mathcal{F} \to X$ is:

$$\mathbb{P}(\mathscr{F}) = \mathbb{P}(\mathcal{F}^\vee),$$

where $\mathscr{F}$ is the sheaf of sections of vector bundle π and $\mathcal{F}^\vee$ is the dual of a vector bundle $\mathcal{F}$.

Note that, we are employing the concepts vector bundle of rank r and locally free sheaf of rank r interchangeably. Given a rank $r+1$ toric vector bundle $\mathscr{F}$ the variety $\mathbb{P}(\mathscr{F})$ can be consider as a complexity r T-variety. Let's review the following findings for future reference.

If a toric vector bundle $\mathscr{F}$ on $X = X_\Sigma$ split as sum of line bundle then $\mathbb{P}(\mathscr{F})$ is a toric variety. As our focus lies on the T-variety of complexity one, we shall now explore the realm of rank two toric vector bundles. Let $\mathscr{F}$ be a rank two toric vector bundle on X_Σ. Given that each toric vector bundle on an affine toric variety splits into toric line bundles then we can express:

$$\mathscr{F}|_{U_\sigma} = \mathscr{O}_X(\mathrm{div}(\chi^{u_\sigma^1})) \oplus \mathscr{O}_X(\mathrm{div}(\chi^{u_\sigma^2})).$$

In [AHS, Example 8.3], authors provides a combinatorial description of $\mathbb{P}(\mathscr{F})$. Define the following polyhedra:

$$\Delta_1 = \left\{ v \in N_{\mathbb{Q}} \,\middle|\, \langle u_\sigma^1 - u_\sigma^2, v\rangle \geq 1 \right\}$$

$$\Delta_2 = \left\{ v \in N_{\mathbb{Q}} \,\middle|\, \langle u_\sigma^2 - u_\sigma^1, v\rangle \geq 1 \right\}$$

$$\nabla_1 = \left\{ v \in N_{\mathbb{Q}} \,\middle|\, \langle u_\sigma^1 - u_\sigma^2, v\rangle \leq 1 \right\}$$

$$\nabla_2 = \left\{ v \in N_{\mathbb{Q}} \,\middle|\, \langle u_\sigma^2 - u_\sigma^1, v\rangle \leq 1 \right\}.$$

Then the following polyhedral divisors are describing $\mathbb{P}(\mathscr{F}|_{U_\sigma})$

$$\mathfrak{D}_1 = \Delta_1 \otimes v_1 + \nabla_2 \otimes v_2$$

$$\mathfrak{D}_2 = \nabla_1 \otimes v_1 + \Delta_2 \otimes v_2.$$

where v_1 and v_2 are one dimensional vectors sub-spaces of fiber of a vector bundle which appears in the Klyachko'S filtration for $\mathscr{F}$ on σ. Note that, v_i is considered as a point in projective line $\mathbb{P}^1$.

3

Equivariant Chow groups and Approximate space

In this chapter, we recall the combinatorial description of Chow groups of toric variety and Chow group of complete rational T-variety of complexity one. The main purpose of this chapter is to construct a fan for approximate space with canonical torus action.

3.1 Chow groups

Let X be a variety, and let $Z_k(X)$ be a free abelian group generated by k-dimensional subvarieties of X. Consider the subgroup of $Z_k(X)$ generated by $\mathrm{div}(f)$, where f is a rational function on $k+1$ dimensional subvariety of X, and we denote it by $R_k(X)$. The k^{th} Chow group of X is denoted by $A_k(X)$ and it is defined as

$$A_k(X) = \frac{Z_k(X)}{R_k(X)}.$$

For a variety X, with a torus action T, the group $A_k^T(X)$ is defined as follows:

$$A_k^T(X) = \frac{Z_k^T(X)}{R_k^T(X)},$$

where $Z_k^T(X)$ is a free abelian group generated by the torus invariant subvarieties of X, and subgroup of $Z_k^T(X)$ generated by $\mathrm{div}(f)$, where f is a semi-invariant rational function on $k+1$ dimensional torus invariant subvariety of X is denoted by $R_k^T(X)$. Since T is connected solvable linear algebraic group, the canonical map $A_k^T(X) \to A_k(X)$ is a group isomorphism(see [FMSS]). The generators of $A_k^T(X)$ are orbit closures. Suppose V is a torus invariant subvariety of dimension k, then $\overline{T \cdot x} = V$, where x is generic point of V.

By [FS, Proposition 2.1], the Chow group $A_k^T(X)$ of a toric variety $X = X(\Sigma)$ is generated by all k-dimensional torus invariant subvarieties $V(\sigma)$. Where $V(\sigma)$ is a orbit closure corresponding to the cone σ in Σ of codimension k. Hence we have the following canonical surjective map

$$J : \bigoplus_{\mathrm{codim}(\sigma)=k} \mathbb{Z}[V(\sigma)] \to A_k^T(X).$$

The kernel of map J is given the divisors $\mathrm{div}(\chi^u)$, where χ^u is a semi-invariant rational function on $k+1$ dimensional torus invariant subvariety. Consider the following set;

$$S = \{\tau \,|\, \tau \subset \sigma,\ \mathrm{codim}(\tau) = k+1\ ,\ \mathrm{codim}(\sigma) = k\}\,.$$

For $\tau \in S$, we denote $M(\tau) = \tau^{\perp} \cap M$. Note that $\dim(\sigma) = \dim(\tau) + 1$. Let $N(\sigma)$ be a lattice generated by the cone σ. For $u \in M(\tau)$;

$$\mathrm{div}(\chi^u) = \sum_{\tau \subset \sigma} \langle u,\, v_{\sigma,\tau} \rangle\, [V(\sigma)],$$

where $v_{\sigma,\tau}$ is a lattice element of σ such that it's image generates the one dimensional lattice $\frac{N(\sigma)}{N(\tau)}$. The Chow group $A_k^T(X)$ is completely determine by the following exact sequence;

$$\bigoplus_{\tau \in S} M(\tau) \longrightarrow \bigoplus_{\mathrm{codim}(\sigma)=k} \mathbb{Z}[V(\sigma)] \xrightarrow{\ J\ } A_k^T(X) \longrightarrow 0\ ,$$

where $u \mapsto [\mathrm{div}(\chi^u)]$ is the definition of first arrow. See [CLS, Chapter 2] for more details.

The torus invariant subvarieties of a toric variety are of the form $V(\sigma)$. Now we will recall a description of torus invariant subvarieties for a complete complexity one T-variety. Let $X = X(\Xi)$ be a complete complexity one T-variety of dimension $d+1$. Let Σ be the tail fan of Ξ, and C be a set of contracted cone. Consider the following diagram;

$$
\begin{tikzcd}
\tilde{X} \arrow[r, "r"] \arrow[d, "\pi"] & X \\
\mathbb{P}^1.
\end{tikzcd}
\tag{3.1}
$$

Consider the set $P = \{p \in \mathbb{P}^1 \,|\, \Xi_p \neq \Sigma\}$. For $p \in \mathrm{Loc}(\mathcal{S}) \subset \mathbb{P}^1$, the fiber $\pi^{-1}(p)$ equals the toric variety $X(\Sigma)$ if $p \notin P$, otherwise it is toric bouquet associated to Ξ_p. Let $\mathrm{orb}(p, F)$ be the orbit of $\pi^{-1}(p)$ corresponding to $F \in \Xi_p$. The Chow group $A_k^T(\tilde{X})$ is generated by the class of k dimensional torus invariant subvarieties of the form $\overline{\mathrm{orb}(p, F)}$, and

$$\dim(\overline{\mathrm{orb}(p, F)}) = \dim(\overline{p}) + \mathrm{codim}(F).$$

For any torus invariant subvarieties $Z \subset \tilde{X}$;

$$
\dim(r(Z)) = \begin{cases} \dim(Z), & \text{if } \dim(\pi(Z)) = 0; \\ \dim(Z) - 1, & \text{if } \dim(\pi(Z)) = 1 \text{ and } Z \text{ is contracted.} \end{cases}
\tag{3.2}
$$

Let us consider the following notations;

1. For any p and $F \in \Xi_p$, $W_{p,F} = r(\overline{\mathrm{orb}(p, F)})$.

2. For $p \notin P$, and $\sigma \in C$, $W_\sigma = r(\overline{\mathrm{orb}(p, \sigma)})$.

3. For $p \notin P$, the non contracted orbit closures are denoted by $B_\sigma = r(\overline{\mathrm{orb}(p, \sigma)})$.

We shall ensure that the size of P is at least 2 by appending extra points, if necessary. For a non-negative integer $k \leq d + 1$, we obtain the k^{th} Chow group of a T-variety of complexity one, consider the following sets

- R_k = Cones of dimension $d+1-k$ corresponding to subvarieties not contracted by r (i.e B_σ).

- V_k = Non-contracted faces of dimension $d-k$ of polyhedral subdivision corresponding to the fiber of points in P (i.e $W_{p,F}$, for $p \in P$, and $\mathrm{tail}(F) \notin C$).

- T_k = Cones of dimension $d - k$ corresponding to subvarieties contracted by r.

Theorem 3.1.1. *[Nød, Theorem 4.1] For a T-variety $X = X(\Xi)$, the Chow group $A_k(X)$ is obtained by the following exact sequence;*

$$\bigoplus_{F\in V_{k+1}} M(F) \bigoplus_{\tau\in R_{k+1}} (M(\tau)\oplus \mathbb{Z}^P/\mathbb{Z}) \bigoplus_{\tau\in T_{k+1}} M(\tau) \to \mathbb{Z}^{V_k}\oplus\mathbb{Z}^{R_k}\oplus\mathbb{Z}^{V_k} \to A_k(X)\to 0\ .$$

Where $M(\tau)=\tau^{\perp}\cap M$, and $M(F)$ is the character lattice of a toric bouquets corresponding to F.

First arrow is given by the following assignments;

1. If $\tau\in T_{k+1}$,

$$u\in M(\tau)\mapsto \sum_{\dim(\sigma)=n-k,\tau\subset\sigma} \langle u,\, v_{\sigma,\tau}\rangle\, W_\sigma,$$

   where $\sigma\in\Sigma$, with $\dim(\sigma)=\dim(\tau)+1$. And $v_{\sigma,\tau}$ is a lattice element of σ such that it's image generates the one dimensional lattice $\frac{N(\sigma)}{N(\tau)}$.

2. An element $m\in M(F)$ maps to the cycle

$$\sum_{\substack{\dim(G)=n-k,\\ F\subset G,\ \mathrm{tail}(G)\notin C}} \langle m,\, v_{F,G}\rangle\, W_{p,G} + \sum_{\substack{\dim(G)=n-k,\\ F\subset G,\ \mathrm{tail}(G)\in C}} \langle m,\, v_{F,G}\rangle\, t_{\mathrm{tail}(G)} W_{\mathrm{tail}(G)},$$

   where $v_{F,G}$ generates $\frac{N(F)}{N(G)}$. The rational number t_G is explained in the following paragraph.

   For $\mathrm{tail}(G)\in C$, by [Nød, Lemma 3.4], the stabilizer of $r(\overline{\mathrm{orb}(p,\mathrm{tail}(G))})$ is group generated by the stabilizers of all $\overline{\mathrm{orb}(q,F)}$ such that $\mathrm{tail}(F)=\mathrm{tail}(G)$. We denote the difference between the ranks of the stabilizer of $r(\overline{\mathrm{orb}(p,\mathrm{tail}(G))})$ and stabilizer of $\overline{\mathrm{orb}(p,\mathrm{tail}(G))}$ by $s_{\mathrm{tail}(G)}$(Please refer to [Nød, Section 3] for more details). For $\mathrm{tail}(G)\in C$, $v_{\mathrm{tail}(G)}$ is the vertex of G corresponding to $B_{\mathrm{tail}(G)}$ (contracted), and $\mu(G)$ is the smallest integer such that $v_{\mathrm{tail}(G)}$ is a lattice point. The number t_G is equal to$\frac{s_{\mathrm{tail}(G)}}{\mu(G)}$.

3. A generator of $\frac{\mathbb{Z}^P}{\mathbb{Z}}$ corresponds to $[p]-[\infty]$ maps to

$$\sum_{F,\mathrm{tail}(F)=\tau} \mu(v_F) W_{p,F} - \sum_{F,\mathrm{tail}(F)=\tau} \mu(v_F) W_{\infty,F}$$

where $\tau \in R_{k+1}$ and v_F is the vertex of F corresponding to B_τ(non contracted), and $\mu(v_F)$ is the smallest integer such that v_F is a lattice point. The divisor $[p] - [\infty]$ is corresponding to a generator of $\frac{\mathbb{Z}^P}{\mathbb{Z}}$.

4. For $\tau \in R_{k+1}$, a point $m \in M(\tau)$ maps to

$$\sum_{F \in V_k, \mathrm{tail}(F)=\tau} \mu(v_F) \langle m, v_F \rangle W_{p,F} + \sum_{\sigma \in R_k, \tau \subset \sigma} \langle m, \bar{\sigma} \rangle B_\sigma$$

where the image of σ in $N(\tau) = \frac{N}{N_\tau}$ is $\bar{\sigma}$.

For a schemes with an action of a group G, Totaro [Tot] and Edidin and Graham [EG] define an equivariant Chow group, which we recall now. For an arbitrary group G, we have the construction of a Δ-complex EG on which G acts freely on the left multiplication. We denote $BG = EG/G$. The spaces EG and BG are usually infinite dimensional spaces. For $G = \mathbb{C}^*$, we have $EG = \mathbb{C}^\infty \setminus \{0\}$ on which G acts freely and $BG = \mathbb{P}^\infty$. But these spaces are inexplicable in algebraic geometry. Nonetheless, we have algebraic varieties $E_m = \mathbb{C}^m \setminus \{0\}$ and $B_m = \mathbb{P}^{m-1}$ which provide approximations to $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^{m-1}$. For two groups G and H, $E(G \times H) \equiv EG \times EH$. For an algebraic torus of dimension d we have $ET = (\mathbb{A}^\infty \setminus \{0\})^d$ and $E_T^N = (\mathbb{A}^N \setminus \{0\})^d$. Thus we denote quotient variety $(X \times (\mathbb{A}^N \setminus \{0\})^d)/T$ by X_T.

Definition 3.1.2 (E_T^N)**.** For T, an n dimensional torus $ET = (\mathbb{A}^\infty \setminus \{0\})^d$ is a space on which T acts freely, we define Nd-dimensional space $E_T^N = (\mathbb{A}^N \setminus \{0\})^d$ approximating the contractible space ET on which T acts freely.

Definition 3.1.3 (k^{th}-equivariant Chow group)**.** For X, an $d+1$ dimensional T-variety of complexity one, and for $k \leq d+1$, the k^{th}-equivariant Chow group is defined as the usual k^{th} Chow group of space X_T i.e $A_k(X_T)$.

3.2 A description of E_T^N

.

In our case, we are lucky that the approximate space for a torus T of dimension d, given by $(\mathbb{A}^N \setminus \{0\})^d$ is a toric variety. The next lemma will give a description of this variety in terms of a fan. The cones of this fan are constructed as follows.

Note that for a d-dimensional torus T, a finite dimensional approximation of the classifying space is given by

$$E_T^N = \left(\mathbb{A}^N \setminus \{0\}\right)^d.$$

E_T^N is a toric variety of dimension Nd, whose dense torus will be denoted by T_E. The fan describing E_T^N as a toric variety can be constructed as follows.

Consider the $\mathbb{Q}$-vector space $\mathbb{Q}^N$ with the standard basis $e_1, \ldots, e_N$. Let θ be the cone generated by $\{e_1, \ldots, e_N\}$. Let $\mathscr{T}_I$ be the set of tuples of numbers between 1 and N:

$$\mathscr{T}_I = \left\{(i_1, \ldots, i_d) \in \mathbb{N}^d \,\middle|\, 1 \leq i_j \leq N, \text{ for each } 1 \leq j \leq d\right\}. \tag{3.3}$$

For each $i \in \{1, 2, \ldots, N\}$, define σ^i to be the face $\theta \cap \rho_i^{\perp}$ where ρ_i is the ray generated by e_i. The cone σ^i is generated by $e_1, \ldots, \widehat{e_i}, \ldots, e_N$, all the basis vectors except e_i. For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, N\}$, let

$$\sigma_I = \sigma_{i_1,\ldots,i_r} = \delta_1 \times \cdots \times \delta_d; \tag{3.4}$$

where $\delta_i = \sigma^i$ if $i \in \{i_1, \ldots, i_r\}$; and $\delta_i = 0$ otherwise. Let Σ_E be the fan generated by these cones in $\mathbb{Q}^{Nd}$. The following lemma is evident.

Lemma 3.2.1. *With notation, as described above, the toric variety corresponding to Σ_E is exactly $E_T^N = \left(\mathbb{A}^N \setminus \{0\}\right)^d$.*

Suppose Y is a curve and $\mathcal{S}$ be a divisorial fan on Y. Suppose $X = X(\mathcal{S})$ be the corresponding affine T-variety (of complexity 1). We wish to describe $X(\mathcal{S}) \times E_T^N$ as a T-variety. Observe that E_T^N is a toric variety under the action of the torus T_E. Being an approximation of the classifying space, E_T^N comes with a natural action of T. Thus we get a diagonal action of T on $X(\mathcal{S}) \times E_T^N$. We wish to compute the geometric quotient

$$(X(\mathcal{S}) \times E_T^N)/T$$

Our idea is that the geometric quotient will be birational to the space of any pp-divisor (see definition 2.2.13) representing $X(\mathcal{S}) \times E_T^N$ is considered as a T-variety under the action of T. We end this chapter by describing $X(\mathcal{S}) \times E_T^N$ as a complexity 1 T-variety under the action of $T \times T_E$.

Lemma 3.2.2. *[AIP+, Propositon 5] For two pp-divisors $\mathfrak{D}' = \sum \Delta_i' \otimes D_i'$ on $(Y', \sigma' \subset N')$ and $\mathfrak{D} = \sum \Delta_i \otimes D_i$ on $(Y, \sigma \subset N)$, define $\mathfrak{D} \times \mathfrak{D}' = \sum(\Delta_i \times \sigma') \otimes (D_i \times Y') + \sum(\sigma \times \Delta_i') \otimes (Y \times D_i')$ as a pp-divisor on $(Y \times Y', N \oplus N')$. Then $X(\mathfrak{D} \times \mathfrak{D}') = X(\mathfrak{D}) \times X(\mathfrak{D}')$.*

Proposition 3.2.3. *Suppose Y is a curve, T a torus and E_T^N the space defined in the definition 3.1.2.*

1. *(Affine case) Suppose $\mathfrak{D} = \sum_{i=1}^n \Delta_i \otimes \{p_i\}$ is a pp-divisor on Y and $X(\mathfrak{D})$ is the corresponding affine T-variety. For $I \in \mathscr{T}_I$ (see equation (3.3)) define*

$$\mathfrak{D}_I = \sum_{i=1}^n \Big(\Delta_i \times \sigma_I\Big) \otimes \{p_i\}$$

where σ_I is the cone defined in equation (3.4). Let $\mathcal{S}_{\mathfrak{D}}$ be the divsorial fan generated by $\{\mathfrak{D}_I \,|\, I \in \mathscr{T}_I\}$. Then, $X(\mathfrak{D}) \times E_T^N$, considered as a complexity 1 T-variety under the action of $T \times T_E$, is described by $\mathcal{S}_{\mathfrak{D}}$.

2. *(General case) For a T-variety $X = X(\mathcal{S})$, described by a divisorial fan $\mathcal{S}$ over a curve Y, $X \times E_T^N$ is desribed by the divisorial fan generated by $\{\mathcal{S}_{\mathfrak{D}} \,|\, \mathfrak{D} \in \mathcal{S}\}$ where for each $\mathfrak{D} \in \mathcal{S}$, $\mathcal{S}_{\mathfrak{D}}$ is defined as in the affine case (item 1).*

Proof. This is an easy consequence of the description of product of T-varieties as a T-variety (see, for example, [AIP+, Section 2.4]) which in turn follows from Künneth formula [Gro, 6.7.8]. □

4

Main Theorem

In chapter 3, we described the combinatorial description of $X(\mathcal{S}) \times E_T^N$ as a complexity one T-variety with $T \times T_E$ action. In the first few sections of this chapter we will describe a combinatorial description of $(X(\mathcal{S}) \times E_T^N)/T$ as a complexity 1 T-variety and it's Chow group. In the last section we will see results on rank 2 vector bundles on toric variety in the equivariant setup.

4.1 Computing the downgrade

Let Y be a smooth curve, and $\mathcal{S}$ be a divisorial fan. Let $X = X(\mathcal{S})$ be a T-variety of codimension 1. Note that $T \times T_E$ acts effectively on $X(\mathcal{S}) \times E_T^N$. By proposition 3.2.2, one can write down the following description of this T-variety. To compute the torus-equivariant Chow group, we study $\left(X(\mathcal{S}) \times E_T^N\right) /\!\!/ T$. We know that the base of the pp-divisor in the description of $X(\mathcal{S}) \times E_T^N$ as a T-variety under the action of T is birational to this quotient i.e. $(X(\mathcal{S}) \times E_T^N\big) /\!\!/ T$ birational to $\widetilde{Y_C}$.

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi_C"] & X \times E_T^N \\
\widetilde{Y_C} \arrow[d, "\pi"] & \\
Y, &
\end{tikzcd}$$

We compute this using downgrading following [IV]. For $X = X(\mathcal{S})$ a T-variety with a d-dimensional torus acting on it as above, let M be the lattice of characters for T and N be the dual of M. Let M_E and its dual N_E correspond to the torus T_E acting on E_T^N. We mention a few maps to make the description of the downgraded T-variety easier. Consider the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M_E \arrow[r, "\iota"] & M \oplus M_E \arrow[r, "\pi"] \arrow[l, bend left, dashed, "t"] & M \arrow[r] & 0 \end{tikzcd}$$

where ι and π are described in terms of the map $I\colon M_E \longrightarrow M$ which is given by the matrix (with respect to the standard basis)

$$I = \begin{pmatrix} I_1 & \cdots & I_d \end{pmatrix}$$

where each I_i for $i = 1, \ldots, d$ is a $d \times N$ matrix given by

$$I_i = \begin{pmatrix} e_i & \cdots & e_i \end{pmatrix}, \qquad i \in \{1, \ldots, d\}$$

where e_i is the i-th vector in the standard basis of $\mathbb{Z}^d$ with 1 at the i-th position and 0 elsewhere. Now,

$$\iota(b) = (-I(b), b) \qquad \text{and} \qquad \pi(a, b) = a + I(b).$$

Dualizing this, we also get a short exact sequence

$$0 \longrightarrow N \xrightarrow{\alpha} N \oplus N_E \xrightarrow{\rho} N_E \longrightarrow 0 \tag{4.1}$$

where the maps are described in terms of $J\colon N \longrightarrow N_E$ which is defined as

$$J = \begin{pmatrix} J_1 \\ \vdots \\ J_d \end{pmatrix}.$$

Here each J_i, $i \in \{1, \ldots, d\}$ is an $N \times d$ matrix of the form

$$J_i^T = \begin{pmatrix} e_i & \cdots & e_i \end{pmatrix}$$

where e_i as above is the i-th element of the standard basis for $\mathbb{Z}^d$. With this notation, we have

$$\alpha(a) = (a, J(a)) \qquad \text{and} \qquad \rho(a,b) = b - J(a).$$

As before, let Y be a curve, T a torus, E_T^N the corresponding approximate space and $X = X(\mathcal{S})$ be a complexity 1 T-variety, where the divisorial fan $\mathcal{S}$ is defined over the curve Y. Let the divisorial fan describing $\tilde{X} := X \times E_T^N$, computed in proposition 3.2.3, be denoted by $\mathcal{S}_{\tilde{X}}$. In this section, we aim to describe $\tilde{X} = X(\mathcal{S}_{\tilde{X}})$ as a T-variety under the action of T in terms of a semiprojective variety $\widetilde{Y_C}$ and a divisorial fan $\mathcal{S}_{\widetilde{Y_C}}$ defined over $\widetilde{Y_C}$. The construction follows [IV, section 5.1] closely. Recall that T_E was the dense open torus in E_T^N (see 3.2). Denote $T \times T_E$ be $\tilde{T}$. The inclusion $T \hookrightarrow \tilde{T} = T \times T_E$ induces a surjective homomorphism $\pi \colon \tilde{M} \longrightarrow M$ of the corresponding lattices of characters. Let $M_E \equiv \ker \pi$ and let us define the choices of sections and cosections of the short exact sequences of lattices using the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & M_E \arrow[r, "\iota"'] & \tilde{M} \arrow[l, bend right, "\tau"'] \arrow[r, "\pi"'] & M \arrow[l, bend right, "\sigma^*"'] \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 & N_E \arrow[l] \arrow[r, bend left, "\tau^*"] & \tilde{N} \arrow[l, "\rho"] \arrow[r, bend left, "\sigma"] & N \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Consider a pp-divisor $\mathfrak{D} = \sum \Delta_p \otimes p \in \mathcal{S}$. Let $\omega_{\mathfrak{D}} \times \sigma_I^\vee \subset \tilde{M}_{\mathbb{Q}}$ be the weight cone of the invariant open subset $X(\mathfrak{D}) \times U_{\sigma_I}$ of $\tilde{X}$ described as a $\tilde{T}$-variety and observe that $\pi(\omega_{\mathfrak{D}} \times \sigma_I^\vee) = M$. For $u = 0 \in M$, define (compare [IV, section 5.1]).

$$\square_u = \tau\Big(\pi^{-1}(u) \cap \tilde{\omega}\Big) \qquad \text{and} \qquad \Psi_u \colon \square_u \longrightarrow \mathrm{CaDiv}_{\mathbb{Q}}(Y)$$

by $\Psi_u(u') = \mathfrak{D}(u' + \sigma^*(u))$. Each Ψ_u is a divisorial polyhedron. For each domain of linearity $\tilde{\omega}_i = \omega_{\mathfrak{D}i} \times \sigma_I \subset \omega_{\mathfrak{D}} \times \sigma_I$ of $\mathfrak{D} \times \sigma_I$, choose $u_i = 0 \in \operatorname{rel\,int} \pi(\tilde{\omega}_i) = M$ in the relative interior of the image. Suppose $\mathcal{S}_{\mathfrak{D} \times \sigma_I}$ be the divisorial fan corresponding to $\sum \Psi_{u_i}$. Let $\tilde{Y}_{\mathfrak{D} \times \sigma_I}$ be the T-variety $X(\mathcal{S}_{\mathfrak{D} \times \sigma_I})$, See [IV, theroem 5.2]. we take Ξ_p to be the coarsest polyhedral subdivision of $\rho(\Delta_p \times \sigma_I)$ containing subdivision of $\pi(\Delta)$ for any face $\Delta \prec \Delta_p \times \sigma_I$.

Proposition 4.1.1. *([IV, Propositon 5.1])* $\Xi_p = \mathcal{S}_{\mathfrak{D}\times\sigma_I p}$

Before we proceed, let us recall some results from [Zie]. Given a polyhedron Δ, we denote the collection of faces of a polyhedron by $L(\Delta)$.

Lemma 4.1.2. *For polyhedron Δ and cone σ, $L(\Delta \times \sigma) = L(\Delta) \times L(\sigma)$, where $L(\Delta) \times L(\sigma) = \{\Delta' \times \sigma' \,|\, \Delta' \prec \Delta$ and $\sigma' \prec \sigma\}$.*

Proof. Let F be a face of $\Delta \times \sigma \subset N_{\mathbb{Q}} \oplus N'_{\mathbb{Q}}$. Consider the canonical projection maps $P : N_{\mathbb{Q}} \oplus N'_{\mathbb{Q}} \to N_{\mathbb{Q}}$ and $P' : N_{\mathbb{Q}} \oplus N'_{\mathbb{Q}} \to N'_{\mathbb{Q}}$. Then $F = P(F) \times P'(F)$ and $P(F) \prec \Delta$, $P'(F) \prec \sigma$ is evident. □

Lemma 4.1.3. *For $\Delta_p \in \mathcal{S}_p$ and $\sigma_I \in \Sigma_E$, $L(\rho(\Delta_p \times \sigma_I)) = \{\rho(\Delta) \,|\, \Delta \prec \Delta_p \times \sigma_I)\}$*

Proof. The map $\rho|_{N\times\sigma_I} : N \times \sigma_I \longrightarrow N_E$ is injective. The projection of polyhedra $\rho|_{\Delta_p\times\sigma_I} : \Delta_p \times \sigma_I \longrightarrow \rho(\Delta_p \times \sigma_I)$ is bijective and ρ is linear. From [Zie, Lemma 7.10], inverse image of a face is a face. □

Lemma 4.1.4. *If $\mathcal{S}_p$ is complete polyhedral complex then*

$$\{\rho(\Delta_p \times \sigma_I) \,|\, \Delta_p \in \mathcal{S}_p \text{ and } \sigma_I \in \Sigma_E\}$$

is complete polyhedral complex.

Proof. Suppose $(a, b) \in \tilde{N}$, where $a = (a_1, a_2 \dots a_d) \in N$ and $b = (b_1, \dots b_{Nd})$ then $(a, b) \mapsto b - J(a)$. Given a element $z = (z_1 \dots z_N, z_{N+1}, \dots z_{2N} \dots z_{Nd}) \in N_E$ we choose a_i minimum in the N-tuple $(z_{(i-1)N+1,\dots z_{iN}})$, for $0 \leq i \leq d$. There is $b \in \sigma_I$ for some I, such that $(-a, b) \mapsto z$ (or $z = b - J(-a))$) and choose appropriate $\Delta_p \in \mathcal{S}_p$, such that $-a \in \Delta_p$. □

We are going to glue $X(\mathcal{S}_{\mathfrak{D}\times\sigma_I})$ for $\mathfrak{D} \in \mathcal{S}$ and $\sigma_I \in \Sigma_I$ to construct $\widetilde{Y_C}$. The map ρ is a linear and injective on each $\Delta_p \times \sigma_I$, and faces map to faces (4.1.3). Consider set $P_{\mathfrak{D}\times\sigma_I} = \{p \in Y \,|\, \Delta_p \neq 0, \phi\}$ and $E_{\mathfrak{D}\times\sigma_I} = \sum_{p \in P_{\mathfrak{D}\times\sigma_I}} p$. Then, $\mathcal{S}_{\mathfrak{D}\times\sigma_I}$ is divisorial fan given by the set of intersections of elements of following set

$$C_{\mathfrak{D}\times\sigma_I} = \{\rho(\Delta_p \times \sigma_I) \otimes p + \phi \otimes (E_{\mathfrak{D}\times\sigma_I} - p) \,|\, p \in P_{\mathfrak{D}\times\sigma_I}\}\,.$$

From [IV, Section 4], $\mathcal{S}_{\mathfrak{D}\times\sigma_I}$ is contraction free. Consider divisorial fan $\widetilde{\mathcal{S}_C}$ set of intersection of elements of set

$$C_{\mathcal{S}\times\Sigma_E} = \bigcup_{\mathfrak{D}\times\sigma_I\in\mathcal{S}\times\Sigma_E} C_{\mathfrak{D}\times\sigma_I}$$

and

Notation 4.2. *Let $\widetilde{Y_C} = X(\widetilde{\mathcal{S}_C})$, and $\pi_C : \widetilde{X \times E_T^N} \to \widetilde{Y_C}$ be the canonical good quotient map.*

$$\begin{array}{ccc} \widetilde{X \times E_T^N} & \xrightarrow{r_E} & X \times E_T^N \\ \downarrow{\scriptstyle \pi_C} & & \\ \widetilde{Y_C} & & \\ \downarrow{\scriptstyle \pi} & & \\ Y. & & \end{array}$$

From the discussion above, we get the following lemma.

Lemma 4.1.5. *The divisorial fan $\widetilde{\mathcal{S}_C}$ over Y constructed above, gives us the T_E-variety $\widetilde{Y_C}$ which corresponds to the good quotient of $\widetilde{X \times E_T^N}$ by the action of T.*

Consider the following setup. Let Ξ be a marked fansy divisor over $\mathbb{P}^1$, and C denotes a collection of marked or contracted cones. For Ξ there is a complete divisorial fan $\mathcal{S}$ such that $\Xi_p = \mathcal{S}_p$ for all $p \in \mathbb{P}^1$. Consider complete complexity one T-variety, $X = X(\mathcal{S}) = X(\Xi)$. Consider the divisorial fan

$$\mathcal{S} \times \Sigma_E = \{\mathfrak{D} \times \sigma_I \,|\, \mathfrak{D} \in \mathcal{S} \text{ and } \sigma_I \in \Sigma_E\}$$

where $\mathfrak{D} = \sum \Delta_p \otimes p \in \mathcal{S}$, we defined pp-divisor,

$$\rho(\mathfrak{D} \times \sigma_I) = \sum \rho(\Delta_p \times \sigma_I) \otimes p.$$

Lemma 4.1.6. *The collection $\mathcal{S}_{Y_C} = \{\rho(\mathfrak{D} \times \sigma_I) \mid \mathfrak{D} \in \mathcal{S}$ and $\sigma_I \in \Sigma_E\}$ is a divisorial fan, Moreover it is complete.*

Proof. First, we are going to prove that $\rho(\mathfrak{D} \times \sigma_I)$ is pp-divisor on the curve $\mathbb{P}^1$. Observe that $\mathfrak{D} \times \sigma_I$ is pp-divisor. For the complexity one case we have an equivalent definition of pp-divisor from [AH, Section 2], and we have $\deg(\rho(\mathfrak{D} \times \sigma_I) = \rho(\deg(\mathfrak{D} \times \sigma_I))$. The following conditions hold.

- $\rho(\mathfrak{D} \times \sigma_I) = \sum \rho(\Delta_p \times \sigma_I) \otimes p$, with $\Delta_p \times \sigma_I$ is polyhedron with tail cone $\mathrm{tail}(\mathfrak{D}) \times \sigma_I$ and the sum runs over all disjoint p's.
- Since $\deg(\mathfrak{D} \times \sigma_I)$ is proper subset of $\mathrm{tail}(\mathfrak{D}) \times \sigma_I$, $\deg(\rho(\mathfrak{D} \times \sigma_I))$ is a proper subset of $\rho(\mathrm{tail}(\mathfrak{D}) \times \sigma_I)$.
- For $u \in (\mathrm{tail}(\mathfrak{D}) \times \sigma_I)^\vee$, we have $\mathrm{eval}_u(\deg(\rho(\mathfrak{D} \times \sigma_I))) = \mathrm{eval}_{u \circ \rho}(\deg(\mathfrak{D} \times \sigma_I))$ and some multiple of $\rho(\mathfrak{D} \times \sigma_I)(u) = (\mathfrak{D} \times \sigma_I)(u \circ \rho)$ is principal.

To prove $\mathcal{S}_{Y_C}$ is a divisorial fan, we take two pp-divisors $\rho(\mathfrak{D} \times \sigma_I)$ and $\rho(\mathfrak{D}' \times \sigma_J)$. Since $\sigma_J \succ \sigma_J \cap \sigma_I \prec \sigma_I$, $\mathfrak{D}' \succ \mathfrak{D}' \cap \mathfrak{D} \prec \mathfrak{D}$ and from 4.1.2, 4.1.3 and 4.1.4, we can conclude that

$$\rho(\mathfrak{D} \times \sigma_I) \succ \rho(\mathfrak{D} \times \sigma_I) \cap \rho(\mathfrak{D}' \times \sigma_J) \prec \rho(\mathfrak{D}' \times \sigma_J).$$

From the criterion of completeness and the above lemma, $\mathcal{S}_{Y_C}$ is complete. □

Now consider the description of $\mathcal{S}_{Y_C}$ as a marked fansy divisor $\Xi_{Y_C} = \sum \mathcal{S}_{Y_C,p} \cdot p$ where, $\mathcal{S}_{Y_C,p} = \{\rho(\Delta_p \times \sigma_I) \mid \Delta_p \in \mathcal{S}_p \text{ and } \sigma_I \in \Sigma_E\}$ and marked cones are $C_{Y_C} = \{\sigma \times \sigma_I \mid \sigma \in C\}$.

Lemma 4.1.7. *From above lemma, we have the following diagram*

$$\begin{array}{ccc} \widetilde{X(\mathcal{S}_{Y_C})} & \xrightarrow{r} & X(\mathcal{S}_{Y_C}) \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{P}^1 & & \end{array},$$

where $\widetilde{X(\mathcal{S}_{Y_C})} = \widetilde{Y_C}$.

Proof. Observe that $\widetilde{X(\mathcal{S}_{Y_C})}$ and $\widetilde{Y_C}$ both are contraction free with $\mathcal{S}_{Y_C p} = \widetilde{\mathcal{S}_{C p}}$. Here the varieties are contraction free, means that the corresponding divisorial fans are contraction free. Then, it is enough to prove that they have the same slices. Hence, by [AHS, Proposition 1.6], $\widetilde{X(\mathcal{S}_{Y_C})} = \widetilde{Y_C}$. $\square$

Notation 4.3. $Y_C := X(\mathcal{S}_{Y_C})$

Remark 4.1.8. *Note that* $X \times E_T^N$ *is a complexity one* T*-variety under the action of* $T \times T_E$*, but not complete. But* $\widetilde{Y_C}$ *is complete complexity one* T*-variety.*

Let G be a reductive group acting on an affine variety $X = \operatorname{Spec}(A)$. Then the ring of G-invariant A^G is finitely generated $\mathbb{C}$-algebra. In our case $G = T$, and action is free hence separated. Therefore the canonical map

$$X \to \operatorname{Spec}(A^T)$$

is geometric quotient, where A^T is the ring of T invariants.

Proposition 4.1.9. *Consider the morphism of pp-divisors* $\mathfrak{D} \times \sigma_I \to \rho(\mathfrak{D} \times \sigma_I)$ *given by the triple* $(i, \rho, 1)$ *where* i *is the identity morphism on* $\mathbb{P}^1$*, and 1 is the unit plurifunction. The map* $q : X \times E_T^N \to Y_C$ *induced by the morphism of pp-divisors is a geometric quotient.*

Proof. The morphism $(i, \rho, 1)$ gives the following ring homomorphism

$$q^{\#}_{\mathfrak{D} \times \sigma_I} : \bigoplus_{u \in \rho(\sigma \times \sigma_I)^\vee} \Gamma(\mathbb{P}^1, \mathcal{O}(\rho(\mathfrak{D} \times \sigma_I)(u))) \to \bigoplus_{v \in (\sigma \times \sigma_I)^\vee} \Gamma(\mathbb{P}^1, \mathcal{O}(\mathfrak{D} \times \sigma_I)(v)).$$

where

$$\Gamma(\mathbb{P}^1, \mathcal{O}(\rho(D))(u)) \to \Gamma(\mathbb{P}^1, \mathcal{O}(D(u \circ \rho)))$$

is the identity map, because $\min\langle \rho(\Delta_P \times \sigma_I), u\rangle = \min\langle \Delta_p \times \sigma_I, u \circ \rho\rangle$. The map $q^{\#}_{\mathfrak{D} \times \sigma_I}$

is the inclusion map, in fact

$$\bigoplus_{u \in \rho(\sigma \times \sigma_I)^\vee} \Gamma(\mathbb{P}^1, \mathcal{O}(\rho(\mathfrak{D} \times \sigma_I)(u)) = \bigoplus_{v \in (\sigma \times \sigma_I)^\vee} \Gamma(\mathbb{P}^1, \mathcal{O}(\mathfrak{D} \times \sigma_I)(v))^T.$$

As T acts freely on $X \times E_T^N$, the induced map $q_{\mathfrak{D} \times \sigma_I} : X(\mathfrak{D} \times \sigma_I) \to X(\rho(\mathfrak{D} \times \sigma_I))$ is a geometric quotient; hence the map q is a geometric quotient. □

Summarizing the above, we get the following lemma.

Lemma 4.1.10. *The T-variety Y_C constructed above is complete and fits into a diagram.*

$$\begin{tikzcd}
\widetilde{X \times E_T^N} \arrow[r, "r_E"] \arrow[d, "\pi_C"] & X \times E_T^N \arrow[d, "q"] \\
\widetilde{Y_C} \arrow[r] \arrow[d, "\pi"] & Y_C \\
\mathbb{P}^1,
\end{tikzcd}$$

where r_E is a T-equivariant birational proper morphism, and the maps q and π_C are geometric quotients.

4.2 Torus equivariant Chow groups of T-varieties

We continue with the above setup, with $\mathrm{tail}(\mathcal{S}) = \Sigma$. Lets us recall the set P of points $p \in \mathbb{P}^1$, such that slice $\mathcal{S}_p \neq \Sigma$. We also ensured that the size of P is at least 2 by appending extra points. For a non-negative integer $k \leq d+1$, let us recall the following sets

- R_k = Cones of dimension $d+1-k$ corresponding to subvarieties not contracted by r.
- V_k = Faces of dimension $d-k$ of polyhedral subdivision corresponding to the fiber of points in P.
- T_k = Cones of dimension $d-k$ corresponding to subvarieties contracted by r.

We obtain the k^{th} Chow group of a rational complete T-variety of complexity one from the following exact sequence

$$K \longrightarrow \mathbb{Z}^{V_k} \oplus \mathbb{Z}^{R_k} \oplus \mathbb{Z}^{V_k} \longrightarrow A_k(X) \longrightarrow 0 \ .$$

where arrows and lattice K are defined in Theorem 3.1.1. Note that in our case, Y_C is a T-variety defined by a pp-divisor over $\mathbb{P}^1$. Thus it contains an open subset, which is a product of an open subset of $\mathbb{P}^1$ with a torus, and hence Y_C is rational. Using the above result, we will give a presentation of the equivariant Chow group of $X(\mathcal{S})$ or, equivalently Chow group of Y_C. For non negative integer $k \leq Nd+1$, we define the following sets:

- $R'_k = \#$ Faces of dimension k in Σ_E.
- r_k = Cones of dimension $Nd+1-k$ corresponding to subvarities not contracted by r.
- v_k = Faces of dimension $Nd-k$ of polyhedral subdivision corresponding to the fiber of points in P.
- t_k = Cones of dimension $Nd-k$ corresponding to subvarities contracted by r.

The cardinalities of sets r_k, v_k, t_k are given by following numbers:

- $|r_k| = R'_{Nd-d}.|R_k| + R'_{Nd-d-1}.|R_{k-1}| \ldots R'_{Nd-d-(k-1)}.|R_1| + R'_{Nd-d-k}.|R_0|$.
- $|v_k| = R'_{Nd-d}.|V_k| + R'_{Nd-d-1}.|V_{k-1}| \ldots R'_{Nd-d-(k-1)}.|V_1| + R'_{Nd-d-k}.|V_0|$.
- $|t_k| = R'_{Nd-d}.|T_k| + R'_{Nd-d-1}.|T_{k-1}| + \ldots R'_{Nd-d-(k-1)}.|T_1| + R'_{Nd-d-k}.|T_0|$.

4.3 $|r_k| + |v_k| + |t_k|$ is constant

For a rank two toric vector bundle $\mathscr{E}$ on a smooth toric variety X_Σ of dimension d. If $\sigma \in \Sigma$ then $\mathscr{E}|_{U_\sigma} = \mathscr{O}(u^1_\sigma) \oplus \mathscr{O}(u^2_\sigma)$. Consider the following partition of Σ. If $u^1_\sigma = u^2_\sigma$ then we say $\sigma \in H$. If σ intersects both $u^1_\sigma - u^2_\sigma > 0$ $u^1_\sigma - u^2_\sigma < 0$ then $\sigma \in I$. Lastly, if $u^1_\sigma - u^2_\sigma \geq 0$ or $u^1_\sigma - u^2_\sigma \leq 0$ on σ then $\sigma \in J$. From [Nød, Proposition 6.3], for $X = \mathbb{P}(\mathscr{E})$ the cycles R_i, V_i, T_i corresponds bijectively to the following sets

- $R_i \leftrightarrow \Sigma(d-i+1) \cap H$
- $V_i \leftrightarrow (\Sigma(d-i+1) \cap J) \cup (\Sigma(d-i) \cap J) \cup 2(\Sigma(d-i) \cap H)$
- $T_i \leftrightarrow (\Sigma(d-i+1) \cap I) \cup (\Sigma(d-i) \cap J) \cup 2(\Sigma(d-i) \cap I)$

. Particularly,

$$S_i = |R_i| + |V_i| + |T_i| = \begin{cases} \#\Sigma(d-i+1) + 2\#\Sigma(d-i), & \text{if } i < d; \\ \#\Sigma(1) + \#P, & \text{if } i = d; \\ 0, & \text{if } i > d; \end{cases} \tag{4.4}$$

and

$$S'_i = \begin{cases} R'_{Nd-d-i} = \Sigma_T(Nd-d-i), & \text{if } i \leq Nd-d; \\ 0, & \text{if } i > Nd-d. \end{cases} \tag{4.5}$$

From above calculation, we can state the following result.

Proposition 4.3.1. *For any rank two toric vector bundle $\mathscr{E}$ on a smooth toric variety X_Σ we have $X = P(\mathscr{E})$ and $X_{\mathscr{E}} = X \times E_T^N/T$, The numbers $|r_k|$, $|v_k|$, and $|t_k|$ are associated with $X_{\mathscr{E}}$ then*

$$|r_k| + |v_k| + |t_k| = \sum_{i=0}^{i=k} S'_i S_{k-i}.$$

Proof. Follows from 4.4 and 4.5 and [Nød, Proposition 6.3] □

Example 4.3.2. *Consider the example [Nød, Example 5.3], for vector bundles $\mathscr{E}$ and $\mathscr{F}$ from [Nød, Remark 6.5] S_j are independent of $\mathscr{E}$ or $\mathscr{F}$. We also fix a large enough value of N, then we have an integer $|r_k| + |v_k| + |t_k|$ is independent of $\mathscr{E}$ or $\mathscr{F}$. Note that corresponding to $\mathscr{E}$, we have space $\mathbb{P}(\mathscr{E}) \times (\mathbb{A}^N \setminus \{0\})^d/T$, and for $\mathscr{F}$, we have $\mathbb{P}(\mathscr{F}) \times (\mathbb{A}^N \setminus \{0\})^d/T$. Consider $\mathbb{P}^2$ with the torus action induced by the fan given by the rays $\rho_1 = (1,0)$,$\rho_2 = (0,1)$, and $\rho_0 = (-1,-1)$ and denote the associated divisors by D_i, for $i = 1, 2, 0$, respectively. Consider rank two vector bundles given by $\mathscr{E} = D_1 \oplus 0$ and $\mathscr{F} = (D_1 + D_2) \oplus D_0$. Then we have that $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{F})$ are complexity one T-varieties of dimension 3. Also $N = 3$, $d = 2$. By [Nød, Proposition 6.3], we have the following table for $\mathbb{P}(\mathscr{E})$(See Section 2.5).*

$u^1_{\sigma_0} = (1,0)$	$u^2_{\sigma_0} = (0,0)$	$u^1_{\sigma_0} - u^2_{\sigma_0} \in J$
$u^1_{\sigma_1} = (0,0)$	$u^2_{\sigma_1} = (0,0)$	$u^1_{\sigma_1} - u^2_{\sigma_1} \in H$
$u^1_{\sigma_2} = (1,-1)$	$u^2_{\sigma_2} = (0,0)$	$u^1_{\sigma_2} - u^2_{\sigma_2} \in J$
$u^1_{\rho_1} = (1,0)$	$u^2_{\rho_1} = (0,0)$	$u^1_{\rho_1} - u^2_{\rho_1} \in J$
$u^1_{\rho_2} = (0,0)$	$u^2_{\rho_2} = (0,0)$	$u^1_{\rho_2} - u^2_{\rho_2} \in H$
$u^1_{\rho_0} = (0,0)$	$u^2_{\rho_0} = (0,0)$	$u^1_{\rho_0} - u^2_{\rho_0} \in H$

Similarly, for $\mathbb{P}(\mathscr{F})$

$u^1_{\sigma_0} = (1,1)$	$u^2_{\sigma_0} = (0,0)$	$u^1_{\sigma_0} - u^2_{\sigma_0} \in J$
$u^1_{\sigma_1} = (-1,1)$	$u^2_{\sigma_1} = (-1,0)$	$u^1_{\sigma_1} - u^2_{\sigma_1} \in J$
$u^1_{\sigma_2} = (1,-1)$	$u^2_{\sigma_2} = (0,-1)$	$u^1_{\sigma_2} - u^2_{\sigma_2} \in I$
$u^1_{\rho_1} = (1,0)$	$u^2_{\rho_1} = (0,0)$	$u^1_{\rho_1} - u^2_{\rho_1} \in J$
$u^1_{\rho_2} = (0,1)$	$u^2_{\rho_2} = (0,0)$	$u^1_{\rho_2} - u^2_{\rho_2} \in J$
$u^1_{\rho_0} = (0,0)$	$u^2_{\rho_0} = (-1,0)$	$u^1_{\rho_0} - u^2_{\rho_0} \in J$

Consider the following tables for $\mathscr{E}$ *and* $\mathscr{F}$*, the numbers* $|R_k|, |V_k|, |T_k|$ *are given below.*

	$\lvert R_2\rvert$	$\lvert V_2\rvert$	$\lvert T_2\rvert$	$\lvert R_1\rvert$	$\lvert V_1\rvert$	$\lvert T_1\rvert$	$\lvert R_0\rvert$	$\lvert V_0\rvert$	$\lvert T_0\rvert$
$\mathbb{P}(\mathscr{E})$	2	3	0	1	7	1	0	4	2
$\mathbb{P}(\mathscr{F})$	0	5	0	0	5	4	0	2	4

	$\lvert R_2\rvert + \lvert V_2\rvert + \lvert T_2\rvert$	$\lvert R_1\rvert + \lvert V_1\rvert + \lvert T_1\rvert$	$\lvert R_0\rvert + \lvert V_0\rvert + \lvert T_0\rvert$
$\mathbb{P}(\mathscr{E})$	5	9	6
$\mathbb{P}(\mathscr{F})$	5	9	6

To obtain the table that demonstrates the values of $|r_1|$, $|v_1|$, $|t_1|$, $|r_2|$, $|v_2|$ *and* $|t_2|$ *consider the following table for* E^3_2*. The following table demonstrates the values of* R'_i*, for* $i = 0, 1, 2, 3, 4$

	R'_4	R'_3	R'_2	R'_1	R'_0
E^3_2	9	18	15	6	1

Consider the following formulas

- $|r_2| = R'_4|R_2| + R'_3|R_1| + R'_2|R_0|$
- $|v_2| = R'_4|v_2| + R'_3|v_1| + R'_2|v_0|$
- $|t_2| = R'_4|T_2| + R'_3|T_1| + R'_2|T_0|$

  and
- $|r_2|+|v_2|+|t_2| = R'_4(|R_2|+|V_2|+|T_2|)+R'_3(|R_1|+|V_1+|T_1|)+R'_2(|R_0|+|V_0|+|T_0|)$.

Also we have

- $|r_1| = R'_4|R_1| + R'_3|R_0|$
- $|v_1| = R'_4|V_1| + R'_3|V_0|$
- $|t_1| = R'_4|T_1| + R'_3|T_0|$

  and
- $|r_1| + |v_1| + |t_1| = R'_4(|R_1| + |V_1| + |T_1|) + R'_3(|R_0| + |V_0| + |T_0|)$.

Similarly,

- $|r_0| = R'_4|R_0|$
- $|v_0| = R'_4|V_0|$
- $|t_0| = R'_4|T_0|$

  and

  $|r_0| + |v_0| + |t_0| = R'_4(|R_0| + |V_0| + |T_0|)$

From above formulas and tables, we have the following table for $(\mathbb{P}(\mathscr{E}) \times E_2^3)/T$ *and* $(\mathbb{P}(\mathscr{F}) \times E_2^3)/T$

	$\lvert r_2\rvert$	$\lvert v_2\rvert$	$\lvert t_2\rvert$	$\lvert r_1\rvert$	$\lvert v_1\rvert$	$\lvert t_1\rvert$	$\lvert r_0\rvert$	$\lvert v_0\rvert$	$\lvert t_0\rvert$
$(\mathbb{P}(\mathscr{E}) \times E_2^3)/T$	*36*	*213*	*48*	*9*	*135*	*45*	*0*	*36*	*18*
$(\mathbb{P}(\mathscr{F}) \times E_2^3)/T$	*0*	*165*	*132*	*0*	*81*	*108*	*0*	*18*	*36*

	$\|r_2\|$ + $\|v_2\|$ + $\|t_2\|$	$\|r_1\|$ + $\|v_1\|$ + $\|t_1\|$	$\|r_0\|$ + $\|v_0\|$ + $\|t_0\|$
$(\mathbb{P}(\mathscr{E}) \times E_2^3)/T$	*36+ 213+ 48=297*	*9+ 135 + 45=189*	*0+ 36 +18=54*
$(\mathbb{P}(\mathscr{F}) \times E_2^3)/T$	*0+ 165 + 132=297*	*0+81 + 108=189*	*0+ 18 +36=54*

5

GIT constructed pp-divisor

In this chapter, we prove that, given an affine T-variety X and T' subtorus of T the (GIT constructed) pp-divisor constructed by Altmann and Hausen is T/T'-invariant. In short there exist a T/T'-variety Y' and torus invariant pp-divisor D' on Y' such that $X(D') \cong X$. We also describe GIT data associated to X with T' action.

5.1 GIT quotients

This section will provide a recapitulation of key findings in the Geometric Invariant Theory of T-varieties, drawing from the works of [AH] and [BH], which we need later. Consider the following setup. Let M be a finite rank lattice, and let A be an integral, finitely generated, M-graded $\mathbb{C}$-algebra

$$A = \bigoplus_{u \in M} A_u.$$

Consider an affine variety $X = \operatorname{Spec}(A)$ with an action of the algebraic torus $T = \operatorname{Spec}(\mathbb{C}[M])$, induced by the M-grading on A. Let L be the trivial line bundle on X with the following torus action:

$$t \cdot (x, c) = (t \cdot x, \chi^u(t) \cdot c),$$

where χ^u is the character corresponding to $u \in M$. Now consider the canonical projection $L \to X$. This map is a torus equivariant map, and is the T-linearization of the trivial line bundle on X with respect to χ^u.

Definition 5.1.1 ([MFK]). A T-linearization of a line bundle on X is a line bundle $L \to X$ along with a fiberwise linear T-action on L such that the projection map is a torus equivariant.

Remark 5.1.2 ([Bri]). *Any T-linearization of a trivial line bundle over X is the linearization corresponding to the unique character, described above.*

Remark 5.1.3. *An invariant section of the linearization of a trivial line bundle with respect to χ^u is precisely an element of some A_{nu} for $n > 0$.*

Definition 5.1.4 (Semistable Points). The set of semistable points associated to a linearization of a trivial line bundle is denoted by $X^{ss}(u)$ and defined as:

$$X^{ss}(u) := \bigcup_{f \in A_{nu},\, n \in \mathbb{Z}_{>0}} X_f.$$

If two linearized line bundles have same set of semistable points, we say that they are *GIT-equivalent*. We recall the description of the GIT-equivalence classes given by a linearization of the trivial line bundle in terms of the orbit cones. We will illustrate this by an example of an affine toric variety. The following definitions are from [BH].

Definition 5.1.5. Consider a point $x \in X$. The *orbit monoid* associated to $x \in X$ is a submonoid $S_T(x) \subset M$ consisting of all $u \in M$ that admit an $f \in A_u$ with $f(x) \neq 0$. A convex cone generated by $S_T(x)$ is called the *orbit cone*, denote it by $\omega_T(x)$. The sublattice generated by the orbit cone $\omega_T(x)$ is called the *orbit lattice*, denote it by $M(x)$.

Definition 5.1.6. The weight cone $\omega \subset M_{\mathbb{Q}}$ is a cone generated by $u \in M$ with $A_u \neq \{0\}$

Definition 5.1.7 (GIT-cone). The GIT-cone associated to an element $u \in \omega \cap M$ is the intersection of all orbit cones containing u, and is denoted by $\lambda(u)$. The collection of GIT-cones forms a fan and called a *GIT-fan*.

For the sake of brevity, we summarized all this data associated with a torus action on an affine variety as follows.

Definition 5.1.8. Suppose X is a T-variety with the action of a torus T. The *GIT-data* associated with (X, T) consists of orbit monoids, orbit cones, orbit lattices, GIT-cones, and set of semistable points.

From [BH, Proposition 2.9], we have an order-reversing one-to-one correspondence between the possible sets of semistable points induced by a linearization of the trivial line bundle and GIT-cones. Consider an affine toric variety $X = \mathrm{Spec}(A)$ with a character lattice M, and dual lattice N. Note :

$$A = \bigoplus_{u \in \sigma^\vee \cap M} \mathbb{C} \cdot \chi^u$$

where σ is the polyhedral cone in N, and $\sigma^\vee$ is its dual. An element $u \in M$ is *saturated* if $A_{(u)} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nu}$ generated by degree one elements. The cone $\sigma^\vee$ is a full dimensional cone, and each $u \in \sigma^\vee$ is saturated. Using 5.1.4, we will compute the $X^{ss}(u)$. Consider a minimal generating set $\{u_1, u_2 \dots u_k\}$ of a semi-group $\sigma^\vee \cap M$. For $u \in \sigma^\vee \cap M$,

$$X^{ss}(u) = \mathrm{Spec}(A_{\chi^u})$$

. If $u = \alpha_1 \cdot u_1 + \alpha_2 \cdot u_2 \cdots + \alpha_k \cdot u_k$ with $\alpha_i \geq 0$ then

$$X^{ss}(u) = \mathrm{Spec}(A_{\chi^u}) = \bigcap_{\alpha_i \neq 0} \mathrm{Spec}(A_{\chi^{u_i}}).$$

5.1.1 Description of GIT-fan and Semistable points.

Lemma 5.1.9 (Collection of all semistable points)**.** *Let $\{u_1, u_2 \dots u_k\}$ be a minimal generating set of a semi-group $\sigma^\vee \cap M$. The collection of a semistable points is $\{X^{ss}(u) \mid u = \alpha_1 \cdot u_1 + \alpha_2 \cdot u_2 \cdots + \alpha_k \cdot u_k, \alpha_i = 0, 1\}$.*

Proof. First observe that above collection is a finite set. If $u = \alpha_1 \cdot u_1 + \alpha_2 \cdot u_2 \cdots + \alpha_k \cdot u_k$ then

$$X^{ss}(u) = X^{ss}(\sum_{\alpha_i \neq 0} u_i) = \bigcap_{\alpha_i \neq 0} X^{ss}(u_i)$$

. □

From [BH, Proposition 2.9], we are going to compute GIT-cone for each $X^{ss}(u)$. To do this we are going to compute $X^{ss}(u_i)$ for each $i \in \{1, 2 \ldots, k\}$. Consider the following poset,

$$\left(S = \left\{ \sum_{i=1}^{k} \alpha_i \cdot u_i \,\middle|\, \alpha_i = 0, 1 \right\}, \geq \right)$$

where, for $v, w \in S$, $v \geq w$ if $X^{ss}(v) \subset X^{ss}(w)$.

Lemma 5.1.10 (Collection of all GIT-cones)**.** *With the above notations continuing, for $v \in S$, GIT-cone $\lambda_T(v)$ is generated by the subset $\{u_i \,|\, v \geq u_i\}$ of $\{u_i \,|\, 1 \leq i \leq k\}$.*

Proof. The cone generated by set $\{u_i \,|\, v \geq u_i\}$ is denoted by $\sigma_T(v)$. First, observe that if $X^{ss}(v) \subset X^{ss}(u_i)$, then for $x \in X$, $\chi^v(x) \neq 0$ if and only if $\forall\ u_i \leq v$, $\chi^{u_i}(x) \neq 0$. Hence $u_i \in \omega_T(x)$ for all x such that $\chi^v(x) \neq 0$ hence $u_i \in \lambda_T(v)$, so $\sigma_T(v) \subset \lambda_T(v)$. If $u \in \lambda_T(v)$, then $X^{ss}(v) \subset X^{ss}(u) \subset X^{ss}(u_i)$ where u_i is a summand of u hence $v \geq u_i$ hence $u \in \sigma_T(v)$.

□

5.2 Description of a GIT-fan with respect to the action of a subtorus

Consider the following setup. Let X be a normal affine variety with an effective T-action. Consider a subtorus T' of the torus T with canonical action on X. First, we have the following exact sequence from the torus inclusion,

$$0 \longrightarrow M'' \longrightarrow M \xrightarrow{\ i\ } M' \longrightarrow 0\ ,$$

where M and M' are the character lattices corresponding to the tori torus T and T', respectively. The lattice M'' is the kernel of the lattice homomorphism i. If $X = \mathrm{Spec}(A)$, then we have a grading $A = \bigoplus_{u \in M} A_u$ with respect to T and, similarly, for the T' action we have an induced grading $A = \bigoplus_{v' \in M'} A_{v'}$. In addition we have,

$$A_{v'} = \bigoplus_{i(u)=v'} A_u.$$

We wish to compute the GIT-data associated with (X, T') from the GIT-data associated with (X, T) and above exact sequence.

Notation 5.1. *We are using the same notation i for lattice homomorphism and vector space homomorphism.*

Notation 5.2. *We denote the elements of M by letters u, v etc. Elements of M' will be denoted by letters with a $'$: e.g. u', v' etc.*

Proposition 5.2.1. *1. Let ω_T and $\omega_{T'}$ be the weight cones associated with the T action and the T' respectively, then $i(\omega_T) = \omega_{T'}$.*

Consider a point $x \in X$.

2. Let $\omega_T(x)$ and $\omega_{T'}(x)$ be the orbit cones associated with the T and T' action respectively, then $i(\omega_T(x)) = \omega_{T'}(x)$.

3. Let $S_T(x)$ and $S'_T(x)$ be the orbit monoid associated with the T and T' action respectively, then $i(S_T(x)) = S'_T(x)$.

Proof. Note, $A_{v'} = \bigoplus_{i(u)=v'} A_u$ then $A_u \neq 0$ for some u if and only if $A_{v'} \neq 0$. Now the results follows from linearity of i and definition of ω_T (resp. $\omega_{T'}$). The statements for orbit monoids and orbit cones follows similarly □

Proposition 5.2.2. *1. Let $X_T^{ss}(u)$ be the semistable point associated with $u \in M$ and let $X_{T'}^{ss}(v')$ be the semistable point associated with $v' \in M'$, then*

$$X_{T'}^{ss}(v') = \bigcup_{\substack{i(u)=nv', \\ n \in \mathbb{Z}_{>0}}} X_T^{ss}(u).$$

Because of the correspondence between the GIT-cones and sets of semistable points, we have the following result.

2. *Let $\lambda_T(u)$ be GIT-cone associated with $u \in M$ (under the T-action) and $\lambda_{T'}(v')$ be GIT-cone associated to $v' \in M'$ (under the T'-action) then*

$$\lambda'(v') = \lambda_{T'}(v') = \bigcap_{i(u)=v'} i(\lambda_T(u)).$$

Proof. The set $X^{ss}_{T'}(v') = \{x \in X \mid f(x) \neq 0 \text{ for some } f \in A_{nv'}\}$, but $A_{nv'} = \bigoplus_{i(u)=nv'} A_u$. If $f(x) \neq 0$ for $x \in X$ and for some $f \in A_{nv'}$ then there is $f_1 \in A_u$ summand of f for some u such that $i(u) = nv'$ and $f_1(x) \neq 0$. Hence

$$X^{ss}_{T'}(v') \subset \bigcup_{i(u)=nv'\ n\in\mathbb{Z}_{>0}} X^{ss}_T(u).$$

Similarly, one can prove $X^{ss}_{T'}(v') \supset \bigcup_{i(u)=nv'\ n\in\mathbb{Z}_{>0}} X^{ss}_T(u)$.

We have $i(\omega_T(x)) = \omega_{T'}(x)$, hence from definition of GIT-cone the statement 2 is evident. □

Example 5.2.3. *Lets take $\sigma = \mathrm{Cone}(\{\bar{e}_1, \bar{e}_2, \bar{e}_1 + \bar{e}_3, \bar{e}_2 + \bar{e}_3\}) \subset \mathbb{Z}^3$, then*

$$\mathbb{C}[S_\sigma] = \bigoplus_{u\in\sigma^\vee\cap M} \mathbb{C}\cdot\chi^u = \mathbb{C}[u, v, w, uvw^{-1}] \equiv \frac{\mathbb{C}[x, y, z, w]}{\langle xy - zw\rangle}$$

where $\sigma^\vee = \mathrm{Cone}(\{e_1, e_2, e_3, e_1 + e_2 - e_3\}) \subset \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z})$. For this example we have GIT-fans shown in the figure 5.1 and semistable points correspondence,

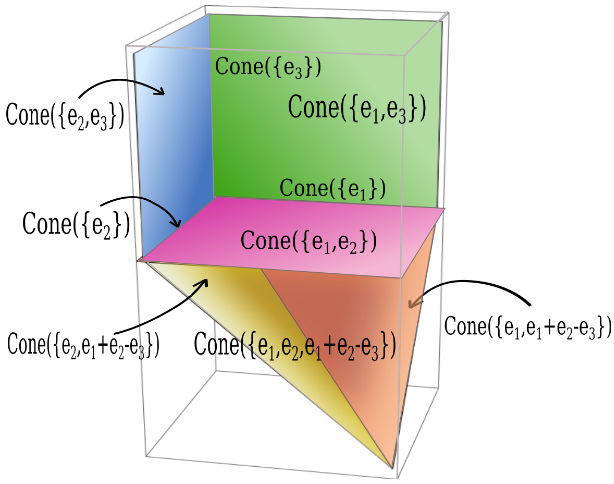

Figure 5.1: $(\mathbb{C}^*)^3$ action

$$
\begin{aligned}
X^{ss}(e_1) &\longleftrightarrow \mathrm{Cone}(\{e_1\})\\
X^{ss}(e_2) &\longleftrightarrow \mathrm{Cone}(\{e_2\})\\
X^{ss}(e_3) &\longleftrightarrow \mathrm{Cone}(\{e_3\})\\
X^{ss}(e_1+e_2) = X^{ss}(e_1+e_2+e_3+e_1+e_2-e_3) &\longleftrightarrow \mathrm{Cone}(\{e_1,e_2\})\\
X^{ss}(e_1+e_2-e_3) = X^{ss}(e_1+e_2+e_1+e_2-e_3) &\longleftrightarrow \mathrm{Cone}(\{e_1,e_2,e_1+e_2-e_3\})\\
X^{ss}(e_1+e_1+e_2-e_3) &\longleftrightarrow \mathrm{Cone}(\{e_1,e_1+e_2-e_3\})\\
X^{ss}(e_2+e_1+e_2-e_3) &\longleftrightarrow \mathrm{Cone}(\{e_2,e_1+e_2-e_3\})\\
X^{ss}(e_1+e_3) &\longleftrightarrow \mathrm{Cone}(\{e_1,e_3\})\\
X^{ss}(e_2+e_3) &\longleftrightarrow \mathrm{Cone}(\{e_2,e_3\}).
\end{aligned}
$$

Consider the above, the torus inclusion map $(\mathbb{C}^*)^2 \to (\mathbb{C}^*)^3$ is given by the following map

$$(t_1,t_2) \mapsto (t_1,t_2,t_1).$$

The lattice homomorphism associated with this inclusion is the map $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^3,\mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2,\mathbb{Z})$ is

$$(a,b,c) \mapsto (a+c,b).$$

From Proposition 5.2.2, The GIT-cones are shown in the figure 5.2 and semistable points

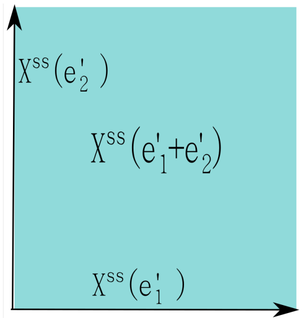

Figure 5.2: $(\mathbb{C}^*)^2$ action

correspondence, $e_1' = (1,0)$ and $e_2' = (0,1)$ are in $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, \mathbb{Z})$

$$X^{ss}(e_1') = X^{ss}(e_1) \cup X^{ss}(e_1+e_3) \cup X^{ss}(e_3) \longleftrightarrow \mathrm{Cone}(\{e_1'\})$$
$$X^{ss}(e_2') = X^{ss}(e_2) \cup X^{ss}(e_1+e_2-e_3) \cup X^{ss}(e_2+e_1+e_2-e_3) \longleftrightarrow \mathrm{Cone}(\{e_2'\})$$
$$X^{ss}(e_1'+e_2') = X^{ss}(e_1+e_2) \cup X^{ss}(e_1+e_1+e_2-e_3) \cup X^{ss}(e_1+e_3) \longleftrightarrow \mathrm{Cone}(\{e_1', e_2'\}).$$

Now, we shall briefly recall the proof of the Theorem 2.2.15. Let X be an affine T-variety, let W_λ be the set of semistable points corresponding to the GIT cone λ. Let Y_λ be the good qoutiont of W_λ is by the action of T. The quotient space Y_λ is given by,

$$Y_\lambda = \mathrm{Proj}(\bigoplus_{n\in\mathbb{Z}_{\geq 0}} A_{nu}), \text{ where } u \in \mathrm{relint}(\lambda).$$

Let Σ be the GIT fan, and $\lambda, \gamma \in \Sigma$, from [AH, Theorem 5.4], if $\gamma \subset \lambda$ then $W_\lambda \subset W_\gamma$, and the converse is also true. Hence we have the following commutative diagram,

$$\begin{tikzcd}
W \arrow[r, "j_\lambda"] \arrow[d, "q"] & W_\lambda \arrow[r, "j_{\lambda\gamma}"] \arrow[d, "q_\lambda"] & W_\gamma \arrow[r, "j_{\lambda 0}"] \arrow[d, "q_\gamma"] & X = W_0 \arrow[dd, "q_0"] \\
Y \arrow[r, "p_\lambda"] \arrow[drrr, "p_0"] & Y_\lambda \arrow[r, "p_{\lambda\gamma}"] \arrow[drr, "p_{\lambda 0}"] & Y_\gamma \arrow[dr, "p_{\gamma 0}"] & \\
 & & & Y_0
\end{tikzcd}$$

The maps j_{--} are inclusion maps, and $W := \varprojlim W_\lambda = \bigcap_{\lambda\in\Gamma} W_\lambda$. The maps q_- are quotient maps. Let Y' be the inverse limit of the induced maps $p_{\lambda\gamma}$ between the quotients space. and let $q' : W \to Y'$ be the inverse limit of the quotient maps(q_λ). Let Y be the normalization of $\overline{q'(W)}$, and q is an induced morphism. If T acts effectively on $X = \mathrm{Spec}(A)$ the weight cone ω is a full dimensional cone. Hence we have a choice of homomorphism

$$s : M \to Q(A)^*, \text{ such that } s(u) \text{ is degree } u \text{ element.}$$

For each saturated $u \in \omega \cap M$, let us consider a GIT-cone $\lambda \in \Sigma$ such that $u \in \mathrm{relint}(\lambda)$. Consider an open cover $\{Y_{\lambda,f} = \mathrm{Spec}([A_f]_0) \,|\, f \in A_u\}$ for Y_λ, hence we have an open cover $\left\{Y_f = p_\lambda^{-1}(Y_{\lambda,f}) \,\middle|\, f \in A_u\right\}$ for Y. Now for each u. Consider the Cartier divisor $\mathfrak{D}(u)$ defined by the local equation $(Y_f, s(u)/f)$. if $u \in \omega$ then there is saturated multiple nu and

$$\mathfrak{D}(u) = \frac{1}{n}\mathfrak{D}(nu).$$

We have the following pp-divisor $\mathfrak{D}$,

$$\omega \to \mathrm{CaDiv}_{\mathbb{Q}}(Y), \; u \mapsto \mathfrak{D}(u)$$

such that $X(\mathfrak{D}) \cong X$.

5.3 Description of the space associated to a T- variety with respect to the action of a subtorus

By [AH], we have a proper polyhedral divisor associated with a normal affine variety with an effective torus action. Let X be an affine T-variety, with a torus T. Let $\mathfrak{D}$ be a pp-divisor on (Y, N), where N is a dual lattice of the lattice $M = \mathrm{Hom}(T, \mathbb{C}^*)$, such that $X \cong X(\mathfrak{D}) = \mathrm{Spec}(A)$. We assume that $\mathfrak{D}$ is a minimal pp-divisor([AH, Definition 8.7]). For a subtorus T' of the torus T, we prove that their exist a base space Y' with an effective T/T' action and a $\frac{T}{T'}$-invariant pp-divisor on (Y', N'), where N' is a dual of a lattice $M' = \mathrm{Hom}(T', \mathbb{C}^*)$.

Theorem 5.3.1. *Consider an affine T-variety X with the action of the torus T, with* *weight cone* $\omega \subset M_{\mathbb{Q}}$. *Let* T' *be a subtorus of* T*, and the associated lattice map be* $i : M \to M'$. *Then, there exists* $\frac{T}{T'}$*-variety* Y' *and a* $\frac{T}{T'}$*-invariant pp-divisor* $\mathfrak{D}'$ *on* (Y', N') *with tail*$(\mathfrak{D}') = i(\omega)^\vee$ *such that* $X(\mathfrak{D}') \cong X$.

The next part of this paper is about the proof of the Theorem 5.3.1. Consider the semistable point $X^{ss}_{T'}(v')$, from [AH, Section 5], the quotient space
$Y_{v'} = X^{ss}_{T'}(v') /\!\!/ T'$ is given by

$$Y_{v'} = \mathrm{Proj}(A_{(v')}),$$

where $A_{(v')} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{nv'}$.

Proposition 5.3.2. *There is effective* $\frac{T}{T'}$ *action on* $Y_{v'}$

Proof. The set of semistable points $X^{ss}_{T'}(v')$ is a T'-invariant open subset of X. Moreover from Proposition 5.2.2, it is T-invariant. On basic open subsets of $Y_{v'}$, the torus $\frac{T}{T'}$ acts effectively. Consider $f \in A_{(v')}$, in particular, we choose $f \in A_u$ for $u \in M$ such that $i(u) = nv'$, for some $n \in \mathbb{Z}_{\geq 0}$. The sets $\mathrm{Spec}([A_f]_0)$ $(0 \in M')$ covers $Y_{v'}$ and it is enough to prove that $\frac{T}{T'}$ acts effectively on each $\mathrm{Spec}([A_f]_0)$, or equivalently, that $[A_f]_0$ $(0 \in M')$ admits an M'' grading such that the weight cone is full dimension. Note that $A_0 = \bigoplus_{u \in M''} A_u$, which induces an M'' grading on $[A_f]_0$. Since $A_0 = \bigoplus_{u \in M\ i(u)=0} A_u \subset [A_f]_0$, and T acts effectively on X, (the dimension of $\mathrm{Cone}(\{u \in M \mid i(u) = 0\}$ is rank of M''). Then, the $\frac{T}{T'}$ action is effective on $\mathrm{Spec}([A_0])$, and hence $\frac{T}{T'}$ acts effectively on $\mathrm{Spec}([A_f]_0)$. □

Using proposition 5.3.2, we are going to prove that there is a $\frac{T}{T'}$ action on Y'. From [BH, Proposition 2.9, Definition 2.8], the collection of all GIT-cones define the GIT-fan, which we denote by $\Sigma_{T'}$. The map $v' \to X^{ss}_{T'}(v')$ is constant on $\mathrm{relint}(\lambda_{T'}(v'))$. So for $\lambda' \in \Sigma_{T'}$ and $w' \in \mathrm{relint}(\lambda')$ if we write $W_{\lambda'} = X^{ss}_{T'}(w')$, $Y_{\lambda'} = Y_{w'}$, we have the following commutative diagram

$$\begin{tikzcd}
W_{T'} \arrow[r, "j'_{\lambda'}"] \arrow[d, "q'"] & W_{\lambda'} \arrow[r, "j'_{\lambda'\gamma'}"] \arrow[d, "q'_{\lambda'}"] & W_{\gamma'} \arrow[r, "j'_{\lambda' 0}"] \arrow[d, "q'_{\gamma'}"] & X = W_0 \arrow[dd, "q'_0"] \\
Y' \arrow[r, "p'_{\lambda'}"] \arrow[drrr, "p'_0"] & Y_{\lambda'} \arrow[r, "p'_{\lambda'\gamma'}"] \arrow[drr, "p'_{\lambda' 0}"] & Y_{\gamma'} \arrow[dr, "p'_{\gamma' 0}"] & \\
 & & & Y_0
\end{tikzcd}$$

All the $j'_{\_\_}$ are inclusion maps, so the inverse limit, $W_{T'}$, is the intersection of sets of semistable points. The Y' is normalization of a canonical component. Let Y_1 is limit $\{Y_{\lambda_{T'}}\}'s$ and

$$Y' = \mathrm{Norm}(\overline{q(W_{T'})}),$$

where Norm(-) denote the normalization. we have the following commutative diagram

$$\begin{tikzcd}
W_{\lambda'} \arrow[r, "j'_{\lambda'\gamma'}"] \arrow[d, "q'_{\lambda'}"] & W_{\gamma'} \arrow[d, "q'_{\gamma'}"] \\
Y_{\lambda'} \arrow[r, "p'_{\lambda'\gamma'}"] & Y_{\gamma'}
\end{tikzcd}$$

from the Proj construction, $q'_{\lambda'}$ and $q'_{\gamma'}$ are torus equivariant maps with respect to the canonical torus map $T \to \frac{T}{T'}$. Also the map $p'_{\lambda'\gamma'}$ is a $\frac{T}{T'}$-equivariant map. Now we shall demonstrate that Y' admits canonical $\frac{T}{T'}$ action. To prove $\frac{T}{T'}$ acts effectively on Y', we required some evident statements which are given below.

Lemma 5.3.3. *Let Y and T' be are two varieties with T actions, the map $v : Y \to Y'$ is a T-equivariant birational map with T acts effectively on Y' then T acts effectively on Y.*

Lemma 5.3.4. *Let Y be a topological space, and $\psi : Y \to Y$ be a continuous map with $A \subset X$ such that $\psi(A) \subset A$, then $\psi(\overline{A}) \subset \overline{A}$.*

Lemma 5.3.5. *Let X be a T-variety, and Norm(X) be its normalization, then Norm(X)*

has a torus action, satisfying the following commutative diagram,

$$\begin{array}{ccc} Norm(X) & \xrightarrow{\bar{t}} & Norm(X) \\ \downarrow g & & \downarrow g \\ X & \xrightarrow{\quad t \quad} & X \end{array}.$$

Lemma 5.3.6. *Consider $q_1 : W_{T'} \to Y_1$ the induced by the commutative diagram 5.3. Then, q_1 is a torus equivariant map, and it defines $\frac{T}{T'}$ action on Y'.*

Proof. From Lemma 5.3.4 and 5.3.5 □

The map $p'_{\lambda'}$ is given by,

$$\mathrm{Norma}(\overline{q(W_{T'})}) \to \overline{q(W_{T'})} \hookrightarrow Y_1 \to Y_{\lambda'}$$

and each arrow is a torus equivariant map.

Proposition 5.3.7. *Y' is a T-variety.*

Proof. We have to prove that Y' is a normal variety and action of $\frac{T}{T'}$-effective. From construction, it is a normal variety and from Lemma 5.3.3 and above arrows 5.3, the action is effective. □

5.3.1 The torus invariant proper polyhedral divisors.

Let M'' be a character lattice associated to $\frac{T}{T'}$. Consider the exact sequence 5.2. Construction of the pp-divisor requires a homomorphism $s' : M' \to Q(A)^*$. Note that $i(\omega)$ is a full dimensional cone, and given a $v' \in i(\omega) \cap M'$, there is a $k \in \mathbb{N}$, such that kv' is saturated. For each $v' \in i(\omega)$ saturated, we will define a Cartier divisor $\mathfrak{D}(v')$. For $v' \in \mathrm{int}(\lambda_{T'})$ saturated, $\left\{Y_{\lambda_{T'},f} \,\middle|\, f \in A_u, \text{where } i(u) = v'\right\}$ is an open cover for $Y_{\lambda_{T'}}$. Consider the open cover $Y'_f = {p'_{\lambda'}}^{-1}(Y_{\lambda_{T'},f})$. Since T acts effectively on X, we have a section $s : M \to Q(A)^*$ such that $s(u)$ is u-homogeneous. Consider the section $s' : M' \to Q(A)^*$ defined:

$$s'(v') = s(u), \text{ For fix } u \in M \text{ such that } i(u) = v'.$$

Now consider the Cartier divisor

$$\mathfrak{D}'(v') = (Y'_f, \frac{s'(i(u))}{f}).$$

Since $p'_{\lambda_{T'}}$ are torus equivariant maps, so Y'_f are torus invariant open subsets. $\frac{s'((i(u))}{f}$ is homogeneous of degree $\deg(s'(i(u))) - \deg(f) \in M''$ (Note that degree of an element $\frac{s'(i(u))}{f}$) is equal to 0 in M'). This defines a torus invariant pp-divisor on Y',

$$\mathfrak{D}' : i(\omega) \to \mathrm{CaDiv}_{\mathbb{Q}}(Y'), \quad \mathfrak{D}'(v') = \frac{1}{k} \cdot \mathfrak{D}'(kv').$$

where kv' is a saturated multiple of v'.

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